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**Addis Ababa University**  
**College of Natural and Computational Sciences**  
**Department of Mathematics**

**Thesis on**

**BESSEL'S DIFFERENTIAL EQUATION WITH INTEGER ORDER**

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## **Abstract**

*Differential equations play an important role in solving real life problems, and we know that solving differential equation is not simple in general. In this thesis we take the Bessel differential equation and discuss the methods to solve the differential equation using Frobenius method. We begin by finding Bessel functions  $J_n(x)$  and  $Y_n(x)$  which are two solutions of Bessel differential equation named as Bessel functions of first and second kind will also be described, when  $n$  is an integer. For our purpose we consider some properties and proofs of recurrence relations, generating functions and special functions. Finally we discuss about the vibration of circular membrane and its solutions.*

**Keyword:** *-Bessel differential equation, Frobenius method, generating functions, Bessel functions and application.*

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## List of Acronym

1. RHS Right hand side.
2. LHS Left hand side.
3. *lim* Limit.
4.  $(X)$  Equation of x
5.  $Y_h$  General solution corresponding to homogeneous equation.
6.  $Y_p$  Specific solution of undetermined coefficient
7.  $W$  Wronskian's of determinant

# CHAPTER ONE

## 1. INTRODUCTION

The Bessel function was defined for the first time by mathematician Daniel Bernoulli and generalized by Friedrich Bessel. The Bessel differential equation is the linear second-order ordinary differential equation of the form

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0, \quad n \in \mathbb{Z} \quad (1)$$

is called a Bessel differential equation and the solutions of Bessel differential equations are known as Bessel functions of the first and the second kinds. The Bessel functions of the first and second kinds are denoted by  $J_n(x)$  and  $Y_n(x)$  respectively and they are linearly independent (it cannot be expressed as linear combination). Its solution is defined as  $y(x) = C_1 J_n(x) + C_2 Y_n(x)$ . Where  $J_n(x)$ , is the solution of the first kind and  $Y_n(x)$  is the solution of the second kind, when  $n$  is an integer.

The applications of the Bessel function in fluid mechanics, vibrating membranes, and heat transformation are described briefly. We described the characteristics of the Bessel function and used Frobenius methods to solve the Bessel differential equation.

There are three chapters in this thesis. The gamma function, power series, second order linear differential equation, ordinary point, and singular point of differential equations are all defined in the first chapter along with their properties. The Bessel equation first kind and second kind of solutions were originally introduced in chapter two. We can also observe some of the Bessel function's fundamental characteristics, and in the final chapter we talked about how Bessel functions were applied to the vibrations of circular membrane.

## 1.1. Preliminaries

### 1.1.1. Power Series

**Definition 1.1** An infinite series of the form

$$\sum_{n=0}^{\infty} C_n(x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + \dots \quad (1.1)$$

The series (1.1) is also called a power series centered at  $a$ . The power series centered at  $a = 0$  is often referred to as the Maclaurian series, that is, the series  $\sum_{n=0}^{\infty} C_n x^n$ .

A power series  $\sum_{n=0}^{\infty} C_n(x-a)^n$  is convergent at a specified value of  $x$  if its sequence of partial sums  $S_N(x)$  converges. That is,  $\lim_{N \rightarrow \infty} S_N(x) = \lim_{N \rightarrow \infty} \sum_{n=0}^N C_n(x-a)^n$  exists. If the limit does not exist at  $x$ , then the series is said to be divergent. The set of points  $x$  at which the power series is convergent is called the interval of convergence of the power series. If the series converges only at  $a$  then  $R = 0$ , and if it converges for all  $x$  then  $R = \infty$ . A power series is called absolutely convergent if the series  $\sum_{n=0}^{\infty} |C_n(x-a)^n|$  converges. A power series converges absolutely within its interval of convergence. By the Ratio test a power series centered at  $a$ , series given in (1.1) is absolutely convergent if  $L = |x-a| \lim_{n \rightarrow \infty} \left| \frac{C_{n+1}}{C_n} \right|$  is less than 1, that is,  $L < 1$ , the series diverges if  $L > 1$  and test fails if  $L = 1$ .

### 1.1.2. Second Order Linear Differential Equation

A second order, linear differential equation is an equation which can be written in the form

$a(x)y'' + p(x)y' + q(x)y = f(x)$ , where  $a(x) \neq 0$  and  $a, p, q$  and  $f$  are continuous functions on some interval  $I$ . If  $f(x) = 0$  then the above equation becomes

$y'' + P(x)y' + Q(x)y = 0$ , where  $P(x) = \frac{p(x)}{a(x)}$  and  $Q(x) = \frac{q(x)}{a(x)}$  and it is called a homogeneous equation. Otherwise, the equation is called non-homogeneous.

**Definition 1.2** The general solution of the second order non homogeneous linear equation

$$y'' + p(x)y' + q(x)y = f(x)$$

Can be expressed in the form  $y = y_h + y_p$

Where,  $y_p$  is any specific function that satisfies the non-homogeneous equation and

$y_h$  is the general solution of the corresponding homogeneous equations of the form

$$y'' + p(x)y' + q(x)y = 0$$

If  $y_1$  and  $y_2$  are a pair of fundamental solutions of the corresponding homogeneous equation with coefficient  $c_1$  and  $c_2$  are arbitrary constants. Then  $y_h = c_1y_1 + c_2y_2$  are called solutions of the homogeneous equations and  $y_p$  is called the particular solutions of the non-homogeneous equations.

To find the specific solution  $y_p$  there are two general approaches; the method of undetermined coefficients, and variation of parameters.

### 1.1.3. Ordinary point and singular point of differential equations

Consider the linear second order differential equation

$$b_2(x)y'' + b_1(x)y' + b_0(x)y = 0 \tag{1.2}$$

In standard form

$$y'' + p(x)y' + q(x)y = 0 \tag{1.3}$$

By dividing leading the coefficients of homogeneous equation by  $b_2(x)$ , and where  $p(x) = \frac{b_1(x)}{b_2(x)}$

and  $q(x) = \frac{b_0(x)}{b_2(x)}$

A point  $x_0$  is said to be an ordinary point of the differential equation (1.2) if both  $p(x)$  and  $q(x)$  in the standard form (1.3) are analytic at  $x_0$ . A point that is not an ordinary point is said to be a singular point of the equation.

A solution of the form  $y = \sum_{n=0}^{\infty} C_n(x - x_0)^n$  is said to be a solution about the ordinary point  $x_0$

**Remark:** If  $x = x_0$  is an ordinary point [ $P(x)$  and  $q(x)$  are analytic at  $t_0$ ] of (1.3), then there exist two linearly independent solutions in the form of a power series centered at  $x_0$ .

That is  $y = \sum_{n=0}^{\infty} C_n(x - x_0)^n$ . A series solution converges at least on some interval defined by

$|x - x_0| < R$ , where  $R$  is the distance from  $x_0$  to the closest singular point.

### Solutions of regular singular points: - The Frobenius method

We say  $x = x_0$  is a regular singular point or irregular singular point, if a singular point  $x_0$  of (1.3) is called a regular singular point of this equation then the functions  $p(x) = (x - x_0)P(x)$  and  $q(x) = (x - x_0)^2Q(x)$  are both analytic at  $x_0$ . A singular point is not regular is said to be an irregular singular point of the equation. These mean that one or both of the functions

$p(x) = (x - x_0)P(x)$ , and  $q(x) = (x - x_0)^2Q(x)$  fail to be analytic at  $x_0$  of (1.3).

In order to solve a differential equation given by (1.2) about a regular singular point we employ the following theorem due to Frobenius

**Theorem 1.1** (Frobenius theorem) if  $x = x_0$  is a regular singular point of the differential equation (1.2), then there exist at least one solution of the form

$y = (x - x_0)^r \sum_{n=0}^{\infty} C_n (x - x_0)^n = \sum_{n=0}^{\infty} C_n (x - x_0)^{n+r}$ , Where  $r$  is constant to be determined.

The series will converge at least on some interval  $0 < x - x_0 < R$ .

**Example** Consider the equation  $x^2(x - 2)^2y'' + (x - 2)y' + 3x^2y = 0$ , then at  $x = 0$  and  $x = 2$ ,  $a_2(x) = x^2(x - 2)^2 = 0$ , so 0 and 2 are its singular points.

