Graduate Seminar Report
on
Non-Integer Order Of Differentiation
and
Its Application On The Dirichlet Problem of Laplace Equation

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CONTENTS

Preface

This seminar report basically consists of two chapters. In the first part, the Sobolev and Nikol Skii Besov functions spaces are investigated and some inclusion theorems are developed as well as some relations of the two function spaces are stated as a particular case. As the main objective of the seminar report we formulate the necessary and sufficient condition for the function (which is equal to the solution of the Dirichlet problem on the boundary) on the boundary of some region, for the solution of the generalized formulation of the Dirichlet problem of Laplace equaton to exist. For the classical formulation of the Dirichlet problem, we, as an assumption, take the value of the function on the boundary of some region in which the solution is defined is continuous on the boundary. But, this does not guarantee the existence of the solution of the problem, which is shown by Hadamard's example. Therefore, as a conclusion, we will see that the formulation of the necessary and sufficient condition for the generalized formulation of the Dirichlet problem of Laplace equaton in the second chapter of this seminar report.

Before all, I thank the almighty God with out His help nothing is happened.

I would like to express my special gratitude to my advisor and teacher Dr. Tsegaye Gedif for his unlimited help for the completion of this seminar report. I am really lucky for his being here with me. I would also like to thank all my teachers for their contribution to bring me to this level.

Finally, I thank my parents for all they have given me and for their prayers. Also, I would like to thank W/o Tobiaw Tefera, for typing this manuscript partially.

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Denotations and Some Definitions

The following standard notations are used this seminar report.

N - the set of all natural numbers,

N_0 - the set of all non-negative integers,

Z - the set of all integers,

R- the set of all real numbers,

N_0^n - $\underbrace{N_o \times N_o \cdots \times N_o}_{n}$ - the set of multi-indices (n is the natural number which will be used exclusively to denote the dimension),

R^n - $\underbrace{R \times R \cdots \times R}_{n}$

A number $\alpha \in R$ is called the exact constant for a given proposition p if and only if there is no a number less than α that satisfies p.

$L_\theta^*(0,H)$ with $1 \le \theta < \infty$, denotes the space of functions g of one variable , measurable on (0, H) , for which

$$\|g\|_{L_\theta(0,H)} = \left(\int_0^H |g(h)|^\theta \frac{dh}{h} \right)^{\frac{1}{\theta}} < \infty \quad L_\infty^*(0,H) = L_\infty(0,H)$$

$\approx$ - "equivalent to"

$\dot{\forall}$ - " for almost all"

a.e - " almost every where"

For a function f, $E_h f := f(x+h)$

For $\alpha \in N_0^n$, $\alpha \neq 0$, we write:

$D^\alpha f \equiv \dfrac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$ the (ordinary) derivative of the function f of order α

For an arbitrary nonempty set $\Omega \subset R^n$ we denote the following:

$C(\Omega)$ - the space of functions continuous on Ω.

$C_b(\Omega)$ - the Banach space of functions f continuous and bounded on Ω with the norm

$$\|f\|_{C(\Omega)} = \sup_{x \in \Omega} |f(x)|,$$

$\overline{C}(\Omega)$ - the Banach space of function uniformly continuously and bounded on Ω with the same norm.

For a measurable[1] nonempty set $\Omega \subset R^n$ we shall denote by:

$L_p(\Omega)$ $(1 \le p \le \infty)$ - the Banach space [2] of functions f measurable on Ω such that the norm

$$\|f\|_{L_p(\Omega)} = \left(\int_\Omega | f | \, dx \right)^{\frac{1}{p}} < \infty$$

$L_\infty(\Omega)$ - the Banach space of functions f measurable on Ω such that the norm

$$\|f\|_{L_\infty(\Omega)} = \operatorname{ess\,sup}_{x\in\Omega} | f(x) | = \inf_{w:\, meas\,(w)=0} \; \sup_{x\in\Omega\setminus w} | f(x) | < \infty$$

for the case where measure$(\Omega) \ne 0$

and if measure$(\Omega) = 0$, then we set $\|f\|_{L_\infty(\Omega)} = 0$

We note that if $\Omega \subset R^n$ is an open set, then for $f \in C(\Omega)$ $\|f\|_{C(\Omega)} = \|f\|_{L_\infty(\Omega)}$.

For an open nonempty set $\Omega \subset R^n$ we shall denote by:

$L_p^{loc}(\Omega)$ $(1 \le p \le \infty)$ - the set of functions defined on Ω such that for each

compact $K \subset \Omega$ $f \in L_p(K)$[3],

$C^l(\Omega)$ $(l \in N)$ - the space of functions f defined on Ω such that $\forall \alpha \in N_0^n$ where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n = l$ *and* $\forall x \in \Omega$ *the derivative* $(D^\alpha f)(x)$ exists and $D^\alpha f \in C(\Omega)$,

$C^\infty(\Omega) = \bigcap_{l=0}^{\infty} C^l(\Omega)$ - the space of infinitely continuously differentiable functions on Ω.

For a function f, supp f := the closure of the set $\{x : f(x) \ne 0\}$

$C_o^\infty(\Omega)$ - The set of all infinitely continuosly differentiable functions on Ω such that supp f is a compact set for all $f \in C_o^\infty(\Omega)$.

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[1] "Measurable" means "measurable with respect to Lebesgue measure." All the integrals in this paper are Lebesgue integrals.

[2] As usual when saying a "Banach space" we ignore here the fact that the condition $\|f\|_{L_p(\Omega)} = 0$ is equivalent to the condition $f \sim 0$ on Ω (i.e. f is equivalent to 0 on $\Omega \Leftrightarrow$ measure $\{x \in \Omega \mid f(x) \ne 0\}$ and not to the condition $f = 0$ on Ω. To be strict we can call it a "semi-Banach space".

[3] $f_k \to f$ in $L_p^{loc}(\Omega)$ *as* $k \to \infty$ means that for each compact $K \subset \Omega$ $f_k \to f$ in $L_p(K)$.

1.Priliminaries

In this preliminary part of this mathematical text , we investigate some elementary concepts that are useful in the next chapters. For instance, since the definitions of the Nikol Skii Besov and the Sobolev Spaces are highly based on the L_p Spaces , we discuss about the L_p spaces and their properties in the following section.

1.1: The L_p Spaces

We know from analysis that the collection of all measurable functions specified on an open set $\Omega \subseteq R^n$ for which the norm

$$\|f\|_{L_p(\Omega)} = \begin{cases} \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}} & , 1 \le p < \infty \\ ess \sup_{x \in \Omega} |f(x)| & , \quad p = \infty \end{cases}$$

is finite will be denoted as $L_p(\Omega)$, $1 \le p \le \infty$ and the norm is called the L_p norm of f and the space of these functions is called the L_p space. In this definition , ess sup $|f(x)|$ denotes the minimum real number among the real numbers those are greater than $|f(x)|$ a.e.

We note that L_p spaces are linear spaces. Indeed,

$$\left(\int_\Omega |\alpha f|^p \, dx \right)^{\frac{1}{p}} = \left(|\alpha| \left(\int_\Omega |f|^p \, dx \right)^{\frac{1}{p}} \right) < \infty \text{ whenever } \left(\int_\Omega |f|^p \, dx \right)^{\frac{1}{p}} < \infty$$

Thus, $\alpha f \in L_p(\Omega)$ whenever $f \in L_p(\Omega)$ .And,

$$|f + g| \le (2 \max \{|f|, |g|\}^p = 2^p (\max\{|f|, |g|\})^p \le 2^p (|f|^p + |g|^p)$$

This implies the sum of two functions in L_p is also in L_p.

The Holder and Minkowski Inequality

To prove some theorems about function spaces in the next chapter, as it is mentioned , we make use of some facts in the L_p space ,specially , the Holder and Minkowski Inequalities. So we state and prove this inequalities. We establish the following lemma before proving these inequalities.

Lemma 1.1: Let α and β be non-negative real numbers, and suppose $0 < \lambda < 1$.Then

$$\alpha^{\lambda}\beta^{1-\lambda} \le \lambda\alpha + (1-\lambda)\beta$$

with equality only if $\alpha = \beta$.

Proof: Consider the function h defined for non-negative real numbers t by

h(t)= $(1-\lambda) + \lambda t - t^{\lambda}$

Then h'(t)= $\lambda - \lambda t^{\lambda-1} = \lambda(1 - t^{\lambda-1})$

Since $\lambda - 1 < 0$ we have $h'(t) < 0$ for $t < 1$ and $h'(t) > 0$ for $t > 1$.

Thus for $t \neq 1$, we have $h(t) < h(1) = 0$

Hence, $(1-\lambda) + \lambda t \ge t^{\lambda}$ with equality only for t=1

If $\beta \neq 0$ then by setting $t = \dfrac{\alpha}{\beta}$, we get $(1-\lambda) + \lambda\dfrac{\alpha}{\beta} \ge \left(\dfrac{\alpha}{\beta}\right)^{\lambda}$.

This implies

$\alpha^{\lambda}\beta^{1-\lambda} \le \lambda\alpha + (1-\lambda)\beta$ while if β=0, the lemma is trivial.

Lemma 1.2: If p and q are non-negative extended real numbers such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and if $f \in L_p$ and $g \in L_p$, then $f.g \in L_1$ and

$$\int |fg| \le \|f\|_p \|g\|_q \tag{1.1}$$

Equality holds if and only if for some non-zero constants α and β,we have $\alpha|f|^p = \beta|g|^q$ a.e.

Inequality (1.1) is called Holder inequality.

Proof: In the case p =1 , $q = \infty$,we have

$$\int |fg| = \int |f||g| \le \int |f| ess\ \sup|g|$$

$$= \int |f| \|g\|_\infty$$

$$= \|g\|_\infty \int |f|$$

$$= \|g\|_\infty \|f\|_1$$

Now assume $1 < p < \infty$, consequently $1 < q < \infty$

Let us first suppose that $\|f\|_p = \|g\|_q = 1$

Thus by lemma 1, with $\alpha = |f(t)|^p$, $\beta = |g(t)|^q$

$\lambda = \frac{1}{p}$, $1 - \lambda = \frac{1}{q}$, we have,

$$|f(t)g(t)| \le \lambda |f(t)|^p + (1 - \lambda)|g(t)|^q \tag{1.2}$$

So, Integrating both sides yields

$$\int |f\,g| \le \lambda \int |f|^p + (1 - \lambda) \int |g|^q = 1 \tag{1.3}$$

If $\|f\| = 0$ and $\|g\| = 0$, the inequality is trivial.

So, let f and g be any elements of L_p and L_q with $\|f\| \ne 0$ and $\|g\| \ne 0$.
Then,

$\frac{f}{\|f\|_p}$ and $\frac{g}{\|g\|_q}$ have norm one.

Thus substituting this in (1.3) yields

$$\frac{1}{\|f\|_p \|g\|_q} \int |fg| = \int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \le 1$$

This results ,

$$\int |fg| \le \|f\|_p \|g\|_q$$

Lemma 1.3: If f and g are in L_p , then so is $f + g$

and

$$\|f + g\|_p \le \|f\|_p + \|g\|_p \tag{1.4}$$

Inequality (1.4)is called the Minkowski Inequality.

Proof: The cases $p = 1$ and $p = \infty$ are straight forward hence we assume

$1 < p < \infty$. Since $|f + g|^p \le 2^p\left(|f|^p + |g|^p\right)$have $f + g \in L_p$

Also ,

$$\int|f + g|^p = \int|f + g|^{p-1}|f + g|$$
$$\le \int|f + g|^{p-1}\left(|f| + |g|\right)$$
$$= \int|f + g|^{p-1}|f| + \int|f + g|^{p-1}|g|$$

By the Hölder Inequality , we have

$$\int|f + g|^{p-1}\left(|f|\right) \le \|f\|_p\|(f + g)^{p-1}\|_q$$

$$\int|f + g|^{p-1}\left(|g|\right) \le \|g\|_p\|(f + g)^{p-1}\|_q$$

$$\|(f + g)^{p-1}\|_q = \left(\int(f + g)^{(p-1)q}\right)^{\frac{1}{q}} = \|f + g\|_p^{\frac{p}{q}}$$

Since $p = q(p\text{-}1)$, we have

$$\|f + g\|_p^p \le (\|f\|_p + \|g\|_p)(\|f + g\|_p^{\frac{p}{q}})$$

or

$$\|f + g\|_p \le \|f\|_p + \|g\|_p$$

The proof is complete.

Lemma 1.4 Consider a sequence of real numbers $\{a_k\}^{\infty}{}_{k=1}$, then for $0 < p < q$,we have

$$\left(\sum_{k=1}^{n}|a_k|^q\right)^{\frac{1}{q}} \le \left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{1}{p}} \tag{1.5}$$

Inequality (1.5) is called Jenssen's Inequality.

Proof: Assume $0 < p < q$

Now for $p = q$ the lemma is trivially true.

Thus ,It remains to show $\left(\sum_{k=1}^{n}|a_k|^q\right)^{\frac{1}{q}} \le \left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{1}{p}}$ for $p < q$.

With out loss of generality let $\alpha = \left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{1}{p}} > 0$ because in case $\alpha = 0$,

the lemma holds trivially. Thus, $\alpha^p = \left(\sum_{k=1}^{n}|a_k|^p\right) \geq |a_k|^p$.

This implies

$$\alpha \geq |a_k| \qquad \text{or} \qquad \frac{|a_k|}{\alpha} \leq 1 \ \forall k \in \{1,2,\ldots,n\}.$$

Now we define a function $Z(t) = C^t \ , \ 0 < C \leq 1$

Obviously , Z is not increasing.

Thus we have $Z(q) \leq Z(p)$since $p < q$ for all C such that $0 < C \leq 1$.

In particular , if we consider $Z_1(t)=\left(\frac{|a_k|}{\alpha}\right)^t$, we get

$$\left(\frac{|a_k|}{\alpha}\right)^q \leq \left(\frac{|a_k|}{\alpha}\right)^p \ , \ \forall k \in \{1,2,\ldots,n\}.$$

This implies $\sum_{k=1}^{n}\left(\frac{|a_k|}{\alpha}\right)^q \leq \sum_{k=1}^{n}\left(\frac{|a_k|}{\alpha}\right)^p = \sum_{k=1}^{n}\frac{|a_k|^p}{\alpha^p} = \frac{1}{\alpha^p}\sum_{k=1}^{n}|a_k|^p = \frac{\sum_{k=1}^{n}|a_k|^p}{\sum_{k=1}^{n}|a_k|^p} = 1$

But , $$\sum_{k=1}^{n}\left(\frac{|a_k|}{\alpha}\right)^q = \frac{1}{\alpha^q}\sum_{k=1}^{n}|a_k|^q \leq 1$$

Thus, $$\left(\sum_{k=1}^{n}|a_k|^q\right) \leq \left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{q}{p}}$$

Hence, $$\left(\sum_{k=1}^{n}|a_k|^q\right)^{\frac{1}{q}} \leq \left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{1}{p}}.$$

So, the proof is complete.

