College of Natural and Computational Sciences

Department of Mathematics

ON THE STRUCTURAL PROPERTIES OF ORDERED WEAK IDEMPOTENT RINGS

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Declaration

I, Tamiru Abera Bayesa, with student ID $GSR/9665/10$, announce that this dissertation is my own original work under the supervision of Prof. Kolluru Venkateswarlu and Dr. Yibeltal Yitayew and that it has not been presented before for completion of any accomplishment to another university, and every foundation of data used for the dissertation has been completely accredited.

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This is to acknowledge that the dissertation prepared by Tamiru Abera Bayesa entitled: "On the Structural Properties of Ordered Weak Idempotent Rings" set forth in attainment of the directives for the Degree of Doctor of Philosophy in Mathematics compiles with the regulations of the university and encounters the acquired standards with regard to originality and quality.

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Abstract

The notion of weak idempotent rings is a new concept and has recently been introduced. It is a ring R of characteristic two, and $a^4 = a^2$ for each a in R. Very few studies have been done on its structural development and the structure of submaximal ideals of weak idempotent rings. So, there are a lot of gaps in its structural development. One of these gaps is the introduction of ordering in it.

In this dissertation, we introduce the concept of partial ordering in a commutative weak idempotent ring R with unity and prove that the introduced partial ordering coincides with the one in Boolean rings whenever it is restricted to the idempotent parts of R. Besides, we introduce the concept of an atom in R and obtain certain results concerning an atom. Further, we study the structure of primary submaximal ideals.

To achieve this, we define partial ordering and an atom in a commutative weak idempotent ring with unity and use the concepts to obtain various results.

Here we introduced the notion of partial ordering in a commutative weak idempotent ring R with unity, studied the role of an atom in the direct product decomposition of R and further on the structure of primary submaximal ideals of R.

We recommend that researchers further study these in detail, shortly prove theorem 4.2.11, and extend the concepts to non-commutative weak idempotent rings with or without unity, and study its potential application in the real world.

List of Notations and Conventions

N	Set of nilpotent elements	Δ	Index set
R_B	Set of idempotent elements	BLR	Boolean-like ring
x_N	Nilpotent element	N_I	set of nilpotent elements annihilated by elements of I
x_B	Idempotent element	Ann(N)	Annihilator of set N
I	Ideal	Spec(R_B)	Set of prime ideals of R_B
N_A	Set of all atoms of N	cWIR	Commutative weak idempotent ring
R/I	Quotient ring	$\langle n_1 \rangle$	An ideal generated by an element n_1
$\bigvee$	Join	X(I)	Set of primary ideals containing I
WIR	Weak idempotent ring	H_4	Four-element Boolean-like ring

Contents

Chapter 1

Introduction

The concept of Boolean rings has been generalized in several ways by various authors (see [8, 21, 30, 31]). For example, p-rings of D. Montgomery and N.H. McCoy [21], p^k-rings presented by Foster A.L. [4], p_1 and p_2 rings established by N.V. Subrahmanyam [26] and A.L.Foster's [1]Boolean-like rings (see [30, 31]) are a few of the ring theoretic broad views. Among these, the Boolean-like rings of A.L. Foster arise naturally from general ring theoretic duality considerations and reserve many of the formal properties of the Boolean rings. A Boolean like ring is a commutative ring with unity and is of characteristic two in which $ab(1+a)(1+b) = 0$ hold for each elements a, b in the ring. Foster provided a technique of creation of Boolean-like rings by combination of Boolean rings and zero rings (a ring of characteristic 2 in which $ab = 0$ for each elements a, b in the ring), utilizing prime ideals of the Boolean rings. Harary [7] generalized Foster's technique of construction of Boolean like rings by considering atomic-based nil radical. Later, Swaminathan [30] provided a general system of creation of every Boolean-like ring.

V.Kolluru etal. [29] introduced the notion of Weak idempotent rings as a generalization to the concept of Boolean like rings.They made an attempt to construct the synthesis of weak idempotent rings by imitating the procedure of Swaminathan. But their synthesis was partial and could obtain the combination of a subclass 2-weak idempotent ring and a weak idempotent ring. They also studied the organization of a 4- and 8-element weak idempotent ring with unity. Further, they proved that every maximal ideal of a weak idempotent ring with unity is nil whenever 0 and 1 are the only idempotent elements of the ring and also obtained some characteristics of submaximal and semiprime ideals of a commutative weak idempotent ring with unity.

Venkateswarlu Kolluru and Dereje Wasihun [27] studied the construction of submaximal ideals in a commutative weak idempotent ring with unity. They established some association among radical,semiprime, primary, and submaximal ideals and also proved that a submaximal ideal of a commutative weak idempotent ring with unity is either primary or semiprime. Furthermore, they proved that there are only two maximal ideas of a commutative weak idempotent ring with unity that covers a submaximal ideal of the ring.

With this backdrop of Boolean like rings and weak idempotent rings we further investigated the study of weak idempotent rings by introducing an ordering. Many of the results were obtained for commutative weak idempotent rings with unity.

Objective

The main objective of this dissertation is to introduce partial ordering in commutative weak idempotent rings with unity.

We addressed the following objectives in this dissertation:

1. To define partial order in commutative weak idempotent rings with unity.
2. To investigate the role of the atom in the decomposition of commutative weak idempotent rings with unity.
3. To study the properties of primary submaximal ideals in commutative weak idempotent rings with unity.

Significance

This dissertation is important for the reason that it has contributed to developing theory of partial ordering, the role of the atom, and investigating further properties of a primary submaximal ideal in a commutative weak idempotent ring with unity.

Scope of the study

The study is limited to commutative weak idempotent rings with unity.

Methodology

We use the following methodologies in order to realize our desired objectives:

1. We define partial ordering in commutative weak idempotent rings with unity.

2. We define an atom in a partially ordered commutative weak idempotent ring studied its role in the decomposition of a cWIR R with unity.

3. We examine further properties of the primary submaximal ideal of a commutative weak idempotent ring with unity.

   In our research work, we first define concepts, state theorems, and give their proofs using the definitions and different results given in the preliminary parts.

Organization

The remaining part of this dissertation is organized as follows:

In chapter 2, we collect definitions, examples, and results concerning the existing literature on Boolean like rings and weak idempotent rings. In the third chapter, we study further continuation on commutative weak idempotent rings with unity. We introduce the definition of partial ordering in commutative weak idempotent ring with unity and obtain that the introduced partial ordering coincides with the ordering in Boolean ring when restricted to the idempotent parts of the ring (see Lemma 3.1.1). We prove that any cyclic R_B submodule of a commutative weak idempotent ring R with unity is a Boolean lattice under the partial ordering defined in Lemma 3.1.1. Further, we obtain that the map $b \mapsto bx$ of $R_B \to R_B x$ is an isotone module epimorphism (see Theorem 3.1.7). Moreover, we establish some properties of commutative weak idempotent rings with unity concerning the partial ordering (see Theorem 3.1.12). We prove that Ann(N) is a prime ideal of commutative weak idempotent ring with unity if and only if every nonzero nilpotent element is an atom (see Theorem 3.2.1). Further, we prove that every nonzero nilpotent element is an atom whenever Ann(N) is a prime ideal of a commutative weak idempotent ring with unity (see Theorem 3.2.1). Also, we prove N_I is an R_B submodule of R and every nonzero element of N_I is an atom. We obtain a relation between a submodule N_I of R and atoms of N. We also establish that every finite nonzero commutative weak idempotent rings with unity admit an atomic basis (see Theorem 3.2.2).

In the fourth chapter, we prove that $bR = \{0, b\}$ or $\{0, b, n, b+n\}$ for idempotent and nilpotent atoms b and n of the ring R respectively, whenever any two nilpotent atoms have an upper bound in R (see Lemma 4.1.7). Further, we show that the subgroup generated by $\{n_i\}_{i\in\Delta}$ in R is a lattice (see Corollary 4.1.11), where $\{n_i\}_{i\in\Delta}$ is the collection of nilpotent atoms of R corresponding to the idempotent atoms $\{b_i\}_{i\in\Delta}$ of R. We also prove that R is atomic if and only if R_B is so (see Lemma 4.2.3), provided that N is nil-free, and any two nilpotent atoms have an upper bound in R. Finally, we obtain the direct product decomposition theorem of a commutative WIR R with unity (see Theorem 4.2.11).

In chapter 5, we study further properties of primary submaximal ideals. We prove that for a commutative weak idempotent ring R with unity and $0 \neq n \in N$, there is a primary ideal P of R so that $n \notin P$ and for nonzero $n_1 \in N$, $n_1 \in P$ or $n + n_1 \in P$, provided that $n \in P + R_B n_1$ (see Theorem 5.1.6). We also prove that if R is a commutative weak idempotent ring with unity and n_1 and n_2 are distinct nonzero nilpotent elements of R, then every primary submaximal ideal of R contains $n_1 n_2$

(see Theorem 5.2.2). We further prove that if P is a primary submaximal ideal of a commutative weak idempotent ring R with unity and $0 \neq n_1 \notin P$ for some $n_1 \in N$, then $P + R_B n_1$ is a maximal ideal of R and $P + \langle n_1 \rangle = P + R_B n_1$ (see Theorem 5.2.3). In the last part, we summarize the general aim of the dissertation with a conclusion and suggestions for further work on Weak idempotent rings.

Chapter 2

Literature Review and Preliminaries

Different researchers defined the term atom in various ways in relation to some basic concepts of commutative rings with unity and found several results using its properties. For instance, D.D. Anderson [2] defined it as an irreducible principal ideal, Sangmin Chun and D.D. Anderson [25] as a nonunit element a in the ring such that $a = bc$ for b, c in the ring implies $\langle a \rangle = \langle b \rangle$ or $\langle a \rangle = \langle c \rangle$ and Moshe Roitman [18] as a nonzero nonunit irreducible element in the ring. Besides, an atomic ring was also studied (see [2, 18, 25]). Atomic ring is defined as a ring in which each nonunit element is a finite product of atoms (see Sangmin Chun and D.D. Anderson [25]), each principal ideal is a finite product of atoms (see D.D. Anderson[2]), each reducible element is a finite product of atoms (see [18]). We also defined it in a commutative weak idempotent ring with unity in relation to partial ordering, and made further study of commutative weak idempotent rings with unity which was introduced by Venkateswarlu et.al [29]. Idempotent, unit and nilpotent elements of a ring plays important role in the theory of rings (see [19, 26]). An element b in a ring is idempotent and nilpotent if $b^2 = b$ and $b^n = 0$ for some $n \in \mathbb{N}$ respectively. An atom of a ring may be an idempotent or nilpotent element of the ring (see Definition 2.1.7). Investigators used the concepts of an atom in their study (see [2, 18, 25]) for further investigation of the properties of the ring under consideration. A lattice is a partially ordered set in which a subset $\{a, b\}$ of the set has both a supremum and an infimum (see [24]). Nilradical of a ring is the set of all nilpotent elements of the ring (see [15, 19, 20]) and the ring is nil-free (or simple) if it has no nonzero nil element. A ring R in which all elements are idempotent is a Boolean ring. Various investigators studied intensively to generalize the concepts of Boolean rings at different times (see [1, 8, 11, 16, 30]). For instance, the study made by A. L. Foster [[1]]: the concept of Boolean-like rings, Montgomery D. and N. H. McCoy [[21]]: a characterization of Boolean rings in broad views, I. Sussman [[8]]: A generalization of Boolean rings are some of the studies done in order to take a broad view of the Boolean rings. Of all these scholars, Foster is the one who introduced the theory of Boolean-like rings as a generalization of Boolean rings with unity. And also, Venkateswarlu et.al [[29]] introduced the structure of weak idempotent rings as a broad view of Boolean-like rings. So, some of the results from [1, 29] are used for further study about the properties of commutative weak idempotent rings with unity.

2.1 Lattice

In this subsection, we collected the definitions of different concepts such as partially ordered sets, upper and lower bounds, maximal element, lattice, atom, and some examples from various literature that are important in our study.

Definition 2.1.1. *[9] Let L be a non-empty set and '<' be a binary relation. We say, L is a partially ordered set (Poset, in short) if, for each a, b and c in L, we have*

1. *$a < a$ (reflexive)*
2. *$a < b$ and $b < a$ implies $a = b$ (anti-symmetry)*
3. *$a < b$ and $b < c$ implies $a < c$ (transitivity)*

Example 2.1.2. *[28] Let P(A) denotes the set of all subsets of a set A. Then $(P(A), \subseteq)$ is a partially ordered set.*

Definition 2.1.3. *[9] Let S be a subset of a Poset (L,<).*

1. *An element x_0 in L is called an upper bound for S if $x < x_0$ for each x in S.*

2. *An element x_0 in $(L,<)$ is a maximal element of L if, for each $b \in L$, $x_0 < b$ implies $x_0 = b$.*

3. *The least upper bound of a and b, for $a, b \in L$, denoted by $a \vee b$, is defined as $a < a \vee b$, $b < a \vee b$ and, for each element x_i in L, $a < x_i$ and $b < x_i$, $a \vee b < x_i$.*

4. *The greatest lower bound of a and b, for $a, b \in L$, denoted by $a \wedge b$, is defined as $a \wedge b < a$, $a \wedge b < b$ and, for each element x_j in L, $x_j < a$ and $x_j < b$, $x_j < a \wedge b$.*

Definition 2.1.4. *[9, 10, 24] Let $(L, <)$ be a partially ordered set. L is called a lattice if, for each pair of elements a, b in L, there exist a greatest lower and a least upper bounds which are denoted by $a \wedge b$ and $a \vee b$ respectively.*

Definition 2.1.5. *[5, 23] A Boolean lattice is a complemented distributive lattice. That is, A structure $(L, \vee, \wedge, \,', \ 0, \ 1)$ which satisfies the following axioms: for each $p, q, r \in L$*

1. *$(p \vee q) \vee r = p \vee (q \vee r)$ and $(p \wedge q) \wedge r = p \wedge (q \wedge r)$ (Associative Laws)*

2. *$p \vee q = q \vee p$ and $p \wedge q = q \wedge p$ (Commutative Laws)*

3. *$(p \vee q) \wedge r = (p \wedge r) \vee (q \wedge r)$ and $(p \wedge q) \vee r = (p \vee r) \wedge (q \vee r)$ (Distributive Laws)*

4. *$p \vee (p \wedge q) = p$ and $p \wedge (p \vee q) = p$ (Absorption Laws)*

5. *$p \vee p' = 1$, $p \wedge p' = 0$ and $(p')' = p$ (Complement Laws)*

6. *$(p \wedge q)' = p' \vee q'$ and $(p \vee q)' = p' \wedge q'$ (De'Morgan's Laws)*

Example 2.1.6. *[5, 23] Let $S = \{a, b, c\}$ and L be the set of all subsets of S. Then $(L, \cup, \cap, \,', \emptyset, S)$ is a Boolean lattice, where for $A \in L$, $A' = S \setminus A$.*

Definition 2.1.7. *[24] An element x_0 in a Boolean lattice $(L, <)$ is*

1. *an atom if $x_0 \neq 0$ and $b < x_0$ implies either $b = 0$ or $b = x_0$.*

2. *simple if $x_0 \neq 0$ and there is an idempotent element b such that $bx_0 = x_0$.*

3. *nil if, for all idempotent element b, $bx_0 = 0$.*

   Further, L is called atomic if, for each nonzero element x in L, there is an atom $m \in L$ with $m < x$ and nil-free if its only nil element is zero.

Example 2.1.8. *[24] From Example 2.1.6, $\{a\}, \{b\}, \{c\}$ are atoms of L and $(L, \subseteq)$ is an atomic lattice.*

Definition 2.1.9. *[24] A Boolean group is a group B in which every element has order two. That is, $b + b = 0$ for each b in B.*

Example 2.1.10. *Let $(R, +, \cdot)$ be a cWIR with unity. Then $(R, +)$ is a Boolean group.*

2.2 Boolean-Like Rings

In this subsection, we collected the definition of different concepts like: Boolean ring[1], symmetric Boolean ring [10], nilpotent element[10], nil-free ring[10], Boolean-like rings[1] and some examples[1, 24] from different literature that are applicable in our study.

Definition 2.2.1. *[10] Let R be a ring and $x \in R$. x is nilpotent if $x^n = 0$ for some $n \in \mathbb{N}$. R is nil-free (simple) if it has no nonzero nilpotent element.*

Definition 2.2.2. *[1] A ring R in which all elements are idempotent is a Boolean ring.*

Definition 2.2.3. *[10] A symmetric Boolean ring is the one that is isomorphic with the ring of all subsets of some set.*

Example 2.2.4. *[1, 24] Integer modulo 2 is a symmetric Boolean ring with unity.*

Definition 2.2.5. *[15, 19, 20] The set of all nilpotent elements of a ring is the Nilradical, N of the ring.*

A Boolean-like ring is introduced as a broad view of Boolean rings with unity.

