Addis Ababa University
College of natural and computational science Department of Mathematics
Thesis on
Colombeau Generalized Functions
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DECLARATION

I declare that this thesis has been composed by me and that no part of the thesis has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship, or any other similar title to me.

Addis Ababa July 2024

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Title:Colombeau Generalized Function

PERMISSION

This is to certify that this thesis is compiled by Mr. TESFAYE KASSA in the Department of Mathematics, Addis Ababa University, under my supervision.

I hereby also confirm that the thesis can be submitted for evaluation by examiners and eventual defense.

Addis Ababa July 2024

ADDIS ABABA UNIVERSITY DEPARTMENT OF MATHEMATICS

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Abstract

Some outcomes regarding products of distributions are presented in this thesis. The results are obtained in the most applicable algebra for nonlinear problems related to Schwartz distributions, which is the Colombeau algebra of generalized functions.

Acknowledgment

I humbly extend my most heartfelt thanks to God, who, in his greatness, gave me this chance to live and go through all such experiences in learning up to this level, of which i never even thought of. Second, my gratitude goes to my advisor, Dr. Tsegaye Gedife, for his constructive comments and a very friendly approach all through the work. Finally, i would also like to express profusely my thanks to all my instructors for the legged support they gave me during my coursework.

Contents

Introduction

Because of the wide application of distributions in the natural sciences and other mathematical fields where products of distributions with coinciding singularities often appear, the problem of multiplication of Schwartz distributions has been for a long time interest of many researchers.

One of the most useful aspects of Schwartz's theory of distributions in applications is that discontinuous functions can be handled as easily as continuous or diferentiable functions which provides a powerful tool in formulating and solving many problems of various fields of science and engineering .

In applications the results sometimes show that multiplications of two generalized functions are not always allowed, so there have been made many attempts to done products of distributions, or rather to enlarge the range of existing products . several attempts have been made to include the distributions into diferential algebras.

Colombeau describes the problem of multiplication of distributions and shows how to overcome it. his theory was primarly been used for dealing with nonlinear partial diferential equations with singularities and was developed during the years and it has now a big appliance in a different fields (physics, geometry, etc).

The diferential Colombeau algebra G(R) as a powerful tool for treating linear and nonlinear problems including singularities has almost the optimal properties while the problem of multiplication of Schwartz distributions is concerned: it is an associative diferential algebra of generalized functions, contains the algebra of smooth functions as a subalgebra (elements of this algebra are some equivalence classes of nets of smooth functions), the distribution space D0 is linearly embedded in it as a subspace and the multiplication is compatible with the operations of diferentiation and products with diferentiable functions.

We evaluate in this thesis some products of distributions with coinciding singularities, as embedded in Colombeau algebra, in terms of associated dis- tributions. the results obtained in this way can be reformulated as regularized products in the classical distribution theory.

Notation and Symbols

N: the set of natural numbers.

N_0 : the set of non-negative integers.

R: the set of all real numbers.

$\mathbb{C}$: the set of all complex numbers.

N_o^n : $N_0 \mathrm{x} \ldots \ldots \mathrm{x} N_0$ (n-times): the set of multi- indices.

$\mathrm{R^n}$: n-dimensional Euclidean space and $\mathrm{x} = (\mathrm{x_1}, \mathrm{x_2}, \ldots \mathrm{x_n})$ be a variable element of $\mathrm{R^n}$.

Domain Ω : an open connected region

A domain Ω in $\mathrm{R^n}$: An open connected subset $\Omega \subseteq \mathrm{R^n}$

$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$:The scalar product in $\mathrm{R^n}$.

For an arbitrary non- empty set $\Omega \subset \mathrm{R^n}$ we shall denote by:

C (Ω) : The set of all continuous functions in Ω.

$C^1(\Omega)$:The set of all once continuously differentiable functions in Ω.

$C^2(\Omega)$: The set of all twice continuously differentiable functions in Ω.

$C^\infty(\Omega)$: The space of infinitely continuously differentiable functions on Ω.

$C_o^\infty(\Omega)$: The space of functions in $C^\infty(\Omega)$ with compact support.

$\mathcal{D}'$:The space of Schwartz distributions.

$\mathcal{G}$: The algebra of Colombeau generalized functions, or $\mathcal{G}$-functions, on Ω.

$\mathcal{F}(f) = \hat{f}\,(f \in S\,(R^n))$:The Fourier transform of a function f.

Chapter 1

Distribution Theory

1.1 Distributions (Generalized functions):

A concept of generalized function (distribution) constitutes a generalization of the classical concept of a function.

It is useful to give precise mathematical meaning for such quantities as mass density or electric charge density of a material point. Such quantities are defined as "average values", what is consistent with experimental approach in which one measures not a density at given point but rather their average values in some sufficiently small neighborhood of the point. As an example we shall consider the electric charge $q > 0$ distributed uniformly in a ball with the radius ε and center at $\boldsymbol{x}_0 = \mathbf{0}$. The charge density of a configuration such as this is given by the following function:

On a material which has point of mass 1 , assume that the point is on the origin of the coordinates, the mean density $f_\varepsilon(x)$ is given by

$$f_\varepsilon(x) = \begin{cases} \frac{3}{4\pi\varepsilon^3} & \text{if } |x| < \varepsilon \\ 0 & \text{if } |x| > \varepsilon \end{cases}$$

As $\varepsilon \to 0$, the mean density $f_{\varepsilon(x)}$ becomes a function

$$\boldsymbol{\delta}(\boldsymbol{x}) = \begin{cases} \infty, \text{ if } & x = 0 \\ 0, \text{ if } & x \neq 0 \end{cases}$$

The integral of the density over the entire space yield the total mass of substance, that is,

$$\int \delta(x)dx = 1 \tag{3}$$

But for the functional $\delta(x)$ defined by (2) $\int \delta(x)dx = 0$. This means that the function does not satisfy the requirement of (3).

Let us now find a somewhat different limit of the sequences of mean densities $f_\varepsilon(x)$, the so called weak limit.

It will readily be seen that for any continuous function $\varphi(x)$,

Formula (1) states that the weak limit of a sequence of function $f_\varepsilon(x), \varepsilon \to 0$ is a functional $\varphi(0)$, that relates to every continuous function $\varphi(x)$, a number $\varphi(0)$, which is its value at the point x = 0.

It is this functional which is taken as the definition of the density $\delta(x)$, and this is the well known Dirac $\boldsymbol{\delta}$ function.

$$\text{So } f_\varepsilon(x) \xrightarrow{\text{weak}} \delta(x) \text{ as } \varepsilon \to 0$$

The value of the function δ on the function φ (the number $\varphi(0)$) will be denoted,

$$\delta(\varphi) = \langle \delta, \varphi \rangle = \varphi(0)$$

This is the value of the functional δ on the function φ.

Definition 1.1.1. *Let f be a complex valued function defined on an open subset $\Omega \subset R^n$.*

We call support of f and denote it by $\operatorname{supp} f = \overline{\{x \in \Omega / f(x) \neq 0\}}$

The support of f is the smallest relative closed set outside of which f is identically zero.

Definition 1.1.2. *Let $\Omega \subset R^n$ be an open set. Included in the set of Test functions* D(Ω) *are all functions which have compact support and infinitely differentiable functions in Ω. That is,*

$$\mathrm{D}(\Omega) = \{\varphi \in C^\infty(\Omega) / \operatorname{supp} \varphi \textit{ is compact } \}$$

Convergence in D($\boldsymbol{\Omega}$) *: A given sequence $\{\varphi_n\}$ in* D(Ω) *converges to φ in* D(Ω) *if*

i.$\exists$ *a compact set* k $\subset\subset \Omega$ *such that* $\operatorname{supp} \varphi_n \subset k, \forall$n.

ii. $\forall \alpha \in N_0^n, D^\alpha \varphi_n(x) \Rightarrow D^\alpha \varphi(x), \mathrm{x} \in \Omega$, as n $\to \infty$, In this case we shall write $\varphi_n \to \varphi$, as n $\to \infty$ in D(Ω).

The linear set D(Ω) equipped with convergence is called the space of test (basic) function D(Ω).

Example 1.1.1. $\varphi(x) = \begin{cases} e^{-\left(\frac{1}{1-|x|^2}\right)}, & when\ |x| < 1 \\ 0, & when\ |x| \geq 1 \end{cases}$ *is in D.*

For $|x| < 1$

$$\frac{\partial \varphi}{\partial x_j} = \frac{-2x_j}{(1-|x|^2)^2}\varphi(x), \quad |x|^2 = x_1^2 + \cdots + x_n^2$$

The $\varphi^n(x)$ partial derivative has the form $\frac{p(x)}{(1-|x|^2)^n} \cdot \varphi(x);$ p(x) is a polynomial.

Now put t = $|x|$

Then $\forall x \in R^n \backslash \{0\}$, x $\mapsto \|x\| =$ t.

Then

$$\varphi(x) = \varphi(t) = \begin{cases} e^{-\left(\frac{1}{1-t^2}\right)}, & \text{when } t < 1 \\ 0, & \text{when } t \geq 1 \end{cases}$$

Then $D^n\varphi(t) = \frac{p(t)}{(1-|x|^2)^n} \cdot \varphi(t)$ which is infinitely differentiable.

Also

$$\begin{aligned} \text{Supp}(\varphi) &= \overline{\{x \in \Omega : \varphi(x) \neq 0\}} \\ &= \{x \in \Omega : |x| \leq 1\} \end{aligned}$$

the supp (φ) is a compact set (since it is closed and bounded) and $\varphi \in \mathrm{C}^\infty(\Omega)$.

1.1.1 The Space of Distributions D'

Definition 1.1.3. *A continuous linear functional on the space of test functions D is called a generalized function /distribution/.The set of all generalized function is denoted by* $D' = D'(\mathrm{R^n})$.

We will write the value of the functional (generalized function) f on the basic function φ as (f, φ).

We say that $f \in D'$ if it satisfies the following conditions.

- A distribution $f \in D'$, is a functional on the space of basic function D, that is, with each basic function φ there is associated a (complex valued) number $\langle f, \varphi \rangle$.
- A distribution $f \in D'$, is a linear functional on D, that is, If $\varphi, \psi \in D$ and $\alpha, \beta \in \mathbb{C}$, then

$$\langle f, \alpha\varphi + \beta\psi\rangle = \alpha\langle f, \varphi\rangle + \beta\langle f, \psi\rangle$$

- A distribution $f \in D'$ is a continuous functional on D, that is, if $\varphi_n \to \varphi$ in D(Ω) as n $\to \infty$, then

$$\langle f, \varphi_n\rangle \to \langle f, \varphi\rangle \text{ as n} \to \infty.$$

The generalized functions f and g specified in Ω are said to be equal in Ω if they are equal as functional on D(Ω), that is, if for any φ in D(Ω), $\langle f, \varphi\rangle = \langle g, \varphi\rangle$. We will write: $f = g$ in Ω or $f(x) = g(x), x \in \Omega$.

Theorem 1.1.1. *The space D' is linear if we define the linear combination $\lambda f + \mu g$ of the generalized function f and g in* D(Ω) *as a functional acting via the formula:*

$$\langle \lambda f + \mu g, \varphi\rangle = \lambda\langle f, \varphi\rangle + \mu\langle g, \varphi\rangle, \varphi \in D.$$

Proof. we show that $\lambda f + \mu g$ is linear and continuous function on D, i. e. belong to D'. Let $\varphi, \psi \in D$ and $\alpha, \beta \in \mathbb{C}$. Then,

$$\begin{aligned}\langle \lambda f + \mu g, \alpha\varphi + \beta\psi\rangle &= \lambda\langle f, \alpha\varphi + \beta\psi\rangle + \mu\langle g, \alpha\varphi + \beta\psi\rangle \\ &= \alpha[\lambda\langle f, \varphi\rangle + \mu\langle g, \varphi\rangle] + \beta[\lambda\langle f, \psi\rangle + \mu\langle g, \psi\rangle] \\ &= \alpha\langle \lambda f + \mu g, \varphi\rangle + \beta\langle \lambda f + \mu g, \psi\rangle\end{aligned}$$

Hence, $\lambda f + \mu g$ is linear

Its continuity follows from the continuity of the functional f and g. If $\varphi_k \to \varphi$ as k $\to \infty$ in D, then

$$\langle \lambda f + \mu g, \varphi_k\rangle = \lambda\,\langle f, \varphi_k\rangle + \mu\,\langle g, \varphi_k\rangle \to \lambda\langle f, \varphi\rangle + \mu\langle g, \varphi\rangle = \langle \lambda f + \mu g, \varphi\rangle \text{ as k} \to \infty$$

Hence, $\lambda f + \mu g$ is continuous.

Therefore, $\lambda f + \mu g$ is linear and continuous on D. i.e. $\lambda f + \mu g \in D'$.

