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College of Natural and Computational Sciences

Department of Mathematics

Harmonic Function and Partial Differential Equations

By:
Wubshet Debebe

Advisor:
Dr.Tesfa Biset

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Approved by the Board Eximiners:

Tesfa Biset(Dr)Advisor	*Signiture* :________________
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"This is to certify that this thesis is written by Wubshet Debebe in the Department of Mathematics ,Addis Ababa University ,under my supervision I here by confirm that the thesis can be submitted for evaluation by examiners and eventual defence.

$Tesfa \quad Biset(Dr)$

$Signiture$:____________________________

$Date$:____________________________ "

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Abstract

This paper is concerned with harmonic function and partial differential equation. Harmonic functions are regarded as solutions of Laplace equation which have a number of properties that are essential in solving partial differential equation.Poisson integral is applied to obtain useful inequalities for positive harmonic functions.

Contents

0.1 Notation

$\mathbb{R}^n$ -real Euclidean space in n dimension

Ω - non-empty open subsets of $\mathbb{R}^n$

$\partial\Omega$ -the boundary of Ω

C(Ω) -the space of continuous function on Ω

$C^1(\Omega)$ -the space of continuously differentiable function on Ω

$C^2(\Omega)$ -the space of twice continuously differentiable function on Ω

$C(\bar{\Omega})$ -the space of continuous function on $\bar{\Omega}$

$B(x,r)$ -ball of radius r about x in $\mathbb{R}^n$

$\partial(B(x,r))$ -the boundary of ball of radius r about x in $\mathbb{R}^n$

$\overline{B(x,r)}$ -A closed ball with radius r centered at x

w_n -volume of unit ball in $\mathbb{R}^n$

nw_n -surface area of unit ball in $\mathbb{R}^n$

0.2 Introduction

Harmonic functions which are solution of Laplace's equation are very special and important class of functions not only in PDE but also in complex analysis, electromagnetic fluids and e.t.c.

The word "harmonic" is commonly used to describe a quality of sound. Harmonic functions derive their name from a roundabout connection.Physisists label the movement of a point on a vibrating string "harmonic motion". such motion may be described using sine and cosine functions are some times called harmonics. In classical Fourier analysis, functions on the unit circle are expand in terms of sines and cosines.

Analogous expansions exist on the sphere in $\mathbb{R}^n$,n>2 in terms of homogeneous harmonic polynomials.Because these polynomials play the same role on the sphere as the harmonics sine and cosine play on the circle. The term "spherical harmonics" was apparently first used in this context by William Thomson(Lord Kelvin) and Peter Tait. By the early 1900's the word "harmonic" was applied not only to homogeneous polynomials with zero Laplacian but also to any solution of Laplace's equation. Harmonic function plays a crucial role in many areas of mathematics,physics and engineering. For different applications it is necessary to extend harmonic function to $\mathbb{R}^n$, where n denotes a fixed positive integer greater than 1.

Harmonic functions, solutions of Laplace equation $\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} ... + \frac{\partial^2 u}{\partial x_n^2} = 0$ or Δu=0 have mean value, converse of mean value, maximum principle properties that are useful to the study of PDEs.

This thesis is organized in to three chapters. The first chapter consists of terminologies, definitions and basic ideas of harmonic functions. The second chapter consists of basic properties of harmonic function,such as : mean value property, converse of mean value property and maximum principle.The third chapter consists of harmonic function and Poisson integral, positive harmonic function and some applications of harmonic functions.

Chapter 1

Preliminaries

1.1 Definition and Terminologies

In this section we will see basic definitions, terminologies and ideas that are important for our study in this paper.

Definition 1. *A partial differential equation (PDE) is an equation involving derivatives of an unknown function* $u:\Omega \to \mathbb{R}$*,where* Ω *is an open subset of* $\mathbb{R}^n$*,*$n\geq 2$ *(or, more generally , of differentiable manifold of dimension* $n\geqslant 2$*)*
A partial derivative of u with respect to x_i *is denoted by*
$u_{x_i}=\frac{\partial u}{\partial x_i}$ *for* $i = 1, 2, 3, ..., n$
In case $n = 2$
we write x, y *in place of* x_1, x_2.
Other wise x is the vector $(x_1, x_2, x_3, ..., x_n)$.

Definition 2. *Let* $u(x) = u(x_1, x_2, ..., x_n)$ *be a twice continuously differentiable function on* Ω *the with Laplace oprator* Δ *is defined as*
$\Delta= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + ... + \frac{\partial^2}{\partial x_n^2}$.
The partial differential equation
$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} ... + \frac{\partial^2 u}{\partial x_n^2} = 0$ *is called the* ***Laplace equation****.*
The Laplace equation is also written as $\Delta u=0$ *or* $\bigtriangledown^2 u = 0$.
If the equation $\Delta u=0$*,where* $\Delta u=\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ *for i=1,2,3,...,n is satisfied for each point of the domain* Ω*, we say that u is a* ***Harmonic function*** *on* Ω.

Example 1. ***i).*** *Let* $\Omega = (x,y) \in \mathbb{R}^2, 0 < x < 1, y \in \mathbb{R}$ *and consider* $n \in \mathbb{N}$ *then,* $u(x,y) = ne^{-nx}\sin ny$ *is a harmonic function. We want to show that* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= -n^2 e^{-nx}\sin ny \\
\frac{\partial u}{\partial y} &= n^2 e^{-nx}\cos ny \\
\frac{\partial^2 u}{\partial x^2} &= n^3 e^{-nx}\sin ny \\
\frac{\partial^2 u}{\partial y^2} &= -n^3 e^{-nx}\sin ny \\
\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= n^3 e^{-nx}\sin ny + (-n^3 e^{-nx}\sin ny) \\
&= 0 \\
\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0
\end{aligned}
$$

$\therefore$ *$u(x,y)$ is harmonic function.*

ii). *$u(z) = ln(|z|^2)$ is harmonic on Ω =$C\backslash\{0\}$.*

Indeed,$ln(z) = ln\sqrt{x^2+y^2}$ and $u_{xx}(z) + u_{yy}(z) = 0$ if $z \neq 0$

Definition 3. *For two independent variables x and y, the inhomogeneous problem where F is a function of the independent variables x and y only, is called the* ***Poisson equation****.*

Definition 4. *(**Analytic function**) Let $f(z) = u(z) + iv(z) = u(x,y) + iv(x,y)$ where u and v are real valued functions,An analytic function is a complex function that is expressible as a power series in z. This means that f is analytic at z ,it has a series of the form $\sum a_n z^n$.*

Thus

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

(a_n is complex constant) Formal differentiation of this series shows that u_x=v_y and u_y=$-v_x$.These are Cauchy-Riemann equation. If we differentiate them,assuming twice differentiability we find that

u_{xx}= v_{yx}= v_{xy}=u_{yy} so that Δu=0, similarly Δv=0 thus real and imaginary parts of an analytic functions are harmonic.