Definition 2.2.6. *[1] A Boolean-like ring (BLR for short) is a commutative ring with unity R such that $ab(1 - a)(1 - b) = 0$ and $a + a = 0$ for all $a, b \in R$.*

Remark 2.2.7. *[1] A Boolean ring is a BLR. But the converse is not.*

Example 2.2.8. *[1] Consider $H_4 = \{0, 1, p, q\}$ and the binary operations $'+'$ and $'\cdot'$ are defined by the following tables.*

Table 2.1: BLR H_4

$+$	0	1	p	q
0	0	1	p	q
1	1	0	q	p
p	p	q	0	1
q	q	p	1	0

$\cdot$	0	1	p	q
0	0	0	0	0
1	0	1	p	q
p	0	p	0	p
q	0	q	p	1

The ring (H_4,+,·) is a BLR, but not a Boolean ring as $p \cdot p = 0 \neq p$.

2.3 Weak Idempotent Rings

This section is concentrated on some definitions, basic results, and examples regarding the concept of a weak idempotent ring.

Definition 2.3.1. *[29] A Weak idempotent ring (WIR, for short) is a ring* $(R, +, \cdot)$ *in which*

1. $b^4 = b^2$ *and*
2. $b + b = 0$ *for each* $b \in R$.

Remark 2.3.2. *[29] For a WIR R,*

1. *The only nilpotent and idempotent element in R is 0.*
2. *Each element in R can be expressed as the sum of idempotent and nilpotent elements of R. That is,* $b = b^2 + (b + b^2)$ *for each* $b \in R$.
3. *If R is commutative, then the representation in number 2 above is unique. So, we denote* b_B *for the idempotent* b^2 *and* b_N *for the nilpotent* $b+b^2$ *in the unique representation. That is,* $b = b_B + b_N$.

Remark 2.3.3. *[29] A Weak idempotent Ring need not be commutative.*

Example 2.3.4. *[29] A set* $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ *with the usual addition and multiplication of matrices over* $\mathbb{Z}_2$ *is a non-commutative WIR with unity.*

Example 2.3.5. *[29] Let* $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$ *and the binary operations* $'+'$ *and* $'\cdot'$ *are defined as follows. Then* $(R, +, \cdot)$ *is a non-commutative WIR.*

Table 2.2: Non-Commutative WIR

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	4	7	2	6	5	3
2	2	4	0	5	1	3	7	6
3	3	7	5	0	6	2	4	1
4	4	2	1	6	0	7	3	5
5	5	6	3	2	7	0	1	4
6	6	5	7	4	3	1	0	2
7	7	3	6	1	5	4	2	0

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	2	0	0	5	3	5
3	0	3	0	0	3	0	3	3
4	0	4	0	0	4	0	4	4
5	0	5	2	3	3	5	0	2
6	0	6	0	0	6	0	6	6
7	0	7	2	3	6	5	4	1

Lemma 2.3.6. *[29] For a WIR R, $0 \neq b \in R$ and $n \in \mathbb{N}$ we have the following.*

1. *$b^n = b$, b^2 or b^3.*

2. *If b is nilpotent, then $b^2 = 0$.*

Definition 2.3.7. *[29] A nonzero element x in a WIR R with unity is said to be a unit element of R if $x^2 = 1$.*

Remark 2.3.8. *[29] Let R be a WIR and n_1, and n_2 be nilpotent elements in R. The product $n_1 n_2$ need not be equal to zero.*

Example 2.3.9. *[29] Let $R = \{r : r = q_0 + q_1 i + q_2 j + q_3 k,$ where $q_i \in \mathbb{Z}_2$ for $0 \leq i \leq 3\}$ such that $i^2 = j^2 = k^2 = 1$, $ij = ji = k, ik = ki = j$ and $jk = kj = i$.*

$(R, \oplus_2, \odot_2)$ is a commutative weak idempotent ring with unity.
Here the elements $n_1 = 1 + i$ and $n_2 = 1 + j$ in R are distinct nilpotent elements of R but $n_1 n_2 = 1 + i + j + k \neq 0$.

2.3.1 Ideals of Weak Idempotent Rings

The concept of ideals of rings has been studied by different investigators, as it is very important to solve several problems in various fields of study. Ideals such as semiprime, prime, maximal, primary, completely prime, submaximal, and so on (see [2, 3, 19, 20, 27]). Of those researchers, Venkateswarlu Kolluru et al. [29] introduced the properties of maximal, semiprime, and submaximal ideals in weak idempotent rings with unity. Venkateswarlu Kolluru and Dereje Wasihun [27] introduced and studied the organization of submaximal ideals in WIR with unity. So, we use some of the results from [27, 29] as follows, which can help us to continue a further study on submaximal ideals of commutative weak idempotent rings (cWIR) with unity.
We begin with the following theorem.

Theorem 2.3.10. *[29] Let R be a WIR and I be an ideal of R. Then the quotient ring, R/I, is a WIR.*

Definition 2.3.11. *[29] A proper ideal M of a WIR R is maximal if there exists an ideal J of R such that $M \subseteq J \subseteq R$ implies either $M = J$ or $J = R$.*

Theorem 2.3.12. *[29] Let I and N respectively be an ideal and the collection of nilpotent elements of a cWIR R with unity. Then, we have the following.*

1. *I is maximal if and only if it is prime.*

2. *N forms an ideal of R.*

3. *If M is a maximal ideal of R, then $N \subseteq M$.*

4. *N is the intersection of all maximal ideals of R.*

Definition 2.3.13. *[29] An ideal I of a cWIR R is called prime if and only if, for $a, b \in R$, $ab \in I$ implies either $a \in I$ or $b \in I$.*

Definition 2.3.14. *[12] Let R be a ring and $S \subseteq R$. Annihilator of S, denoted by Ann(S), is defined as Ann(S) $= \{r \in R : rs = 0$ for all $s \in S\}$.*

Definition 2.3.15. *[29] In a commutative weak idempotent ring R with unity, a proper ideal P of R is primary if for $a, b \in R$, $ab \in P$ implies $a \in P$ or $b^n \in P$ for some $n \in \mathbb{N}$.*

Theorem 2.3.16. *[29] Let $x = x_B + x_N$ and I be an ideal of a cWIR R with unity such that $x \notin I$.*

1. *If $x_B \notin I$, then there is a maximal ideal J of R such that $I \subset J$ and $x \notin J$.*

2. *If $x_N \notin I$, then there is a primary ideal P of R such that $I \subset P$ and $x \notin P$.*

Definition 2.3.17. *[29] Let I be an ideal of a WIR R. I is submaximal if there is exactly one maximal ideal of R which covers I.*

Example 2.3.18. *[29] From Example 2.3.9, $Q_2 = \{0, j + k, 1 + i, 1 + i + j + k\}$, $Q_3 = \{0, i + k, 1 + j, 1 + i + j + k\}$, $Q_4 = \{0, i + j, 1 + k, 1 + i + j + k\}$ are all submaximal ideals of R.*

Theorem 2.3.19. *[29] Let I be an ideal of a cWIR R with unity. For an idempotent element $b \in R$, either $b \in I$ or $1 + b \in I$ if and only if I is primary.*

Theorem 2.3.20. *[29] The intersection of primary ideals of a cWIR with unity is $\{0\}$.*

Definition 2.3.21. *[27] Let I be a proper ideal of a cWIR R with unity. I is semiprime if, for $a \in R$, $a^2 \in I$ implies $a \in I$.*

Theorem 2.3.22. *[27] Let I be a submaximal ideal of a cWIR R with unity. I is covered by exactly two maximal ideals of R if and only if it is semiprime.*

Theorem 2.3.23. *[27] Let I be a submaximal ideal of a cWIR R with unity. There are at most two maximal ideals of R which covers I.*

Theorem 2.3.24. *[27] A primary submaximal ideal of a cWIR R with unity is covered by exactly one maximal ideal of R.*

Theorem 2.3.25. *[29] A submaximal ideal of a cWIR with unity is either primary or semiprime.*

Theorem 2.3.26. *[29] Let M_1 and M_2 be two different maximal ideals of a cWIR with unity. Then $I = M_1 \cap M_2$ is submaximal. Furthermore, M_1 and M_2 are the only maximal ideals of R containing I.*

Theorem 2.3.27. *[29] Let R be a cWIR with unity and $I \neq R$ be an ideal of R. Then I is the intersection of each primary ideals of R which contains I.*

Example 2.3.28. *[29] From Example 2.3.9, the ideal of R such as $Q = \{0\}$, $Q_1 = \{0, 1+i+j+k\}$, $Q_2 = \{0, j+k, 1+i, 1+i+j+k\}$, $Q_3 = \{0, i+k, 1+j, 1+i+j+k\}$, $Q_4 = \{0, i+j, 1+k, 1+i+j+k\}$ are all primary.*

2.4 Module

This section deals with the definition of a module and its examples, and about the basis of a vector space.

Definition 2.4.1. *[10] An R-module of a commutative ring R with unity is an abelian group (M,+) in which, for each elements m_1, m_2 and m in M, and r_1, r_2 and 1 in R, a scalar multiplication $R \times M \to M$, denoted by $(r, m) \mapsto rm$ satisfies the following axioms:*

1. $r(m_1 + m_2) = rm_1 + rm_2$
2. $(r_1 + r_2)m = r_1m + r_2m$
3. $(r_1r_2)m = r_1(r_2m)$
4. $1m = m$

Definition 2.4.2. *[10] A module map is called an isotone if it preserves order.*

Example 2.4.3. *[10]*

1. *An abelian group is a $\mathbb{Z}$-module.*

2. *A commutative ring R with unity is an R-module.*

Definition 2.4.4. *[10] Let M be an R-module and $S \subseteq M$ be an additive subgroup of M. We say S is a submodule of M if, for $s \in S$ and $r \in R$, $sr \in S$.*

Example 2.4.5. *An ideal of a ring is a submodule of the ring.*

Definition 2.4.6. *[10] Let R be a ring and M be an R-module. We say that M is a cyclic module if there exists an element $x \in M$ such that $M = Rx = \{rx : r \in R\}$.*

Definition 2.4.7. *[10] Let V be a vector space over a field F and $X = \{x_1, x_2, ..., x_m\}$ be a collection of elements of V. If, for some nonzero scalars $\alpha_1, \alpha_2, ..., \alpha_m \in F$, $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_m x_m = 0$, X is called linearly dependent and independent if $\alpha_1 = \alpha_2 = ... = \alpha_m = 0$. Further, if $v = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_m x_m$ for each element v in V, we say X is a basis of V.*

Example 2.4.8. *[10] $\mathbb{R}^2$ is a vector space over $\mathbb{R}$ and $\{(1,0),(0,1)\}$ is its standard basis.*

Chapter 3

Ordered Weak Idempotent Rings

3.1 Partial Ordering

This section is a further continuation of the work on commutative WIRs (cWIR, for short) with unity. We introduce the partial ordering in cWIRs with unity and prove that it coincides with the partial ordering in a Boolean ring when restricted to the idempotent parts of the ring and find certain results on a Poset $(R, <)$. We prove that any cyclic module over the collection of Boolean elements of a cWIR with unity is a Boolean lattice. We also prove that Ann(N) is a prime ideal of a cWIR R with unity if and only if every nonzero nilpotent element of R is an atom. Further, we prove N_I is an R_B submodule of R, and every nonzero element of N_I is an atom. We obtain a relation between a submodule N_I of R and atoms of N. We also establish that every finite nonzero cWIR admits an atomic basis.

We now begin with the following.

Lemma 3.1.1. *Let R be a cWIR with unity and R_B be the set of all idempotent elements of R. For $x,\ y \in R$, define $y < x$ if and only if there exists $b \in R_B$ such that $bx = y$. Then, $(R, <)$ is a poset. If the relation is restricted to R_B, the partial ordering " $<$ " coincides with the definition of the partial ordering over the Boolean ring.*

Proof. Assume that R is a cWIR with unity and R_B is the set of all idempotent elements of R.

1. Let $x \in R$. Claim $<$ is reflexive. $x < x$ as $x = 1x$ and 1 in R. Hence $<$ is reflexive.

2. Let $x < y$ and $y < x$ for $x, y \in R$. Claim that $x = y$. Then, there exist elements $b_1, b_2 \in R_B$ such that $b_1 y = x$ and $b_2 x = y$ by Lemma 3.1.1 and our assumption. So, as R is commutative and $b_2 \in R_B$, $x = b_1 y = b_1 b_2^2 x = b_2 b_1 b_2 x = b_2 b_1 y = b_2 x = y$. Hence, $<$ is anti-symmetric.

3. For $x, y, z \in R$, let $x < y$ and $y < z$. We claim that $x < z$. Then, there exist elements b_1 and b_2 in R_B by our assumption and Lemma 3.1.1 so that $x = b_1 y$ and $y = b_2 z$. So, $x = b_1 y = b_1 b_2 z$. Thus, $x < z$ by Lemma 3.1.1 which implies $<$ is transitive.

Therefore, $(R, <)$ is a Poset by (1), (2) and (3) above.

Let $a, b \in R_B$ such that $a < b$. Claim that $ab = a$. Then there exists $x \in R_B$ such that $bx = a$ by our assumption and definition of $<$. So, $ab = b^2 x = bx = a$ as $b^2 = b$. If $ab = a$, then $a < b$ as $a \in R_B$. Therefore, the partial ordering " $<$ " restricted to R_B coincides with the definition of partial ordering over Boolean rings. □

Note: From now on wards $(R, <)$ represents a partially ordered cWIR R with unity as defined in Lemma 3.1.1.

Example 3.1.2. *Let R be a cWIR with unity. Then, the direct product $R \times R$ is also a cWIR with unity $(1,1)$. At least $(1,0), (0,1)$ and $(1,1)$ are idempotent elements of $R \times R$. Consider $(1,0)(a,b) = (a,0)$ and $(0,1)(a,b) = (0,b)$ for $a, b \in R$. Thus, $(a,0) < (a,b)$ and $(0,b) < (a,b)$.*

Remark 3.1.3. *Let R be a cWIR with unity and R_B be the set of all idempotent elements of R. Then R is an R_B-module.*

Proof. Let R be a cWIR with unity and R_B be the set of all idempotent elements of R. For each $r_1, r_2, r \in R$ and $b, b_1, b_2 \in R_B$ the following hold.

1. $b(r_1 + r_2) = br_1 + br_2$ and $(b_1 + b_2)r = b_1r + b_2r$ by distributive property of R.

2. $(b_1b_2)r = b_1(b_2r)$ by associative property of R.

3. $1r = r$ by property of unity in R.

Therefore R is an R_B- module. $\square$

Remark 3.1.4. *Every cWIR with unity is a vector space over the field* $\{0, 1\}$.

Proof. Let R be a cWIR with unity and $F = \{0, 1\}$. Claim that R is a vector space over F. As $F = \{0, 1\} \subset R$, R is an F-module by Remark 3.1.3. So, as F is a field, R is a vector space over F. $\square$

Remark 3.1.5. *If (R,+,$\cdot$) is a cWIR with unity, then*

1. *(R,+) is a Boolean group.*

2. *(R,+) is a vector space over* $\{0, 1\}$ *and hence has a basis.*

Proof. Let (R,+,$\cdot$) be a cWIR with unity.

1. Claim that (R,+) is a Boolean group. As R is a ring (R,+) is an abelian group. As R is a cWIR with unity $r + r = 0$ for each $r \in R$. Thus (R,+) is a group of order two and hence a Boolean group.

2. (R,+) is a vector space over $\{0, 1\}$ by Remark 3.1.4 and hence has a basis $\{1\}$.