Convergence in D$'$: Let $\{f_n\}$ be a sequence of distributions. Then f_n is said to be convergent to the distribution f if and only if

$$\langle f_n, \varphi\rangle \xrightarrow{n\to\infty} \langle f, \varphi\rangle \text{ for all } \varphi \in D(\Omega)$$

In this case we shall write $f_n \to f$ as n $\to \infty$ in D'

This convergence is called weak convergence.

□

Definition 1.1.4. *A measurable function* $f : \Omega \to \mathbb{C}$ *is said to be locally integrable if*

$$\int_{\mathrm{K}} |f(x)|dx < \infty \text{ for all compact } \mathrm{K} \subset \Omega.$$

1.1.2 Regular Distributions

A regular distribution is a distribution that can be represented as the integral of the product of a locally integrable function and a test function over the domain Ω. i.e functional generated by the function f(x) locally integrable in R^n :

$$(f, \varphi) = \int_{\Omega} f(x)\varphi(x)dx, \quad \varphi \in D$$

From the property of linearity of the integral follows the linearity of this functional:

$$\begin{aligned}(f, \lambda\varphi + \beta\psi) &= \int f(x)[\lambda\varphi(x) + \beta\psi(x)]dx \\ &= \lambda \int f(x)\varphi(x)dx + \beta \int f(x)\psi(x)dx \\ &= \lambda(f, \varphi) + \beta(f, \psi)\end{aligned}$$

While from the theorem which concerns proceeding to the limit under the integral sign follows the continuity of this functional on D:

$$(f, \varphi_n) = \int f(x)\varphi_n(x)dx \to \int f(x)\varphi(x)dx = (f, \varphi)$$

as n $\to \infty$ if $\varphi_n \to \varphi$ as n $\to \infty$ in D.

To show (1) is functional,we have f is locally integrable and φ is compact support then $(f, \varphi) = |\int_{\Omega} f(x)\varphi(x)dx| \leq \int_{\Omega} |f(x)\varphi(x)dx| \leq \int_{\Omega} |f(x)||\varphi(x)|dx \leq sup \int_{\Omega} |f(x)|dx \leq \infty$

Thus the functional defined in (1) defines a generalized function belonging to D'.

Examples of regular distributions are:

- The Heaviside step function ;H(x) defined by

$$H(x) = \begin{cases} 1; x > 0 \\ 0; x < 0 \end{cases}$$

This function is locally integrable and corresponding to regular distribution is defined by: $\mathrm{H}(\varphi) = \int_\Omega H(x)\varphi(x)\mathrm{dx}$

1.1.3 Singular distributions

A singular distribution is a distribution that cannot be represented as the integral of a locally integrable function multiplied by a test function.

examples of singular distribution are:

- The dirac delta function,denoted as δ
- The function $\frac{1}{x}$ is singular distribution because $\int_{-\infty}^{\infty} \frac{\varphi}{x} dx$ is not convergent.

Definition 1.1.5. *For $T_f \in D'(\Omega)$, if there is $f \in L^1_{loc}(\Omega)$ via the formula:*

$(T_f, \varphi) = \int_\Omega f(x)\varphi(x)dx, \quad \varphi \in D$, are called regular distributions and all other generalized functions are called singular distributions.

Example 1.1.2. *Let $x_0 \in \Omega_0$ be a fixed element. we consider the function*

$$\delta_{x_0} : \mathrm{D}(\Omega) \to \mathbb{C} \text{ defined by } \varphi \mapsto \varphi(x_0).$$

We have for each compact $\mathrm{K} \subset \Omega$,

$$|\delta_{x_0}(\varphi)| = |\varphi(x_0)| \leq \|\varphi\|$$

This implies that δ_{x_0} is continuous. we call δ_{x_0} the Dirac mass, distribution or delta function at x_0. in the case $x_0 = 0$ we will write simple δ instead of δ_0.

The distribution δ_{x_0} is not a regular distribution. this we will prove now.

Let $\Omega_0 = \Omega \backslash \{x_0\}$.

Let $\mathrm{K} \subset \Omega_0$ be a compact set.

Then (by definition) we have that $D_K(\Omega_0)$ is the set of all
$\varphi : \Omega_0 \to \mathbb{C}, \varphi \in C^\infty(\Omega_0)$, supp $(\varphi) \subseteq K \subset \Omega_0$.
Now we consider $\varphi \in \delta_{x_0} : D_K(\Omega_0)$.
For this φ we have $\varphi : \Omega \to \mathbb{C}$, $\varphi \in C^\infty(\Omega)$, supp $(\varphi) \subseteq K \subset \Omega_0$.
Therefore, we have $\varphi(x_0) = 0$. since $x_0 \notin \Omega_0$, i.e. $x_0 \notin \mathrm{supp}(\varphi)$.
This implies $\delta_{x_0} : D_K(\Omega_0) = 0$, i.e. $\langle \delta_{x_0}, \varphi \rangle = 0$ for all $\varphi \in D_K(\Omega_0)$.
Now we assume that there is an $f \in L^1_{loc}(\Omega)$ such that $\delta_{x_0} = f$.

$$\langle \delta_{x_0}, \varphi \rangle = \langle f, \varphi \rangle = \int_{\Omega} f(x)\varphi(x)dx = \int_{\Omega_0} f(x)\varphi(x)dx = 0 \text{ for all } \varphi \in D_K(\Omega_0).$$

immediately

$$f(x) = 0 \text{ for almost all } x \in \Omega_0,$$

and therefore,

$$f(x) = 0 \text{ for almost all } x \in \Omega.$$

But this is not true, because for a function $\varphi \in D(\Omega)$ for which $x_0 \in \operatorname{supp}(\varphi)$ we get

$$\langle \delta_{x_0}, \varphi \rangle = \varphi(x_0) \neq 0, \text{ i.e } \delta_{x_0} \neq 0$$

Lemma 1.1.2. *The support of a generalized function f is the completion of Ω_f to Ω,*

so that, $\operatorname{Supp} f = \Omega \backslash \Omega_f$: $\operatorname{supp} f$ is closed set in Ω. Where Ω_f is the largest open set in which f vanishes. Also, the support of f is the smallest closed subset of Ω out side of which the distribution f is zero.

Lemma 1.1.3. *Let $\{f_k\}$ be a sequence of distributions.*

If there is a linear mapping $f : \mathrm{D}(R^n) \to \mathbb{C}$ such that

$$\langle f_k, \varphi \rangle \xrightarrow{k \to \infty} \langle f, \varphi \rangle \text{ for all } \varphi \in \mathrm{D}(R^n),$$

then f is a distribution.

Proof: We have only to show that f is linear and continuous, i.e. $f \in D'(R^n)$. From

$$\langle f, \varphi \rangle = \lim_{k \to \infty} \langle f_k, \varphi \rangle$$

We get $\langle f, \varphi_1 + \varphi_2 \rangle = \lim_{k \to \infty} \langle f_k, \varphi_1 + \varphi_2 \rangle$

$$\begin{aligned} &= \lim_{k \to \infty} (\langle f_k, \varphi_1 \rangle + \langle f_k, \varphi_2 \rangle) \\ &= \lim_{k \to \infty} \langle f_k, \varphi_1 \rangle + \lim_{k \to \infty} \langle f_k, \varphi_2 \rangle \\ &= \langle f, \varphi_1 \rangle + \langle f, \varphi_2 \rangle \text{ and} \end{aligned}$$

$$\langle f, \alpha\varphi\rangle = \lim_{k\to\infty} \langle f_k, \alpha\varphi\rangle$$
$$=\alpha \lim_{k\to\infty} \langle f_k, \varphi\rangle \quad = \alpha\langle f, \varphi\rangle$$

i.e. f is linear.

For the proof of the continuity of f we consider a sequence (φ_v), where $\varphi_v \in D(R^n)$ and $\varphi_v \to 0$ as $v \to \infty$. we have to prove that $\langle f, \varphi_V\rangle \to 0$ as $v \to \infty$, i.e. we have to prove that f is continuous at 0 .

We assume that the last is not true. then there is an $\alpha > 0$ such that

$$|\langle f, \varphi_v\rangle| \geq 2\alpha \text{ for all v} \in N.$$

Since $\langle f, \varphi\rangle = \lim_{k\to\infty} \langle f_k, \varphi\rangle$ for each $\varphi \in D(R^n)$ we get particularly $\langle f, \varphi_v\rangle = \lim_{k\to\infty} \langle f_k, \varphi_v\rangle$ and therefore

$$2\alpha \leq |\langle f, \varphi_v\rangle| = \left|\lim_{k\to\infty} \langle f_k, \varphi_v\rangle\right| = \lim_{k\to\infty} |\langle f_k, \varphi_v\rangle| \text{ for all v} \in N$$

From this it follows that for each $\epsilon > 0$ and for each v $\in N$ there is an index k_v such that

$$2\alpha - \epsilon \leq |\langle f_{k_v}, \varphi_v\rangle| \text{ for all v} \in N$$

To make it easier we choose $\epsilon = \alpha$. then we have that for each v $\in N$ there is an index k_v such that

$$\alpha \leq |\langle f_{k_v}, \varphi_v\rangle| \text{ for all v} \in N$$

since $\varphi_v \to 0$ as v $\to \infty$ in D(Ω),then $\langle f, \varphi_v\rangle \to 0$ as v $\to \infty$. then we get that $\langle f_{k_v}, \varphi_v\rangle \to 0$ as v $\to \infty$ for all k $\in N$, since $f_k \in D'(R^n)$.this implies that $\langle f_{k_v}, \varphi_v\rangle \to 0$ as v $\to \infty$.

But this is a contradiction .

Example 1.1.3. *We consider Cauchy's principal value of*

$$\int_R \frac{\varphi(x)}{x}\mathrm{dx}, \textit{ where } \varphi \in D(R).$$

Cauchy's principal value is defined as

$$\text{CPV} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x} dx = \lim_{\varepsilon\to 0} \left[\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x} dx + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x} dx\right]$$

We define the mapping,

$$\langle p, \varphi \rangle = \text{CPV} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x} dx$$

And we show that p is a distribution. first we show that the mapping p_k defined by

$$\langle p_k, \varphi \rangle = \int_{|x|>k} \frac{\varphi(x)}{x} dx, \varphi \in D(R), \text{ is a regular distribution for each k} \in N$$

For this we define,

$$f_k(x) = \begin{cases} 0, \text{ if } |x| \leq \frac{1}{k} \\ \frac{1}{x}, \text{ if } |x| > \frac{1}{k} \end{cases}, \quad \text{k} \in N$$

Then f_K is locally integrable. to see this we consider an arbitrary compact set k $\subset R$. let a $= \min k$, b $= \max k$ and c $= \max \left\{ |\text{a}|, |\text{b}|, \frac{1}{k} \right\}$. Then we have

$$\int_K |f_k(x)|\, dx \leq \int_{-c}^{c} |f_k(x)|\, dx = \int_{-c}^{-\frac{1}{k}} |f_k(x)|\, dx + \int_{-\frac{1}{k}}^{\frac{1}{k}} |f_k(x)|\, dx + \int_{\frac{1}{k}}^{c} |f_k(x)|\, dx$$

$=$-$\ln(-x)|_c^{-\frac{1}{k}} + \ln x|_c^{\frac{1}{k}}$=-$\left(\ln \frac{1}{k} - \ln c\right) + \ln c - \ln \frac{1}{k}$.

$=2\left[\ln c - \ln \frac{1}{k}\right] < \infty$.

Then we get that

$$\langle T_{f_k}, \varphi \rangle = \int_{-\infty}^{\infty} f_k(x)\varphi(x) dx = \int_{-\infty}^{\frac{-1}{k}} \frac{\varphi(x)}{x} dx + \int_{\frac{1}{k}}^{\infty} \frac{\varphi(x)}{x} dx$$
$$= \int_{|x|>\frac{1}{k}} \frac{\varphi(x)}{x} dx = \langle p_k, \varphi \rangle \text{ is}$$

a regular distribution.

Now we have,

$$\begin{aligned}\lim_{\mathrm{k}\to 0}\langle T_{f_k},\varphi\rangle &= \lim_{\mathrm{k}\to 0}\langle p_k,\varphi\rangle \\ &= \lim_{\mathrm{k}\to 0}\left[\int_{-\infty}^{-\frac{1}{k}}\frac{\varphi(x)}{x}dx + \int_{\frac{1}{k}}^{\infty}\frac{\varphi(x)}{x}dx\right] \\ &= \mathrm{CPV}\,\frac{\varphi(x)}{x}dx = \langle p,\varphi\rangle\end{aligned}$$

i.e p is the (weak) limit of a sequence of (regular) distributions. Therefore, p is a distribution.

1.1.4 Multiplication of a distribution by a function $f \in \boldsymbol{C}^{\infty}(\Omega)$

Definition 1.1.6. *Let* $\mathrm{T} \in D'(\Omega), \varphi \in \mathrm{D}(\Omega)$ *and* $f \in C^{\infty}(\Omega)$*.*

Then $\langle f\cdot \mathrm{T},\varphi\rangle = \langle \mathrm{T}, f\cdot\varphi\rangle \quad \varphi \in D(\Omega)$ is called the product of f and T.