And  $a_1(x) = \frac{(x-x_0)a_1(x)}{a_2(x)}$  and  $a_0(x) = \frac{(x-x_0)^2 a_0(x)}{a_2(x)}$

$\Rightarrow a_1(x) = \frac{(x-x_0)(x-2)}{x^2(x-2)^2}$  and  $a_0(x) = \frac{(x-x_0)^2 3x^2}{x^2(x-2)^2}$ , now at  $x_0=0$  then

$a_1(0) = \frac{1}{x(x-2)} = \frac{1}{0}$ , is undefined and  $a_0(0) = \frac{3x^2}{(x-2)^2} = 0$ .

Therefore: 0 is an irregular singular point. And

At  $x_0 = 2$ ,  $a_1(2) = \frac{1}{x^2} = \frac{1}{4} \neq 0$  and  $a_0(2) = 3 \neq 0$

**Therefore**  $a_1(x)$ , and  $a_0(x)$  are none zero at  $x = 2$ , so 2 is regular point.

### 1.1.4. Gamma Function

The Gamma function is first introduced by the Swiss mathematician Leonhard Euler (1707-1783) in his goal to generalize the factorial to non-integer values. For any complex  $x$  with real part greater than zero the Gamma function is defined by  $\Gamma(x) = \int_{t=0}^{\infty} e^{-t} t^{x-1} dt$ .

#### 1.1.4.1. Property of Gamma functions.

- i.  $\Gamma(x + 1) = x\Gamma(x)$ , where  $x$  is positive real number.
- ii.  $\Gamma(1) = 1$
- iii.  $\Gamma(x + 1) = x!$
- iv.  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- v.  $\Gamma\left(x + \frac{1}{2}\right) = \frac{(2x)!\sqrt{\pi}}{2^{2x}x!}$

#### Proof

- i. 
$$\begin{aligned}\Gamma(x + 1) &= \int_{t=0}^{\infty} e^{-t} t^x dt \\ &= -[e^{-t} t^x]_0^{\infty} + x \int_{t=0}^{\infty} e^{-t} t^{x-1} dt \text{ (Integrating by part)} \\ &= x \int_{t=0}^{\infty} e^{-t} t^{x-1} dt = x\Gamma(x) \\ \Gamma(x + 1) &= x \int_{t=0}^{\infty} e^{-t} t^{x-1} dt = x\Gamma(x)\end{aligned}$$

**Therefore:**  $\Gamma(x + 1) = x\Gamma(x)$

- ii. By definition  $\Gamma(1) = \int_{t=0}^{\infty} e^{-t} t^{1-1} dt = \int_{t=0}^{\infty} e^{-t} dt = 1$

**Therefore:**  $\Gamma(1) = 1$

- iii. From (i) it is evident that if the value of  $\Gamma(x)$  is known for  $x$  between two successive integers the value  $\Gamma(x)$  for any positive value of  $x$  can be determined by successive application of (i). we get

$$\begin{aligned}\Gamma(x + 1) &= (x)\Gamma(x) \\ &= x(x - 1)\Gamma(x - 1) \\ &= x(x - 1)(x - 2)\Gamma(x - 2) \\ &= x(x - 1)(x - 2)(x - 3)\Gamma(x - 3)\end{aligned}$$

$$\begin{aligned}
&= x(x-1)(x-2)(x-3)(x-4)\dots 3.2.1\Gamma(1) \\
&= x(x-1)(x-2)(x-3)(x-4)\dots 3.2.1 = x!
\end{aligned}$$

**Therefore:**  $\Gamma(x+1) = x!$  for all positive integer  $x$ .

iv. We start the proof by defining the gamma function,  $\Gamma(x) = \int_{x=0}^{\infty} e^{-t}t^{x-1} dt$

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \int_{x=0}^{\infty} e^{-t}t^{\frac{1}{2}-1} dt \\
&= \int_{x=0}^{\infty} e^{-t}t^{\frac{-1}{2}} dt, \text{ (integrate by part and let } u = t^{\frac{-1}{2}}, \rightarrow t = u^2 \text{ )} \\
&= 2 \int_{u=0}^{\infty} e^{-u^2} du
\end{aligned}$$

We also write this integral in terms of another new variable  $u=v$  and both side square. We get

$$\begin{aligned}
\left(\Gamma\left(\frac{1}{2}\right)\right)^2 &= 2 \int_0^{\infty} e^{-u^2} du \cdot 2 \int_0^{\infty} e^{-v^2} dv, \\
\left(\Gamma\left(\frac{1}{2}\right)\right)^2 &= 4 \iint_0^{\infty} e^{-(u^2+v^2)} du dv,
\end{aligned}$$

If we change to standard polar coordinates, we have  $du dv = r dr d\theta$ , where  $u = r \cos(\theta)$  and  $v = r \sin \theta$  and hence

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4 \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} r dr d\theta, \text{ (integrate with respect to } \theta \text{ )}$$

The limit of integration give as the positive quadrant of (u,v)-plane as required performing the integration over u. we have

$$\begin{aligned}
\left(\Gamma\left(\frac{1}{2}\right)\right)^2 &= 2\pi \int_0^{\infty} e^{-r^2} r dr, \text{ (integrate with respect to } r \text{), we get} \\
\left(\Gamma\left(\frac{1}{2}\right)\right)^2 &= 2\pi \left[ \frac{1}{2} e^{-r^2} \right]_0^{\infty} = 2\pi \frac{1}{2} = \pi
\end{aligned}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \text{ (both side square root )}$$

**Therefore:**  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

V. We start from the duplication (or legender duplication ) formula states that for any  $x > 0$

$$\Gamma(x)\Gamma(x + \frac{1}{2}) = 2^{1-2x}\sqrt{\pi} \Gamma(2x)$$

We can rearrange the duplication formula to isolate  $\Gamma(x)$

$$\Gamma(x + \frac{1}{2}) = \frac{2^{1-2x}\sqrt{\pi} \Gamma(2x)}{\Gamma(x)}$$

To apply the factorial properties,  $\Gamma(2x) = (2x - 1)!$  and  $\Gamma(x) = (x - 1)!$

$$\Rightarrow \Gamma(2x) = (2x - 1)! = \frac{(2x)!}{2x}, \text{ and } \Gamma(x) = \frac{x!}{x}$$

$$\Gamma(x + \frac{1}{2}) = \frac{2^{1-2x}\sqrt{\pi} \Gamma(2x)}{\Gamma(x)} = \frac{2^{1-2x}\sqrt{\pi}(2x)!}{2 \cdot x!}$$

$$\Gamma(x + \frac{1}{2}) = \frac{\sqrt{\pi}(2x)!}{2^{(2x)}x!}$$

**Therefore:**  $\Gamma(x + \frac{1}{2}) = \frac{\sqrt{\pi}(2x)!}{2^{(2x)}x!}$

### 1.1.5. Recurrence relation (formula) for $J_n$

Some of the recurrence relations as follow

- i.  $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$
- ii.  $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$
- iii.  $J'_n(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$
- iv.  $J'_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$
- v.  $J'_n(x) = \frac{1}{2} [J_{n-1}(x) - J_{n+1}(x)]$
- vi.  $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$