$\square$

Definition 3.1.6. *Let S be an additive subgroup of a cWIR R with unity. If there is* $a \in R$ *such that* $S = R_Ba$*, then we say that S is a cyclic* R_B*-submodule of R.*

Theorem 3.1.7. *Any cyclic* R_B*-submodule of a cWIR with unity is a Boolean lattice under the ordering defined in Lemma 3.1.1. Further, the map* $b \mapsto bx$ *of* $R_B \to R_Bx$ *is an order-preserving module epimorphism.*

Proof. Let $R_Bx = \{bx : b \in R_B\}$ be a cyclic R_B-submodule of a cWIR with unity and partially ordered as in Lemma 3.1.1. Claim that $(R_Bx, <)$ is a lattice. For b_1x and b_2x in R_Bx, $b_1(b_1 + b_2 + b_1b_2)x = b_1^2x + b_1b_2x + b_1^2b_2x = b_1x + b_1b_2x + b_1b_2x = b_1x$

as $b_1, b_2 \in R_B$ and R is of characteristic two. Similarly, $b_2(b_1 + b_2 + b_1b_2)x = b_2x$. Hence $b_1x < (b_1 + b_2 + b_1b_2)x$ and $b_2x < (b_1 + b_2 + b_1b_2)x$. Suppose that bx in R_Bx is an upper bound of b_1x and b_2x. Then $b_1x < bx$ and $b_2x < bx$. It follows that there exist $c, d \in R_B$ such that $cbx = b_1x$ and $dbx = b_2x$ by Lemma 3.1.1. As $(c + d + cb_2)bx = (b_1 + b_2 + b_1b_2)x$ and $c + d + cb_2 \in R_B$, $(b_1 + b_2 + b_1b_2)x < bx$ and hence $(b_1 + b_2 + b_1b_2)x$ is the least upper bound of b_1x and b_2x. $b_1b_2x < b_1x$ and $b_1b_2x < b_2x$ as $b_2b_1x = b_1b_2x$, b_1 and $b_2 \in R_B$ and by Lemma 3.1.1. Suppose that bx in R_Bx is a lower bound of b_1x and b_2x. That is, $bx < b_1x$ and $bx < b_2x$. So, there exist elements c and $d \in R_B$ such that $cb_1x = bx$ and $db_2x = bx$ by Lemma 3.1.1. Thus, $cb_1x = db_2x$ and $cb_1b_2x = db_2x = bx$. Hence $bx < b_1b_2x$ and b_1b_2x is the greatest lower bound of b_1x and b_2x. Therefore, $(R_Bx, <)$ is a lattice.
Define a map $\phi : R_B \to R_Bx$ by $\phi(b) = bx$ for each $b \in R_B$.

Claim ϕ is well defined module epimorphism. Assume that $b_1 = b_2$ for $b_1, b_2 \in R_B$. Then $b_1x = b_2x$ and hence $\phi(b_1) = \phi(b_2)$ for each $b_1 = b_2$ in R_B. Therefore, ϕ is well defined. For $b_1, b_2 \in R_B$ and $r \in R$, we have $\phi(b_1 + b_2) = (b_1 + b_2)x = b_1x + b_2x = \phi(b_1) + \phi(b_2)$, $\phi(rb_1) = (rb_1)x = r(b_1x) = r\phi(b_1)$ and $\phi(1) = 1x = x$. Thus, ϕ is a module homomorphism. Thus ϕ is a module epimorphism by its definition.

Again we claim that ϕ is an order preserving map. Assume that $b_1,\ b_2 \in R_B$ and $b_1 < b_2$. Then, $b_1b_2 = b_1$ by Lemma 3.1.1 and our assumption. So, $\phi(b_1) = b_1x = b_1b_2x = b_1\phi(b_2)$. Therefore, $\phi(b_1) < \phi(b_2)$ by Lemma 3.1.1 and hence ϕ is an order preserving (an isotone) map. □

Remark 3.1.8. *Let R be a cWIR with unity, $b \in R_B$ and $x \in R$ be an atom of R. Let $(R, <)$ be as defined in Lemma 3.1.1. Then $bx = x$ or $bx = 0$.*

Proof. Assume that the hypotheses hold. Then $bx = bx$ implies $bx < x$ as $b \in R_B$ and by Lemma 3.1.1. Hence $bx = 0$ or $bx = x$ as x is an atom of R. □

Lemma 3.1.9. *Let a cWIR $(R, <)$ with unity have idempotent atoms. Then, the following hold.*

1. *Every simple element of R is an atom of R.*

2. *Every atom of R is either nil or simple.*

Proof. Let a cWIR R with unity have idempotent atoms.

1. Let $r \in R$ be a simple element of R. Then, there exists an atom $b \in R_B$ such that $br = r$ by hypothesis. Assume that $x < r$, for $x \in R$.

   Claim that $x = r$ or $x = 0$. We get $b_1 r = x$ for some $b_1 \in R_B$ by our assumption and Lemma 3.1.1. Hence $bx = bb_1 r = b_1 br = b_1 r = x$ as $br = r$ and R is commutative. Thus, $x = b_1 br = br = r$ or $x = rb_1 b = 0$ as $b_1 b = b$ or 0 for the idempotent atom b. Therefore, r is an atom of R.

2. Let r in R be an atom. Claim that r is either nil or simple. We have that $br < r$ for each $b \in R_B$ by Remark 3.1.8. As r is an atom of R, $br = r$ or $br = 0$ for each $b \in R_B$. If $br = 0$ for all $b \in R_B$, then r is a nil element of R. Otherwise, there exists some $b \in R_B$ such that $br = r$. Hence r is either a nil or simple element of R.

□

Lemma 3.1.10. *Every nonzero nil element in a partially ordered cWIR (R, $<$)(as defined in Lemma 3.1.1) with unity is an atom.*

Proof. Let $x \neq 0$ be a nil element of a partially ordered cWIR R with unity such that $y < x$ for $y \in R$. Claim that x is an atom of R. It follows that there exists $b \in R_B$ such that $bx = y$ by Lemma 3.1.1. Hence $y = bx = 0$ for all $b \in R_B$ as x is a nil element of R. Therefore, x is an atom of R. □

Lemma 3.1.11. *Let R be a cWIR with unity and b be an idempotent element of R. If $1 + b \in Ann(N)$, then N is nil-free.*

Proof. Suppose that b is an element of R_B in a cWIR R with unity such that $1 + b \in Ann(N)$. Claim that $bn \neq 0$ for each $n \in N$. As $1 + b \in Ann(N)$ $bn = n$ for all nonzero n in N. This implies every nonzero element of N is not nil and hence N is nil-free. □

Theorem 3.1.12. *In a partially ordered cWIR (R, $<$) with unity, we have the following properties.*

1. *$0 < a$ for all $a \in R$.*

2. *$b < a$ implies that $bc < ac$ for each $a, b, c \in R$.*

3. *For each $a \in R$, $a^3 < a$ and $a_N(1 + a_B) < a$.*

4. *For $a, b \in R, b < a$ implies*

   (a) *$ab = b_B = a_Bb_B$.*

   (b) *$a_Bb_N = a_Nb_B = b_Bb_N$.*

   (c) *$ab_B = a_Bb = bb_B$.*

   (d) *$xa = xb = b$ for some $x \in R_B$.*

   (e) *$b_B < a_B$ and $b_N < a_N$.*

5. *$a < 1$ if and only if $a \in R_B$.*

6. *For $a \in R$ and $b \in R_B, b < a$ if and only if $ab = b$.*

7. *Let a be a unit element in R and $b \in R$. $b < a$ if and only if $ab = b_B$.*

8. *For $a \in R_B$ and $b \in R$, $b < a$ implies $b \in R_B$.*

9. *For $b \in R$ and $a \in N$, $b < a$ implies $b \in N$ (set of all nilpotent elements of R).*

10. *Units in R are maximal elements in $(R, <)$.*

Proof. 1. As $a0 = 0$ for all $a \in R$, $0 < a$ for all $a \in R$.

2. Suppose that $b < a$ for $a, b \in R$.

   Claim that $bc < ac$ for each $c \in R$. It follows that $ax = b$ for some $x \in R_B$ by Lemma 3.1.1 and $acx = bc$ for all $c \in R$. Hence $bc < ac$ for each $a, b, c \in R$ as $x \in R_B$.

3. For any $a \in R$, $aa^2 = a^3$ and a^2 idempotent. Hence $a^3 < a$ by Lemma 3.1.1.

Consider that $a(1+a^2) = a(1+a^2)$ and $(1+a^2)$ idempotent. This gives us $a(1+a^2) < a$ by Lemma 3.1.1. It follows that $a + a^2 + a^3 + a^4 < a$. But $a + a^2 + a^3 + a^4 = a + a^2 + a^2(a + a^2) = (a + a^2)(1 + a^2) = a_N(1 + a_B)$. Hence $a_N(1 + a_B) < a$.

4. For $a, b \in R, b < a$ implies

   (a) $ax = b$ for some $x \in R_B$ by Lemma 3.1.1. This gives us $ab = a^2x = a_Bx$ as a^2 is idempotent and $b_B = b^2 = a^2x = a_Bx = ab$ as $x \in R_B$. Hence $b_B = b^2 = (ab)^2 = a^2b^2 = a_Bb_B$ as $a^2, b^2 \in R_B$.

   (b) As $ab = b_B = b^2$ by (4a) above and R is commutative, we have $a^2b = aba = ab^2 = b^2b$. As R is of characteristic two, $b = b^2 + (b + b^2) = b_B + b_N$ hence $a_Bb_N = a^2(b+b^2) = a^2b+a^2b^2$ But $a^2b = a(ab) = ab^2$ by (4a) above. So, $a_Bb_N = ab^2 + a^2b^2 = (a + a^2)b^2 = a_Nb_B$ and $a_Nb_B = (a + a^2)b^2 = ab^2 + a^2b^2 = b^2b + b^2 = b^2(b + b^2) = b_Bb_N$ as $ab = b^2$ from (4a) above and $b^2 = b^4$ for each $b \in R$.

   (c) $ab_B = ab^2 = (ab)b = b^2b = bb_B$ and $a_Bb = a^2b = ab^2 = b_Ba$.

   (d) $xa = b$ implies that $xa = x^2a = xb = b$ for some $x \in R_B$.

   (e) $b < a$ implies $ax = b$ for some $x \in R_B$. As $a^2x = ab = b^2,\ b_B < a_B$. The result $ax = b$ implies $ax + a^2x = b + b^2$. Hence $b_N < a_N$.

5. Let $a < 1$ for $a \in R$. Then $1x = a$ for some $x \in R_B$ and hence $a \in R_B$. For $a \in R_B$, $1.a = a$ which implies $a < 1$.

6. Let $a \in R$ and $b \in R_B$. If $b < a$, then $ax = b$ for some $x \in R_B$. Thus, $a^2x = b$ and hence $ab = b$. If $ab = b$, then $b < a$ as $b \in R_B$.

7. Let a be a unit element in R and $ab = b_B$ for $b \in R$. It follows that $b = a^2b = ab_B$ as a is a unit element of R. Hence $b < a$. The converse follows from 4(a).

8. Let $a \in R_B$ such that $b < a$ for $b \in R$. Then $ax = b$ for some $x \in R_B$ and $b^2 = (ax)^2 = ax = b$ which implies $b \in R_B$.

9. Let $a \in N$ and $b \in R$ such that $b < a$. Then $ax = b$ for some $x \in R_B$ and $b^2 = (ax)^2 = a^2x^2 = 0$ which implies $b \in N$.

10. For a unit element a in R, let $a < b$ for $b \in R$. So, $bx = a$ for some $x \in R_B$ and $b^2x = 1$ which implies x is a unit element. Thus $x = 1$ as it is both a unit and an idempotent element. Hence $a = b$. i.e. a is a maximal element in $(R, <)$.

□

From properties number 8 and 9 above, we observe that nonzero idempotent and nilpotent elements of R are incomparable.

Remark 3.1.13. *A unit element of R is maximal in the poset* $(R, <)$. *However, the converse fails. Consider the following.*

Example 3.1.14. *Let* $H_4 = \{0, 1, p, 1+p\}$ *from example 2.2.8 and* $B = \{0, a, b, a+b\}$ *be a Boolean group of 4 elements. Define a unitary* H_4*-module system on B where the multiplication is generated from the following:* $pa = a$ *and* $pb = 0$. *Consider the ring* $R = (H_4 \times B, +, .)$ *where the operations are defined as:* $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$ *and* $(a_1, b_1)(a_2, b_2) = (a_1a_2, a_2b_1 + a_1b_2)$. *Then R is a cWIR with unity. The element* $(p, a+b)$ *is a maximal element with the above order relation but it is not a unit element.*

Theorem 3.1.15. *Let (R, <) be a partially ordered cWIR with unity as defined in Lemma 3.1.1 and* a, b, x *be elements in R. If* $b < a$ *and* $b+x < a+x$, *then* $ax = bx$.

Proof. Suppose that $b < a$ and $b + x < a + x$ for a, b, x in R. It follows that $ab = b_B$ and $(a + x)(b + x) = (b + x)_B = b_B + x_B$ by Theorem 3.1.12(4a). Simplifying the equation we have $ab + x_B + ax + bx = b_B + x_B$ which gives us $ax = bx$. □

But $b + x < a + x$ is not in general true even though $b < a$ and $ax = bx$.
In Example 3.1.14, $(0, 0) < (p, 0)$ and $(0, 0)(p, 0) = (p, 0)(p, 0)$ but $(p, 0) \not< (0, 0)$.

Remark 3.1.16. *In a partially ordered cWIR (R, <) with unity as defined in Lemma 3.1.1, if for any* $a, b \in R, b < a$ *implies* $b + x < a + x$ *for all* $x \in R$, *then* $R = \{0\}$.

Proof. Suppose that for $a, b \in R$, $b < a$ implies $b + x < a + x$ for all $x \in R$.

We claim that $a = b = x$.

Then $b + a < a + a \Rightarrow a + b < 0$ as R is of characteristic two. But $0 < a + b$ by Theorem 3.1.12(1). As $<$ is anti-symmetric, $a + b = 0$. Thus $a = b$ as R is of characteristic two. As $b + x < a + x$ for all $x \in R$, $b + x = y(a + x)$ for some $y \in R_B$ by Lemma 3.1.1. This implies $b_B + x_B = y(a_B + x_B)$ by squaring both sides. So, $b_B + x_B = yb_B + yx_B$ as $a_B = b_B$. Again we have $b_B(1 + y) = x_B(1 + y)$. So, $b_B(1 + y) < x_B$ as $1 + y \in R_B$ by Lemma 3.1.1. From this and our hypothesis we have $b_B(1 + y) + x_B < 0$. From Theorem 3.1.12(1) and as $<$ is anti-symmetric, we have $b_B(1 + y) + x_B = 0$. As R is of characteristic two we have $b_B(1 + y) = x_B$ and hence $x_B < b_B$ by Lemma 3.1.1 as $1 + y \in R_B$. Using our hypothesis and this we have $x_B + b_B < 0$. So, by Theorem 3.1.12(1) and this gives us $b_B = x_B$ for each $x \in R$. Thus $a = b = x$ for all x in R. Therefore $R = \{0\}$. □

3.2 Atoms and Atomic Basis

This section deals with different results in relation to an idempotent and (or nilpotent) atoms of a cWIR R with unity and the properties of a partially ordered non zero cWIR R (as defined in Lemma 3.1.1) in relation to its atomic basis (see Theorem 3.2.2).

We begin with the following.

Theorem 3.2.1. *Let N be the nilradical of a cWIR R with unity. $Ann(N)$ is a prime ideal if and only if each nonzero element of N is an atom.*

Proof. ($\Rightarrow$:) Suppose that $Ann(N)$ is a prime ideal of R and $0 \neq n \in N$.

Claim that n is an atom of N.

Then, for each $b \in R_B$, $b \in Ann(N)$ or $1 + b \in Ann(N)$ as $b(1 + b) \in Ann(N)$ and our hypothesis. That is, $bn = 0$ or $(1+b)n = 0$ for all $n \in N$. These gives us $bn = n$ or 0 for all $n \in N$.But $bn < n$ and either $bn = n$ or 0 for each $0 \neq n \in N$. Thus, every nonzero nilpotent element is an atom of R by Remark 3.1.8.

