Addis Ababa University Office of Graduate Program

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Graduate Project Report

On

A Survey of the Riordan Group

Submitted to the Department of Mathematics in Partial fulfillment of the requirements for Master's degree of Science in Mathematics

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Summary of the Project

This project is all about a Survey of the Riordan group which is intimately related to the Riordan arrays in particular to the Fundamental Theorem of Riordan Arrays (FTRA) in solving enumerative problems. We focus on counting the average number of points on the $x-$axis of Dyck paths and the average number of hills in Dyck paths using the Catalan numbers, Fine numbers, and Schrőder numbers by switching between sequences and generating functions. The project also gives a unified presentation about tackling combinatorial identities, and finally introduce the group nature of Riordan arrays under matrix multiplication $(*)$ defined by $\left(g(z), f(z)\right) * \left(h(z), l(z)\right) = (g(z)h\left(f(z)\right), l(f(z)))$, where $g(z), f(z), h(z)$ and $l(z)$ are generating functions in a proper Riordan array $A = (d(z), h(z))$ such that the k^{th} column of a combinatorial sequence $(a_{n,k})_{n,k\geq o}$ of A defined by $a_{n,k} = [z^n]d(z)h(z)^k$.

Declaration

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

Esubalew Getie

Signature------------------------------ Date ---------------------------------------

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Section One

1. Introduction

The Riordan group, first described in 1991, and with the recent death of John Riordan this seems appropriate to name. It is a special collection of infinite dimensional matrices, whose entries are associated with the combinatorial sequences. We notice that we switch freely between column vectors and their generating functions as well as switching back and forth freely between sequences and the exponential generating functions via the Riordan group concept for the effective and efficient computation of typical applications in connection with

i. inverting identities;
ii. proving identities;
iii. developing combinatorial interpretations, and recursion properties in some combinatorial objects like paths(i.e. Dyck paths, Motzkin paths, Schrőder paths, etc) , permutations, walks, and trees.

Generating functions provide a surprisingly strong tool for counting that give a formal calculus for enumerative sequences, close to the theory of power series in real and complex analysis. A generating function counts the number of objects using an additional parameter n which classifies the instances of the problem according to their ‘complexity’. Let $(a_n)_{n\geq 0}$ be a sequence of numbers from the field $\mathbb{R}$. The elements a_n are usually nonnegative integers when we count something, but we start from more general numbers. The generating function of this sequence is the function

$f : \mathbb{N} \longrightarrow \mathbb{R}$ for which $f(n) = a_n$. we redefine this as follows:

the generating function of $(a_n)_{n\geq 0}$ is the formal power series

$$G(z) = \sum_{n=0}^{\infty} a_n z^n$$

where z is a letter that will be called a **variable**. Such a power series is called '**formal'** because we are not interested in its convergence, or its sum for any specific values of z. Indeed, $G(z)$ – despite its name – is not a function at all; it is just a way of writing the sequence $(a_n)_{n\geq 0}$.

Example: Let $a_0 = 1$ and $a_1 = 1$. The sequence $(a_n)_{n\geq 0}$ defined by the recurrence relation $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$ gives the Fibonacci numbers: $1, 1, 2, 3, 5, 8, 13, ...$ Their generating function is thus $G(z) = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + \ldots$, where the next values are easy to compute, but it is more difficult to give a formula for the general value a_n.

This project focuses on the connection of enumeration of average points on the x-axis of Dyck paths and average number of hills with the Riordan group. Riordan arrays act on a column array of complex numbers. For an infinite matrices M= $\left(a_{i,j}\right)_{i,j\geq 0}$ with entries in complex numbers $C_i(z) = \sum_{n=0}^{\infty} a_{n,i} z^n$ be the generating function of the i^{th} column of M as in Shapiro et al [9, p. 230]. We now make the crucial assumption that

$$C_k(z) = g(z)\left(f(z)\right)^k \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(*)$$

where $g(t) = 1 + g_1 z + g_2 z^2 + g_3 z^3 + \cdots$ and $f(z) = z + f_2 z^2 + f_3 z^3 + \cdots$ according to Shapiro et al[9] notation, in which we prefer to use in this project ,while an Italian known combinatorist R. Sprugnoli defines (*) being described in [10] as follows:

$$C_k(z) = g(z)\left(zf(z)\right)^k.$$

We write M = $\left(g(z), f(z)\right)$ and say that $\left(g(z), f(z)\right)$ is a Riordan array.

Now multiply M on the right by a column vector A = $\left(a_{0,}a_{1,}a_{2,\ldots}\right)^T$, where $A(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ and note that the resulting column vector $\left(b_{0,}b_{1,}b_{2,\ldots}\right)^T$ has the generating function

$$\begin{aligned} B(z) &= a_0 C_0(z) + a_1 C_1(z) + a_2 C_2(z) + \cdots \\ &= a_0\, g(z) + a_1 g(z) f(z) + a_2 g(z)[f(z)]^2 + \cdots \\ &= g(z)[a_0 + a_1 f(z) + a_2 [f(z)]^2 + \cdots] \\ &= g(z) A\left(f(z)\right) \end{aligned}$$

Thus $(g(z), f(z)) * A(z) = g(z)A\left(f(z)\right) = B(z)$ **-------------------------------- (**)**

The project is basically organized based on a Survey of the Riordan Group by Shapiro [8], in which most concepts are summarized, on the Riordan Group by Shapiro et al[9] , and Riordan Arrays by Sprugnoli [10].

In section three, we would like to point out, in the Combinatorics part of the concept of Riordan arrays; how to apply the Fundamental Theorem of Riordan Arrays (FTRA) in typical applications related to Dyck paths and proving combinatorial identities.

In the Algebra part, we consider how Riordan arrays form a group under matrix multiplication including inverses of the Riordan arrays and types of subgroups of the Riordan group.

Section Two

2. Preliminary concepts

- A **path**, p, of length n from (x, y) to (x', y') with step set S is a sequence of points in the plane $(x, y), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ such that all $(x_{i+1}, y_{i+1}) \epsilon$ S.
- **Dyck paths** are lattice paths starting and ending on the horizontal axis using steps $(1,1) and$ $(1, -1)$, and never going below the horizontal axis. The number of Dyck paths of length 2n equals the Catalan numbers C_n. In short, Dyck paths are positive paths from $(0,0)$ to $(2n, 0)$ with $S = \{(1,1)(1, -1)\}$.
- **Fine numbers** are $1, 0, 1, 2, 6, 18, 57, \dots$ and generated by $F(z) = \frac{1}{z}\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}$.
- **Hill** in a Dyck path is a pair of consecutive steps giving a peak of height 1.
- **Lattice point** is a point on the Cartesian plane with integral coordinates.
- **Peak** at height k on a Dyck path is a point on the path with ordinate y=k that is preceded by a (1,1) step and immediately followed by a (1,-1) step.
- **Schrőder** paths are paths from $(0,0)$ to $(2n, 0)$ with S' $=\{(1,1), (2,0), (1, -1)\}$. These are counted by the big Schrőder numbers 1, 2, 6, 22, 90, …
- A **return** is defined as a non-origin vertex having ordinate 0.
- A **Motzkin** path of length n is a path going from $(0,0)$ to $(n, 0)$ consisting of up steps U $=(1,1)$, down steps D$=(1, -1)$and horizontal steps H=(1,0)which never goes below the x $-$axis. The (k, t)-Motzkin path is a Motzkin path such that each horizontal step is weighted by k, each down step weighted by t and each up step is weighted by 1.
- **Catalan numbers**: the sequence$\{C_n\}_{n\geq 0} = \{1,1,2,5,14,42,132,429,1430, \dots\}$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is called the n^{th} Catalan number. The generating function for the Catalan numbers is denoted by $C(z) = \frac{1-\sqrt{1-4z}}{2z}$.
- **A Riordan array** is an infinite lower triangular matrix of the form

$$D = \left(d_{n,k}\right)_{n,k\geq 0}$$

where $d_{n,k} = [z^n]\, d(z)\big(h(z)\big)^k$; $d(z), h(z)$ are two formal power series with $\big(\mathbf{d}(\mathbf{z}), \mathbf{h}(\mathbf{z})\big)^{-1} = \left(\frac{1}{\mathbf{g}(\bar{h}(\mathbf{z}))}, \bar{h}(z)\right)$; for $\bar{h}\big(h(z)\big) = h\left(\bar{h}(z)\right) = z$ and, we have $d_{n,k} = 0$, for all $k > n$. The entry of a Riordan matrix, R can be expressed as

$$d_{n,k} = a_0\, d_{n,k} + a_1\, d_{n,k+1} + a_2\, d_{n,k+2} + a_3\, d_{n,k+3} + \cdots$$

where $\{a_j\}_{j\geq 0}$ is a sequence that is independent of n and k. The sequence $\{a_j\}_{j\geq 0}$ is referred to as the ***A*-sequence** of R and its generating function is denoted $A_R(z)$.

Similarly, the elements of the first column of R (excluding $d_{0,0}$), can be expressed as

$$d_{n+1,0} = z_0\, d_{n,0} + z_1\, d_{n,1} + z_2\, d_{n,2} + z_3\, d_{n,3} + \cdots$$

where $\{z_j\}_{j\geq 0}$ is a sequence that is independent of n and k. The sequence $\{z_j\}_{j\geq 0}$ is referred to as the **Z**-sequence of R and its generating function is denoted $\mathbf{Z_R}(z)$.

- The main **Rules** for the '**coefficient of**` functionals:

1) $[z^n](af(z) + bg(z)) = a[z^n]f(z) + b[z^n]g(z)$

   [Linearity].

2) $[z^n]zf(z) = [z^{n-1}]f(z)$

   [Shifting] .

3) $[z^n]f'(z) = (n+1)[z^{n+1}]f(z)$

   [Differentiation].

4) $[z^n]f(z)g(z) = \sum_{k=0}^{n}[z^k]f(z)\,.\,[y^{n-k}]g(y)$,

   [Convolution].

5) $[z^n]f\big(g(z)\big) = \sum_{k=0}^{n}[z^k]f(z)\,.\,[y^{n-k}]g(y)^k$,

   [Composition].

6) $[z^n]\bar{f}(z) = \frac{k}{n}[z^{n-k}](\frac{z}{f(z)})^n$, here $\bar{f}(z)$ is the compositional inverse of $f(z)$,i.e.

   $f\left(\bar{f}(z)\right) = \bar{f}\big(f(z)\big) = z.$

   [Inversion (Lagrange Inversion Formula - LIF)].

7) $[z^n]f(az) = a^n[z^n]f(z)$

[The translation $z \longmapsto az$].

8) $[z^{kn}]f(z^k) = [z^n]f(z)$.

[The translation $z^n \longmapsto z$].