#### Proof

- i. By definition of  $J_n(x)$  we have

$$\begin{aligned} x^n J_n(x) &= x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+n+1)} \left(\frac{x}{2}\right)^{n+2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^n}{k! \Gamma(k+n+1)} \left(\frac{x}{2}\right)^{n+2k} \end{aligned}$$

$$\begin{aligned}
\text{Then, } \frac{d}{dx} [x^n J_n(x)] &= \frac{d}{dx} \sum_{k=0}^{\infty} \frac{(-1)^k x^n}{k! \Gamma(k+n+1)} \left(\frac{x}{2}\right)^{n+2k} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)^{n+2k}}{k! \Gamma(k+n+1)} \frac{d}{dx} (x^{2n+2k}) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)^{n+2k}}{k! \Gamma(k+n+1)} (2n+2k) x^{2n+2k-1} \\
&= x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(n-1+k+1)} \left(\frac{x}{2}\right)^{n+2k-1} \\
&= x^n J_{n-1}(x)
\end{aligned}$$

$$\begin{aligned}
\text{ii. } \frac{d}{dx} [x^{-n} J_n(x)] &= \frac{d}{dx} \left[ x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+n+1)} \left(\frac{x}{2}\right)^{n+2k} \right] \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)^{n+2k}}{k! \Gamma(k+n+1)} \frac{d}{dx} (x^{2k}) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)^{n+2k} 2k x^{2k-1}}{k(k-1)! \Gamma(k+n+1)} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)^{n+2k-1}}{(k-1)! \Gamma(k+n+1)} x^{2k-1} \cdot \frac{x^n}{x^n} \\
&= \sum_{k+1=1}^{\infty} \frac{(-1)^{(k+1)} \left(\frac{1}{2}\right)^{n+2k+1} x^{n+2k+1}}{k! \Gamma(k+n+2) x^n} \\
&= -x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(n+1+k+1)} \left(\frac{x}{2}\right)^{n+1+2k} \\
&= -x^{-n} J_{n+1}(x)
\end{aligned}$$

iii. From recurrence formula (i)

$$\begin{aligned}
\frac{d}{dx} [x^n J_n(x)] &= x^n J_{n-1}(x) \\
n x^{n-1} J_n(x) + x^n J_n'(x) &= x^n J_{n-1}(x) \\
J_n'(x) &= J_{n-1}(x) - \frac{n}{x} J_n(x)
\end{aligned}$$

iv. From recurrence formula (ii)

$$\begin{aligned}\frac{d}{dx}[x^{-n}J_n(x)] &= -x^{-n}J_{n+1}(x) \\ -nx^{-n-1}J_n(x) + x^{-n}J'_n(x) &= -x^{-n}J_{n+1}(x) \\ J'_n(x) &= -J_{n+1}(x) + \frac{n}{x}J_n(x)\end{aligned}$$

v. From recurrence formula (iii) and (iv) i.e.

$$\begin{aligned}+ \begin{cases} J'_n(x) = J_{n-1}(x) - \frac{n}{x}J_n(x) \\ J'_n(x) = \frac{n}{x}J_n(x) - J_{n+1}(x) \end{cases} \\ 2J'_n(x) = J_{n-1}(x) - J_{n+1}(x) \\ J'_n(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]\end{aligned}$$

vi. From recurrence formula (iii) and (iv)

$$\begin{aligned}- \begin{cases} J'_n(x) = J_{n-1}(x) - \frac{n}{x}J_n(x) \\ J'_n(x) = \frac{n}{x}J_n(x) - J_{n+1}(x) \end{cases} \\ 0 = \frac{-2n}{x}J_n(x) + J_{n-1}(x) + J_{n+1}(x)\end{aligned}$$

**Therefore:**  $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x)$

## CHAPTER TWO

### 2. Bessel equation and Bessel functions

**Definition 2.1.1** A differential equation of the form

$$x^2 y'' + xy' + (x^2 - n^2)y = 0 \quad (2.1)$$

is called **Bessel equation** of order  $n$ , where  $n$  is constant. The solution of this Bessel equation is called **Bessel function**. In standard form equation (2.1) can be rewrite as:

$$y'' + p(x)y' + q(x)y = 0 \quad (2.2)$$

With  $p(x) = \frac{1}{x}$  and  $q(x) = 1 - \frac{n^2}{x^2}$

The function  $p(x)$  and  $q(x)$  are analytic everywhere except at the point  $x = 0$ , the singularity occurs in the Bessel equation due to the  $x^2$  sitting in the denominator.

The behavior of the coefficients  $p(x)$  and  $q(x)$  at  $x = 0$  is

$$\lim_{x \rightarrow 0} p(x) = \lim_{x \rightarrow 0} \frac{1}{x} = \pm\infty, \quad \lim_{x \rightarrow 0} q(x) = \lim_{x \rightarrow 0} \left(1 - \frac{n^2}{x^2}\right) = -\infty$$

$$\text{But } \lim_{x \rightarrow 0} (x - 0) p(x) = \lim_{x \rightarrow 0} (x - 0) \frac{1}{x} = 1, \text{ finite}$$

$$\lim_{x \rightarrow 0} (x - 0)^2 q(x) = \lim_{x \rightarrow 0} (x - 0)^2 \left(1 - \frac{n^2}{x^2}\right) = -n^2, \text{ finite}$$

So we conclude that  $x = 0$  is a regular singular point and we may apply the Frobenius method at  $x = 0$ . The solution of equation (2.1) is Bessel functions. These functions appear frequently in applications involving cylindrical geometry and have been extensively studied. Bessel functions are the most widely used functions in science and engineering.

#### 2.1. Bessel Functions of the First Kind

The Bessel functions of the first kind of order  $n$  are the solution of the Bessel differential equation (2.1)

To find the first solution we begin by taking a power series,

$$y(x) = \sum_{k=0}^{\infty} C_k x^{k+r} \quad (2.3)$$

For convenience, the first and second derivatives of this power series are:

$$y'(x) = \sum_{k=0}^{\infty} (k+r)C_k x^{k+r-1} \quad (2.4)$$

$$y''(x) = \sum_{k=0}^{\infty} (k+r)(k+r-1)C_k x^{k+r-2} \quad (2.5)$$

These are substitute into the Bessel equation (2.1) and solve for its necessary component equation (2.3), (2.4) and (2.5), then we have

$$\begin{aligned} x^2 \sum_{k=0}^{\infty} (k+r)(k+r-1)C_k x^{k+r-2} + x \sum_{k=0}^{\infty} (k+r)C_k x^{k+r-1} + (x^2 - n^2) \sum_{k=0}^{\infty} C_k x^{k+r} &= 0 \\ \sum_{k=0}^{\infty} (k+r)(k+r-1)C_k x^{k+r} + \sum_{k=0}^{\infty} (k+r)C_k x^{k+r} + \sum_{k=0}^{\infty} C_k x^{k+r+2} - \sum_{k=0}^{\infty} C_k n^2 x^{k+r} &= 0 \\ \sum_{k=0}^{\infty} (k+r)(k+r-1)C_k x^{k+r} + \sum_{k=0}^{\infty} (k+r)C_k x^{k+r} + \sum_{k=2}^{\infty} C_{k-2} x^{k+r} - \sum_{k=0}^{\infty} C_k n^2 x^{k+r} &= 0 \end{aligned}$$

Writing the terms corresponding to  $k = 0$  and  $k = 1$  separately gives

$$C_0(r^2 - n^2)x^r + C_1((r+1)^2 - n^2)x^{r+1} + \sum_{k=2}^{\infty} (C_k[(r+k)^2 - n^2] + C_{k-2})x^{k+r}$$

Equating coefficients of the series to zero gives

$$C_0(r^2 - n^2) = 0 \quad (k = 0) \quad (2.6)$$

$$C_1((r+1)^2 - n^2) = 0 \quad (k = 1) \quad (2.7)$$

$$(C_k[(r+k)^2 - n^2] + C_{k-2}) = 0 \quad (k \geq 2) \quad (2.8)$$

From equation (2.6),  $C_0 \neq 0$ , then We get the indicial equation with

$r^2 - n^2 = 0$ , implies  $r_1 = n$  and  $r_2 = -n$  are indicial roots.

Setting  $r = n$ , equation (2.8) gives the recurrence relation.

$$C_k = \frac{-1}{k(k+2n)} C_{k-2}, \text{ for } k \geq 2$$

This is one of the recurrence relations, and then to check the pattern, the even and odd indexed terms are determined independently (separately).

We deal with the odd-indexed terms first with  $r = n$  equation (2.7) becomes

$$C_1((r+1)^2 - n^2) = 0$$

This implies that  $C_1 = 0$ , recall that  $n \geq 0$  in the equation (2.1) and so

$$C_3 = C_5 = \dots = 0$$

To find a pattern for the even-indexed terms we rewrite the recurrence relation with  $k = 2m$  and get

$$C_{2m} = \frac{-1}{2m(2m+2n)} C_{2m-2} = \frac{-1}{2^2 m(m+n)} C_{2(m-1)}, \text{ for } m \geq 1$$

This gives  $C_2 = \frac{-1}{2^2(1+n)} C_0$

$$C_4 = \frac{-1}{2^2 2(2+n)} C_2 = \frac{1}{2^4 2! (1+n)(2+n)} C_0$$

$$C_6 = \frac{-1}{2^2 3(3+n)} C_4 = \frac{-1}{2^6 3! (1+n)(2+n)(3+n)} C_0$$

⋮

$$C_{2m} = \frac{(-1)^m}{2^{2m} m! (1+n)(2+n)(3+n)\dots(m+n)} C_0$$

Now we need to choose a convenient value for  $c_0$ . we know that the summation in the final power series equation needs to be convergent in order for it to be a solution to the Bessel equation (2.1).