($\Leftarrow$:)Conversely, suppose that every nonzero nilpotent element of R is an atom and assume that $Ann(N)$ is not prime. It follows that there is an element $b \in R_B$ with $b \notin Ann(N)$ and $1 + b \notin Ann(N)$. That is, there exist nonzero nilpotent elements n_1 and n_2 such that $bn_1 \neq 0$ and $(1 + b)n_2 \neq 0$. As n_1 and n_2 are atoms, $bn_1 = n_1$ and $(1 + b)n_2 = n_2 \Rightarrow bn_2 = 0$. So, $b(n_1 + n_2) = bn_1 + bn_2 = n_1$. So, $n_1 + n_2$ is not an atom by definition which is a contradiction to our hypothesis. Therefore, the ideal, $Ann(N)$ is prime. □

Theorem 3.2.2. *Let $(R, <)$ be a partially ordered non zero cWIR with unity. If R is finite, then it has an atomic basis.*

Proof. Let R be a nonzero finite partially ordered cWIR with unity. Considering the partial ordering over R, $(R_B r, <)$ is Boolean lattice for $0 \neq r \in R$. Assume that br is an atom of $R_B r$ and $x < br$ for $x \in R$. Claim that $x = br$ or 0. There exists $b_1 \in R_B$ such that $b_1 br = x$ by Lemma 3.1.1 and hence $x \in R_B r$. As br is an atom of $R_B r$, $x = br$ or $x = 0$. Thus, br is an atom of R. As $R_B r$ is a finite Boolean lattice, it is atomic. Hence r is the join of all atoms of $R_B r$. Let $b_1 r$ and $b_2 r$ be two distinct atoms of $R_B r$. We know that $b_1 b_2 r < b_1 r$ and $b_1 b_2 r < b_2 r$ and hence $b_1 b_2 r = 0$ because $b_1 r \neq b_2 r$. Thus $b_1 r \vee b_2 r = (b_1 \vee b_2)r = (b_1 + b_2)r$ as $R_B r$ is a Boolean lattice. Let $b_1 r, b_2 r, b_3 r, ..., b_k r$ be distinct atoms of $R_B r$. Then $b_1 r \vee b_2 r \vee b_3 r \vee ... \vee b_k r = (b_1 + b_2 + b_3 + ... + b_k)r$. Therefore, $r = \sum_{i=1}^{n} b_i r$, where $\{b_i r\}_{1 \leq i \leq n}$ are all atoms of $R_B r$. That is, r is the sum of atoms of R. Let A be the set of all atoms of R. A is nonempty as R is finite. As we proved above, every nonzero element of R is the sum of elements of A. Thus, A generates R as a vector space over $\{0, 1\}$. That is, R has a basis contained in A. Hence R has an atomic basis. □

Corollary 3.2.3. *If a partially ordered cWIR (R,<) with unity has a finite nilradical N, then N has an atomic basis.*

Proof. Let N be a nonzero finite nilradical if a cWIR R with unity. Consider the partial ordering given by Lemma 3.1.1 over N and the Boolean lattice $(R_B n, <)$ for $0 \neq n \in N$. Assume that bn is an atom of $R_B n$ and $x < bn$ for $x \in N$. There exists $b_1 \in R_B$ such that $b_1 bn = x$ and hence $x \in R_B n$. As bn is an atom of $R_B n$, $x = bn$ or $x = 0$. Thus, bn is an atom of N. As $R_B n$ is a finite Boolean lattice, it is atomic.

Hence n is the join of all atoms of R_Bn. Let b_1n and b_2n be two distinct atoms of R_Bn. We know that $b_1b_2n < b_1n$ and $b_1b_2n < b_2n$ and hence $b_1b_2n = 0$ because $b_1n \neq b_2n$. Thus $b_1n \vee b_2n = (b_1 \vee b_2)n = (b_1 + b_2)n$ as R_Bn is a Boolean lattice. Let $b_1n, b_2n, b_3n, ..., b_kn$ be distinct atoms of R_Bn. Then $\bigvee_{i=1}^{k} b_in = (b_1+b_2+b_3+...+b_k)n$. Therefore, $n = \sum_{i=1}^{n} b_in$, where $\{b_in\}_{1\leq i\leq k}$ are all atoms of R_Bn. That is, n is the sum of atoms of N. Let A be the set of all atoms of N. A is nonempty as N is finite. As we proved above, every nonzero element of N is the sum of elements of A. Thus, A generates N as a vector space over $\{0, 1\}$. That is, N has a basis contained in A. Hence N has an atomic basis. □

Theorem 3.2.4. *Let n be a nonzero nilpotent element of a cWIR (R, <) with unity. n is an atom of R if and only if $Ann(n) \cap R_B$ is a prime ideal of R_B, where $Ann(n) = \{x \in R : xn = 0\}$.*

Proof. Let n be a nonzero nilpotent element of a cWIR (R, <) with unity.
($\Rightarrow$:) Assume that n is an atom. Claim that $Ann(n) \cap R_B$ is a prime ideal of R_B. It follows that $bn = 0$ or $bn = n$ for every $b \in R_B$. Thus $b \in Ann(n) \cap R_B$ or $1 + b \in Ann(n) \cap R_B$. Let a and b be in R_B such that $ab \in Ann(n) \cap R_B$. Assume that $a \notin Ann(n) \cap R_B$. Then $1 + a \in Ann(n) \cap R_B$ as n is an atom of R. As $Ann(n) \cap R_B$ is an ideal of R_B, $b + ab = b(1 + a) \in Ann(n) \cap R_B$. Hence $b \in Ann(n) \cap R_B$.Therefore, $Ann(n) \cap R_B$ is a prime ideal of R_B.

($\Leftarrow$:) Conversely, suppose that $Ann(n) \cap R_B$ is a prime ideal of R_B and $x < n$ for $x \in R$. Claim that $x = n$ or 0. Then $bn = x$ for some $b \in R_B$ by Lemma 3.1.1. If $bn = x \neq 0$, then $b \notin Ann(n) \cap R_B$. So, $1 + b \in Ann(n) \cap R_B$ by Theorem 2.3.19 as $R_B \cap Ann(n)$ is prime. Thus $(1 + b)n = 0$ which implies $x = bn = n$. Hence n is an atom of R. □

Theorem 3.2.5. *Let n be a nonzero nilpotent element of a cWIR (R, <) with unity. $Ann(n)$ is a prime ideal of R if and only if n is an atom of R.*

Proof. Let n be a nonzero nilpotent element of a cWIR (R, <) with unity.

($\Rightarrow$:) Suppose that n is an atom of R. For every $b \in R_B$, $b(1 + b) \in Ann(n)$. Claim that either b or $1 + b$ belongs to $Ann(n)$. As n is an atom we have $bn = 0$ or $bn = n$

for each $b \in R_B$. That is, $b \in Ann(n)$ or $1 + b \in Ann(n)$. Therefore, $Ann(n)$ is a prime ideal of R.

($\Leftarrow$:) Conversely, suppose that $Ann(n)$ is a prime ideal of R and $x < n$ for $x \in R$. Then $bn = x$ for some $b \in R_B$. If $bn = x \neq 0$, then $b \notin Ann(n)$ and hence $1 + b \in Ann(n)$. Thus, $(1 + b)n = 0$ and hence n is an atom of R. $\square$

Theorem 3.2.6. *For every nilpotent atom n of a cWIR $(R, <)$ with unity, if $n < x$ for some $x \in R$, then $xn = 0$.*

Proof. Let n be a nilpotent atom of R and $n < x$ for some $x \in R$. Then, $xn = n_B = 0$ by Theorem 3.1.12 4(a) which is the result. $\square$

Theorem 3.2.7. *For a prime ideal I of R_B in a cWIR R with unity, define $N_I = \{n \in N : bn = 0$ for all $b \in I\}$. Then:*

1. *N_I is an R_B-submodule of R.*
2. *Every nonzero element of N_I is an atom.*
3. *For every $0 \neq n \in N_I, Ann(n) \cap R_B = I$.*

Proof. Suppose that I is a prime ideal of R_B in a cWIR R with unity.

1. As R is of characteristic two, N_I is an additive subgroup of R. Suppose that $n \in N_I$ for a prime ideal I of R_B and $b \in R_B$. Claim that $bn \in N_I$. $b_i bn = bb_i n = b0 = 0$ for each $b_i \in I$ as $b_i n = 0$ by Definition of N_I. Thus $bn \in N_I$. Therefore, N_I is an R_B-submodule of R.

2. As I is prime, for each $b \in R_B$, either $1 + b \in I$ or $b \in I$. Thus, for every $0 \neq n \in N_I$ and for each $b \in R_B$, either $bn = n$ or $bn = 0$. Hence, every nonzero element of N_I is an atom.

3. For every $0 \neq n \in N_I$, $x \in I$ implies that $xn = 0$. Thus $x \in Ann(n)$ and hence $x \in Ann(n) \cap R_B$ as I is prime. So, $I \subseteq Ann(n) \cap R_B$. But every prime ideal is maximal in a Boolean ring and hence for every $0 \neq n \in N_I$, $Ann(n) \cap R_B = I$ by hypothesis.

□

Theorem 3.2.8. *Let $I \subseteq R_B$ be a prime ideal of a cWIR R with unity. There is no nilpotent atom if and only if $N_I = \{0\}$ for every prime ideal I of R_B.*

Proof. ($\Rightarrow$:) Suppose $N_I \neq \{0\}$ for some prime ideal I of R_B. It follows that N has a nonzero atom by Theorem 3.2.7(2). Hence N has no atom if $N_I = \{0\}$ for each prime ideal I.

($\Leftarrow$:) Conversely, suppose that $N_I = \{0\}$ for every prime ideal I of R_B. Assume that $0 \neq n \in N$ is an atom. Then $Ann(n) \cap R_B$ is a prime ideal of R_B by Theorem 3.2.4 and $0 \neq n \in N_I$ for $I = Ann(n) \cap R_B$ which is a contradiction to our hypothesis. Hence N has no atom. □

Corollary 3.2.9. *Let $I \subseteq R_B$ be a prime ideal of a cWIR R with unity and $0 \neq n_1, n_2 \in N_I$, where $n_1 \neq n_2$. Then n_1, n_2 and $n_1 + n_2$ are all atoms of R and $Ann(n_1) \cap R_B$=$Ann(n_2) \cap R_B$=$Ann(n_1 + n_2) \cap R_B = I$.*

Proof. Suppose that n_1, n_2 are distinct nonzero nilpotent elements in N_I for a prime ideal of R_B. Then $b(n_1 + n_2) = 0$ for all $b \in I$ as $n_1, n_2 \in N_I$ and $n_1 + n_2 \neq 0$ as $n_1 \neq n_2$. Hence $n_1, n_2, n_1 + n_2$ are nilpotent atoms of R by Theorem 3.2.7(2). Therefore, Ann$(n_1) \cap R_B$=Ann$(n_2) \cap R_B$=Ann$(n_1 + n_2) \cap R_B = I$ by Theorem 3.2.7(3). □

Corollary 3.2.10. *If I and J are two distinct prime ideals of R_B in a cWIR R with unity, then $N_I \cap N_J = \{0\}$.*

Proof. Suppose that I and J are two distinct prime ideals of R_B in a cWIR R with unity and $0 \neq n \in N_I \cap N_J$. Then $I = Ann(n) \cap R_B = J$ by Corollary 3.2.9 which is a contradiction. Therefore, $N_I \cap N_J = \{0\}$. □

Lemma 3.2.11. *Let b be an idempotent atom of a cWIR (R, $<$) with unity and $n \in N$. If $bn \neq 0$, then bn is a nilpotent atom of R.*

Proof. Assume that b is an idempotent atom of R and $n \in N$, where $bn \neq 0$. bn is a nilpotent element of R. Let $x < bn$ for $x \in R$. Then $cbn = x$ for some $c \in R_B$ by

Lemma 3.1.1. As b is an idempotent atom of R, $cb = 0$ or $cb = b$. If $x \neq 0$, then $x = cbn = bn$. Hence bn is a nilpotent atom of R. $\square$

Lemma 3.2.12. *If b_1 and b_2 are two distinct idempotent atoms of a cWIR R with unity, then $b_1b_2 = 0$.*

Proof. Suppose that b_1 and b_2 are two distinct idempotent atoms of a cWIR R with unity such that $b_1b_2 \neq 0$. Then $b_1b_2 = b_1$ and $b_1b_2 = b_2$ as b_1 and b_2 are atoms of R. Thus $b_1 = b_2$ which is a contradiction to our hypothesis. Therefore, $b_1b_2 = 0$. $\square$

Theorem 3.2.13. *Let N_I be as defined in Theorem 3.2.7 and has only one nonzero element for each prime ideal I of R_B. Then Ann(N)$\cap R_B = \{0\}$. Additionally, N is finite if and only if R_B is finite.*

Proof. Let b be an element in $Ann(N) \cap R_B$. Then $bn = 0$ for every $n \in N$ and hence $b \in Ann(n) \cap R_B$ for every $n \in N$. Let n_I be the only nonzero element of N_I. Then $b \in Ann(n_I) \cap R_B = I$ for each prime ideal I of R_B. Therefore, $b = 0$ as the intersection of all prime ideals of a Boolean ring is $\{0\}$. Hence $Ann(N) \cap R_B = \{0\}$.
($\Rightarrow$:) Suppose that R_B is infinite. Then the number of distinct prime ideals of R_B is infinite. Define a map $\phi : Spec(R_B) \to N_A$ by $\phi(I) = n_I$, where n_I is the nonzero element in N_I and N_A is the collection of each atoms of N. Assume that $n_I = n_J$ and $I \nsubseteq J$. There exists $b_i \in I$ such that $b_in_J = n_J$ as n_J is an atom. $0 = b_in_I = b_in_J = n_J$ which is a contradiction as n_J is an atom. Therefore $I \subseteq J$. Similarly, $J \subseteq I$ and hence ϕ is well defined. And also, ϕ is a one-to-one mapping by Corollary 3.2.10. Thus, N_A is infinite and hence N is infinite.
($\Leftarrow$:) Conversely, let R_B be finite and n_1 and n_2 be two distinct nilpotent atoms. Let $Ann(n_1) \cap R_B = I$ and $Ann(n_2) \cap R_B = J$. Then I and J are prime ideals of R_B by Theorem 3.2.5. As N_I has only one nonzero element, $I \neq J$. Define a map $\theta : N_A \to Spec(R_B)$ by $\theta(n_i) = Ann(n_i) \cap R_B$ for every $n_i \in N_A$, where Spec(R_B) is the set of all prime ideals of R_B. So, θ is a one-to-one mapping. Thus, N_A is finite as Spec(R_B) is so. As R_B is finite, there exist distinct atoms $b_1, b_2, ..., b_k$ in R_B such that $\bigvee_{i=1}^{k} b_i = 1$. Consider the set $\{b_in\}_{n\in N}, b_in \neq 0, 1 \leq i \leq k$. It is a collection of distinct atoms and hence it is finite. For any $n \in N$, $n = (b_1 + b_2 + ... + b_k)n$. Hence, $\{b_in\}_{n\in N}, b_in \neq 0, 1 \leq i \leq k$ generates N as a vector space over $\{0, 1\}$. That is, N

has a finite atomic basis $\{b_i n\}_{n\in N}, b_i n \neq 0, 1 \leq i \leq k$ as $b_i's$ are finite. Therefore, N is finite. $\square$

Theorem 3.2.14. *Let N be the nilradical of a partially ordered cWIR $(R, <)$ with unity. For any two distinct nonzero elements n_1 and n_2 of N, $n_2 < n_1$ implies $n_1 + n_2 < n_1$, but n_2 and $n_1 + n_2$ are incomparable.*

Proof. If $n_2 < n_1$, then $bn_1 = n_2$ for some $b \in R_B$ by Lemma 3.1.1 which gives us $(b+1)n_1 = bn_1 + n_1 = n_1 + n_2$. Hence, $n_1 + n_2 < n_1$ by Lemma 3.1.1 as $1 + b \in R_B$. Assume that n_2 and $n_1 + n_2$ are comparable. If $n_2 < n_1 + n_2$, then $b(n_1 + n_2) = n_2$ for some $b \in R_B$. It gives us $(b + 1)(n_1 + n_2) = n_1$ and hence $n_1 < n_1 + n_2$. Thus, $n_1+n_2 = n_1$ which is a contradiction as $n_2 \neq 0$, $n_1+n_2 < n_1$ and $(R, <)$ is symmetric. If $n_1 + n_2 < n_2$, then $bn_2 = n_1 + n_2$ for some $b \in R_B$. It gives us $(b + 1)n_2 = n_1$ and hence $n_1 < n_2$. Thus, $n_1 = n_2$ which is a contradiction. Therefore, n_2 and $n_1 + n_2$ are incomparable. $\square$

Theorem 3.2.15. *Let n_1 and n_2 be two distinct atoms of N in a cWIR R with unity. $n_1 + n_2$ is an atom if and only if $[bn_1 = 0 \Leftrightarrow bn_2 = 0]$ for all $b \in R_B$.*

Proof. $(\Rightarrow:)$ Suppose that n_1 and n_2 are two distinct atoms of N in a cWIR R with unity such that $n_1 + n_2$ is an atom. Assume that $bn_1 = 0$. It follows that $b(n_1 + n_2) = bn_1 + bn_2 = bn_2$. As n_2 and $n_1 + n_2$ are atoms, we have either $bn_2 = 0$ or $bn_2 = n_2$ and $b(n_1 + n_2) = 0$ or $b(n_1 + n_2) = n_1 + n_2$ respectively. If $bn_2 \neq 0$, then $n_1 + n_2 = n_2$ which is a contradiction. Hence $bn_2 = 0$. Similarly, we can show that $bn_2 = 0 \Rightarrow bn_1 = 0$ for all $b \in R_B$.
$(\Leftarrow:)$ Conversely, suppose that $bn_1 = 0 \Leftrightarrow bn_2 = 0$ for all $b \in R_B$. Let $b \in R_B$. Then $b(n_1 + n_2) = bn_1 + bn_2 = n_1 + n_2$ or 0 as $bn_1 = 0 \Leftrightarrow bn_2 = 0$ and n_1, n_2 are atoms. Hence $n_1 + n_2$ is an atom. $\square$