Lemma 1.5: Let f be a non-negative measurable function, the there is a non-negative sequence of simple functions $\{\varphi_n\}_{n=1}^{\infty}$ vanishing out side a set of finite measure such that

1) $\varphi_1 \leq \varphi_2 \leq \ldots . \leq$
2) $\varphi_n(x) \longrightarrow f$ as $n \longrightarrow \infty$

Proof: For each positive integer n, we define sets

$$E_{nk} = \{x : (k-1)2^{-n} \leq f(x) \leq k2^{-n} \qquad k = 0, 1, 2, \ldots\ldots, n2^n$$

Since f is measurable E_{nk}s are measurable.

Moreover, E_{nk}s are pair wise disjoint.

Now, we define $\varphi_n(x)$ as follow.

$$\varphi_n(x) = \begin{cases} 2^{-n}\sum_{k=1}^{n2^n}(k-1)\chi_{E_{nk}}(x) & , \quad 0 \leq f(x) \leq n \\ n & , \quad f(x) > n \end{cases}$$

we show φ_ns are increasing.

i.e. $\varphi_n(x) \leq \varphi_{n+1}(x)$ for all x.

If $f(x) \leq n$, then there is k such that $(k-1)2^{-n} \leq f(x) \leq k2^{-n}$

Now, $[(k-1)2^{-n}, k2^{-n}] = [\,2(k-1)2^{-n+1}, (2k+1)\,2^{-n+1}) \cup$

$$[(2k+1)\,2^{-n+1}, (2k+2)2^{-(n+1)}]$$

Thus, if $f(x) \in [\,2(k-1)2^{-n+1}, (2k+1)\,2^{-n+1})$, then $\varphi_n(x) = (k-1)2^{-n}$

and $\varphi_{n+1}(x) = 2(k-1)2^{-n+1} = (k-1)2^{-n}$

Therefore, $\varphi_n(x) = \varphi_{n+1}(x)$

If $f(x) \in [(2k+1)\,2^{-n+1}, (2k+2)2^{-(n+1)}]$, then $\varphi_{n+1}(x) = (2k+1)\,2^{-(n+1)}$

$$\geq (k-1)2^{-n}$$

$$= \varphi_n(x)$$

Thus, $\quad \varphi_{n+1}(x) \geq \varphi_n(x)$

If $f(x) > n$, then $\varphi_n(x) = n$, Clearly, $\varphi_{n+1}(x) \geq n$

If $f(x) = \infty$, then $\varphi_n(x) = n$ for all $n \in N$

Therefore, $\varphi_n(x)$ approaches infinity as n goes to infinity.

or ,

$\varphi_n(x)$ approaches $f(x)$ as n goes to infinity.

If $f(x) < \infty$, then there is $m \in N$ such that $0 \leq f(x) \leq m$

This implies $0 \leq f(x) \leq n$ for all $n \geq m$

Thus, $0 \leq f(x) - \varphi_n(x) \leq 2^{-n}$ for all $n \geq m$

Letting n sufficiently large, we have $f(x) \longrightarrow \varphi_n(x)$

Therefore, we have the lemma.

Lemma1.6: Let f be an integrable function on $(-\infty, \infty)$. Then

$$\int f(x)dx = \int f(x+h)dx \qquad (1.6)$$

Proof:

Case i: $f(x) = \chi_E(x)$ where E is measurable and $mE < \infty$

f is integrable. Then $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \chi_E(x)dx = mE$

and $\int_{-\infty}^{\infty} f(x+h)dx = \int_{-\infty}^{\infty} \chi_E(x+h)dx = \int_{-\infty}^{\infty} \chi_{(E-h)}(x)dx = m(E-h)$

By translation invariance of Lebesgue measure, $mE = m(E-h)$

Hence,

$$\int f(x)dx = \int f(x+h)dx$$

Case ii: Suppose $f(x) = \sum_{i=1}^{m} a_i \chi_{E_i}(x)$, where each E_i is measurable with $mE_i < \infty$ as f is integrable.

Thus, $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \sum_{i=1}^{m} a_i \chi_{E_i}(x) = \sum_{i=1}^{m} \int_{-\infty}^{\infty} a_i \chi_{E_i}(x)$

$$= \sum_{i=1}^{m} a_i \int_{-\infty}^{\infty} \chi_{E_i}(x)$$

$$= \sum_{i=1}^{m} a_i \int_{-\infty}^{\infty} \chi_{E_i}(x)$$

$$= \sum_{i=1}^{m} a_i \int_{-\infty}^{\infty} \chi_{E_i}(x+h) \quad , \qquad \text{by case i above.}$$

$$= \sum_{i=1}^{m} \int_{-\infty}^{\infty} a_i \chi_{E_i}(x+h)$$

$$= \int_{-\infty}^{\infty} \sum_{i=1}^{m} a_i \chi_{E_i}(x+h)$$

$$= \int_{-\infty}^{\infty} f(x+h)dx$$

Case iii: Suppose f is a non-negative function. Then , since f is integrable and f is non-negative, by lemma 1.5 there exist an increasing sequence of non-negative measurable simple functions vanishing out side a set of finite measure and such that $\varphi_n(x)$ goes to f as n goes to ∞ .

Thus, by Monotone convergence theorem , we have

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi_n(x)dx = \int_{-\infty}^{\infty} f(x)dx$$

Then, $\int_{-\infty}^{\infty} f(x+h)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi_n(x+h)dx$, by Monotone convergence theorem.

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi_n(x)dx \quad , \text{ by case ii.}$$

$$= \int_{-\infty}^{\infty} f(x)dx \qquad , \text{by Monotone convergence theorem.}$$

Case iv: Let f be integrable , thus $f = f^{+} - f^{-}$ and f^{+} and f^{-} are integrabe and moreover

$$f^{+} \geq 0 \ , \quad f^{-} \geq 0$$

Therefore, $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f^{+}(x)dx - \int_{-\infty}^{\infty} f^{-}(x)dx$

$$= \int_{-\infty}^{\infty} f^{+}(x+h)dx - \int_{-\infty}^{\infty} f^{-}(x+h)dx \quad \text{,by case iii.}$$

$$= \int_{-\infty}^{\infty} [f^{+}(x+h) - f^{-}(x+h)]dx$$

$$= \int_{-\infty}^{\infty} f(x+h)dx$$

Therefore, we have the lemma for all arbitrary functions.

1.2 Difference of a function

In this text , as far as difference of a function is concerned , we consider a function $f: R^{n} \to R$.
Now the differences of functions (namely, the first , the second, the third and so on) are given as follows.

The first difference of a function $f: R^{n} \to R$ with respect to x_j with step $h \in R$ is denoted by (Δ_h, jf) and given by

$$(\Delta_h, jf)(x) = f(x+hej) - f(x).$$

where $h \in R$ and $he_j = (0 , \ldots , 0 , h , 0 , \ldots , 0)$

j^{th} -component

The second difference of a function $f: R^{n} \to R$ with respect to x_j with step $h \in R$ is denoted by (Δ_h^2, jf) and given by

$$(\Delta_h^2, jf)(x) = f(x+2hej) - 2f(x+hej) + f(x)$$

where $h \in R$ and $mhe_j = ((0, \dots, 0, h, 0, \dots, 0)$, m = 1 , 2
j^{th} -component

This definition is derived from the first difference .

Indeed ,

$$
\begin{aligned}
(\Delta_h^2, jf) &= (\Delta_h, j(\Delta_h, jf)) \\
&= (\Delta_h . j(f(x+he) - f(x)) \\
&= [f(x+he_j+he_j) - f(x+he_j)] - [f(x+he_j) - f(x)] \\
&= f(x+2hej) - 2f(x+hej) + f(x)
\end{aligned}
$$

The third difference of a function $f: R^n \to R$ with respect to x_j with step $h \in R$ is denoted by (Δ_h^3, jf) and given by

$$
\begin{aligned}
(\Delta_h^3, jf) &= (\Delta_h, j(\Delta^2{}_h, jf)) \\
&= [f(x+2he_j+he_j) - 2f(x+he_j+he_j) + f(x+he_j)] - \\
&\quad - [f(x+2he_j) - 2f(x+he_j) + f(x)] \\
&= f(x+3hej) - 3f(x+2hej) + 3f(x+hej) - f(x)
\end{aligned}
$$

Continuing this process , we have by induction that the α^{th} difference of a function $f: R^n \to R$ with respect to x_j with step $h \in R$ which is denoted by (Δ_h^α, jf) is given by

$$
(\Delta_h^\alpha, jf) = \sum_{k=0}^{\alpha} (-1)^k \binom{\alpha}{k} f(x+(\alpha-k)he_j) \text{ for } \alpha \in \mathrm{N}
$$

If we are given the β^{th} derivative $D_j^\beta(f)$ of f in the j^{th} direction , then the first and the second difference of $D_j^\beta(f)$ are respectively given as

$$
(\Delta_h, jD_j^\beta f) = D_j^\beta f(x+hej) - D_j^\beta f(x)
$$

and

$$
(\Delta^2{}_h, jD_j^\beta f) = D_j^\beta f(x+2hej) - 2D_j^\beta f(x+hej) + D_j^\beta f(x)
$$

Lemma1.7: Let $\alpha \in \mathrm{N}$, f: $R^n \to R$ be a function and $1 \le p \le \infty$, then

$$\left\|\left(\Delta^{\alpha}{}_{h}, jf\right)\right\|_{L_p(R)} \le 2^{\alpha}\|f\|_{L_p(R)}$$

Proof: $$\left\|\left(\Delta^{\alpha}{}_{h}, jf\right)\right\|_{L_p(R)} = \left\|\sum_{k=0}^{\alpha}(-1)^k\binom{\alpha}{k}f(x+(\alpha-k)he_j)\right\|_{L_p(R)}$$

$$\le \sum_{k=0}^{\alpha}\left|(-1)^k\binom{\alpha}{k}\right|\left\|f(x+(\alpha-k)he_j\right\|_{L_p(R)}$$

$$= \sum_{k=0}^{\alpha}\left|\binom{\alpha}{k}\right|\|f(x)\|_{L_p(R)} \quad \text{, by lemma 1.6.}$$

$$= \|f\|_{L_p(R)}\sum_{k=1}^{\alpha}\binom{\alpha}{k}$$

$$= 2^{\alpha}\|f\|_{L_p(R)}$$

Therefore , we have the lemma.

1.3 Notion of the Trace of a function

Let $f \in L_1^{loc}(R^n)$ where $n > 1$. We would like to define the trace $\mathrm{tr}\, f \equiv tr_{R^m} f \equiv f|_{R^m}$ of the function f on R^m where $1 \le m < n$.

<u>*Denotation*</u>: For each $x \in R^n$, we put $x = (u , v)$ where $u = (x_1 , \dots , x_m)$ and

$v = (x_{m+1}, \dots ,x_n)$.

Now, suppose that $R^m(v)$ is the m-dimensional subspace of points (u , v) , where v is fixed and u runs through all possible values. we represent $R^m = R^m(0)$.

If f is continuos , the trace $\mathrm{tr}\, f$ is defined as a restriction of f: $\mathrm{tr}\, f = f(u ,0)$, $u \in R^m$. However, this way of defining the trace does not make sense for arbitrary function $f \in L_1^{loc}(R^n)$, since actually it is defined only up to a set of n-dimensional measure zero. In fact , one can easily construct two functions f , $h \in L_1^{loc}(R^n)$, which are equivalent on R^n , but $f(u ,0) \ne h(u ,0)$ for all $u \in R^m$.

Finally , it is natural to define the traces themselves up to a set of m-dimensional measure zero. The above is a motivation for the following requirements for the notion of the trace on R^n of a function $f \in L_1^{loc}(R^n)$:

1) a trace $g \in L_1^{loc}(R^n)$,
2) if $g \in L_1^{loc}(R^m)$ is a trace of f, then $\varphi \in L_1^{loc}(R^m)$ is also a trace of f, if and only if φ is equivalent to g on R^m.
3) if g is a trace of f and h is equivalent to f on R^n, then g is also a trace of h,
4) if f is continuous , then $f(u,0)$ is a trace of f.

Definition 1.1 : Let $f \in L_1^{loc}(R^n)$ and $g \in L_1^{loc}(R^n)$. The function g is said to be a trace of the function f if there exists a function h equivalent to f on R^n, which is such that

$$h(.\,,v) \to g(.) \text{ in } L_1^{loc}(R^m) \text{ as } v \to 0 \tag{1.7}$$

Clearly the requirements 1) - 4) are satisfied. In fact, if g is a trace of f and φ is equivalent to g, then (1.7) implies $h(.\,,v) \to \varphi(.)$ in $L_1^{loc}(R^m)$ and φ is also a trace of f. Next suppose that both g and φ are traces of f, then we have (1.7) and also $H(.\,,v)) \to \varphi(.)$ in $L_1^{loc}(R^m)$ as $v \to 0$ for some $H \approx f$ on R^m. We note that for each compact $K \subset R^m$

$$\|g-\varphi\|_{L_1(K)} \le \|h(.,v)-g\|_{L_1(K)} + \|h(.,v)-H(.,v)\|_{L_1(K)} + \|H(.,v)-\varphi\|_{L_1(K)}.$$

Since $h \approx H$ on R^n, $h(.,v) \approx H(.,v)$ on R^m for almost all $v \in R^{n-m}$. Hence , there exists a sequence $\{v_s\}_{s \in N}$, $v_s \in R^{n-m}$, such that $v_s \to 0$ as $s \to \infty$ and

$$\|g-\varphi\|_{L_1(K)} \le \|h(.,v_s)-g\|_{L_1(K)} + \|H(.,v_s)-\varphi\|_{L_1(K)}.$$

on letting $s \to \infty$, we establish that $g \approx \varphi$ on R^m.
Finally if f is continuous, then $\|f(u,v)-f(u,0)\|_{L_{1,u}(K)} \le (mK)\max_{u \in K}|f(u,v)-f(u,0)|$.

Hence, $\|f(.,v)-f(.,0)\|_{L_{1,u}(K)} \to 0$ as $v \to 0$ because f is uniformly continuous on

$K \times \tilde{B}$, where $\tilde{B}$ is the unit ball in R^{n-m}. Thus, $f(.\,,0)$ is a trace of f.