Example 1.1.4. *Let* $\mathrm{T} = \delta$*. then*

$$\begin{aligned}\langle f\cdot \mathrm{T},\varphi\rangle &= \langle f\cdot\delta,\varphi\rangle \\ &= \langle \delta, f\cdot\varphi\rangle \\ &= f(0)\varphi(0) \\ &= f(0)\delta(\varphi)\end{aligned}$$

Therefore, $(f\cdot\delta) = f(0)\delta$

Example 1.1.5. *If* f_1 *and* f_2 *are two distributions, we cannot assign a distribution* $\Pi(f_1, f_2)$ *in the same way that it amount of the usual product for regular distribution and is at the same time continuous in both distribution. Let* $f_k(\mathrm{x}) = \sin(\mathrm{kx})$ *in* $\mathrm{D}'(\Omega)$*. then*

$$\pi(f_k, f_k) = \sin^2(\mathrm{kx}) \ \ (i.e.\ \pi(f_k,f_k) = \pi(f_k)(f_k) = f_k(\mathrm{x})f_k(\mathrm{x}))$$

Although $f_k \to 0$ *in* D' *as* $\mathrm{k}\to\infty$

$$\pi(f_k, f_k) \to \frac{1}{2} \ \ as\ \mathrm{k}\to\infty$$

Solution:-

For any test function φ, we have

$$\langle f_k, \varphi\rangle = \int_R \sin(kx)\varphi(x)dx$$
$$= -\frac{1}{k}\int_R \cos(kx)\varphi'(x)dx, \text{ using integration by part}$$

So that $|\langle f_k, \varphi\rangle| \xrightarrow{\mathrm{k}\to\infty} 0$, for all $\varphi \in D(\Omega)$
This implies that $f_k \to 0$ in $\mathrm{D}'(\Omega)$.
Similarly

$$\langle \pi(f_k, f_k), \varphi\rangle = \int_{\mathrm{R}} \sin^2(\mathrm{kx})\varphi(\mathrm{x})\mathrm{dx}$$
$$= \int_R \frac{1-\cos(2kx)}{2}\varphi(x)dx$$
$$= \int_R \frac{\varphi(x)}{2}dx - \int_{\mathrm{R}} \frac{\cos(2\mathrm{kx})}{2}\varphi(x)dx$$
$$= \frac{1}{2}\int_R \varphi(x)dx - \frac{1}{2}\int_{\mathrm{R}} \cos(2\mathrm{kx})\varphi(\mathrm{x})\mathrm{dx}$$
$$= \frac{1}{2}\int_R \varphi(x)dx - \frac{1}{4k}\int_{\mathrm{K}} \sin(2\mathrm{kx})\varphi'(\mathrm{x})\mathrm{d}x \text{ using integration by part.}$$

So that, $|\langle \pi(f_k, f_k), \varphi\rangle| \xrightarrow{\mathrm{k}\to\infty} \frac{1}{2}\int_{\mathrm{R}} \varphi(\mathrm{x})\mathrm{dx} = \left\langle \frac{1}{2}, \varphi\right\rangle$
This implies that

$$\pi(f_k, f_k) \to \frac{1}{2} \text{ as } \mathrm{k} \to \infty$$

Theorem 1.1.4. *Let* $f \in \mathrm{D}'(\Omega)$ *and* $\varphi \in D(\Omega)$. *Then*
$\mathrm{Supp}(f\varphi) \subset \mathrm{supp}\, f \cap \mathrm{supp}\, \varphi$

Proof. Let $\Omega_{f\varphi}$ be the largest open set in $\mathbf{\Omega}$ which contains both Ω_f and Ω_φ.
That is $\Omega_f \cup \Omega_\varphi \subset \Omega_{f\varphi}$.

$$\mathrm{Supp}\, f\varphi = \Omega\backslash\Omega_{f\varphi} \subset \Omega\backslash(\Omega_f \cup \Omega_\varphi)$$
$$\subset (\Omega\backslash\Omega_\varphi) \cap (\Omega\backslash\Omega_\varphi)$$
$$\subset \mathrm{Supp}\, f \cap \mathrm{Supp}\, \varphi$$

□

1.1.5 The Space of basic (rapidly decreasing) functions

Definition 1.1.7. *Let $f : R^n \to \mathbb{C}$ and $x^\alpha = {x_1}^{\alpha_1}...{x_n}^{\alpha_n}$. the function f is said to be rapidly decreasing if and only if*

$$\lim_{|x|\to\infty} x^\alpha f(x) = 0 \text{ for all } \alpha \in {N_0}^n$$

The space $\mathrm{S}(R^n) = \left\{ f \in C^\infty(\Omega)/D^\beta f \text{ is rapidly decreasing for all } \beta \in N_0^n \right\}$ *is called the space of rapidly decreasing functions.*

Therefore, the space $\mathrm{S} = \mathrm{S}(R^n)$ of all functions infinitely differentiable in R^n that decrease together with all their derivatives, as $|x| \to \infty$, faster than any power of $|x|^{-1}$ is called the space of rapidly decreasing functions. For instance, $\varphi(x) = e^{-|x|^2} \in S$.

Convergence in S: The sequence of function $\varphi_1, \varphi_2...$ belonging to S converges to a function φ in S, denoted by $\varphi_k \to \varphi$ as k $\to \infty$ in S. if for all α and β

$$x^\beta D^\alpha \varphi_k(x) \Rightarrow x^\beta D^\alpha \varphi(x), \text{ as k} \to \infty$$

1.1.6 The Space of Tempered Distributions S'

Definition 1.1.8. *A generalized function of slow growth is any continuous linear functional on the space* S *of test functions. We denote by* $\mathrm{S}' = \mathrm{S}'(R^n)$ *the set of all generalized functions of slow growth.*

Convergence in S' : A sequence of generalized functions $f_1, f_2...$ taken from S' converges to the generalized function $f \in S'$, denoted by $f_k \to f$, as k $\to \infty$ in S', if for any $\varphi \in S$, $\langle f_k, \varphi\rangle \to \langle f, \varphi\rangle$, k $\to \infty$. this convergence is called a weak convergence of a sequence of functional.

The linear set $S'(\Omega)$ equipped with convergence is termed the spaces of generalized functions of slow growths S'

The space $\mathrm{S}'(R^n)$ is a sub space of $\mathrm{D}'(R^n)$, that is, let $f \in S'$, the $\langle f, \varphi\rangle$ is defined for all $\varphi \in S$ but D $\subset$ S.

Let $\varphi \in$ D, which implies $\varphi \in$ S the $\langle f, \varphi\rangle$ is defined. Since φ is arbitrary in D, $\langle \mathrm{f}, \varphi\rangle$ is defined for all $\varphi \in$ D.

$$\therefore f \in D', \text{ hence } \mathrm{S}' \subset \mathrm{D}'$$

Convergence S' implies convergence D', that is,

Let $f_1, f_2...$ be a sequence function in S' converges to $f \in S'$.

i.e. $\langle f_n, \varphi \rangle \to \langle f, \varphi \rangle$ as n $\to \infty$ for all $\varphi \in S$
As DC S, $\langle f_n, \varphi \rangle \to \langle f, \varphi \rangle$ as n $\to \infty$ for all $\varphi \in D$.
Then it follows that convergence in D'.

Chapter 2

Schwartz distribution

2.1 Schwartz distributions

The space $\mathcal{D}'$ of Schwartz distributions contains, besides the ordinary (i.e., $\mathcal{C}, \mathcal{C}^m$ and $\mathcal{C}^\infty$) functions, generalized functions corresponding to discontinuous functions and unbounded functions. While these functions cannot be differentiated in the classical sense, they can be indefinitely differentiated in the sense of distributions. That is $f \in C^p\left(R^n\right)$.

Then whenever $\alpha, |\alpha| \leq p$, and $\varphi \in \mathcal{D}$ the formula for integration by parts,

$$(D^\alpha f, \varphi) = \int D^\alpha f(x)\varphi(x)dx = (-1)^{|\alpha|} \int f(x) D^\alpha \varphi(x)dx = (-1)^{|\alpha|}\left(f, D^\alpha \varphi\right)$$

......(2.1) is valid.

We shall also take this equation as the definition of the (generalized) derivative $D^\alpha f$ of the generalized function $f \in D'$:

$(D^\alpha f, \varphi) = (-1)^{|\alpha|}\left(f, D^\alpha \varphi\right), \quad \varphi \in D$

We shall check that $D^\alpha f \in D'$. In fact, since $f \in D'$, the functional $D^\alpha f$, definable by the right-hand side is **linear**:

Let $\varphi, \psi \in D'$ and $\lambda, \beta \in \mathbb{C}$,then

$$\begin{aligned}
\left(D^\alpha f, \lambda\varphi + \beta\psi\right) &= (-1)^{|\alpha|}\left(f, D^\alpha(\lambda\varphi + \beta\psi)\right) \\
&= (-1)^{|\alpha|}\left(f, \lambda D^\alpha \varphi + \beta D^\alpha \psi\right) \\
&= \lambda(-1)^{|\alpha|}\left(f, D^\alpha \varphi\right) + \beta(-1)^{|\alpha|}\left(f, D^\alpha \psi\right) \\
&= \lambda\left(D^\alpha f, \varphi\right) + \beta\left(D^\alpha f, \psi\right)
\end{aligned}$$

Hence $D^{\alpha} f$ is linear.

To show **continuous**:

$$\left(D^{\alpha} f, \varphi_k\right) = (-1)^{|\alpha|} \left(f, D^{\alpha} \varphi_k\right) \rightarrow (-1)^{|\alpha|} \left(f, D^{\alpha} \varphi\right) = \left(D^{\alpha} f, \varphi\right)$$

for, if $\varphi_k \rightarrow \varphi$ as $k \rightarrow \infty$ in D, then also $D^{\alpha} \varphi_k \rightarrow D^{\alpha} \varphi$ as $k \rightarrow \infty$ in D.

We shall denote by $\{D^{\alpha} f(x)\}$ the classical derivative (where it exists). It follows from the definition of the generalized derivative that if the generalized function $f \in C^p(G)$, then

$$D^{\alpha} f = \{D^{\alpha} f(x)\}, \quad x \in G, \quad |\alpha| \leq p$$

$D^{\alpha} f \in D'$

Example 2.1.1. *The Dirac measure δ on R^n defined by*

$$\langle \delta, \varphi \rangle = \varphi(0), \textit{ for all } \varphi \in D(\Omega) \textit{ is a distribution.}$$

Let $\Omega = \mathrm{R}$ and consider that Heaviside function

$$H(x) = \begin{cases} 1, & \textit{if } x > 0 \\ 0, & \textit{if } x < 0 \end{cases}$$

Then the derivative of H(x) *in the sense of distribution is defined by:*

$$\begin{aligned}
\langle DH, \varphi \rangle &= -\langle H, \varphi' \rangle = -\int_{-\infty}^{\infty} H(\mathrm{x}) \varphi'(x) dx \\
&= -\int_{-\infty}^{0} H(\mathrm{x}) \varphi'(x) dx - \int_{0}^{\infty} H(\mathrm{x}) \varphi'(x) dx \\
&= -\int_{-\infty}^{0} 0 \varphi' dx - \int_{0}^{\infty} 1 \varphi'(x) dx \\
&= -\int_{0}^{\infty} \varphi'(x) dx \\
&= -\lim_{b \rightarrow \infty} \int_{0}^{b} \varphi'(x) dx = -\lim_{b \rightarrow \infty} \left[\varphi(x)_0^b\right] \\
&= -\lim_{b \rightarrow \infty} [\varphi(b) - \varphi(0)] \\
&= \varphi(0) \\
&= \langle \delta, \varphi \rangle
\end{aligned}$$

Therefore, $H' = \delta$ the Dirac measure.