Remark. *i) Due to the linearity of Δ, sum of any finite number of harmonic function is harmonic and scalar multiple of a harmonic function is harmonic.*
*ii) In two dimensions,one can associate with a harmonic function u(x,y), a conjugate harmonic function v(x,y) which satisfy the first order system of PDE called **Cauchy-Riemann equation** .u_x=v_y and u_y=$-v_x$*

Definition 5. *A complex valued function f is said to be **holomorphic** on a domain Ω, if it is differentiable in a nbhd of any point in Ω and satisfy Cauchy-Riemann equation.*

Theorem 1. *Let u(x,y) be harmonic in some nbhd of the point (x_o, y_o),then there exist a conjugate harmonic function v(x,y) defined in the nbhd and $f(z) = u(x, y)+iv(x, y)$ is analytic function.*

Proof. The harmonic function u(x,y) and its conjugate harmonic function v(x,y) will satisfy Cauchy-Riemann equation u_x=v_y and u_y=$-v_x$.
From $v_y(x, y)$ =$u_x(x, y)$
Integrating both sides with respect to y

$$\int v_y(x, y)dy = \int u_x(x, y)dy \tag{1.1}$$

where $\varphi(x)$ is a function of x only

$$v(x, y) = \int u_x(x, y)dy + \varphi(x)$$

differentiating with respect to x and replace v_x by $-u_y$ to obtain

$$-u_y(x, y) = \frac{d}{dx}(\int u_x(x, y)dy) + \varphi'(x) \tag{1.2}$$

since u is harmonic, all terms except those involving x(only)in (1.2) cancel out and a formula for $\varphi'(x)$ will purely be a function of x.Now the integration of $\varphi'(x)$ gives $\varphi(x)$ and obtain $v(x, y)$.
Thus $f(z) = u + iv$ is an analytic function. □

1.2 Boundary value problems associated to Laplace equation

A boundary value problem is a problem of finding a function which satisfies a given partial differential equation and particular boundary condition. There are at least three common boundary value problems associated with Laplace and Poisson equations.For simplicity, we describe them only for Laplace equation and the corresponding notations can be easily extended to Poisson equation.

1.2.1 Dirichlet Problem

Let Ω be bounded domain in $\mathbb{R}^2$ with piecewise smooth boundary and f be a continuous function on the closure of Ω. Finding a solution u_{xx} $+u_{yy} = 0$ in the domain Ω such that
i) u is continuous on closure of Ω and
ii) $u|\partial\Omega =$ f, is Dirichlet Problem If Ω is the unit disk,then the corresponding Dirichlet Problem is called interior Dirichlet Problem. If Ω is a complement of the closed unit disk, then the corresponding Dirichlet Problem is called exterior Dirichlet Problem.

1.2.2 Neumann problem (second BVP)

To find a solution u_{xx} $+u_{yy} = 0$ in the domain Ω such that
i) u, u_x, u_y are continuous on closure of Ω and
ii) at every point of $\partial\Omega$, the directional derivative of u in the direction of the normal(denoted by $\frac{\partial u}{\partial n} = \nabla u.\overrightarrow{n} = \partial_n u$) satisfies $\frac{\partial u}{\partial n} =$ f
where $\overrightarrow{n}$ denotes the outward unit normal vector to the boundary $\partial\Omega$.

1.2.3 Robin problem (Third BVP)

To find a solution u_{xx} $+u_{yy} = 0$ in the domain Ω such that
i) u, u_x, u_y are continuous on closure of Ω and
ii) at every point of $\partial\Omega$, $u + \alpha\frac{\partial u}{\partial n}$ = f where α is a given constant.

Remark. *There are other kinds of boundary problems for example on a part of the* *boundary $\partial\Omega$ one of the three BVP discribed above is imposed and on the remaining part, another one of the above three. we do not consider such BVP in this paper.*

Definition 6. *Suppose Ω is an open subset. A function $u \in C^2(\Omega)$ is sub-harmonic if $\triangle u \geq 0$ in Ω and super harmonic if $\triangle u \leq 0$ in Ω.*

Example 2. *The function $u(x) = |x|^4$ is sub-harmonic in $\mathbb{R}^n$*
since $\triangle u = 4(n+2)|x|^2 \geq 0$

Chapter 2

Properties of harmonic function

2.1 Mean value property

In this section we prove mean value property which all harmonic functions satisfy. First we give some definitions: for a function u defined $B(x,r)$,the average of u on $B(x,r)$ is given by

$$\fint_{B(x,r)} u(y)dy = \frac{1}{w_n r^n} \int_{B(x,r)} u(y)dy \tag{2.1}$$

For a function u defined on $\partial B(x,r)$,the average of u on $\partial B(x,r)$ is given by

$$\fint_{\partial B(x,r)} u(y)ds(y) = \frac{1}{n w_n r^{n-1}} \int_{\partial B(x,r)} u(y)ds(y) \tag{2.2}$$

where $w_n = \frac{2(\pi^{\frac{n}{2}})}{\Gamma(\frac{n}{2}+1)} = \frac{(\pi^{\frac{n}{2}})}{n\Gamma(\frac{n}{2})}$(volume of a ball $B(0,1)$ in $\mathbb{R}^n$)

2.1.1 Surface area and volume of disk on $\mathbb{R}^n$

To find the surface area and volume of a disk first let us define Gamma function and Properties of gamma function.

Definition 7. *The Gamma function* $\Gamma : (0,\infty) \to \mathbb{R}$ *is defined as* $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$,$\forall x \in (0,\infty)$

2.1.2 Properties of gamma function

i) $\Gamma(t+1) = t\Gamma(t)$

ii) $\Gamma(n) = (n-1)!$

iii) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Proof. i) From the right hand side

$$t\Gamma(t) = t(\int_0^\infty x^{t-1}e^{-x}dx)$$

$$\text{let} \quad u = e^{-x}dv = x^{t-1}$$

$$\text{let} \quad t > 0$$

$$\begin{aligned} t\Gamma(t) &= t(e^{-x}\frac{x^t}{t}/_0^\infty - \int_0^\infty \frac{x^t}{t}(-1)e^{-x}dx \\ &= t\int_0^\infty \frac{x^t}{t}e^{-x}dx \\ &= \int_0^\infty x^t e^{-x}dx \end{aligned}$$

$$\Rightarrow \Gamma(t+1) = t\Gamma(t) \tag{2.3}$$

ii) To proof by induction

$$\text{From} \quad \Gamma(t+1) = \int_0^\infty x^t e^{-x}dx$$

$$\begin{aligned} \Gamma(1) &= \int_0^\infty x^0 e^{-x}dx \\ &= \int_0^\infty e^{-x}dx \\ &= -e^{-x}/_0^\infty \\ &= 1 \end{aligned}$$

$$Hence \quad \Gamma(0+1) = 1 = 0! \quad for \quad n \in N$$

iii) $\Gamma(\frac{1}{2}) = \int_0^\infty u^{\frac{-1}{2}} e^{-u} du$

Let $u = t^2$,

we have; $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-t^2} dt$

but

$$\begin{aligned} \int_0^\infty e^{-t^2} dt &= \int_0^\infty e^{-v^2} dv \\ (\Gamma(\frac{1}{2}))^2 &= 2(\int_0^\infty e^{-t^2} dt)(2\int_0^\infty e^{-v^2} dv) \\ &= 4\int_0^\infty \int_0^\infty e^{-(t^2+v^2)} dt dv \end{aligned}$$