Let $\Omega \subset R^n$ be a domain . we say that Ω is a bounded elementary domain with a resolved boundary with parameters d , D ,satisfying $0 < d \leq D < \infty$, if

$$\Omega = \{x \in R^n : a_n < x_n < \varphi(\bar{x}), \bar{x} \in W\}$$

where diam$\Omega \leq D$, $\bar{x} = (x_1,\ldots,x_{n-1})$, $W = \{\bar{x} \in R^{n-1} : a_i < x_i < b_i, i = 1,\ldots n-1\}$

$-\infty \leq a_i < b_i \leq \infty$, and $a_n + d \leq \varphi(\bar{x})$, $\bar{x} \in W$.

If in addition, $\varphi \in C^l(\overline{W})$ for some $l \in N$ and $\|D^\alpha \varphi\|_{C(\overline{W})} \leq M$ if $1 \leq |\alpha| \leq l$ where $0 \leq M < \infty$, then we say that Ω is a bounded elementary domain with a C^l - boundary with the parameters d , D , M.

Moreover, we say that an open set $\Omega \subset R^n$ has a resolved boundary with parameters d ($0 < d < \infty$) , D ($0 < d \leq \infty$) and $\in N$ if there exists open parallelepipeds V_j , $j = 1, \ldots$, s where s $\in N$ for bounded Ω and s = ∞ for unbounded Ω such that

1) $(V_j)_d \cap \Omega \neq \phi$ and diam $V_j \leq D$
2) $\Omega \subset \bigcup_{j=1}^{s} (V_j)_d$,
3) the multiplicity of the covering $\{V_j\}_{j=1}^{s}$ does not exceed ,
4) there exist maps λ_j , $j = 1, \ldots$, s , which are compositions of rotations, reflections and translations and are such that

$$\lambda_j(V_j) = \{x \in R^n : a_{ij} < x_i < b_{ij}, i = 1,\ldots n\}$$

and

$$\lambda_j(\Omega \cap V_j) = \{x \in R^n : a_{nj} < x_n < \varphi_j(\bar{x}), \bar{x} \in W\}$$

where $\bar{x} = (x_1,\ldots,x_{n-1})$, $W_j = \{\bar{x} \in R^{n-1} : a_{ij} < x_i < b_{ij}, i = 1,\ldots n-1\}$,

and $a_{nj} + d \leq \varphi_j(\bar{x}) \leq b_{nj}$ -d , $\bar{x} \in W_j$, if $V_j \cap \partial\Omega \neq \phi$

If $V_j \subset \Omega$, then $\varphi_j(\bar{x}) \equiv b_{nj}$

Let $\Omega \subset R^n$ be an open set with a C^1- boundary . we would like to extend Definition 1.1 to the case , in which R^n , R^m are replaced by Ω , $\partial\Omega$ respectively where Ω is a bounded elementary domain with a C^1- boundary with parameters d ,D, M.

Definition 1.2 : Let $\Omega \subset R^n$ be an open set with a C^1-boundary and $f \in L_1(\Omega \cap B)$ for each ball $B \subset R^n$. Suppose that $f = \sum_{j=1}^{s} f_j$, where supp $f_j \subset V_j$ and $f_j \in L_1(\Omega \cap V_j)$ (where V_j's are open parallelepipeds satisfying conditions 1) - 4) in the definition of a bounded elementary domain above) . If the functions g_j are traces of the functions f_j on $V_j \cap \partial\Omega$, j = 1, 2, . . . , s ,then the function $\sum_{j=1}^{s} g_j$ is said to be the trace of the function f on $\partial\Omega$.

1.4 The Euler-lagrange Equation

Let's consider the integral

$$I = \int_a^b F(y, y', x)dx \tag{1.8}$$

Where a , b and the form of the function F are fixed by given considerations but the curve $y(x)$ has to be chosen so as to make stationary the value of I ,which is clearly a function (more accurately a functional) of this curve, i.e. $I = I\ (x\ ,\ y)$. Referring to the figure , we wish to find the function $y(x)$ (given say the solid line)such that first order small changes in it (for the two broken lines) will make only second order changes in the value of I .

Writing this in a more mathematical form , let us suppose that $y(x)$ is the function required to make I stationary and consider making the replacement

$$y(x) \rightarrow y(x) + \alpha\eta(x) \tag{1.9}$$

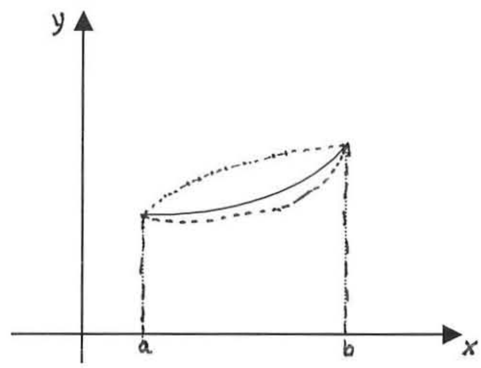


Fig. Possible paths for the integral (1.8). The solid line is the curve along which the integral is assumed stationary . The broken curves represent small variation from this path.

Where the parameter α is small and $\eta(x)$ is an arbitrary function with sufficiently amenable mathematical properties. For the value of I to be stationary with respect to these variations , we require

$$\frac{dI}{d\alpha}\Big|_{\alpha=0} = 0 \text{ for all } \eta(x). \qquad (1.10)$$

Substituting (1.9) into (1.8) and expanding as a Taylor series in α ,

we obtain $$I(y,\alpha) = \int_a^b F(y+\alpha\eta, y'+\alpha\eta', x)dx$$

$$= \int_a^b F(y, y', x)dx + \int_a^b \left(\frac{\partial F}{\partial y}\alpha\eta + \frac{\partial F}{\partial y'}\alpha\eta'\right)dx + O(\alpha^2)$$

With this form for I (y, α) , the condition (1.10) implies that for all $\eta(x)$, we require

$$\delta(I) = \int_a^b \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right)dx = 0$$

Where $\delta(I)$ denotes the first-order variation in the value of I due to the variation (1.5) in the function $y(x)$. Integrating the second term by parts this becomes

$$\left[\eta\frac{\partial F}{\partial y'}\right]_a^b + \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)dx = 0 \qquad (1.11)$$

In order to simplify the result we will, for the moment, assume that the end points are fixed, i.e. not only a and b are given but also $y(a)$ and $y(b)$. This restriction means that we require $\eta(a)=\eta(b)=0$, in which case the first term on the right hand side of (1.11) equals zero at both end points. Since (1.8) must be satisfied for arbitrary $\eta(x)$, it is easy to see that we require

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

This is known as the Euler-Lagrange (EL) equation, and is a differential equation for y(x), since the function F is known.
Moreover, it can also be calculated when n independent variables are involved to extremise the integral

$$I = \int_{R^n} F(y, \frac{\partial y}{\partial x_1}, \ldots, \frac{\partial y}{\partial x_n}, x)dx$$

using the same analysis as above, we find the extremising function

$y(x) = y(x_1, \ldots, x_n)$ must satisfy $\frac{\partial F}{\partial y} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial y_{x_i}}\right)$ where $y_{x_i} = \frac{\partial y}{\partial x_i}$

2. *Function Spaces*

In this chapter, we see two function spaces which are very useful for the investigation of the Dirichlet problem in the next chapter.

2.1 Sobolev Spaces

Definition 2.1 Let $\Omega \subset R^n$ be an open set , $l \in$ N, $1 \le p \le \infty$.The function f belongs to the sobolev space $W_p^l(\Omega)$ if $f \in L_p(\Omega)$, if it has weak derivatives $D_w^\alpha f$ on Ω for all $\alpha \in N_0^n$ satisfying $|\alpha| = l$ and

$$\|f\|_{W_p^l(\Omega)} = \|f\|_{L_p(\Omega)} + \sum_{|\alpha|=l} \|D_w^\alpha f\|_{L_p(\Omega)} < \infty \tag{2.1}$$

In the one dimensional case , $\|f\|_{W_p^l(\Omega)} = \|f\|_{L_p(\Omega)} + \|f_w^{(l)}\|_{L_p(\Omega)}$

Theorem 2.1: Let $\Omega \subset R^n$ be an open set, $l \in$ N, $1 \le p \le \infty$. Then $W_p^l(\Omega)$ is a Banach space.

Idea of the proof: Obviously $W_p^l(\Omega)$ is a normed space. To prove completeness, starting with the cauchy sequence $\{f_k\}_{k\in N}$ in $W_p^l(\Omega)$, deduce using the completeness of $L_p(\Omega)$ that there exist $f \in L_p(\Omega)$ and $f_\alpha \in L_p(\Omega)$, where $\alpha \in N_0^n$, $|\alpha| = l$, such that $f_k \to f$ and $D_w^\alpha f_k \to f_\alpha$ in $L_p(\Omega)$. From the closed ness of the weak differentiation it follows that $f_\alpha = D_w^\alpha f$.

Hence, $f_k \to f$ in $W_p^l(\Omega)$.[1]

[1] Two norms $\|.\|_1$ and $\|.\|_2$ defined on a vector space X are said to be equivalent if and only if there are two positive reals c_1 , c_2 such that $c_1 \|x\|_2 \le \|x\|_1 \le c_2 \|x\|_2$ for all $x \in X$

Remark 2.1: Norm (2.1) is equivalent to

$$\|f\|^{*}_{W_p^l(\Omega)} = \left(\int_{\Omega} \left(|f|^p + \sum_{|\alpha|=l} |D_w^{\alpha} f|^p \right) dx \right)^{\frac{1}{p}} \quad \text{for } 1 \le p < \infty \tag{2.2}$$

and to

$$\|f\|^{*}_{W_\infty^l(\Omega)} = \max\{ \|f\|_{L_\infty(\Omega)}, \max_{|\alpha|=l} \|D_w^{\alpha} f\|_{L_\infty(\Omega)} \} \quad \text{for } p = \infty. \tag{2.3}$$

i.e., $\forall f \in W_p^l(\Omega)$, $c_3 \|f\|^{*}_{W_p^l(\Omega)} \le \|f\|_{W_p^l(\Omega)} \le c_4 \|f\|^{*}_{W_p^l(\Omega)}$ where $c_3, c_4 > 0$ are

independent of f.

This follows, with c_3, c_4 depending only on n, p and l, from Holder's and Jenssen's inequalities for finite sums. If $p = 2$, then $W_2^l(\Omega)$ is a Hilbert space with the inner product

$$(f, g)_{W_p^l(\Omega)} = \int_{\Omega} \left(f\,\overline{g} + \sum_{|\alpha|=l} D_w^{\alpha} f\, \overline{D_w^{\alpha} g} \right) dx \tag{2.4}$$

and $\|f\|^{*}_{W_2^l(\Omega)}$ is a Hilbert norm. i.e., $\|f\|^{*}_{W_2^l(\Omega)} = (f, f)^{\frac{1}{2}}_{W_2^l(\Omega)}$.

Definition 2.2: Let $\Omega \subset R^n$ be an open set, $l \in N$, $1 \le p \le \infty$. The function f belongs to the semi-normed Sobolev space $w^{*l}_{p}(\Omega)$ if $f \in L_1^{loc}(\Omega)$, if it has weak derivatives $D_w^{\alpha} f$ on Ω for all $\alpha \in N_0^n$ satisfying $|\alpha| = l$ and

$$\|f\|_{w^{*l}_p(\Omega)} = \sum_{|\alpha|=l} \|D_w^{\alpha} f\|_{L_p(\Omega)} < \infty. \tag{2.5}$$

The space $w^{*l}_{p}(\Omega)$ is also complete space (the proof is similar to the proof of Theorem 2.1). Thus $w^{*l}_{p}(\Omega)$ is a semi-Banach space.

Remark 2.2: $W_p^l(\Omega) \subset w^{*l}_{p}(\Omega)$ but in general $W_p^l(\Omega) \ne w^{*l}_{p}(\Omega)$.

i.e.

$$\|f\|_{w^{*l}_p(\Omega)} \le C \|f\|^{*}_{W_p^l(\Omega)} \quad \text{for some } C > 0. \tag{2.6}$$

Remark 2.3: In particular for $l = 1$ and $p = 2$, we have the semi-Hilbert space denoted by $w_2^1(\Omega)$ and given by

$$w_2^1(\Omega) = \left\{u : \sqrt{D(u)} < \infty \quad \text{where } D(u) = \int_\Omega \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 dx\right.$$

Remark 2.4: Let $l, m, n \in \mathrm{N}$, $m < n$ and $1 \le p \le \infty$ .Then traces on R^m exist for all $f \in W_p^l(R^n)$ if, and only if,

$$l > \frac{n-m}{p} \quad \text{for } 1 < p \le \infty, \quad l \ge n-m \quad \text{for } p = 1, \tag{2.7}$$

i.e. ,if, and only if,

$$W_p^l(R^{n-m}) \subset\!> C(R^{n-m}).$$

Remark 2.5: Assume (2.7) is satisfied. It follows that for each $f \in W_p^l(R^n)$

the tr $f \in L_p(R^m)$ and

$$\|trf\|_{L_p(R^m)} \le c_3 \|f\|_{W_p^l(R^n)} \text{ for some } c_3 > 0.$$

Now we consider the trace space

$$tr_{R^m} W_p^l(R^n) = \{trf, f \in W_p^l(R^n)$$
$$= \{g \in L_1^{loc}(R^n) : g = trf \ , \exists f \in W_p^l(R^n)\}$$

If this is considered then $tr_{R^m} W_p^l(R^n) \subset L_p(R^n)$.

2.2 The Nikol Skii Besov Spaces

Definition 2.3: Let $l > 0$, $1 \le p \le \infty$, $1 \le \theta \le \infty$, , where l is the index of smoothness, p is the index of summability, θ is additional index characterizing smoothness and consider a function $f: R^n \to R$. Then we say that f belongs to the Nikol Skii Besov Space $B_{p,\theta}^l(R^n)$ if, and only if $f \in L_p(R^n)$ and

$$\|f\|_{B^l_{p,\theta}(R^n)} = \|f\|_{L_p(R^n)} + \sum_{j=1}^{n} \|f\|_{\beta^l_{p,\theta,j}(R^n)} < \infty$$

where

$$\|f\|_{\beta^l_{p,\theta,j}(R^n)} = \left[\int_0^\infty \left(h^{-(l-\bar{l})} \left\| \left(\Delta_h^\alpha, jD_j^{\bar{l}} f \right) \right\|_{L_p(R^n)} \right)^\theta \frac{dh}{h} \right]^{\frac{1}{\theta}} \quad \text{if } \theta < \infty$$

and

$$\|f\|_{\beta^l_{p,\theta,j}(R^n)} = \sup_{h>0} h^{-(l-\bar{l})} \left\| \left(\Delta_h^\alpha, jD_j^{\bar{l}} f \right) \right\|_{L_p(R^n)} \quad \text{if } \theta = \infty$$

In this definition ,

$$\bar{l} = [\, l \,] \text{ and } \alpha = 1 \text{ whenever } l \notin \mathrm{N}$$

and

$$\bar{l} = l - 1 \text{ and } \alpha = 2 \quad \text{whenever } l \in \mathrm{N}.$$

This is a general definition of the Nikol Skii Besov Spaces for all set of values of l , p , θ and n. For a function f: $R^n \to R$, the norm of f changes as we take different set of values of l, p, θ, and n. Now We are going to see some particular cases by taking some different set of values of l, p, θ, and n.