In addition, it would be advantageous to use the factorial  $(m+n)!$  in the denominator of  $c_k$  for these reasons, we will choose  $C_0 = \frac{1}{2^n n!}$ . we have

$$\begin{aligned} C_{2m} &= \frac{(-1)^m}{2^{2m} m! (1+n)(2+n)(3+n)\dots(m+n)} \frac{1}{2^n n!} \\ &= \frac{(-1)^m}{2^{n+2m} m! (m+n)!}, \text{ now substitute in to equation (2,3), for } r=n \end{aligned}$$

$$\begin{aligned}
y(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{2^n 4^m m! (m+n)!} x^{2m+n} \\
y(x) &= \left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (m+n)!} \left(\frac{x}{2}\right)^{2m}
\end{aligned} \tag{2.9}$$

This power series is only meaningful if it is convergent. We will check by the well-known Ratio Test. If it passes, we have solved Bessel's equation. We use

$$\begin{aligned}
\rho &= \lim_{m \rightarrow \infty} \left| \frac{C_{m+1}}{C_m} \right| \\
&= \lim_{m \rightarrow \infty} \left| \frac{\frac{(-1)^{m+1}}{(m+1)! (m+n+1)!} \left(\frac{x}{2}\right)^{2m+2}}{\frac{(-1)^m}{m! (m+n)!} \left(\frac{x}{2}\right)^{2m}} \right| \\
&= \lim_{m \rightarrow \infty} \left| \frac{-1}{(m+1)(m+n+1)} \left(\frac{x}{2}\right)^2 \right| \\
&= \lim_{m \rightarrow \infty} \frac{1}{(m+1)(m+n+1)} \left(\frac{x}{2}\right)^2 = 0
\end{aligned}$$

Therefore  $\rho = 0$

For  $\rho < 1$ , the series converges. We have found a solution to equation (2.1)

The Gamma function is defined as

$$\Gamma(n) = (n-1)!$$

Making this modification to equation (2.9) by gamma functions, we now have

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m}$$

This is the Bessel function of first kind of order  $n$ . This is not valid for  $-n \in \mathbb{N}$ , where the Gamma function is undefined. In this case, we will begin the summation at  $m = n$  to any undefined Gamma terms (since  $m = n$  corresponds to  $\Gamma(-n+n+1)$ ):

$$J_{-n}(x) = \left(\frac{x}{2}\right)^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(-n+m+1)} \left(\frac{x}{2}\right)^{2m}, \text{ and we substitute } m, \text{ by } m+n, \text{ we get}$$

$$\begin{aligned}
&= \left(\frac{x}{2}\right)^{-n} \sum_{m=0}^{\infty} \frac{(-1)^{(m+n)}}{\Gamma(n+m+1)\Gamma(-n+n+m+1)} \left(\frac{x}{2}\right)^{2(m+n)} \\
&= \left(\frac{x}{2}\right)^{-n} \left(\frac{x}{2}\right)^{2n} \sum_{m=0}^{\infty} \frac{(-1)^m (-1)^n}{\Gamma(n+m+1)\Gamma(m+1)}, \\
&= (-1)^n J_n(x)
\end{aligned} \tag{2.10}$$

This will also solve the Bessel differential equation (2.1).

## 2.2 . Bessel Functions of the Second Kind

The Bessel functions of the second kind of order  $n$ , when  $n$  is an integers, are the solution of the Bessel differential equation (2.1), We will determine a second linearly independent solution for the Bessel differential equation (2.1), in terms of the first kind solution  $J_n(x)$ .

Let us begin by examining the behavior of  $J_n(x)$  as  $x$  approaches to zero. Consider  $\lim_{x \rightarrow 0} J_n(x)$  where

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m}. \text{ Then}$$

$$1. \text{ when } n > 0 \text{ we have } \lim_{x \rightarrow 0} \left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m} = 0$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(\frac{x}{2}\right)^n = 0$$

$$2. \text{ when } n = 0, \text{ we have } \lim_{x \rightarrow 0} \left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+1)} \left(\frac{x}{2}\right)^{2m} = 1$$

$$\Rightarrow \lim_{x \rightarrow 0} \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+1)} \left(\frac{x}{2}\right)^{2m} = 1, \text{ since } \lim_{x \rightarrow 0} \left(\frac{x}{2}\right)^0 = 1$$

$$3. \text{ when } n < 0, \text{ we have } \lim_{x \rightarrow 0} \left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m}$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(\frac{2}{x}\right)^{-n} = \pm \infty$$

$$\textbf{Therefore: } \lim_{x \rightarrow 0} J_n(x) = \begin{cases} 0, & \text{if } n > 0 \\ 1, & \text{if } n = 0 \\ \pm \infty, & \text{if } n < 0 \end{cases}$$

We consider for  $R$  non-integer and  $n$  is an integer, then the solutions of second kind Bessel functions are given by

$$Y_R(x) = \frac{\cos(R\pi)J_R(x) - J_{-R}(x)}{\sin(R\pi)} \quad (2.11)$$

For non-integer  $R$ , if the limit  $Y_n(x) = \lim_{R \rightarrow n} Y_R(x) = \lim_{R \rightarrow n} \frac{J_R(x) \cos(R\pi) - J_{-R}(x)}{\sin(R\pi)}$  exists, then the function  $Y_n(x)$  is called the Bessel functions of the second kind of order  $n \in \mathbb{Z}$ .

Since  $\cos(n\pi) = (-1)^n$ ,  $\sin(n\pi) = 0$  for  $n \in \mathbb{Z}$  and  $J_{-n}(x) = (-1)^n J_n(x)$  we have

$$Y_n(x) = \lim_{R \rightarrow n} \frac{J_R(x) \cos(R\pi) - J_{-R}(x)}{\sin(R\pi)} = \frac{(-1)^n J_n - J_{-n}}{\sin(n\pi)} = \frac{0}{0}, \text{ form (or is indeterminate).}$$

Thus by L Hospital's rule differentiate with respect to  $R$

$$\begin{aligned} Y_n(x) &= \lim_{R \rightarrow n} \frac{\partial / \partial R [J_R(x) \cos(R\pi) - J_{-R}(x)]}{\frac{\partial}{\partial R} [\sin(R\pi)]} \\ Y_n(x) &= \lim_{R \rightarrow n} \left[ \frac{-\pi \sin(R\pi) J_R(x) + \left(\frac{\partial}{\partial R} J_R(x)\right) \cos(R\pi) - \frac{\partial}{\partial R} J_{-R}(x)}{\pi \cos R\pi} \right] \\ &= \left[ \frac{-\pi \sin(n\pi) J_n(x) + \left(\frac{\partial}{\partial R} J_R(x)\right) \cos(n\pi) - \frac{\partial}{\partial R} J_{-R}(x)}{\pi \cos n\pi} \right] \\ &= \left[ \frac{-\pi \sin(n\pi) J_n(x) + J_n'(x) \cos(n\pi) - J_{-n}'(x)}{\pi \cos n\pi} \right] \\ &= \frac{1}{\pi} [J_n'(x) - (-1)^n J_{-n}'(x)] \end{aligned}$$

Let us show that  $J_n$  and  $J_{-n}$  are solutions of the Bessel equation that satisfies the following

$$x^2 J_n''(x) + x J_n'(x) + (x^2 - n^2) J_n(x) = 0$$

$$x^2 J_{-n}''(x) + x J_{-n}'(x) + (x^2 - n^2) J_{-n}(x) = 0$$

Differentiate with respect to  $n$ , we get

$$x^2 \frac{d^2}{dx^2} \left( \frac{\partial}{\partial n} J_n(x) \right) + x \frac{d}{dx} \left( \frac{\partial}{\partial n} J_n(x) \right) + (x^2 - n^2) \frac{\partial}{\partial n} J_n(x) - 2n \frac{\partial}{\partial n} J_n(x) = 0 \quad (2,12)$$

$$x^2 \frac{d^2}{dx^2} \left( \frac{\partial}{\partial n} J_{-n}(x) \right) + x \frac{d}{dx} \left( \frac{\partial}{\partial n} J_{-n}(x) \right) + (x^2 - n^2) \frac{\partial}{\partial n} J_{-n}(x) - 2n \frac{\partial}{\partial n} J_{-n}(x) = 0 \quad (2,13)$$

We multiply the second equation (2,13) by  $(-1)^n$  and subtract from the first equation (2,12).