Bivariate transformation

$$\begin{aligned} t &= r\cos\theta \\ v &= r sin\theta \end{aligned}$$

The region r which define first quadrant of the region of the integral disk of

$$(\Gamma(\frac{1}{2}))^2 = 4\int_0^\infty \int_0^\infty e^{-(t^2+v^2)} dt dv.$$

$t = r\cos\theta$ and $v = r sin\theta$ will transform in the integral problem from cartesian coordinate in polar coordinate. The new variable (r, θ) will range $0 \le r \le \infty$ and $0 \le \theta \le \frac{\pi}{2}$

$$
\begin{aligned}
(\Gamma(\frac{1}{2}))^2 &= 4(\int_0^{\frac{\pi}{2}} d\theta)(\int_0^{\infty} e^{-r^2} rdr \\
&= 4(\frac{\pi}{2})(\frac{-1}{2}\int_0^{\infty} e^t dt) \\
&= 4(\frac{-\pi}{4})(\int_0^{\infty} e^t dt) \\
&= -\pi(0-1) \\
&= \pi \\
Since \quad e^{-r^2} > 0, \forall r \geq & \; 0,
\end{aligned}
$$

$$
\begin{aligned}
\Gamma(\frac{1}{2}) &\geq 0 \\
Hence \quad (\Gamma(\frac{1}{2}))^2 &= \pi
\end{aligned}
$$

$$\therefore \Gamma(\frac{1}{2}) = \sqrt{\pi} \tag{2.4}$$

by induction $\Gamma(\frac{1}{2}+1) = \Gamma(\frac{3}{2})$ using equation (2.3)

$$t\Gamma(t) = \Gamma(t+1), \quad (t > 0)$$

Let $t = \frac{1}{2}$ using equation (2.4),$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi} = \Gamma(\frac{1}{2}+1) = \Gamma(\frac{3}{2})$$

$$\Rightarrow \Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi} \tag{2.5}$$

In similar way $\Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi}$ and so on...
In general for any positive integer n,$\Gamma(1+\frac{1}{2}+n) = \frac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi}$ for all n=0,1,2,...
Formulation of area and volume of a ball will be convenient to calculate in polar coordinates
Thus

$$\int_{\mathbb{R}^n} f(x)dx = \int_0^{\infty}\int_{S(x_o,r)} f(y)d\sigma(y) \tag{2.6}$$

for each $x_o \in \mathbb{R}^n$
In particular for each $r > 0$
Where $d\sigma$ represents surface measure on the $n-1$ dimensional sphere $S(x_o, r)$ of radius r centered at x_o.
The total surface measure of sphere /ball is proportional to r and the constant will be taken
so that it is by definition, $nw_n r^{n-1}$
Thus for example nw_n,in dimensional n=1,2,3 has the values 2,2π and 4π.In dimensional n=1,2,3 these numbers represent the count of two points of the length of a unit circle, and the surface area of a unit sphere.
By Euler's integral definition of the gamma function (definition 7) convergence of the integral requires that $t-1 > -1$ or $x > 0$
As an example we can take $f(x) = e^{-x^2}$ and x_o=0
Then $\int_{\mathbb{R}^n} e^{-x^2} dx = nw_n \int_0^\infty e^{-r^2} r^{n-1} dr$
Here the total surface measure of the ball is defined to be $nw_n r^{n-1}$
we can also write this as

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-x^2} dx &= nw_n \frac{1}{2} \int_0^\infty u^{\frac{n}{2}-1} e^{-u} du \\ &= nw_n \frac{1}{2} \int_0^\infty u^{\frac{n}{2}} u^{-1} (e^{-u}) du \\ &= nw_n \frac{1}{2} \Gamma(\frac{n}{2}) \end{aligned} \tag{2.7}$$

When n= 2 this says that the value of the integral is π.
The differential of Volume of a hypersphere of radius $r = R$ is

$$V_n = \upsilon_n r^n \Rightarrow dV_n = \upsilon_n . n r^{n-1} dr \tag{2.8}$$

where, υ_n is equal to w_n.
By Gausian integral we have $\int_{-\infty}^{\infty} e^{-x^2} dx = \pi^{\frac{1}{2}}$ multiplying this integral by it self n times subscripting each dummy variable x by a different index i

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} e^{(-\sum_{i=1}^{n} x_i^2)} \Pi_{i=1}^{n} dxi = \pi^{\frac{n}{2}} \tag{2.9}$$

however the summation is simply equal to r^2 in n-dimension and the product of differentials is just the n-dimensional volume element. Writing the volume element in spherical coordinates and performing all of the angular integrals, that element becomes (2.8)
so that equation (2.9) reduces to

$$\int_0^\infty e^{-r^2} w_n n r^{n-1} dr = \pi^{\frac{n}{2}} \tag{2.10}$$

pull the constants w_n and n out of the integral and change the variables to $u = r^2$ to get

$$\frac{1}{2} w_n n \int_0^\infty e^{-u} u^{\frac{n}{2}-1} du = \pi^{\frac{n}{2}} \tag{2.11}$$

but from definition of gamma function $\int_0^\infty e^{-u} u^{\frac{n}{2}-1} du = \Gamma(\frac{n}{2})$
Thus, $\pi^{\frac{n}{2}} = n w_n \frac{1}{2} \Gamma(\frac{n}{2})$
This proves the fact that area of the unit $n-1$ sphere is $n w_n = \frac{2(\pi^{\frac{n}{2}})}{\Gamma(\frac{n}{2})} = A(R)$

The volume of the unit ball is

$$w_n = \frac{2(\pi^{\frac{n}{2}})}{n\Gamma(\frac{n}{2})} = \frac{(\pi^{\frac{n}{2}})}{\Gamma(\frac{n}{2}+1)} \tag{2.12}$$

□

2.1.3 Green's Identity

Consider Divergence theorem,
Let Ω be a C^1 domain and $w \in C^1(\Omega)$ be vector field then

$$\int_{\partial\Omega} w.nds = \int_\Omega div.w(x)dx \tag{2.13}$$

where ds is (n-1)dimensional Lebesgue measure of $\partial\Omega$, $dx = dx_1, dx_2, dx_3, ..., dx_n$

put $w = v\nabla u$ and assume that $u, v \in C^2(\Omega) \bigcap C^1(\Omega)$

$$\begin{aligned} Then \quad w.n &= (v\nabla u).n = v(\nabla u.n) = v\frac{\partial u}{\partial n} \quad and \\ div(w(x)) &= div(v\nabla u) \\ &= \nabla(v\nabla u) \\ &= \nabla v\nabla u + v\nabla^2 u \\ &= \nabla v\nabla u + v\triangle u, \quad where \quad \nabla^2 = \triangle \end{aligned}$$

Thus using equation (2.13)

$$\int_{\partial\Omega} v\frac{\partial u}{\partial n} ds = \int_{\Omega} \nabla v\nabla u + v\triangle u dx (G1) \tag{2.14}$$

This is Green's first identity.

$$\int_{\partial\Omega} u\frac{\partial v}{\partial n} ds = \int_{\Omega} (\nabla u.\nabla v + u\triangle v) dx \tag{2.15}$$

Subtracting (2.14) from (2.15) we have

$$\int_{\partial\Omega} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}) ds = \int_{\Omega} (u\triangle v - v\triangle u) dx (G2) \tag{2.16}$$

This is Green's 2^{nd} identity .
If we set $u = v$ in G_1,we obtain

$$\int_{\partial\Omega} u\frac{\partial u}{\partial n} ds = \int_{\Omega} ((\nabla u)^2 + u\triangle u) dx \tag{2.17}$$

This is Green's 3^{rd} identity.