Corollary 3.2.16. *Let n_1 and n_2 be two distinct atoms of N in a cWIR R with unity. n_1+n_2 is an atom if and only if $Ann(n_1)\cap R_B = Ann(n_2)\cap R_B = Ann(n_1+n_2)\cap R_B$.*

Proof. $(\Rightarrow:)$ Suppose that $n_1 + n_2$ is an atom. Let $b \in Ann(n_1) \cap R_B$. Claim that $b \in Ann(n_2)$. As $b \in Ann(n_1)$, $bn_1 = 0$. This implies $bn_2 = 0$ by Theorem 3.2.15. So, $b \in Ann(n_2) \cap R_B$ and hence $b \in Ann(n_1 + n_2) \cap R_B$. Thus $Ann(n_1) \cap R_B \subseteq$

$Ann(n_2) \cap R_B$ and $Ann(n_1) \cap R_B \subseteq Ann(n_1+n_2) \cap R_B$. Similarly, $Ann(n_2) \cap R_B \subseteq Ann(n_1) \cap R_B$ and $Ann(n_1+n_2) \cap R_B \subseteq Ann(n_2) \cap R_B$. Therefore, $Ann(n_1) \cap R_B = Ann(n_2) \cap R_B = Ann(n_1+n_2) \cap R_B$.
($\Leftarrow$:) Conversely suppose that $Ann(n_1) \cap R_B = Ann(n_2) \cap R_B = Ann(n_1+n_2) \cap R_B$ and assume that $n_1 + n_2$ is not an atom. Then $b(n_1+n_2)$ is neither equal to zero or $n_1 + n_2$ for each $b \in R_B$. This implies $bn_1 \neq 0 \neq bn_2$ and $bn_1 \neq n_1, bn_2 \neq n_2$. Hence n_1 and n_2 are not atoms which contradict our hypothesis. Thus, the result holds. □

Theorem 3.2.17. *Let N be the nilradical of a partially ordered cWIR $(R, <)$ with unity. Any two distinct atoms n_1 and n_2 of N have an upper bound in R if and only if $n_1 + n_2$ is not an atom of N.*

Proof. ($\Rightarrow$:) Let n_1 and n_2 be two distinct atoms of N in a cWIR R with unity that have an upper bound in R. Assume that $n_1 + n_2$ is an atom of N. Let $x \in R$ be an upper bound of n_1 and n_2. Then $n_1 < x$ and $n_2 < x$ and $x \in N$ by Theorem 3.1.12(9). This implies $b_1x = n_1$ and $b_2x = n_2$ for some $b_1, b_2 \in R_B$. Thus, $b_1x = b_1n_1 = n_1$ and $b_2x = b_2n_2 = n_2$. $b_2n_2 = n_2 \neq 0$ implies that $b_2n_1 = n_1 \neq 0$ by Theorem 3.2.15. Now, we get that $b_2b_1x = b_2n_1 = n_1$ and $b_2b_1x = b_1n_2 = n_2$ and hence $n_1 = n_2$ which is a contradiction as n_1 and n_2 are distinct. Therefore, n_1+n_2 is not an atom.
($\Leftarrow$:) Conversely, suppose that $n_1 + n_2$ is not an atom of N. Then $bn_1 = 0$ but $bn_2 \neq 0$ for some $b \in R_B$ by Theorem 3.2.15. So, $b(n_1 + n_2) = n_2$ which implies $n_2 < n_1 + n_2$ and $(1 + b)(n_1 + n_2) = n_1$ which implies $n_1 < n_1 + n_2$. Therefore, n_1 and n_2 have an upper bound in R. □

Theorem 3.2.18. *Let N be the nilradical of a partially ordered cWIR $(R, <)$ with unity and n_1 and n_2 be two distinct atoms of N. If $n_1 + n_2$ is not an atom, then it is the least upper bound of n_1 and n_2.*

Proof. Suppose that n_1 and n_2 are two distinct atoms of N such that $n_1 + n_2$ is not an atom. Then $n_1 + n_2$ is an upper bound of n_1 and n_2 by Theorem 3.2.17. Let $n_1 < x < n_1 + n_2$ and $n_2 < x < n_1 + n_2$. If $n_1 = x$, then $n_2 < n_1$ which is a contradiction. Thus, $n_1 \neq x$ and similarly $n_2 \neq x$. We have $n_1 = b_1x$ and $n_2 = b_2x$

for some $b_1, b_2 \in R_B$ by Lemma 3.1.1. Then $b_1x + b_2x = n_1 + n_2$ which implies $(b_1 + b_2)x = n_1 + n_2$ and hence $n_1 + n_2 < x$. Thus, $x = n_1 + n_2$. Therefore, $n_1 + n_2$ is the least upper bound of n_1 and n_2. □

Theorem 3.2.19. *Let N be the set of all nilpotent elements of a cWIR $(R, <)$ with unity and $n_1, n_2, n_3, ..., n_k$ be a finite number of distinct atoms of N. If any two elements of $\{n_1, n_2, n_3, ..., n_k\}$ have an upper bound, then $n_1 + n_2 + n_3 + ... + n_k$ is the least upper bound of $\{n_1, n_2, n_3, ..., n_k\}$.*

Proof. Suppose that any two elements of $\{n_1, n_2, n_3, ..., n_k\}$ have an upper bound. Then $Ann(n_i) \cap R_B \neq Ann(n_j) \cap R_B$ for each $i \neq j$ and $i, j \in \{1, 2, ..., k\}$. Thus, there exists $b_i \in R_B$ such that $b_i \in Ann(n_i)$ but $b_i \notin Ann(n_j)$ for $i \neq j$ and $i, j \in \{1, 2, ..., k\}$. Consider $b_2b_3...b_k \in Ann(n_j) \cap R_B$ for all $j \in \{2, ..., k\}$ but $b_2b_3...b_k \notin Ann(n_1) \cap R_B$ as $Ann(n_1) \cap R_B$ is a prime ideal of R_B. Hence, $b_2b_3...b_kn_1 = n_1$ as n_1 is an atom of N. We obtained that $b_2b_3...b_k(n_1 + n_2 + n_3 + ... + n_k) = n_1$ and hence $n_1 < n_1+n_2+n_3+...+n_k$. Similarly, we can show that $n_1+n_2+n_3+...+n_k$ is an upper bound of n_i for all $i \in \{2, ..., k\}$. Let n_o be an upper bound of $\{n_1, n_2, n_3, ..., n_k\}$. Then for each $i = 1, 2, 3, ..., k$, there exists $c_i \in R_B$ such that $c_in_o = n_i$. Thus, $(\sum_{i=1}^{k} c_i)n_o = \sum_{i=1}^{k} n_i$ and hence $n_1+n_2+n_3+...+n_k < n_o$. Therefore, $n_1+n_2+n_3+...+n_k$ is the least upper bound of $\{n_1,\ n_2,\ n_3, ..., n_k\}$. □

Chapter 4

Role of Atoms in Weak Idempotent Rings

The notions of idempotent and nilpotent elements and atoms of a ring play an important role in the theory of rings. An atom of a ring may be an idempotent or nilpotent element of the ring. Using these concepts in a cWIR with unity, we investigate further results.

4.1 Lattice on the Nil-radical

This section mainly aims at dealing with a lattice of the collection of nilpotent atoms $(\{n_i\}_{i\in I}, <)$ corresponding to the idempotent atoms $\{b_i\}_{i\in I}$, where the partial ordering $<$ is defined as in Lemma 3.1.1(see Corollary 4.1.11).
We begin with the following Lemma.
Lemma 4.1.1. *Let b be a nonzero element of R_B in a cWIR $(R, <)$ with unity. The following are equivalent.*

1. *b is an atom of R.*

2. *b is an atom of R_B.*

3. *For every $x \in R_B$, either $b < x$ or $bx = 0$.*

Proof. $[1 \Rightarrow 2]$: is obvious.

$[2 \Rightarrow 3]$: Suppose that b is an atom of R_B such that $bx \neq 0$ for $x \in R_B$. Then $bx = y$ for some $y \in R_B$ and hence $y < b$. As b is an atom of R_B, $y = 0$ or $y = b$. Hence $bx = b$. Therefore, $b < x$.

$[3 \Rightarrow 1]$: Suppose for every $x \in R_B$, either $b < x$ or $bx = 0$. Let $r < b$ for $r \in R$. Then $bz = r$ for some $z \in R_B$. But $b < z$ or $bz = 0$ as $z \in R_B$ by (3). So, $bz = b$ or $bz = 0$ from (3) and by definition. Thus, $r = 0$ or $r = b$ and hence b is an atom of R. $\square$

Lemma 4.1.2. *If x is an atom of a cWIR R with unity, then x_B and x_N are atoms of R provided that $x_B \neq 0$, $x_N \neq 0$.*

Proof. Let x be an atom of a cWIR R with unity. If $x_B \neq 0$ and $x_N = 0$, then $x = x_B$ is an atom of R as x is so. Again, if $x_B = 0$ and $x_N \neq 0$, then $x = x_N$ and it is an atom of R as x is so. $\square$

The following example justifies that the converse of the above Lemma is not true.

Example 4.1.3. *Consider the ring $H_4 = \{0, 1, p, 1 + p\}$ and let B=$\{$0,a,b,a+b$\}$ be a Boolean group of 4 elements. Define a unitary $(H_4)^2$-module system on B^2 where the multiplication is based on the following: $pa = a$ and $pb = 0$. Consider the ring $((H_4)^2 \times B^2, +, .)$ where the operations are defined as:*
$((a_1, a_2), (b_1, b_2)) + ((c_1, c_2), (d_1, d_2)) = ((a_1 + c_1, a_2 + c_2), (b_1 + d_1, b_2 + d_2))$ and $((a_1, a_2), (b_1, b_2)).((c_1, c_2), (d_1, d_2)) = ((a_1c_1, a_2c_2), (a_1, a_2)(d_1, d_2) + (c_1, c_2)(b_1, b_2))$, where the operation between $(H_4)^2 = H_4 \times H_4$ and $B^2 = B \times B$ is component wise. Then $((H_4)^2 \times B^2, +, .)$ is a cWIR with unity.

Let $x = ((1, 0), (0, a))$. Then $x_B = ((1, 0), (0, 0))$ is an atom of R and $x_N = ((0, 0), (0, a))$ is an atom of R. But x is not an atom of R.

Lemma 4.1.4. *Let a be an atom of a cWIR (R, $<$) with unity and $r \in R$. Then ra is an atom of R provided $ra \neq 0$.*

Proof. Let a be an atom of R such that $ra \neq 0$ for $r \in R$. Then, for all $b \in R_B, ba < a$ which implies either $ba = a$ or $ba = 0$. So, $bra = ra$ or $bra = 0$. Hence ra is an atom of R. $\square$

Lemma 4.1.5. *Let b be an atom of R_B and any two nilpotent atoms have an upper bound in a cWIR $(R, <)$ with unity. Then either $bN = \{0\}$ or $bN = \{0, n\}$, where n is an atom of N. Further $bN = \{0, n\}$, $bn_1 = 0$ for all atoms $n_1 \neq n$.*

Proof. Let bn_1 and bn_2 be nonzero elements of bN. Then, bn_1 and bn_2 are atoms of R by Lemma 4.1.4 and hence atoms of N by Lemma 4.1.1. Thus, there exists $x \in R$ such that $bn_1 < x$ and $bn_2 < x$ as both bn_1 and bn_2 are nonzero by Lemma 4.1.1(3). $bn_1 < bx$ and $bn_2 < bx$ by Theorem 3.1.12 (2) so that $bx \neq 0$ and it is an atom of N by Lemma 4.1.4. Hence $bn_1 = bn_2 = bx$ as both bn_1 and bn_2 are nonzero. Therefore, either $bN = \{0\}$ or $bN = \{0, n\}$. Suppose that $bN \neq \{0\}$. There exists $n_1 \in N$ such that $bn_1 \neq 0$. Let $bn_1 = n$. It follows that $bn_1 = bn = n$ and hence $bN = \{0, n\}$. $bn = n$ is an atom of N by Lemma 4.1.4. Let n_2 be an atom of N and $n \neq n_2$ such that $bn_2 \neq 0$. So, $bn_2 = n$. But $bn_2 = n_2$ as n_2 is an atom of N. Thus $n = n_2$ which is a contradiction. Therefore, $bn_2 = 0$ for all atoms $n_2 \neq n$. $\square$

Lemma 4.1.6. *Let any two nilpotent atoms have an upper bound in a cWIR R with unity. Then, for every two distinct idempotent atoms b_1 and b_2, $b_1N \cap b_2N = \{0\}$.*

Proof. Suppose that b_1 and b_2 are two distinct idempotent atoms with $b_1N = \{0, n_1\}$ and $b_2N = \{0, n_2\}$. Assume that $b_1N \cap b_2N \neq \{0\}$. It follows that $b_2n_1 = b_2n_2 = n_1 = n_2$ by Lemma 4.1.5. So, $b_1b_2n_1 = n_1 = n_2$. But $b_1b_2 = 0$ by Lemma 3.2.12 as b_1 and b_2 are distinct idempotent atoms. Hence $n_1 = n_2 = 0$ which is a contradiction. Therefore, $b_1N \cap b_2N = \{0\}$. $\square$

Lemma 4.1.7. *Let any two nilpotent atoms have an upper bound in a cWIR R with unity. For idempotent atom b, $bR = \{0, b\}$ or $bR = \{0, b, n, b + n\}$, where n is a nilpotent atom.*

Proof. In a cWIR R with unity, $R = \{r \in R : r = r_B + r_N\}$. Thus, $bR = \{br_B + br_N : r \in R\}$. But $br_N = 0$ or $br_N = n$ for all r_N. As b is an idempotent atom, $br_B = b$ or $br_B = 0$. Hence, $bR = \{0, b\}$ or $bR = \{0, b, n, b + n\}$. $\square$

Definition 4.1.8. *Let any two nilpotent atoms have an upper bound in a cWIR R with unity and $\{b_i\}_{i \in I}$ be the set of all idempotent atoms such that $b_iN \neq \{0\}$. If*

$b_iN = \{0, n_i\}$, *then* $\{n_i\}_{i\in I}$ *is called the set of nilpotent atoms corresponding to the idempotent atom* b_i *of* $\{b_i\}_{i\in I}$.

Lemma 4.1.9. *If nilpotent atoms* n_1 *and* n_2 *have an upper bound in a cWIR with unity, then* $n_1n_2 = 0$.

Proof. Suppose that n_1 and n_2 have an upper bound in a cWIR R with unity. It follows that $n_1 + n_2$ is the least upper bound of n_1 and n_2 by Theorem 3.2.18. So, $n_1 < n_1 + n_2$. But $n_1(n_1 + n_2) = 0$ by Theorem 3.2.2(4(a)). Thus, $n_1^2 + n_1n_2 = 0$ and hence $n_1n_2 = 0$. □

Note.

1. Now and then we use $\{n_i\}_{i\in I}$ to denote the set of nilpotent atoms corresponding to each idempotent atom b_i for all $i \in I$, $\{b_i\}_{i\in I}$ denotes the set of all idempotent atoms.

2. For r_1 and $r_2 \in R$, if there exists an upper and a lower bound of r_1 and r_2, then the least upper and greatest lower bounds of r_1 and r_2 are denoted by $r_1 \vee r_2$ and $r_1 \wedge r_2$ respectively.

