



Addis Ababa University  
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Analysis of Constrained Optimal Control Problem

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Addis Ababa University  
Department of Mathematics

The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **Analysis On Constrained Optimal Control Problem** by Desta Hundessa in partial fulfillment of the requirements for the degree of master of Science.

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# Abstract

This paper gives a review for optimal control problems with state inequality constraints. To solve the problem with pure state inequality constraint, we use different approaches with complementary slackness (first order constraints), the indirect adjoining approach for higher order constraints and the indirect adjoining approach with continuous adjoint functions. Furthermore, the application of optimal control problems conditions is demonstrated by solving illustrative examples.

# Notation

$F[t]$	Objective function
$\dot{X}$	State equation
$U, U(t)$	Control vector
$X, X(t)$	State vector
$\lambda(t)$	Co-state vector(adjoint vector)
$H[t]$	Hamiltonian
$L[t]$	Lagrangian
$u^*, u^*(t)$	Optimal control vector
$u(x(t), t)$	Optimal region
$(x^*, u^*)$	Feasible or admissible pair
$x^*, x^*(t)$	Optimal state vector
$h[t]$	Mixed constarint
$k[t]$	Pure state constarint

# Chapter 1

## Basic Concepts And Definitions

### 1.1 Introduction

Optimal control is one of the important branches of optimization. Optimal control deals with the problem of finding a constraint law for a given system such that a certain optimality criteria is achieved. A control problem includes a cost functional that is a function of state and control variables. An optimal control is set of differential equations describing the paths of the control variables that minimize the cost functional. the optimal control can be derived using pontryagin's maximum principle (a necessary condition also known as pontryagin's minimal principle or simply pontryagin's principle), or by solving the Hamiltonian Jacobin Bellman equation (a sufficient conditions). Suppose that the state of a certain system is described by a number of parameters  $y = (y_1, y_2, \dots, y_n)$  which evolve according to the state equation  $\dot{y}(t) = g(t, y, u)$  where  $u(t) = (u_1(t), u_2(t), \dots, u_m(t))$ , represents the control (input or decision) vector exercised on the system. We Assume that  $u(t) \in U_{ad}$  is from some suitable vector space.

The initial and /or final conditions of the state of system be given by  $y(0) = y_0, y(T) = y_T$  where  $T$  is the final time of consideration. The cost functional is given by:

$$F(y, u) = \int_0^T (f(t, y(t), u(t)))dt, \quad (1.1)$$

where  $f$  measures how good a choice of the control  $u$ . Then a pair  $(y, u)$  is said to be feasible or admissible if the following are satisfied

- i). Control constraints  $u(t) \in U_{ad}, t \in [0, T]$ .
- ii). State law:  $\dot{y}(t) = g(t, y, u)$ .
- iii). The end-point conditions:  $y(0) = y_0, y(T) = y_T$  are satisfied

optimal control problem is:

$$\begin{aligned} \max F(y, u) &= \int_0^T (f(t, y(t), u(t)))dt \\ \text{Subject to : } u(t) &\in U_{ad} \\ \dot{y}(t) &= g(t, y, u) \\ y(0) &= Y_0, y(T) = Y_T \end{aligned} \quad (1.2)$$

## 1.2 Lagrange multipliers and the Hamiltonian

First assume that  $U_{ad}$  the entire vector space. Thus the only constraint of the problem will be the differential equation together with the appropriate boundary or initial conditions  $\dot{y}(t) = g(t, y(t), u(t)), y(0) = y_0, y(T) = y_T$   
 $\Rightarrow g(t, y, u) - \dot{y}(t) = 0$

Corresponding to this *equality* constraint (pointwise) we define a multiplier function  $\lambda : [0, T] \rightarrow \mathfrak{R}^n$

Then the Lagrangian of the problem will be

$$L(y, u, \lambda, \dot{y}) = \int_0^T [f(t, y, u) + \lambda(t)(g(t, y, u) - \dot{y}(t))]dt \quad (1.3)$$

### Theorem 1.1

If  $(y^*, u^*, \lambda^*, \dot{y}^*)$  is a minimizer of L, then  $(y^*, u^*)$  is also a minimizer of F.