Lemma 1. ***(Consequence of Green's identity)***
Let $F \in C^(\bar{\Omega})$ If u solves $\triangle u$ =F on the domain Ω then

$$\int_{\Omega} F(x,y) dx dy = \int_{\partial\Omega} \partial n u ds \tag{2.18}$$

Proof. Integrating both sides of the equation $\triangle u$ =F on Ω

$$\int_{\Omega} F(x,y) dx dy = \int_{\partial\Omega} \triangle u(x,y) dx dy \tag{2.19}$$

Applying Green's first identity, the integral on the right hand side of equation(2.19) becomes $\int_{\Omega} \triangle u(x,y)dxdy = \int_{\partial\Omega} \partial nuds$ □

Theorem 2. *Let* $\Omega \subset \mathbb{R}^n$.*If* $u \in C^2(\Omega)$ *is harmonic, then*
$u(x) = ⨍_{\partial B(x,r)} u(y)ds(y) = ⨍_{B(x,r)} u(y)dy$, $\forall B(x,r) \subset \Omega$

Proof. Assume that $u \in C^2(\Omega)$ is harmonic for $r > 0$
define

$$\phi(r) = ⨍_{\partial B(x,r)} u(y)ds(y) \tag{2.20}$$

For $r = 0$ define $\phi(0) = u(x)$.

If we can show that $\phi'(r) = 0$, we can conclude that ϕ is constant
And therefore $u(x) = ⨍_{\partial B(x,r)} u(y)ds(y)$

To prove $\phi'(r) = 0$
First making a change of variables $y = x + rz, \Rightarrow z = \frac{y-x}{r}$

$$\begin{aligned}
\phi(r) &= ⨍_{\partial B(x,r)} u(y)ds(y) \\
&= ⨍_{\partial B(0,1)} u(x+rz)ds(z) \\
\Rightarrow \phi'(r) &= ⨍_{\partial B(0,1)} \nabla u(x+rz).zds(z) \\
&= ⨍_{\partial B(x,r)} \nabla u(y).\frac{y-x}{r}ds(y) \\
&= ⨍_{\partial B(x,r)} \frac{\partial u}{\partial n}(y)ds(y) \\
&= \frac{1}{nw_n r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial n}(y)ds(y) (the \quad average \quad of \quad u \quad on \quad \partial B(x,r))
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{nw_n r^{n-1}} \int_{B(x,r)} \nabla.(\nabla u) dy (by \quad divergence \quad theorem) \\
&= \frac{1}{nw_n r^{n-1}} \int_{B(x,r)} \triangle u(y) dy \\
&= 0
\end{aligned}$$

since u is harmonic $\triangle u = 0$

$\Rightarrow \phi'(r) = 0$

ϕ is constant.

claim $\lim_{r \to 0^+} \phi(r)$ =u(x)
since ϕ is constant

Let r=t

$$\begin{aligned}
\phi(r) &= \lim_{t \to 0^+} \phi(t) \\
&= \lim_{t \to 0} ⨍_{\partial B(x,t)} u(y) ds(y) \\
&= u(x)
\end{aligned}$$

Therefore

$$\begin{aligned}
|u(x) - ⨍_{\partial B(x,t)} u(y) ds(y)| &= |⨍_{\partial B(x,r)} (u(x) - u(y)) ds(y)| \\
&\leq \int_{\partial B(x,r)} |u(x) - u(y)| ds(y) \\
&\leq \max_{y \in \partial B(x,t)} \{|u(x) - u(y)|\}
\end{aligned}$$

Next to prove that $u(x) = ⨍_{B(x,r)} u(y) dy$

using the first result and taking sphere with radius s centered at x inside sphere with radius r centered at x $(s < r)$

$$\begin{aligned}
\int_{B(x,r)} u(y)dy &= \int_0^r \int_{\partial B(x,r)} u(y)ds(y \\
&= \int_0^r n w_n s^{n-1} \fint_{\partial B(x,r)} u(y)ds(y)ds \\
&= \int_0^r n w_n s^{n-1} u(x)ds \\
&= n w_n u(x) \int_0^r s^{n-1}ds \\
&= n w_n u(x) \frac{s^{n-1+1}}{n-1+1} /_{s=0}^{s=r} \\
&= w_n u(x) s^n /_{s=0}^{s=r} \\
&= w_n u(x) r^n
\end{aligned}$$

Therefore, $\int_{B(x,r)} u(y)dy = w_n r^n u(x)$ which imlies

$u(x) = \frac{1}{w_n r^n} \int_{B(x,r)} u(y)dy$ (dividing both sides by $w_n r^n$)

$$= \fint_{B(x,r)} u(y)dy$$

from equation (2.20)

$$u(x) = \fint_{\partial B(x,r)} u(y)ds(y)$$

Hence $u(x) = \fint_{B(x,r)} u(y)dy = \int_{\partial B(x,r)} u(y)ds(y)$

□

2.2 Converse of mean value property

Theorem 3. *If $u \in C^2(\Omega)$ satisfies $u(x) = \fint_{B(x,r)} u(y)dy = \fint_{\partial B(x,r)} u(y)ds(y)$,$\forall B(x,r) \subset \Omega$ then u harmonic.*

Proof. suppose not, then $\triangle u(x) \neq 0$ somewhere

$\triangle u(x) > 0$ at 0

so $\triangle u(x) > 0$ on $B(x, r)$ for some $r > 0$

Let $\varphi(r) = ⨍_{\partial B(x,r)} u(y)ds(y) = u(x)$

$\varphi'(r) = 0$ from theorem (2)

But $0 = \varphi'(r) = \frac{1}{nw_n r^{n-1}} \int_{B(x,r)} \triangle u(y)dy > 0$, since $\triangle u(y) > 0$

$0 > 0$ contradiction

$\therefore \triangle u(y) = 0$

Hence u is harmonic. □

2.3 Maximum principle

The maximum principle states that a non-constant harmonic function can not attain a maximum or a minimum at an interior point of its domain.This result implies that the value of harmonic function in a bounded domain are bounded by its maximum or minimum values on the boundary.

Theorem 4. ***(Weak maximum principle)***
Suppose Ω is a bounded domain and $u \in C^2(\Omega) \bigcap C(\bar{\Omega})$ is harmonic,then the maximum of u on $\bar{\Omega}$ is attained on the $\partial\Omega$.
i.e

$$\max_{x\in\bar{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x)$$

Proof. Introduce M= $\max\limits_{x\in\bar{\Omega}} u(x)$, m= $\max\limits_{x\in\partial\Omega} u(x)$

Obviously $m \leq M$

Assume that $m < M$

put $u(x_o) = M$ as u is continuous on $\bar{\Omega}$

$$v(x) = u(x) + \frac{M-m}{2d^2}||x - x_o||^2$$

where d is the diameter of Ω

$$d = sup||x - y||$$

Then ,v is a C^2 function on Ω and $v(x_o) = u(x_o)$
we have $v(x) \geq u(x)$
Hence , $\max_{x \in \bar{\Omega}} v(x) \geq M$ on the boundary $\partial\Omega$

we have

$$v(x) \leq m + \frac{M-m}{2d^2}d^2 = m + \frac{M-m}{2} = \frac{M+m}{2} \leq M$$

$\Rightarrow v(x) \leq M, \forall x \in \partial\Omega$

Therefore, there must be a point $y \in \Omega$

such that

$$\max_{x \in \bar{\Omega}} v(x) = v(y)$$

At this point y since v takes a maximum at y, we all first derivatives $\partial x_i v$ vanish at y and the second derivatives $\partial x_i x_i v$ are at y non-positive that is $\frac{\partial^2 v}{\partial^2 x_i}(y) \leq 0$

Thus

$$\triangle v(y) \leq 0 \tag{2.21}$$

But

$$\begin{aligned}\triangle v &= \triangle u + \frac{M-m}{2d^2}\triangle ||x - x_o||^2 \\ &= \frac{M-m}{2d^2}2n, as \triangle ||x - x_o||^2 = 2n \\ &= \frac{n(M-m)}{d^2} > 0\end{aligned}$$

$$\Rightarrow \triangle v(x) > 0 \tag{2.22}$$

we obtained $\triangle v(y) \leq 0$ and $\triangle v > 0$ which is a contradiction.
$\therefore m = M$

$$\Rightarrow \max_{x\in\bar{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x)$$