9) $[z^n][F(w)|w = z\phi(w)] = \frac{1}{n}[z^{n-1}]F'(z)\phi(z)^n$

[a useful formulation of Lagrange Inversion Formula].

10) $[z^n][F(w)|w = z\phi(w)] = \frac{1}{n}[z^{n-1}]F'(z)\phi(z)^{n-1}(\phi(z) - z\phi'(z))$

[another useful LIF].

11) $[z^n]\frac{1}{1-tz} = t^n$.

12) $[z^n]\frac{1}{(1+rz)(1+sz)} = \frac{r^{n+1}-s^{n+1}}{r-s}(-1)^n$.

13) $[z^n]\frac{1}{1+bz+cz^2} = \frac{2(\sqrt{c})^{n+1}\sin(n+1)\theta}{\sqrt{4c-b^2}}$ $,\theta = \tan^{-1}\frac{\sqrt{4c-b^2}}{-b} + r$; [$b > 0$]

where denominator is an irreducible polynomial of second degree.

Section Three

3.1. Fundamental Theorem of Riordan Arrays (FTRA)

Definition: An infinite lower triangular matrix, $L = (\mathrm{l}_{n,k})_{n,k\geq 0}$ is a Riordan matrix if there exist generating functions $g(z) = \sum_{n=0}^{\infty} g_n \, \mathrm{z}^{\mathrm{n}}, f(z) = \sum_{n=0}^{\infty} f_n \, \mathrm{z}^{\mathrm{n}}, f_0 = 0, f_1 \neq 0$ such that $l_{n,0} = g_n \; and \; \sum_{n\geq k} l_{n,k} \, \mathrm{z}^{\mathrm{n}} = g(z) \, (f(z))^k$.

From the definition it is clear that a Riordan matrix L is completely defined by the functions $g(z)$ and $f(z)$. Hence L is called Riordan and we write $L = (g(z), f(z))$, or simply $L = (g, f)$.

The bivariate generating function of a Riordan array is given by

$$\begin{aligned} \mathrm{d}(\mathrm{z},\mathrm{t}) = \sum_{n,k\geq 0} \mathrm{d}_{n,k} \mathrm{z}^{n} \mathrm{t}^{k} &= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty} \mathrm{d}_{n,k} \mathrm{z}^{n}\right)\mathrm{t}^{k} \\ &= \sum_{k=0}^{\infty}\left[\mathrm{d}(\mathrm{z}).\left(\mathrm{zh}(\mathrm{z})\right)^{k}\right]\mathrm{t}^{k}, \text{ where } \mathrm{d}_{n,k} = \mathrm{d}(\mathrm{z})\left(\mathrm{zh}(\mathrm{z})\right)^{k} \\ &= \mathrm{d}(\mathrm{z}) \sum_{k=0}^{\infty}\left(\mathrm{z.t.h}(\mathrm{z})\right)^{k} \\ &= \frac{\mathrm{d}(\mathrm{z})}{1-\mathrm{tz}\,\mathrm{h}(\mathrm{z})} \end{aligned}$$

Therefore $\mathrm{d}(\mathrm{z},\mathrm{t}) = \frac{\mathrm{d}(\mathrm{z})}{1-\mathrm{tz}\,\mathrm{h}(\mathrm{z})}$

In the squel we always assume that $\mathrm{d}(\mathrm{z}) \neq 0$; if we also have $\mathrm{h}(\mathrm{z})$ then the Riordan array is said to be **proper**. In the proper case the diagonal elements $\mathrm{d}_{n,n}$ are different from zero for all $n \in \mathbb{N}$.

Definition: Proper Riordan arrays can also be defined in terms of the two sequences, $A = \{a_i\}_{i\in\mathbb{N}}$ with $a_0 \neq 0$, called the A-sequence and $Z = \{z_0, z_1, z_2, \ldots\}$ is the Z-sequence such that every element $\mathrm{d}_{n+1,k+1}$ can be expressed as a linear combination with coefficients in A, of the elements in the preceding row, starting from the preceding columns:

$$\mathrm{d}_{n+1,k+1} = a_0 \mathrm{d}_{n,k} + a_1 \mathrm{d}_{n,k+1} + a_2 \mathrm{d}_{n,k+2} + \cdots$$

and such that every element in column 0 can be expressed as a linear combination ,with coefficient in Z, of all the elements of the preceding row:

$$d_{n+1,0} = z_0 d_{n+1,0} + z_1 d_{n+1,1} + z_2 d_{n+1,2} + \cdots.$$

The generating functions $A(z)$and $Z(z)$ of these sequences are related to the pair$(d(z), h(z))$ by the formulae:

$$h(z) = A(h(z))$$

$$d(z) = \frac{d_{0,0}}{1 - zZ(h(z))}$$

Example: For the Pascal Triangle P, we have:

$$A(z) = 1 + z$$

$$Z(z) = 1 \ .$$

Actually, we have

$$P = \left(\frac{1}{1-z}, \frac{z}{1-z}\right) = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \\ & & \dots & & & \ddots \end{bmatrix}$$

the A- sequence is $(1,1,0,0, \dots)$ and the Z-sequence is $(1,0,0,0, \dots)$.

Theorem 3.1.1: An infinite lower triangular array $D = (d_{n,k})_{n,k \geq 0}$ is a Riordan array if and only if a sequence $A = (a_0 \neq 0 , a_1, a_2, \dots)$ exists such that for every $n, k \in \mathbb{N}$

$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots = \sum_{j=0}^{\infty} a_j d_{n,k+j}$ ------------------ (1)

holds;

where its proof is given in Sprugnoli [10] and let us use with a modified definition of Riordan arrays, i.e. using $C_k(z) = g(z)(f(z))^{k}$, for each column C_i , $i = 0,1,2, \dots$ instead of $C_k(z) = g(z)(zf(z))^{k}$ along with He et al[6].

Proof: Let us suppose that D is the Riordan array $(d(z), h(z))$ and let us consider the Riordan array $\left(\frac{d(z)h(z)}{z}, h(z)\right)$;

We define the Riordan array $(A(z), B(z))$ by the relation:

$$\big(A(z), B(z)\big) = \big(d(z), h(z)\big)^{-1} * \left(\frac{d(z)h(z)}{z}, h(z)\right)$$

Or $\quad \big(d(z), h(z)\big) * \big(A(z), B(z)\big) = \left(\frac{d(z)h(z)}{z}, h(z)\right)$

By performing the matrix product, we find:

$d(z)A(h(z)) = \frac{d(z)h(z)}{z}$ and $B(h(z)) = h(z)$…………………………… (2)

The latter identity implies $B(z) = z$.

Therefore, we have

$\big(d(z), h(z)\big) * (A(z), z) = \left(\frac{d(z)h(z)}{z}, h(z)\right)$.

The element $f_{n,k}$ of the left hand member is

$f_{n,k} = [z^n]\, d(z)h(z)^k\, A\big(h(z)\big) = \sum_{j=0}^{\infty} d_{n,j}\, a_{j-k} = \sum_{j=0}^{\infty} d_{n,j+k}\, a_j$, by composition rule of coefficient of functionals; if as usual we interpret a_{j-k} as 0 when $j < k$.

The same element in the right hand member is:

$$[z^n]\frac{d(z)h(z)h(z)^k}{z} = [z^{n+1}]d(z)h(z)^{k+1} = d_{n+1,k+1}$$

By equating these two quantities, we have the identity in (1). We remark that the first relation in (2) is equivalent to $zA(h(z)) = h(z)$ since $d(z)A(h(z)) = \frac{d(z)h(z)}{z}$ and $d(z)$ is a formal power series.

For the converse, let us observe that (1) uniquely defines the array D when the elements of column 0 $(i.e.\, d_{0,0}, d_{1,0}, d_{2,0}, \ldots)$ are given.

Let $d(z)$ be the generating function of this column and $A(z)$ the generating function of the sequence A and define $h(z)$ as the solution of the functional equation $h(z) = zA(h(z))$, which is uniquely determined because of our hypothesis $a_{0,} \neq 0$.

We can therefore consider the proper Riordan array $\widehat{D} = (d(z), h(z))$ by the first part of the theorem, D satisfies relation (1) above, for every $n, k \in \mathbb{N}$.

Therefore, by our observation, it must coincide with D. ■

Example: Consider a Riordan array F defined by $d_F(z) = (1 - z - z^2)^{-1}$,

so that column 0 is composed by Fibonacci numbers, shifted by one place. Besides, the A-sequence is A_F = (1,1,1,1,…), that is, any element $F_{n+1,k+1}$ is obtained by summing all the elements in the previous row, starting from column k. The Riordan array F is

$$h_F(z) = zA_F(h(z)) = \frac{z}{1-h_F(z)}:$$

this equation has two solutions, but we know that $A_F(0) \neq 0$ so that we should consider the solution with the minus sign:

$$h_F(z) = \frac{1-\sqrt{1-4z}}{2} = z + z^2 + 2z^3 + 5z^4 + 14z^5 + 42z^{67} + 132z \quad + \cdots$$

the well-known Catalan numbers. Therefore we have:

$$F = \left(\frac{1}{1-z-z^2}, \frac{1-\sqrt{1-4z}}{2}\right).$$

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	2	2	1				
3	3	5	3	1			
4	5	12	9	4	1		
5	8	31	26	14	5	1	
6	13	85	77	46	20	6	1

Table 3.1 The Fibonacci triangle F

Questions; Prove the following identity;

$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$.

Proof : to prove the identity

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1} \qquad ,$$

let there are n students in our school. We could select a group of k students to serve in the council of our school and then choose the president from the council:

i. Choose k students from the n students as the council, where $1 \leq k \leq n$. This can be done in $\binom{n}{k}$ ways;
ii. Select a president from the k students;
iii. Sum over all the possible numbers k, and we will get the left hand side (LHS) of the identity.

There is also another way as follows:

Choose one of the n students as the president. There are n possible choices. Then there are 2^{n-1} different choices to select the rest of the council from the other n-1 students. Therefore, there are $n2^{n-1}$ distinct ways to do this which is exactly the right hand side of the identity.

Thus, the two methods applied for doing the same thing and hence they are equivalent to each other, that is, we get the above identity.

Let us see an important theorem, its proof is provided in Shapiro et al [8, p.7] , [9, p.230].)

Theorem 3.1.2: The Fundamental Theorem of Riordan Arrays (FTRA).

$$\text{Let} \quad (g(z), f(,z)) \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \text{----------------------------- (\#)}$$

where the generating functions of the two column vectors are $A(z)$ and $B(z)$, respectively.

Then the identity in (#) above is true if and only if the following equation holds:

$$g(z)A(f(z)) = B(z)$$

Example: Let $b_n = \sum_{k=0}^{n} \binom{n-k}{k} 6^k$, and we want to find a closed form expression for $\boldsymbol{b_n}$.

