ADDIS ABABA UNIVERSITY

DEPARTMENT OF MATHEMATICS

On System of Ordinary Differential Equations and the Trace-Determinant Plane

A thesis Submitted to the Department of Mathematics of Addis Ababa University in Partial fulfillment of the Requirements of the Master of Science Degree in Mathematics.

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Were, the undersigned, here by that have read and examined this thesis paper, a thesis on the System of ordinary differential equations (ODEs) and the trace-determinant plane: which is done by **TEMESGEN BEKELE** in partial fulfillment of the requirements for the degree of Master of Science and recommend to the school of graduate studies for acceptance of the thesis.

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ABSTRACT

This thesis provides an advanced analysis of systems of ordinary differential equations (ODEs) using the trace-determinant plane as a central framework. The trace-determinant plane, defined by the trace and determinant of the Jacobean matrix of a linear system, serves as a powerful geometric tool for understanding and classifying the qualitative behavior of dynamical systems.

The research begins with a detailed examination of the trace-determinant plane in the context of linear systems. By exploring the relationships between the trace (sum of Eigen-values) and determinant (product of Eigen-values) of the Jacobean matrix, the thesis elucidates the various types of equilibrium points and their stability characteristics. The trace-determinant plane is shown to map regions of distinct dynamic behaviors, including nodes, saddles, spirals, and centers, offering a comprehensive classification framework.

Building on this foundation, the thesis extends the analysis to nonlinear systems by employing linearization techniques. It demonstrates how the trace-determinant plane can be utilized to approximate the local dynamics around equilibrium points and provides insights into the global behavior of nonlinear systems. The impact of parameter variations on the trace-determinant plane is also investigated, revealing how bifurcations and stability changes can be visualized and interpreted.

A significant contribution of this thesis is the integration of computational methods with theoretical analysis. Algorithms for plotting the trace-determinant plane and analyzing system stability are developed and applied to a range of practical examples, including control systems, ecological models, and mechanical systems. These case studies illustrate the practical utility of the trace-determinant plane in both theoretical and applied contexts.

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CHAPTER 1

INTRODUCTION

Mathematical frameworks known as systems of ordinary differential equations (ODEs) are used to explain large dynamic systems using a system of connected differential equations. Comprehending these systems is essential for assessing and forecasting the behavior of complex systems that emerge in diverse scientific and technical fields. Many dependent variables and their rates of change in relation to an independent variable x, usually time t, are included in an ODE system. The dynamics of a two dimensional system can be written as follows:

$$\frac{dX}{dt}\text{=Ax,}$$

Where x= $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is a vector of dependent variables, and A is a 2x2 matrix representing the system's dynamics.

The linear relationships between the variables and their rates of change are represented A. Relevance to Dynamic System Modeling- Engineering and Control Systems: By simulating the behavior of electrical circuits, mechanical systems, and control systems, systems of ODEs can provide light on stability, responsiveness, and optimal design. Biology and Medicine: They facilitate predictions and interventions in biological systems by modeling disease transmission, population dynamics, and biochemical processes. Economics and Social Sciences: ODEs facilitate the analysis and forecasting of economic trends and social phenomena by aiding in the knowledge of economic growth, market dynamics, and social behavior. The fields of physics and environmental science aid in our understanding of natural and artificial systems by describing physical processes including fluid dynamics, heat transfer, and ecological interactions.

The Plane Determined by Trace: One important analytical technique for analyzing the behavior of linear systems of ODEs, especially those with two variables, is the trace-determinant plane. By concentrating on two essential matrix invariants, the trace and the determinant of the coefficient matrix A. this approach streamlines the analysis of system dynamics.
Trace:-The total of matrix A's diagonal members is its **trace.**

$$\text{Tr(A)}=a_{11} + a_{22}$$

The trace provides information about the average rate of change of the system and It influences the stability of the equilibrium points.

Determinant (det(A): The determinant of matrix A is given by:

$$\text{det(A)}=a_{11}a_{22} - a_{12}\, a_{21}\ .$$

The determinant measures the scaling factor of the area in the phase plane due to the system's dynamics.

It helps determine the nature of the equilibrium points. In the trace-determinant plane, these two quantities are plotted to analyze the stability and type of equilibrium points for the system. The plane allows for a classification of the system based on the Eigen-values of A. By analyzing the trace and determinant, one can classify the equilibrium points and their stability as follows;

1) **Stable Node**:

Trace: $Tr(A)<0$

Determinant: $det(A)>0$

Trajectories converge to the equilibrium point, indicating stability.

2) **Unstable Node**:

Trace: $Tr(A)>0$

Determinant: $det(A)>0$

Trajectories diverge from the equilibrium point, indicating instability.

3) **Saddle Point**:

Trace: $Tr(A)\neq 0$

Determinant: $det(A)<0$

Trajectories diverge along one axis and converge along another, indicating a saddle point.

4) **Center (or Spiral)**:

Trace: $Tr(A)=0$

Determinant: $det(A)>0$

Trajectories form closed orbits or spirals around the equilibrium point, indicating a center or spiral behavior.

The Eigen-values are purely imaginary, indicating that trajectories form closed orbits around the equilibrium point, which corresponds to a center or spiral (in this case, a center).

In applying the trace-determinant plane to analyze planar systems of ODEs provides a structured approach to understanding system dynamics. By combining analytical methods with phase portraits, one can derive a comprehensive understanding of stability and behavior. This approach not only facilitates theoretical insights but also enhances practical applications in diverse fields, from engineering design to ecological modeling. **The other one is Phase Portraits** is a plot that shows the trajectories of a dynamical system in the phase space. Each point in the phase space represents a possible state of the system, and the trajectories represent the evolution of the system's state over time. There components**;** are Trajectories**,** Equilibrium Points (Critical Points), and Direction Field (Vector Field)**.**

Planar Systems is a system of ODEs that involves two variables. Mathematically, it can be expressed as:

$$\begin{cases} \dfrac{dx}{dt} = \mathrm{f}(\mathrm{x}, \mathrm{y}) \\ \dfrac{dy}{dt} = \mathrm{g}(\mathrm{x}, \mathrm{y}) \end{cases}$$

Here, x and y are the state variables, and f and g are functions that define the system's dynamics.

Example of a Planar System;

Consider the following planar system:

$$\begin{cases} \dfrac{dx}{dt} = \mathrm{x} + 2\mathrm{y} \\ \dfrac{dy}{dt} = -\mathrm{x} - 2\mathrm{y} \end{cases}$$

This can be represented in matrix form as:

$$\frac{d}{dX}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Here, the matrix A is:

$$\mathrm{A} == \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix}$$

Phase Portraits for Planar Systems focusing on planar systems of ODEs and their phase portraits, you would typically explore the following aspects: such as System Definition and Matrix Representation ;

$$\frac{dX}{dt} = \mathrm{Ax}, \quad \text{where A is a } 2\times 2 \text{ matrix.}$$

Find the equilibrium points by setting;

$$\begin{cases} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = 0 \end{cases}, \text{ and solve the resulting equations.}$$

In a planar system, the mathematical formulation involves defining a system of first-order ODEs and representing it in matrix form. The matrix representation facilitates the analysis of system behavior through Eigen-values, eigenvectors, and stability criteria. By examining the matrix A, its trace, determinant, and Eigen-values, one can determine the nature and stability of equilibrium points and gain insights into the overall dynamics of the system.

Systems of ordinary differential equations are fundamental for modeling dynamic phenomena across various scientific and engineering disciplines. The trace-determinant plane offers a valuable analytical tool for understanding the qualitative behavior of these systems. For this research, integrating phase portrait analysis with the trace-determinant approach provides a robust framework for exploring system dynamics, stability, and real-world applications.

1.1 Objectives

1.1.1 General Objectives

- Explore the foundational principles and theoretical aspects of planar systems of ODEs, focusing on their solutions, stability, and qualitative behavior.
- Examine how trace determinants influence the dynamics of planar systems and their phase portraits.
- Formulate and refine techniques for constructing and analyzing phase portraits of planar systems, utilizing trace-determinants as a tool for understanding system behavior.
- Apply the developed theories and methods to real-world planar systems to demonstrate practical implications and validate theoretical results.

1.1.2 Specific Objectives

- ✓ Define trace determinants in the context of 2-dimensional systems of ODEs, derive their mathematical properties, and understand their impact on the system's dynamics.
- ✓ Investigate how trace determinants affect the nature of fixed points (equilibrium points) in planar systems and their stability properties, including node, saddle, and spiral types.
- ✓ Develop methods for constructing phase portraits of planar systems, including techniques for sketching trajectories and identifying qualitative features like limit cycles, equilibria, and invariant sets.
- ✓ Analyze how varying system parameters influence the phase portraits of planar ODE systems, including changes in stability, bifurcations, and the shape of trajectory patterns.
- ✓ Apply the developed techniques to specific examples of planar systems in fields such as biology (e.g., predator-prey models), engineering (e.g., electrical circuits), or economics (e.g., economic models).

CHAPTER 2:

PRELIMINARIES

An ordinary differential equation (ODEs) is mathematical equation that relates a function with its derivatives. In the context of ODEs, the term ''ordinary'' distinguishes these equation from partial differential equations(PDEs), which involves partial derivatives of functions with more than one independent variables.

More formally, ODE is an equation of the form:

$$F(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2},\ldots,\frac{d^ny}{dx^n})=0,$$

Where y=y(x) is the unknown function of the independent variable (x).

The order of the ODE is determined by the highest derivatives of the function (y) that appears in the equation..

Second-Order ODEs:-A second-order ODE involves the second derivative of the unknown function. It has the general form:-

$$\frac{d^2y}{dx^2} = f(x, y,\frac{dy}{dx})$$

A planar system of ordinary differential equations **(ODEs)** refers to a system consisting of two coupled first-order differential equations that describe the evolution of two variables over time. These systems are typically represented in the following form:

$$\frac{dX}{dt} = Ax \quad ,$$

Where, $X = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is the state vector of the system and A is a 2x2 matrix containing constants that determine the system's dynamics. In matrix form, the planar system can be written as:

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Where, $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, is the coefficient matrix.

Here are several examples illustrating different types of planar systems of **ODEs;**

(a) Stable Node;

Consider the planar system described by:

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \text{ for this system:}$$

The matrix A is $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$.