We get

$$x^2 \frac{d^2}{dx^2} \left( \frac{\partial}{\partial n} J_n - (-1)^n \frac{\partial}{\partial n} J_{-n} \right) + x \frac{d}{dx} \left( \frac{\partial}{\partial n} J_n - (-1)^n \frac{\partial}{\partial n} J_{-n} \right) + (x^2 - n^2) \left[ \frac{\partial}{\partial n} J_n - (-1)^n J_{-n} \right] - 2n[J_n - (-1)^n J_{-n}] = 0$$

Taking the limit as  $R$  approaches to  $n$ ,  $J_n - (-1)^n J_{-n} = 0$

To show  $J_n(x)$  and  $Y_n(x)$  are linearly independent for all integer  $n$ , we check the wronskians determinant by Abel's theorem.

**Abel's Theorem** says that if  $y_1(x)$  and  $y_2(x)$  are two solutions of the differential equation  $p(x)y'' + q(x)y' + r(x)y = 0$ , then the Wronskian of the two solutions is of the form  $\frac{C}{p(x)}$ , where  $C$  is a constant which does not depend on  $x$ . Notice that we can write Bessel's equation in the form  $x^2 y'' + xy' + (x^2 - n^2)y = 0$ . Then the Wronskian determinant of  $J_n(x)$  and  $Y_n(x)$  is of the form  $\frac{A_v}{p(x)}$ , where  $A_v$  does not depend on  $x$ . we omit the detailed calculation here, but the Wronskian determinant of  $J_n(x)$  and  $Y_n(x)$  given by

$$\begin{aligned} W(J_n(x), Y_n(x)) &= \begin{vmatrix} J_n(x) & Y_n(x) \\ J_n'(x) & Y_n'(x) \end{vmatrix} \\ &= ce^{-\int \frac{q(x)}{p(x)} dx}, \text{ Where } c = e^{-c_0} \\ &= ce^{-\int \frac{1}{x} dx} = ce^{-\ln(x) + c_1} = e^{\ln \frac{1}{x}} ce^{c_1} = \left(\frac{1}{x}\right) k = \frac{k}{x}, \text{ where } k = ce^{c_1} \end{aligned}$$

$$W(J_n(x), Y_n(x)) = \frac{k}{x} \neq 0 \quad (2.14)$$

**Therefore:** the functions  $J_n(x)$  and  $Y_n(x)$  are linearly independent.

Hence,  $y(x) = C_1 J_n(x) + C_2 Y_n(x)$ .

## 2.3. Properties of Bessel Functions

### 2.3.1. The Generating Function

**Definition1.3** A generating function of another function  $a_n$  is the function whose power series has  $a_n$  as the coefficient of  $x^n$ . That is the generating function of  $a_n$  is the function  $G(a_n; x)$ , where  $G(a_n; x) = \sum_{n=-\infty}^{\infty} a_n x^n$

**Theorem1.2:**  $e^{\frac{x}{2}(z-z^{-1})}$  is the generating function of  $J_n(x)$ . That is  $e^{\frac{x}{2}(z-\frac{1}{z})} = \sum_{n=0}^{\infty} J_n(x) z^n$ .

**Proof** we have  $e^{\frac{x}{2}z} e^{\frac{-x}{2}z^{-1}} = e^{\frac{x}{2}z} e^{\frac{-x}{2}z^{-1}}$

$$\begin{aligned} e^{\frac{x}{2}z} e^{\frac{-x}{2}z^{-1}} &= \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^m}{m!} z^m \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^k}{k!} z^{-k} \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{m+k}}{k! m!} z^{m-k} \end{aligned}$$

$$\begin{aligned} \text{Put } m = n + k, &= \sum_{k=0}^{\infty} \sum_{n+k=0}^{\infty} \frac{(-1)^k}{k! (n+k)!} \left(\frac{x}{2}\right)^{n+k+k} z^{n+k-k} \\ &= \sum_{k=0}^{\infty} \left( \sum_{n=-k}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{m+k}}{k! (n+k)!} \right) z^n \quad m, k \geq 0 \\ &= \sum_{n=-\infty}^{\infty} \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)! k!} \left(\frac{x}{2}\right)^{2k+n} \right) z^n \\ &= \sum_{n=-\infty}^{\infty} J_n(x) z^n \end{aligned}$$

**Therefore:**  $e^{\frac{x}{2}(z-\frac{1}{z})} = \sum_{n=0}^{\infty} J_n(x) z^n$

**Lemma1.1.** we have

- i.  $J_n(-x) = J_{-n}(x) = (-1)^n J_n(x) \quad \forall n \in \mathbb{Z}$
- ii.  $\sum_{n \in \mathbb{Z}} J_n(x) = 1$
- iii.  $J_n(x+y) = \sum_{k \in \mathbb{Z}} J_k(x) J_{n-k}(y)$

**Proof:** i. we make the change of variables  $x \rightarrow -x$  and  $z \rightarrow z^{-1}$  and insert into the generating function.  $\sum_{n=-\infty}^{\infty} J_n(-x) z^{-n} = e^{\frac{-x}{2}(z^{-1}-z)} = e^{\frac{-x}{2z}} e^{\frac{x}{2}z}$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} \frac{\left(\frac{-x}{2z}\right)^m}{m!} \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2z}\right)^k}{k!} \\
&= \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^m (-1)^m}{m!} z^{-m} \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^k}{k!} z^k \\
\text{Put } -n &= k - m, \\
&= \sum_{m=0}^{\infty} \left( \sum_{k=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{m+k}}{m!k!} \right) z^{m-k} \\
&= \sum_{n=-\infty}^{\infty} \left( \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m-n)!} \left(\frac{x}{2}\right)^{2m-n} \right) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} J_{-n}(x) z^{-n}
\end{aligned}$$

So we have

$$\sum_{n=-\infty}^{\infty} J_n(-x) z^{-n} = \sum_{n=-\infty}^{\infty} J_{-n}(x) z^{-n}$$

Comparing the coefficients, we get

$$J_n(-x) = J_{-n}(x)$$

$$\Rightarrow J_n(-x) = J_{-n}(x) = (-1)^n J_n(x)$$

ii. Set  $z = 1$  in the generating function. Then we have

$$\begin{aligned}
e^{\frac{x}{2}(1-1^{-1})} &= 1 = \sum_{n=-\infty}^{\infty} J_n(x) (1)^n \\
&= \sum_{n \in \mathbb{Z}} J_n(x)
\end{aligned}$$

iii. Observe that  $\sum_{n \in \mathbb{Z}} J_n(x+y) t^n = e^{\frac{1}{2}(x+y)(t-t^{-1})}$

$$\begin{aligned}
&= e^{\frac{1}{2}x(t-t^{-1})} e^{\frac{1}{2}y(t-t^{-1})} \\
&= \sum_{k \in \mathbb{Z}} J_k(x) t^k \sum_{m \in \mathbb{Z}} J_m(y) t^m \\
&= \sum_{n \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} J_k(x) J_{n-k}(y) \right) t^n
\end{aligned}$$

Comparing the coefficients both side yields the result

$$\textbf{Therefore: } J_n(x+y) = \sum_{k \in \mathbb{Z}} J_k(x) J_{n-k}(y)$$

**Lemma1.2.** we have

- i.  $\cos(x) = J_0(x) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(x)$   
 ii.  $\sin(x) = 2 \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(x)$

**Proof:** Take  $z = e^{i\theta} = \cos \theta + i \sin \theta$  and set  $i \sin \theta = \frac{1}{2} \left( z - \frac{1}{z} \right)$  from Euler's formula