Proof: we have to follow the important procedures in tackling this identity. These are

Step 1. The summation can be written as row sum for $k = 0,1,2, \dots, n$.

$$\binom{0}{0} = 1$$

$$\binom{1}{0} = 1$$

$$\binom{2}{0} + \binom{1}{1}6 = 7$$

$$\binom{3}{0} + \binom{2}{1}6 = 13$$

$$\binom{4}{0} + \binom{3}{1}6 + \binom{2}{0}6^2 = 55$$

Step2.Set them up as a matrix product

$$\begin{bmatrix} 1 & & & & \\ 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & & & \\ 1 & 3 & 1 & & \\ 1 & 4 & 3 & & \\ & \cdots & & & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ 6^2 \\ 6^3 \\ 6^4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 7 \\ 13 \\ 55 \\ \vdots \end{bmatrix}$$

Step3. Here, we can see from the columns of the matrices at LHS that the generating functions can be identified as

$$g(z) = \frac{1}{1-z}, f(z) = \frac{z^2}{1-z} \text{ and } A(z) = \frac{1}{1-6z};$$

Step4. By FTRA, we have

$$\left(\frac{1}{1-z}, \frac{z^2}{1-z}\right) * \left(\frac{1}{1-6z}\right) = \frac{1}{1-z} \frac{1}{(1-6\frac{z^2}{1-z})} = \frac{1}{1-z-6z^2}$$

$$= \frac{1}{(1-3z)(1+2z)} = \frac{1}{5}\left(\frac{3}{(1-3z)} + \frac{2}{1+2z}\right).$$

Then $\quad b_n = [z^n]\left(\frac{1}{5}\left(\frac{3}{(1-3z)} + \frac{2}{1+2z}\right)\right) = \frac{1}{5}\left(3^{n+1} + (-1)^n 2^{n+1}\right),$

which is the sequence of the right hand side.

For the general case, we look at the Riordan arrays $(g(z), f(z))$ column by column, and multiply it by the column vector on the left hand side

$$\begin{bmatrix} | & & & & & \\ | & | & & & & \\ g & gf & gf^2 & & & \\ | & | & | & | & & \\ | & | & | & | & | & \\ & \cdots & & & & \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ . \\ . \\ . \end{bmatrix}$$

Table 1: Column of Riordan arrays

This yields

$$\begin{aligned} a_0 g + a_1 gf + a_2 gf^2 + \cdots \quad &= g(a_0 + a_1 f + a_2 f^2 + \cdots) \\ &= g(z). A(f(z)) = B(z), \end{aligned}$$

and we have our result as shown in Shapiro et al [9]. This allows us to switch easily between the matrix form and generating functions.

Example: Let

$$M = \begin{bmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 5 & 4 & 1 & & & \\ 14 & 14 & 6 & 1 & & \\ 42 & 48 & 27 & 8 & 1 & \\ & & \cdots & & & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 4^2 \\ 4^3 \\ 4^4 \\ \vdots \end{bmatrix}$$

be a matrix identity. For this matrix identity, let $g(z)$be the generating function for Catalan numbers (1, 2, 5, 14, ...):

$$g(z) = 1 + 2z + 5z^2 + 14z^3 + \cdots.$$

We can rewrite this multiplying by z and adding 1 on both sides of the equation:

$$1 + zg(z) = 1 + z + 2z^2 + 5z^3 + \cdots = C(z).$$

Solving for $g(z)$, we obtain: $g(z) = \frac{C(z)-1}{z} = C^2(z) = (\frac{1-\sqrt{1-4z}}{2z})^2 = \frac{1-2z-\sqrt{1-4z}}{2z^2}$,

since $C(z) = 1 + zC(z)^2$.

Therefore $g(z) = \frac{1-2z-\sqrt{1-4z}}{2z^2}$.

From the second column of M and given $g(z)$, we get $f(z) = zg(z)$,

where $f(z) = z + f_2 z^2 + f_3 z^3 + \cdots$,and $f_0 = 0,\ f_1 = 1$.

From the recurrence relation

$$a_{i+1,j+1} = a_{i,j} + 2a_{i,j+1} + a_{i,j+2},$$

We can derive the generating function for the sequence in the $j^{th}, (j \geq 0)$ column in the matrix given above as

$$M_{j+1}(z) = g(z)(zg(z))^j.$$

Now we have the Riordan array M= $(g(z), zg(z))$.Since the generating function of $(1,2,3,4, \ldots)$ equals $A(z) = \frac{1}{(1-z)^2}$, it follows that $B(z) = g(z)A\big(zg(z)\big) = \frac{1}{1-4z}$ is the generating function for the right hand side of M. Thus we obtain the identity M above.

Example: When we return back to the above Question, in student council problem in applying the FTRA, let us follow certain steps:

Step 1: For n=3 and 4, we have (i.e. using $\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$ above)

$$\binom{3}{0} \times \mathbf{0} + \binom{3}{1} \times \mathbf{1} + \binom{3}{2} \times \mathbf{2} + \binom{3}{3} \times \mathbf{3} = \mathbf{3} \times \mathbf{2^2}$$

$$\binom{4}{0}\times 0+\binom{4}{1}\times 1+\binom{4}{2}\times 2+\binom{4}{3}\times 3+\binom{4}{4}\times 4=4\times 2^3$$

respectively.

Step 2: Then we will change it as a lower triangular matrix times a column vector equals a column vector:

$$\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ & & \cdots & & & \ddots \end{bmatrix}\begin{bmatrix} 0\\ 1\\ 2\\ 3\\ 4\\ \vdots \end{bmatrix}=\begin{bmatrix} 0\\ 1\times 2^0\\ 2\times 2^1\\ 3\times 2^2\\ 4\times 2^3\\ \vdots \end{bmatrix}$$

Step 3:The generating functions of the first three columns are

$$g(z)=\frac{1}{1-z}\ ,\quad g(z)f(z)=\frac{z}{(1-z)^2}\quad \text{and}\quad g(z)f(z)^2=\frac{z^2}{(1-z)^2}\,.$$

So, $\quad f(z)=\frac{z}{1-z}$

Here, also the generating functions of the two column vectors on the left hand side (LHS) and the right hand side (RHS) above will be

$$A(z)=\frac{z}{(1-z)^2}\quad \text{and}\quad B(z)=\frac{z}{(1-2z)^2}\ ,$$

respectively.

Now, by the Fundamental Theorem of Riordan Arrays

$$\big(g(z),f(z)\big)*A(z)=B(z)$$

$$\left(\frac{1}{1-z}\ ,\frac{z}{1-z}\right)*\frac{z}{(1-z)^2}=\frac{1}{1-z}\,\frac{\frac{z}{1-z}}{\left(1-\frac{z}{1-z}\right)^2}$$

$$=\frac{z}{(1-2z)^2}$$

Since $B(z)=\frac{z}{(1-2z)^2}=z+2(2z)+3(2z)^2+\cdots+n2^{n-1}z^n+\cdots=\sum n2^{n-1}z^n$,

we have the generating function for the column vectors on the right hand side.

Question: In a matrix L, how can we find $f=f(z)$?

i.e. Having been given a matrix L and $g(z)$ in which we suspect is Riordan, how can we find $f(z)$?

Method1: Comparing the coefficients of gf and g .

Example. Let

$$L = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \\ & & \cdots & & & \ddots \end{bmatrix}.$$

In this matrix, the first two columns with generating functions are gf and g. As we know, $g(z) = \frac{1}{1-z}$.Then, we find $f = f(z)$ by comparing the coefficients of gf and g as follows:

$$g(z)f(z) = 1 + 1z + 1z^2 + 1z^3 + \cdots \text{----- (1)}$$

$$f(z) = \quad z + z^2 + z^3 + \cdots \text{------------(2), (1) and (2)implies (3)}$$

$$g(z)f(z) = 0 + 1z + 2z^2 + 3z^3 + \cdots \text{ ------ (3)}$$

Therefore, we can guess that $f = f(z) = \frac{z}{1-z} = zf(z)$.We can then compute $g(z)f(z)^k$ for k=2,3,… to see if we are getting the coefficients of the next column.

We can identify a Riordan matrix by its generating functions or by its dot diagram. A dot diagram is a symbolic representation of the recursions which define the matrix.

Let R be a Riordan matrix with entries $r_{n,k}$ for $n, k \geq 0$.

Definition: We say that $[b_1, b_2, b_3, \ldots; a_0, a_{1,} a_2, \ldots]$ is the dot diagram for R if

$$r_{n,0} = b_1 r_{n-1,0} + b_2 r_{n-1,1} + b_3 r_{n-1,2} + \cdots \text{ for } n \geq 1$$

and

$$r_{n,k} = a_0 r_{n-1,k-1} + a_1 r_{n-1,k} + a_2 r_{n-1,k+1} + \cdots \text{ for } n, k \geq 1$$

provided that $r_{0,0} = g(0) = 1.$

Method 2: From the recurssion or ‘‘dot diagram’’.

Example. (dot diagram for Pascal triangle) For

$$P = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 & \ddots \\ & & \cdots & & \end{bmatrix} = \left(\frac{1}{1-z}, \frac{z}{1-z}\right),$$

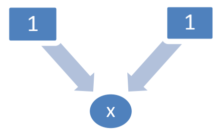


Figure 3.1(a) A dot diagram for Pascal triangle. P has the dot diagram [1;1,1].

Remark: A small Schrőder path is a lattice path starting at (0,0) and ending at (2n,0) and using steps H= (2,0), U=(1,1) and D=(1,-1) such that no step is below the x-axis and there are no peaks at level one. Imposing this condition gives us small Schrőder numbers while without it we would have the large Schrőder numbers.

Example: Consider the following small Schrőder numbers arrays

$$L=\begin{bmatrix} 1 & & & & \\ 3 & 1 & & & \\ 11 & 6 & 1 & & \\ 45 & 31 & 9 & 1 & \\ 197 & 1\ 56 & 60 & 12 & 1 \\ & & \cdots & & \ddots \end{bmatrix}$$

We might be interested to compute $f(z)$ in L since 1, 3, 11, 45, 197, … are the small Schrőder numbers . There is a recursion between four entries of any two adjacent rows as follows:

$$a_{n,k} = a_{n-1,k-1} + 3a_{n-1,k} + 2a_{n-1,k+1}.$$

The picture for this is

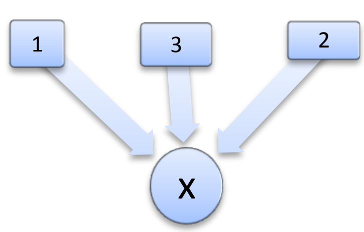


Figure 3.1(b) A dot diagram for small Schrőder numbers

From this recursion, for $k \geq 2$, the $a_{n,1}$ is the n^{th} small Schrőder number, which counts small Schrőder path of length 2n.