□

Corollary 1. *(**Weak minimum principle**) Suppose that Ω is a bounded domain, u is harmonic in Ω and u is continuous on $\bar{\Omega}$*

Then $\min_{x\in\bar{\Omega}} u(x) = \min_{x\in\partial\Omega} u(x)$

Proof. $\min_{x\in\bar{\Omega}} u = \max_{x\in\partial\Omega} -u$

$\min_{x\in\partial\Omega} u = \max_{x\in\partial\Omega} -u$
By replacing u by -u in theorem (4)
we have

$$\min_{x\in\bar{\Omega}} u(x) = \min_{x\in\partial\Omega} u(x)$$

□

Theorem 5. *(**Uniqueness theorem**) suppose Ω is bounded domain and u_1 and u_2 are harmonic in Ω continuous in $\bar{\Omega}$ and $u_1|\partial\Omega = u_2|\partial\Omega$, Then $u_1 = u_2$ in Ω*

Proof. Define $u = u_1 - u_2$ in Ω,

Then u is harmonic and $u|\partial\Omega = 0$

by weak minimum principle

$\max_{x\in\bar{\Omega}} u = 0$ as $u|\partial\Omega = 0$

by weak minimum principle

$\min_{x\in\bar{\Omega}} u = 0$

$\Rightarrow u(x) = 0$ on Ω and

Hence $u_1(x) = u_2(x)$,$\forall x \in \Omega$

$$\therefore u_1 = u_2$$

□

Corollary 2. *Let Ω be a bounded domain, $f \in C(\bar{\Omega})$ and $\varphi \subset (\partial\Omega)$ then the Dirichlet problem*

$$\begin{cases} \triangle u = f(x,y) & (x,y) \in \Omega \\ u(x,y) = g(x,y) & (x,y) \in \partial\Omega \end{cases} \tag{2.23}$$

has no more than one solution

Proof. Suppose u_1 and u_2 are solutions of (2.23) in Ω , and $u = u_1 - u_2$
then $u \in C(\Omega)$ is harmonic and $u \in C(\bar{\Omega})$
$u = 0$ on $\partial\Omega$ by uniqueness theorem.
By weak maximum principle $u = 0$ in Ω.
$\therefore u = 0$
$\Rightarrow u_1 = u_2$. □

Chapter 3

Harmonic function and Poisson Integral

3.1 Poisson integral

Since the boundary values of harmonic functions determine these functions uniquely(under the assumption of continuity on the boundary), it is natural to ask a question about reconstructing the harmonic function from its boundary values. A slightly more general problem is known as the **Dirichlet problem**.

Definition 8. *Let Ω be a domain in $\mathbb{C}$ and let $\phi : \partial\Omega \to \mathbb{R}$ be a continuous function.The Dirichlet problem is to find a function f harmonic on Ω and such that* $\lim_{z\to\xi} f(z) = \phi(\xi), \forall\xi \in \Omega$

One important case on a circular domain (disk) is when the positive answer comes via an explicit construction. It uses the so-called ***Poisson kernel****. Now we shall set about finding the harmonic function on a disk from its values on the disk boundary.*

Definition 9. *The following real valued function of two complex variables z and ξ,*
$p_k(z,\xi) = Re(\frac{z+\xi}{\xi-z}) = \frac{1-|z|^2}{|\xi-z|^2}(|z| < 1, |\xi| = 1)$
is known as poisson kernel.
where (P_k) denotes poisson kernel.

Definition 10. *Let $\phi : D(w,a) \to \mathbb{R}$ be continuous function. Then its Poisson integral P_I is defined by*

$p_I(x) \; = \frac{1}{2\pi}(\int_0^{2\pi} p_k(\frac{z-w}{a}, e^{i\theta})\phi(w + ae^{i\theta}))d\theta$

In polar coordinates the disk about w is in the form of

$p_I(w + re^{it}) = \frac{1}{2\pi}(\int_0^{2\pi} \frac{a^2-r^2}{a^2-2arcos(\theta-t)+r^2} h(w + re^{i\theta})d\theta$

since the boundary of the region $\partial\Omega$ is a circle, using polar coordinates r, θ defined by

$$\begin{aligned} x &= rcos\theta \\ y &= rsin\theta \end{aligned}$$

$\frac{\partial x}{\partial r} = cos\theta,\ \frac{\partial y}{\partial r} = sin\theta$

$\frac{\partial x}{\partial \theta} = -rsin\theta,\ \frac{\partial y}{\partial \theta} = rcos\theta$

$\frac{\partial \theta}{\partial x} = \frac{-sin\theta}{r},\ \frac{\partial \theta}{\partial y} = \frac{cos\theta}{r}$

Since $\quad p = p(r, \theta)$

$$\begin{aligned}
\frac{\partial p}{\partial x} &= \frac{\partial p}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \theta}{\partial x}\frac{\partial p}{\partial \theta} \\
&= cos\theta\frac{\partial p}{\partial r} - \frac{sin\theta}{r}\frac{\partial p}{\partial \theta} \\
\frac{\partial p}{\partial y} &= \frac{\partial p}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \theta}{\partial y}\frac{\partial p}{\partial \theta} \\
&= sin\theta\frac{\partial p}{\partial r} + \frac{cos\theta}{r}\frac{\partial p}{\partial \theta} \\
\frac{\partial^2 p}{\partial x^2} &= \frac{\partial}{\partial x}(cos\theta\frac{\partial p}{\partial r} - \frac{sin\theta}{r}\frac{\partial p}{\partial \theta}) \\
&= (cos\theta\frac{\partial^2 p}{\partial r^2} - \frac{sin\theta}{r}\frac{\partial p}{\partial r\partial \theta})cos\theta + (cos\theta\frac{\partial p}{\partial \theta\partial r} - \frac{\partial p}{\partial r}sin\theta - \frac{\partial p}{\partial \theta^2}\frac{sin\theta}{r} \\
&- \frac{\partial p}{\partial \theta}\frac{cos\theta}{r})(\frac{-sin\theta}{r})
\end{aligned}$$

$$
\begin{aligned}
&= cos^2\theta\frac{\partial^2 p}{\partial r^2} - 2\frac{(sin\theta cos\theta)}{r}\frac{\partial^2 p}{\partial r\partial\theta} + \frac{sin^2\theta}{r^2}\frac{\partial^2 p}{\partial\theta^2} + \frac{2sin\theta cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{sin^2\theta}{r}\frac{\partial p}{\partial r} \\
\frac{\partial^2 p}{\partial y^2} &= \frac{\partial}{\partial y}(sin\theta\frac{\partial p}{\partial r} + \frac{cos\theta}{r}\frac{\partial p}{\partial\theta}) \\
&= (sin\theta\frac{\partial p}{\partial r^2} + \frac{cos\theta}{r}\frac{\partial p}{\partial r\partial\theta} - \frac{\partial p}{\partial\theta}\frac{cos\theta}{r^2})sin\theta + (sin\theta\frac{\partial p}{\partial\theta\partial r} + \frac{\partial p}{\partial r}cos\theta \\
&+ \frac{\partial p}{\partial\theta^2}\frac{cos\theta}{r} - \frac{\partial p}{\partial\theta}\frac{sin\theta}{r})(\frac{cos\theta}{r}) \\
&= sin^2\theta\frac{\partial^2 p}{\partial r^2} + 2\frac{(sin\theta cos\theta)}{r}\frac{\partial^2 p}{\partial r\partial\theta} + \frac{cos^2\theta}{r^2}\frac{\partial^2 p}{\partial\theta^2} + \frac{2sin\theta cos\theta}{r^2}\frac{\partial p}{\partial\theta} + \frac{cos^2\theta}{r}\frac{\partial p}{\partial r} \\
\triangle p_I &= \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \\
&= \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial\theta^2}
\end{aligned}
$$

Using Laplacian in polar coordinates (r, θ) in the plane

$$\triangle p(r, \theta) = p_{rr} + \frac{1}{r}p_r + \frac{1}{r^2}p_{\theta\theta} = 0 \tag{3.1}$$

Which is a Laplace equation in polar coordinates.