<u>Case i</u>: Let $n = 1$, $0 < l < 1$ and $1 \le \theta < \infty$, thus the norm of a function $f: R \to R$ in this Nikol Skii Besov Space can be analyzed as follows.

Since $0 < l < 1$, we have $\bar{l} = 0$ and $\alpha = 1$

So,

$$\left\| \left(\Delta_h^\alpha, jD_j^{\bar{l}} f \right) \right\|_{L_p(R)} = \|(\Delta_h, f)\|_{L_p(R)} = \|f(x+h) - f(x)\|_{L_p(R)}$$

$$\|f\|_{B^l_{p,\theta}(R^n)} = \|f\|_{B^l_{p,\theta}(R)}$$

$$= \|f\|_{L_p(R)} + \sum_{j=1}^{1} \|f\|_{\beta^l_{p,\theta,j}(R)}$$

$$= \|f\|_{L_p(R)} + \|f\|_{\beta^{l}_{p,\theta,1}(R)}$$

$$= \|f\|_{L_p(R)} + \left(\int_0^\infty \|f(x+h) - f(x)\|^{\theta}{}_{L_p(R)} \frac{dh}{h^{1+l\theta}} \right)^{\frac{1}{\theta}}$$

Therefore in this case f to be in this space, f should be in $L_p(R)$

and also $\|f\|_{L_p(R)} + \left(\int_0^\infty \|f(x+h) - f(x)\|^{\theta}{}_{L_p(R)} \frac{dh}{h^{1+l\theta}} \right)^{\frac{1}{\theta}} < \infty$

<u>Case ii</u>: Let $n = 1,\ 0 < l < 1\ , \theta = \infty$

These imply $\bar{l} = 0$ and $\alpha = 1$

Thus for a function $f: R \to R$,the norm $\|f\|_{B^{l}_{p,\infty}(R)}$ is given by

$$\|f\|_{B^{l}_{p,\infty}(R)} = \|f\|_{L_p(R)} + \underset{h>0}{Sup}\, h^{-l} \|f(x+h) - f(x)\|_{L_p(R)}$$

This case is the same as the first case above except that $\theta < \infty$ in the first and $\theta = \infty$ in the second.

<u>Case iii</u>: Let $n = 1\ ,\ 0 < l < 1\ ,\ p = \infty\ ,\ \theta = \infty$

These imply $\bar{l} = 0$ and $\alpha = 1$

In this case , as $p = \infty$, we have

$$\|f\|_{L_p(R)} = \|f\|_{L_\infty(R)} = \underset{x\in R}{ess\ \sup} |f(x)| \quad ,$$

where $\underset{x\in R}{ess\ \sup} |f(t)|$ is the infremum of Sup g(t) as g ranges over all

functions which are equal to f a.e.

i.e. $\underset{x\in R}{ess\ \sup} |f(t)| =$ Inf { M : m{ t : f (t) > M } = 0 }, m denotes the lebesgue

measure of a set.

Thus, $\|f\|_{B^l_{p,\theta}(R)} = \|f\|_{B^l_{\infty,\infty}(R)}$

$$= \|f\|_{L_\infty(R)} + \sup_{h>0} h^{-l}\|f(x+h)-f(x)\|_{L_\infty(R)}$$

$$= \|f\|_{L_\infty(R)} + \sup_{h>0} h^{-l} \operatorname*{ess\,sup}_{x\in R}|f(x+h)-f(x)|$$

In this case, $f\in \|f\|_{B^l_{\infty,\infty}(R)}$ implies $f\in \|f\|_{L_\infty(R)}$ and

$$\sup_{h>0} h^{-l} \operatorname*{ess\,sup}_{x\in R}|f(x+h)-f(x)| < \infty$$

This implies $\|f\|_{L_\infty(R)} < \infty$ and thus there is $M_1\in R$ such that $\forall x \in R$,

$|f(x)| \le M_1$ and $\exists M_2 \in R$ such that $\forall h>0,\ h^{-l} \operatorname*{ess\,sup}_{x\in R}|f(x+h)-f(x)| \le M_2$

$$\Rightarrow \quad \operatorname*{ess\,sup}_{x\in R}|f(x+h)-f(x)| \le h^l M_2$$

$$\Rightarrow \quad \forall x\in R,\ |f(x+h)-f(x)| \le h^l M_2$$

Now, let $M = \text{Max}\{M_1, M_2\}$

Therefore, We have the following:

$f\in \|f\|_{B^l_{\infty,\infty}(R)}$ if and only if there is a number M such that $\forall x\in R$,

$|f(x)| \le M$ and for all $h>0$, $\forall x\in R$, we have $|f(x+h)-f(x)| \le h^l M$

<u>Case iv:</u> Let $n=1,\ 0<l<1,\ 1\le p<\infty,\ \theta=\infty$

In this case, $\bar{l}=0$

Thus, $f \in B^l_{p,\theta}(R)$ if and only if $f\in L_p(R)$ and

$$\|f\|_{L_p(R)} + \sup_{h>0} h^{-l}\|f(x+h)-f(x)\|_{L_p(R)} < \infty$$

This implies, there is a real number M such that, for all $h>0$

$$h^{-l}\|f(x+h)-f(x)\|_{L_p(R)} \le M$$

or

$$\|f(x+h)-f(x)\|_{L_p(R)} \le M\, h^l$$

Thus we conclude that f belongs to $B^{l}_{p,\theta}\,(R)$ if and only if $f \in L_p(R)$ and there is a real number M such that $\|f(x+h)-f(x)\|_{L_p(R)} \le M\,h^{l}$ for all positive number h.

]

Case v: Let $n = 1$, $l = 1$, $\theta = \infty$

The above four cases are taking l such that $0 < l < 1$, but in this case we take $l = 1$

Since l is an integer , we have $\bar{l} = l - 1 = 0$ and $\alpha = 2$

Therefore , we take the second difference of the function f.

Thus ,

$$\|f\|_{B^{l}_{p,\theta}(R)} = \|f\|_{B^{1}_{p,\infty}(R)} = \|f\|_{L_p(R)} + \sup_{h>0} h^{-l} \|(\Delta^2_h, f)\|_{L_p(R)}$$

$$= \|f\|_{L_p(R)} + \sup_{h>0} h^{-1} \|f(x+2h) - 2f(x+h) + f(x)\|_{L_p(R)}$$

$$= \|f\|_{L_p(R)} + \sup_{h>0} \frac{\|f(x+2h) - 2f(x+h) + f(x)\|_{L_p(R)}}{h}$$

Therefore , f belongs to $B^{l}_{p,\theta}(R)$ if and only if $f \in L_p(R)$ and for some positive real number M, for all $h > 0$ $\|(\Delta^2_h, f)\|_{L_p(R)} \le M\,h$.

In case if $p = \infty$, we get f belongs to $B^{l}_{p,\theta}(R)$ if and only if for some positive real number M,

$\forall x \in R$ $|f(x)| \le M$ and $\forall h > 0$, $\forall x \in R$ $|(\Delta^2_h, f)| \le M\,h$.

$$|f(x+2h) - 2f(x+h) + f(x)| = |f(x+2h) - f(x+h) - [f(x+h) - f(x)]|$$

$$\le |f(x+2h) - f(x+h)| + |[f(x+h) - f(x)]|$$

Thus , $\forall x \in R$, $\forall h > 0$, $|f(x+h) - f(x)| \le M\,h$ implies

$$|f(x+2h) - 2f(x+h) + f(x)| \le |f(x+2h) - f(x+h)| + |[f(x+h) - f(x)]|$$

$$\le M\text{h} + M\,h$$

$$= 2M\,h$$

If we put $M' = 2M$, then we get

$$|f(x+2h) - 2f(x+h) + f(x)| \le M'h$$

where $M'h = 2M$

But for some real number M, $\forall x \in R$, $\forall h > 0$ such that $|(\Delta_h^2, f)| \le Mh$

does not imply

for some $M' > 0$, $\forall x \in R$, $\forall h > 0$, $|f(x+h) - f(x)| \le 2M h$.

Case vi: Let $n = 1$, $0 < l < 1$, $1 \le p < \infty$, $1 \le \theta < \infty$

In this case, f belongs to $B_{p,\theta}^{l}(R)$ if and only if $f \in L_p(R)$ and

$$\|f\|_{L_p(R)} + \left(\int_0^\infty (h^{-l}\|f(x+h) - f(x)\|_{L_p(R)})^\theta \frac{dh}{h}\right)^{\frac{1}{\theta}} < \infty$$

or

$$f \in L_p(R) \text{ and } \left(\int_0^\infty \|f(x+h) - f(x)\|^\theta{}_{L_p(R)} \frac{dh}{h^{1+l\theta}}\right) < \infty$$

Fact: $\left(\int_0^\infty (h^{-l}\|f(x+h) - f(x)\|_{L_p(R)})^\theta \frac{dh}{h}\right) < \infty$ if and only if

$$\left(\int_0^1 (h^{-l}\|f(x+h) - f(x)\|_{L_p(R)})^\theta \frac{dh}{h}\right) < \infty .$$

Proof: ($\Rightarrow$)Since $[0,1] \subseteq [0, \infty)$, we have that

$$\left(\int_0^1 (h^{-l}\|f(x+h) - f(x)\|_{L_p(R)})^\theta \frac{dh}{h}\right) \le \left(\int_0^\infty (h^{-l}\|f(x+h) - f(x)\|_{L_p(R)})^\theta \frac{dh}{h}\right)$$

Thus, $\left(\int_0^\infty (h^{-l}\|f(x+h) - f(x)\|_{L_p(R)})^\theta \frac{dh}{h}\right) < \infty$ implies

$$\left(\int_0^1 (h^{-l}\|f(x+h) - f(x)\|_{L_p(R)})^\theta \frac{dh}{h}\right) < \infty.$$

($\Leftarrow$) Suppose $\left(\int_0^1 (h^{-l}\|f(x+h)-f(x)\|_{L_p(R)})^{\theta}\frac{dh}{h}\right) < \infty$

Now, $\left(\int_0^\infty (h^{-l}\|f(x+h)-f(x)\|_{L_p(R)})^{\theta}\frac{dh}{h}\right) = \left(\int_0^\infty \|f(x+h)-f(x)\|^{\theta}{}_{L_p(R)}\frac{dh}{h^{1+l\theta}}\right)$

$$= \left(\int_0^1 \|f(x+h)-f(x)\|^{\theta}{}_{L_p(R)}\frac{dh}{h^{1+l\theta}}\right)$$

$$+ \left(\int_1^\infty \|f(x+h)-f(x)\|^{\theta}{}_{L_p(R)}\frac{dh}{h^{1+l\theta}}\right)$$

Now we consider $\left(\int_1^\infty \|f(x+h)-f(x)\|^{\theta}{}_{L_p(R)}\frac{dh}{h^{1+l\theta}}\right)$.

$$\left(\int_1^\infty \|f(x+h)-f(x)\|^{\theta}{}_{L_p(R)}\frac{dh}{h^{1+l\theta}}\right) \leq \left(\int_1^\infty (2\|f\|_{L_p(R)})^{\theta}\frac{dh}{h^{1+l\theta}}\right)$$

$$= (2\|f\|_{L_p(R)})^{\theta}\int_1^\infty \frac{dh}{h^{1+l\theta}}$$

But $\int_1^\infty \frac{dh}{h^{1+l\theta}} = \lim_{b\to\infty}\frac{-h^{-l\theta}}{l\theta}\Big|_1^\infty = \lim_{b\to\infty}(\frac{1}{l\theta}-\frac{b^{-l\theta}}{l\theta}) = \frac{1}{l\theta}$

Therefore , $\left(\int_1^\infty \|f(x+h)-f(x)\|^{\theta}{}_{L_p(R)}\frac{dh}{h^{1+l\theta}}\right) \leq \frac{2^{\theta}\|f\|^{\theta}{}_{L_p(R)}}{l\theta}$

Now since $\|f\|_{L_p(R^n)} < \infty$, $\left(\int_1^\infty \|f(x+h)-f(x)\|^{\theta}{}_{L_p(R)}\frac{dh}{h^{1+l\theta}}\right)$ is always finite.

Therefore, since $\left(\int_1^\infty \|f(x+h)-f(x)\|^{\theta}{}_{L_p(R)}\frac{dh}{h^{1+l\theta}}\right)$ is always finite and by

hypothesis

$\left(\int_0^1 (h^{-l}\|f(x+h)-f(x)\|_{L_p(R)})^{\theta}\frac{dh}{h}\right)$ is finite , we have that

$\left(\int_0^1 (h^{-l}\|f(x+h)-f(x)\|_{L_p(R)})^{\theta}\frac{dh}{h}\right)$ converges implies

$$\left(\int_0^{\infty} (h^{-l} \| f(x+h) - f(x) \|_{L_p(R)})^{\theta} \frac{dh}{h} \right) \text{ converges.}$$

Case vii: Let $n = 1$, $p = \infty$, $0 < l < 1$

These implies $\bar{l} = 0$ and $\alpha = 1$

Denotation: If $p = \theta$, then $B^l_{p,\theta}(R^n)$ is denoted by $B^l_p(R^n)$.