Thus, we look at the k^{th} column and we find that

$gf^k = z(gf^{k-1} + 3gf^k + 2gf^{k+1})$,

Because we need to move down one row in the matrix, the equation contains the term z.

Dividing by gf^{k-1}, we obtain

$$f = z(1 + 3f + 2f^2)$$
$$= \frac{1-3z-\sqrt{1-6z+z^2}}{4z}$$
$$= z(1 + 3z + 11z^2 + 45z^3 + \cdots).$$

If we denote $1 + 3z + 11z^2 + 45z^3 + \ldots$ by $g(z)$, then $f(z) = zg(z)$ and

$$g = 1 + 3f + 2f^2 = 1 + 3zg + 2zg\,f$$
$$= 1 + z(3g + 2gf)$$
$$g - z(3g + 2gf) = 1$$
$$g = \frac{1}{1-3z-2zf} = \frac{1}{1-3z-2z(\frac{1-3z-\sqrt{1-6z+z^2}}{4z})} = \frac{1-3z-\sqrt{1-6z+z^2}}{4z^2}$$

According to this, we get the following Riordan Array

$$\mathrm{L} = (g(z), zg(z)),$$

where $g(z) = \frac{1-3z-\sqrt{1-6z+z^2}}{4z^2}$.

Remark: we recall that **Dyck paths** are lattice paths starting and ending on the horizontal axis using steps $(1,1)$ *and* $(1,-1)$, and never going below the horizontal axis.

For Dyck paths of length 2n, we have

i) the minimum number of points on the x-axis is 2;

ii) the maximum number of points on the x-axis is n+1.

Question: What is the average number of points on the x-axis of Dyck paths?

Here, before we go to answer the Question,

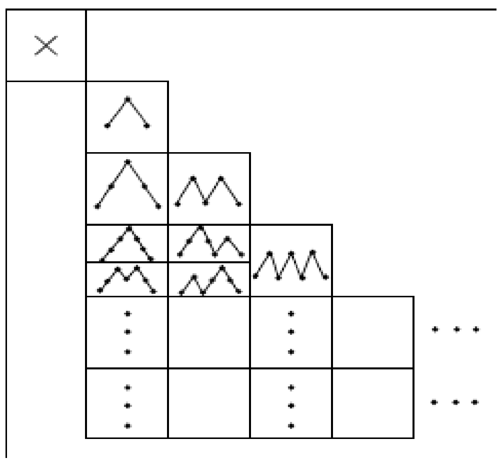
What is the average number of points on the x-axis of Dyck paths?

on the application section below.

3.2 Applications

3.2.1 The average number of points on the x-axis of Dyck paths

Let k denote the number of points on the x-axis of Dyck paths.



n \ k	1	2	3	4	5		
0	1						
1		1					
2		1	1				
3		2	2	1			
4		5	5	3	1		
5		14	14	9	4	1	...
		...	...	...	...	...	...

Figure 3.2: Dyck paths and the number of paths with points on the x-axis.

In the table on the right hand side, the numbers in the left most column denote n; the numbers in the top most row are the number of points on the x-axis corresponding to such paths. After we fill in the blanks with 0,we get a matrix which we denote $M = C(z)$, since the row sum as well as two of the columns are the Catalan numbers.

Question: Is M a Riordan array?

Proof. **i)** we can observe that the generating functions for the columns in M are

$1, zC, z^2C^2, z^3C^3, \ldots$ (<u>Note</u>, $C \coloneqq C(z)$)

Thus we have that $g(z) = 1$

and $g(z)f(z) = zC$, using generating function of the second column(method 1)

implies $f(z) = zC$

where C represents the generating function for the Catalan numbers.

Thus

$$(g(z), f(z)) = (1, \mathrm{zC})$$

is a Riordan array.

ii) Now we want to show that

$$(1, \mathrm{zC})\mathrm{A}(\mathrm{z}) = \mathrm{C}\ ,$$

where $\mathrm{A(z)}$ is the generating function for some number sequence.

From the matrix M, it is easy to verify that this is possible, since

$$\begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 1 & 1 & & & & \\ 0 & 2 & 2 & 1 & & & \\ 0 & 5 & 5 & 3 & 1 & & \\ 0 & 14 & 14 & 9 & 4 & 1 & \ddots \\ & & & \cdots & & & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \\ 14 \\ 42 \\ \vdots \end{bmatrix}$$

Let us denote the generating functions of the two column vectors by $\mathrm{A(z)}$ and $\mathrm{B(z)}$ respectively, then $\mathrm{A(z)} = \frac{1}{1-\mathrm{z}}$; $\mathrm{B(z)} = \mathrm{C(z)}$.

Thus $\quad g(z) * \mathrm{A}\big(f(z)\big) = 1\,\frac{1}{1-\mathrm{zC}} = \mathrm{C} = \mathrm{B(z)},$ since $\mathrm{C} = 1 + \mathrm{zC}^2$.

By the Fundamental Theorem of Riordan arrays, we have the required result, that is, the right hand side column has C as its generating function as required.

iii) Now let us compute the average number of the points on the x-axis of Dyck paths of length2n. From the table above, we obtain

$$\begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 1 & 1 & & & & \\ 0 & 2 & 2 & 1 & & & \\ 0 & 5 & 5 & 3 & 1 & & \\ 0 & 14 & 14 & 9 & 4 & 1 & \ddots \\ & & & \cdots & & & \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 14 \\ 42 \\ 132 \\ \vdots \end{bmatrix}.$$

The entries of the vector in the right hand side (RHS) of this equation are the number of points on the x-axis $[z^n]\mathrm{B(z)}$, which are the Catalan Numbers, $\mathrm{C}_{\mathrm{n}+1}$.

The proof of this is similar to the proof of (ii) except now $\mathrm{A(z)} = \frac{1}{(1-\mathrm{z})^2}$.

By the FTRA,

$$\mathrm{B(z)} = (1, \mathrm{zC}) * \frac{1}{(1-\mathrm{z})^2} = 1\ \frac{1}{(1-\mathrm{zC})^2} = \frac{1}{(1-\mathrm{zC})^2}$$

$= C^2(z)$, the generating function for the column at the right most hand side.

$= \frac{C-1}{z}$, since $zC^2(z) + 1 = C(z)$.

which is the generating function for $\sum_{n=0}^{\infty} C_{n+1} z^n$.

Thus the average number of points on the x-axis is

$$\frac{C_{n+1}}{C_n} = \frac{\frac{1}{n+2}\binom{2n+2}{n+1}}{\frac{1}{n+1}\binom{2n}{n}} = \frac{4n+2}{n+2}.$$ As n→ ∞ , the limit is 4 ∎

3.2.2. The average number of hills in Dyck paths

Definition: A hill is a UD [up step-down step] pair of steps that form a peak at height 1, and we denote the number of hills in Dyck paths of length 2n by H_n.

Example: For n=3, $C_3 = 5$

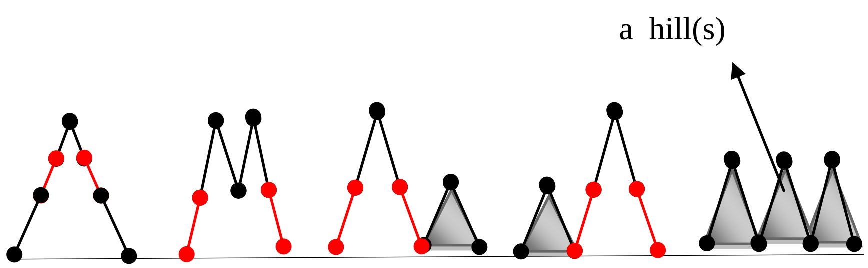


Figure 3.3(a) Dyck paths with their corresponding hills

Thus $H_3 = 5$ and the average number of hills is $\frac{5}{5} = 1$.

If n=4, $C_4 = 14$

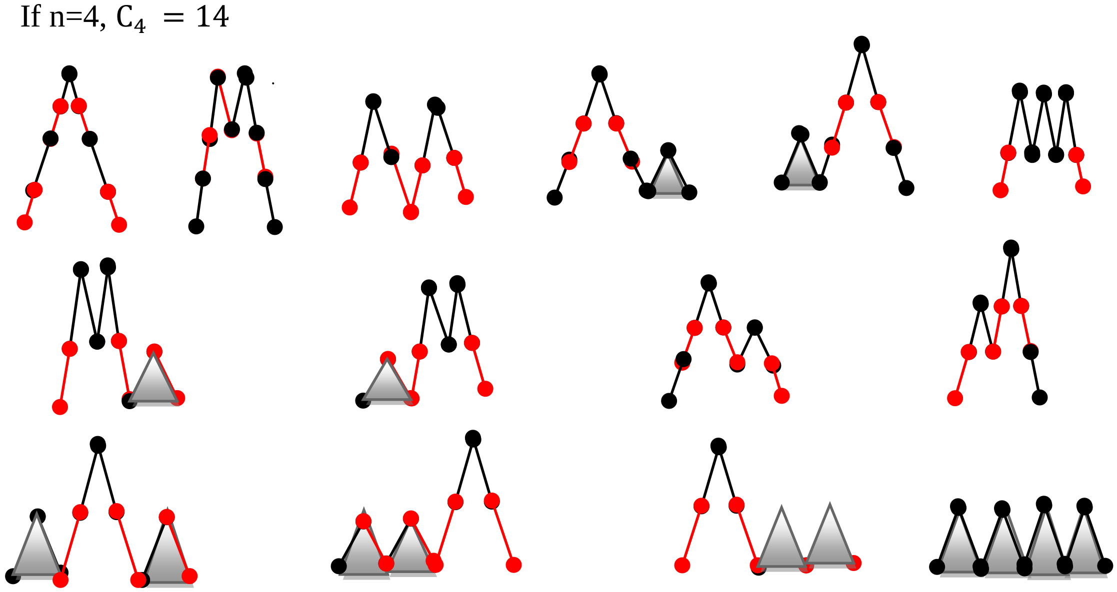


Figure 3.3 (b) Dyck paths with their corresponding hills

Thus H_4=14 and the average number of hills is $\frac{14}{14} = 1$.

Question: Can we generalize that $H_n = C_n$?