The Eigen-values of A are negative real numbers, indicating that trajectories converge to the the equilibrium point, classifying the equilibrium point as a stable node.

(b) Unstable Node

Consider the planar system described by:

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$, for this system:

The matrix A is $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

The Eigen-values of A are positive real numbers, indicating that trajectories diverge from the origin, classifying the equilibrium point as an unstable node.

(c) Saddle Point

Consider the planar system described by:

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$, for this system:

The matrix A is $\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$,for this system.

The Eigen-values of A have opposite signs, indicating that the equilibrium point is a saddle point where trajectories diverge along one direction and converge along another.

(d) Center

Consider the planar system described by:

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$, for this system:

The matrix A is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The Eigen-values of A are purely imaginary, indicating that trajectories form closed orbits around the origin, classifying the equilibrium point as a center.

Planar systems, characterized by their two-dimensional state space, offer a simplified framework for analyzing dynamical systems compared to higher-dimensional systems. This simplicity makes them particularly useful for theoretical understanding and practical applications .These are; Two-Dimensional State Space and Simplified Analysis.

A planar system is described by two first-order ordinary differential equations (ODEs) involving two state variables. This system can be written in matrix form as:

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Where A is a 2 x 2 matrix of constants;

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The state vector $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ represents the system's configuration at time t, and A determines the system's dynamics.

A Simplified Analysis is which uses a Matrix Representation**.** The dynamics of planar systems are governed by a 2 x 2 matrix A, which simplifies several aspects of analysis:

Eigen-values and Eigenvectors: The Eigen-values of A are found by solving the characteristic polynomial:

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$$

Where Tr(A) is the trace and det(A) is the determinant of A. These Eigen-values determine the nature of equilibrium points (e.g., stable nodes, saddle points).

Planar Systems: A planar dynamical system is a system of differential equations involving two state variables. Mathematically, such a system can be described by the following set of ordinary differential equations (ODEs):

$$\frac{dx}{dt} = f(x,y) \qquad \text{and} \qquad \frac{dy}{dt} = g(x,y)$$

where x and y are the state variables representing the state of the system at any given time t, and f(x,y) and g (x,y) are functions that describe the rates of change of x and y, respectively. The system has exactly two state variables, x and y, which define the state of the system in a two-dimensional plane. These variables can represent quantities such as position and velocity, or other pairs of related variables.

Phase Portraits is a graphical representation that shows trajectories, equilibrium points, and vector fields in the state space. Phase portraits help visualize how the system evolves over time, the nature of equilibrium points, and the overall behavior of trajectories.

Equilibrium Points; Points in the state space where the system's state does not change,

$$\text{These; } \frac{dx}{dt} = 0 \qquad \text{and} \qquad \frac{dy}{dt} = 0 \ .$$

Analysis: The stability and type of equilibrium points (e.g., nodes, foci, and saddle points) can be determined by linearizing the system around these points and analyzing the Jacobean matrix.

Linear and Nonlinear Systems:

Linear Systems:-Linear systems of differential equations are among the most fundamental types of dynamical systems. A general planar linear system can be written in matrix form as:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{where} \quad \frac{dX}{dt} = a_{11}x + a_{12}y$$

Where, $x = \begin{pmatrix} x \\ y \end{pmatrix}$ is the state vector and A is a 2×2 constant matrix. The matrix A determines the behavior of the system, with its Eigen-values and eigenvectors playing a key role in shaping the dynamics.

Linear Systems: Systems where f(x,y) and g(x,y) are linear functions of x and y,

$$\frac{dx}{dt} = ax + by \qquad \frac{dy}{dt} = cx + dy.$$

These systems can be analyzed using matrix methods and often exhibit simpler behavior that can be easily visualized in phase portraits.

Nonlinear Systems: Systems where f(x, y) and g(x,y) are nonlinear functions.

For example:

$$\frac{dx}{dt} = x - x^3 - y,$$

$$\frac{dy}{dt} = x + y - y^3$$

Nonlinear systems can exhibit more complex behaviors such as limit cycles or chaotic dynamics.

The phase portrait of a nonlinear system is typically more complex than that of a linear system. Key features include:

1. **Fixed Point**: A fixed point (or equilibrium point) is a point in the phase space where the system remains at rest if it starts there. Mathematically, if x^* is a fixed point, then $\dot{x}=0$ at x^*. In the phase portrait, this fixed point might be represented as a point where trajectories either converge to or diverge from.

2. Limit Cycle: A limit cycle is a closed trajectory in the phase space that is periodic. Trajectories that start near the limit cycle will tend to orbit around it, either approaching it or diverging away depending on their initial conditions.

Example: Consider the Van der Pol oscillator:

$$\dot{x}=y$$

$$\dot{y}=\mu\,(1-x^2)\,y-x$$

For $\mu>0$, the system exhibits a limit cycle. In the phase portrait, you'll see closed orbits around the fixed point. The trajectory spirals into or away from the closed orbit depending on whether μ is positive or negative.

3. Bifurcation: A bifurcation occurs when a small change in the parameters of a system causes a qualitative change in its behavior. The system's phase portrait changes, often leading to the creation or destruction of fixed points or limit cycles.

Example: Consider the bifurcation in the following system:

$$\dot{x}=\mu-x^2$$

$$\dot{y}=y$$

The parameter μ affects the behavior.

- For $\mu<0$, there are no fixed points for x, and the system exhibits behavior driven primarily by y.
- For $\mu>0$, there are two fixed points at $x=\pm\sqrt{\mu}$.

As μ crosses zero, a bifurcation occurs. In the phase portrait, this can be seen as the creation or destruction of fixed points.

4. Chaotic Behavior: Chaotic behavior in a system refers to dynamics that are highly sensitive to initial conditions and exhibit a complex, a periodic behavior. This means that trajectories never repeat and can be very irregular.

Example: The Lorenz system is a classic example of chaos:

$$\dot{x}=\sigma\,(y-x)$$

$$\dot{y}=x(\rho-z)-y$$

$$\dot{z}=xy-\beta z$$

For certain parameter values (σ, ρ, β), the Lorenz system exhibits chaotic behavior. In the phase portrait, this appears as a complex, non-repeating pattern of trajectories. The system's sensitivity to initial conditions means small differences in starting points lead to vastly different trajectories..

Linearization and Stability Analysis: A common technique in analyzing nonlinear systems is to linearize the system around its fixed points. This involves approximating the system by a linear one using the Jacobean matrix J evaluated at the fixed point x*

$$J=\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\ , \text{ at a fixed point } x^*,$$

is the Eigen-values of the Jacobean matrix.

The study of phase portraits serves as a bridge between the simple yet powerful theory of linear systems and the intricate, often surprising behavior of nonlinear systems. It highlights how small changes in system structure can lead to significant differences in system behavior, making it an indispensable tool in the analysis of dynamical systems. This introduction sets the stage for a detailed exploration of phase portraits and their implications in the study of dynamical systems, emphasizing the transition from linear to nonlinear systems.

Homogeneous Planar Systems of ODEs

Homogeneous Second-Order ODE:

Form:

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

A Solution involves finding the characteristic equation, and the general solution depends on the roots of the characteristic equation (real, repeated, or complex).the characteristic equation obtained by assuming a solution of the form; y=e^{rx} , Leading to $r^2+pr+q=0$. solving the quadratic equation yields root ($r_{1,}\, r_2$), then

The general solution based on the nature of the roots.

Distinct real roots: - y(x) =$c_1.e^{xr_1}+c_2.e^{xr_2}$

Repeated real roots: - y(x) =$(c_1+xc_{2)}e^{xr}$

Complex real roots:- $y(x)=e^{\alpha x}(c_1.\cos(\beta x)+c_2.\sin(\beta x))$

Non-Homogeneous Second-Order ODE:

These ODEs have the Form: - $\frac{d^2y}{dx^2}$ + P(x) $\frac{dy}{dx}$ + Q(x) y= R(x)

The general solution is the sum of the general solution of the corresponding homogeneous equation and a particular solution of the non-homogeneous equation.

Thus are; - The complementary solution y_c(x) and particular solution y_p(x).

Then, y(x) =y_c(x) +y_p(x).

Special Forms:

Euler-Cauchy Equation: is a type of linear differential equation with variable coefficients.

It generally has form:-

$$x^n y^{(n)} + a_{n-1}x^{n-1}y^{n-1} + \dots + a_1 x y' + a_0 y = 0.$$ And

The simplest case is the second-order Euler –Cauchy equation;

$$x^2 \frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + by = 0.$$

To solve such equation of Euler –Cauchy equation use,

y(x) =x^r.then we can find roots.

Harmonic Oscillator:

$\frac{d^2x}{dt^2} + {\omega_0}^2 x = 0$ where ω_0 is angular frequency of the oscillator.

The general solution to the differential equation is

$$X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t).$$

Alternatively, it can be written in the form;

$$X(t) = C.\cos(\omega_0 t + \phi)$$

Where C, is the amplitude and ϕ is the phase of the oscillation.

Examples: Simple harmonic oscillator:

$m\frac{d^2x}{dt^2} + {\omega_0}^2 kx = 0$, where m is mass and k is spring constant.

Solution: y= A cos (ω_0)+ B sin (ω_0).

Damped harmonic oscillator:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} kx=0, \text{ where b is damping coefficients.}$$

A homogeneous planar system of ordinary differential equations (ODEs) refers to a system where the rate of change of the state vector x (t) is directly proportional to the matrix A and the state vector itself. In mathematical terms, the system is described by:

$$\frac{dx}{dt} = Ax$$

Where, x(t) is the state vector, $x(t)= \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$,

A is a 2 x 2 constant matrix that defines the dynamics of the system.

This form is called **homogeneous** because the system does not include any explicit functions of t or additional external inputs; the change in x(t) depends only on the state vector and the matrix A

Matrix Representation of matrix A in the homogeneous system has the general form:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The system equations can be expanded as:

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

This gives two coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = a_{11}x + a_{12}y \\ \dfrac{dy}{dt} = a_{21}x + a_{22}\, y \end{cases}$$

To solve the homogeneous system, we often use the following steps:

a) Find Eigen-values and Eigenvectors

Solve the characteristic polynomial:

$$de\ (A-\lambda\ I)=0$$

Where, I is the identity matrix. The characteristic polynomial for A is:

$$\lambda^2 - Tr(A)\lambda + \det(A) = 0.$$

Where $Tr(A)=a_{11} + a_{22}$ is the trace and

$\det(A)= a_{11}a_{22} - a_{12}a_{21}$ is the determinant of A.