Thus in our case ,

$$\|f\|_{B^l_p(R)} = \|f\|_{L_p(R)} + \left(\int_0^{\infty} (h^{-l} \| f(x+h) - f(x) \|_{L_p(R)})^{p} \frac{dh}{h} \right)^{\frac{1}{p}}$$

$$= \|f\|_{L_p(R)} + \left(\int_0^{\infty} \| f(x+h) - f(x) \|^{p}{}_{L_p(R)} \frac{dh}{h^{1+lp}} \right)^{\frac{1}{p}}$$

$$= \|f\|_{L_p(R)} + \left(\int_0^{\infty} \left[\left(\int_R |f(x+h) - f(x)|^p \, dx \right)^{\frac{1}{p}} \right]^p \frac{dh}{h^{1+lp}} \right)^{\frac{1}{p}}$$

$$= \|f\|_{L_p(R)} + \left(\int_0^{\infty} \int_{-\infty}^{\infty} |f(x+h) - f(x)|^p \, dx \frac{dh}{h^{1+lp}} \right)^{\frac{1}{p}}$$

$$= \|f\|_{L_p(R)} + \left(\int_{-\infty}^{\infty} dx \int_0^{\infty} |f(x+h) - f(x)|^p \frac{dh}{h^{1+lp}} \right)^{\frac{1}{p}}$$

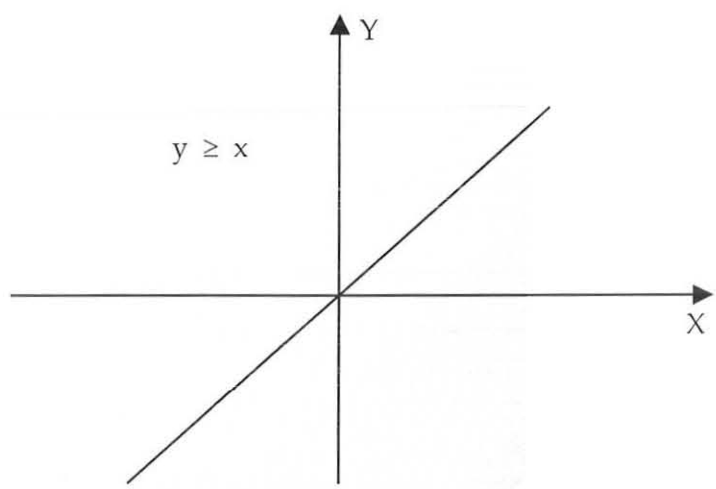
This is because f is integrable and the terms for the variable of integration h and x are separable.

Now we introduce a new variable as follows.

Let $h + x = y$, then $h = y - x$ and $dh = dy$

Claim: $\int_{-\infty}^{\infty} dx \int_{x}^{\infty} \frac{|f(y)-f(x)|^{p}}{|y-x|^{1+lp}} dy = \frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{|f(y)-f(x)|^{p}}{|y-x|^{1+lp}} dy$

Proof: Consider the following graph.



From the graph , we can see that

$$\int_{-\infty}^{\infty} dx \int_{x}^{\infty} \frac{|f(y)-f(x)|^{p}}{|y-x|^{1+lp}} dy = \int_{-\infty}^{\infty} dy \int_{-\infty}^{y} \frac{|f(y)-f(x)|^{p}}{|y-x|^{1+lp}} dx \quad \ldots\ldots\ldots\ldots\ (*)$$

But by symmetry , we have

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{y} \frac{|f(y)-f(x)|^{p}}{|y-x|^{1+lp}} dx = \int_{-\infty}^{\infty} dx \int_{-\infty}^{x} \frac{|f(y)-f(x)|^{p}}{|y-x|^{1+lp}} dy \quad \ldots\ldots\ldots\ (**)$$

Thus, $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{|f(y)-f(x)|^{p}}{|y-x|^{1+lp}} dy = \int_{-\infty}^{\infty} dx \int_{-\infty}^{x} \frac{|f(y)-f(x)|^{p}}{|y-x|^{1+lp}} dy + \int_{-\infty}^{\infty} dx \int_{x}^{\infty} \frac{|f(y)-f(x)|^{p}}{|y-x|^{1+lp}} dy$

$$= \int_{-\infty}^{\infty} dx \int_{x}^{\infty} \frac{|f(y)-f(x)|^p}{|y-x|^{1+lp}} dy + \int_{-\infty}^{\infty} dx \int_{x}^{\infty} \frac{|f(y)-f(x)|^p}{|y-x|^{1+lp}} dy$$

by (*) and (**).

$$= 2 \int_{-\infty}^{\infty} dx \int_{x}^{\infty} \frac{|f(y)-f(x)|^p}{|y-x|^{1+lp}} dy$$

Therefore we have the claim.

Hence, $\|f\|_{B_p^l(R)} = \|f\|_{L_p(R)} + \frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{|f(y)-f(x)|^p}{|y-x|^{1+lp}} dy$, where $x + h = y$.

Case viii: When $n \in \mathrm{N}, p = \theta,\ 0 < l < 1$

In this case, like case vii it can be shown that

$$\|f\|_{B_p^l(R^n)} \approx \|f\|_{L_p(R^n)} + \left(\int_{R^n} \int_{R^n} \frac{|f(x)-f(y)|^p}{|x-y|^{1+lp}} dxdy \right)^{\frac{1}{p}}$$

Case ix: When $n \in \mathrm{N}, p = \theta,\ l = 1$

In this case, we have

$$\|f\|_{B_p^1(R^n)} \approx \|f\|_{L_p(R^n)} + \left(\int_{R^n} \int_{R^n} \frac{\left|f(x) - f(\frac{x+y}{2}) + f(y)\right|^p}{|x-y|^{1+p}} dxdy \right)^{\frac{1}{p}}$$

Case x: Let $n = 1,\ 1 < l < 2,\ 1 \le p,\ \theta \le \infty$

In this case, $\bar{l} = 1$ and $\alpha = 0$

$$\|f\|_{B_{p,\theta}^l(R)} \approx \|f\|_{L_p(R)} + \left(\int_0^{\infty} (h^{-(l-1)} \|(\Delta_h, f'(x))\|_{L_p(R)})^{\theta} \frac{dh}{h^1} \right)^{\frac{1}{\theta}}$$

<u>*Denotation:*</u> $B_{p,\infty}^l(\mathrm{R}) \equiv H_p^l(R^n)$

In this case, $f \in H_p^l(R^n)$ if and only if $f \in L_p(R)$ and for some number M, $\|\Delta_h, f'(x)\|_{L_p(R)} \le M\, h^{l-1}$

2.3 Some theorems on Nikol Skii Besov Spaces

In this section, we see some inclusion theorems to develop the relation between Nikol Skii Besov Spaces of different Index of smoothness l as well as different additional index of smoothness θ. And, finally, we will put important theorems which will be useful for the application on the generalized formulation of Dirichlet problem in the last Chapter. Before we prove theorems we establish the following important theorem.

Theorem 2.2:(Equivalent norms in Nikolskii Besov spaces)

Let $1 \le p, \theta \le \infty$, $\ell > 0$, then

$$\|f\|_{B^{\ell}_{p,\theta}(R^n)} \approx \|f\|_{L_p(R^n)} + \sum_{j=1}^{n} \left(\int_0^\infty \left(h^{-(l-m_j)} \left\| (\Delta_h^{\alpha_j}, jD_{j,w}^{m_j} f) \right\|_{L_p(R^n)} \right)^{\theta} \frac{dh}{h} \right)^{1/\theta}$$

when $\theta < \infty$

$$\|f\|_{B^{\ell}_{p,\theta}(R^n)} \approx \|f\|_{L_p(R^n)} + \sum_{j=1}^{n} \sup_{h>0} \left(h^{-(l-m_j)} \left\| (\Delta_h^{\alpha_j}, jD_{j,w}^{m_j} f) \right\|_{L_p(R^n)} \right.$$

when $\theta = \infty$

where $\alpha_j \in N$, $m_j \in N_0$ and $\alpha_j + m_j > \ell > m_j$ which is called condition of permissibility.

In the definition of the Nikol Skii Besov spaces α is the minimum of α_j's satisfying the condition of permissibility.

For example, for $\alpha_j > \ell$ and $m_j = 0$, we have

$$\|f\|_{B^{\ell}_{p,\theta}(R^n)} \approx \|f\|_{L_p(R^n)} + \sum_{j=1}^{n} \left(\int_0^\infty \left(h^{-\ell} \left\| (\Delta_h^{\alpha_j}, jf) \right\|_{L_p(R^n)} \right)^{\theta} \frac{dh}{h} \right)^{1/\theta}$$

Proof:(for $n = 1$, $0 < \ell < 1$, $\theta = \infty$)

Thus it is to show

$$\|f\|_{H^{\ell}_p(R)} = \|f\|_{L_p(R)} + \sup_{h>0} \frac{\left\| (\Delta_h, f) \right\|_{L_p(R)}}{h^{\ell}}$$

$$\approx \|f\|_{L_p(R)} + \sup_{h>0} \frac{\left\| (\Delta_h^2, f) \right\|_{L_p(R)}}{h^{\ell}} \quad \text{i.e for } \alpha_j = 2 \text{ and } 2 > \ell > 0$$

$$:= \|f\|^*_{H_p^\ell(R)}$$

Now we start with

$$\left\|(\Delta_h^2, f)\right\|_{L_p(R)} \le 2\ \|(\Delta_h, f)\|_{L_p(R)}$$

Consequently, $\|f\|^*_{H_p^\ell(R)} \le 2\ \|f\|_{H_p^\ell(R)}$ .... (*)

$$(\Delta_h f) = \frac{1}{2}(\Delta_{2h}, f) - \frac{1}{2}((\Delta_h^2, f)$$

$$E_h - I = \frac{1}{2}(E_{2h} - I) - \frac{1}{2}(E_h - I)^2$$

This is analogous to $(x - 1) = \frac{1}{2}(x^2 - 1) - \frac{1}{2}(x - 1)^2$

Thus, $\|(\Delta_h, f)\|_{L_p(R)} \le \frac{1}{2}\ \|(\Delta_{2h}, f)\|_{L_p(R)} + \frac{1}{2}\ \left\|(\Delta_h^2, f)\right\|_{L_p(R)}$

$$\Rightarrow \underbrace{\frac{\|(\Delta_h, f)\|_{L_p(R)}}{h^\ell}}_{:=\varphi(h)} \le \frac{1}{2^{1-\ell}} \underbrace{\frac{\|(\Delta_{2h}, f)\|_{L_p(R)}}{(2h)^\ell}}_{:=\varphi(2h)} + \frac{1}{2}\frac{\|(\Delta_h^2, f)\|_{L_p(R)}}{h^\ell}$$

Let $M = \|f\|^*_{h_p^\ell(R)} = \sup\limits_{h>0} \frac{\|(\Delta_h^2, f)\|_{L_p(R)}}{h^\ell}$ and $C = \frac{1}{2^{1-\ell}}$

$$\varphi(h) \le C\ \varphi(2h) + \frac{M}{2}$$

$$\varphi(h) \le C\ (C\ \varphi(4h) + \frac{M}{2}) + \frac{M}{2}$$

$$= C^2\ \varphi(4h) + (C + 1)\ \frac{M}{2}$$

$$\le C^3\ \varphi(8h) + (C^2 + C + 1)\ \frac{M}{2}$$

$\le C^k\ \varphi(2^k h) + \frac{M}{2}\ (C^{k-1} + \ldots + C + 1)$, by induction

$\le C^k\ \varphi(2^k h) + \frac{M}{2(1 - C)}$ since $1 \ge 1 - \left(\frac{1}{2^{1-\ell}}\right)^k = 1 - C^k$

$$\varphi(h) = \frac{\|(\Delta_h, f)\|_{L_p(R)}}{h^\ell}$$

Now If $h \geq 1$, then $\varphi(h) \leq \|(\Delta_h, f)\|_{L_p(R)} \leq 2\,\|f\|_{L_p(R)}$,

choose k such that $2^k h \geq 1$, then

$$\varphi(h) \leq 2C^k\,\|f\|_{L_p(R)} + \frac{M}{2(1-C)} \text{ as } 2^k h \geq 1$$

$$\Rightarrow \quad \varphi(h) \leq 2\,\|f\|_{L_p(R)} + \frac{M}{2(1-C)},\ \forall\, h > 0 \text{ as } C^k = \left(\frac{1}{2^{1-\ell}}\right)^k \leq 1$$

$$\|f\|_{h_p^l(R)} = \sup_{h>0} \frac{\|(\Delta_h, f)\|_{L_p(R)}}{h^\ell} \leq 2\,\|f\|_{L_p(R)} + \frac{M}{2(1-C)}$$

$$\Rightarrow \quad \|f\|_{H_p^\ell(R)} \leq 3\|f\|_{L_p(R)} + \frac{\|f\|^*{}_{h_p^l(R)}}{2(1-C)}$$

$$\leq \max\left\{3, \frac{1}{2(1-C)}\right\}\left(\|f\|_{L_p(R)} + \|f\|^*{}_{h_p^\ell(R)}\right) \quad \ldots\ldots\ldots\ldots\ (**)$$

From (*) and (**) , we can see that the two norms are equivalent.

And, $\|f\|_{h_p^l(R)} \leq \frac{1}{2(1-C)}\|f\|^*{}_{h_p^\ell(R)}$

$$\frac{1}{2}\|f\|^*{}_{h_p^\ell(R)} \leq \|f\|_{h_p^l(R)}$$

This inequality is true for $f \in L_p(R)$

For instance, consider $f(x) = x$

$$\left(\Delta_h^2, f\right) = 0,\ (\Delta_h, f) = h$$

$$\frac{\|(\Delta_h, f)\|_{L_p(R)}}{h^\ell} \leq C\,\frac{\|(\Delta_{2h}, f)\|_{L_p(R)}}{(2h)^\ell} + \frac{M}{2}$$

$$\sup_{h>0}\frac{\|(\Delta_h, f)\|_{L_p(R)}}{h^\ell} \leq C\sup_{h>0}\frac{\|(\Delta_{2h}, f)\|_{L_p(R)}}{(2h)^\ell} + \frac{M}{2}, \qquad 0 < C < 1$$

$$\sup_{h>0}\frac{\|(\Delta_h, f)\|_{L_p(R)}}{h^\ell} \leq \frac{M}{2(1-C)} \text{ when } \sup_{h>0}\frac{\|(\Delta_h, f)\|_{L_p(R)}}{h^\ell} < \infty$$

Now we see that for $n = 1, 0 < \ell < 1, 1 \le \theta < \infty$

$$\frac{\|(\Delta_h, f)\|_{L_p(R)}}{h^\ell} \le C\, \frac{\|(\Delta_{2h}, f)\|_{L_p(R)}}{(2h)^\ell} + \frac{1}{2}\,\frac{\|(\Delta_h^2, f)\|_{L_p(R)}}{h^\ell} \quad \text{where } C = \frac{1}{2^{1-\ell}} < 1$$

$$\|g(h)\| = \left(\int_u^\infty g(h)^\theta \frac{dh}{h}\right)^{1/\theta}, u > 0$$

The norm, $\left(\int_u^\infty \left(\frac{\|(\Delta_h, f)\|_{L_p(R)}}{h^\ell}\right)^\theta \frac{dh}{h}\right)^{1/\theta}$

$$\le C\left(\int_u^\infty \left(\frac{\|(\Delta_{2h}, f)\|_{L_p(R)}}{(2h)^\ell}\right)^\theta \frac{dh}{h}\right)^{1/\theta} + \frac{1}{2}\left(\int_u^\infty \left(\frac{\|(\Delta^2{}_h, f)\|_{L_p(R)}}{h^\ell}\right)^\theta \frac{dh}{h}\right)^{1/\theta}$$

$$\le 2\, \|f\|_{L_p(\Re)} \left(\int_u^\infty \frac{dh}{h^{1+\theta\ell}}\right)^{1/\theta}$$

$$\le C_1\, \|f\|_{L_p(\Re)}\, u^{-\ell} \quad C_1 > 0$$

$$M = \left(\int_0^\infty \left(\frac{\|(\Delta^2{}_h, f)\|_{L_p(\Re)}}{h^\ell}\right)^\theta \frac{dh}{h}\right)^{1/\theta}$$

$$\varphi(u) \le C\, \varphi(2u) + \frac{1}{2} M$$

$$\Rightarrow \varphi(u) \le \frac{1}{-2(c-1)} M, u > 0, u \to 0^+$$

The first part of the proof

$$\left(\int_0^\infty \left(\frac{\|(\Delta_h, f)\|_{L_p(R)}}{h^\ell}\right)^\theta \frac{dh}{h}\right)^{1/\theta} \le \frac{1}{2(1-c)}\left(\int_0^\infty \left(\frac{\|(\Delta_h, f)\|_{L_p(R)}}{h^\ell}\right)^\theta \frac{dh}{h}\right)^{1/\theta}$$

$1 \le \theta < \infty \quad \forall f \in L_p(R)$

We use the following lemma 2.1 to 2.4 to prove the inclusion theorems.