Proof: By the Fundamental Theorem of Riordan arrays, let

$$\mathcal{F} = \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ 2 & 2 & 0 & 1 & & \\ 6 & 4 & 3 & 0 & 1 & \\ & & \cdots & & & \ddots \end{bmatrix} = (\mathrm{F}, \mathrm{zF}),$$

$\mathcal{F}$ is Riordan, because every column (excluding $f_{n,0}$, $the\ first\ column$) can be expressed as a linear combination of the preceding column with $\mathcal{F}(z) = (g(z), f(z))$, where $g(z) = \mathrm{F(z)} = 1 + \mathrm{z}^2 + 2\mathrm{z}^3 + 6\mathrm{z}^4 + \cdots$, and $f(z) = zg(z)$.

The numbers in the k^{th} column are the numbers of Dyck paths of length 2n with k hills, for $k = 0, 1, 2, \ldots$.

Let $F(z)$ denote the generating function of the number of Dyck paths with no hills .Then

$$\mathrm{F(z)} = 1 + 0.\mathrm{z} + 1.\mathrm{z}^2 + 2.\mathrm{z}^3 + 6.\mathrm{z}^4 + \cdots.$$

($F = \mathrm{F(z)}$: Generating function for the Fine numbers)

We can decompose Dyck paths by occurrence of hills, and we get

$$\mathrm{C} = F + FzF + FzFzF + \cdots = F + zF^2 + z^2F^3 + \cdots.$$

$$= \frac{F}{1-zF}$$

Where C & F are abbreviations of C(z) &F(z) , respectively, that is ,

C=GF (no hills) + GF(one hill) + GF(two hills) + …

$\{Notation\text{: GF} \coloneqq \text{Generating Function}\}$.

Obviously,

$$\mathcal{F} * \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \\ 14 \\ \vdots \end{bmatrix} = \mathrm{C},$$

since every Dyck path has some number of hills.

By the Fundamental Theorem of Riordan arrays, we have

$$(\mathrm{F}, \mathrm{zF}) * \frac{1}{1-\mathrm{z}} = \mathrm{F}\frac{1}{1-\mathrm{zF}} = \mathrm{C}(\mathrm{z}) .$$

We want to find the total number of hills, so

$$\mathcal{F} * \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ 2 & 2 & 0 & 1 & & \\ 6 & 4 & 3 & 0 & 1 & \\ & & \cdots & & & \ddots \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \\ 14 \\ \vdots \end{bmatrix} .$$

The entries of the last column vector are the coefficients of the generating function $\mathrm{H}(\mathrm{z})$ for the total number of hills. Hence

$$\begin{aligned} \mathrm{H}(\mathrm{z}) = (\mathrm{F}, \mathrm{zF}) * \frac{\mathrm{z}}{(1-\mathrm{z})^2} &= \mathrm{F}\frac{\mathrm{zF}}{(1-\mathrm{zF})^2} \\ &= \mathrm{z}\left(\frac{\mathrm{F}}{1-\mathrm{zF}}\right)^2 \\ &= \mathrm{zC}(\mathrm{z})^2 = \mathrm{C}(\mathrm{z}) - 1 \quad \text{, since} \quad \mathrm{C}(\mathrm{z}) = 1 + \mathrm{zC}(\mathrm{z})^2. \end{aligned}$$

Thus the total number of hills is given by the *Catalan* numbers except when $n = 0$.

As a result, the average number of hills is exactly 1 for $\mathrm{n} \geq 1$.

Example: Let $\mathrm{f}(\mathrm{z}) = \mathrm{z}$, $\mathrm{g}(\mathrm{z}) = \frac{\mathrm{e}^{-\mathrm{z}}}{1-\mathrm{z}}$, and its Riordan array is

$$\begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ 2 & 3 & 0 & 1 & & & \\ 9 & 8 & 6 & 0 & 1 & & \\ 44 & 45 & 20 & 10 & 0 & 1 & \\ & & \cdots & & & & \ddots \end{bmatrix} = \begin{bmatrix} 1 & & & & & & \\ 0 \times 1 & 1 & & & & & \\ 1 \times 1 & 0 \times 2 & 1 & & & & \\ 2 \times 1 & 1 \times 3 & 0 \times 3 & 1 & & & \\ 9 \times 1 & 2 \times 4 & 1 \times 6 & 0 \times 4 & 1 & & \\ 44 \times 1 & 9 \times 5 & 2 \times 10 & 1 \times 10 & 0 \times 5 & 1 & \\ & & \cdots & & & & \ddots \end{bmatrix}$$

$$= \left(\frac{\mathrm{e}^{-\mathrm{z}}}{1-\mathrm{z}}, \mathrm{z}\right) .$$

This $g(z)$ is the generating function of the number of derangements of n ordered objects. Derangements are permutations without fixed points (i.e., having no cycles of length one).

The number of derangements of length n, say $d(n)$, satisfies the recurrence relations:

$$d_n = (n-1)(d_{n-1} + d_{n-2})$$

With $d_2 = 1,\ d_1 = 0$ or with $d_1 = 0,\ d_0 = 1$.

This recurrence relation can be rewritten in a better form.

Bringing nd_{n-1} to the left gives:

$$d_n - nd_{n-1} = -d_{n-1} + (n-1)d_{n-2}.$$

Multiply by $(-1)^n$ gives:

$$(-1)^n(d_n - nd_{n-1}) = (-1)^{n-1}(d_{n-1} - (n-1)d_{n-2}).$$

Now iterate this formula, so that ultimately on the right when n is reduced to 3 we obtain:

$$(-1)^2(d_2 - (2)d_1) = 1$$

Thus:

$$(-1)^n(d_n - nd_{n-1}) = 1$$

Solving for d_n gives a recurrence relation which much better than the first one above:

$$d_n = nd_{n-1} + (-1)^n$$

With $d_1 = 0$, and $d_0 = 1$.

In derangements, the k^{th} column counts permutations with k fixed points.

Remark: The product of two Riordan arrays

Question: What happens when two Riordan matrices are multiplied?

According to matrix multiplication formula , the product of two Riordan arrays is a Riordan array; therefore, a natural question is: how do the A- and Z-sequences of the product depend on the analogous sequences of the two factors? In order to answer this question, let us consider two proper Riordan arrays

$$D_1 = (d_1(z), h_1(z))$$

and

$$D_2 = (d_2(z), h_2(z))$$

and their product $D_3 = D_1 * D_2 = (d_1(z)d_2\big(h_1(z)\big), h_2\big(h_1(z)\big))$

so that $d_3(z) = d_1(z)d_2\big(h_1(z)\big)$, and $h_3(z) = h_2(h_1(z))$

Example: Let us compute the product $Q = P * F$, where

$$\mathrm{P} = (\frac{1}{1-z}, \frac{z}{1-z})$$

$$\mathrm{F} = (\frac{1}{1-z-z^2}, \frac{1-\sqrt{1-4z}}{2}),$$

(Riordan array of Fibonacci numbers).

Solution: $Q = (d_Q(z), h_Q(z)) = (d_P(z), h_P(z)) * (d_F(z), h_F(z))$

$$= (d_P(z)(d_F(h_P(z))), h_F(h_P(z)))$$

$$= \left[\left(\frac{1}{1-z}, \frac{z}{1-z}\right) * \left(\frac{1}{1-z-z^2}, \frac{1-\sqrt{1-4z}}{2}\right)\right]$$

$$= \left(\frac{1}{1-z}\ \frac{(1-z)^2}{1-3z+z^2}\ , \frac{1-\sqrt{\frac{1-5z}{1-z}}}{2}\right)$$

$$= \left(\frac{1-z}{1-3z+z^2}\ , \frac{1}{2}(1-\sqrt{\frac{1-5z}{1-z}})\right)$$

$d_Q(z) = \frac{1-z}{1-3z+z^2}$, corresponding to Fibonacci numbers in odd positions; besides $h_Q(z) = \frac{1}{2}(1-\sqrt{\frac{1-5z}{1-z}})$.

The product is **not** commutative, so we also define

$G = F * P = \big(d_G(z), h_G(z)\big) = (d_F(z)d_P(h_F(z)), h_P(h_F(z)))$,

for which we find

$d_G(t) = d_F(z)d_P(h_F(z)) = \frac{1-\sqrt{1-4z}}{2z(1-z-z^2)} = 1 + 2z + 5z^2 + 12z^3 + 31z + \cdots$, and

$\big(h_G(t)\big) = h_P\big(h_F(t)\big) = \frac{1-2z-\sqrt{1-4z}}{2t} = z + 2z^2 + 5z^3 + 14z^4 + \cdots = C^2(z)$

is another version of the Catalan numbers,. The upper part of these two triangles is given below.

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	2	1					
2	5	4	1				
3	13	14	6	1			
4	34	48	27	8	1		
5	89	166	111	44	10	1	
6	233	587	443	210	65	12	1

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	2	1					
2	5	4	1				
3	12	14	6	1			
4	31	46	27	8	1		
5	85	150	108	44	10	1	
6	248	493	410	206	65	12	1

Table 2: The triangles $Q = P * F$ and $G = F * P$

Note: the sequence 1, 2 , 5, 13, 34, 89, 233, 610,…, a(n)= $3a(n-1) - a(n-2)$, with a(0)=a(1)=1, and its generating function $G(z) = \frac{1-2z}{1-3z+z^2}$.

It counts:

- Number of ordered trees with n+1 edges and height at most 3 (height=number of edges on a maximal path starting at the root);
- Number of directed column-convex polyominoes of area $n+1$;
- Number of 31-avoiding words of length n on alphabet {1,2,3} which do not end in 3; (e.g. n=3, we have 111,112,121,122,132,211,212,221,222,232,321,322 and 332).
- Number of permutations of $[n+1]$ avoiding 321 and 3412. E.g. a_3 = 13 because the permutations of [4] avoiding 321 and 3412 are: 1234, 2134, 1324, 1243, 3124, 2314, 2143, 1423, 1342, 4123, 3142, 2413, 2341.
- Number of 1324-avoiding circular permutations on [n+1];

On the other hand, from the second table above at the right, the sequence of the first column counts the number of n-node binary trees fixed by the corresponding automorphism (s), which is the convolution of Catalan and Fibonacci numbers, with generating function , $G(z) = \frac{\sqrt{(1-(1-4z)}}{2(1-z-z^2)}$.