Find Eigenvectors: For each Eigen-value λ, solve:

$$(A-\lambda I)\ v=0$$

Where, v is the eigenvector corresponding to λ.

b) Construct the General Solution

If the Eigen-values λ_1 and λ_2 are real and distinct, the general solution is:

$$X(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t},$$

Where v_1 $_{and}$ v_2 the eigenvectors are associated with λ_1 $_{and}$ λ_2 , respectively, and c_1 $_{and}$ c_2 are constants determined by initial conditions.

Repeated Real Eigen-values: If there is a repeated real Eigen-value, say λ, the solution involves:

$$X(t) = (c_1 + c_2 t)\, v_1\, e^{\lambda t},$$

Where v_1 are the eigenvector and c_1 $_{and}$ c_2 are constants.

Complex Eigen-values: If the Eigen-values are complex, say $\lambda = \alpha \pm i\beta$, the general solution is:

$X(t) = e^{\alpha t}(c_1 \cos(\beta t)\, v_1 + c_2 \sin(\beta t)\, v_2)$ Where, v_1 $_{and}$ v_2 are complex vectors associated with the real and imaginary parts of the Eigen- Values.

Homogeneous planar systems of ODEs are characterized by their simplicity and fundamental importance in modeling and analysis. The system is described by;

$\frac{dx}{dt}$=Ax, where A is a 2 x2 matrix.

The analysis involves finding Eigen-values and eigenvectors of A, which determine the behavior and stability of the system.

Role of the Jacobean Matrix

The Jacobean matrix is a fundamental concept when dealing with dynamical systems, particularly for understanding the local behavior of the system near equilibrium points.The Jacobean matrix J of a system is the matrix of partial derivatives of the system's equations with respect to its variables. For a planar system, where the system is described by:

$$\frac{dx}{dt} = \begin{pmatrix} f_{1\,(x,y)} \\ f_{2(x,y)} \end{pmatrix}$$

The Jacobean matrix J is defined as:

$$J = \begin{pmatrix} \frac{\vartheta f_1}{\vartheta x} & \frac{\vartheta f_1}{\vartheta y} \\ \frac{\vartheta f_2}{\vartheta x} & \frac{\vartheta f_2}{\vartheta y} \end{pmatrix}$$

b. Application to Linear Systems:

For a linear planar system;

$\frac{dx}{dt}$ =Ax, the Jacobean matrix J at any point x is simply the matrix A itself.

Thus: J=A

This is because the partial derivatives of the linear functions with respect to x and y is constants, and the Jacobean is constant throughout the state space.

CHAPTER 3

Analyzing System Of Ordinary Differential Equations; Eigen value-eigenvector, Phase Portraits And The Trace-Determinant Plane

Analytical methods for planar systems refer to techniques used to find exact solutions or properties of a system of two ordinary differential equations (ODEs) that describe the evolution of a system in a two-dimensional phase plane. These methods involve solving the system of equations without resorting to numerical approximations. Commonly we focus on linear and nonlinear systems of analytical methods for planar systems.

3.1. Eigen values and Eigenvectors Analysis

The Eigen values and Eigen-vectors method is a fundamental analytical tool for solving linear systems of differential equations. To understand the behavior of linear systems, we analyze the Eigen-values and eigenvectors of matrix (A). The Eigen-values determine the nature of the equilibrium points (nodes, saddles, spirals, centers), while the eigenvectors give the directions along which the system evolves.

Eigen value (λ):- are scalars value associated with a square matrix that satisfy the equation

AV=λV, where A is the matrix; V is non-zero vector and λ is the Eigen-value.

To find the Eigen-value of matrix A , use by solving characteristics polynomial

det(A - $\lambda I) = 0$.where I is identity matrix of the same as A.

So, the Eigen-value determines behavior solution to the linear systems.

Thus, describe the rates of growth or decay (real part) and oscillatory behavior (imaginary part) of the solutions.

Eigenvectors: Is non-zero vector that satisfies the equation;

AV=λV, for some Eigen valueλ.

To find eigenvector corresponding to an Eigen-value λ, we solve the homogeneous system of equations;

(A-$\lambda I)V = 0$, for non-zero V.

4. Example

Consider the planar system:

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \text{ for this system the matrix A is } \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}.$$

Find Eigen-values: Compute the characteristic polynomial:

$$\det(A-\lambda I)=\det\begin{pmatrix} 1-\lambda & 1 \\ 3 & 3-\lambda \end{pmatrix}=0$$

$$(1-\lambda)(3-\lambda)-(3.1)=\lambda^2-4\lambda$$

$$\lambda(\lambda-4)=0, \lambda_1=0 \text{ and } \lambda_2=4.$$

Find Eigenvectors: Solve (A−λI) v=0 for each Eigen-value to get eigenvectors.

3.1.1 Real distinct Eigen-values:

Consider $X'=AX$ and suppose that A has two real Eigen-value, $\lambda_{1<}\lambda_2$.

Assuming for the moment that $\lambda_i \neq 0$, there are three cases to consider.

1) Opposite sign; $\lambda_{1<} 0 \ and \ \lambda_2 > 0$
2) Both are negative; $\lambda_{1<}\lambda_2 < 0$
3) Both are positive; $\lambda_{1>}\lambda_2 > 0$. I give specific example for each case.

Example; 1(**Saddle Point)**: Occurs when the Eigen-values are real and have opposite sign.

The equilibrium point is unstable,

Some trajectories are attracted to the point along one direction,

While, others are repelled in another direction.

Example 1:

Consider the matrix: A = $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$

Trace: Tr(A)=2+ 1= 3

Determinant: det(A)=(2.1)−(O.1)= 2

To adjust for the saddle point, let’s use: A= $\begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix}$

Trace: Tr(A)=1+(−3)=−2

Determinant: det(A)=(1⋅−3)−(2⋅−1)=−3+2=−1

The Eigen-values are:

$$\det\begin{pmatrix} 1-\lambda & 2 \\ -1 & -3-\lambda \end{pmatrix}$$

$$(1-\lambda)(-3-\lambda)-(-1)(2) = 0$$

$$\lambda_1 = -1+\sqrt{2} \ \text{ and } \ \lambda_2 = -1-\sqrt{2} \ .$$

The Eigen-values have opposite signs, indicating a saddle point where trajectories diverge along one axis and converge along another.

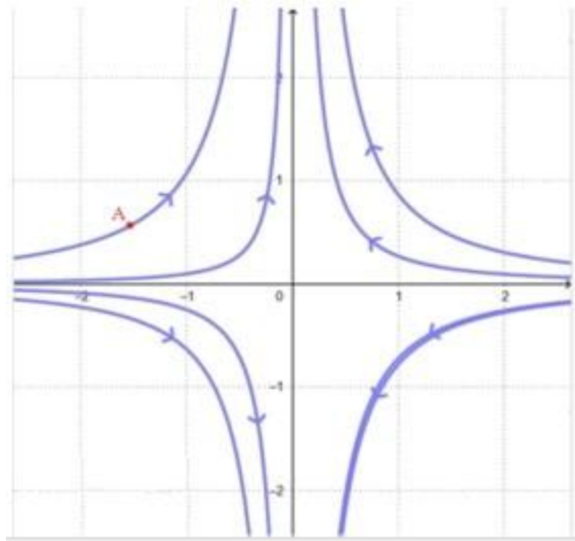

Figure 3.1 saddle phase portrait for x'=-x and y' =y.

Example 2; both are positive; $\lambda_{1>} \lambda_2 > 0$(**Node)**: Occurs when both Eigen values are real and have the same sign.

If both are negative; - The node is stable (attracting)

The equilibrium point is stable.

All nearby trajectories are attracted to the equilibrium point.

If both are positive;-The **node** is unstable (repelling).

The equilibrium point is unstable.

All nearby trajectories move away from the equilibrium point.

Matrix A, has both positive

$$A=\begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

Characteristic Polynomial:

$$\det(A\ \boldsymbol{\lambda I})=\left(\begin{pmatrix} \mathbf{2-\lambda} & \mathbf{1} \\ \mathbf{0} & \mathbf{1-\lambda} \end{pmatrix}\right)$$

$$=(\mathbf{2-\lambda})(\mathbf{1-\lambda})\text{-}(0)(1) = \boldsymbol{\lambda}^2\text{-}3\boldsymbol{\lambda} +2 = 0$$

So, the Eigen -values, $\boldsymbol{\lambda_1}$ =1 and $\boldsymbol{\lambda_2}$ = 2, both are positive real number.

Where I is the identity matrix and $\boldsymbol{\lambda}$ represents the Eigen-values.

Eigenvectors:

For, λ_1=1:

$$\det(A\text{-}\ \lambda_1\mathbf{I}) =\begin{pmatrix} \mathbf{2-1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1-1} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}= \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\left(\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{v_1} + \boldsymbol{v_2} = 0$$

$$\boldsymbol{v_1} = -\boldsymbol{v_2}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -v_1 \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \mathbf{-1} \end{pmatrix}.$$

$$\boldsymbol{v_1} = \begin{pmatrix} \mathbf{1} \\ \mathbf{-1} \end{pmatrix}.$$

2. For $\lambda_2=2$:

$$\det(A - \lambda_1 I) = \begin{pmatrix} \mathbf{2-2} & \mathbf{1} \\ \mathbf{0} & \mathbf{1-2} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\begin{cases} \mathbf{0}.\boldsymbol{v_1} + \mathbf{v_2} = \mathbf{0} \\ \mathbf{0}.\mathbf{v_1} - \mathbf{v_{2=0}} \end{cases}$$

Which implies, that; $-\boldsymbol{v_2}=0$

This gives us: $v_2=0$

Thus, an eigenvector for $\lambda_2=2$ is $v_2=\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}$,

The general solution $x(t)= e^{\alpha t}(c_1 v_1 + c_2\, v_2)$

$$= c_1 e^{\lambda_1} t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{\lambda_2} t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example 3:

Consider the matrix: $A = \begin{pmatrix} -3 & 1 \\ 0 & -1 \end{pmatrix}$

Trace: $\mathrm{Tr}(A) = (-3)+(-1) = -4$

Determinant: $\det(A)=(-3)(-1)-(1\cdot 0)=3$.