**Proof**

$$\text{Define } p(y, u, \lambda) = \int_0^T \lambda(g(t, y, u) - \dot{y}(t))dt$$

Let  $\lambda^*$  corresponds to the pair  $(y^*, u^*)$  such that  $(y^*, u^*, \lambda^*, \dot{y}^*)$  is a minimum for L.

Then  $L(y^*, u^*, \lambda^*, \dot{y}^*) = F(y, u) + P(y, u, \lambda)$  and  $L(y^*, u^*, \lambda^*, \dot{y}^*) \leq L(y, u, \lambda, \dot{y})$ , for all  $(y, u)$ .

if the pair  $(y^*, u^*)$  is feasible, the state condition must be satisfied. That is:

$$\dot{y}_*(t) = g(t, y^*, u^*) \text{ implies } P(y^*, u^*, \lambda) = 0$$

$$F(y^*, u^*) + P(y^*, u^*, \lambda^*) \leq F(y, u) + P(y, u, \lambda^*) \text{ for all } (y, u) \text{ feasible.}$$

Implies  $F(y^*, u^*) \leq F(y, u)$  for all feasible pair  $(y, u)$ . i.e.  $F(y^*, u^*)$  is minimum. ■

$$\text{Let } G(y, u, \lambda, \dot{y}, \dot{u}, \dot{\lambda}, t) = f(t, y, u) + \lambda(g(t, y, u) - \dot{y}) \text{ then } L(y, u, \dot{\lambda}, \dot{y}).$$

becomes a variational problem. Hence  $(y^*, u^*, \lambda^*)$  is an extremal.  
 This implies that the Euler Lagrangian Differential Equation (ELDE):

$$i). \quad \frac{d}{dt} \left[ \frac{\partial G}{\partial \dot{y}} \right] = \frac{\partial G}{\partial y} \Rightarrow -\dot{\lambda} = f_y + \lambda g_y$$

$$ii). \quad \frac{d}{dt} \left[ \frac{\partial G}{\partial \dot{u}} \right] = \frac{\partial G}{\partial u} \Rightarrow 0 = f_u + \lambda g_u$$

$$iii). \quad \frac{d}{dt} \left[ \frac{\partial G}{\partial \dot{\lambda}} \right] = \frac{\partial G}{\partial \lambda} \Rightarrow 0 = g(t, y, u) - \dot{y}$$

Now define the Hamiltonian of the problem by:

$$H = f + \lambda g$$

$$H_y = f_y + \lambda g_y$$

$$0 = f_u + \lambda g_u$$

In terms of the Hamiltonian, the necessary conditions for optimality can be written as:

$$i). \quad \text{The Adjoint Condition: } \dot{\lambda}(t) = -\frac{\partial H}{\partial y}, \quad \lambda(0) = \lambda(T) = 0, \quad (TR)$$

$$ii). \quad \text{The Optimality Condition: } 0 = \frac{\partial H}{\partial u}$$

$$iii). \quad \text{The State Condition: } \dot{y}(t) = g(t, y, u), \quad y(0) = y_0, \quad y(T) = y_T \quad (\text{End point condition})$$

### Example-1

If  $F(y, u) = \int_0^T u^2(t) dt$  and  $\dot{y} = u + \alpha y$ , for some  $\alpha \in \mathfrak{R}$ , determine the optimal control, under the initial condition  $y(0)=1$ .