Lemma 4.1.10. *In a cWIR (R, <) with unity, for* $n_1, n_2 \in \{n_i\}_{i\in I}$, $n_1 \vee n_2$ *and* $n_1 \wedge n_2$ *exist and are equal to* $n_1 + n_2$ *and* 0 *respectively.*

Proof. Let n_1 and n_2 be in $\{n_i\}_{i\in I}$. It follows that $b_1n_1 = n_1$ and $b_1n_2 = 0$ by Lemma 4.1.5. Consider $b_1(n_1 + n_2) = b_1n_1 + b_1n_2 = n_1$ which implies $n_1 + n_2$ is not an atom. Hence $n_1 + n_2$ is the least upper bound of n_1 and n_2 by Theorem 3.2.18. Let $x < n_1$ and $x < n_2$ for x in R. If $x \neq 0$, then $x = n_1 = n_2$ as both n_1 and n_2 are in $\{n_i\}_{i\in I}$ which is a contradiction and hence $x = 0$. Therefore, 0 is the greatest lower bound of n_1 and n_2. □

Corollary 4.1.11. *The subgroup generated by* $\{n_i\}_{i\in I}$ *in* N *is a lattice.*

Proof. Let A be a subgroup generated by $\{n_i\}_{i\in I}$ and $a, b \in A$. Then $a = \sum\limits_{j\in F_a} n_j$ and $b = \sum\limits_{k\in F_b} n_k$, where F_a and F_b are finite subsets of I. We point out that $a + b = \sum\limits_{t\in F_a\cup F_b} n_t$. Let $F_a = \{\alpha_1, \alpha_2, ..., \alpha_r\}$ and $F_b = \{\beta_1, \beta_2, ..., \beta_s\}$, where $F_a \cap F_b =$

$\{\gamma_1, \gamma_2, ..., \gamma_p\}$ and $p \leq r$ and $p \leq s$. Then $(b_{\alpha_1} + b_{\alpha_2} + ... + b_{\alpha_r})(a+b) = a$ and $(b_{\beta_1} + b_{\beta_2} + ... + b_{\beta_s})(a+b) = b$. Hence $a < a+b$ and $b < a+b$, that is, $a+b$ is an upper bound of a and b. Let x be an upper bound of a and b. Then there exist elements c and $d \in R_B$ such that $cx = a$ and $dx = b$. Thus, $(c+d)x = a+b$ and hence $a+b < x$. Therefore, $a+b$ is the least upper bound of a and b. Let $e = n_{\gamma_1} + n_{\gamma_2} + ... + n_{\gamma_p}$. Then $(b_{\gamma_1} + b_{\gamma_2} + ... + b_{\gamma_p})a = e$ and $(b_{\gamma_1} + b_{\gamma_2} + ... + b_{\gamma_p})b = e$.Thus, e is the lower bound of a and b. Let x be the lower bound of a and b. Then there exist $c, d \in R_B$ such that $ca = x$ and $db = x$. If $x \neq 0$, then $c(n_{\alpha_1} + n_{\alpha_2} + ... + n_{\alpha_r}) = x$ and $d(n_{\beta_1} + n_{\beta_2} + ... + n_{\beta_s}) = x$ implies $cn_{\alpha_j} \neq 0$ for some $\alpha_j \in F_a$ and $dn_{\beta_k} \neq 0$ for some $\beta_k \in F_b$. Let $c(n_{\alpha_{j_1}} + n_{\alpha_{j_2}} + ... + n_{\alpha_{j_q}}) = x$ and $d(n_{\beta_{k_1}} + n_{\beta_{k_2}} + ... + n_{\beta_{k_v}}) = x$. Thus, $n_{\alpha_{j_1}} + n_{\alpha_{j_2}} + ... + n_{\alpha_{j_q}} = x$ and $n_{\beta_{k_1}} + n_{\beta_{k_2}} + ... + n_{\beta_{k_v}} = x$ since $b_{\alpha_j} n_{\alpha_j} = n_{\alpha_j}$ and $b_{\beta_k} n_{\beta_k} = n_{\beta_k}$. If $b_{\alpha_{j_i}} \notin \{b_{\beta_{k_1}}, b_{\beta_{k_2}}, ..., b_{\beta_{k_v}}\}$ for some $i = 1, 2, ..., q$, then $b_{\alpha_{j_i}} x = b_{\alpha_{j_i}}(n_{\beta_{k_1}} + n_{\beta_{k_2}} + ... + n_{\beta_{k_v}}) = 0$. Thus, $n_{\alpha_{j_i}} = 0$ which is a contradiction. Hence, $b_{\alpha_{j_1}}, b_{\alpha_{j_2}}, ..., b_{\alpha_{j_q}} \subseteq \{b_{\beta_{k_1}}, b_{\beta_{k_2}}, ..., b_{\beta_{k_v}}\}$. Similarly, $\{b_{\beta_{k_1}}, b_{\beta_{k_2}}, ..., b_{\beta_{k_v}}\} \subseteq \{b_{\alpha_{j_1}}, b_{\alpha_{j_2}}, ..., b_{\alpha_{j_q}}\}$. This gives us $\{b_{\beta_{k_1}}, b_{\beta_{k_2}}, ..., b_{\beta_{k_v}}\} = \{b_{\alpha_{j_1}}, b_{\alpha_{j_2}}, ..., b_{\alpha_{j_q}}\} \subseteq \{b_{\gamma_1}, b_{\gamma_2}, ..., b_{\gamma_p}\}$. Thus, $(b_{\beta_{k_1}} + b_{\beta_{k_2}} + ... + b_{\beta_{k_v}})e = x$. Therefore, e is the greatest lower bound of a and b. □

4.2 Nil-free Radical

The main concentration of this section on the role of idempotent and nilpotent atoms in the decomposition of a cWIR R with unity (See Theorem 4.2.11). We proved different results regarding the nil-free nilradical N of R, and this concept is used in relation to the symmetric properties of the Boolean parts R_B of R, assuming that the set of all nilpotent atoms of R is complete.

We begin with the following Lemma.

Lemma 4.2.1. *Let (R, $<$) be a partially ordered cWIR with unity, the collection, R_B has atoms and any two nilpotent atoms have an upper bound in R. Then the following are equivalent.*

1. *N is nil-free.*

2. For any $0 \neq n \in N$, there exists an atom b of R_B such that $bn \neq 0$.

3. N is atomic and the set of all atoms of N is precisely $\{n_i\}_{i \in I}$.

Proof. i) ($1 \Rightarrow 2$) Assume that N is nil-free and $0 \neq n \in N$. Then $bn \neq 0$ for each $b \in R_B$. Thus there exists an atom b of R_B such that $bn \neq 0$.

ii) ($2 \Rightarrow 3$) Assume that (2) holds. So, there exists a nilpotent atom $n_j \in \{n_i\}_{i \in I}$ corresponding to an atom b of R_B by Lemma 4.1.5. Hence, $\{n_i\}_{i \in I}$ is non-empty. If n is an atom of N such that $n \notin \{n_i\}_{i \in I}$, then $bn = 0$ for all atoms b of R_B by Lemma 4.1.5 and Definition 4.1.8 which contradicts our hypothesis (2). So, the set of all atoms of N is $\{n_i\}_{i \in I}$. For $0 \neq x \in N$ there exists an atom $b \in R_B$ such that $bx \neq 0$. Therefore, bx is an atom of N and hence N is atomic as $bx < x$.

iii) ($3 \Rightarrow 2$) Suppose that (3) holds and $0 \neq n \in N$. There exists $n_k \in \{n_i\}_{i \in I}$ such that $n_k < n$ for some $k \in I$. So, $b_k n_k < b_k n$ by Theorem 3.1.12(2) which implies $b_k n \neq 0$ and hence (2) holds.

iv) ($2 \Rightarrow 1$) Assume that N is not nil-free. Then, there exists $0 \neq n \in N$ such that $bn = 0$ for all $b \in R_B$ which contradicts (2). So, our assumption is false and (1) holds.

□

Remark 4.2.2. *In a cWIR R with unity*
If every nonzero element of N is simple, then N is nil-free.

Proof. Let R be a cWIR with unity.

Assume that every nonzero element of N is simple and $b \in R_B$. Then $bn = n$ for each $0 \neq n \in N$. So, zero is the only nil element of N. Therefore, N is nil-free. □

Lemma 4.2.3. *Let N be nil-free and any two nilpotent atoms have an upper bound in a partially ordered cWIR (R, <) with unity. R is atomic if and only if R_B is so.*

Proof. Assume that N is nil-free and any two nilpotent atoms have an upper bound in a partially ordered cWIR (R, $<$) with unity.

($\Rightarrow$:) Suppose that R is atomic and b is a nonzero element of R_B. Then $b \in R$ and there exists an atom say a of R with $a < b$. This implies $a \in R_B$ by Theorem 3.1.12(8) and is an atom of R_B. Thus, R_B is atomic.

($\Leftarrow$:) Conversely, suppose that R_B is atomic. N is atomic by the hypothesis and Lemma 4.2.1. Let x be a nonzero element of R. Then, $x = x_B + x_N$ and either $x_B \neq 0$ or $x_N \neq 0$. If $x_B \neq 0$ and $x_N = 0$, then there is an atom y of R_B with $y < x$. If $x_B = 0$ and $x_N \neq 0$, then there exists an atom n of N such that $n < x$. If both $x_B \neq 0$ and $x_N \neq 0$, then there exists an atom b of R_B such that $b < x_B$. So, $bx = bx_B + bx_N = b + bx_N \neq 0$ by Lemma 3.1.1 and Theorem 3.1.12(4a). Hence bx is an atom of R by Lemma 4.1.4 and $bx < x$. Therefore, R is atomic. □

Lemma 4.2.4. *Let R_B be atomic and any two nilpotent atoms have an upper bound in a cWIR (R, $<$) with unity. If N is nil-free and $0 \neq n \in N$, then*

1. *Every atom of $R_B n$ is of the form $b_j n$ for some atom $b_j \in \{b_i\}_{i \in I}$.*

2. *$R_B n$ is atomic.*

Proof. Suppose that N is nil-free and $0 \neq n \in N$.

1. There is an atom b of R_B so that $bn \neq 0$ by definition. Thus, bn is an atom of $R_B n$ and hence an atom of N by Lemma 4.1.4. But $bn = n_j$ for some $n_j \in \{n_i\}_{i \in I}$ by the hypothesis and Lemma 4.2.1(3). Therefore, $bn = n_j = b_j n_j$ and $bb_j \neq 0$. If $bb_j = 0$, then $0 = bb_j n = b_j bn = b_j n_j = n_j$ which contradicts that n_j is an atom of N. Thus, $bb_j \neq 0$ and $bb_j = b_j$. Hence, $b_j n = bb_j n = b_j n_j = n_j = bn$. That is, any atom of $R_B n$ has the form of $b_j n$ for some atom b_j of $\{b_i\}_{i \in I}$.

2. Let $0 \neq bn \in R_B n$. Then $n_j < bn$ for some $n_j \in \{n_i\}_{i \in I}$ by the hypothesis and Lemma 4.2.1(3) and $b_j n_j < b_j bn$ by Theorem 3.1.12(2). So, $b_j b \neq 0$ and hence $b_j b = b_j$. $n_j = b_j n \in R_B n$ by Lemma 4.1.5 and hence $b_j n < bn$. Therefore, $R_B n$ is atomic by definition.

□

Lemma 4.2.5. *Let $(R_B, <)$ be atomic and any two nilpotent atoms have an upper bound in a cWIR R with unity. If N is nil-free, then for all $n \in N$, $n = \bigvee_{j\in J} n_j$, where J is some subset of I.*

Proof. Let $\{b_i\}_{i\in I}$ be a set of atoms of R_B and $K = \{k \in I : b_k n_k \neq 0$ and $b_k n_k \neq n_k, \forall k\}$. If $n = 0$, then $0 = \bigvee_{k\in K} n_k$. Let $0 \neq n \in N$. From the hypothesis and Lemma 4.2.4, $R_B n$ is atomic and every atom of $R_B n$ is of the form $b_j n$ for some atom b_j of $\{b_i\}_{i\in I}$. Therefore, $n = \bigvee_{i\in I} b_i n$. If $b_i n$ is an atom, then $b_i n = n_i$ by Lemma 4.1.5 and hence $n = \bigvee_{j\in J\subset I} n_j$, where $J = \{j \in I : b_j n \neq 0\}$. □

Lemma 4.2.6. *Let R_B be atomic and $\{b_k\}_{k\in\Delta}$ be the set of all atoms of R_B in a cWIR (R, <) with unity. If $\{b_k\}_{k\in\Delta}$ is join-complete, then, for any $b \in R_B$,*
$b \wedge (\bigvee_{j\in J\subset\Delta} b_j) = \bigvee_{j\in J}(b \wedge b_j)$

Proof. Suppose that $\{b_k\}_{k\in\Delta}$ is join-complete and $b \in R_B$. Then there exists $x \in R_B$ such that $x = \bigvee_{j\in J\subset\Delta} b_j$ and $b_j < x$ for every $j \in J$. Thus, $bb_j < bx$ for all $j \in J$ by Theorem 3.1.12(2). So, bx is an upper bound of $\{bb_j\}_{j\in J\subset\Delta}$. Let u be an upper bound of $\{bb_j\}_{j\in J\subset\Delta}$. Then $bb_j < u$ for all $j \in J$. Claim that $bx < u$. But $bb_j u = bb_j$ by Lemma 3.1.1 and so $b_j = bb_j u + bb_j + b_j = bb_j u + bb_j x + b_j x$. Thus, $b_j(bu + bx + x) = b_j$ which gives us $b_j < bu + bx + x$ for all $j \in J$. So, $x < bu + bx + x$ as $x$is lub$\{b_j\}_{j\in J\subset\Delta}$. This gives us $bxu = bx$ and hence $bx < u$ by Lemma 3.1.1. Therefore, $bx = \bigvee_{j\in J\subset\Delta} bb_j$ which is required. □

Lemma 4.2.7. *Let R_B be atomic, N be nil-free and any two nilpotent atoms have an upper bound in a cWIR (R, <) with unity. Let $\{n_i\}_{i\in I}$ be join-complete. If b is an idempotent atom of R and $J \subset I$, then*

1. *$b(\bigvee_{j\in J} n_j) = 0$ if $b \notin \{b_i\}_{i\in I}$.*
2. *$b_i(\bigvee_{j\in J} n_j) = n_i$ if and only if $n_i \in \{n_j\}_{j\in J}$.*

Proof. Suppose that $\{n_i\}_{i\in I}$ is join-complete, b is an idempotent atom of R and $J \subset I$. Then $n = \bigvee_{j\in J} n_j$ exists in N by the hypothesis and Lemma 4.2.5.

1. If $b \notin \{b_i\}_{i\in I}$, then $bN = \{0\}$ by Definition 4.1.8. Therefore, $bn = 0$ and hence $b\left(\bigvee_{j\in J} n_j\right) = 0$.

2. Let $n_i \in \{n_j\}_{j\in J}$. Then $n_i < n$ for each $i \in I$. This implies $n \neq 0$ and $b_j n \neq 0$ for some idempotent atom b_j. Also, there exists an idempotent element $b \in R$ such that $bn = n_i$ by Lemma 3.1.1 as $n_i < n$. From this, we have $bb_i n = b_i n_i = n_i$ so that $n_i < b_i n \neq 0$. But $b_i N = \{0, n_i\}$. Therefore, $n_i = b_i n_i = b_i n$. i.e. $b_i(\bigvee_{j\in J} n_j) = n_i$. Also, if $b_i n = n_i$, then $n_i < n$ and hence n covers the atom n_i. Suppose $n_i \notin \{n_j\}_{j\in J}$. Then $n_i = b_i n = (b_i n) \wedge (\bigvee_{j\in J} b_j n)$ as $n_j = b_j n$ and $R_B n$ is a Boolean lattice. Therefore, $n_i = \bigvee[(b_i n) \wedge (b_j n)]$ by Lemma 4.2.6. That is, $n_i = \bigvee(b_i \wedge b_j)n = 0$ as $b_i \neq b_j$ for any $j \in J$ which is a contradiction and hence $n_i \in \{n_j\}_{j\in J}$.

□

Remark 4.2.8. *Let b be the non-trivial idempotent element of a cWIR R with unity. Then $R = bR\bigoplus(1+b)R$. For instance, in example 4.1.3, $b_1 = ((1,0),(0,0))$ and $b_2 = ((0,1),(0,0)) = ((1,1),(0,0)) + b_1$ are non-trivial idempotent elements of R. Here, $R = b_1R\bigoplus b_2R = b_1R\bigoplus(1+b_1)R$, where $\bigoplus$ represents the direct product.*

Theorem 4.2.9. *Let R be a partially ordered cWIR with unity and b_1 and b_2 be two distinct idempotent atoms of R. Then $b_1 \vee b_2$ and $b_1 \wedge b_2$ exist in R and are equal to $b_1 + b_2$ and 0 respectively.*

Proof. Suppose that b_1 and b_2 are two distinct idempotent atoms of R. Then $b_1 < b_1 + b_2$, $b_2 < b_1 + b_2$ as $b_1(b_1 + b_2) = b_1$ and $b_2(b_1 + b_2) = b_2$ by Lemma 3.2.12. Let x be an upper bound of b_1 and b_2. Then there are elements say y and $z \in R_B$ so that $b_1 = yx$ and $b_2 = zx$ by Lemma 3.2.12 from which we get $b_1 + b_2 < x$. Thus, $b_1 + b_2$ is supremum of $\{b_1, b_2\}$ and $b_1 \vee b_2 = b_1 + b_2$. Let z be the lower bound of b_1 and b_2. Assume, if possible that $z \neq 0$. Then there exist elements say y and $x \in R_B$ such that $b_1 y = z$ and $b_2 x = z$. So, we get $b_1 z = b_1 y = z$ and $b_2 z = b_2 x = z$.These give us $z = b_2 z = b_2 b_1 z = 0$ by Lemma 3.2.12 which is a contradiction. Therefore, $z = 0$ and $b_1 \wedge b_2 = 0$. □

Remark 4.2.10. *Let R be a cWIR with unity and $\{b_1, b_2, ..., b_m\}$ be a finite set of distinct idempotent atoms of R. Then sup $\{b_1, b_2, ..., b_m\} = b_1 + b_2 + ... + b_m$ and inf*

$\{b_1, b_2, ..., b_m\} = 0$.