Analysis: The Eigen-values are found by solving;

$$\det(A-\lambda I)=0$$

$$\det\begin{pmatrix} -3-\lambda & 1 \\ 0 & -1-\lambda \end{pmatrix}$$

$$(-3-\lambda)(-1-\lambda) - (1.0) = 0$$

$$\lambda_1=-3 \quad \text{and} \quad \lambda_2=-1$$

Both Eigen-values are negative, so the system is a stable node.

Trajectories converge to the equilibrium point.

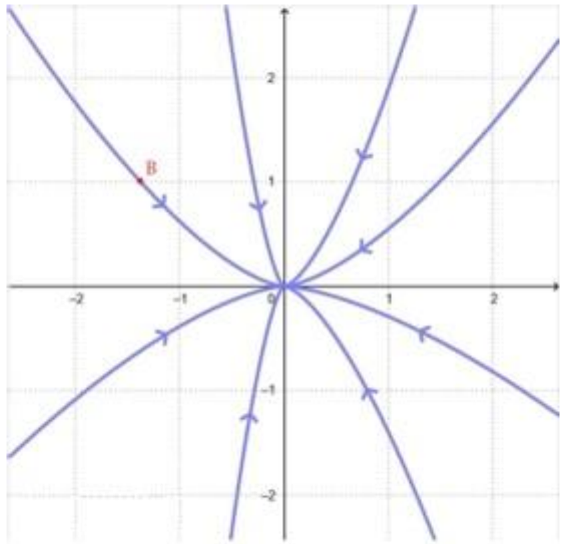

Figure 2: Phase portrait of nodal sink, where $\lambda_1 < \lambda_2 < 0$

Example 2:

Consider the matrix: A = $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$

Trace: Tr(A)=2+1=3

Determinant: det(A)=(2·1)−(1·0) =2

Analysis: The Eigen-values are found by solving;

det(A−λI)=0;

$$\det\begin{pmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix}$$

$$(2-\lambda)(1-\lambda)-(1)(0)=(2-\lambda)(1-\lambda)=0$$

$$\lambda_1 = 2 \text{ and } \lambda_2 = 1$$

Both Eigen-values are positive, so the system is an unstable node. Trajectories diverge from the equilibrium point.

3.1.2 Complex Eigen-values

The systems exhibits spiral behavior, either as a spiral sink (stable) or spiral source (unstable).

Spiral: occurs when Eigen-values are complex,

When > 0 (positive real part) which is called Unstable Spiral:-

The equilibrium point is unstable.

Trajectories spiral outwards from the point.

When < 0 (negative real part): which is called Stable spiral:

The equilibrium point is stable.

Trajectories spiral inwards toward the point.

Center: : Occurs when Eigen-values are purely imaginary, leading to closed orbits around the equilibrium point.

When ($\lambda = 0$), (purely imaginary Eigen-values)

- The equilibrium point has neutral stability.

- Trajectories form closed orbits around the equilibrium point

Example:

Consider the matrix: A= $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Trace: Tr(A)=0+0=0

Determinant: det(A)=(0·0)−(1·−1)=1

The Eigen-values are:

$$\det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = (-\lambda)(-\lambda) - (-1).(1) = \lambda^2+1$$

$$\lambda = \pm\, i$$

The figure below shows that spiral sink, spiral source and center on on xy plane.

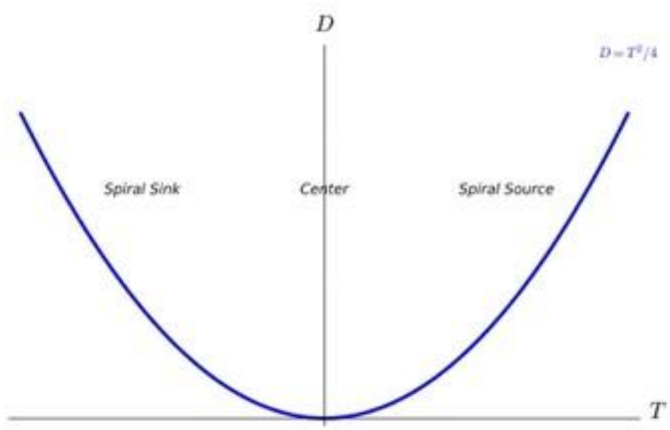

Figure 3: two complex Eigen-value case.

3.1.3 Real repeated Eigen-values

This case can result in a degenerate node, which may stable or unstable depending on the Eigen-values.

EXERCISE:- 1.suppose that a matrix A has repeated real Eigen-values λ and there exists a pair of linearly independent eigenvectors associated to A. prove that A=$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

Solutions: - to prove that a matrix A with repeated real Eigen-values λ and a pair of linearly independent eigenvectors is of the form A=$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

Given,

A matrix A which has 2x2 matrix.

Matrix A has repeated real Eigen-values λ

There exists a pair of linearly independent eigenvectors associated with A.

PROOF: - Eigen-values and eigenvectors definition.

Let A be a 2x2 matrix. Since A has a repeated Eigen-valuesλ, it means that the characteristics equation of A is: det(A- λI)=0.

The characteristics polynomials; det(A- λI)=(λ_1- λ).(λ_2- λ)=0

Given that $\lambda_1 = \lambda_2 = \lambda$, the char.polynomials is (λ_1- λ)^2 =0

So, the Eigen-values λ have multiplicity 2.

2) Linearly independent eigenvectors,

Since there exists a pair of linearly independent eigenvectors v_1 and v_2, the matrix A is diagonalizable. This means that there is a matrix P composed of the linearly independent eigenvectors v_1 and v_2 such that:

$$A=PAP^{-1}$$

Where A is a diagonal matrix of the Eigen-values of A, i.e.,

$$A=\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} .$$

Step 2: Since A has repeated Eigen-values $\boldsymbol{\lambda}$ and two linearly independent eigenvectors, the diagonal matrix A must have $\boldsymbol{\lambda}$ on the diagonal for both entries:

$$A=\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

Thus, the matrix A can be written as:

$$A=P\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} P^{-1} .$$

Step 3: Because A is a scalar multiple of the identity matrix (i.e., A=λI), any similarity transformation with P results in:

$$A= P(\lambda I)P^{-1} = \lambda PP^{-1} = \lambda I .$$

Thus, the matrix A must be of the form: $A= \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

Finally the matrix A is of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ if it has repeated real eigenvalues and a pair of linearly independent eigenvectors.

EXERCISE: - 2.find the general solution of the following harmonic oscillator equations:

$$x'' + x' + x=0$$

$$x'' + 2x' + x=0$$

Solution for: - (a) $x'' + x' + x=0$

The second-order linear homogeneous differential equation with constant coefficients;

Characteristics equation:

Assume a solution of the form x (t) =e^{rt}, where r is constant.

$$\Longrightarrow r^2 + r + 1 = 0$$

Solve the characteristics, by using quadratic equation.

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

From our equation a=1 ,b=1 , c=1 then

$$r = \frac{-1 \pm \sqrt{1^2-4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{1-4}}{2}$$

$r = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}}{2}$i, the roots are complex number.

$r_1 = \frac{-1+\sqrt{3}}{2}$i and $r_2 = \frac{-1-\sqrt{3}}{2}$i

General solution .since the roots are complex, the general solution is:

$$X(t) = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Where $\alpha = Re(r) = \frac{-1}{2}$ and $\beta = Im(r) = \frac{\sqrt{3}}{2}$

Therefore, the general solution is;

$$X(t) = e^{-t/2}(c_1 \cos(\frac{\sqrt{3}}{2} t) + c_2 \sin(\frac{\sqrt{3}}{2} t))$$

Where, c_1 and c_2 are arbitrary constants.

(b) x'' + 2x' + x=0, is the second-order linear homogeneous differential equation with constant coefficients. To solve it, follow these steps:-

Assume a solution of the form x (t) =e^{rt}, where r is constant.

$$\Longrightarrow r^2 + 2r + 1 = 0$$

Solve the characteristics, by using quadratic equation.

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

From our equation a=1 ,b=2 , c=1 then

$$r = \frac{-2 \pm \sqrt{(-2)^2-4(1)(1)}}{2(1)} = \frac{-2 \pm \sqrt{4-4}}{2} = \frac{-2 \pm 0}{2}$$

r =-1

$\Longrightarrow$, $r_1 = r_2$= -1 (repeated roots).

General solution; since the roots are real number then, the general solution is:

x(t) = (c_1 + $c_2 t$)) e^{rt}, where r=-1

x(t) = (c_1 + $c_2 t$)) e^{-t}.

Therefore, the general solution is;

$$x(t) = (c_1 + c_2 t))\, e^{-t}$$

Where, c_1 and c_2 are arbitrary constants.

3.2 Analytical Methods for Nonlinear Systems

Nonlinear systems are more complex and can exhibit a wide variety of behaviors that are not possible in linear systems. The analysis of nonlinear systems often involves approximating them as linear systems near equilibrium points or employing other techniques to understand their global behavior.

3.2.1 Linearization (Near Equilibrium points)

Linearization involves approximating a nonlinear system by a linear one near an equilibrium point. This is done by computing the Jacobean matrix of the system at the equilibrium point.

Given a nonlinear system:

$$\frac{dx}{dt} = f(x), \text{ Where f(x) is a nonlinear function,}$$

We can approximate it near an equilibrium point x(t) by:

$$\frac{dx}{dt} \approx A(x-x^*),$$

Where, A is the Jacobean matrix of f evaluated at x*. The behavior near the equilibrium point can then be studied using the linear system A.

A system of nonlinear differential equations, typically expressed in the form;

$$\frac{dx}{dt} = f(x, y)$$

$$\frac{dy}{dt} = g(x, y)$$

Example 1:-consider a nonlinear system:

$$\frac{dx}{dt} = x - y - x^2$$

$$\frac{dy}{dt} = x + y - y^2$$

1. Find the equilibrium points:-

Set f(x, y) =0 and g(x, y) = 0.

Solving these gives equilibrium points, for instance, (0, 0).