3.1.1 Dirichlet Problem for a circle

Dirichlet Problem for a circle is defined as

$$\begin{cases} PDE : \nabla^2 p = 0, \quad 0 \le r \le a, \quad 0 \le \theta \le 2\pi \\ BC : p(r, \theta) = h(\theta), \quad 0 \le \theta \le 2\pi \end{cases} \tag{3.2}$$

Where $h(\theta)$ is a continuous function on $\partial\Omega$.
From polar coordinate of Laplace equation $\nabla^2 p = 0$ in equation (3.1) $\triangle p(r, \theta) = p_{rr} + \frac{1}{r}p_r + \frac{1}{r^2}p_{\theta\theta} = 0$.
If $p(r, \theta) = R(r)H(\theta)$ the above equation reduces to
$R''H + \frac{1}{r}R'H + \frac{1}{r^2}RH'' = 0$. This equation can be witten as

$$\frac{r^2 R'' + rR'}{R} = -\frac{H''}{H} = k \tag{3.3}$$

which means that a function of r is equal to a function of θ and, therefore each must be equal to a constant k (separation of constant).

Case1

Let $k = \lambda^2$ *then*

$r^2R'' + rR' - \lambda^2 R = 0$ *which is Euler type of equation and can be solved by setting* $r = e^z$*.Its solution is,*

$R = c_1e^{\lambda z} + c_2e^{-\lambda z} = c_1r^{\lambda} + c_2r^{-\lambda}$

Also, $H'' + \lambda^2 H = 0$

Whose solution is $H = c_3cos\lambda\theta + c_4sin\lambda\theta$

Therefore,

$$p(r, \theta) = (c_1r^{\lambda} + c_2r^{-\lambda})(c_3cos\lambda\theta + c_4sin\lambda\theta) \tag{3.4}$$

Case2

Let $k = -\lambda^2$ *then*

$r^2R'' + rR' + \lambda^2 R = 0$,$H'' - \lambda^2 H = 0$

Their respective solutions are

$R = c_1cos(\lambda lnr) + c_2sin(\lambda lnr),\ H = c_3e^{\lambda\theta} + c_4e^{-\lambda\theta}$

Thus,

$$p(r, \theta) = (c_1cos(\lambda lnr) + c_2sin(\lambda lnr))(c_3e^{\lambda\theta} + c_4e^{-\lambda\theta}) \tag{3.5}$$

Case3

Let k=0 then we have $rR'' + R' = 0$ *setting* $R'(r) = v(r)$

we obtain $r\frac{dv}{dr} + v = 0$*, that is* $\frac{dv}{dr} + \frac{dr}{r} = 0$

Integrating we get $ln(vr) = lnc_1$*,therefore* $v = \frac{c_1}{R} = \frac{dr}{dr}$

On integration $R = c_1lnr + c_2$

Also $H'' = 0$

After integrating twice , we get $H = c_3\theta + c_4$

Thus

$$p(r, \theta) = (c_1lnr + c_2)(c_3\theta + c_4) \tag{3.6}$$

Now for the interior problem ,r=0 is a point in the domain Ω and since lnr is not defined at r=0, the solution (3.5) and (3.6) are not acceptable, Thus the required solution is obtained from (3.4)

The periodicity condition in θ implies,

$c_3cos\lambda\theta + c_4sin\lambda\theta = c_3cos(\lambda(\theta + 2\pi)) + c_4sin(\lambda(\theta + 2\pi))$
That is $c_3(cos\lambda\theta - cos(\lambda\theta + 2\lambda\pi)) + c_4(sin\lambda\theta - sin(\lambda\theta + 2\lambda\pi)) = 0$
or $2sin\lambda\pi[c_3sin(\lambda\theta + 2\pi) - c_4cos(\lambda\theta + \lambda\pi)] = 0 \Rightarrow \lambda\pi = 0, \lambda\pi = n\pi, \lambda = n(n = 0, 1, 2, 3, ...)$, *using the principle of supper position and renaming the acceptable general solution can be written as*

$$p(r,\theta) = \sum_{0}^{\infty}(c_nr^n + c_2r^{-n})(a_ncosn\theta + b_nsinn\theta) \tag{3.7}$$

Thus,the appropriate solution assumes the form

$$p(r,\theta) = \sum_{0}^{\infty} r^n(A_ncosn\theta + B_nsinn\theta) \tag{3.8}$$

For n=0 let the constant A_o *be* $\frac{A_o}{2}$*.Then the solution is:*

$$p = \frac{1}{2}A_o + \sum_{n=1}^{\infty} r^n(A_ncosn\theta + B_nsinn\theta) \tag{3.9}$$

Formally differentiating this series term by term, we verify that (3.8) is indeed harmonic.

$$A_o = \frac{1}{\pi}\int_0^{2\pi} h(\phi)d\phi \tag{3.10}$$

$$A_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)cosn\phi d\phi \tag{3.11}$$

$$B_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)sinn\phi d\phi \tag{3.12}$$

Writing (3.9) using (3.10),(3.11)and(3.12)
$p(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} h(\phi)d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}(\frac{r}{a})^n \int_0^{2\pi} h(\phi)(cosn\phi cosn\theta + sin\phi sinn\theta)d\phi$
consider $r < \tilde{a} < a$*.since the series converges uniformly here,we can interchange the order of summation and integration*
we obtain
$p(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)d\phi + \frac{1}{\pi}\int_0^{2\pi} h(\phi)\sum_{n=1}^{\infty}(\frac{r}{a})^n(cosn\phi cosn\theta + sin\phi sinn\theta)d\phi$
By cosine property
$cosn\phi cosn\theta + sinn\phi sinn\theta = cosn(\theta - \phi)$
and $cosn(\theta - \phi) = \frac{e^{in(\theta-\phi)} + e^{-in(\theta-\phi)}}{2}$

$p(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)d\phi + \frac{1}{\pi}\int_0^{2\pi} h(\phi)\sum_{n=1}^{\infty}(\frac{r}{a})^n(cos(\theta-\phi)d\phi$

$$\Rightarrow p(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)[1 + 2\sum_{n=1}^{\infty}(\frac{r}{a})^n(cos(\theta-\phi)]d\phi \tag{3.13}$$

Now

$$\{1 + 2\sum_{n=1}^{\infty}(\frac{r}{a})^n(cos(\theta-\phi)\} = \{1 + 2\sum_{n=1}^{\infty}\frac{r^n}{a^n}[\frac{e^{in(\theta-\phi)} + e^{-in(\theta-\phi)}}{2}]\}$$

$$= 1 + \{\sum_{n=1}^{\infty}(\frac{re^{i(\theta-\phi)}}{a})^n + \sum_{n=1}^{\infty}(\frac{re^{-i(\theta-\phi)}}{a})^n\}$$

$$= 1 + \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} + \frac{re^{i(\theta-\phi)}}{a - re^{-i(\theta-\phi)}}$$

$$= \frac{a^2 - r^2}{a^2 - 2arcos(\theta-\phi) + r^2}$$

Therefore,

$$p(r,\theta) = \int_0^{2\pi} h(\phi)\frac{a^2 - r^2}{a^2 - 2arcos(\theta-\phi) + r^2}\frac{d\phi}{2\pi} \tag{3.14}$$

this single formula is known as ***Poisson formula***.