Proposition 2.1.1. *If $f \in D'(\Omega)$ and α and β are multi-indices, then*

$$D^{\alpha}\left(D^{\beta} f\right)=D^{\beta}\left(D^{\alpha} f\right)=D^{\alpha+\beta} f.$$

Proof. $\left\langle D^{\alpha+\beta} f, \varphi\right\rangle=(-1)^{|\alpha|+|\beta|}\left\langle f, D^{\alpha+\beta} \varphi\right\rangle$

$$=(-1)^{|\beta|}\left\langle D^{\alpha} f, D^{\beta} \varphi\right\rangle=\left\langle D^{\beta}\left(D^{\alpha} f\right), \varphi\right\rangle$$
$$=(-1)^{|\alpha|}\left\langle D^{\beta} f, D^{\alpha} \varphi\right\rangle=\left\langle D^{\alpha}\left(D^{\beta} f\right), \varphi\right\rangle$$

Hence $D^{\alpha+\beta} f=D^{\beta}\left(D^{\alpha} f\right)=D^{\alpha}\left(D^{\beta} f\right)$ □

Lemma 2.1.2. *The operation of differentiation D^{α} is linear and continuous from D' into D', that is,*

Linearity: Let $\mathrm{f}, \mathrm{g} \in D'(\Omega)$ and $\lambda, \beta \in \mathbb{C}$, we have

$$\langle\lambda \mathrm{f}+\beta \mathrm{g}, \varphi\rangle=\lambda\langle f, \varphi\rangle+\beta\langle\mathrm{g}, \varphi\rangle, \forall \varphi \in \mathrm{D}(\Omega).$$

then $D^{\alpha}\langle\lambda f+\beta g, \varphi\rangle=\langle D^{\alpha}(\lambda f+\beta g), \varphi\rangle$

$$=(-1)^{|\alpha|}\langle\lambda \mathrm{f}+\beta \mathrm{g}, \mathrm{D}^{\alpha} \varphi\rangle$$
$$=(-1)^{|\alpha|} \lambda\langle f, D^{\alpha} \varphi\rangle+(-1)^{|\alpha|} \beta\langle\mathrm{g}, \mathrm{D}^{\alpha} \varphi\rangle$$
$$=\lambda\langle D^{\alpha} f, \varphi\rangle+\beta\langle D^{\alpha} \mathrm{g}, \varphi\rangle$$

Continuity: suppose $f_k \rightarrow 0, \mathrm{k} \rightarrow \infty$ in $D'(\Omega)$. Then for all $\varphi \in \mathrm{D}(\Omega)$, we have

$$\langle\mathrm{D}^{\alpha} \mathrm{f}_{\mathrm{k}}, \varphi\rangle=(-1)^{|\alpha|}\langle\mathrm{f}_{\mathrm{k}}, \mathrm{D}^{\alpha} \varphi\rangle \rightarrow 0, \mathrm{k} \rightarrow \infty, \text{ and this implies that}$$
$$D^{\alpha} f_k \rightarrow 0, \text{ as } \mathrm{k} \rightarrow \infty \text{ in } D'(\Omega).$$

Lemma 2.1.3. *(Leibniz Rule).Let $f \in C^{\infty}(\Omega), \mathrm{u} \in D'(\Omega)$, and α a multi-index. Then*

$$D^{\alpha}(f u)=\sum_{\beta \leq \alpha}\left(\begin{array}{l}\alpha \\ \beta\end{array}\right)(D^{\alpha-\beta} f)(D^{\beta} u) \in D'(\Omega)$$

Where

$$\left(\begin{array}{l}\alpha \\ \beta\end{array}\right)=\frac{\alpha!}{(\alpha-\beta)!\beta!}$$

$\alpha!=\alpha_1!\alpha_2!\cdots\alpha_n!$ *and $\beta \leq \alpha$ means that β is a multi-index with $\beta_i \leq \alpha_i$ for* $\mathrm{i}=1, \ldots \mathrm{n}$.

If $\mathrm{u} \in C^{\infty}(\Omega)$*, this is just the product rule for differentiation.*

Proof. By the previous proposition, we have the theorem if it is true for multi-indices that have a single non zero component, say the first component. We proceed by induction on n = α. □

The result holds for n = 0*, but we will need the result for* n = 1*.Denote D^α by D_1^α.*

When n = 1*, for any $\varphi \in D(\Omega)$*

$$\begin{aligned} \langle D_1(uf), \varphi\rangle &= -\langle fu, D_1\varphi\rangle \\ &= -\langle u, fD_1\varphi\rangle \\ &= -\langle u, D_1(f\varphi) - D_1 f\varphi\rangle \\ &= \langle D_1 u, f\varphi\rangle + \langle u, D_1 f\varphi\rangle \\ &= \langle fD_1 u + D_1 f u, \varphi\rangle, \end{aligned}$$

and the result holds.

Now assume the result for derivative up to order n − 1. Then

$$\begin{aligned} D_1^n(fu) &= D_1 D_1^{n-1}(fu) \\ &= D_1 \sum_{j=0}^{n-1} \binom{n-1}{j} D_1^{n-1-j} f D_1^j u \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} \left(D_1^{n-j} f D_1^j u + D_1^{n-1-j} f D_1^{j+1} u \right) \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} D_1^{n-j} f D_1^j u + \sum_{j=1}^{n} \binom{n-1}{j-1} D_1^{n-j} f D_1^j u \\ &= \binom{n-1}{0} D^0 fu \sum_{j=1}^{n-1} \binom{n-1}{j} D_1^{n-j} f D_1^j u + \end{aligned}$$

$\sum_{j=1}^{n-1} \binom{n-1}{j-1} D_1^{n-j} f D_1^j u + \binom{n-1}{n-1} f D^n u$

Now combine the two sume,we have

$$= \binom{n}{0} D_1^n f D_1^0 u + \sum_{j=1}^{n-1} \binom{n}{j} D_1^{n-j} f D_1^j u + \binom{n}{n} D_1^{n-n} f D_1^n u$$

$$= \sum_{j=0}^{n} \binom{n}{j} D_1^{n-j} f D_1^j u$$

where the last equality follows from the combinatorial identify

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}$$

and so the induction proceeds.

Theorem 2.1.4. *If* $\langle f_k, \varphi \rangle \xrightarrow{k\to\infty} \langle f, \varphi \rangle$ *for all* $\varphi \in \mathrm{D}(\Omega)$, *then*

$\langle D^\alpha f_k, \varphi \rangle \xrightarrow{k\to\infty} \langle \mathrm{D}^\alpha f, \varphi \rangle$ for all $\varphi \in \mathrm{D}(\Omega)$, for all multi indices.

Proof: $\langle D^\alpha f_k, \varphi \rangle = (-1)^{|\alpha|} \langle f_k, D^\alpha \varphi \rangle \xrightarrow{k\to\infty} (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle = \langle D^\alpha f, \varphi \rangle$ for all $\varphi \in D(\Omega)$.

Generalized function $f \in C^p(G)$, then

$$D^\alpha f = \{D^\alpha f(x)\}, \quad x \in G, \quad |\alpha| \le p$$

2.1.1 Properties of Generalized Derivatives

The following properties of the operation of differentiation of generalized functions are true.

(a) Any generalized function is infinitely differentiable.

In fact, if $f \in D'$, then $\partial f / \partial x_i \in D'$; in its turn $(\partial / \partial x_j)(\partial f / \partial x_i) \in D'$; and so on.

(b) The result of differentiation does not depend on the order of differentiation; for example,

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right) = \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right) = D^{(1,1)} f \tag{2.2}$$

In fact, for any $\varphi \in D$ we obtain

$$\left(D^{(1,1)} f, \varphi\right) = \left(f, \frac{\partial^2 \varphi}{\partial x_1 \partial x_2}\right) = \left(\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right), \varphi\right) = \left(\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right), \varphi\right)$$

from which Eq. (2.2) .

In general,

$$D^{\alpha+\beta} f = D^\alpha \left(D^\beta f\right) = D^\beta \left(D^\alpha f\right) \tag{2.3}$$

(c) If $f \in D'$ and $a \in C^{\infty}(R^n)$, then Leibnitz' formula for differentiation of the product af is valid. For example:

$$\frac{\partial(af)}{\partial x_1} = \frac{\partial a}{\partial x_1} f + a \frac{\partial f}{\partial x_1} \tag{2.4}$$

In fact, if φ is any basic function, then

$$\begin{aligned}
\left(\frac{\partial(af)}{\partial x_1}, \varphi\right) &= -\left(af, \frac{\partial \varphi}{\partial x_1}\right) = -\left(f, a\frac{\partial \varphi}{\partial x_1}\right) \\
&= -\left(f, \frac{\partial(a\varphi)}{\partial x_1} - \frac{\partial a}{\partial x_1}\varphi\right) = -\left(f, \frac{\partial(a\varphi)}{\partial x_1}\right) + \left(f, \frac{\partial a}{\partial x_1}\varphi\right) \\
&= \left(\frac{\partial f}{\partial x_1}, a\varphi\right) + \left(\frac{\partial a}{\partial x_1} f, \varphi\right) = \left(a\frac{\partial f}{\partial x_1}, \varphi\right) + \left(\frac{\partial a}{\partial x_1} f, \varphi\right) \\
&= \left(a\frac{\partial f}{\partial x_1} + \frac{\partial a}{\partial x_1} f, \varphi\right)
\end{aligned}$$

(d) If the generalized function $f(x) = 0$ for $x \in G$, then also $D^{\alpha} f(x) = 0$ for $x \in G$, so that $\operatorname{supp} D^{\alpha} f \subset \operatorname{supp} f$.

In fact, if $\varphi \in D(G)$ then $D^{\alpha}\varphi \in D(G)$ and so

$$(D^{\alpha} f, \varphi) = (-1)^{|\alpha|} (f, D^{\alpha}\varphi) = 0, \quad \varphi \in D(G)$$

which shows that $D^{\alpha} f = 0$ for $x \in G$.

(e) The operation of differentiation is continuous from D' into D', that is, if $f_k \to f$ as $k \to \infty$ in D', then $D^{\alpha} f_k \to D^{\alpha} f$ as $k \to \infty$ in D'.

Indeed, according to the definition of convergence in the space D' , for all $\varphi \in D$ we have

$$(D^{\alpha} f_k, \varphi) = (-1)^{|\alpha|} (f_k, D^{\alpha}\varphi) \to (-1)^{|\alpha|} (f, D^{\alpha}\varphi) = (D^{\alpha} f, \varphi), \quad k \to \infty$$

which shows that $D^{\alpha} f_k \to D^{\alpha} f$ as $k \to \infty$ in D'.

2.1.2 The Fourier transform of test functions belonging to S.

Definition 2.1.1. *The Fourier transform of a function $f \in S(R^n)$ is defined by the integral.*

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) =) = (2\pi)^{-n/2} \int_{R^n} f(x) e^{-i\langle x, \xi\rangle} \mathrm{dx}$$

Where $\langle x,\xi\rangle = x\xi = x_1\xi_1 + \cdots + x_n\xi_n$.

Definition 2.1.2. *The inverse Fourier transform of a function* $f \in S\left(R^n\right)$ *is defined by the integral*

$$\left(\mathcal{F}^{-1}f\right)(\xi) = \check{f}(\xi) = (2\pi)^{-n/2}\int_{R^n} f(x)e^{i\langle x,\xi\rangle}\mathrm{dx}$$

Lemma 2.1.5. *Let* $f \in \mathrm{S}\left(R^n\right)$ *and* α *be a multi-index.then*

i) $\mathrm{D}^\alpha(\mathcal{F}f)(\xi) = (-i)^{|\alpha|}\mathcal{F}\left(\mathrm{x}^\alpha f\right)(\xi)$

ii) $\mathcal{F}\left(D^\alpha f\right)(\xi) = (i)^{|\alpha|}\xi^\alpha(\mathcal{F}f)(\xi)$

Proof. i) $\mathrm{D}^\alpha(\mathcal{F}f)(\xi) = \frac{\partial^\alpha}{\partial\xi^\alpha}\left[(2\pi)^{-n}/2\int_{R^n} f(x)e^{-i\langle \mathrm{x},\xi\rangle}\mathrm{dx}\right]$

$$= (2\pi)^{-n/2}\int_{R^n} f(x)\frac{\partial^\alpha}{\partial\xi^\alpha}e^{-i\langle x,\xi\rangle}\mathrm{dx}$$
$$= (2\pi)^{-n/2}\int_{R^n} f(x)\left(-ix_1\right)^{\alpha_1}\left(-ix_2\right)^{\alpha_2}\ldots\left(-ix_n\right)^{\alpha_n}e^{-i\langle x,\xi\rangle}dx$$
$$= (-\mathrm{i})^{\alpha_1+\ldots+\alpha_\mathrm{n}}(2\pi)^{-\mathrm{n}/2}\int_{R^n} f(x)\mathrm{x_1}^{\alpha_1}\ldots\mathrm{x}_\mathrm{n}^{\alpha_\mathrm{n}}e^{-i\langle \mathrm{x},\xi\rangle}\mathrm{dx}$$
$$= (-\mathrm{i})^{|\alpha|}(2\pi)^{-\mathrm{n}/2}\int_{R^n} f(x)\mathrm{x}^\alpha e^{-i\langle \mathrm{x},\xi\rangle}\mathrm{dx}$$
$$= (-i)^{|\alpha|}\mathcal{F}\left(\mathrm{x}^\alpha f\right)(\xi)$$

ii)$\mathcal{F}\left(\mathrm{D}^\alpha f\right)(\xi) = (2\pi)^{-n/2}\int_{R^n} e^{-i\langle x,\xi\rangle}\mathrm{D}^\alpha f(\mathrm{x})\mathrm{dx}$

$$= (-1)^{|\alpha|}(2\pi)^{-n/2}\int_{R^n} f(x)\frac{\partial^\alpha}{\partial x^\alpha}e^{-i\langle x,\xi\rangle}dx$$
$$= (-1)^{|\alpha|}(2\pi)^{-n/2}\int_{R^n} f(x)\left(-i\xi_1\right)^{\alpha_1}\cdot\ldots\cdot\left(-i\xi_n\right)^{\alpha_n}e^{-i\langle x,\xi\rangle}dx$$
$$= (-1)^{|\alpha|}(2\pi)^{-n/2}\int_{R^n} f(x)(-i)^{|\alpha|}\xi^\alpha e^{-i\langle x,\xi\rangle}dx$$
$$= (-1)^{|\alpha|}(-i)^{|\alpha|}\xi^\alpha(2\pi)^{-n/2}\int_{R^n} f(\mathrm{x})e^{-i\langle \mathrm{x},\xi\rangle}\mathrm{dx}$$
$$= i^{|\alpha|}\xi^\alpha[\mathcal{F}f](\xi)$$

We note that the space of test functions D is not mapped in to itself by the fourier transform since the fourier transform of a function with compact support is an analytic function, and consequently is either not of compact support or zero.