Lemma 2.1: Let $\alpha \in R$. Then for all monotone continuous function f, $\exists c_1, c_2 > 0$ such that

$$c_1 \sum_{k=-\infty}^{\infty} 2^{k(\alpha+1)} f(2^k) \le \int_0^{\infty} x^{\alpha} f(x)\,dx \le c_2 \sum_{k=-\infty}^{\infty} 2^{k(\alpha+1)} f(2^k)$$

$$\sum_{k=1}^{\infty} f(k) \le \int_0^{\infty} f(x)dx \le \sum_{k=0}^{\infty} f(k) \text{ for a decreasing function } f.$$

Proof: Consider $\int_0^{\infty} x^{\alpha} f(x)dx$

Now let $x = 2^y \Rightarrow dx = 2^y \ln 2\, dy$

and $x \to 0^+$ implies $y \to -\infty$, $x \to \infty$ implies $y \to \infty$

Thus, $\int_0^{\infty} x^{\alpha} f(x)dx = \ln 2 \int_{-\infty}^{\infty} 2^{\alpha y} f(2^y)\,dy$

$$= \ln 2 \sum_{k=-\infty}^{\infty} \int_k^{k+1} 2^{(\alpha+1)y} f(2^y)dy$$

then using monotonocity of function f and property of the integral, we have

$$A_2\, 2^{(\alpha+1)(k-1)} f(2^{k-1}) \le \int_{k-1}^{k} 2^{(\alpha+1)y} f(2^y)dy$$

$$\le \begin{cases} 2^{(\alpha+1)k} f(2^k) & ,\alpha+1 \ge 0 \\ 2^{-(\alpha+1)}\, 2^{(\alpha+1)k} f(2^k), & \alpha+1 < 0 \end{cases} = A_1\, 2^{(\alpha+1)k} f(2^k)$$

There fore, $\int_0^{\infty} x^{\alpha} f(x)dx \le A_3 \sum_{k=-\infty}^{\infty} 2^{(\alpha+1)k} f(2^k)$

and

$$\int_0^{\infty} x^{\alpha} f(x)dx \ge A_4 \sum_{k=-\infty}^{\infty} 2^{(\alpha+1)(k-1)} f(2^{k-1})$$

$$= A_4 \sum_{k=-\infty}^{\infty} 2^{(\alpha+1)k} f(2^k) \quad \text{substituting } k-1 \to k$$

From convergence of the integral, it follows the convergence of the series So, the first part of the proof is complete.

To prove the second part, we follow the same procedure as part one.

Indeed, consider $\int_0^\infty f(x)dx$

So, we can write $\int_0^\infty f(x)dx$ as follows,

$$\int_0^\infty f(x)dx = \sum_{k=0}^\infty \int_k^{k+1} f(x)dx$$

But $\int_k^{k+1} f(x)dx \leq f(k)$ as f is decreasing $f(k)$ is the maximum value of f in $[k, k+1]$

and $\int_k^{k+1} f(x)dx \geq f(k+1)$ as f is decreasing and $f(k+1)$ is the minimum value of f in $[k,k+1]$

This implies $\sum_{k=0}^\infty f(k+1) \leq \int_0^\infty f(x)dx$

$$\leq \sum_{k=0}^\infty f(k) \quad \text{by changing } k+1 \to k,$$

we have

$$\sum_{k=1}^\infty f(k) \leq \int_0^\infty f(x)dx \leq \sum_{k=0}^\infty f(k)$$

So, we are done.

Remark 2.6: Let $g \geq 0$ and g changes slowly .That is for $c_3, c_4 > 0$, $\forall$ x, y

$$\frac{1}{2} \leq \frac{x}{y} \leq 2$$

$c_3 g(x) \leq g(y) \leq c_4 g(x)$, then $\exists c_5, c_6$ such that

$$c_5 \sum_{k=-\infty}^\infty 2^k g(2^k) \leq \int_0^\infty g(x)dx \leq c_6 \sum_{k=-\infty}^\infty 2^k g(2^k)$$

Lemma 2.2: Let $\alpha \in N, h, \eta \in R$ and $f \in L_1^{loc}(R)$. Then for almost all $x \in R$,

$$\left(\Delta_h^\alpha, f\right)(x) = \sum_{k=1}^{\alpha} (-1)^{\alpha-k} \binom{\alpha}{k} \Bigg((\Delta^\alpha_{\frac{k}{\alpha}\eta}, f)\ (x + (\alpha-k)h)$$

$$+ (-1)^{\alpha+1} (\Delta^\alpha_{h-\frac{k}{\alpha}\eta}, f)(x + k\eta)\Bigg) \tag{2.8}$$

Proof: For $\alpha = 1$

The left hand side of (2.8) becomes (Δ_h, f) and thus $(\Delta_h, f) = f(x + h) - f(x)$

The right hand side of (2.8) becomes $(\Delta_\eta E_{(\alpha-k)h} + \Delta_{h-\eta} E_\eta)$ but $\alpha - k = 0$

Thus,

$$(\Delta_\eta, E_{(\alpha-k)h} f) + (\Delta_{h-\eta}, E_\eta f) = (\Delta_\eta, f) + (\Delta_{h-\eta}, E_\eta f)$$
$$= f(x + \eta) - f(x) + f(x + h) - f(x + \eta)$$
$$= f(x + h) - f(x)$$

So, it is true for $\alpha = 1$

Let $\alpha = 2$, Then

The left hand side of (2.8) becomes $\left(\Delta_h^2, f\right)$,

and thus

$$\left(\Delta_h^2, f\right) = f(x + 2h) - 2f(x + h) + f(x)$$

The right hand side of (2.8) becomes

$$\sum_{k=1}^{2} (-1)^{2-k} \binom{2}{k} \left[\left(\Delta^2_{\frac{k}{2}\eta}, E_{(2-k)}f\right) + (-1)^3 \left(\Delta^2_{h-\frac{k}{2}\eta}, E_{k\eta}f\right)\right]$$

$= -2 f(x + \eta + h) + 4f(x + \frac{1}{2}\eta + h) - 2f(x + h) +$

$+ 2f(x + 2h) - 4f(x + h + \frac{1}{2}\eta) + 2f(x + \eta) +$

$+ f(x + 2\eta) - 2f(x + \eta) + f(x)$

$- \quad f(x + 2h) + 2 f(x + h + \eta) - f(x + 2\eta)$

$= f(x + 2h) - 2f(x + h) + f(x)$

So, we are done for $\alpha = 2$

we can prove the theorem by induction.

Corollary: Let $\alpha \in N, h, \eta \in R, 1 \leq p \leq \infty$ and $f \in L_p(R)$, then

$$\left\|(\Delta_h^\alpha, f)\right\|_{L_p(R)} \leq \sum_{k=1}^{\alpha} \binom{\alpha}{k} \left[\left\|(\Delta^\alpha_{\frac{k}{\alpha}\eta} f)\right\|_{L_p(R)} + \left\|(\Delta^\alpha_{h-\frac{k}{\alpha}\eta,} f)\right\|_{L_p(R)} \right] \tag{2.9}$$

Proof: By using the above Lemma 2.2 and applying the invariance of the norm $\|\cdot\|_{L_p(R)}$ with respect to translations, we have the corollary.

Lemma 2.3: Let $1 \leq p \leq \infty$, then for all $\alpha \in N$, there is $A = A(\alpha)$ such that $\forall\, h > 0$ and for all functions f in $L_p(R)$, we have

$$\left\|(\Delta_h^\alpha, f)\right\|_{L_p(R)} \leq \frac{A(\alpha)}{h} \int_0^h \left\|(\Delta_\eta^\alpha, f)\right\|_{L_p(R)} d\eta \tag{2.10}$$

In particular,

$$\left\|(\Delta_h, f)\right\|_{L_p(R)} \leq \frac{2}{h} \int_0^h \left\|(\Delta_\eta, f)\right\|_{L_p(R)} d\eta \tag{2.11}$$

where 2 is the exact constant.

Proof:

i. From the above corollary, we get for $\alpha \in N, h, \eta \in R$

$$\left\|(\Delta_h^\alpha, f)\right\|_{L_p(R)} \leq \sum_{k=1}^{\alpha} \binom{\alpha}{k} \left[\left\|(\Delta^\alpha_{\frac{k}{\alpha}\eta}, f)\right\|_{L_p(R)} + \left\|(\Delta^\alpha_{h-\frac{k}{\alpha}\eta}, f)\right\|_{L_p(R)} \right]$$

Integrating this from 0 to h by η, we have

$$h\left\|(\Delta_h^\alpha, f)\right\|_{L_p(R)} \leq \sum_{k=1}^{\alpha} \binom{\alpha}{k} \left[\int_0^h \left\|(\Delta^\alpha_{\frac{k}{\alpha}\eta}, f)\right\|_{L_p(R)} d\eta + \int_0^h \left\|(\Delta^\alpha_{h-\frac{k}{\alpha}\eta}, f)\right\|_{L_p(R)} d\eta \right]$$

$$\leq \sum_{k=1}^{\alpha} \frac{\alpha}{k} \binom{\alpha}{k} \left[\int_0^{\frac{k}{\alpha}h} \left\|(\Delta_\eta^\alpha, f)\right\|_{L_p(R)} d\eta + \int_{h-\frac{k}{\alpha}n}^{h} \left\|(\Delta_\eta^\alpha, f)\right\|_{L_p(R)} d\eta \right]$$

This is done by the changes of variables

$\frac{k}{\alpha} \rightarrow \eta$ and $h\text{-}\frac{k}{\alpha}\eta \rightarrow \eta$

$$\leq 2\alpha \sum_{k=1}^{\alpha} \frac{\binom{\alpha}{k}}{k} \int_0^h \left\|(\Delta_\eta^\alpha, f)\right\|_{L_p(R)} d\eta$$

$$= 2\alpha \int_0^h \left\|(\Delta_\eta^\alpha, f)\right\|_{L_p(R)} d\eta \sum_{k=1}^{\alpha} \frac{\binom{\alpha}{k}}{k}$$

$$\Rightarrow \qquad \left\|(\Delta_h^\alpha, f)\right\|_{L_p(R)} \leq \frac{2\alpha \sum_{k=1}^{\alpha} \binom{\alpha}{k}}{h} \int_0^h \left\|(\Delta_\eta^\alpha, f)\right\|_{L_p(R)} d\eta$$

Letting $A(\alpha) = 2\alpha \sum_{k=1}^{\alpha} \frac{\binom{\alpha}{k}}{k} \leq 2\alpha 2^\alpha$, we have the lemma.

ii. The proof of Inequality (2.11)

Let $0 < \eta < h$, Thus

$$f(x+h) - f(x) = f(x+\eta) - f(x) + f(x+\eta+h-\eta) - f(x+\eta)$$
$$= (\Delta_\eta, f) + (\Delta_{h-\eta}, E_\eta f)$$

This implies that

$$\left\|(\Delta_h, f)\right\|_{L_p(R)} \leq \left\|(\Delta_\eta, f)\right\|_{L_p(R)} + \left\|(\Delta_{h-\eta}, f)\right\|_{L_p(R)} \tag{2.12}$$

Integrating this by η from 0 to h, we get

$$h\left\|(\Delta_h, f)\right\|_{L_p(R)} \leq \int_0^h \left\|(\Delta_\eta, f)\right\|_{L_p(R)} d\eta + \int_0^h \left\|(\Delta_{h-\eta}, f)\right\|_{L_p(R)} d\eta$$

$$= 2\int_0^h \left\|(\Delta_\eta, f)\right\|_{L_p(R)} d\eta,$$ by change of variable of $h - \eta \to \eta$.

Therefore,

$$\left\|(\Delta_h, f)\right\|_{L_p(R)} \leq \frac{2}{h} \int_0^h \left\|(\Delta_\eta, f)\right\|_{L_p(R)} d\eta$$

So, we are done.