$e^{\frac{x}{2} \left( z - \frac{1}{z} \right)} = e^{ix \sin \theta} = \sum_{n=-\infty}^{\infty} J_n(x) e^{in\theta}$ , from the generating function

Now, put  $\theta = \frac{\pi}{2}$ , then for all  $x$

$$e^{ix \sin \frac{\pi}{2}} = \sum_{n=-\infty}^{\infty} J_n(x) e^{(i\frac{\pi}{2})n}$$

$$\cos x + i \sin x = \sum_{n=-\infty}^{\infty} J_n(x) \left[ \cos \left( \frac{n\pi}{2} \right) + i \sin \left( \frac{n\pi}{2} \right) \right]$$

$\cos x = \sum_{n=-\infty}^{\infty} J_n(x) \cos \left( \frac{n\pi}{2} \right)$ , and  $\sin x = \sum_{n=-\infty}^{\infty} \sin \left( \frac{n\pi}{2} \right) J_n(x)$

Using  $\cos \left( \frac{n\pi}{2} \right) = 0$  if  $n$  is odd and  $(-1)^m$  if  $n = 2m$  (even)

And  $J_{-2n}(x) = (-1)^{2n} J_{2n}(x) = J_{2n}(x)$ ,

We obtain  $\cos x = \sum_{n=-\infty}^{\infty} J_n(x) \cos \left( \frac{n\pi}{2} \right)$

$$= J_0(x) + \sum_{n=1}^{\infty} (-1)^n J_{2n}(x) + \sum_{n=-1}^{-\infty} (-1)^n J_{2n}(x)$$

$$= J_0(x) + \sum_{n=1}^{\infty} (-1)^n J_{2n}(x) + \sum_{n=1}^{\infty} (-1)^n J_{-2n}(x)$$

$$= J_0(x) + \sum_{n=1}^{\infty} (-1)^n J_{2n}(x) + J_{-2n}(x)$$

$$= J_0(x) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(x), \text{ since } J_{-2n}(x) = J_{2n}(x) (-1)^{2n} = J_{2n}(x)$$

$$\cos(x) = J_0(x) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(x)$$

To show that  $\sin x = 2 \sum_{n=1}^{\infty} (-1)^n J_{2n+1}(x)$

Since  $\sin \left( \frac{n\pi}{2} \right) = \begin{cases} (-1)^n, & \text{if } n \text{ is odd, that is } n = 2n - 1, n \in \mathbb{N} \\ 0, & \text{if } n \text{ is even, that is } n = 2n, n \in \mathbb{N} \end{cases}$

$$\Rightarrow \sin x = \sum_{n=-\infty}^{\infty} \sin \left( \frac{(2n+1)\pi}{2} \right) J_{2n+1}(x)$$

$$\sin(x) = 2 \sum_{n=1}^{\infty} (-1)^n J_{2n+1}(x)$$

**Therefore:**  $\sin x = 2 \sum_{n=1}^{\infty} (-1)^n J_{2n+1}(x)$

## 2.4. Some Special Values

In this section we will examine what the Bessel functions look like when some particular values are chosen.

**Lemma 2.3.1** If  $n = \frac{1}{2}$  or  $n = -\frac{1}{2}$ , we have

$$J_{\frac{1}{2}}(x) = Y_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x), \text{ and}$$

$$J_{-\frac{1}{2}}(x) = -Y_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x)$$

**Proof** Observe the series representation of  $J_n$  and apply the properties of the gamma function. We

begin with  $J_{\frac{1}{2}}(x) = \sqrt{\frac{x}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\frac{3}{2})} \left(\frac{x}{2}\right)^{2k}$ , since  $\Gamma\left(k + \frac{3}{2}\right) = \frac{(2k+1)!\sqrt{\pi}}{k!2^{2k+1}}$ , we have

$$\begin{aligned} &= \sqrt{\frac{x}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \frac{(2k+1)!\sqrt{\pi}}{k!2^{2k+1}}} \left(\frac{x}{2}\right)^{2k} \\ &= \sqrt{\frac{x}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+1)!\sqrt{\pi}} \left(\frac{x}{2}\right)^{2k}, \text{ since } \Gamma(k+1) = k! \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+1)!\sqrt{\pi}} \left(\frac{x}{2}\right)^{2k+1-\frac{1}{2}}, \text{ since } \frac{1}{2} = 1 - \frac{1}{2} \\ &= \sqrt{\frac{2}{x}} \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+1)!\sqrt{\pi}} \left(\frac{x}{2}\right)^{2k+1} \\ &= \sqrt{\frac{2}{x\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \end{aligned}$$

Since we know the Taylor Series  $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$

We can conclude that  $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{x\pi}} \sin(x)$

We have that  $Y_{\frac{-1}{2}} = \frac{\cos\left(\frac{-\pi}{2}\right)J_{\frac{1}{2}}(x)-J_{\frac{1}{2}}(x)}{\sin\left(\frac{-\pi}{2}\right)}$ , since  $Y_n(x) = \frac{\cos(n\pi)J_n(x)-J_{-n}(x)}{\sin(n\pi)}$

$$Y_{\frac{-1}{2}} = \frac{(0)J_{\frac{1}{2}}(x)-J_{\frac{1}{2}}(x)}{-1}$$

$Y_{\frac{-1}{2}} = J_{\frac{1}{2}}(x)$ , Thus we have proven the first one.

To prove the second by consider the gamma function

Now we begin to put  $\Gamma\left(x + \frac{1}{2}\right) = \frac{(2x)!\sqrt{\pi}}{2^{2x}x!}$  and  $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$  we have

$$\begin{aligned} J_{-\frac{1}{2}}(x) &= \left(\frac{x}{2}\right)^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma\left(k+\frac{1}{2}\right)} \left(\frac{x}{2}\right)^{2k} \\ &= \sqrt{\frac{2}{x}} \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k}}{(2k)!\sqrt{\pi}} \left(\frac{x}{2}\right)^{2k} \\ &= \sqrt{\frac{2}{x\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \\ &= \sqrt{\frac{2}{x\pi}} \cos x \end{aligned}$$

Also  $Y_{\frac{1}{2}} = \frac{\cos\left(\frac{\pi}{2}\right)J_{\frac{1}{2}}(x)-J_{-\frac{1}{2}}(x)}{\sin\left(\frac{\pi}{2}\right)}$

$$\begin{aligned} &= \frac{(0)J_{\frac{1}{2}}(x)-J_{-\frac{1}{2}}(x)}{1} \\ &= -J_{-\frac{1}{2}}(x) \end{aligned}$$

## 2.5. Integral Representation of Bessel Function

The integral representation of the Bessel function has the form

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x \sin \theta) d\theta, \quad \forall n = 0, 1, 2, \dots$$

**Proof** Let  $x$  and  $\theta$  be real numbers and set  $t = e^{i\theta}$ . Then the integral representation of Bessel function as shown

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(n\theta - x \sin\theta) d\theta \quad \forall n = 0, 1, 2, \dots \quad (1)$$

We start from the generating function

$$e^{\frac{x}{2}(t - \frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

So that LHS and RHS can be expressed as

$$e^{\frac{x}{2}(t - \frac{1}{t})} = e^{\frac{x}{2}(e^{i\theta} - e^{-i\theta})} = e^{ix \sin\theta} = \cos(x \sin\theta) + i \sin(x \sin\theta) \quad (2)$$

$$\begin{aligned} \sum_{n=-\infty}^{\infty} J_n(x) t^n &= \sum_{n=-\infty}^{\infty} J_n(x) (e^{i\theta})^n = \sum_{n=-\infty}^{\infty} J_n(x) [\cos(n\theta) + i \sin(n\theta)] \\ \sum_{n=-\infty}^{\infty} J_n(x) t^n &= J_0(x) + [J_{-1}(x) + J_1(x)] \cos\theta + [J_{-2}(x) + J_2(x)] \cos(2\theta) + \dots \\ &\quad + i \{ [J_1(x) - J_{-1}(x)] \sin\theta + [J_2(x) - J_{-2}(x)] \sin 2\theta + \dots \} \end{aligned}$$