2.1.3 The Fourier transform of distribution belonging to S'.

Definition 2.1.3. *Let $f(\mathrm{x})$ be an integrable function on R^n. Then its Fourier transform defined as $(\mathcal{F}f)(\xi) = (2\pi)^{-n/2}\int_{R^n} f(\mathrm{x})e^{-i\langle \mathrm{x},\xi\rangle}\mathrm{dx}$ for all $f \in S'$, $|(\mathcal{F}f)(\xi)| \leq \int |f(x)|\mathrm{dx} < \infty$ is a continuous function bounded in R^n and, consequently, defines a generalized function from S',*

$$
\begin{aligned}
\langle \mathcal{F}f(\xi), \varphi\rangle = \langle \hat{f}, \varphi\rangle &= \int \hat{f}(\xi)\varphi(\xi)d\xi \\
&= \int \left[(2\pi)^{-n/2}\int f(x)e^{-i\langle x,\xi\rangle}dx\right]\varphi(\xi)\mathrm{d}\xi \\
&= \int f(x)\left[(2\pi)^{-n/2}\int \varphi(\xi)e^{-i\langle x,\xi\rangle}d\xi\right]dx \\
&= \int f(x)(\mathcal{F}\varphi)(x)dx \\
&= \langle f, \mathcal{F}\varphi\rangle \\
&= \langle f, \hat{\varphi}\rangle \quad \varphi \in \mathrm{S}
\end{aligned}
$$

Example 2.1.2. *If $f = \delta$, then*

$$
\begin{aligned}
\langle \mathcal{F}\delta, \varphi \rangle &= \langle \delta, \hat{\varphi}\rangle \\
&= (\hat{\varphi})(0) \\
&= (2\pi)^{-n/2}\int_{R^n} \varphi(\mathrm{x})e^{-i\langle \mathrm{x},0\rangle}\mathrm{dx} \\
&= (2\pi)^{-n/2}\int_{R^n} \varphi(\mathrm{x})\mathrm{dx} \\
&= (2\pi)^{-n/2}\langle 1, \varphi\rangle \\
&= \left\langle (2\pi)^{-n/2}, \varphi\right\rangle, \quad \varphi \in \mathrm{S} \\
&\therefore \mathcal{F}(\delta) = (2\pi)^{-n/2}
\end{aligned}
$$

We shall prove that the operation $\mathcal{F}^{-1}$ is the inverse operation to the Fourier transform $\mathcal{F}$, that is,

$$\mathcal{F}^{-1}[\mathcal{F}[f]](\xi) = f, \quad \mathcal{F}\left[\mathcal{F}^{-1}[f]\right](\xi) = f, \quad f \in S'$$

In fact, by virtue of equation (1)-(3), for all $\varphi \in S$ we obtain the equations

$$\begin{aligned}\left\langle \mathcal{F}^{-1}[\mathcal{F}[f]](\xi), \varphi\right\rangle &= \langle \mathcal{F}[\mathcal{F}[f](-\xi)], \varphi\rangle \\ &= \langle \mathcal{F}[f](-\xi), \mathcal{F}[\varphi]\rangle \\ &= \langle \mathcal{F}[f], \mathcal{F}[\varphi](-\xi)\rangle \\ &= \left\langle \mathcal{F}[f], \mathcal{F}^{-1}[\varphi]\right\rangle \\ &= \left\langle f, \mathcal{F}\left[\mathcal{F}^{-1}[\varphi]\right]\right\rangle = \langle f, \varphi\rangle^{\cdot} \\ &= \left\langle f, \mathcal{F}^{-1}[\mathcal{F}[\varphi]\right\rangle \\ &= \left\langle \mathcal{F}^{-1}[f], \mathcal{F}[\varphi]\right\rangle \\ &= \left\langle \mathcal{F}\left[\mathcal{F}^{-1}[f]\right], \varphi\right\rangle\end{aligned}$$

2.1.4 Properties of the Fourier transform

i)Differentiating the Fourier transform

If $f \in S'$, then $D^{\alpha}\mathcal{F}[\mathrm{f}] = \mathcal{F}\left[(\mathrm{ix})^{\alpha}\mathrm{f}\right]$
i.e. Let $\varphi \in \mathrm{S}$, then $\langle \mathrm{D}^{\alpha}\mathcal{F}[\mathrm{f}], \varphi\rangle = (-1)^{|\alpha|}\langle \mathcal{F}[\mathrm{f}], \mathrm{D}^{\alpha}\varphi\rangle$

$$\begin{aligned}&= (-1)^{|\alpha|}\langle \mathrm{f}, \mathcal{F}\left[\mathrm{D}^{\alpha}\varphi\right]\rangle \\ &= (-1)^{|\alpha|}\langle \mathrm{f}, (-\mathrm{ix})^{\alpha}\mathcal{F}[\varphi]\rangle \\ &= (-1)^{|\alpha|}\langle (-\mathrm{ix})^{\alpha}\mathrm{f}, \mathcal{F}[\varphi]\rangle \\ &= \langle (\mathrm{ix})^{\alpha}\mathrm{f}, \mathcal{F}\varphi\rangle \\ &= \langle \mathcal{F}(\mathrm{ix})^{\alpha}\mathrm{f}, \varphi\rangle\end{aligned}$$

$$\therefore \mathrm{D}^{\alpha}[\mathrm{f}](\xi) = \mathcal{F}\left[(\mathrm{ix})^{\alpha}\mathrm{f}\right](\xi)$$

ii) The Fourier transform of the derivatives

$$\mathcal{F}\left[D^{\alpha}f\right] = (-i\xi)^{\alpha}\mathcal{F}[f]$$

Proof. For all $\varphi \in \mathcal{S}$ (from properties of Section 1.1), we obtain

$$\begin{aligned}\left(\mathcal{F}\left[D^{\alpha}f\right], \varphi\right) &= \left(D^{\alpha}f, \mathcal{F}[\varphi]\right) \\ &= (-1)^{|\alpha|}\left(f, D^{\alpha}\mathcal{F}[\varphi]\right) \\ &= (-1)^{|\alpha|}\left(f, \mathcal{F}\left[(i\xi)^{\alpha}\varphi\right]\right) \\ &= (-1)^{|\alpha|}\left(\mathcal{F}[f], (i\xi)^{\alpha}\varphi\right) \\ &= \left((-i\xi)^{\alpha}\mathcal{F}[f], \varphi\right)\end{aligned}$$

Thus, $\mathcal{F}\left[D^{\alpha}f\right] = (-i\xi)^{\alpha}\mathcal{F}[f]$ □

iii) The Fourier transform of a translation

$$\mathcal{F}\left[f\left(x-x_0\right)\right]=e^{-i\langle x,\xi\rangle}\mathrm{dx}\mathcal{F}[f]$$

Proof. Using defination of fourier tansformation

$$(\mathcal{F}f)(\xi)=\hat{f}(\xi)=)=(2\pi)^{-n/2}\int_{R^n}f(x)e^{-i\langle x,\xi\rangle}\mathrm{dx}$$

Now substitute the translated function f(x-x_0)

$$\mathcal{F}[f(x-x_0)]=\hat{f}(x-x_0))=(2\pi)^{-n/2}\int_{R^n}f(x)e^{-i\langle x,\xi\rangle}\mathrm{dx}$$

$$=(2\pi)^{-n/2}\int_{R^n}f(y)e^{-i\langle y+x_0,\xi\rangle}\mathrm{dy}$$

$$=(2\pi)^{-n/2}\int_{R^n}f(y)e^{-i\langle y,\xi\rangle}.e^{-i\langle x_0,\xi\rangle}\mathrm{dy}$$

$$=e^{-i\langle x_0,\xi\rangle}(2\pi)^{-n/2}\int_{R^n}f(y)e^{-i\langle y,\xi\rangle}\mathrm{dy}$$

$$Hence,\mathcal{F}\left[f\left(x-x_0\right)\right]=e^{-i\langle x,\xi\rangle}\mathrm{dx}\mathcal{F}[f]$$

□

It is not possible to represented non-trivial distributions, such as Dirac's δ-function, by simple algebraic formulas or even by ordinary limiting processes.

They can however be represented by sequences of smooth functions

$$D(x):=\lim_{\epsilon\to 0}D_\epsilon(x),\qquad \text{where } D_\epsilon(x)\in\mathcal{D},...(2.5)$$

for which ordinary pointwise convergence is not required. Instead, what is required is 'weak convergence' for the scalar product of $D(x)$ with any test functions $\varphi(x)\in\mathcal{D}$, i.e., the existence of the limit

$$\forall\varphi\in\mathcal{D},D(\varphi):=\lim_{\epsilon\to 0}\int_\Omega D_\epsilon(x)\varphi(x)dx\in\mathbb{R}...........(2.6)$$

The meaning of operating 'in the sense of distributions' is then that all operations on distributions are actually performed on $D_\epsilon(x)$, while $D(x)$ can be seen as a convenient symbol to designate a given distribution.

Eq. (2.6) shows that distributions can be interpreted as linear functionals $D(\varphi) = D(\varphi)$ defined by their effect on test functions. Moreover, since many different sequencies may converge weakly to the same limit, each distribution corresponds to an equivalence classes of such sequencies, which all together form the Schwartz distribution space $\mathcal{D}'$.

Definition 2.1.4. *Two distributions D and $E \in \mathcal{D}'$, of respective representatives D_ϵ and E_ϵ, are said to be equal (or equivalent), and one write $D = E$, iff*

$$\forall \varphi \in \mathcal{D}, \quad \lim_{\epsilon \to 0} \int_\Omega (D_\epsilon(x) - E_\epsilon(x))\, \varphi(x) dx = 0 \text{.........(2.7)}$$

Example 2.1.3. *Dirac's δ-function is defined by the property*

$$\delta(\varphi) = \int_\Omega \delta(x)\varphi(x) dx = \varphi(0), \text{.........(2.8)}$$

so that all sequences which have this property form an equivalence class corresponding to Dirac's δ-function distribution, conventionally denoted by the symbol ' $\delta(x)$.' Two examples of such sequences are

$$\delta_\epsilon(x) = \frac{1}{\pi\epsilon} \frac{\epsilon^2}{\epsilon^2 + x^2}, \quad \text{and} \quad \delta_\epsilon(x) = \frac{1}{\sqrt{\pi}\epsilon} \exp\left(-\frac{x^2}{\epsilon^2}\right) \text{.......(2.9)}$$

More generally, any normalizable $\mathcal{C}^\infty$ function $\rho(y)$ with compact support can be used to define δ-sequencies, i.e.,

$$\forall \rho \in \mathcal{C}_0^\infty, \quad \int_\Omega \rho(y) dy = 1 \quad \Rightarrow \quad \delta_\epsilon(x) := \frac{1}{\epsilon}\rho\left(\frac{x}{\epsilon}\right) \text{........(2.10)}$$

The requirement that both δ_ϵ and φ are $\mathcal{C}_0^\infty(\Omega)$ functions and the definition (2.1) enable to derive at once a number of useful properties. For instance, the equations

$$(x\delta(x)\varphi(x)) = 0, \quad \text{and } \frac{1}{x}\delta(x)\varphi(x) = -\varphi'(0), \text{...........(2.11)}$$

which are often symbolically written ' $x\delta(x) = 0$,' and ' $x^{-1}\delta(x) = -\delta'(x)$ ' or ' $x\delta'(x) = -\delta(x)$ ' are the fundamental formulas of calculus with distributions. In summary distributions are not functions in the usual sense but equivalence classes of weakly convergent sequencies of smooth functions.

All operations on distributions are therefore made on these sequencies, which are thus added, differentiated, etc., according to the operation in question.

It is remarkable that distributions enjoy essentially all properties of $\mathcal{C}^\infty$ functions, including multiplication by a $\mathcal{C}^\infty$ function, with a few exceptions such as the impossibility to multiply two distributions in the general.

As will be seen, the Colombeau formalism provides a non-ambiguous notation for these formulas, i.e., $x\delta(x) \asymp 0$, and $x^{-1}\delta(x) \asymp -\delta'(x)$ or $x\delta'(x) \asymp -\delta(x)$.

Theorem 1. (Schwartz local structure theorem) Any distribution is locally a partial derivative of a continuous function .