The A- and Z-sequences of these triangles can be computed by means of the formulas in the previous section; however, we will now find them in a more direct way by using the corresponding sequences of P and F. In general, we know that $h(z) = zA(h(z))$ and consequently $\bar{h}(z) = z/A(z)$, where $\bar{h}(z)$ represents the compositional inverse of $h(z)$. We recall that the typical column of $\left(\mathrm{h(z)}, \mathrm{l(z)}\right)$ is $\mathrm{h(z)[l(z)]^k}$ and yields the matrix multiplication

$$\left(\mathrm{g(z)}, \mathrm{f(z)}\right) * \left(\mathrm{h(z)}, \mathrm{l(z)}\right) = (\mathrm{g(z)h}\left(\mathrm{f(z)}\right), \mathrm{l(f(z))}).$$

This is a multiplication with identity $I = (1, z)$ and group inverse

$$\left(\mathbf{g(z)}, \mathbf{f(z)}\right)^{-1} = \left(\frac{1}{\mathbf{g}(\bar{f}(\mathbf{z}))}, \bar{f}(\mathrm{z})\right)$$

where $\mathbf{f}(\bar{f}(\mathbf{z})) = \bar{f}\left(\mathrm{f}(\mathbf{z})\right) = \mathrm{z}$, $\bar{f}(\mathrm{z})$ is the compositional inverse of $\mathrm{f(z)}$. The existence of a unique compositional inverse in $\mathbb{C}[z]$, set of complex numbers is guaranteed by the formal power series $(\mathrm{z}) = \mathrm{z} + \mathrm{f}_2\mathrm{z}^2 + \mathrm{f}_3\mathrm{z}^3 + \cdots$for $\mathrm{f}_0 = 0$ and $\mathrm{f}_1 = 1$. Hence, the set of Riordan matrices, $\mathcal{R}$ forms a group under matrix multiplication, termed as **the Riordan group** $(\mathcal{R},*)$ which we denote by R.

3.3. The Riordan group

Definition: The Riordan group

$$R = \left\{ \begin{array}{c} \big(g(z), f(z)\big) : \big(g(z), f(z)\big) \text{ is a Rirdan array}, g(z) = g_o + g_1 z + g_2 z^2 + \cdots, \\ g_o = 1, \text{and} \\ f(z) = f_o + f_1 z + f_2 z^2 + \cdots, \\ \text{where } f_o = 0, f_o = 1 \end{array} \right\}, \text{ i.e.}$$

each member of R is a lower triangular matrix with 1's on the main diagonal.

The matrix multiplication $(*)$ in R is

$\big(g(z), f(z)\big) * \big(h(z), l(z)\big) = (g(z)h\big(f(z)\big), l(f(z)))$**……………(1)**

The identity is again $I = (1, z)$ as mentioned above.

The inverse of $\big(g(z), f(z)\big)$,$\big(g(z), f(z)\big)^{-1} = \left(\frac{1}{\mathbf{g}\left(\bar{f}(\mathbf{z})\right)}, \bar{f}(z)\right)$,

where $\bar{f}(z)$is the compositional inverse of f(z), i.e., $f\left(\bar{f}(z)\right) = \bar{f}\big(f(z)\big) = z$.

Example: From Shapiro et al[3] let us take the Pascal triangle ,

$$P = \begin{bmatrix} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ 3\ 1 \\ 1\ 4\ 6\ 4\ 1\ \ddots \\ \cdots \end{bmatrix} = \left(\frac{1}{1-z}, \frac{z}{1-z}\right) = (\,g(z), f(z)).$$

The generating function for the first column of P is

$$g(z) = \frac{1}{1-z} = 1 + 1.z + 1.z^2 + 1.z^3 + \cdots \text{ ----------------------------- } (2)$$

Let us recall that if $A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$, then

$A(z)\frac{1}{1-z} = a_0 + (a_0 + a_1)z + (a_0 + a_1 + a_2)z^2 + \cdots = \sum_{n=0}^{\infty}(\sum_{i=0}^{n} a_i)z^n$------ (3)

$c_k(z) = g(z)\, f(z)^k = \frac{1}{1-z}\,(\frac{z}{1-z})^k$, hence a Riordan array.

Since $P = (g(z), f(z))$ and $\big(g(z), f(z)\big)^{-1} = \left(\frac{1}{g(\bar{f}(z))}, \bar{f}(z)\right)$, where

$f\left(\bar{f}(z)\right) = \bar{f}\big(f(z)\big) = z$, implies that $\bar{f}(z) = \frac{z}{1+z}$.

Thus $P^{-1} = \left(\frac{1}{\mathbf{g}(\bar{f}(\mathbf{z}))}, \bar{f}(z)\right) = \left(\frac{1}{\mathbf{g}(\frac{z}{1+z})}, \frac{z}{1+z}\right) = \left(\frac{1}{1-\frac{z}{1+z}}, \frac{z}{1+z}\right) = \left(\frac{1}{1+z}, \frac{z}{1+z}\right)$.

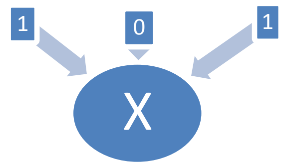
Moreover,

$$PP^{-1} = \left(\frac{1}{1-z}, \frac{z}{1-z}\right) * \left(\frac{1}{1+z}, \frac{z}{1+z}\right) = \left(\frac{1}{1+z}, \frac{z}{1+z}\right) * \left(\frac{1}{1-z}, \frac{z}{1-z}\right) = P^{-1}P = (1, z),$$

the identity element of R.

The next **example** is the other version of the Pascal triangle

$$B = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 2 & 0 & 1 & & & & \\ 0 & 3 & 0 & 1 & & & \\ 6 & 0 & 4 & 0 & 1 & & \\ 0 & 10 & 0 & 5 & 0 & 1 & \ddots \\ & & & \cdots & & & \end{bmatrix} \Longleftrightarrow$$



To compute the generating function $f(z)$, we have to form the rule of B in which each entry is the sum of the elements to the left and right in the row above, that is $b_{n+1,j} = b_{n,j-1} + b_{n,j+1}$ $, j \geq 1$.

From the first column, which are central binomial coefficients except alternating zeros and hence we use z^2 instead of z in the other generating function $g(z) = \frac{1}{\sqrt{1-4z^2}}$.

Thus

$$C_k(z) = zC_{k-1}(z) + zC_{k+1}(z).$$

That means

$$g(z)[f(z)]^k = zg(z)[f(z)]^{k-1} + zg(z)[f(z)]^{k+1},$$

implies that

$$f(z) = z + zf(z)^2 .$$

Solving for $f(z)$, we get

$$f(z) = \frac{1-\sqrt{1-4z^2}}{2z}.$$

From $$f(z) = z + z\, f(z)^2 = z(1 + f(z)^2),$$

it follows immediately using compositional inverse $\bar{f}(z)$ substitution on z that

$$z = \bar{f}(z)\{1 + z^2\}$$

So

$$\bar{f}(z) = \frac{z}{1+z^2}$$, where $\bar{f}(z)$ is the compositional inverse of $f(z)$,

and $\frac{1}{g(\bar{f}(z))} = \sqrt{1 - 4\left(\frac{z}{1+z^2}\right)^2} = \frac{1-z^2}{1+z^2} = 1 - 2z^2 + 2z^2 - 2z^4 - \cdots.$

Therefore,

$$B^{-1} = \langle \frac{1-z^2}{1+z^2}, \frac{z}{1+z^2} \rangle = \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ -2 & 0 & 1 & & & & & \\ 0 & -3 & 0 & 1 & & & & \\ 2 & 0 & -4 & 0 & 1 & & & \\ 0 & 5 & 0 & -5 & 0 & 1 & & \\ -2 & 0 & 9 & 0 & -6 & 0 & 1 & \ddots \\ & & & \cdots & & & & \end{bmatrix}.$$

Example: prove the following identity using the Riordan group

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof: In Pascal triangle, in the RHS the sum of the numbers in Row n equals 2^n, as shown below.

Row 0 : 1 = 1 = 2^0
Row 1 : 1 + 1 = 2 = 2^1
Row 2 : 1 + 2 + 1 = 4 = 2^2
Row 3 : 1 + 3 + 3 + 1 = 8 = 2^3
Row 4 : 1 + 4 + 6 + 4 + 1 = 16 = 2^4
Row 5 : 1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5
Row 6 : 1 + 6 + 15 + 20 + 15 + 6 + 1 = 64 = 2^6
Row 7 : 1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128 = 2^7
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1

On the other hand, the left hand side (LHS) can be shown as

$$\begin{bmatrix} 1 & & & & & & & \\ 1 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ 1 & 3 & 3 & 1 & & & & \\ 1 & 4 & 6 & 4 & 1 & & & \\ 1 & 5 & 10 & 10 & 5 & 1 & & \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & \ddots \\ & & & \cdots & & & & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ 32 \\ 64 \\ \vdots \end{bmatrix}$$

$$\left(\frac{1}{1-\mathrm{z}}, \frac{\mathrm{z}}{1-\mathrm{z}}\right) * \left(\frac{1}{1-\mathrm{z}}\right) = \left(\frac{1}{1-\mathrm{z}} \; \frac{1}{1-\frac{\mathrm{z}}{1-\mathrm{z}}}\right)$$

$$= \left(\frac{1}{1-2\mathrm{z}}\right) = g(z) \ ,$$

This $g(z) = \frac{1}{1-2\mathrm{z}}$ is the generating function of the column of the matrix product at the right hand side, that is the row sum of numbers in the Pascal triangle ■

3.3.1 Subgroups of the Riordan Group

The Riordan group, R is a mathematical structure which lies in the intersection of Algebra and Combinatorics. In essence, it is a special collection of infinite dimensional matrices whose entries are associated with the combinatorial sequences.

Example: Let D= $\{L \in R : L = (g(z), f(z) | f'(z) = g(z)\}$ a subset of the Riordan group. Show that D is a subgroup of R.

Proof: Since in $(1, t)$, t´=1 and $I = (1, t)\epsilon D$ implies $D \neq ø$.

Closure: For any $(g_1(z), f_1(z))$, $(g_2(z), f_2(z))$ ϵD,

we have $f_1{}'(z) = g_1(z)$, $f_2{}'(z) = g_2(z)$.

Because $\big(g_1(z), f_1(z)\big) * \big(g_2(z), f_2(z)\big) = (g_1(z) g_2\big(f_1(z)\big), f_2(f_1(z)))$

and $\big[f_2\big(f_1(z)\big)\big]' = f_2'\big(f_1(z)\big) f_1'(z) = g_2\big(f_1(z)\big) g_1(z)$,

so $(g_1(z), f_1(z))$ $(g_2(z), f_2(z))$ ϵD.Hence D is closed under matrix multiplication.

Associativity: Because D is a subset of a group, it is obvious.

Identity: Since $z' = 1$, $I = (1, z)\epsilon D$.

Inverse: For any$\big(g(z), f(z)\big)\epsilon D$, the inverse is $\left(\frac{1}{g\left(\bar{f}(z)\right)}, \bar{f}(z)\right)$.