$J_{-1}(x) = -J_1(x)$ ,  $J_{-2}(x) = (-1)^2 J_2(x) = J_2(x)$ , ..., in general  $J_{-n}(x) = (-1)^n J_n(x)$

$$\begin{aligned} \sum_{n=-\infty}^{\infty} J_n(x) t^n &= [J_0(x) + 2J_2(x) \cos 2\theta + 2J_4(x) \cos 4\theta + \dots] \\ &\quad + i [2J_1(x) \sin\theta + 2J_3(x) \sin 3\theta + \dots] \end{aligned} \quad (3)$$

$$\sum_{n=-\infty}^{\infty} J_n(x) t^n = J_0(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x) \cos(2n\theta) + 2i \sum_{n=1}^{\infty} J_{2n-1}(x) \sin(2n-1)\theta \quad (4)$$

Form equation (2) and (4) we get

$$\cos(x \sin\theta) + i \sin(x \sin\theta) = J_0(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x) \cos 2n\theta + 2i \sum_{n=1}^{\infty} J_{2n-1}(x) \sin(2n-1)\theta \quad (5)$$

$$\cos(x \sin\theta) = J_0(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x) \cos(2n\theta) \quad (6)$$

$$\sin(x\sin\theta) = 2 \sum_{n=1}^{\infty} J_{2n-1}(x) \sin(2n-1)\theta \quad (7)$$

Multiply equation (6) by  $\cos(n\theta)$  and equation (7) by  $\sin(n\theta)$  and integrate each function with respect to  $\theta$  from 0 to  $\pi$ .

$$\int_0^{\pi} \cos k\theta \cos n\theta d\theta = \int_0^{\pi} \sin k\theta \sin n\theta d\theta = 0; k \neq n \quad (8)$$

$$\int_0^{\pi} \cos^2 k\theta d\theta = \int_0^{\pi} \sin^2 k\theta d\theta = \frac{\pi}{2}, \text{ for } k = n$$

From equation (6)

$$\int_0^{\pi} \cos(n\theta) \cos(x\sin\theta) d\theta = J_0(x) \int_0^{\pi} \cos n\theta d\theta + 2 \sum_{n=1}^{\infty} J_{2n}(x) \int_0^{\pi} \cos 2n\theta \cos n\theta d\theta \quad (9)$$

The first integrals disappears for all values of integral n. That is  $\int_0^{\pi} \cos(n\theta) d\theta = 0$

The second integral disappears only if  $n \neq 2m$  but if  $n = 2m$  or  $n$  is even.

$$\int_0^{\pi} \cos(n\theta) \cos(n\theta) d\theta = \int_0^{\pi} \cos^2(n\theta) d\theta = \frac{\pi}{2} \text{ for } n = 2m$$

$$\int_0^{\pi} \cos(n\theta) \cos(x\sin\theta) d\theta = \begin{cases} \pi J_n(x) & \forall n: \text{even} \\ 0 & \forall n: \text{odd} \end{cases} \quad (10)$$

From equation (7)

$$\int_0^{\pi} \sin(n\theta) \sin(x\sin\theta) d\theta = 2 \sum_{n=1}^{\infty} J_{2n-1}(x) \int_0^{\pi} \sin(2n-1)\theta \sin(n\theta) d\theta$$

$$\int_0^{\pi} \sin(n\theta) \sin(x\sin\theta) d\theta = \begin{cases} \pi J_n(x) & \forall n: \text{odd} \\ 0 & \forall n: \text{even} \end{cases} \quad (11)$$

Adding the two expansion equation (10) and equation (11) and both side dividing by  $\pi$ , we have for all integrals values of n

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} [\cos(n\theta) \cos(x\sin\theta) + \sin(n\theta) \sin(x\sin\theta)] d\theta \quad (12)$$

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x \sin\theta) d\theta, \quad \forall n = 0, 1, 2, \dots$$

In special case for  $n = 0$ ,  $J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\theta) d\theta$ .

## 2.6. Orthogonal Properties of Bessel's functions

If  $\alpha$  and  $\mu$  are the solutions of Bessel functions and  $J_n(\alpha) = 0$ , then the condition of orthogonality of Bessel's function over the interval  $(0, 1)$  is given by

$$\int_0^1 x J_n(\alpha x) J_n(\mu x) dx = 0, \text{ For } \alpha \neq \mu \quad (1)$$

With the condition of normalization

$$\int_0^1 x [J_n(\alpha x)]^2 dx = \frac{1}{2} J_{n+1}^2(\alpha x), \text{ for } \alpha = \mu \quad (2)$$

From equation (1) and (2), we get

$$\int_0^1 x [J_n(\alpha x)]^2 dx = \frac{1}{2} J_{n+1}^2(\alpha x) \delta_{\alpha\mu} \quad (3)$$

**Proof:** The second order Bessel's differential Equation is

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0 \quad (4)$$

Let us change the independent variable  $x$  by  $\alpha x$  and  $\mu x$ , the resulting equation is

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (\alpha^2 x^2 - n^2)y = 0, \text{ and } x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (\mu^2 x^2 - n^2)y = 0 \quad (5)$$

$y(x) = c_1 J_n(\alpha x) + c_2 Y_n(\alpha x)$ , we see that  $y_1 = J_n(\alpha x)$  and  $y_2 = J_n(\mu x)$  are the solution of the equation (4)

$$x^2 \frac{d^2 y_1}{dx^2} + x \frac{dy_1}{dx} + (\alpha^2 x^2 - n^2)y_1 = 0 \quad (6)$$

$$x^2 \frac{d^2 y_2}{dx^2} + x \frac{dy_2}{dx} + (\mu^2 x^2 - n^2)y_2 = 0 \quad (7)$$

Multiply equation (6) by  $y_2$  and equation (7) by  $y_1$  and subtract equation (7) from (6). we get

$$(\mu^2 - \alpha^2)xy_1y_2 = x \frac{d}{dx} \left[ y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx} \right] + \left[ y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx} \right] = \frac{d}{dx} \left[ x \left( y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx} \right) \right] \quad (8)$$

Integration as we omit the constant of integration we get

$$(\mu^2 - \alpha^2) \int_0^1 xy_1y_2 dx = \int_0^1 \frac{d}{dx} \left[ x \left( y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx} \right) \right] dx = \left[ x \left( y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx} \right) \right]_0^1 \quad (9)$$

Using  $y_1 = J_n(\alpha x)$  and  $y_2 = J_n(\mu x)$  and dividing by  $(\mu^2 - \alpha^2)$  we get

$$(\mu^2 - \alpha^2) \int_0^1 x J_n(\alpha x) J_n(\mu x) dx = \left[ x \left( J_n(\mu x) \frac{dJ_n(\alpha x)}{dx} - J_n(\alpha x) \frac{dJ_n(\mu x)}{dx} \right) \right]_0^1 \quad (10)$$

$$\int_0^1 x J_n(\alpha x) J_n(\mu x) dx = \frac{\alpha J_n(\mu) J_n'(\alpha) - \mu J_n(\alpha) J_n'(\mu)}{\mu^2 - \alpha^2} \quad (11)$$

Thus to determine the orthogonality condition. If  $\mu$  and  $\alpha$  are two zeros of  $J_n(x)$ , implies that  $J_n(\alpha) = 0$  and  $J_n(\mu) = 0$ ,

**Case I**, when  $\alpha \neq \mu$

$$\begin{aligned} \int_0^1 x J_n(\alpha x) J_n(\mu x) dx &= \frac{\alpha J_n(\mu) J_n'(\alpha) - \mu J_n(\alpha) J_n'(\mu)}{\mu^2 - \alpha^2} \\ &= \frac{\alpha(0) J_n'(\alpha) - \mu(0) J_n'(\mu)}{\mu^2 - \alpha^2} = \frac{0 - 0}{\mu^2 - \alpha^2} = 0 \end{aligned}$$

$$\int_0^1 x J_n(\alpha x) J_n(\mu x) dx = 0, \text{ for } \alpha \neq \mu \quad (12)$$

**Therefore:** The orthogonality condition satisfies, for  $\alpha \neq \mu$ .

**Case II**, when  $\alpha = \mu$

To reach the normalization condition let us reconsider equation (11). If  $\alpha = \mu$ , and we take limits to get the form  $\frac{0}{0}$ , then on RHS we apply L'Hopital rule (differentiate with respect to  $\mu$  keeping  $\alpha$  as constant) and find the limits as  $\mu$  approaches to  $\alpha$ .