Differentiation induces therefore the following remarkable cascade of relationships: continuously differentiable functions $\rightarrow$ continuous functions $\rightarrow$ distributions.

This gives a unique position to Schwartz distributions because they constitute the smallest space in which all continuous functions can be differentiated any number of times. For this reason it is best to reserve the term 'distribution' to them, and to use the expression 'generalized function' for any of their generalizations.

2.1.5 Multiplication and regularization of distributions

There are two kinds of problems with the multiplication of distributions:

(i) The product of two distributions is, in general, not defined.

For example, the square of Dirac's δ function is not a weakly converging sequence. that means

$\delta^2 = \lim_{\epsilon\rightarrow 0} \delta_\epsilon^2 = \infty$

(ii) Differentiation is inconsistent with multiplication because the Leibniz rule, or even associativity, can fail under various circumstances. For example, while Dirac's δ-function is related to Heaviside's step function through differentiation as $\delta(x) = \mathrm{H}'(x)$, the algebraic identity $\mathrm{H}^2(x) = \mathrm{H}(x)$ leads to inconsistencies. Indeed,

$$\mathrm{H}^2 = \mathrm{H} \Rightarrow 2\mathrm{H}\delta = \delta \Rightarrow 2\mathrm{H}^2\delta = \mathrm{H}\delta \Rightarrow 2\mathrm{H}\delta = \mathrm{H}\delta! \qquad (2.12)$$

Over the years many methods for solving these problems have been proposed. One of the simplest and most effective is 'regularization,' which consists of modifying the functions to be multiplied or differentiated in such a way that they become more regular (i.e., continuous, differentiable, finite, etc.).

All operations are then done with the regularized functions until the end of the calculation, and the final result is obtained by the inverse process which returns the function from its regularization.

A particularly convenient regularization technique is based on the convolution product. For instance, if $f(x)$ is any function on $\mathbb{R}$, its regularization $f_\epsilon(x)$ is

$$f_\epsilon(x) = (f * \rho_\epsilon)(x) := \int_\Omega f(x-y)\rho_\epsilon(y)dy ......(2.13)$$

Here $\rho_\epsilon(x)$ is a smoothing kernel (also called regularizer or 'mollifier') which in its simplest form is a δ-sequence as defined in Eq. (2.10), and $\epsilon \in (0,1)$ is the (so called) regularization parameter. Consequently, in the limit $\epsilon \to 0$, the mollifier becomes equal to the δ-function, which by Eq. (2.8) acts as the unit element in the convolution product, i.e., $f*\delta = f$. Thus, when $\epsilon \neq 0$ the regularization is such that f is 'mollified' by the convolution, while f can be retrieved by taking the limit $\epsilon \to 0$. The power of convolution as a regularization technique stems from the theorem

Theorem 2. The convolution $(D * \rho)(x)$ of a distribution $D \in \mathcal{D}'$ by a function $\rho \in \mathcal{D}$ is a $\mathcal{C}^\infty$ function in the variable x .

Regularized functions $f * \rho_\epsilon$ and distributions $D * \rho_\epsilon$ can therefore be freely multiplied and differentiated. Moreover, the mollified sequence $D_\epsilon = D * \rho_\epsilon$ provides a representative sequence of the type (2.5) of any distribution $D \in \mathcal{D}'$.

2.1.6 Applications of distributions

Fundamental Solutions of Linear Differential Operators

The Fourier transform is applied to construct the fundamental solutions of linear differential operators having constant coefficients. Naturally, only fundamental solutions of slow growth can be obtained by this method.

1. Generalized Solutions of Linear Differential Equations.

Let

$$\sum_{|x|=0}^{m} a_\alpha(x) D^\alpha u = f(x), \quad f \in D' \tag{2.14}$$

be a linear differential equation of order m with coefficients $a_\alpha \in C^\infty(R^n)$. Introducing the differential operator

$$L(x,D) = \sum_{|\alpha|=0}^{m} a_\alpha(x) D^\alpha, \quad D = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)$$

we shall rewrite this equation in the form

$$L(x,D)u = f(x)$$

Each generalized function $u \in D'$ which satisfies this equation in the region G in a generalized sense, that is, for any $\varphi \in D$, supp $\varphi \subset G$

$$(L(x,D)u, \varphi) = (f, \varphi) \tag{2.15}$$

is known as the generalized solution of Eq. (2.14) in the region G. Equation (2.15) is equal in effect to the equation

$$(u, L^*(x,D)\varphi) = (f, \varphi), \quad \varphi \in D(G)$$

where

$$L^*(x,D)\varphi = \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^\alpha (a_\alpha \varphi) \tag{2.16}$$

In fact,

$$\begin{aligned} (L(x,D)u, \varphi) &= \left(\sum_{|\alpha|=0}^{m} a_\alpha D^\alpha u, \varphi \right) = \sum_{|\alpha|=0}^{m} (a_\alpha D^\alpha u, \varphi) \\ &= \sum_{|\alpha|=0}^{m} (D^\alpha u, a_\alpha \varphi) = \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} (u, D^\alpha (a_\alpha \varphi)) \\ &= \left(u, \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^\alpha (a_\alpha \varphi) \right) = (u, L^*(x,D)\varphi) \end{aligned}$$

We shall formulate the converse result as the following lemma.

Lemma 2.1.6. *If the generalized solution $u(x)$ of Eq. (2.14) in the region G belongs to the class $C^m(G)$ and $f \in C(G)$, then it is also the classical solution of this equation in the region G.*

Proof. Since $u \in D' \cap C^m(G)$, the classical and generalized derivatives of the function u up to and including the order m coincide in the region G . Since u is the generalized solution of Eq. (2.14) in the region G, then the function $L(x, D)u - f$ which is continuous in G vanishes in the region G in the sense of generalized functions. According to Du Bois Reymond's lemma , $L(x, D)u(x) - f(x) = 0$ at all points of the region G, so that u satisfies Eq. (2.14) in the region G in the classical sense. The lemma is proved.

2. Fundamental Solutions. Let L be an operator with constant coefficients $a_\alpha(x) = a_\alpha$:

$$L(D) = \sum_{|\alpha|=0}^{m} a_\alpha D^\alpha, \quad L^*(D) = L(-D) \tag{2.17}$$

The generalized function $E \in D'$ which satisfies equation

$$L(D)E = \delta(x) \tag{2.18}$$

in R^n is said to be the fundamental solution (the function of influence) of the differential operator $L(D)$.

The fundamental solution $E(x)$ of the operator $L(D)$, generally speaking, is not unique; it is defined accurately as far as the term $E_0(x)$, which is an arbitrary solution of the homogeneous equation $L(D)E_0 = 0$.

In fact, the generalized function $E(x) + E_0(x)$ is also a fundamental solution of the operator $L(D)$,

$$L(D)(E + E_0) = L(D)E + L(D)E_0 = \delta(x)$$

Fundamental Solution of a Linear Differential Operator with Ordinary Derivatives

$$\frac{d^n E}{dt^n} + a_1(t)\frac{d^{n-1}E}{dt^{n-1}} + \cdots + a_n(t)E = \delta(t)$$

the fundamental solution of this operator is expressed by the formula

$$E(t) = H(t)Z(t)$$

where $Z(t)$ satisfies the homogeneous equation $LZ = 0$ and the initial conditions

$$Z(0) = Z'(0) = \cdots = Z^{(n-2)}(0) = 0, \quad Z^{(n-1)}(0) = 1$$

Specifically, the functions

$$E_1(t) = H(t)e^{-at}, \quad E_2(t) = H(t)\frac{\sin at}{a} \tag{2.19}$$

are fundamental solutions, respectively, of the operators

$$\frac{d}{dt} + a, \quad \frac{d^2}{dt^2} + a^2$$

Fundamental solutions of Laplace's equation

Fundamental solution of Laplace's equation in n dimensions is defined as a solution of the following distributional equation

$$\left(\nabla^2 D_n(\boldsymbol{z}), \varphi(\boldsymbol{z})\right) = (\delta^n(\boldsymbol{z}), \varphi(\boldsymbol{z})) \tag{2.20}$$

Applying the Fourier's transform method we get

$$\begin{aligned} \left(\nabla^2 D_n(\boldsymbol{z}), \varphi(\boldsymbol{z})\right) &= \left(\nabla^2 F^{-1}\left[\tilde{D}_n(\boldsymbol{k})\right], \varphi(\boldsymbol{z})\right) \\ &= \left(F^{-1}\left[(i\boldsymbol{k})^2 \tilde{D}_n(\boldsymbol{k})\right], \varphi(\boldsymbol{z})\right) \end{aligned}$$

and for RHS of equation (2.20)

$$(\delta^n(\boldsymbol{z}), \varphi(\boldsymbol{z})) = \left(F^{-1}[1(\boldsymbol{k})], \varphi(\boldsymbol{z})\right)$$

The transformed equation is then of the form

$$\left(F^{-1}\left[-\boldsymbol{k}^2 \tilde{D}_n(\boldsymbol{k})\right](\boldsymbol{z}), \varphi(\boldsymbol{z})\right) = \left(F^{-1}[1](\boldsymbol{z}), \varphi(\boldsymbol{z})\right) \tag{2.21}$$

or equivalently

$$\left(\boldsymbol{k}^2 \tilde{D}_n(\boldsymbol{k}) + 1, F^{-1}[\varphi](\boldsymbol{k})\right) = 0 \tag{2.22}$$

where $\boldsymbol{k}^2 \tilde{D}_n(\boldsymbol{k}) \in \mathcal{S}'$ and $F^{-1}[\varphi](\boldsymbol{k}) \in \mathcal{S}$. Solutions of equation (2.22) strongly depends on dimensionality of the problem. In fact, expression

$$\tilde{D}_n(\boldsymbol{k}) = -\frac{1}{|\boldsymbol{k}|^2} \tag{2.23}$$

is a correct solution of distributional equation $|\boldsymbol{k}|^2\tilde{D}_n(\boldsymbol{k}) = -1$ for $n = 3$. In lower dimensions $n = 1, 2$ expression (2.23) has a non-integrable pole at $|\boldsymbol{k}| = 0$, hence it must be substituted by properly regularized function.

Solution in $n = 2$

Expression $-\frac{1}{|k|^2}$ in $n = 2$ is not a distributional solution! A correct solution can be introduced by means of regularization scheme:

$$\tilde{D}_2(\boldsymbol{k}) = \mathrm{reg}\left[-\frac{1}{|\boldsymbol{k}|^2}\right] = -\mathcal{P}\frac{1}{|\boldsymbol{k}|^2} \tag{2.24}$$

where

$$\left(\mathcal{P}\frac{1}{|\boldsymbol{k}|^2}, \varphi\right) := \int_{|\boldsymbol{k}|<1} \frac{\varphi(\boldsymbol{k}) - \varphi(\boldsymbol{0})}{|\boldsymbol{k}|^2} d^2k + \int_{|\boldsymbol{k}|>1} \frac{\varphi(\boldsymbol{k})}{|\boldsymbol{k}|^2} d^2k \tag{2.25}$$

Plugging this expression to $(D_2(\boldsymbol{z}), \varphi(\boldsymbol{z}))$ one gets

$$\begin{aligned}(D_2(\boldsymbol{z}), \varphi(\boldsymbol{z})) &= \left(F^{-1}\left[\tilde{D}_2(\boldsymbol{k})\right](\boldsymbol{z}), \varphi(\boldsymbol{z})\right) \\ &= \dots \\ &= \left(\frac{1}{2\pi}\ln|\boldsymbol{z}|, \varphi(\boldsymbol{z})\right) \end{aligned} \tag{2.26}$$

and finally

$$D_2(\boldsymbol{z}) = \frac{1}{2\pi}\ln|\boldsymbol{z}| \tag{2.27}$$

Solution in $n = 3$

The inverse Fourier transform of $\tilde{D}_3$ reads

$$D_3(\boldsymbol{z}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{i\boldsymbol{k}\cdot\boldsymbol{z}}\left[-\frac{1}{|\boldsymbol{k}|^2}\right] d^3k \tag{2.28}$$

This integral can be easily evaluated in spherical coordinates in space of the vector $\boldsymbol{k}$

$$\boldsymbol{k}\cdot\boldsymbol{z} = kz\cos\theta, \quad d^3k = k^2\sin\theta d\theta d\phi \tag{2.29}$$

where $k \equiv |\boldsymbol{k}|, z \equiv \boldsymbol{z}$. One gets

$$
\begin{aligned}
D_3(\boldsymbol{z}) &= -\frac{1}{(2\pi)^3} \int_0^\infty k^2 dk \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi e^{ikz\cos\theta} \frac{1}{k^2} \\
&= -\frac{1}{(2\pi)^2} \int_0^\infty dk \int_{-1}^{1} du e^{ikzu} \\
&= -\frac{1}{(2\pi)^2} \int_0^\infty \frac{1}{ikz} \left(e^{ikz} - e^{-ikz}\right) dk \\
&= -\frac{1}{(2\pi)^2} 2 \int_0^\infty \frac{\sin(kz)}{kz} dk = -\frac{1}{2\pi^2} \frac{1}{z} \underbrace{\int_0^\infty \frac{\sin(x)}{x} dx}_{\frac{\pi}{2}} \qquad (2.30)
\end{aligned}
$$

and finally

$$
D_3(\boldsymbol{z}) = -\frac{1}{4\pi|\boldsymbol{z}|} \qquad (2.31)
$$

Chapter 3

Colombeau generalized functions

3.1 Colombeau generalized functions

A Colombeau algebra $\mathcal{G}$ is an associative differential algebra in which multiplication, differentiation, and integration are similar to those of $\mathcal{C}^\infty$ functions.