Denotation : $\left\|(\Delta_h^\alpha, f)\right\|^*_{L_p(R)} := \frac{1}{h} \int_0^h \left\|(\Delta_\eta^\alpha, f)\right\|_{L_p(R)} d\eta$

Lemma 2.4: Let $\ell > 0, 1 \le p, \theta \le \infty$, Then

$$\|f\|_{B^{\ell}_{p,\theta}(R)} \approx \|f\|^*_{B^{\ell}_{p,\theta}(R)}$$

$$\text{where } \|f\|^*_{B^{\ell}_{p,\theta}(R)} = \|f\|_{L_p(R)} + \left[\left(\int_0^{\infty} h^{-(\ell-\bar{l})} \left\|(\Delta_h^{\alpha}, jD^{\bar{l}} f)\right\|^*_{L_p(R)}\right)^{\theta} \frac{dh}{h}\right]^{1/\theta}$$

Proof: By Lemma 2.3, taking $\alpha \in N$

$$\|f\|_{B^{\ell}_{p,\theta}(R)} \le A(\alpha)\|f\|^*_{B^{\ell}_{p,\theta}(R)} \tag{2.13}$$

To show the other side ,

$$\|f\|^*_{B^{\ell}_{p,\theta}(R)} = \|f\|_{L_p(R)} + \left[\left(\int_0^{\infty} h^{-(\ell-\bar{\ell})} \left\|(\Delta_h^{\alpha} jD^{\bar{l}} f)\right\|^*_{L_p(R)}\right)^{\theta} \frac{dh}{h}\right]^{1/\theta}$$

If ℓ is non-integer $\bar{\ell} = [\ell]$ and $\alpha = 1$

If ℓ is integer $\bar{\ell} = \ell - 1$ and $\alpha = 2$

This implies $\|f\|^*_{B^{\ell}_{p,\theta}(R)} = \|f\|_{L_p(R)} + \left[\int_0^{\infty}\left(h^{-(\ell-\bar{\ell})} \left\|(\Delta_h^{\alpha} jD^{\bar{l}} f)\right\|^*_{L_p(R)}\right)^{\theta} \frac{dh}{h}\right]^{1/\theta}$

$$= \|f\|_{L_p(R)} + \left[\int_0^{\infty}\left(h^{-(\ell-\bar{\ell})-\frac{1}{\theta}} \frac{1}{h}\int_0^h \varphi(\eta)d\eta \right)^{\theta} dh\right]^{1/\theta}$$

where $\varphi(\eta) = \left\|(\Delta_{\eta}^{\alpha}, D^{\bar{l}} f)\right\|_{L_p(R)}$

$$= \|f\|_{L_p(R)} + \|h^a H_1 \varphi\|_{L^*_{\theta}(0,\infty)}, \text{ where } a = -(\ell-\bar{\ell}) - \frac{1}{\theta}$$

$$\le C\left(\|f\|_{L_p(R)} + \|h^a \varphi\|_{L^*_{\theta}(0,\infty)}\right), \quad C > 0$$

$$= C\left(\|f\|_{L_p(R)} + \left[\left(\int_0^{\infty} h^{-(\ell-\bar{\ell})} \left\|(\Delta_h^{\alpha}, jD^{\bar{l}} f)\right\|_{L_p(R)}\right)^{\theta} \frac{dh}{h}\right]^{1/\theta}\right.$$

$$= C\, \|f\|_{B^{\ell}_{p,\theta}(R)} \tag{2.14}$$

where $1 \le \theta \le \infty$ and $\alpha < \frac{1}{\theta}; -(\ell - \bar{\ell}) - \frac{1}{\theta} < \frac{1}{\theta}$

Consequently from (2.13) and (2.14) , we get

$$\|f\|_{B^{\ell}_{p,\theta}(R)} \approx \|f\|^*_{B^{\ell}_{p,\theta}(R)}$$

Now , once we have the above four lemmas and theorem (2.1) , we prove the following inclusion theorems.

Theorem 2.2: Let $\ell > 0, 1 \le p \le \infty, 1 \le \theta_2 < \theta_1 \le \infty$ .Then

$$B^{\ell}_{p,\theta_2}(R^n) \subset B^{\ell}_{p,\theta_1}(R^n)$$

Proof: For simplicity we take $n = 1$

$$\|f\|_{B^{\ell}_{p,\theta_1}(R)} = \|f\|_{L_p(R)} + \left(\int_0^{\infty} \left(h^{-(\ell-\bar{\ell})} \|(\Delta^{\alpha}_h, D^{\bar{\ell}} f)\|_{L_p(R)} \right)^{\theta_1} \frac{dh}{h} \right)^{1/\theta_1}$$

$$\approx \|f\|_{L_p(R)} + \left(\int_0^{\infty} \left(h^{-\ell} \|(\Delta^{\alpha}_h, f)\|_{L_p(R)} \right)^{\theta_1} \frac{dh}{h} \right)^{1/\theta_1} \tag{2.15}$$

This is by theorem 2.1 and taking $m_j = 0$ and $\alpha > \ell$

What we want to show is

$$\|f\|_{B^{\ell}_{p,\theta_1}(R)} \le C\, \|f\|_{B^{\ell}_{p,\theta_2}(R)} \quad \text{, for some } C > 0$$

$$(2.15) \approx \|f\|_{L_p(R)} + \left(\int_0^{\infty} \left(h^{-\ell-1} \int_0^{h} \|(\Delta^{\alpha}_{\eta}, f)\|_{L_p(R)} d\eta \right)^{\theta_1} \frac{dh}{h} \right)^{1/\theta_1}$$

$$= \|f\|_{L_p(R)} + \left(\int_0^{\infty} h^{-(l+1)\theta_1 - 1} \left(\int_0^{h} \|(\Delta^{\alpha}_{\eta}, f)\|_{L_p(R)} d\eta \right)^{\theta_1} dh \right)^{1/\theta_1}$$

$$\le \|f\|_{L_p(R)} + A_1 \left(\sum_{k=-\infty}^{\infty} \left(2^{-(\ell+1)k} \varphi(2^k) \right)^{\theta_1} \right)^{1/\theta_1} ,$$

where $h^a = h^{-(\ell+1)\theta_1 - 1}$, $A_1 > 0$

and $\varphi(h) = \int_0^h \left\|(\Delta_\eta^\alpha, f)\right\|_{L_p(R)} d\eta$

which is a monotone continuous function.

$$\leq \|f\|_{L_p(R)} + A_1 \left(\sum_{k=-\infty}^{\infty} \left(2^{-(\ell+1)k} \varphi(2^k)\right)^{\theta_2} \right)^{1/\theta_2} \text{,by Jenssen's}$$

inequality as $\theta_1 \leq \theta_2$

$$\leq \|f\|_{L_p(R)} + A_2 \left(\int_0^\infty \left(h^{-\ell-1} \int_0^h \left\|(\Delta_\eta^\alpha, f)\right\|_{L_p(R)} d\eta \right)^{\theta_2} \frac{dh}{h} \right)^{1/\theta_2} \text{,} A_2 > 0$$

$$\approx \|f\|_{L_p(R)} + \left(\int_0^\infty \left(h^{-\ell} \left\|(\Delta_h^\alpha, f)\right\|_{L_p(R)} \right)^{\theta_2} \frac{dh}{h} \right)^{1/\theta_2}$$

$$\approx \|f\|_{L_p(R)} + \left(\int_0^\infty \left(h^{-(\ell-\bar{\ell})} \left\|(\Delta_h^\alpha, D^{\bar{\ell}} f)\right\|_{L_p(R)} \right)^{\theta_2} \frac{dh}{h} \right)^{1/\theta_2} \text{ bytheorem 2.1}$$

$$= \|f\|_{B^\ell_{p,\theta_1}(R)}$$

This implies $\|f\|_{B^\ell_{p,\theta_1}(\Re)} \leq C\|f\|_{B^\ell_{p,\theta_2}(R)}$ for some $C \in R$

Thus, If a function *f* is in $B^\ell_{p,\theta_2}(R)$. Then

$\|f\|_{B^\ell_{p,\theta_2}(R)} < \infty$ implies by the above argument $\|f\|_{B^\ell_{p,\theta_1}(R)} < \infty$

This implies *f* belongs to $B^\ell_{p,\theta_1}(R)$

Therefore,

$$B^\ell_{p,\theta_2}(R) \subset B^\ell_{p,\theta_1}(R)$$

Theorem 2.3: Let $0 < \ell_1 < \ell_2 \leq \infty$, $1 \leq p \leq \infty$, $1 \leq \theta_1, \theta_2 \leq \infty$, Then

$$B^{\ell_2}_{p,\theta_2}(R^n) \subset B^{\ell_1}_{p,\theta_1}(R^n)$$

Proof: We want to show that for $\ell_1 < \ell_2$, there exists a constant $C \in R$ such that

$$\|f\|_{B^{\ell_1}_{p,\theta_1}(R^n)} \leq C\|f\|_{B^{\ell_2}_{p,\theta_2}(R^n)}$$

for each $f \in B^{\ell_2}_{p,\theta_2}(R^n)$.

Let $n = 1$ for simplicity , we consider two cases

Case I: $\theta_2 \geq \theta_1$, then

$$\|f\|_{B^{\ell_1}_{p,\theta_1}(R)} = \|f\|_{L_p(R)} + \left(\int_0^\infty \left(h^{-\ell_1+\bar{\ell}_1}\|(\Delta_h^\alpha, D^{\bar{\ell}_1}f)\|_{L_p(R)}\right)^{\theta_1}\frac{dh}{h}\right)^{1/\theta_1}$$

$$\approx \|f\|_{L_p(R)} + \left(\int_0^\infty \left(h^{-\ell_1}\|(\Delta_h^\alpha, f)\|_{L_p(R)}\right)^{\theta_1}\frac{dh}{h}\right)^{1/\theta_1}$$

$$= \|f\|_{L_p(R)} + \left(\int_0^1 \left(h^{-\ell_1}\|(\Delta_h^\alpha, f)\|_{L_p(R)}\right)^{\theta_1}\frac{dh}{h} + \int_1^\infty (h^{-\ell_1}\|(\Delta_h^\alpha, f)\|_{L_p(R)})^{\theta_1}\frac{dh}{h}\right)^{1/\theta_1}$$

This is by theorem 2.1 taking $\alpha_j \in N$ such that $\alpha_j > \ell_1, \ell_2$ and $m_j = 0$

But $\displaystyle\int_1^\infty \left(h^{-\ell_1}\|(\Delta_h^\alpha, f)\|_{L_p(R)}\right)^{\theta_1}\frac{dh}{h} \leq \int_1^\infty \|(\Delta_h^\alpha, f)\|^{\theta_1}_{L_p(R)}\frac{dh}{h^{1+\ell_1\theta_1}}$

$$\leq \int_1^\infty 2^{\theta_1\alpha}\|f\|^{\theta_1}_{L_p(R)}\frac{dh}{h^{1+\ell_1\theta_1}}$$

$$\leq 2^{\theta_1\alpha}\|f\|^{\theta_1}_{L_p(R)}\int_1^\infty \frac{dh}{h^{1+\ell_1\theta_1}}$$

$$= \frac{\left[2^{\theta_1\alpha}\|f\|^{\theta_1}_{L_p(R)}\right]^{\theta_1}}{\theta_1\ell_1}$$

Since $\|f\|_{L_p(R)}$ is finite, we have

$$\int_1^\infty \left(h^{-\ell_1}\|(\Delta_h^\alpha, f)\|_{L_p(R)}\right)^{\theta_1}\frac{dh}{h} \text{ is finite.}$$

This implies the convergence or non-convergence of $\|f\|_{B^{\ell}_{p,\theta_1}(R)}$ is determined by the convergence or non-convergence of

$$\int_0^1 \left(h^{-\ell_1} \|(\Delta_h^\alpha, f)\|_{L_p(R)} \right)^{\theta_1} \frac{dh}{h}$$

Thus,

$$\left(\int_0^1 \left(h^{-\ell_1} \|(\Delta_{h,}^\alpha f)\|_{L_p(R)} \right)^{\theta_1} \frac{dh}{h} \right)^{1/\theta_1} = \left(\int_0^1 \|(\Delta_h^\alpha, f)\|_{L_p(R)}^{\theta_1} \frac{dh}{h^{1+\theta_1 \ell_1}} \right)^{1/\theta_1}$$

$$= \left(\int_0^1 [\left(h^{-\ell_2} \|(\Delta_h^\alpha, f)\|_{L_p(R)} \right)^{\theta_2}]^{\theta_1/\theta_2} h^{(\ell_2-\ell_1)\theta_1} \frac{dh}{h} \right)^{1/\theta_1}$$

Using the Holder inequality with exponent $\frac{\theta 2}{\theta 1} \geq 1$ as $\theta_2 \geq \theta_1$ and with measure $\frac{dh}{h}$ we have that

$$\left(\int_0^1 \left(\left[h^{-\ell_2} \|(\Delta_h^\alpha, f)\|_{L_p(R)} \right]^{\theta_2} \right)^{\frac{\theta 1}{\theta 2}} h^{(\ell_2-\ell_1)\theta_1} \frac{dh}{h} \right)^{1/\theta_1} \leq$$

$$\left(\int_0^1 \left(\left[h^{-\ell_2} \|(\Delta_h^\alpha, f)\|_{L_p(R)} \right]^{\theta_2} \right)^{\frac{\theta 1}{\theta 2} \cdot \frac{\theta 2}{\theta 1}} \frac{dh}{h} \right)^{\frac{\theta 1}{\theta 2} 1/\theta_1} \left(\int_0^1 h^{(\ell_2-\ell_1)\theta_1 \frac{\theta_2/\theta_1}{\frac{\theta 2}{\theta 1}-1}} \frac{dh}{h} \right)^{\left(1-\frac{\theta 1}{\theta 2}\right) 1/\theta_1}$$

Since $(\ell_2 - \ell_1)\theta_1 \dfrac{\theta 2/\theta_1}{\theta 2/\theta_1 - 1} > 0$, we have

$$\left(\int_0^1 h^{(\ell_2-\ell_1)\theta_1 \frac{\theta_2/\theta_1}{\frac{\theta 2}{\theta 1}-1}} \frac{dh}{h} \right)^{\left(1-\frac{\theta 1}{\theta 2}\right) 1/\theta_1} = A_1 \text{ for some } A_1 \in R.$$

Therefore, we conclude that

$$\|f\|_{B^{\ell_1}_{p,\theta_1}(R)} \le A_2 \left(\|f\|_{L_p(R)} + \int_0^\infty \left(h^{-\ell_2} \|(\Delta_h^\alpha, f)\|_{L_p(R)} \right)^{\theta_2} \frac{dh}{h} \right)^{1/\theta_2}$$

$$\le A_3 \|f\|_{B^{\ell_2}_{p,\theta_2}(R)} \text{ by theorem 2.1}$$

for some $A_2, A_3 \in R$.

Case II: If $\theta_2 < \theta_1$

$$\|f\|_{B^{\ell_1}_{p,\theta_1}(R)} \le C_1 \|f\|_{B^{\ell_1}_{p,\theta_2}(R)} \text{ by theorem 2.2}$$

$$\le C_2 \|f\|_{B^{\ell_2}_{p,\theta_2}(R)} \text{ by case (i) for } C_1, C_2 \in R$$

Therefore, in any case we have $\|f\|_{B^{\ell_1}_{p,\theta_2}(R)} \le A \|f\|_{B^{\ell_2}_{p,\theta_2}(R)}$, for $A > 0$

Hence,

$$B^{\ell_2}_{p,\theta_2}(R) \subset B^{\ell_1}_{p,\theta_1}(R)$$

Remark 2.7: Let $l > 0$, $0 < \varepsilon < l$, $1 \le p, \theta, \theta_1, \theta_2 \le \infty$, then

$$B^{\ell+\varepsilon}_{p,\theta_2}(R^n) \subset B^{\ell}_{p,\theta}(R^n) \subset B^{\ell-\varepsilon}_{p,\theta_1}(R^n)$$

Hence the parameter θ is a weaker parameter compared with the main smoothness parameter ℓ.