$$\int_0^1 x J_n^2(\alpha x) dx = \lim_{\mu \rightarrow \alpha} \frac{\alpha J_n'(\mu) J_n'(\alpha) - J_n(\alpha) J_n'(\mu) - \mu J_n(\alpha) J_n''(\mu)}{2\mu} \quad (13)$$

Now we can substitute  $\alpha$ , in to the differentiation above equation (13) we get

$$= \frac{\alpha J'_n(\alpha) J'_n(\alpha) - J_n(\alpha) J'_n(\alpha) - \mu J_n(\alpha) J''_n(\alpha)}{2\alpha}$$

Now since  $J_n(\alpha) = 0$ , we have

$$\int_0^1 x J_n^2(\alpha x) dx = \frac{\alpha J'_n(\alpha) J'_n(\alpha)}{2\alpha} = \frac{J'_n(\alpha) J'_n(\alpha)}{2} = \frac{(J'_n(\alpha))^2}{2} \quad (14)$$

We use the fourth recurrence relation

$$x J'_n(x) = n J_n(x) - x J_{n+1}(x)$$

Change the independent variable  $x$  by  $\alpha x$  and we get

$$\alpha x J'_n(\alpha x) = n J_n(\alpha x) - \alpha x J_{n+1}(\alpha x)$$

For  $x = 1$  we have

$$\alpha J'_n(\alpha) = n J_n(\alpha) - \alpha J_{n+1}(\alpha) = -\alpha J_{n+1}(\alpha) \quad \text{Since } J_n(\alpha) = 0 \quad (15)$$

Using  $J'_n(x) = -J_{n+1}(\alpha)$  substitute in equation (14). we have

$$\int_0^1 x J_n^2(\alpha x) dx = \frac{1}{2} J_{n+1}^2(\alpha) \quad (16)$$

Now since  $J_{n+1}(\alpha) \neq 0$  because  $\alpha$  is the zero of  $J_n(\alpha)$ .

**Therefore:** The above equation satisfies the normalization condition.

## CHAPTER THREE

### 3. Application of Bessel functions

#### 3.1. The vibration of circular membrane

This chapter seeks to show how Bessel functions can be used to solve issues involving physical conditions. Bessel equation is a second order linear differential equations that arises in many areas of applied mathematics and physics. Bessel equations are many applications in real life, including electromagnetic, heat transfer, signal propagation, fluid mechanics and acoustics. Bessel functions are used in one of the most straightforward physical phenomena, the vibrations of a circular membrane. This membrane vibrates when it is gently (gradually) deformed from its equilibrium position and then released because of the restoring force brought on by the distortion. The challenge is to understand this vibration of motion and how to solve this moment.

We consider the vibration of circular membrane. Assume that an elastic circular membrane (radius  $a$ ) can vibrate and the material has a uniform density. Let  $u(x, y, t)$  denote the displacement of the membrane at time  $t$  from its equilibrium position. We use polar coordinates in the  $xy$ -plane of the variables  $x$  and  $y$  with corresponding magnitude  $r$  and angle  $\theta$  is given by,

$$x = r \cos(\theta), \quad y = r \sin(\theta) \quad (3.1)$$

The function  $U$  doesn't depend on the angle  $\theta$  the corresponding wave equation satisfies to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (3.2)$$

If the displacement  $U(r, t)$  depends upon the radial distance  $r$  from the center of the membrane and on time  $t$ , then equation (3.2) can be written in the form of the wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right) \quad (3.3)$$

Since the membrane has uniform density, so  $u = u(r, t)$  that is, it is independent of the  $\theta$ . We have

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \quad (3.4)$$

The boundary condition is  $u(a, t) = 0$ , and

The initial conditions take the form

$$f(r) = U(r, 0), \text{ and } g(r) = \frac{\partial u(r, 0)}{\partial t}, \text{ for } 0 < r < 1 \quad (3.5)$$

Now we will look solutions in Separation of variable method yields

$$U(r, t) = R(r)T(t) \quad (3.6)$$

At the boundary condition  $u(a, t) = 0$ , and twice differentiate equation (3.6) with respect to  $r$  &  $t$ , and substitute in to wave equation (3.4). We get

$$RT'' = c^2(R''T + \frac{1}{r}R'T) \quad (3.7)$$

After dividing both side by  $c^2R(r)T(t)$ , we have

$$\frac{R'' + \frac{1}{r}R'}{R} = \frac{T''}{c^2T}, \text{ thus RHS of this equation does not depend on } r \text{ and LHS also doesn't depend on } t.$$

so it follows that both side equals some constant  $\lambda$ . we get

$$\frac{R'' + \frac{1}{r}R'}{R} = \frac{T''}{c^2T} = -\lambda^2 \quad (3.8)$$

Where  $\lambda$  is constant. Thus

$$\begin{cases} R'' + \frac{1}{r}R' + \lambda^2R = 0 \\ T'' + c^2\lambda^2T = 0 \end{cases} \quad (3.9)$$

We notice that the general solution is given by

$$R(r) = c_1J_0(\lambda r) + c_2Y_0(\lambda r) \quad (3.10)$$

Since  $Y_0$ , is unbounded when  $r = 0$ , so  $c_2 = 0$ . Thus on the boundary condition  $U(a, t) = 0$ ,

So  $R(a) = c_1J_0(\lambda a) = 0$  and  $c_1 = 1$  implies that

$$J_0(\lambda a) = 0 \quad (3.11)$$

To change variables put  $\lambda a = \mu$  is a zero of  $J_0(\mu)$  and implies that  $\lambda = \frac{\mu}{a}$ . Setting  $c_1 = 1$ , we obtain for each integer  $n = 1, 2, 3, \dots$

$$R_n(r) = J_0(\lambda_n r) = J_0\left(\frac{\mu_n}{a} r\right) \quad (3.12)$$

Where  $\mu_n = \lambda_n a$  is the zero's of  $J_0(\mu_n)$ .

Again we describe solutions of the second equation(3.10) is given by

$$T(t) = A_n \cos(\lambda c t) + B_n \sin(\lambda c t) \quad (3.14)$$

Now we assemble equation(3.13) and (3.14) in to separation variable equation(3.6).

we get,  $u_n(r, t) = T_n(t)R_n(\lambda_n r)$

$$= [A_n \cos(c \lambda_n t) + B_n \sin(c \lambda_n t)].J_0(\lambda_n r) \quad (3.15)$$

For any natural number  $n$ , the general solution is given by

$$u(r, t) = \sum_{n=1}^{\infty} (A_n \cos c \lambda_n t + B_n \sin c \lambda_n t).J_0(\lambda_n r)$$

Where

$$f(r) = u(r, 0) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r), \text{ and}$$

$$g(r) = \frac{\partial u(r, 0)}{\partial t} = \sum_{n=1}^{\infty} B_n C \lambda_n J_0(\lambda_n r)$$

Fourier–Bessel series theorem implies that

$$A_n = \frac{2}{a^2 J_1^2(\mu_n)} \int_0^a r f(r) J_0(\lambda_n r) dr$$

$$B_n = \frac{2}{\lambda_n a^2 J_1^2(\mu_n)} \int_0^a r g(r) J_0(\lambda_n r) dr$$

## SUMMARY

In this thesis, Bessel's equation was introduced and we have discussed the Frobenius method solutions of the Bessel differential equations of the Bessel functions the first and second kinds.

We applied the Frobenius method. The solutions of Bessel differential equations are the Bessel functions of the first kind of order  $n$ . When  $n$  is an integer  $J_n(x)$  and  $J_{-n}(x)$  are linearly dependent. The second linearly independent solution  $Y_n(x)$  can be written as a linear combination of the first kind solutions  $J_n(x)$  and  $J_{-n}(x)$ . So we described the general solution of Bessel differential equations of the form

$$y(x) = C_1 J_n(x) + C_2 Y_n(x), \text{ where } C_1 \text{ and } C_2 \text{ are arbitrary constants.}$$

We talked about the generating function, recurrence relation, some special values, integral representation of Bessel's function, and orthogonal properties of Bessel functions.

Finally, we discussed the applications of the Bessel function on vibrations of circular membranes on the physical conditions at initial and boundary conditions and found a solution to it.

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