Colombeau and others have introduced a number of variants of $\mathcal{G}$ but all 'Colombeau algebras' have in common one essential feature: The $\mathcal{C}^\infty$ functions are a faithful differential subalgebra of $\mathcal{G}$, a feature that Colombeau discovered to be essential to overcome Schwartz's multiplication-impossibility theorem.

With hindsight it is easy to understand why: If we suppose that $\mathcal{G}$ is an algebra containing the distributions and such that all its elements can be freely multiplied and differentiated just like $\mathcal{C}^\infty$ functions (i.e., in a way respecting commutativity, associativity, and the Leibniz rule), then $\mathcal{C}^\infty$ must be a subalgebra of $\mathcal{G}$ because $\mathcal{C}^\infty \subset \mathcal{D}'$.

Thus, to define $\mathcal{G}$, it suffices to start from a differential algebra $\mathcal{E}$ containing the distributions, and then to define $\mathcal{G}$ as a subalgebra of $\mathcal{E}$ such that the embedding of the $\mathcal{C}^\infty$ functions in $\mathcal{G}$ is an identity. In formulas: if $[g] \in \mathcal{G}$ represents an object g embedded in $\mathcal{G}$, then for all $f \in \mathcal{C}^\infty$ we want that $[f] = f$, whereas for any other function or distribution $D \in \mathcal{D}'$ we may have $[D] \neq D$.

This simple observation gives a powerful hint for an elementary construction of $\mathcal{G}$ because by Definition 3 there is a one to one correspondence between any arbitrary distribution $D(x)$ and a class of weakly convergent sequence of $\mathcal{C}^\infty$ functions $D_\epsilon(x)$, and by Theorem 2 any representative of that class can be written as a convolution of the form (2.13). Thus, the starting point is to consider for $\mathcal{E}$ the set of mollified sequences

$$\mathcal{E} := \{f_\epsilon : (\eta, x) \mapsto f_\epsilon(\eta, x)\} \text{....(3.1)}$$

which are $\mathcal{C}^\infty$ functions in the variable x for any given Colombeau mollifier η, and depend on the parameter $\epsilon \in (0,1)$ through the scaled mollifier.

$$\eta_\epsilon(x) := \frac{1}{\epsilon}\eta\left(\frac{x}{\epsilon}\right), \quad \text{normalized as} \quad \int \eta(y)dy = 1 \text{.........(3.2)}$$

The distributions $f \in \mathcal{D}'$ are then embedded in $\mathcal{E}$ as the convolution

$$f_\epsilon(x) := \eta_\epsilon(-x) * f(x) = \int \frac{1}{\epsilon}\eta\left(\frac{y-x}{\epsilon}\right) f(y)dy$$
$$= \int \eta(z) f(x+\epsilon z)dz \text{.....(3.3)}$$

where, in order to define $\mathcal{G} \subset \mathcal{E}$, the Colombeau mollifier η may need to have specific properties in addition to the normalization (3.2).

To find these additional properties we have to study the embeddings of $\mathcal{C}^\infty$ functions. We therefore calculate (3.3) for $f \in \mathcal{C}^\infty$, which enables to apply Taylor's theorem with remainder to obtain at once

$$f_\epsilon(x) = f(x)\int \eta(z)dz + \ldots \text{....(3.4)}$$
$$+ \frac{\epsilon^n}{n!} f^{(n)}(x) \int z^n \eta(z)dz + \ldots \text{.....(3.5)}$$
$$+ \frac{\epsilon^{(q+1)}}{(q+1)!} \int z^{q+1}\eta(z) f^{(q+1)}(x+\vartheta\epsilon z)dz, \text{.......(3.6)},$$

where $f^{(n)}(x)$ is the n-th derivative of $f(x)$, and $\vartheta \in (0,1)$. Since $f \in \mathcal{C}^\infty$ and η has compact support, the integral in (3.6) is bounded so that the remainder is of order $\mathrm{O}\left(\epsilon^{q+1}\right)$ at any fixed point x.

Then, if following Colombeau the mollifier η is chosen in the set

$$\left\{\int \eta(z)dz = 1, \quad \text{and} \quad \int z^n \eta(z)dz = 0, \quad \forall n = 1, \ldots, q \in \mathbb{N}\right\}, \text{.......(3.7)}$$

all the terms in (3.5) with $n \in (1,q)$ are zero and we are left with

$$\forall f \in \mathcal{C}^\infty, \quad f_\epsilon(x) = f(x) + \mathrm{O}\left(\epsilon^{q+1}\right) \text{.......(3.8)}$$

Therefore, provided η is a Colombeau mollifier and q can take any value in $\mathbb{N}$, it is possible to make the difference $f_\epsilon(x) - f(x)$ as small as we please even if $\epsilon \in (0,1)$ is kept finite.

In terms of the embeddings (3.3) the condition $[f] = f$ insuring that the smooth functions are identically embedded in $\mathcal{G}$ is that $[f_\epsilon](x) = [f](x) = f(x)$ for all $f \in \mathcal{C}^\infty$. Thus, comparing with (3.8), we are led to consider the set $\mathcal{N}$ of the so-called negligible functions, which correspond to the differences between the $\mathcal{C}^\infty$ functions and their embeddings in $\mathcal{E}$, i.e.,

$$\forall f \in \mathcal{C}^\infty, \quad \forall q \in \mathbb{N}, \quad f_\epsilon(x) - f(x) = \mathrm{O}\left(\epsilon^q\right) \quad \in \mathcal{N} \text{........(3.9)}$$

To define $\mathcal{G}$ we need a prescription such that the differences (3.9) can be neglected, i.e., equated to zero in $\mathcal{G}$. Moreover, for $\mathcal{G}$ to be an algebra, that prescription must be stable under multiplication.

That means that all elements $g \in \mathcal{G}$ have to have the property that any of their representatives $g_\epsilon \in \mathcal{E}$ multiplied by a negligible function are negligible.

In mathematical language, $\mathcal{N}$ has to be an ideal of the subset $\{g_\epsilon\} = \mathcal{E}_\mathrm{M} \subset \mathcal{E}$ of all representatives of all elements of $\mathcal{G}$. Or, in simple language, the negligible functions have to behave as the 'function zero' when multiplying any function of $\mathcal{E}_\mathrm{M}$.

It is however very simple to characterize this subset: Following Colombeau we call the elements of $\mathcal{E}_\mathrm{M}$ moderate (or multipliable) functions, and we define

$$\forall g_\epsilon \in \mathcal{E}_\mathrm{M}: \quad \exists N \in \mathbb{N}_0, \quad \text{such that} \quad g_\epsilon(x) = \mathrm{O}\left(\epsilon^{-N}\right) \text{.......(3.10)}$$

Indeed, as q in (3.8) is as large as we please, and N in (3.10) a fixed integer, the product of a negligible function by a moderate one will always be a negligible function: $\mathcal{N}$ is an ideal of $\mathcal{E}_\mathrm{M}$. Moreover, the product of two moderate functions is still moderate: They are multipliable.

For example, the Colombeau embeddings (3.3) of the δ and Heaviside functions are

$$\delta_\epsilon(x) = \frac{1}{\epsilon}\eta\left(-\frac{x}{\epsilon}\right), \quad \text{and} \quad \mathrm{H}_\epsilon(x) = \int_{-\infty}^{x/\epsilon} \eta(-z)dz, \text{..........(3.11)}$$

which are moderate functions with $N = 1$ and 0 , respectively. More generally, it can easily be proved using Schwartz's local structure theorem that:

Theorem 3.1.1. *(Colombeau local structure theorem) Any distribution is locally a moderate (i.e., multipliable) generalized function.*

Therefore, $\mathcal{N} \subset (\mathcal{C}^\infty)_\epsilon \subset (\mathcal{C})_\epsilon \subset (\mathcal{D}')_\epsilon \subset \mathcal{E}_\mathrm{M}$. It is also a matter of elementary calculation to verify that $\mathcal{E}_\mathrm{M}$ and $\mathcal{N}$ are algebras for the usual pointwise operations in $\mathcal{E}$. Moreover, $\mathcal{E}_\mathrm{M}$ is a differential algebra, (i.e., stable under partial differentiation) of $\mathcal{E}$ in which $\mathcal{N}$ is a differential ideal.

The fact that $\mathcal{N}$ is an ideal of $\mathcal{E}_\mathrm{M}$ is the key to defining $\mathcal{G}$. Indeed, if we conventionally write $\mathcal{N}$ for any negligible function, then

$$\forall g_\epsilon, h_\epsilon \in \mathcal{E}_\mathrm{M}, \quad (g_\epsilon + \mathcal{N}) \cdot (h_\epsilon + \mathcal{N}) = g_\epsilon \cdot h_\epsilon + \mathcal{N} \text{..........(3.12)}$$

Thus, it suffices to define the elements of $\mathcal{G}$ as the elements of $\mathcal{E}_\mathrm{M}$ modulo $\mathcal{N}$, i.e., to define the Colombeau algebra as the quotient

$$\mathcal{G} := \frac{\mathcal{E}_\mathrm{M}}{\mathcal{N}} \text{............(3.13)}$$

That is, an element $g \in \mathcal{G}$ is an equivalence class $[g] = [g_\epsilon + \mathcal{N}]$ of an element $g_\epsilon \in \mathcal{E}_\mathrm{M}$, which is called a representative of the generalized function g.

If ' $\odot$ ' denotes multiplication in $\mathcal{G}$, the product $g \odot h$ is defined as the class of $g_\epsilon \cdot h_\epsilon$ where g_ϵ and h_ϵ are (arbitrary) representatives of g and h; similarly $\mathrm{D}g$ is the class of $\mathrm{D}g_\epsilon$ if D is any partial differentiation operator.

Therefore, when working in $\mathcal{G}$, all algebraic and differential operations (as well as composition of functions, etc.) are performed component-wise at the level of the representatives g_ϵ.

$\mathcal{G}$ is an associative and commutative differential algebra because both $\mathcal{E}_\mathrm{M}$ and $\mathcal{N}$ are such.

The two main ingredients which led to its definition are the primacy given to $\mathcal{C}^\infty$ functions, and the use of Colombeau mollifiers for the embeddings.

In fact, Colombeau proved that the set (3.7) is not empty and provided a recursive algorithm for constructing the corresponding mollifiers for all $q \in \mathbb{N}$. He also showed that the Fourier transformation provides a simple characterization of the mollifiers. But,in most applications of the Colombeau algebras, the explicit knowledge of the form of the Colombeau mollifiers is not necessary: It is sufficient to know their defining properties (3.7).

For example, let us verify that $\delta_\epsilon(x)$ given by (3.11) has indeed the sifting property expected for Dirac's δ-function. Starting from (3.11) and employing Taylor's theorem we can write

$$\delta_\epsilon(\varphi) = \int \delta_\epsilon(x)\varphi(x)dx = \int_{-\infty}^{\infty} \frac{1}{\epsilon}\eta\left(-\frac{x}{\epsilon}\right)\varphi(x)dx = \int_{-\infty}^{\infty} \eta(-z)\varphi(\epsilon z)dz$$
$$= \int_{-\infty}^{\infty} \eta(-z)\left(\varphi(0) + \varphi z\varphi'(0) + \frac{(\varphi z)^2}{2!}\varphi''(0) + \dots\right) dz.......(3.14)$$

Then, in Schwartz theory, we take the limit (2.6), i.e.,

$$\eta_\epsilon(x) := \frac{1}{\epsilon}\eta\left(\frac{x}{\epsilon}\right), \quad \text{normalized as} \quad \int \eta(y)dy = 1.......(3.15)$$

which is the expected result by the normalization (3.2). However, in Colombeau theory, there is no need to take a limit to get the sifting property because in the development (3.14) the conditions (3.7) imply that all terms in z^n with $1 < n < q+1$ are identically zero. Thus

$$\delta_\epsilon(\varphi) = \varphi(0) + \mathrm{O}\left(\epsilon^{q+1}\right), \quad \forall q \in \mathbb{N}......(3.16)$$

where the remainder is an element of $\mathcal{N}$ so that in $\mathcal{G}$ the sifting property of δ_ϵ is an equality rather than a limit. It is this kind of qualitative difference between the Schwartz and Colombeau theories which makes it possible in $\mathcal{G}$ to go beyond distribution theory.