2. Compute the Jacobean;

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 1-2x & -1 \\ 1 & 1-2y \end{bmatrix}$$

3. Evaluate at equilibrium (0, 0)

$$J(0,0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Find Eigen values:- the characteristics polynomial is given:

$$\det(J - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$$

$$\det\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = 0$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

Then $(1-\lambda)(1-\lambda) - 1(-1) = 0$

$\lambda^2 - 2\lambda + 2 = 0$, where a=1,b=-2 and c=2

By using quadratic formula;

$$X = \frac{-(b) \pm \sqrt{(b)^2 - 4ac}}{2a},$$

We get;

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}, \quad \lambda = \frac{2 \pm \sqrt{4-8}}{2}$$

$$\lambda = \frac{2 \pm \sqrt{-4}}{2}$$

$\lambda = \frac{2 \pm 2i}{2}$, thus λ_1=1+i and λ_2=1-i.

Therefore, our Eigen-values are complex. So, solution is unstable **Spiral**:

Since the real part of λ positive:

The equilibrium point is unstable.

Trajectories spiral outwards from the point.

3.2.2 Global Phase Portraits

A global phase portraits are visual representations used in dynamical systems theory to understand the behavior of differential equation or discrete system across their entire phase space. These portraits illustrate the trajectories of system state variables over time and provide insights into the systems stability, attractors and overall dynamics.

Example one on Global Phase Portraits:

Consider a simple 2D (two-dimensional) non-linear system;

$$\frac{dx}{dt} = y - x^3$$

$\frac{dy}{dt} = -x - y^3$, to analyze this system;

Find fixed point.

Set right hand sides to zero

$$\text{Implies} \quad \begin{cases} y - x^3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1) \\ -x - y^3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2) \end{cases}$$

From the first equation substituting, $y = x^3$, into (2)

$$\Longrightarrow -x - \left(x^{3)3}\right) = 0$$

$$\Longrightarrow -x - x^9 = 0$$

$$\Longrightarrow \quad x^9 + x = 0$$

Factoring, $x\,(x^8+1) = 0$

Yields x=0(the only real solution) and y=0. So the only fixed points is (0, 0).

To analyze stability first linearize the system around the fixed point by finding the Jacobean matrix.

$$J = \begin{bmatrix} \frac{\partial}{\partial x}(y - x^3) & \frac{\partial}{\partial y}(y - x^3) \\ \frac{\partial}{\partial x}(-x - y^3) & \frac{\partial}{\partial y}(-x - y^3) \end{bmatrix} = \begin{bmatrix} -3x^2 & 1 \\ -1 & -3y^2 \end{bmatrix}$$

At (0, 0);

$$J\,(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Then the Eigen-values of this Jacobean matrix is found by solving;

$$\Longrightarrow \quad det(J - \lambda I) = 0$$

$$\Longrightarrow \quad \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\Longrightarrow \quad \lambda^2 - 1 = 0$$

$$\Longrightarrow \quad \lambda^2 = 1$$

The Eigen-value are $\lambda = \pm 1$, indicating a saddle point at (0, 0). Such that $\lambda_1 = 1$ and $\lambda_2 = -1$

Which is the equilibrium point is unstable and the point is saddle point.

Example two on Global Phase Portraits:

Consider a system of two differential equations:

$$\frac{dx}{dt} = x - y - x(x^2 + y^2)$$

$$\frac{dy}{dt} = x + y - y(x^2 + y^2) \quad \text{to analyze this system}$$

Let's solve it through the global phase portraits approach.

Step 1: Find fixed point(equilibrium point)

Set right hand sides to zero

Implies $\begin{cases} x - y - x(x^2 + y^2) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1) \\ x + y - y(x^2 + y^2) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2) \end{cases}$

Solve these equation simultaneously, for x=0 and y=0.

$$\Rightarrow \begin{cases} 0 - 0 - 0 = 0 \\ 0 + 0 - 0 = 0 \end{cases}$$

Thus, (0, 0) is an equilibrium point.

To find other equilibrium points, analyze the systems further or use numerical methods.

Here, we have focus on (0,0) for simplicity.

Linearize the systems around the equilibrium point. By compute the Jacobean matrix J, at the equilibrium point (0, 0);

$$J = \begin{bmatrix} \frac{\partial}{\partial x}(x - y - x(x^2 + y^2) & \frac{\partial}{\partial y}(x - y - x(x^2 + y^2) \\ \frac{\partial}{\partial x}(x + y - y(x^2 + y^2) & \frac{\partial}{\partial y}(x + y - y(x^2 + y^2) \end{bmatrix}$$

By compute the partial derivatives;

$$J = \begin{bmatrix} 1 - (x^2 + y^2) - 2x^2 & -1 - 2xy \\ 1 - 2xy & 1 - (x^2 + y^2) - 2y^2 \end{bmatrix}$$

At (0, 0);

$$J = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Find the Eigen-values of the Jacobean matrix;

$$\det(J - \lambda I) = \det\begin{pmatrix} 1 - \lambda & -1 \\ 1 & 1 - \lambda \end{pmatrix}$$

$$\Rightarrow \quad (1 - \lambda)^2 - (-1)(1) = \lambda^2 - 2\lambda + 2 = 0$$

Solving by using quadratic equation, we get;

$$\lambda = 1 \pm i$$

$$\Rightarrow \lambda_1 = 1 + i \quad and \quad \lambda_2 = 1 - i.$$

The Eigen-values ($\lambda = 1 \pm i$) has positive real parts, indicating that (0, 0) is an unstable spiral point.

3.2.3 Nonlinear Dynamics and Chaos

Nonlinear systems unlike linear systems, where superposition holds and solution can be easily analyzed, nonlinear systems are governed by equations where the variables are raised to powers other than one or multiplied together.

Chaos: - chaos refers to the sensitive dependence on initial conditions, often illustrated by the ''butterfly effect,'' where small changes in initial conditions can lead to vastly different outcomes, making long – term predictions difficult. Chaotic systems are deterministic but appear random due to their complexity. This can lead to complex, non-repeating phase portraits that require numerical methods to analyze and visualize.

Solved Example: the logistic Map

The logistic map is a classic example of a nonlinear dynamical system, defined by the equation:

$$x_{n+1} = rx_n(1 - x_n)$$

Where, $x_n$is the population at generation n and r is a parameter that represents the growth rate.

Example ; Let us consider the logistic map with r=3.5 and an initial population$x_0 = 0.5$. We will compute a few iterations to observe the behavior of the system.

Iteration 1;

$$x_1 = 3.5.0.5\ (1 - 0.5)$$

$$= \mathbf{0.875.}$$

Iteration 2:

$$x_2 = 3.5.0.875\ (1\text{-}0.875)$$

$$=3.5.0.875.0.125$$

$$\mathbf{=0.3828125}$$

Iteration 3:

$$x_3 =3.5.0.3828125.(1\text{-}0.3828125)$$

$$=3.5.0.3828125.0.6171875 \approx \mathbf{0.825.}$$

Iteration 4:

$$x_4 = 3.5.0.825.(1\text{-}0.825)$$

$$= 3.5.0.825.0.175= 0.501.$$

As we iterate then logistic map, we can observe that the values of x_n, fluctuate and do not settle down to a single value, indicating the potential for chaotic behavior. Specially, as r approaches values greater than 3.57.

3.2.4 Numerical Methods for Phase Portraits of Planar Systems

In this chapter, we focus on the numerical methods used to analyze and construct phase portraits of planar systems. These methods are particularly useful for systems where analytical solutions are difficult or impossible to obtain, especially in nonlinear cases. We will explore numerical integration techniques, methods for constructing phase portraits, and present case studies that illustrate these concepts in practice. For instance, numerical integration techniques are essential for solving differential equations that define the trajectories in a phase portrait. These methods approximate the solutions over discrete time steps, enabling the study of the system's dynamics.

(a) Euler's Method

Euler's Method is the simplest numerical integration technique. It uses a straightforward iterative approach to approximate the solution:

$$x_{n+1} = x_n + \Delta h. f(x_n) \qquad \text{and}$$

$$y_n = y_n - \Delta h. f(y_n)$$

Where x_n and y_n is the state vector at time h_n, and Δ h is the time step.

To apply Euler's methods in phase portraits analysis: the following step are help full.

Choose a time step h: select a small h to balance accuracy and computational efficiency.

Initialize conditions: set up the initial conditions $(x_{0\,,}y_0)$ for your trajectories.

Use Euler's method to compute the trajectories by iterating the update rules.

Plot the computing points on the (x, y) plane to visualize the phase portraits.

Example; simple harmonic oscillator:

Consider a simple harmonic oscillator with equations:

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -x$$

For Euler methods;

Choose; h=0.01

Start with: $(x_{0\,,}y_0) = (1{,}0)$.

Compute trajectories: using update rules

$$x_{n+1} = x_n + \Delta h. f(y_n) \qquad \text{and}$$

$$y_{n+1} = y_n - \Delta h. f(x_n)$$

Iterate these updates to get the trajectory.

Step 0:

$$x_{0=1} \quad \text{and} \quad y_0 = 0.$$

Step 1:

$$x_1 = x_0 + \Delta h. f(y_0) = 1 + (0.01).(0) = 1$$

$$y_1 = y_0 + \Delta h. f(x_0) = 0-(0.01).(1) = -0.01.$$

Step 2:

$$x_2 = x_1 + \Delta h. f(y_1) = 1 + (0.01)(-0.01) = 1-0.0001 = 0.9999$$

$$y_2 = y_1 + \Delta h. f(x_1) = -0.01-(0.01)(1) = -0.02$$

Step 3: $x_3 = x_2 + \Delta h. f(y_2)$ = 0.9999 + (0.01).(-0.02 = 0.9999-0.0002 =0.9997

$y_3 = y_2 + \Delta h. f(x_2)$ = -0.02-(0.01).(0.9999)

= -0.02-0.00999 =-0.029999.

Final iteration: - to get the final values after N=1000, iterations .we continue applying the update rules iteratively.

Advantages: Easy to implement and computationally inexpensive.

Limitations: Low accuracy and stability issues, especially for stiff systems or when larger time steps are used.

(b) Runge-Kutta Methods

The Runge-Kutta methods, particularly the fourth-order Runge-Kutta (RK4) method, are more accurate than Euler's Method. The RK4 method calculates intermediate steps to achieve a higher order of accuracy:

$$x_{n+1} = x_n + \frac{1}{6}(k_1+2k_2+2k_3+k_4),$$

Where k_1, k_2, k_3, and k_4 are intermediate calculations based on the function f(x).

Example: consider the differential equation below;

$$\frac{dy}{dx} = t-y$$

With the initial condition; $y_0 = 1$.we want to approximate the value of y at t=0.2.using the RK4, method with a step size h=0.1.