Proof. Suppose that $\{b_1, b_2, ..., b_m\}$ is a finite set of distinct idempotent atoms of R. Then $b_i < \sum_{i=1}^{m} b_i$ as $b_i(\sum_{i=1}^{m} b_i) = b_i$ for each i. Let x in R be an upper bound of each b_i for each $1 \leq i \leq m$. This implies $b_i < x$ for each $1 \leq i \leq m$. There exist $y_i \in R_B$ such that $b_i = y_i x$ for each $1 \leq i \leq m$ by Lemma 3.1.1. Thus $\sum_{i=1}^{m} b_i = \sum_{i=1}^{m} y_i x$. Thus $\sum_{i=1}^{m} b_i < x$ by Lemma 3.1.1. Therefore, $\sum_{i=1}^{m} b_i$ is the least upper bound of $\{b_1, b_2, ..., b_m\}$.

Let z be the lower bound of $\{b_i\}_{i=1}^{m}$. Assume, if possible that $z \neq 0$. Then there exist elements say $y_i \in R_B$ such that $b_i y_i = z$ for each $1 \leq i \leq m$. So, we get $b_i z = b_i y_i = z$ and $b_j z = b_j y_j = z$ for each $1 \leq i, j \leq m$.These give us $z = b_j z = b_j b_i z = 0$ by Lemma 3.2.12 which is a contradiction. Therefore, $z = 0$ and $\bigwedge_{i=1}^{m} b_i = 0$.

□

Theorem 4.2.11. *Let N and R_B be a nilradical and a Boolean sub-ring of a cWIR R with unity respectively. R_B is a symmetric Boolean ring, N is nil-free and the set of all nilpotent atoms is complete if and only if R is isomorphic to the direct product of WIRs each of which is a copy of a 2-elements field or a four-element BLR H_4 (i.e. R_B is isomorphic to the Boolean ring of all subsets of some set).*

Proof. ($\Rightarrow$:) Let R_B be a symmetric Boolean ring, N nil-free and the set of all nilpotent atoms of R be complete. Assume that $\{b_k\}_{k\in\Delta}$ is the set of all idempotent atoms, $\{b_i\}_{i\in I\subset\Delta}$ is the set of all idempotent atoms for which $b_i N \neq \{0\}$ and $\{n_i\}_{i\in I}$ is the set of all nilpotent atoms corresponding to $\{b_i\}_{i\in I}$. Then Rb_r is either $\{0, b_r\}$ or $\{0, b_r, n_r, b_r + n_r\}$ and $Rb_{r_1} \cap Rb_{r_2} = \{0\}$ for all $r_1, r_2 \in \Delta$, $r_1 \neq r_2$ by Lemma 4.1.7 and Lemma 4.1.6. Hence, Rb_r is either a copy of the 2 element field or a copy of the four-element BLR H_4. Consider the direct product $\bigoplus_{r\in\Delta} Rb_r$. Define $\psi : R \to \bigoplus_{r\in\Delta} Rb_r$ by $\psi(x) = (xb_r)_{r\in\Delta}$.

Then $\psi(x + y) = ((x + y)b_r)_{r\in\Delta} = (xb_r)_{r\in\Delta} + (yb_r)_{r\in\Delta} = \psi(x) + \psi(y)$.

$\psi(xy) = (xyb_r)_{r\in\Delta} = (xb_r)_{r\in\Delta} \cdot (yb_r)_{r\in\Delta} = \psi(x)\psi(y)$ as $b_r \in R_B$ and R is commutative.

$\psi(1) = (b_r)_{r\in\Delta} = \bigvee_{r\in\Delta} b_r = 1$ as R_B is a symmetric Boolean ring. Therefore, ψ is a homomorphism. Let $\psi(x) = \psi(y)$. So, we get $(xb_r)_{r\in\Delta} = (yb_r)_{r\in\Delta}$. Then $xb_r = yb_r$

for all $r \in \Delta$. i.e. $x_B b_r = y_B b_r$ and $x_N b_r = y_N b_r$ for all $r \in \Delta$.

Now, $x_B = x_B(\bigvee_{r\in\Delta} b_r) = \bigvee_{r\in\Delta} x_B b_r = \bigvee_{r\in\Delta} y_B b_r = y_B(\bigvee_{r\in\Delta} b_r) = y_B$.

$x_N = \bigvee_{i\in I} b_i x_N$ by Lemma 4.2.5 and hence

$x_N = \bigvee_{r\in\Delta} b_r x_N = \bigvee_{r\in\Delta} b_r y_N = (\bigvee_{r\in\Delta} b_r) y_N = y_N$ as $1 = \bigvee_{r\in\Delta} b_r$. Therefore, $x = y$ and so ψ is one-to-one. Let $(x_r b_r)_{r\in\Delta}$ be any element of $\bigoplus_{r\in\Delta} Rb_r$. Then $x_r b_r = s_r b_r + t_r n_r$ where $s_r, t_r \in \{0,1\}$ as Rb_r is either a copy of the 2 elements field or a copy of the four-element BLR H_4 and $n_r = 0$ if $r \in \Delta - I$. Consider $\bigvee_{r\in\Delta} t_r n_r + \bigvee_{r\in\Delta} s_r b_r$ in R. $\psi(\bigvee_{r\in\Delta} t_r n_r + \bigvee_{r\in\Delta} s_r b_r) = ([\bigvee_{r\in\Delta} s_r b_r + \bigvee_{r\in\Delta} t_r n_r] b_r)_{r\in\Delta} = (\bigvee_{r\in\Delta}[s_r b_r + t_r n_r] b_r)_{r\in\Delta} = \bigvee_{r\in\Delta}[(x_r b_r)_{r\in\Delta}] = (x_r b_r)_{r\in\Delta}$ by Lemma 4.2.6 and Lemma 4.2.7. Hence, ψ is onto and so that R is isomorphic to $\bigoplus_{r\in\Delta} Rb_r$.

($\Leftarrow$:) Conversely let R be isomorphic to $\bigoplus_{i\in I} R_i$, where R_i is a copy of either two element field or the four-elements Boolean-like ring H_4. Let us denote the multiplicative identity of R_i by b_i and if a nonzero nilpotent element exists in R_i, it can be denoted by n_i. That is, $R_i = \{0, b_i\}$ or $\{0, b_i, n_i, b_i + n_i\}$. Let $x \in \bigoplus_{i\in I} R_i$. Then $x = (x_i)_{i\in I}$ where $x_i \in R_i$. x is an idempotent if and only if $(x_i^2)_{i\in I} = (x_i)_{i\in I} \Leftrightarrow x_i^2 = x_i, \forall i \in I \Leftrightarrow x_i = 0$ or b_i. x is nilpotent if and only if $x^2 = 0 \Leftrightarrow x_i = 0$ or n_i. Hence, the collection of each idempotent elements of $\bigoplus_{i\in I} R_i$ is $R_B = \{(x_i)_{i\in I} : x_i = 0 \text{ or } b_i\}$ and the collection of nilpotent elements of $\bigoplus_{i\in I} R_i$ is $N = \{(x_i)_{i\in I} : x_i = 0 \text{ or } n_i\}$. Let $0 \neq x = (x_i)_{i\in I} \in R_B$. x is an atom if and only if $x_i = b_i$ for some $i \in I$ and $x_j = 0$ for all $j \neq i$ and $j \in I$. For: if x is an atom, then $x_i \neq 0$ for at least one $i \in I$. If $x_i \neq 0$ and $x_j \neq 0$ with $i \neq j$ and $i, j \in I$, then we have $x_i = b_i$ and $x_j = b_j$ from the hypothesis. So, $x = (x_i)_{j\neq i\in I} + x_j = (b_i)_{j\neq i\in I} + b_j$. Let $y = (y_i)_{i\in I}$ with $y_i = b_i, y_j = 0$ for all $j \neq i$ and $i, j \in I$. Then y is neither equal to zero nor x.

Considering $xy = (x_i)_{i\in I}(b_i)_{i\in I} + x_j(b_i)_{j\neq i\in I} = (x_i b_i)_{i\in I} + (x_j b_i)_{j\neq i\in I} = (b_i)_{i\in I} = y$ by Lemma 3.2.12. This contradicts the hypothesis, x is an atom. Hence, $x_i \neq 0$ holds for at most one $i \in I$. Therefore, if x is an atom and $x \in R_B$, then $x_i = b_i$ for some $i \in I$ and $x_j = 0$ for all $j \neq i$ with $i, j \in I$. Conversely, let $0 \neq x \in R_B$ be of this form. That is, $x_i = b_i$ for some $i \in I$ and $x_j = 0$ for all $j \neq i$ with $i, j \in I$. Let $y < x$. Then $y \in R_B$ by Theorem 3.1.12(8) and $y_i = b_i$ for some $i \in I$ and $y_j = 0$ for all $j \neq i$ with $i, j \in I$ by the hypothesis. Thus $xy = (x_i)_{i\in I}(y_i)_{i\in I} = (b_i)_{i\in I} = x$. Hence, x is an atom. Let $0 \neq x \in R_B$ and $x = (x_i)_{i \in I}$. Then there exists at least one $i \in I$ such that $x_i \neq 0$. Let $y = (y_i)_{i \in I}$ with $y_i = b_i$ and $y_j = 0$ for all $j \neq i, j \in I$. Then y is an atom and $xy = y$. ie. $y < x$ and hence R_B is atomic. Let $\{y_k\}_{k \in \Delta}$ be any set of atoms of R_B such that $y_i = b_i$ if $y_{k_i} = b_i$ for some $k \in \Delta$ and $y_i = 0$ if $y_{k_i} = 0$ for all $k \in \Delta$. Then, considering $y = (y_k)_{k \in \Delta}$ we have $y = (y_k)_{k \in \Delta} = (y_{k_i})_{i \in I} = (y_i)_{i \in I}$. Hence y is the upper bound of $\{y_k\}_{k \in \Delta}$. Let x be an upper bound of $\{y_k\}_{k \in \Delta}$. Then there exists an element $z_k \in R_B$ such that $y_k = z_k x$ by Lemma 3.1.1. So, we have $y = (y_k)_{k \in \Delta} = (z_k x)_{z_k \in R_B} = x(z_k)_{z_k \in R_B}$ by Lemma 4.2.6 which gives us $y < x$ by Lemma3.1.1. Thus y is the supremum of $\{y_k\}_{k \in \Delta}$. Therefore, R_B is complete. Let $0 \neq x \in N$ and $x = (x_i)_{i \in I}$. Then, there exists an $i \in I$ such that $x_i \neq 0$. Therefore, $x_i = n_i$. Consider $y = (y_i)_{i \in I}$ with $y_i = b_i$ and $y_j = 0$ for all $j \neq i, j \in I$. Then y is an atom of R_B and $yx \neq 0$. Hence, N is nil-free. Also, atoms of n are of the form $(x_i)_{i \in I}$ with $x_i = n_i$ for some $i \in I$ and $x_j = 0$ for all $j \neq i$ and $i, j \in I$. If $\{z_k\}_{k \in \Delta}$ be any set of atoms of N, then $z = (z_i)_{i \in I}$ such that $z_i = n_i$ if $z_{k_i} = n_i$ for some $k \in \Delta$ and $z_i = 0$ if $z_{k_i} = 0$ for all $k \in \Delta$ is the supremum of $\{z_k\}_{k \in \Delta}$. And 0 is the infimum of $\{z_k\}_{k \in \Delta}$. Hence the set of all nilpotent atoms of $\bigoplus_{i \in I} R_i$ is complete. $\square$

Chapter 5

Primary Ideals in Commutative Weak Idempotent Rings

In this chapter, we discuss further results on primary submaximal ideals in commutative weak idempotent rings with unity.

5.1 Primary Ideals

In this section, we give examples in relation to primary ideals of a cWIR R with unity not containing a nilpotent element n of R and discuss results on primary ideals of R in relation to nonzero nilpotent elements of R.
We begin with the following.
Remark 5.1.1. *Let R be a cWIR with unity, P be a primary ideal of R and $0 \neq n \in N$ such that $n \notin P$. Then*

1. *$P + \langle n \rangle$ is an ideal of R which is neither maximal nor submaximal.*

2. *$P + R_B n$ may not be an ideal of R, where $R_B n = \{bn : b \in R_B\}$.*

We give the following example to clarify the above remark.
Example 5.1.2. *From Example 2.3.28, $Q = \{0\}$ is a primary ideal of a cWIR R with unity and $0 \neq n = 1+i+j+k$ is a nilpotent element of R which does not belong*

to Q. Here the ideal $Q+\langle n\rangle = Q_1 = \{0, 1+i, j+k, 1+i+j+k\}$ is neither maximal nor submaximal. Again $n_1 = i+j$ is a nonzero nilpotent element of R which does not belong to Q and $Q+R_Bn_1 = \{0, i+j\}$ is not an ideal of R, where $R_B = \{0, 1\}$.

Theorem 5.1.3. *Let P be a nonzero primary ideal of a cWIR R with unity and n be a nonzero nilpotent element of R which does not belong to P. Then, $P+\langle n\rangle$ is a primary ideal of R.*

Proof. Let $Q = P+\langle n\rangle$ and $z+Q$ be a nonzero idempotent element of R/Q. Then, $z^2+Q = z+Q$ from which we obtained that $z(1+z) \in Q = P+\langle n\rangle$. So, $z(1+z) = w+rn$ for some $w \in P$ and $r \in R$ as $\langle n\rangle = \{rn : r \in R\}$. This implies $z^2(1+z^2) \in P$ and hence $1+z^2 \in P$. For if $z^2 \in P$, then $z^2 \in Q$ which implies $z+Q = Q$ and hence $z \in Q$ which is a contradiction. Then, $z \in R_B$. For if $z \in N$, then $1 \in P$ and this is a contradiction as P is a primary ideal. So, $1+z \in P$ and hence $1+z = y$ for some $y \in P$. Now, considering $z+Q = 1+y+Q = 1+Q$ as $y \in Q$. Thus, Q and $1+Q$ are the only idempotent elements of R/Q. Therefore, R/Q is local and hence Q is a primary ideal of R. □

Remark 5.1.4. *Let n be a nonzero nilpotent element in a cWIR R with unity which does not belong to a primary ideal P of R. If n_1 is a nonzero nilpotent element of R with $n_1 \neq n$, then both n_1 and n_1+n may not belong to P.*

For justification of the above remark see the following example.

Example 5.1.5. *From Example 2.3.28, Q_1 is a primary ideal of a cWIR R with unity with $n = 1+i \notin Q_1$. Take $n_1 = 1+j \notin Q_1$ which gives $n_1+n = i+j \notin Q_1$.*

Theorem 5.1.6. *Let R be a cWIR with unity and n be a nonzero element in N. Then*

1. *There is a primary ideal P of R with $n \notin P$.*
2. *For $0 \neq n_1 \in N$, either $n_1 \in P$ or $n+n_1 \in P$, provided that P is the primary ideal of R in (1) and $n \in P+R_Bn_1$.*

Proof. Suppose that R is a cWIR with unity and $0 \neq n \in N$.

1. Let $\Sigma_n = \{I$: I is a proper ideal of R such that $n \notin I\}$. Then Σ_n is a non-

empty set as $\{0\}$ belongs to it. Considering Σ_n ordered by inclusion, it is a Poset. Let C be a chain of ideals in Σ_n and $J = \cup_{I \in C} I$. Then $I \subseteq J$ for all $I \in C$, J is an ideal of R and $n \notin J$. Hence $J \subseteq P$ and $n \notin P$ for some primary ideal P of R.

2. For a nonzero element $n \in N$, there is a primary ideal P of R with $n \notin P$ and n belongs to all ideals of R that properly contain P by (1). Let $n_1 \in N$ such that $n_1 \notin P$. Then $P \subset P + Rn_1$. We claim that $n + n_1 \in P$. If $n_1 = n$, clearly $n + n_1 \in P$. Suppose that $n_1 \neq n$. Then $n \in P + Rn_1$ as $n_1 \notin P$ and $P + Rn_1 \notin \Sigma_n$. Let $n \in P + R_B n_1$ and hence $n = x + bn_1$ for some $x \in P$ and $b \in R_B$. This implies $n + bn_1 = x \in P$. Then $b \notin P$. For: $b \in P$, $bn_1 \in P$ and hence $n \in P$ which is a contradiction. So, $b \notin P$. As P is a primary ideal of R and $b \notin P$, $1 + b \in P$ since $b(1 + b) = 0 \in P$ and $(1+b)^m = 1+b$ for all positive integer m. Thus, $n_1 + bn_1 = (1+b)n_1 \in P$ and hence $n + n_1 = n + bn_1 + n_1 + bn_1 \in P$ as $n + bn_1 \in P$. Hence the theorem follows.