3.1.1 Interpretation and multiplication of distributions

To construct the Colombeau algebra we have been led to embed the distributions as the representative sequences $\gamma_\epsilon \in \mathcal{E}$ defined by (3.3) where η is a Colombeau mollifier (3.6). We can therefore recover any distribution γ by means of (2.7), i.e., as the equivalence class

$$\gamma(\varphi) := \lim_{\epsilon\to 0} \int \gamma_\epsilon(x)\varphi(x)dx, \quad \forall\varphi(x) \in \mathcal{D}, ...(3.17)$$

where γ_ϵ can be any representative of the class $[\gamma] = [\gamma_\epsilon + \mathcal{N}]$ because negligible elements are zero in the limit $\epsilon \to 0$. Of course, as we work in $\mathcal{G}$ and its elements get algebraically combined with other elements, there can be generalized functions $[g_\epsilon]$ different from the class $[\gamma_\epsilon]$ of an embedded distribution which nevertheless correspond to the same distribution γ. This leads to the concept of association, which is defined as follows

Definition 3.1.1. *Two generalized functions g and $h \in \mathcal{G}$, of respective representatives g_ϵ and h_ϵ, are said to be associated, and one write $g \asymp h$, iff*

$$\lim_{\epsilon\to 0} \int \left(g_\epsilon(x) - h_\epsilon(x)\right)\varphi(x)dx = 0, \quad \forall\varphi(x) \in \mathcal{D} \text{..........(3.18)}$$

Thus, if g is a generalized function and γ a distribution, the relation $g \asymp \gamma$ implies that g admits γ as 'associated distribution,' and γ is called the 'distributional shadow' (or 'distributional projection') of g.

Objects (functions, numbers, etc.) which are equivalent to zero in $\mathcal{G}$, i.e., equal to $\mathrm{O}\left(\epsilon^q\right), \forall q \in \mathbb{N}$, are called 'zero.' On the other hand, objects associated to zero in $\mathcal{G}$, that is which tend to zero as $\epsilon \to 0$, are called 'infinitesimals.' Definition (3.18) therefore means that two different generalized functions associated to the same distribution differ by an infinitesimal.

The space of distributions is not a subalgebra of $\mathcal{G}$. Thus we do not normally expect that the product of two distributions in $\mathcal{G}$ will be associated to a third distribution: In general their product will be a genuine generalized function.

For example, the square of Dirac's δ-function, Eq. (3.11), which corresponds to $\left(\delta^2\right)_\epsilon(x) = \left(\delta_\epsilon\right)^2(x) = \epsilon^{-2}\eta^2(-x/\epsilon)$, has no associated distribution. Indeed, making a Taylor development as in (3.14),

$$\begin{aligned}
&\lim_{\epsilon\to 0}\delta_\epsilon^2(T) = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{1}{\epsilon^2}\eta^2\left(-\frac{x}{\epsilon}\right)T(x)dx = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{-\infty}^{\infty}\eta^2(-z)T(\epsilon z)dz \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{-\infty}^{\infty}\eta^2(-z)\left(T(0) + \epsilon z T'(0) + \frac{(\epsilon z)^2}{2!}T''(0) + \dots\right)dz \\
&= \lim_{\epsilon\to 0}\frac{T(0)}{\epsilon}\int_{-\infty}^{\infty}\eta^2(-z)dz + T'(0)\int_{-\infty}^{\infty}z\eta^2(-z)dz = \infty.
\end{aligned} \text{............(3.19)}$$

But, referring to (3.10), $\left(\delta^2\right)_\epsilon(x)$ is a moderate function with $N=2$. The square of $\delta(x)$ makes therefore sense in $\mathcal{G}$ as a 'generalized function' with representative $\left(\delta^2\right)_\epsilon(x)=\eta^2(-x/\epsilon)/\epsilon^2$. Moreover, its point-value at zero, $\eta^2(0)/\epsilon^2$, can be considered as a 'generalized number.'

On the other hand, we have in $\mathcal{G}$ elements like the n-th power of Heaviside's function, Eq. (3.11), which has an associated distribution but is such that $[\mathrm{H}^n](x)\neq[\mathrm{H}](x)$ in $\mathcal{G}$, whereas $\mathrm{H}^n(x)=\mathrm{H}(x)$ as a distribution in $\mathcal{D}'$. Similarly, we have $[x]\odot[\delta](x)\neq 0$ in $\mathcal{G}$, whereas $x\delta(x)=0$ in $\mathcal{D}'$. In both cases everything is consistent: Using (3.18) one easily verifies that indeed $[\mathrm{H}^n](x)\asymp[\mathrm{H}](x)$ and $[x]\odot[\delta](x)\asymp 0$.

These differences between products in $\mathcal{G}$ and in $\mathcal{D}'$ stem from the fact that distributions embedded and multiplied in $\mathcal{G}$ carry along with them infinitesimal information on their 'microscopic structure.'

That information is necessary in order that the products and their derivatives are well defined in $\mathcal{G}$, and is lost when the factors are identified with their distributional projection in $\mathcal{D}'$.

For example, since $[\mathrm{H}^n](x)\neq[\mathrm{H}](x)$ the inconsistencies displayed in (2.12) do not arise in $\mathcal{G}$.

Nevertheless, if at some point of a calculation it is desirable to look at the intermediate results from the point of view of distribution theory, one can always use the concept of association to retrieve their distributional content.

In fact, this is facilitated by a few simple formulas which easily derive from the definition (3.18). For instance,

$$\forall f_1,\forall f_2\in\mathcal{C} \quad\Rightarrow\quad [f_1]\odot[f_2]\asymp[f_1\cdot f_2]\,,\ldots\ldots\ldots(3.20)$$
$$\forall f\in\mathcal{C}^\infty,\forall\gamma\in\mathcal{D}' \quad\Rightarrow\quad [f]\odot[\gamma]\asymp[f\cdot\gamma],\ldots\ldots\ldots\ldots(3.21)$$
$$\forall\gamma_1,\forall\gamma_2\in\mathcal{D}' \quad\Rightarrow\quad [\gamma_1]\odot[\gamma_2]\not\asymp[\gamma_1\cdot\gamma_2]\,,\ldots\ldots\ldots(3.22)$$
$$\forall g_1,\forall g_2\in\mathcal{G},\quad g_1\asymp g_2 \quad\Rightarrow\quad \mathrm{D}^\alpha g_1\asymp\mathrm{D}^\alpha g_2\ldots\ldots\ldots(3.23)$$

For example, applying the last equation to $\left[\mathrm{H}^2\right](x)\asymp[\mathrm{H}](x)$ one proves the often used distributional identity $2[\delta](x)[\mathrm{H}](x)\asymp[\delta](x)$.

3.1.2 Results on some products of Distributions

Theorem 3.1.2. *The product of the generalized functions* $\ln|x|$ *and* $\delta^{(s-1)}(x)$ *for* $s = 0, 1, 2, \ldots$ *in* $\mathcal{G}(\mathbf{R})$ *admits associated distributions and it holds:*

$$\widetilde{\ln|x|} \cdot \widetilde{\delta^{(s-1)}(x)} \approx \frac{(-1)^s}{s}\delta^{(s-1)}(x) \tag{3.24}$$

Proof. For given $\varphi(x) \in \mathcal{D}(R)$ we suppose that $\operatorname{supp}\varphi(x) \subseteq [-l, l]$, without lost of generality. Then using the embedding rule and the substitution $v = (y-x)/\varepsilon$ we have the representatives of the distribution $\ln|x|$ in Colombeau algebra:

$$\widetilde{\ln|x|}\,(\varphi_\varepsilon, x) = \varepsilon^{-1}\int_{x-l\varepsilon}^{x+l\varepsilon} \ln|y|\varphi\left(\frac{y-x}{\varepsilon}\right) dy = \int_{-l}^{l} \ln|x+\varepsilon v|\varphi(v)dv$$

Similar,

$$\widetilde{\delta^{(s-1)}}\,(\varphi_\varepsilon, x) = \frac{(-1)^{s-1}}{\varepsilon^s}\varphi^{(s-1)}\left(-\frac{x}{\varepsilon}\right)$$

Then, for any $\psi(x) \in \mathcal{D}(\mathbf{R})$ we have:

$$\begin{aligned}
\left\langle \widetilde{\ln|x|}\,(\varphi_\varepsilon, x) \cdot \widetilde{\delta^{(s-1)}}\,(\varphi_\varepsilon, x)\,, \psi(x)\right\rangle &= \int_{-\infty}^{\infty} \widetilde{\ln|x|}\,(\varphi_\varepsilon, x)\,\widetilde{\delta^{(s-1)}}\,(\varphi_\varepsilon, x)\,\psi(x)dx \\
&= \frac{(-1)^{s-1}}{\varepsilon^s}\int_{-l\varepsilon}^{l\varepsilon}\left(\int_{-l}^{l} \ln|x+\varepsilon v|\varphi(v)dv\right)\varphi^{(s-1)}(-x/\varepsilon)\psi(x)dx \\
&= \frac{(-1)^s}{\varepsilon^{s-1}}\int_{-l}^{l}\varphi^{(s-1)}(u)\psi(-\varepsilon u)\int_{-l}^{l}\ln|\varepsilon v - \varepsilon u|\varphi(v)dvdu. \qquad (3.25)
\end{aligned}$$

using the substitution $u = -x/\varepsilon$. By the Taylor theorem we have that

$$\psi(-\varepsilon\omega) = \sum_{k=0}^{s-1}\frac{\psi^{(k)}(0)}{k!}(-\varepsilon\omega)^k + \frac{\psi^{(s)}(\eta\omega)}{(s)!}(-\varepsilon\eta)^s \tag{3.26}$$

for $\eta \in (0, 1)$. Using this for (3.25) we have:

$$\left\langle \widetilde{\ln|x|}\,(\varphi_\varepsilon, x) \cdot \widetilde{\delta^{(s-1)}}\,(\varphi_\varepsilon, x)\,, \psi(x)\right\rangle = \sum_{i=0}^{s-1}\frac{(-1)^{s+i}\psi^{(i)}(0)}{i!\varepsilon^{s-i-1}}J_i + O(\varepsilon)$$

where $J_i = \int_{-l}^{l} \varphi(v)dv \int_{-l}^{l} \ln|\varepsilon v - \varepsilon u| u^i \varphi^{(s-1)}(u)du$ and $i = 0, 1, \ldots, s-1$. Next using integration by part we have:

$$\begin{aligned}
J_i =& \int_{-l}^{l} \varphi(v)dv \int_{-l}^{l} \ln|\varepsilon v - \varepsilon u| u^i \varphi^{(s-1)}(u)du \\
=& \frac{1}{i+1} \int_{-l}^{l} \varphi(v)dv \int_{-l}^{l} \ln|\varepsilon v - \varepsilon u| \varphi^{(s-1)}(u) d\left(u^{i+1} - v^{i+1}\right) \\
=& -\frac{1}{i+1} \int_{-l}^{l} \varphi(v)dv \int_{-l}^{l} \Big[\left(u^{i+1} - v^{i+1}\right) \ln|\varepsilon v - \varepsilon u| \varphi^{(s)}(u)du \\
& - \frac{1}{i+1} \int_{-l}^{l} \varphi(v)dv \int_{-l}^{l} \frac{u^{i+1} - v^{i+1}}{u - v} \varphi^{(s-1)}(u)du \Big].
\end{aligned}$$

The first term above is zero, and we have

$$(i+1)J_i = \sum_{k=0}^{i} \int_{-l}^{l} v^{i-k} \varphi(v)dv \int_{-l}^{l} u^k \varphi^{(s-1)}(u)du = \int_{-l}^{l} u^i \varphi^{(s-1)}(u)du.$$

So, the only non-zero term we have it for $i = s - 1$ and that is $J_{s-1} = \frac{(-1)^{s-1}(s-1)!}{s}$.

$$\begin{aligned}
\left\langle \widetilde{\ln|x|} \left(\varphi_\varepsilon, x\right) \cdot \widetilde{\delta^{(s-1)}} \left(\varphi_\varepsilon, x\right), \psi(x) \right\rangle &= \frac{(-1)^s \psi^{(s-1)}(0)}{s} + O(\varepsilon) \\
&= \frac{(-1)^s}{s} \left\langle \delta^{(s-1)}(x), \psi(x) \right\rangle + O(\varepsilon).
\end{aligned}$$

Therefor passing to the limit, as $\varepsilon \to 0$, we obtain equation (3.24) proving the theorem. When nonlinear operations are used, a more general theory of nonlinear generalized functions is needed.

Chapter 4

Summary

We have evaluated some products of generalized functions, involving derivatives of the Dirac delta function, in Colombeau algebra in terms of associated distributions. this is significant because products of this type are very often used not only in physics, especially in quantum physics, but in other natural sciences and engineering, too, Colombeau differential algebra of generalized functions contains the space of schwartz distributions as a subspace, and the product of elements in it is generalization of the product of distributions, and thus all the results obtained in this way can be reformulated as regularized products in the classical distribution theory.

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