But $f(\bar{f}(z) = z$ yields $z' = \frac{d}{dz}(f\left(\bar{f}(z)\right) = f'\left(\bar{f}(z)\right)\bar{f}'(z) = 1.$

Thus $\bar{f}'(z) = \frac{1}{f`(\bar{f}(z))} = \frac{1}{g(\bar{f}(z))}$ and $(g(z), f(z))^{-1} \epsilon D$ ■

There are several subgroups of the Riordan group ***R*** which are of particular combinatorial importance :

I) The Appell subgroup,

$$\mathcal{M} = \{A \in \mathcal{R}: A = (g(z), z), for\ some\ g\}$$

We have a Riordan matrix collection $\mathcal{M}$ under matrix multiplication$(*)$.

Claim: $(\mathcal{M},*)$ is a subgroup of $R = (\mathcal{R},*)$, where $\mathcal{R}$ is set of Riordan matrices.

Proof: For any $A = (g(z), z)$with $g(z) \neq 0$ (i.e. 0 column matrix) is in $\mathcal{M}$, we have $\mathcal{M} \neq \emptyset$.

1) Closure**:** Let $(g(z), z), (d(z), z)$ be two elements in $\mathcal{M}$. We want to show that $(g(z), z) * (d(z), z)$ is in $\mathcal{M}$ for some generating functions $g(z)$ and $d(z)$.

Now, by using the matrix product we have

$$(g(z), z) * (d(z), z) = (g(z)d(z), \mathrm{z}) = (\mathrm{a}(\mathrm{z}), \mathrm{z}),$$

$(\mathrm{a}(\mathrm{z}), \mathrm{z})$ is in $\mathcal{M}$ for some $\mathrm{a}(\mathrm{z})$ since the product of two Riordan matrices is again Riordan as mentioned earlier. Thus closurity holds.

2) Associativity: obviously holds since matrix multiplication is associative.

3) Identity: we want to show that $(1, \mathrm{z})$ is an identity element in $\mathcal{M}$. i.e.

For any $(g(z), z) \in \mathcal{M}$, $(g(z), z) * (1, \mathrm{z}) = (1, \mathrm{z}) * (g(z), z) = (g(z), z)$.

Here**,** $(g(z), z) * (1, \mathrm{z}) = (g(z). 1, \mathrm{z}) = (g(z), z)$

and

$(1, \mathrm{z}) * (g(z), z) = (1. g(z), \mathrm{z}) = (g(z), z)$.Thus $(1, \mathrm{z})$ is the identity element in $\mathcal{M}$.

4) Inverse**:** We recall that for any $(g(z), \mathrm{f}(z))$ in ***R*, its inverse** can be defined as

$$(\mathrm{g}(\mathrm{z}), \mathrm{f}(\mathrm{z}))^{-1} = \left(\frac{1}{\mathbf{g}(\bar{\mathrm{f}}(\mathbf{z}))}, \bar{\mathrm{f}}(\mathrm{z})\right).$$

Similarly, for $(d(z), z)$ in $\mathcal{M}$, $(d(z), z)^{-1} = \left(\frac{1}{\mathrm{d}(\mathrm{z})}, \mathrm{z}\right)$ is also in $\mathcal{M}$ and

$(d(z), z) * \left(\frac{1}{\mathrm{d}(\mathrm{z})}, \mathrm{z}\right) = \left(\frac{1}{\mathrm{d}(\mathrm{z})}, \mathrm{z}\right) * \left(\frac{1}{\mathrm{d}(\mathrm{z})}, \mathrm{z}\right) = (1, \mathrm{z})$.

Hence every element in $\mathcal{M}$ is invertible for $(d(z), z)$ being arbitrary ■

Therefore, $(\mathcal{M},*)$ is an Appell subgroup of ***R*.**

Example. Let

$$A = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 1 & 1 & & & \\ 4 & 2 & 1 & 1 & & \\ 8 & 4 & 2 & 1 & 1 & \\ 16 & 8 & 4 & 2 & 1 & 1 \\ & & \cdots & & & \ddots \end{bmatrix}.$$

Claim: A is an Appell subgroup of the Riordan group R.

Proof: Let $\mathcal{R}$ be set of Riordan arrays with matrix multiplication $(*)$ satisfying

R= { $(\mathcal{R},*)$}, is a Riordan group.

We want to show that

i)A is subset of a Riordan array $\mathcal{R}$;

$ii)$\{(A,$*$)\}.is a group, i,e. Appell subgroup.

i) Now, let $\mathrm{A} = (\mathrm{a}_{\mathrm{n,k}})_{\mathrm{n,k}\geq 0} = (g(z), f(\mathrm{z}))$ be an infinite matrix with

$\mathrm{a}_{\mathrm{n,k}} = [\mathrm{z}^{\mathrm{n}}]\mathrm{g}(\mathrm{z})\mathrm{f}(\mathrm{z})^{\mathrm{k}}, n, k \geq 0,$

Where g(z) and f(z) are two generating functions of the k^{th} column of A in which $\mathrm{a}_{0,0} \neq 0$.

From the matrix columns (the first) given above,

$$\begin{aligned} \mathrm{g}(\mathrm{z}) &= 1 + 1.\mathrm{z} + 2.\mathrm{z}^2 + 4.\mathrm{z}^3 + 8.\mathrm{z}^4 + 16.\mathrm{z}^5 + \cdots \\ &= 1 + \mathrm{z}(1 + \mathrm{z} + 2\mathrm{z} + 4\mathrm{z}^2 + 8\mathrm{z}^3 + 16\mathrm{z}^4 + \cdots) \\ &= 1 + \frac{z}{1-2z} \\ &= \frac{1-z}{1-2z}. \end{aligned}$$

And to find $f = f(\mathrm{z})$ by comparing g(z) and g(z)f(z) from the second column ($k = 1$),

$$\begin{aligned} \mathrm{g}(\mathrm{z})f(\mathrm{z}) &= 0 + 1.\mathrm{z} + 2.\mathrm{z}^2 + 4.\mathrm{z}^3 + 8.\mathrm{z}^4 + 16.\mathrm{z}^5 + \cdots \\ &= \mathrm{z}\left(\frac{1-z}{1-2z}\right) \end{aligned}$$

Implies that $f = f(\mathrm{z}) = \mathrm{z}$.Hence $\mathrm{A} = \big(g(z), f(\mathrm{z})\big) = (\frac{1-z}{1-2z}, \mathrm{z})$ is a subset of the Riordan array $\mathcal{R}$.

ii)Then we have left to show that { (A,$*$)} is a group.

Associativity: holds because A is a subset of R.

Identity: $I = (1, \mathrm{z})$ is the identity element in A with $g(z) = 1$ since for any $\left(g(z), f(\mathrm{z})\right)$ in A,

$\left(g(z), f(\mathrm{z})\right) * (1, \mathrm{z}) = \left(\frac{1-z}{1-2z}, \mathrm{z}\right)(1, \mathrm{z}) = \left(\frac{1-z}{1-2z}. 1, z\right) = \left(g(z), f(\mathrm{z})\right)$and

$(1, \mathrm{z}) * \left(g(z), f(\mathrm{z})\right) = (1, \mathrm{z}) * \left(\frac{1-z}{1-2z}, \mathrm{z}\right) = \left(1. \frac{1-z}{1-2z}, \mathrm{z}\right) = \left(g(z), f(\mathrm{z})\right).$

Inverse: For any $\left(g(z), f(\mathrm{z})\right) \in$ A, the inverse is $\left(\frac{1}{\left(g(z), \bar{f}(\mathrm{z})\right)}, \bar{f}(\mathrm{z})\right)$. But $f\left(\bar{f}(z)\right) = z$ yields $\bar{f}(z) = z$. Thus $\left(\frac{1}{\left(g(\bar{f}(\mathrm{z})\right)}, \bar{f}(\mathrm{z})\right) = \left(\frac{1}{g(z)}, z\right) = (h(z), z) \in$ A.

Then we get what desired, A is an Appell subgroup ■

II) The Associated subgroup

$$\mathcal{L} = \{L \in \mathcal{R}: L = (1, f(z)), for\ some\ f\}.$$

Proof: To show that $(\mathcal{L},*)$ is the Associated subgroup of R, for $\mathcal{L} \neq \emptyset$.

1)Closure: Let $(a(z), b(z)), (d(z), h(z)) \in$ A such that $a(z) = d(z) = 1$. Then

$\left(1, b(z)\right) * \left(1, h(z)\right) = \left(1, h\big(b(z)\big)\right)$, is in $\mathcal{L}$ in which $h\big(b(z)\big) := f(z) \in \mathcal{R}$.

Thus $(*)$ is closed in $\mathcal{L}$.

2)Associativity: with the same argument as in (I) above.

3)Identity: For any $(1, f(z)) \in \mathcal{L}$ for some f, we want to show that $(1, z)$ is an identity element of $\mathcal{L}$ under $(*)$. First, in case $f(z) = z$, $(1, z)$ is in $\mathcal{L}$.

Next, $\left(1, f(z)\right) * (1, z) = (1, f(z))$ and $(1, z) * \left(1, f(z)\right) = \left(1, f(z)\right)$.

Thus $(1, z)$ is the identity element of $\mathcal{L}$ under $(*)$.

4) Inverse: For any $(1, b(z)) \in \mathcal{L}$, $(1, b(z))^{-1} = (1, \bar{b}(z))$,

where $b\left(\bar{b}(z)\right) = \bar{b}\big(b(z)\big) = z$.

Therefore, $(\mathcal{L},*)$ is the Associated subgroup of R ■

Example: Let

$$L = \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 0 & 2 & 1 & 1 & & \\ 0 & 5 & 5 & 3 & 1 & \\ 0 & 14 & 14 & 9 & 4 & 1 \\ & & \cdots & & & \ddots \end{bmatrix}.$$

In this case, $L = (1, f(z))$ where $f(z) = zC(z)$.

III) The Bell subgroup

$$\mathcal{B} = \{B \in \mathcal{R}: B = \big(g(z), zg(z)\big), for\ some\ g\}$$

Claim: $(\mathcal{B},*)$ is a subgroup of R, the Riordan group.

Proof: 1)Closure: Let $\big(g(z), zg(z)\big), \big(d(z), zd(z)\big)$ *be two elements of a non − empty set* $\mathcal{B}$. We want to show that $\big(g(z), zg(z)\big) * \big(d(z), zd(z)\big)$is in $\mathcal{B}$.

Then by matrix product

$\big(g(z), zg(z)\big) * \big(d(z), zd(z)\big) = (g(z)\Big(d\big(zg(z)\big)\Big), zg(z)d(zg(z)))$, which has a form $(a(z), za(z)$ for some generating function $a(z) = g(z)\Big(d\big(zg(z)\big)\Big)$.

Thus the result is in $\mathcal{B}$.

2) associativity : holds obviously with similar case for (I)and (II) above.

3)Identity: For $g(z) = 1$ in $\big(g(z), zg(z)\big)$ of $\mathcal{B}$, $(1, z) \in \mathcal{B}$.