RK4 method steps:

Initialize value: $t_0 = 0$, $y_0 = 1$, h=0.1.

RK4 iteration formula

Compute k_1, k_2 k_3 and k_4 as follows;

$$k_1 = h\,.f(t_n, y_n)$$

$$k_2 = h.\ f\,(t_n + \frac{h}{2}, y_{n+\frac{k_1}{2}})$$

$$k_3 = h.\ f\,(t_n + \frac{h}{2}, y_{n+\frac{k_2}{2}})$$

$$k_4 = h.\ f\,(t_{n+h}, y_n + k_3)$$

Update y_{n+1}:

$$y_{n+1}:\ y_n + \frac{1}{6}(k_1 + 2k_2\ + 2k_3\ + k_4)$$

Update t_{n+1}:

$$t_{n+1} = t_n\ + h$$

Solving the example;

First iteration:

Calculate k_1;

$$k_1 = h\,.f\,(t_0, y_0)$$
$$= 0.1(0-1)$$
$$=-0.1$$

Calculate k_2;

$$k_2 = \text{h. f}\,(t_n + \tfrac{h}{2}, y_{n+\frac{k_1}{2}})$$
$$= 0.1\ (0.05-(1-\tfrac{0.1}{2}))$$
$$= 0.1(0.05-0.95)$$
$$= 0.1(-0.9)$$
$$= -0.09$$

Calculate k_3;

$$k_3 = \text{h. f}\,(t_n + \tfrac{h}{2}, y_{n+\frac{k_2}{2}})$$
$$= 0.1(0.05-(1-\tfrac{0.09}{2}))$$
$$= 0.1(0.05-(1-0.045))$$
$$= 0.1(0.05-0.95)$$
$$= -0.0905$$

Calculate k_4;

$$k_4 = \text{h. f}\,(t_{0+h}, y_0 + k_3)$$
$$= 0.1(0.1-(1-0.0905))$$
$$= 0.1(0.1-0.9095)$$
$$= 0.1(-0.8095)$$
$$= -0.08095.$$

Update y_1;

$$y_1 = y_0 + \tfrac{1}{6}\,(k_1 + 2k_2 + 2k_3 + k_4$$
$$= 1 + \tfrac{1}{6}\,(-0.1 + 2(-0.09) + 2(-0.0905) - 0.08095$$
$$= 1 + \tfrac{1}{6}\,(-0.1-0.18-0.181-0.08095)$$
$$= 1 + \tfrac{1}{6}\,(-0.54195) \quad \approx 1 - 0.090325$$
$$\approx 0.909675.$$

Update t_1;

$$t_1 = t_0 + \text{h} = 0 + 0.1 = 0.1.$$

Second iteration:

Calculate k_1;

$$k_1 = \text{h.f}\,(t_1, y_1) = 0.1(0-1-0.909675)$$
$$= 0.1(-0.809675)$$

$$= -0.0809675$$

Calculatek_2;

$$k_2 = \text{h. f}(t_1 + \frac{h}{2}, y_{1+\frac{k_1}{2}})$$

$$= 0.1(0.15\text{-}(0.909675\text{-}\frac{0.0809675}{2}) \quad = 0.1.$$

Calculatek_3;

$$k_3 = \text{h. f}(t_1 + \frac{h}{2}, y_{1+\frac{k_2}{2}})$$

$$= 0.1(0.15\text{-}(0.909675\text{-}\frac{0.071916625}{2}))$$

$$= -0.063731171875$$

Updatey_2;

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$= 0.909675 + \frac{1}{6}$(-0.0809675+ 2(-0.071916625) + 2(-0.07236328125) – 0.063731171875)

$= 0.909675 + \frac{1}{6}$(-0.0809675- 0.14383325-0.1447265625-0.063731171875)

$=0.909675 + \frac{1}{6}$(-0.433258507375) $\approx$0.909675-0.07220975123

$\approx$ 0.83746524877.

Updatet_2:

$$t_2 = t_1 + \text{h} = 0.1 + 0.1$$

$$= 0.2.$$

After two iteration, the approximate value of y at t=0.2, is y $\approx$ 0.8375.

Advantages: High accuracy, suitable for a wide range of problems.

Limitations: More computationally intensive than Euler’s method, but the accuracy gain typically outweighs the cost.

3.3 The Trace-Determinant plane

3.3.1 Introduction to the Trace-Determinant Plane

In the context of a 2 x2 matrix A, which represents the system dynamics in planar systems, the trace and determinant are fundamental scalar quantities that provide insight into the system's behavior,

- **Trace** of a matrix A, denoted as Tr(A), is the sum of the diagonal elements:

  $$\text{Tr(A)}=a_{11}+a_{22}$$

  The trace reflects the sum of the Eigen-values of the matrix.

- **Determinant** of a matrix A, denoted as det(A), is computed as:

  $$\det(\text{A})= a_{11}a_{22} - a_{12}a_{21}$$

  The determinant reflects the product of the Eigen-values of the matrix.

Theorem: Stability Criteria by Trace-Determinant Plane

Consider a linear system given by $\frac{dX}{dt}$=Ax, where A is a 2×2 matrix. The stability of the equilibrium point at the origin can be determined using the trace Tr(A) and determinant Det(A) of A as follows:

1. **Asymptotic Stability**: If Tr(A)<0 and Det(A)>0, then the origin is asymptotically stable.
2. **Instability**: If Tr(A)>0 and Det(A)>0, then the origin is unstable.
3. **Saddle Point**: If Det(A)<0, then the origin is a saddle point and hence unstable.
4. **Degenerate Case**: If Det(A)=0, the system might have a degenerate equilibrium and requires further analysis.

Proof:

1) **Eigen-value of A;**

   To analysis the stability of the equilibrium point, we need to find the Eigen-values of the matrix A. for a 2x2 matrix A, the Eigen values λ are are determined by solving the characteristics polynomial:

   Det(A- λI) =0 , where I is the identity matrix,the characteristics polynomial is;

   $$\text{Det(A-}\,\lambda I) = det\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix},$$

   Expanding this determinant we get,

   $$\text{Det(A-}\,\lambda I)=(a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21}$$
   $$=\lambda^2\text{-}(a_{11}+a_{22})\lambda + (a_{11}a_{22}-a_{12}a_{21})$$
   $$= \lambda^2\text{-Tr(A)}\lambda + \text{Det(A)},$$

Where, Tr(A) $=a_{11}+a_{22}$ is the trace of A and Det(A) $=a_{11}a_{22}-a_{12}a_{21}$ is the determinant of A.

Then the Eigen-values $\lambda_{1,2} = \frac{Tr(A)\pm\sqrt{(Tr(A))^2-4Det(A)}}{2}$,

2) **Asymptotic stability**

To have an asymptotically stable equilibrium point, both Eigen values must have negative real parts.

- Real part of Eigen values: the real part of the Eigen-values $\lambda_{1,2}$ is,

$$\text{Re}\,(\lambda_{1,2}) = \frac{Tr(A)}{2}$$

Therefore, If Tr(A)<0, then Re $(\lambda_{1,2})$<0, meaning both Eigen-values have negative real parts.

- **Condition for Real Eigen-values**: For real Eigen-values, the discriminant

  $(Tr(A))^2 - 4Det(A)$ must be non-negative. If Det(A)>0, the discriminant is non-negative, and the Eigen-values are real.

  Hence, if Tr(A)<0 and Det(A)>0, the Eigen-values are real and negative, making the equilibrium point asymptotically stable.

3) **Instability**

- **Real Parts of Eigen values**: If Tr(A)>0, then:

$$\text{Re}\,(\lambda_{1,2}) = \frac{Tr(A)}{2} > 0$$

  Thus, both Eigen-values have positive real parts, leading to instability.

- **Condition for Real Eigen-values**: If Det(A)>0, the Eigen values are real and positive, confirming instability.

4) **Saddle Point**

- **Eigen values with Opposite Signs**: If Det(A)<0, the discriminant $(\text{Tr(A)})^2 - 4\text{Det(A)}$ is positive, leading to real Eigen-values with opposite signs. This creates a saddle point.
- **Stability**: A saddle point is always unstable because it has Eigen-values with opposite signs, meaning one direction is stable while the other is unstable.

5. **Degenerate Case**

- **Determinant Zero**: If Det(A)=0, the characteristic polynomial becomes:

  $\lambda^2 - \text{Tr(A)}\lambda = \lambda\,(\lambda - \text{Tr(A)})$ One Eigen-value is zero, and the other is Tr(A).The system's stability depends on the value of Tr(A), and further analysis is needed to determine stability.

  By analyzing the trace and determinant of the matrix A, we can determine the stability of the equilibrium point at the origin in a 2x2 linear system.

Significance of Trace and Determinant

- **Trace** provides information about the trace of the matrix which relates to the sum of the Eigen-values. It is associated with the divergence or convergence of trajectories in a linear system. Specifically:
    - If $Tr(A)<0$, the system generally has trajectories that converge to the equilibrium point.
    - If $Tr(A)>0$, the system generally has trajectories that diverge from the equilibrium point.
- **Determinant** provides information about the product of the Eigen-values and influences the nature of the equilibrium point:
    - If $\det(A)>0$, the Eigen-values are either both real with the same sign or complex with zero real part.
    - If $\det(A)<0$, the Eigen-values are real and have opposite signs.

3.3.2 Geometric Interpretation of the Trace-Determinant plane

The trace-determinant plane is a useful tool for visualizing the stability and dynamics of planar systems. The plane is defined by plotting the trace and determinant of the matrix A, and the position of a point on this plane helps classify the type of equilibrium and stability:

. **Stable System (NODE) (Both Eigen-values Negative)**

Condition: $Tr(A)<0$ and $\det(A)>0$

- A system is considered stable if all Eigen-values have negative real parts. For a 2x2 matrix, this typically means both Eigen-values are negative real numbers. The trajectories converge to the equilibrium point.