□

5.2 Primary Submaximal Ideals

The structure of primary submaximal ideals of a WIR is introduced by V.Kolluru et.al.[29] and studied more by Venkateswarlu Kolluru and Dereje Wasihun [27] in WIR. In this section, we further studied about primary submaximal ideals in commutative weak idempotent rings with unity.

We begin with the following examples of primary submaximal ideals of a cWIR R with unity.

Example 5.2.1. *From Example 2.3.9,* $Q_1 = \{0, 1+i+j+k\}$, $Q_2 = \{0, j+k, 1+i, 1+i+j+k\}$, $Q_3 = \{0, i+k, 1+j, 1+i+j+k\}$, $Q_4 = \{0, i+j, 1+k, 1+i+j+k\}$ *are all primary submaximal ideals. Here* $n = 1+i+j+k \in N$ *belongs to every primary submaximal ideal of* R *and* $n = 1+i+j+k = (1+i)(1+j)$ *where both* $1+i$ *and* $1+j$ *are nilpotent elements of* R*.*

Theorem 5.2.2. *Let R be a cWIR with unity and n_1, n_2 are distinct nonzero nilpotent elements of R. Then every primary submaximal ideal of R contains $n_1 n_2$.*

Proof. Let n_1 and n_2 be two distinct nonzero nilpotent elements of a cWIR R with unity. If $n_1 n_2 = 0$, it is obvious. Let $n_1 n_2$ be a nonzero nilpotent element of R. Suppose that there is a primary submaximal ideal P of R so that $n_1 n_2 \notin P$. It follows that $P \subset P + \langle n_1 n_2 \rangle$. Moreover, $P \subset P + \langle n_1 n_2 \rangle \subset P + \langle n_1 \rangle$. Now we show that $P + \langle n_1 \rangle$ is a proper ideal of R. Assume that $P + \langle n_1 \rangle$ is not a proper ideal of R. Then $1 \in P + \langle n_1 \rangle$ and $1 = x + bn_1$ for some $x \in P$ and some $b \in R$. Thus, $1 + x$ is nilpotent So, $(1 + x)^2 = 0$ which implies $x^2 = 1$ and hence x is a unit element in P which contradicts P is a proper ideal of R. Therefore, $P + \langle n_1 \rangle$ is a proper ideal of R. Now, we claim that $P + \langle n_1 n_2 \rangle \neq P + \langle n_1 \rangle$. For if $P + \langle n_1 n_2 \rangle = P + \langle n_1 \rangle$, then $n_1 = x + bn_1 n_2$ for some $x \in P$ and $b \in R$ as $n_1 \in P + \langle n_1 \rangle$. This implies $(1 + bn_2)n_1 = x \in P$ from which we get that $(1 + bn_2)^2 \in P$ as $n_1 \notin P$ and P is primary. Thus, $1 \in P$ which is a contradiction as P is a proper ideal of R. Therefore, $P + \langle n_1 n_2 \rangle \neq P + \langle n_1 \rangle$. As both of them are proper ideals of R containing P, it is a contradiction to the definition primary submaximal ideal. Therefore, our assumption $n_1 n_2 \notin P$ is false and hence the theorem follows. □

Theorem 5.2.3. *If P is a primary submaximal ideal of a cWIR R with unity which does not contain a nonzero nilpotent element n of R, then $P + R_B n$ is the maximal ideal of R.*

Proof. Let n be a nonzero nilpotent element of R which does not belong to a primary submaximal ideal P of R. Then $P \subset P + R_B n$ as $nN \subseteq P$ by Theorem 5.2.2. Let $x, y \in P + R_B n$. Then $x = m_1 + b_1 n$ and $y = m_2 + b_2 n$ for some $m_1, m_2 \in P$ and $b_1, b_2 \in R_B$. So, $x + y = (m_1 + m_2) + (b_1 + b_2)n \in P + R_B n$. For $r \in R$, $rx = rm_1 + rb_1 n$. If $r \in N$, then $rx \in P \subset P + R_B n$ by Theorem 5.2.2 as P is an ideal. If $r \in R_B$, then $rb_1 \in R_B$ and hence $rx \in P + R_B n$. Thus in any case $rx \in P + R_B n$ and hence $P + R_B n$ is an ideal of R. And also $P + R_B n$ is a proper ideal of R. Assume, if possible that, it is not a proper ideal. Then $1 \in P + R_B n$. This implies $1 = m + bn$ for some $m \in P$ and $b \in R_B$ which also gives us $1 + m^2 = 0 \in P$. Thus, $1 \in P$ as $m \in P$ which is a contradiction as P is a proper ideal of R. Therefore, our assumption is false and hence $P+R_Bn$ is a proper ideal of R. This and remark **??** implies that $P+R_Bn$ is a maximal ideal of R. $\square$

Theorem 5.2.4. *Let P be a primary submaximal ideal of a cWIR R with unity and $0 \neq n_1 \notin P$ for some $n_1 \in N$. Then $P+\langle n_1\rangle = P+R_Bn_1$.*

Proof. Let $0 \neq n_2 \in N$ and $n_2 \neq n_1$. Then $(1+n_2)n_1 \in P+\langle n_1\rangle$ by Theorem 5.2.2 as P is a primary submaximal ideal of R. Then $n_1+n_2n_1 = x+yn_1$ for some $x \in P$ and $y \in R$. This implies $n_1+yn_1 = x+n_2n_1 \in P$ by Theorem 5.2.2 and hence $(1+y)n_1 \in P$. As P is primary and $n_1 \notin P$, $(1+y)^2 \in P$ which implies $1+y^2 \in P$. Then $y \in R_B$. For: if $y \in N$, then $y^2 = 0$ and hence $1 \in P$ which is a contradiction as P is a proper ideal of R. Therefore, $y \in R_B$ and hence $P+\langle n_1\rangle = P+R_Bn_1$. $\square$

Theorem 5.2.5. *Let P be a primary submaximal ideal of a cWIR R with unity and n_1 and n_2 are distinct nonzero nilpotent elements of R such that both do not belong to P. Then $P+R_Bn_1 = P+R_Bn_2$.*

Proof. Let P be a primary submaximal ideal of R and $n_1 \neq n_2$ are nonzero nilpotent elements of R with $n_1, n_2 \notin P$.Assume that $P+R_Bn_1 \neq P+R_Bn_2$. Then $P+\langle n_1\rangle$ and $P+\langle n_2\rangle$ are distinct proper ideals of R containing P which contradicts the fact that there is exactly one maximal ideal that covers P. Thus, our assumption is false and $P+R_Bn_1 = P+R_Bn_2$. $\square$

Theorem 5.2.6. *Let P be a primary submaximal ideal of a cWIR R with unity and n_1 be a nonzero nilpotent element of R with the property that $n_1 \notin P$. If there exists a nonzero nilpotent element n_2 of R such that $n_2 \notin P$, then $n_1+n_2 \in P$.*

Proof. Suppose that P is a primary submaximal ideal of R and n_1 and n_2 are nonzero nilpotent elements of R that do not belong to P. If $n_1 = n_2$, then clearly $n_1+n_2 \in P$. Suppose that $n_1 \neq n_2$. Then $P+R_Bn_1 = P+R_Bn_2$ and so that $n_1 = x+bn_2$ for some $x \in P$ and $b \in R_B$. If $b \in P$, then $n_1 \in P$ which is a contradiction. Thus, $b \notin P$ and hence $1+b \in P$ as P is a primary ideal of R. As $n_2 \in P+R_Bn_2 = P+R_Bn_1$, $n_2 = y+b_2n_1$ for some $y \in P$ and $b_2 \in R_B$. So, $n_1+n_2 = x+bn_2+y+b_2n_1 = x+(1+b)(y+b_2n_1) \in P$ as $x, 1+b \in P$. Therefore, $n_1+n_2 \in P$. $\square$

Remark 5.2.7. *The notion of primary submaximal and maximal ideals of a cWIR with unity is different.*

We clarify this by the following example.

Example 5.2.8. Q_2, Q_3 *and* Q_4 *are primary submaximal ideals of* R *in Example 2.3.28 which are not maximal.* N *is a maximal ideal of* R *which is primary but not submaximal. Here* $0 \neq n = 1 + i + j + k \notin Q$ *and* Q *is the only maximal primary ideal that is not submaximal.*

Theorem 5.2.9. *For a submaximal ideal* I *of a cWIR* R *with unity, define* $X(I) = \{P : P \text{ is a primary ideal of } R \text{ so that } I \subseteq P\}$. *Then X(I) has exactly two elements.*

Proof. Suppose that I is a submaximal ideal of R. Then I is either semiprime or primary. If I is a semiprime, then it is contained in exactly two maximal ideals of R both of which cover I.

Both of them are also prime and hence primary. So, these are the only elements of $X(I)$. If I is a primary ideal, then it is contained in a unique maximal ideal say M of R and both I and the unique maximal ideal M belong to $X(I)$. Hence, in either case, $X(I)$ has exactly two elements. □

Remark 5.2.10. *Let* R *be a cWIR with unity,* N *and* X *be the nilradical and the collection of all primary submaximal ideals of* R *respectively. For* $n \in N$, *define* $\overline{X}_n = \{P : P \in X \text{ and } n \notin P\}$. *Then,* $(N, +)$ *is not isomorphic to* $(\{\overline{X}_n\}_{n \in N}, \Delta)$, *where* Δ *denotes the symmetric difference.*

For this, look at the following example.

Example 5.2.11. *From example 2.3.28* $\{\overline{X}_n\}_{n \in N} = \{\emptyset, Q_2, Q_3, Q_4\}$ *whereas* N *has eight elements. Here,* $\overline{X}_n$ *is an empty set for* $n = 1 + i + j + k \in N$.

Conclusions and Recommendations

We introduce the partial ordering in a cWIR R with unity (see Lemma 3.1.1) and prove that it coincides with the natural properties of the ordering in Boolean rings whenever R is restricted to its idempotent elements. Also, we introduce the notion of an atom and found certain results on the poset $(R, <)$ (see Theorem 3.1.12). We prove that any cyclic module over the collection of Boolean elements of a cWIR with unity is a Boolean lattice (see Theorem 3.1.7). We also establish that every finite nonzero cWIR with unity admits an atomic basis (see Theorem 3.2.2). Using the concepts of an atom, we prove the direct product decomposition theorem: R_B is a symmetric Boolean ring, N is nil-free, and the set of all nilpotent atoms is complete if and only if R is isomorphic to the direct product of WIRs each of which is a copy of a two-element field or a four-element BLR H_4 (see Theorem 4.2.11). Finally, we studied the properties of primary submaximal ideals in a cWIR R with unity about nonzero nilpotent elements of R. The product of two distinct nonzero nilpotent elements of a commutative weak idempotent ring R with unity is contained in every primary submaximal ideal of R (see Theorem 5.2.2).

In general, the partial ordering in commutative weak idempotent rings with unity holds as it works in Boolean rings with unity. The concepts of an atom in a commutative weak idempotent ring R with unity have been used to prove the direct product decomposition of R under special conditions mentioned in the Theorem 4.2.11. And also, we have verified the existence of a primary submaximal ideal containing a nonzero nilpotent element of a commutative weak idempotent ring R with unity and studied some structure of it.

Here we introduced the notion of partial ordering in commutative weak idempotent rings with unity, studied the role of atoms in the direct product decomposition of a commutative weak idempotent ring R with unity, and studied the structure of the primary submaximal ideal of R to some extent. We recommend that researchers further study these in detail, shortly prove Theorem 4.2.11 and extend the concepts to non-commutative weak idempotent rings with or without unity.

Bibliography

[1] A.L. Foster, *The Theory of Boolean-like Ring.* Trans. Amer. Math. Soc. Vol.59,1946.

[2] D.D.Anderson, *Some Finiteness Conditions on a Commutative Ring.* Housten Journal of Mathematics, Volume 4, No.3, 1978.

[3] Dereje Wasihun, Tilahun Abebaw, Yibeltal Yitayew and Venkateswarlu Kolluru. Weak idempotent rings.Sinet: Ethiop. J. Sci., 45(1): 97–104, 2020.

[4] Foster A.L. p^k-rings and ring logics. Ann.Scu. Norm.Pisa, Vol.5,(1951) 279-300.

[5] G. Grätzer, *General Lattice Theory*, 2nd Edition, Birkhäuser (1998).

[6] G. F. Birkenmeier, H. E. Heatherly and E. K. Lee, *Completely prime ideals and associated radicals.* in Proc. Biennial Ohio State-Denison Conf., 1992, eds. S. K. Jain and S. T. Rizvi (World Scientific, Singapore, 1993), pp. 102–129.

[7] Harary.F.,*Atomic Boolean-like rings with finite radicals.* Duke Math.Journal, Vol.17, 1950.

[8] I. Sussman, *A generalization of Boolean rings.* Math. Ann. 136, (1958), 326-338.

[9] Jayanthi Ganapathy, *Maximal Elements and Upper Bounds in Posets.* Pi Mu Epsilon Journal Vol.9, No.7 (FALL 1992), pp. 462-464.

[10] Joseph J. Rotman, *Advanced Modern Algebra.* Prentice Hall, 1st edition(2002).

[11] K. Hailu and K. Venkateswarlu Boolean-like semiring Modules.Int.J.Contemp.Math. Sciences, Vol.9(2014), 395-402.

[12] K. Venkateswarlu and B. V. N. Murthy, *Primary Ideals in Boolean-like Rings.* Int. J. Contemp. Math. Sciences, Vol. 6(2011), no.28, 1367-1377.

[13] K. Venkateswarlu and B. V. N. Murthy, *semiprime Ideals and Annihilators in Boolean-like Semi Rings.* International Journal of Algebra, Vol. 5, 2011,no. 28, 1363 - 1370.

[14] L. Bhaskar, A.K.S. Chandrasekhar Rao and T. Srinivas, *Some Theorems on SS-Elements of a Ring*, Palestine Journal of Mathematics. Vol. 12(1)(2023), 537-543.

[15] Matej Brešar, *Introduction to Noncommutative Algebra*, Springer.

[16] M. Nagata, On the theory of radicals in a ring, J. Math. Soc. Japan vol. 3 (1951) pp. 330-344.

[17] Michael Holloway, *Some Characterizations of Finite Commutative Nil Rings*, Palestine Journal of Mathematics, Vol. 2(1) (2013), 6-8.

[18] Moshe Roitman, *On the atomic property for power series rings.* Elsevier Journal of Pure and Applied Algebra 145 (2000), 309-319.

[19] Nahid Ashrafi, Marjan Shebani and Huanyin Chen, *Rings Involving Idempotent, Units and Nilpotent Elements.* https://www.researchgate.net/publication/267759872, Nov 2014.

[20] Nam Kyun Kima, Yang Lee, *On rings whose prime ideals are completely prime.* Elsevier Journal of Pure and Applied Algebra 170 (2002), 255–265.

[21] N.H. McCoy and Montgomery D. *A representation of generalized Boolean rings.* Duke Math. J. Vol. 3, (1937), 456-459.

[22] Rodney Coleman, Laurent Zwald. *Primary Ideals* 2020.

[23] R. Sikorski, *Boolean Algebras.* 2nd Edition, Springer-Verlag, New York (1964).

[24] S. Givant and P. Halmos, *Introduction to Boolean Algebras.* Springer.

[25] Sangmin Chun and D.D.Anderson, *A Class of Atomic Rings.* Communications in Algebra, 40: 1086-1095, 2012.

[26] Subrahmanyam N.V. *The lattice theory of certain classes of rings.* Math.Ann. 139(1960), 275-286.

[27] Venkateswarlu Kolluru and Dereje Wasihun, *Submaxiamal ideals of Weak Idempotent Rings.* Bull.Int. Math. Virtual Inst., Vol. 11(1)(2021),169-176.

[28] Stanley Burris H.P. Sankappanavar, *A Course in Universal Algebra*, 2012.

[29] Venkateswarlu Kolluru, Dereje Wasihun and Tilahun Abebaw, *Structure of Weak Idempotent Rings.* Bull. Int. Math. Virtual Inst., Vol. 10(3)(2020), 535-545.

[30] V. Swaminathan, *Boolean-like Rings.* PhD dissertation, Andhra University, India, 1982.

[31] V. Swaminathan, *On Foster's Boolean-like Ring.* Math.Seminar Notes, Kobe University, Japan, Vol. 8,(1980), 347-367.