### Solution

The Hamiltonian of the problem is:  $H = f + \lambda g = u^2 + \lambda(u + \alpha y)$

1. Adjoint Condition

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial y}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial y} = -\alpha \lambda$$

Transversality condition (TR condition)  $\lambda(0) = 0$

$$\dot{\lambda} + \alpha \lambda = 0$$

$$\lambda(t) = K e^{-\alpha t}$$

With  $\lambda(0) = 0 \Rightarrow K = 0 \Rightarrow \lambda(t) = 0$

2. Optimality Condition

$$\frac{\partial H}{\partial u} = 0$$

$$\begin{aligned}
2u + \lambda &= 0 \\
u &= -\frac{1}{2}\lambda = 0 \\
u &= 0
\end{aligned}$$

### 3.State Condition

$$\begin{aligned}
\dot{y} &= u + \alpha y = \alpha y \\
y(t) &= Ce^{\alpha t}, \text{ with } y(0) = 1 \\
&\Rightarrow C = 1 \\
\therefore y^*(t) &= e^{\alpha t}; \\
u^*(t) &= 0, \forall t \in [0, 1].
\end{aligned}$$

with,

$$\lambda^*(t) = 0, \forall t \in [0, 1].$$

i.e. The optimal control (if it exists) must be  $u^*(t) = 0$ .

### Example 1.2

$$\begin{aligned}
\min F(x, u) &= \int_0^1 u^2(t) dt \\
\text{Subject to : } &\ddot{y}(t) = u(t)
\end{aligned}$$

End point conditions:  $y(0) = \dot{y} = 1, y(1) = 0$

#### Solution:

To solve the problem first reduce the second order equations to first order system with components

$$y_1 = y$$

$$y_2 = \dot{y}$$

So that  $\dot{y}_1 = \dot{y} = y_2, \dot{y}_2 = \ddot{y} = u$  and

end point conditions:  $y_1(0) = 0 = y_1(1)$  and  $y_2(0) = 1$

The Hamiltonian is :

$$H(t, y(t), u(t), \lambda(t)) = u^2(t) + \lambda_1(t)y_2(t) + \lambda_2(t)u(t)$$

#### 1. The Optimality Condition

$$\begin{aligned}
\frac{\partial}{\partial u} H(t, y(t), u(t)) &= 0 \\
2u + \lambda_2 &= 0 \\
u &= -\frac{1}{2}\lambda_2
\end{aligned}$$

## 2. The Adjoint Conditions

$$\begin{aligned}
 -\frac{\partial}{\partial y_1}H(t, y(t), u(t)) &= -\dot{\lambda}_1 = 0 \\
 -\frac{\partial}{\partial y_2}H(t, y(t), u(t)) &= \dot{\lambda}_2 = 0 \\
 \dot{\lambda}_2 &= -\dot{\lambda}_1
 \end{aligned}$$

Therefore the optimality equations together with end points are:

$$\begin{aligned}
 u &= -\frac{1}{2}\lambda_2 \\
 \dot{\lambda}_1 &= 0 \\
 \dot{\lambda}_2 &= -\dot{\lambda}_1
 \end{aligned}$$

$$y_1 = y_2, \dot{y}_2 = u, y_1(0) = 1, y_1(1) = 0, y_2(0) = 1, \text{ and } \lambda_2(1) = 0$$

Now from the above conditions:

$$\begin{aligned}
 \dot{\lambda}_1(t) &= \lambda_1 = C_1 \text{ and } \dot{\lambda}_1(t) = C_1, \text{ but } \dot{\lambda}_2 = -\dot{\lambda}_1 \\
 \dot{\lambda}_2(t) &= -C_1
 \end{aligned}$$

$$\lambda_2(t) = -C_1(t) + C_2$$

$$\text{Therefore, } u(t) = -\frac{1}{2}(-c_1(t) + c_2) = -\frac{1}{2}c_1t - \frac{1}{2}c_2$$