Example:

Consider the matrix:

$$A=\begin{pmatrix} -3 & 2 \\ -1 & -2 \end{pmatrix}$$

Trace and Determinant:

- Trace: $tr(A)=-3+(-2)=-5$
- Determinant: $\det(A)=(-3\times-2)-(2\times-1)=6+2=8$

Eigen-value Calculation:

The characteristic polynomial is:

$$\lambda^2-(Tr(A))\lambda+\det(A) = \lambda^2+5\lambda+8$$

Solving for Eigen-values:

$$\lambda = \frac{-5\pm\sqrt{5^2-4(1)(8)}}{2(1)}$$

$$\frac{-5\pm\sqrt{-7}}{2}$$

$$\lambda = \frac{-5}{2} \pm i\frac{\sqrt{7}}{2}$$

Here, the Eigen-values are:

$$\lambda_1 = \frac{-5}{2} + i\frac{\sqrt{7}}{2} \quad \text{and} \quad \lambda_2 = \frac{-5}{2} - i\frac{\sqrt{7}}{2}$$

Trace-Determinant Plane:

- **Point**: (8,−5)
- This point lies inside the ellipse (real part negative for complex Eigen-values).

Behavior: The Eigen-values are complex with a negative real part, indicating the system is stable but exhibits oscillatory behavior with decaying amplitude (spiral inward in the phase portrait).

Unstable System (Node:**) (Both Eigen-values Positive)**

- **Condition**: Tr(A)>0 and det(A)>0
- The trajectories diverge from the equilibrium point.
- A system is unstable if all Eigen-values have positive real parts. For a 2x2 matrix, this means both Eigen-values are positive real numbers.

Example:

Consider the matrix:

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

Trace and Determinant:

Trace**:** tr(A)=3+ 2 =5

Determinant: det(A) =(2.3)−(1.0)=6+0=6

Eigen-value Calculation:

The characteristic polynomial is:

$$\lambda^2\text{-(Tr(A))}\lambda + \det(A) = \lambda^2 - 5\lambda + 6$$

Solving for Eigen-values:

$$\lambda = \frac{5 \pm \sqrt{5^2 - 4(1)(6)}}{2(1)}$$

$$\frac{-5 \pm \sqrt{25 - 24}}{2}$$

$$\frac{-5 \pm 1}{2} = \frac{5 \pm 1}{2}$$

Here, the Eigen-values are:

$$\lambda_1 = 3 \quad \text{and} \quad \lambda_2 = 2$$

Trace-Determinant Plane:

Point: (6, 5)

This point is inside the ellipse (both Eigen-values are real and positive).

Behavior: both Eigen-values are positive, indicating that trajectories move away from the origin, leading to instability (source in the phase portrait).

Saddle Point (One Positive and One Negative Eigen-value)

Condition: Tr(A)≠0 and det(A)<0

Trajectories diverge along one direction and converge along another.

A system is a saddle point if it has one Eigen-value with a positive real part and one with a negative real part. This results in trajectories diverging along one axis and converging along another.

Consider the matrix:

$$A= \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

Trace and Determinant:

Trace: Tr(A)=1+(-1) = 0

Determinant: det(A) =(1.(-1))−(2.(-1))=-1+2=1

Eigen-value Calculation:

The characteristic polynomial is:

$$\lambda^2 - (\mathrm{Tr}(A))\lambda + \det(A) = \lambda^2 - 0.\lambda + 1$$

$$\lambda^2 + 1$$

Solving for Eigen-values:

$$\lambda = \pm i$$

Here, the Eigen-values are:

$$\lambda_1 = \mathrm{i} \quad \text{and} \qquad \lambda_2 = -i$$

Trace-Determinant Plane:

- **Point**: (1,0)
- This point is outside the elliptical region (indicating complex Eigen-values).

The Eigen values are purely imaginary, which means the system exhibits oscillatory behavior but is stable in terms of not growing exponentially. However, in this example, the actual Eigen-values are complex, so it would be more accurately described as exhibiting oscillations around an unstable saddle point in the phase portrait.

1. **Centers**:
   - **Centers or Spirals** are plotted where Tr(A)=0 and det(A)>0.

     The Eigen-values are purely imaginary.

The trajectories form closed orbits or spirals around the equilibrium point

Example;

Consider the matrix:

$$A=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Trace: $Tr(A)=0+0=0$

- **Determinant**: $\det(A)=(0\times0)-(-1\times1)=1$

On the Trace-Determinant Plane:

- **Position**: $Tr(A)=0$ and $\det(A)>0$
- **Interpretation**: This is a center, with trajectories forming closed orbits around the equilibrium point.

The trace-determinant plane provides a powerful way to quickly assess and classify the behavior of planar systems, facilitating a deeper understanding of their dynamics and stability.

Notice that;

- **Stable System**: Both Eigen values are negative, leading to trajectories converging to the equilibrium point.
- **Unstable System**: Both Eigen values are positive, leading to trajectories diverging from the equilibrium point.
- **Saddle Point**: One positive and one negative Eigen value, leading to trajectories diverging along some directions and converging along others.

By analyzing the trace and determinant of the matrix, we can predict and understand the system's stability and behavior. The trace-determinant plane provides a graphical method to visualize these characteristics and interpret the dynamics of the system based on the Eigen values.

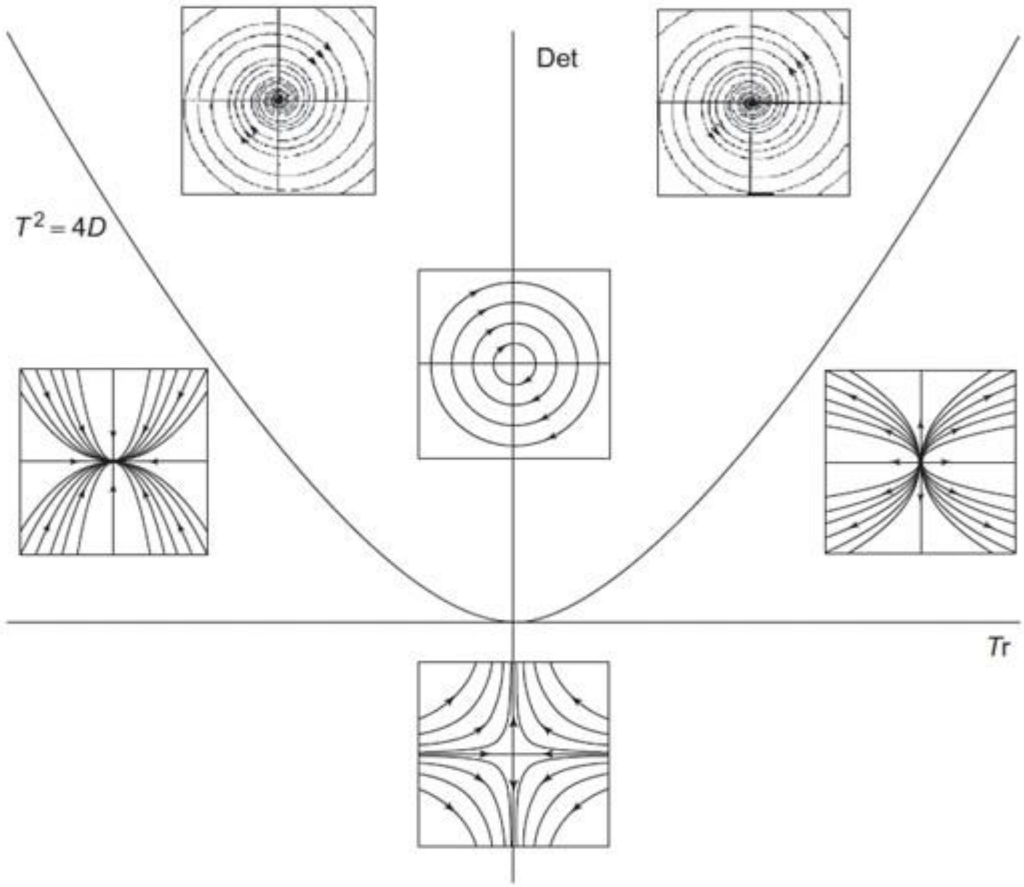

Figure 1: The Trace-Determinant Plane

CHAPTER 4

Applications of Systems of ODEs and the Trace-Determinant Plane

4.1 Control Theory and the Trace-Determinant Plane

In control theory, engineers use systems of ordinary differential equations (ODEs) to model the dynamics of various systems, such as mechanical, electrical, or aerospace systems. The primary goal is to design controllers that ensure the system remains stable and performs as desired under various conditions.

The dynamics of a linear time-invariant (LTI) system can be described by a system of ODEs of the form;

$$x' = Ax,$$

Where, x is the state vector and A is the system matrix. The matrix A encapsulates the system's dynamics, and its Eigen-values provide crucial information about the system's behavior.

The trace and determinant of the matrix A are important for understanding the stability and nature of the equilibrium points of the system.

- **Trace** Tr(A): The sum of the Eigen-values of A. It provides information about the growth rate of perturbations. A positive trace typically indicates that the equilibrium point is unstable.
- **Determinant** det (A): The product of the Eigen-values of A. It helps in understanding the nature of the Eigen-values (real or complex) and their impact on the system's stability.

Example: Inverted Pendulum;

- An inverted pendulum is a classic example used in control theory to demonstrate concepts of stability and control. The pendulum consists of a rod pivoting at one end with a base that can move horizontally.
- The goal is to control the base's movement to keep the pendulum balanced in the upright position.

Mathematical Model:

- The dynamics of the inverted pendulum can be represented by a system of ODEs derived from Newton's laws or Lagrangian mechanics. The state-space representation of the system can be written as;

  $$x' = Ax,$$

  Where x includes the pendulum's angle and angular velocity, and A is a matrix derived from the system's linearization around the equilibrium point.
- **Linearization:** Around the upright equilibrium position (where the pendulum is perfectly vertical), linearize the equations of motion to obtain the matrix A.
- **Trace and Determinant Calculation:** Compute the trace and determinant of A:
  - **Trace** Tr(A) determines the sum of the Eigen-values, which affects the system's response to perturbations.

- **Determinant** det(A) determines the product of the Eigen-values, which affects whether the system's Eigen-values are real or complex.

- **Trace-Determinant Plane Analysis:** Plot the trace and determinant on the trace-determinant plane to analyze the stability:
    - **Stable System:** If the trace is negative and the determinant is positive, the system has a stable equilibrium.
    - **Unstable System:** If the trace is positive, or the determinant is negative, the equilibrium is unstable.
- **Proportional-Derivative (PD) Controller:** To stabilize the inverted pendulum, engineers often design controllers (e.g., PD controllers) that adjust the base's movement based on the pendulum's angle and angular velocity.
    - By tuning the controller parameters, the trace and determinant of the closed-loop system matrix (which includes the effects of the controller) can be adjusted to achieve desired stability and performance.