The Poisson formula can be written in a geometric way as follows;

write $X = (x,y)$ *with polar coordinate* (r,θ)

$X' = (x',y')$ *with polar coordinate* (a,ϕ)

The origin of points X *and* X' *forms triangle with sides* $r = |X|$,$a = |X'|$ *and* $|X - X'|$.

By the law of cosine

$|X - X'|^2 = a^2 + r^2 - 2arcos(\theta-\phi)$

The arc length element on the circumference is

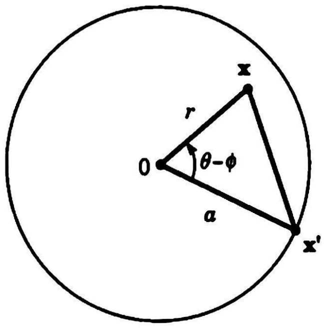
$ds' = ad\phi$. *Therefore,Poisson formula takes the alternative form*

$p(X) = \frac{a^2-|X|^2}{2\pi a}\int_{|X'|=a}\frac{p(X')}{|X-X'|^2}ds'$ *for* $X \in \Omega$ *and*

where $p(X') = h(\phi)$,*this a line integral with respect to arc length* $ds' = ad\phi$.

since $s' = a\phi$ *for a circle.*

Thus the Poisson kernel (p_k) *for the exterior of the disk is*



$p_k(X, X') = \frac{1}{2\pi a} \frac{a^2 - |X'|^2}{|X - X'|^2} (|X| < a, |X'| = a)$

In the disk about w

$D(w, r)$=$\{z : |z - w| < r\}$,

The boundary ∂D *of* $D(w, r)$ *is the set*

$\partial D(w, r)$=$\{z : z = w + re^{i\theta}, \theta \in [0, 2\pi]\}$

where $z = w + re^{i\theta}$ *is parametrization for the integration* θ *runs from* 0 *to* 2π *of equation(3.13) with* $dz = ire^{i\theta}d\theta$

This takes

$$h(w + re^{it}) = \frac{1}{2\pi} (\int_0^{2\pi} \frac{a^2 - r^2}{a^2 - 2arcos(\theta - t) + r^2} h((w + re^{i\theta}))d\theta. \qquad (3.15)$$

Lemma 2. *Let* $\phi : D(w, a) \rightarrow R$ *be continuous function.*P_I *is harmonic on* D

Proof. It follows from the definition of the Poisson kernel that PI is the real part of a harmonic function is harmonic(real and imaginary part of analytic functions are harmonic function).since z is real part of holomorphic function it is harmonic.

□

Theorem 6. ***(Poisson Integral Formula)*** *If h is harmonic in a nbhd of the disk* $\bar{D}(w, a)$ *then for* $r < a$ *that is inside the disk* $D(w, r)$

$h(w + re^{it})$=$\frac{1}{2\pi} (\int_0^{2\pi} \frac{a^2 - r^2}{a^2 - 2arcos(\theta - t) + r^2} h((w + re^{i\theta}))d\theta$

Proof. Its proof is already in definition (11) of equation (3.14) above
Writing the Poisson integral of the disk, that is for $r = 0$ one recovers the mean value property of harmonic function.
Thus the Poisson integral formula can be viewed as a generalization of mean value property. □

3.2 Positive harmonic functions

The Poisson integral formula allows to obtain useful inequalities for positive harmonic function.

Remark. *If a non negative harmonic function attains a minimum value zero on a domain,It is zero throughout the domain. so the class of non-negative harmonic function on a domain consists of all positive and zero functions.As an application of Poisson integral we prove Harnak inequality.The Harnak inequality refers to a control of the maximum of a non-negative solution of an equation by its minimum.*

Theorem 7. ***(Harnak's Inequality)*** *Let h be a positive harmonic function on the disk(w,a) of the radius a about w, then for $|z-w|{=}r < a$ and $\forall t$*
$\frac{a-r}{a+r}h(w) \leq h(z) \leq \frac{a+r}{a-r}h(w)$.
That is the values of h in disk $D(w,a)$ are bounded below and above by multiples of the values of h at the center w of the disk,with both bounds only depending on the distance to the center.

Proof. Take any s with $r < s < a$. since h is continuous on $\bar{D}(w,s)$ we know that it is the Poisson integral of its restriction to the boundary circle $D(w,s)$

$$h(z) = \frac{1}{2\pi}(\int_0^{2\pi} \frac{s^2 - |z-w|^2}{|se^{it} - (z-w)|^2} h(w + se^{it})dt \tag{3.16}$$

for $z \in D(w,s)$
using the positivity of $h(z)$ and the inequality $\frac{s-|z-w|}{s+|z-w|} \leq \frac{s^2-|z-w|^2}{|se^{it}-(z-w)|^2} \leq \frac{s+|z-w|}{s-|z-w|}$
If $|z-w| = r < s$ then
$\frac{s-r}{s+r} \leq \frac{s^2-|z-w|^2}{|se^{it}-(z-w)|^2} \leq \frac{s+r}{s-r}$.
Using equation (3.16),the assumption $h \geq 0$,and the mean value property of h we

conclude that when $|z-w|=r$,

$$h(z) \le \frac{s+r}{s-r} \cdot \frac{1}{2\pi} (\int_0^{2\pi} h(w+se^{it}) dt = \frac{s+r}{s-r} h(w)$$

$$\text{and } h(z) \ge \frac{s-r}{s+r} \cdot \frac{1}{2\pi} (\int_0^{2\pi} h(w+se^{it}) dt = \frac{s-r}{s+r} h(w)$$

The result now follows by letting $s \to a$

$$\frac{a-r}{a+r} h(w) \le h(z) \le \frac{a+r}{a-r} h(w).$$

□

Corollary 3. ***(Liouville theorem)***
Every harmonic function $h : \mathbb{C} \to \mathbb{R}^n$ which is bounded from above or below must be constant.

Proof. Suppose h is bounded above by M
Then $\psi = M - h$ is a non-negative harmonic function in the plane
By Harnak's Inequality for $\frac{a-|z|}{a+|z|}\psi(0) \le \psi(z) \le \frac{a+|z|}{a-|z|}\psi(0)$ if $|z| \le a$
Letting $a \to \infty$ we obtain $\psi(z) \le \psi(0)$
This shows that $\psi(z) \equiv \psi(0)$, hence h is constant. The case where h is bounded below follows from this by considering $-h$

□

3.3 Application of harmonic function

Harmonic function plays a crucial role in many real applications of mathematics,physics and engineering. In this section we consider some examples as an application of harmonic functions.