We get the matrix product

$$(a(z), za(z)) * (1, z) = (a(z)\,1, za(z) = (a(z), za(z)$$

and

$(1, z) * \big(a(z), za(z)\big) = \big(1\,a(z), za(z)\big) = \big(a(z), za(z)\big)$.

Thus $(1, z)$ is the identity element of $\mathcal{B}$ under $(*)$.

4) Inverse: For any $\big(g(z), zg(z)\big) \in \mathcal{B}$,

$\big(g(z), zg(z)\big)^{-1} = \left(\frac{1}{g(z\bar{g}(z))}, z\bar{g}(z)\right) = (\bar{g}(z), z\bar{g}(z)) \in \mathcal{B}$.

Therefore, $(\mathcal{B},*)$ is a subgroup of R ∎

Example: The Pascal triangle, $P = \left(\frac{1}{1-z}, \frac{z}{1-z}\right)$, i.e.

$$P = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ & & \cdots & & & \ddots \end{bmatrix}$$

is an example.

Another important subgroup is hitting time subgroup,

$$\mathcal{H} = \left\{(d(z), h(z)) \middle| d(z) = \frac{zh'(z)}{h(z)}\right\}.$$

Example: The Pascal triangle P belongs to the hitting time subgroup.

Proof: Here we have from the preceding Example that

$$h(z) = \frac{z}{1-z} \quad ,$$

and compute $d(z)$ in terms of $h(z)$.

So $d(z) = \frac{1}{1-z} = \left(\frac{1}{1-z}\right)\left(\frac{1-z}{1-z}\right)\frac{z}{z} = \frac{z}{(1-z)^2}\frac{1-z}{z}$, and $\frac{zh'(z)}{h(z)} = z\frac{1}{(1-z)^2}\frac{1}{\frac{1}{1-z}} = \frac{1}{1-z}$;

That is $d(z) = \frac{zh'(z)}{h(z)}$, satisfies the definition of a hitting time subgroup.

Hence, the Pascal triangle is an example of hitting time subgroup.

Claim: We can decompose R as the semi direct product of the Appell subgroup and the Associated (Lagrange) subgroup.

Proof: $(g, f) = (g, z)(1, f)$

$= \left(\frac{gz}{f}, z\right)\left(\frac{z}{f}, f\right)$, since in the Appell subgroup $f = z$ while $g = 1 = \frac{z}{f}$ in the Associate subgroup.

$= (\frac{gz}{f}\frac{f}{z}, f)$ ■

Example:

$$R = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 2 & 9 & 6 & 1 & & & \\ 9 & 44 & 42 & 12 & 1 & & \\ 44 & 265 & 320 & 130 & 20 & 1 & \\ & & \cdots & & & & \ddots \end{bmatrix} = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ 2 & 3 & 0 & 1 & & & \\ 9 & 8 & 6 & 0 & 1 & & \\ 44 & 45 & 20 & 10 & 0 & 1 & \\ & & \cdots & & & & \ddots \end{bmatrix} \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 2 & 1 & & & & \\ 0 & 6 & 6 & 1 & & & \\ 0 & 24 & 36 & 12 & 1 & & \\ 0 & 120 & 240 & 120 & 20 & 1 & \\ & & \cdots & & & & \ddots \end{bmatrix}$$

(In the Appell subgroup) (In the Associate subgroup)

3.3.2 Generating Functions and the Riordan Arrays

A generating function counts the number of objects using an additional parameter n which classifies the instances of the problem according to their ‘complexity’.

Definition: Let $(a_n)_{n\geq 0}$ be a sequence of numbers from the field $\mathbb{R}$. The elements a_n are usually nonnegative integers when we count something, but for generality we (must) start from more general numbers. The generating function of this sequence is the function $f : \mathbb{N} \longrightarrow \mathbb{R}$ for which $f(n) = a_n$. we redefine this as follows:

the generating function of $(a_n)_{n\geq 0}$ is the formal power series

$$G(z) = \sum_{n=0}^{\infty} a_n z^n$$

where z is a letter that will be called a variable. Such a power series is called ‘formal’ because we are not interested in its convergence, or its sum for any specific values of z. Indeed, $G(z)$ – despite its name – is not a function at all; it is just a way of writing the sequence $(a_n)_{n\geq 0}$.

Example: Let $a_0 = 1$ an $a_1 = 1$. The sequence $(a_n)_{n\geq 0}$ defined by the recurrence relation $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$ gives the Fibonacci numbers: $1, 1, 2, 3, 5, 8, 13, ...$ Their generating function is thus $F(z) = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + \ldots$, where the next values are easy to compute, but it is more difficult to give a formula for the general value a_n.

Remark: Riordan Array transformations are among the most important transformations related to Riordan Arrays:

Rule (A): The general summation rule of Riordan Arrays (d(z), h(z)):

$$\sum_k d_{n,k} f_k = [z^n] d(z) f(h(z))$$

Proof**:**
$\sum_k d_{n,k} f_k = \sum_k [z^n] d(z) h(z)^k f_k = [z^n] d(z) \sum_k f_k\, (h(z))^k = [z^n] d(z) f(h(z))$.

Rule (B):

$$\sum_k \binom{n+ak}{m+bk} t^{m+bk} f_k = [z^m](1+tz)^n f(z^{-b}(1+tz)^a) \quad ,b<0.$$

Proof:

$\binom{\mathrm{n+ak}}{\mathrm{m+bk}} \mathrm{t^{m+bk}} = [z^{m+bk}](1+tz)^{n+ak} = [z^m](1+tz)^n\,(z^{-b}(1+tz)^a)^k.$

Therefore we have a Riordan Array and we apply the general rule (A):

$\sum_k \binom{\mathrm{n}+ak}{m+bk} t^{m+bk} f_k = [z^m](1+tz)^n \sum_k f_k\,(z^{-b}(1+tz)^a)^k$

$= [z^m](1+tz)^n f(z^{-b}(1+tz)^a) \quad ,b<0$ ■

This rule can also be applied when b = 0, provided the generating function f(z) is a polynomial; in that case, the rule will be denoted by (B*).

Example: Prove the identity

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{t}{k}\binom{y}{k}^{-1} = \binom{y-t}{n}\binom{y}{n}^{-1}.$$

Proof: To show the identity, we simplified

$\binom{n}{k}\binom{y}{k}^{-1} = \frac{n!(y-k)!}{(n-k)!y!} = \binom{y}{n}^{-1}\binom{y-k}{n-k}.$

So

$$\sum\nolimits_{k=0}^{n}(-1)^k \binom{n}{k}\binom{t}{k}\binom{y}{k}^{-1} = \binom{y}{n}^{-1}\sum\nolimits_{k=0}^{n}\binom{y-k}{n-k}\binom{t}{k}(-1)^k$$

$= \binom{y}{n}^{-1}[z^{\mathrm{n}}](1+z)^y\left[(1-u)^t|_{u=\frac{z}{1+z}}\right],$ by applying Rule (B).

$= \binom{y}{n}^{-1}[z^{\mathrm{n}}](1+z)^{y-t} = \binom{y-t}{n}\binom{y}{n}^{-1}$ ■

3.3.3 Exponential Generating Functions

The Riordan group is a useful tool for many combinatorial enumeration problems. The two main enumeration techniques involve ordinary and exponential generating functions.

Until recently, the theory in the ordinary case was a bit richer but recently Emeric Deutsch has introduced a pair of first order differential equations

$$\left.\begin{array}{l} R(f(z)) = f'(z) \\ C\big(f(z)\big) = \frac{g(z)}{g'(z)} \end{array}\right\} \text{E.Deutch's differential equations}$$

Where , $R(z) = r_0 \ + r_1 y + r_2 y^2 + \cdots$, and $C(z) = c_0 \ + c_1 y + c_2 y^2 + \cdots$,

for the exponential case which parallels some of the developments in the ordinary case as in [7].

The exponential Riordan group is a set of infinite lower-triangular integer matrices where each matrix is defined by a pair of generating functions $g(z) = g_0 + g_1 z + g_2 z^2 + \cdots$, and $f(z) = f_1 z + f_2 z^2 + \cdots$, where $g_0 \neq 0$ and $f_1 \neq 0$. In what follows, we shall assume $g_0 = f_1 = 1$.

The associated matrix is the matrix whose k^{th} column has exponential generating function $\frac{g(x)f(x)^k}{k!}$ (the first column being indexed by 0). The matrix corresponding to the pair f, g is denoted by (g, f). The group law is given by

$$(g, f) * (h, l) = (g(hof), lof).$$

The identity for this law is $I = (1, z)$ and the inverse of (g, f) is $(g, f)^{-1} = (\frac{1}{go\bar{f}}, \bar{f})$ where $\bar{f}$ is the compositional inverse of f. We use the notation eR to denote this group. If M is the matrix (g, f), and $U = (u_n)_{n\geq 0}$ is an integer sequence with exponential generating function $U(z)$, then the sequence Mu has exponential generating function $g(\text{z})U(f(\text{z}))$.

Thus the row sums of the array (g, f) have exponential generating function given by $g(z)e^{f(z)}$since the sequence $1, 1, 1, \ldots$ has exponential generating function e^z.

Example: Consider the binomial matrix B is given by $\text{B} = (e^z, \text{z})$.

By the definition above, the exponential generating function of its row sums is given by $e^z * e^z = e^{z+z} = e^{2z}$ as expected since $e^{2z}$is the exponential generating function of 2^{n}.

Conclusion: The Riordan arrays are special collection of infinite dimensional matrices, whose entries are associated with the combinatorial sequences.

Proper Riordan arrays are more likely related to the Riordan group under matrix multiplication.

Generating functions provide a surprisingly strong tool for counting combinatorial objects (such as Dyck paths, hills, etc) by Riordan arrays using the Catalan numbers with generating function $C(z) = \frac{1-\sqrt{1-4z}}{2z}$., the Fine numbers with generating function $F(z) = \frac{1}{z}\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}$. and others few auxiliary results are:

a) $C_k(z) = g(z)\left(f(z)\right)^k$, the k^{th} column of the Riordan array $\left(g(z), f(z)\right)$.

b) $\left(g(z), f(z)\right)A(z) = g(z)A\left(f(z)\right) = B(z)$, the Fundamental Theorem of Riordan Arrays, FTRA.

c) $\left(g(z), f(z)\right) * \left(h(z), l(z)\right) = (g(z)h\left(f(z)\right), l(f(z)))$, the Riordan matrix Product.

d) $\left(\mathbf{g}(\mathbf{z}), \mathbf{f}(\mathbf{z})\right)^{-1} = \left(\frac{1}{\mathbf{g}(\bar{f}(\mathbf{z}))}, \bar{f}(z)\right)$, the inverse of the Riordan matrix $\left(g(z), f(z)\right)$.

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