The trace-determinant plane is a powerful tool in control theory for analyzing the stability of linear systems, including the inverted pendulum. By examining the trace and determinant of the system matrix A, engineers can gain insights into the stability and nature of equilibrium points, which is essential for designing effective controllers. The application of these concepts helps ensure that systems remain stable and perform optimally under various conditions.

4.2 Population Dynamics and the Trace-Determinant Plane

1. Ecological Models

- In ecology, systems of ODEs are frequently used to model interactions between different species. These models help ecologists understand how populations of species interact with each other and how these interactions affect their long-term behavior.
- Typical models include predator-prey dynamics, competition between species, and the spread of diseases within populations.

Example:

- The Lotka-Volterra model is a classic example that describes the dynamics of predator-prey interactions using a system of ODEs. The model provides a framework to study how populations of predators and prey evolve over time and how their interactions influence their stability.

Lotka-Volterra Predator-Prey Model

Model Description:

- The Lotka-Volterra model consists of a pair of first-order nonlinear differential equations:

$$\frac{dN}{dt} = \alpha N - \beta NP$$

$$\frac{dP}{dt} = \delta NP - \gamma P$$

Where;

- N(t) is the prey population
- P(t) is the predator population
- α is the prey birth rate
- β is the predation rate
- δ is the rate at which predators increase by consuming prey,
- γ is the predator death rate

Equilibrium Points:

- To analyze the model, find the equilibrium points by setting, $\frac{dN}{dt} = 0$ and $\frac{dP}{dt} = 0$;

$$\alpha N - \beta NP = 0$$

$$\delta NP - \gamma P = 0$$

Solving these equations yields equilibrium points where:

- $(N^*, P^*) = (0,0)$ — Extinction equilibrium.
- $(N^*, P^*) = \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$— Non-trivial equilibrium.

Trace-Determinant Analysis:

- **Trace and Determinant:**
  - **Trace** Tr(J) is the sum of the Eigen-values, calculated as Tr(J)=−(α+ γ)
    Determinant det(J)is the product of the Eigen-values, calculated as

$$\det(J) = \alpha\gamma - \beta\delta\frac{\alpha}{\beta} = \alpha\gamma - \delta\alpha = \alpha(\gamma - \delta)$$

- **Stability Analysis:**
  - **Stable Node/Centre:** If both Eigen-values have negative real parts, the equilibrium point is stable. For the Lotka-Volterra model, Eigen-values are purely imaginary, indicating a center where the populations oscillate in cycles.
  - **Unstable Node/Saddle:** If the trace is positive or the determinant is negative, the equilibrium is unstable, leading to divergent or chaotic behavior.

Predicting Population Cycles:

- Using the trace-determinant plane, ecologists can predict and visualize the cyclical behavior of predator and prey populations. Understanding these cycles helps in predicting how changes in parameters like birth rates or predation rates affect the system.

Effects of Environmental Changes:

- Changes in environmental parameters (e.g., changes in food availability or habitat destruction) can be incorporated into the model to study their impact on equilibrium points and stability. By modifying the parameters and analyzing the new trace and determinant, ecologists can assess how such changes influence population dynamics.

Conservation and Management:

- Insights gained from the trace-determinant analysis help in designing conservation strategies and managing ecosystems to maintain balanced populations and prevent species extinction.

The Lotka-Volterra model is a fundamental tool in ecology for studying predator-prey dynamics. By applying the trace-determinant plane analysis to this model, researchers can understand the stability of equilibrium points and predict the cyclical nature of population interactions. This approach provides valuable insights into the long-term behavior of populations and the impact of environmental changes, aiding in effective ecological management and conservation efforts.

4.3 Economic Models and the Trace-Determinant Plane

1. Economic Dynamics

- In economics, systems of ODEs are used to model dynamic processes like capital accumulation, economic growth, and market fluctuations.
- These models help economists understand how economic variables evolve over time and how different factors or policies influence these dynamics.

Example:

- The Solow Growth Model is a widely used economic model that describes how capital accumulation, population growth, and technological progress drive economic growth.

2. Solow Growth Model;

- The Solow Growth Model describes the long-term behavior of an economy using a system of ODEs. It focuses on the accumulation of capital and its effects on output growth.
- The basic form of the model is:

$$\frac{dk}{dt} = sf(k) - (n+\delta)k$$

where ,

- k(t) is the capital per worker.
- s is the savings rate,
- f(k) is the production function (often assumed to be f(k)= k^{α}.
- n is the population growth rate,
- δ is the depreciation rate of capital.

Equilibrium Analysis:

- To find the steady-state (equilibrium) level of capital, set;

$$\frac{dk}{dt} = 0;$$

$$sf(k^*) - (n+\delta)k^*$$

Solving this equation yields the steady-state capital per worker k^*.

Linearization:

- To analyze the stability of the steady state, linearize the system around k^* by calculating the Jacobean matrix. The Jacobean matrix in this context is derived from the linearized version of the ODE:

  $$J=\frac{\partial}{\partial k}[sf(k)-(n+\delta)k]|_{k=k^*}$$

  For the Cobb-Douglas production function $f(k)=k^{\alpha}$,

  The Jacobean matrix is: $J=s\alpha k^{\alpha-1}-(n+\delta)$ at the steady-state k*, this simplifies to:

  $$J= s\alpha(k^*)^{\alpha-1} - (n+\delta)$$

Trace-Determinant Analysis:

  - **Trace**: Since the system is one-dimensional in this case, the trace is just the value of the Jacobean matrix.
  - **Determinant**: For a one-dimensional system, the determinant is simply the value of the Jacobean matrix.

- **Stability Analysis:**
  - **Stable Steady State**: If the Jacobean matrix at k* has a negative value, the steady state is stable. This implies that small deviations from k* will decay over time, and the system will return to equilibrium.
  - **Unstable Steady State**: If the Jacobean matrix has a positive value, the steady state is unstable, leading to divergence from the equilibrium point.

3. Implications of Trace-Determinant Analysis:

- By analyzing how changes in parameters (e.g., savings rate s, population growth rate n, and depreciation rate δ affect the trace and determinant, economists can assess the impact of different economic policies on the stability of capital accumulation and growth.
- Understanding the stability of the steady state helps in predicting how different economic interventions or shocks might affect long-term growth. For instance, an increase in the savings rate s generally shifts the steady-state capital level upward, potentially leading to higher long-term output growth.
- The trace-determinant plane can be used to compare different economic scenarios and policies, such as changes in savings rates or technological advancements, and their effects on the stability and dynamics of the economic system.
- The insights gained from trace-determinant analysis help in making long-term economic projections and understanding how economies adjust to new steady states following changes in economic conditions.

In the context of economic models, such as the Solow Growth Model, the trace-determinant plane provides valuable insights into the stability and dynamics of economic equilibria. By analyzing the trace and determinant of the Jacobean matrix, economists can assess the stability of steady-state capital accumulation and output growth, understand the effects of different economic policies, and make informed predictions about long-term economic behavior.

CONCLUSION

This thesis has delved into the theoretical framework of systems of ordinary differential equations (ODEs) and the trace-determinant plane, focusing on their implications for analyzing stability and dynamics of linear systems. The key findings are:

1. **Theoretical Insights into Systems of ODEs:**
   - Systems of ODEs provide a foundational framework for modeling dynamic processes across various fields. The behavior of such systems, particularly their stability and response to perturbations, is intrinsically linked to the properties of the system matrix.
2. **Trace-Determinant Plane:**
   - The trace-determinant plane offers a crucial tool for understanding the stability of linear systems. By analyzing the trace (the sum of Eigen-values) and the determinant (the product of Eigen-values) of the system matrix, one can determine the nature of equilibrium points and predict system behavior.
3. **Stability Analysis:**
   - The trace-determinant plane allows for a clear assessment of stability conditions. Specifically, for a 2D linear system, the trace-determinant plane helps classify equilibrium points as stable nodes, unstable nodes, saddles, or centers based on the Eigen-values derived from the trace and determinant.
4. **Phase Portraits:**
   - While the trace-determinant plane provides a quantitative measure, phase portraits offer a qualitative visualization of system behavior. Together, these tools offer a comprehensive understanding of system dynamics, including trajectories and the nature of equilibrium points.

Overall, these tools are integral to both theoretical analysis and practical applications of dynamical systems. They offer a robust framework for understanding and designing systems with desirable stability characteristics.

Future Research Directions:

- Extend the analysis of the trace-determinant approach to nonlinear systems. While the trace-determinant plane is effective for linear systems, nonlinear systems exhibit more complex behaviors that require advanced techniques for stability analysis.
- Investigate methods for applying the trace-determinant analysis in higher-dimensional systems. Developing generalizations for systems with more than two dimensions can enhance the understanding of complex dynamics and stability in multi-dimensional contexts.
- Explore advanced stability analysis techniques, such as Lyapunov’s methods or numerical simulations, to complement the trace-determinant approach. These techniques can provide deeper insights into system behavior and stability, especially in nonlinear or high-dimensional systems.

- Develop and refine computational tools for analyzing large-scale systems. Efficient algorithms for computing Eigen-values, trace, and determinant in high-dimensional spaces will facilitate more effective and scalable stability analysis.
- Promote interdisciplinary research that integrates dynamical systems theory with other fields, such as biology, economics, and machine learning. This integration can lead to innovative applications and deeper insights into complex systems.
- Create educational resources and software tools to help students and practitioners better understand and apply the trace-determinant plane and phase portraits in their work. Effective teaching tools can enhance the accessibility and application of these concepts.
- Encourage the use of the trace-determinant plane and phase portraits in practical problem-solving scenarios across various domains. Applying these concepts to real-world systems can validate theoretical findings and guide effective system design.

In summary, while the trace-determinant plane and phase portraits are powerful tools for analyzing linear systems, expanding their application to nonlinear, higher-dimensional, and hybrid systems presents valuable opportunities for future research. Advancing computational methods and fostering interdisciplinary collaborations will further enhance the understanding and application of these theoretical concepts.

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    - Useful for computational analysis of differential equations.
2. **MathWorks.** *MATLAB Documentation*. Retrieved from MATLAB.

Provides resources for numerical simulation and analysis of differential equations