In this sequel , we have two important theorems about the relations of the Sobolev and Nikolskii Besove spaces and about the space of traces functins and the spaces of functions.

Theorem 2.4: Let $l \in N$, $1 \le p \le \infty$. Then

$$B^{\ell}_{p,\theta_1}\left(R^n\right) \subset W^{\ell}_p(R^n) \subset B^{\ell}_{p,\theta_2}\left(R^n\right)$$

Where $\theta_1 = \min\{p , 2\}$ and $\theta_2 = \max\{p , 2\}$

In particular, if $p = 2$ then $\theta_1 = \theta_2 = 2$

Therefore, we get $B^{\ell}_{2,2}\left(R^n\right) = W^{\ell}_2(R^n$

Theorem 2.5 Let l , m, $n \in N$, $m < n$, $1 \le p \le \infty$ and $l > \dfrac{n-m}{p}$

$$tr_{R^m} W^l_p(R^n) = B^{\ell-\frac{n-m}{p}}_{p,p}\left(R^m\right) = B^{\ell-\frac{n-m}{p}}_{p}\left(R^m\right)$$

Remark 2.8: Let $\ell \in$ N, $1 \le p \le \infty$ and $\Omega \subset R^n$ be an open set with a C^l -boundary. Then

$$tr_{\partial\Omega} W^{\ell}_p(\Omega) = B^{\ell-\frac{1}{p}}_p(\partial\Omega), \qquad \ell > \frac{1}{p}$$

In particular , if $l = 1$ and $p = 2$,we get

$$tr_{\partial\Omega} W^1_2(\Omega) = = B^{\frac{1}{2}}_2(\partial\Omega) = W^{1/2}_2(\partial\Omega)$$

Moreover, for a function f in $w^1_2(\Omega)$, we have

$$\left\| f|_{\partial\Omega} \right\|_{W^{1/2}_2(\partial\Omega)} \le C\|f\|_{w^1_2(\Omega)} \text{ for some } C > 0$$

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[2] we refer for the above theorems and facts the book " Sobolev Spaces on Domains, V.I.Burenkov "

3.Appliction

In the last chapter we have developed the basic concepts of the two function spaces and we have seen some relation between them which are useful for our application purpose. In this sequel, we will see that the two function spaces have a significant role for a generalized formulation of the Dirichlet problem for which to have the necessary and sufficient condition for the function φ to determine the existence of the solution of the problem. First we give the generalized and classical formulations of the Dirichlet problem in the following section.

3.1 The Dirichlet problem

Before we go to the generalized formulation of the Dirichlet problem , it is better to remind the classical formulation of the problem.

Classical formulation

Let $\Omega \subset R^n$ be an open set and $\varphi : \partial\Omega \rightarrow R$ be a continuous function then find a continuous function u on Ω such that

1. u_{xi} exists for each i $\in \{1, 2, \ldots\ldots, n\}$ and $\Delta u = 0$ on Ω ,

   where Δ is the Laplace operator. (3.1)
2. $u = \varphi$ on $\partial\Omega$
3. u is continuous on $\overline{\Omega}$.

This is the well known Dirichlet problem of great importance in Physics.

Generalized formulation

1. $\Delta u = 0$ on Ω a.e , the weak derivatives $\frac{\partial^2 u}{\partial x_i^2}$ exist for each i $\in$ {1, 2 , ...,n}
2. $u = \varphi$ on $\partial\Omega$ (3.2)
3. $u \in W_2^1(\Omega)$

Where $\Omega \subset R^n$ is an open set with C^1 -boundary

Now we are going to formulate the necessary and sufficient condition for φ for a solution of problem (3.2) to exist.

3.2 Variational method of solving Dirichlet problem for Laplace equation

In classical formulation of the Dirichlet problem for Laplace equation (problem 3.1) $\varphi \in C(\partial\Omega)$. But Hadamard's example shows that there is $\varphi \in C(\partial\Omega)$ such that u is not the solution of the generalized formulation of the Dirichlet problem for Laplace equation in problem (3.2). Thus, our aim is to find necessary and sufficient condition for φ insuring the existence of the solution of problem (3.2) using the theory of function spaces and the notion of traces of functions. We give the following Hadamard's remark before we go to the formulation.

Remark 3.1: The generalised formulation involves the assumption that the solution belongs to the sobolev space $W_2^1(\Omega)$. In Hadamards example , the classical solution exists but does not belong to this space. Hence it is not the generalized solution. So Hadamards example shows that not each continouos function belongs to the space $W_2^{1/2}(\partial\Omega)$

The integral $D(u)$ given by the formula

$$D(u) = \int_\Omega \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 dx \tag{3.3}$$

is called the Dirichlet integral. If $D(u) < \infty$, it has a physical interpretation that it describes a potential energy.

If u is in $L_2(\Omega)$ and $D(u) < \infty$ then u is from $W_2^1(\Omega)$ because

$$W_2^1(\Omega) = \left\{ u : u \in L_2(\Omega), \|u\|_{W_2^1(\Omega)} = \|u\|_{L_2(\Omega)} + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L_2(\Omega)} < \infty \right\}$$

The position of equilibrium corresponds to the minimum of the potential energy.

$$\min_{v} D(v) = D(u)$$

From the functional

$$\int_{\Omega} F(u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, x)dx = \int_{\Omega} \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 dx$$

by using Euler-Lagrange equation, it follows that

$$\frac{\partial F}{\partial u} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial F}{\partial u_{x_i}} \right) = 0 \quad , \text{ where } u_{x_i} = \frac{\partial u}{\partial x_i}$$

Which implies

$$0 - 2\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial u}{\partial x_i} \right) = -2\Delta u = 0$$

i.e.

$$\Delta u = 0$$

Thus Problem (3.2) can be formulated as

$$\left.\begin{array}{l} 1.\ \inf_{v \in W^1_{2,\varphi}(\Omega)} D(v) = D(u) \\ 2.\ u|_{\partial\Omega} = \varphi \\ 3.\ u \in W^1_2(\Omega) \quad , \text{where } W^1_{2,\varphi}(\Omega) = \{ u \in W^1_2(\Omega) : u|_{\partial\Omega} = \varphi \} \end{array}\right\} \quad (3.4)$$

Remark 3.2: The formulation of the problem is valid if and only if $W^1_{2,\varphi}(\Omega) \neq \phi$ and

$$W^1_{2,\varphi}(\Omega) \neq \phi \text{ if and only if } \varphi \in W_2^{1/2}(\partial\Omega)$$

Indeed, suppose $W^1_{2,\varphi}(\Omega) \neq \phi$ and hence let $u \in W^1_{2,\varphi}(\Omega)$

Thus $u|_{\partial\Omega} = \varphi$ and $u \in W^1_2(\Omega)$. This implies $\varphi \in tr_{\partial\Omega} W^1_2(\Omega) = B_2^{1/2}(\partial\Omega) = W_2^{1/2}(\partial\Omega)$

So, we are done . To show the other side let $\varphi \in W_2^{1/2}(\partial\Omega)$. Thus $\varphi \in tr_{\partial\Omega} W^1_2(\Omega)$ which implies $W^1_{2,\varphi}(\Omega) \neq \phi$.

The condition $u|_{\partial\Omega} = \varphi$ for all $u \in W_2^1(\Omega)$ follows from the condition

$$\inf_{v \in W_{2,\varphi}^1(\Omega)} D(v) = D(u).$$

Theorem 3.1: Let $\Omega \subset R^n$ be a region with C^1 -boundary . Then problem (3.4) has a solution if and only if $\varphi \in W_2^{1/2}(\partial\Omega)$.

Proof:

($\Rightarrow$) The problem is valid if $W_{2,\varphi}^1(\Omega) \neq \phi$, then by the above remark 3.1

$\varphi \in W_2^{1/2}(\partial\Omega)$.

($\Leftarrow$) Let $\varphi \in W_2^{1/2}(\partial\Omega)$, then by the above remark $W_{2,\varphi}^1(\Omega) \neq \phi$.

Let d = $\inf_{v \in W_{2,\varphi}^1(\Omega)} D(v)$

Then we want to show that , there exists $u \in W_{2,\varphi}^1(\Omega)$ such that $D(u)$ = d.

From the definition of infremum it follows that there is a sequence $\{v_k\}_{k \in N}$ from $W_{2,\varphi}^1(\Omega)$ such that $D(v_k) \to$ d as $k \to \infty$.

Thus, $D(v_k) \geq$ d for all $k \in N$.

Using parallelogram equality for semi-Hilbert space

$$\|v - w\|^2 + \|v + w\|^2 = 2(\|v\|^2 + \|w\|^2)$$

Where,

$$\langle v, w \rangle = \int_\Omega \left(\sum_{i=1}^n \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \right) dx$$

and

$$\|v\|^2 = \langle v, v \rangle = D(v) = \int_\Omega \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i} \right)^2 dx$$

Now, letting $v = \frac{v_k}{2}$ and $w = \frac{v_m}{2}$,

we have

$$D(\frac{v_k - v_m}{2}) + D(\frac{v_k + v_m}{2}) = \frac{1}{2}(D(v_k) + D(v_m))$$

Then,

$$0 \le D(\frac{v_k - v_m}{2}) \le \frac{1}{2}(D(v_k) + D(v_m)) - D(\frac{v_k + v_m}{2})$$

Since $W_2^1(\Omega)$ is linear , $\frac{v_k + v_m}{2} \in W_{2,\varphi}^1(\Omega)$.

And,

$$\frac{v_k + v_m}{2}\,|_{\partial\Omega} = \frac{1}{2}\left(v_k|_{\partial\Omega} + v_m|_{\partial\Omega}\right) = \varphi$$

Therefore,

$$0 \le D(\frac{v_k - v_m}{2}) \le \frac{1}{2}(D(v_k) + D(v_m)) - \text{d} \text{ as } D(\frac{v_k + v_m}{2}) \ge \text{d}.$$

$$\Leftrightarrow \qquad \frac{1}{4}D(v_k - v_m) \to 0 \text{ as } k, m \to \infty.$$

Thus , because of the completeness of the space $w_2^1(\Omega)$

where

$$w_2^1(\Omega) = \left\{u : \|u\|_{w_2^1(\Omega)} = \sqrt{D(u)} < \infty \right\}$$

there exists $u \in w_2^1(\Omega)$ such that $\|u - v_k\|_{w_2^1(\Omega)} \to 0$ as $k \to \infty$.

And also because of continuity of semi norms

$$\|v_k\|_{w_2^1(\Omega)} \to \|u\|_{w_2^1(\Omega)} \text{ as } k \to \infty,$$

which means $D(v_k) \to D(u)$ as $k \to \infty$.

Thus we have $D(u) = \lim_{k\to\infty} D(v_k) = \text{d}$ and $D(u) = \inf_{v \in W_{2,\varphi}^1(\Omega)} D(v)$.

On the other hand according to the theorem about traces, we have

$$\|f|_{\partial\Omega}\|_{W_2^{1/2}(\partial\Omega)} \le C\|f\|_{w_2^1(\Omega)} \text{ for some } C > 0.$$

If $f = u - v_k$ then

$$\|u|_{\partial\Omega} - v_k|_{\partial\Omega}\|_{W_2^{1/2}(\partial\Omega)} \le C\|u - v_k\|_{w_2^1(\Omega)} \to 0 \text{ as } k \to \infty.$$

which implies that

$$\|u|_{\partial\Omega} - \varphi\|_{W_2^{1/2}(\partial\Omega)} = 0$$

$$u|_{\partial\Omega} = \varphi$$

There fore we proved that

$$u \in W^{1}_{2.\varphi}(\Omega)$$

Theorem 3.2: Let $\Omega \subset R^n$ be a region with C^1-boundary, $\varphi \in W_2^{1/2}(\partial\Omega)$. Then if u is the solution of problem (3.3), then for all $\psi \in C_o^\infty(\Omega)$ we have

$$\int_\Omega \Big(\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_i} \Big) dx = 0.$$

Proof: Let $\psi \in C_0^\infty(\Omega)$. Then since $C_0^\infty(\Omega) \subset W_2^1(\Omega)$, we have $u + t\psi \in W^1_{2,\varphi}(\Omega)$, $t \in R$.

$$u + t\psi \,|_{\partial\Omega} = u\,|_{\partial\Omega} + t\psi\,|_{\partial\Omega} = \varphi \text{ as } t\psi\,|_{\partial\Omega} = 0$$

Then $$D(u) = \min_{t\in R} D(u + t\psi)$$

and

$$\min_{t\in R} D(u+t\psi) = \min_{t\in R} \phi(t)$$

$$= \phi(0)$$

where

$$\phi(t) = D(u+t\psi) \text{ and } D(u+t\psi) = \int_\Omega \sum_{i=1}^{n} \left(\frac{\partial(u+t\psi)}{\partial x_i} \right)^2 dx.$$

Then $\phi'(0) = 0$ as 0 is the minimum point.

$$D(u+t\psi)'|_{t=0} = \frac{d}{dt} \int_\Omega \sum_{i=1}^{n} \left(\frac{\partial(u+t\psi)}{\partial x_i} \right)^2 dx \,|_{t=0}$$

$$= \frac{d}{dt} \left(\int_\Omega \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 dx + 2t \int_\Omega \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_i} \right) dx + t^2 \int_\Omega \sum_{i=1}^{n} \left(\frac{\partial \psi}{\partial x_i} \right)^2 dx \Big|_{t=0} \right) = 0$$

This implies

$$2\int_\Omega \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_i} \right) dx = 0.$$

Thus we have the theorem.

References

1. V. I. Burenkov, *Sobolev spaces on domains,*
   Moscow / Cardiff, 1997

2. Ayele T. G. , *Short notes on Nikol Skii Besov spaces*

3. K. F. Riley and S. J. Bence, M. P. Hobson , *Mathematical Methods for Physics and Engineering,*
   Cambridge Low - Price Edition Cambridge , 1997

4. H. L. Royden, *Real Analysis,*
   Macmillan Publisher, New York, 1963

5. V. S. Vladmirov, *Generalized Functions in Mathematical Physics*
   Mir publishers, Moscow, 1979