Example 3. *(An application of the maximum principle for subharmonic functions)*
Suppose that $\Omega \subseteq \mathbb{R}^2$ is a bounded domain, suppose that $V : \Omega \to \mathbb{R}$ is continuous and suppose that for some constant $C \in \mathbb{R}$
$V_{xx} + V_{yy} = C$ *on* Ω
show that the function $U = |\triangle V|^2$ is subharmonic on Ω

Solution 1. *We compute*
U_x=$(V_x^2+V_y^2)_x$=$2V_xV_{xx}+2V_yV_{yx}$,$V\in C(\bar{\Omega})$
U_{xx} =$(2V_xV_{xx}+2V_yV_{yx})_x$ $=2V_{xx}^2+2V_xV_{xxx}+2V_{yx}^2+2V_yV_{yxx}$
U_y=$(V_x^2+V_y^2)_y$=$2V_yV_{yy}+2V_xV_{xy}$
U_{yy} =$(2V_xV_{xy}+2V_yV_{yy})_y$ $=2V_{xy}^2+2V_xV_{xyy}+2V_{yy}^2+2V_yV_{yyy}$
$\triangle U$=U_{xx}+U_{yy}= $2v_{xx}^2+2V_x(V_{xxx}+V_{xyy})+2V_{yx}^2+2V_y(V_{yxx}++V_{yyy})+2V_{xy}^2+2V_{yy}^2$
Since $V_{xx}+V_{YY}=C$ *we can differentiate both sides with respect to x and with respect to y respectively, we deduce that* $V_{xxx}+V_{xyy}$= $V_{yxx}+V_{yyy}=0$
In particular,it follows that:
$\triangle U$=$V_{xx}^2+4V_{xy}^2+2V_{yy}^2\geq 0$
$\Rightarrow U\geq 0$
$\therefore U$ *is a subharmonic function.*

Example 4. *(Mean value property) If U is a harmonic in a disk, and continuous in a closed disk, the value at the center equals the average of the boundary values. This follows directly from the fact that the value of Poisson kernel at the origin equals* $(2\pi a)^{-1}$. *If U is harmonic in the exterior of a disk, continuous up to the boundary and bounded, then it is given by integration of the boundary values with respect to the Poisson kernel,and then we have*

$lim_{|x|\to\infty}u(x)$= *average of u over* $\partial\Omega_a$

This follows directly from the from the fact that for the exterior Poisson kernel $P_k(x,y)$

$lim_{|x|\to\infty}P_k(x,y)$=$\frac{1}{2\pi a}$,
Uniformly over $y\in\partial\Omega_a$ *as the expression*
$P(x,y)=\frac{1}{2\pi a}\frac{|X|^2-a^2}{|X-Y|^2}$, *for* $|X|>a,|Y|=a$

Remark. *The mean value property can be used to define "harmonic functions in a way that makes sense of continuous functions. u is harmonic if its average value over any circle equals the values at the center.*

Example 5. *(Liouville's theorem) Suppose that u is harmonic in* $\mathbb{R}^n$ *and that there is a constant c such that*

$\int_{B(0,1)} |u(y)|dy < c$, $\forall x \in \mathbb{R}^n$
show that u is constant

Solution 2. *Consider Green's first identity formula*

$\int_\Omega \nabla u.\nabla v dx = \int_{\partial\Omega} v\frac{\partial u}{\partial n} ds - \int_\Omega v\triangle u dx$

for v=1 we have

$\int_{\partial\Omega} \frac{\partial u}{\partial n} ds = \int_\Omega \triangle u dx$
Let $B(x_o, r)$ *be a ball in* $\mathbb{R}^n$ *we have*

$$\begin{aligned} 0 &= \int_{B(x_o,r)} \triangle u dx \\ &= \int_{\partial B(x_o,r)} \frac{\partial u}{\partial n} ds \\ &= r^{n-1} \int_{\partial B(0,1)} \frac{\partial u}{\partial r}(x_o + rx) ds \\ &= r^{n-1} w_n \frac{\partial}{\partial r} \frac{1}{w_n} \int_{\partial B(0,1)} \frac{\partial u}{\partial r}(x_o + rx) ds \end{aligned}$$

Thus $\frac{1}{w_n}\int_{\partial B(0,1)} \frac{\partial u}{\partial r}(x_o + rx)ds$ *is independent of* r
Hence it is constant
By continuity as $r \to 0$ *,we obtain mean value property*
$u(x_o) = \frac{1}{w_n}\int_{\partial B(0,1)} \frac{\partial u}{\partial r}(x_o + rx)ds$
If $\int_{B(0,1)} |u(y)|dy < c$, $\forall x \in \mathbb{R}^n$
we have $|u(x)| < c$ *in* $\mathbb{R}^n$
since u is harmonic and bounded in $\mathbb{R}^n$
By Liouville's theorem, u is constant.

Chapter 4

Conclusion and summary

In this thesis we discussed some basic properties of harmonic functions and solution of Laplace equation $\triangle u = 0$

•. Consider Laplacian differential operator

$\triangle u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial {x_i}^2}, u \in C^2(\Omega)$.

i) $u \in C^2(\Omega)$ is a harmonic in Ω,iff $\triangle u = 0$ in Ω.

ii) $u \in C^2(\Omega)$ is a sub harmonic in Ω,iff $\triangle u \leq 0$ in Ω.

iii) $u \in C^2(\Omega)$ is a supper harmonic in Ω,iff $\triangle u \geq 0$ in Ω.

• For $u \in C^2(\Omega) \bigcap u \in C^1(\bar{\Omega})$ with Ω bounded in $\mathbb{R}^n$

$\Rightarrow$ If $\triangle u \geq 0$ in Ω then, $u \leq \max\limits_{\partial\Omega} u (\Leftrightarrow \max\limits_{\bar{\Omega}} u = \max\limits_{\partial\Omega} u)$

$\Rightarrow$ If $\triangle u \leq 0$ in Ω then, $u \geq \min\limits_{\partial\Omega} u (\Leftrightarrow \min\limits_{\bar{\Omega}} u \min\limits_{\partial\Omega} u)$

$\Rightarrow$ If $\triangle u = 0$ in Ω then, $\min\limits_{\partial\Omega} u \leq u \leq \max\limits_{\partial\Omega} u$

• Dirichlet problem ,separation of variable and Laplace solution in polar coordinates for a circle results the Poisson integral formula;

$p(r, \theta) = \int_0^{2\pi} h(\phi) \frac{a^2 - r^2}{a^2 - 2arcos(\theta - \phi) + r^2} \frac{d\phi}{2\pi}$

♣ The Poisson integral formula allows to obtain Harnak's inequalities for positive harmonic function to a control of the maximum of a non-negative solution of an equation by its minimum.

▹Generally it is important to understand the power of properties of harmonic function and harmonic functions are solutions of laplace equation in the study of PDEs.

Bibliography

[1] L.C. Evans partial differential equation Graduate studies in mathematics,American Mathematical society, 1899.

[2] L.C. Evans An introduction to stochastic partial differential equation,American Mathematical Society, 2013.

[3] D.Gilbarg and N.S.Trudinger,Elliptic partial differential equation second order, Springer Verlag, 1977.

[4] Thomas Ranshford potential theory in the complex plane, CPU in 1995.

[5] F.Jhon, partial differential equations, Applied mathematical sciences,Springer Newyork,1991.

[6] Sheldon Axler,Harmonic functions from a complex analyisis viewpoint,American Mathematical monthly, 93(1986).

[7] S.Axler.P. Bourdon and W.Ramy,"Harmonic functions Theory", GMT 137, Springer Newyork,1992.

[8] U.Kuran, On the mean value property of harmonic functions, Bull, London math, Soc. 4(1972).

[9] W.A An introduction Struss,Partial Differential Equations, second edition in Brown University,John Wiley and Sons,Ltd(2007).

[10] K.Sankara Rao,Introductio To partial differential equations, third edition in Anna University,Asoke.K.PHI,Ltd(2011).