$$\text{But } \dot{y}_2(t) = u(t)$$

$$\dot{y}_2(t) = -\frac{1}{2}(C_1(t) + C_2)$$

$$\int (\dot{y}_2(t)) = -\int \left(\frac{1}{2}(-C_1(t) + C_2)\right) dt$$

$$y_2(t) = \frac{1}{4}(C_1t^2 - \frac{1}{2}C_2t + C_3$$

$$y_1(t) = \int y_2(t) = \frac{1}{12}C_1t^3 - \frac{1}{4}C_2t^2 + C_3t + C_4$$

where  $C_1, C_2, C_3$  and  $C_4$  are constant that will be determined. Now let solve all constant using given conditions

$$y_1(0) = 1 \text{ and } y_1(1) = 0$$

$$y_1(0) = \frac{1}{12}C_1(0)^3 - \frac{1}{4}C_2(0)^2 + C_3(0) + C_4 = 1, C_4 = 1$$

$$\begin{aligned}
y_1(1) &= \frac{1}{12}C_1(1)^3 - \frac{1}{4}C_2(1)^2 + C_3(1) + 1 = 0 \\
\frac{1}{12}C_1 - \frac{1}{4}C_2 + C_3 + 1 &= 0, \text{ where } y_2(0) = 1 \\
y_2(0) &= \frac{1}{4}C_1(1)^2 - \frac{1}{4}C_2(0) + C_3 = 1, C_3 = 1
\end{aligned}$$

Therefore

$$\frac{1}{12}C_1 - \frac{1}{4}C_2 + 2 = 0$$

(TR), but  $\lambda_2(1) = 0$ ,  $\lambda_2(1) = -C_1 + C_2 = 0$

We can solve  $C_1$  and  $C_2$  simultaneously

$$\begin{aligned}
\frac{1}{12}C_1 - \frac{1}{4}C_2 &= -2 \\
C_1 - 3C_2 &= -24 \\
-C_1 + C_2 &= 0 \\
\Rightarrow C_1 = 12 = C_2
\end{aligned}$$

The optimal solutions are:

$$\begin{aligned}
u(t) &= 6(t + 1) \\
y_1(t) &= t^3 - 3t^2 + t + 1 \\
y_2(t) &= 3t^2 - 6t + 1
\end{aligned}$$

The optimality conditions are necessary for all optimal solution.

**Theorem 1.2**

let  $f$  and  $g$  be convex functionals with respect to  $(y, u)$  for each fixed  $t \in [0, T]$ , then every solution of the optimal control problem (1.1) that satisfy the necessary conditions will be an optimal solution of the optimal control.

**Proof**

Assume that the pair  $(y^*, u^*)$  satisfy all the necessary conditions and let  $(y, u)$  be any other admissible pair.

$$\begin{aligned}
F(y, u) - F(y^*, u^*) &= \int [f(t, y, u) - f(t, y^*, u^*)] dt, F \text{ is convex implies} \\
f(t, y, u) - f(t, y^*, u^*) &\geq f_y(y - y^*) - f_u(u - u^*) \\
&\geq \int_0^T \left[ \frac{\partial}{\partial y} f(t, y, u)(y - y^*) + \frac{\partial}{\partial u} f(t, y, u)(u - u^*) \right] dt
\end{aligned}$$

$$\begin{aligned}
&= \int_0^T [-\dot{\lambda} - \lambda \frac{\partial g}{\partial y}(t, y^*, u^*)](y - y^*) - \lambda \frac{\partial g}{\partial u}(t, y^*, u^*)(u - u^*)] dt \\
&= - \int_0^T \lambda g_y(t, y^*, u^*)(y - y^*) dt - \int_0^T \dot{\lambda}(y - y^*) dt - \int_0^T \lambda g_u(t, y^*, u^*)(u - u^*) dt \\
&= - \int_0^T \lambda g_y(y - y^*) dt - [\lambda y]_0^T + \int_0^T \lambda y' dt + [\lambda y^*]_0^T - \int_0^T \dot{\lambda} y^* dt - \int_0^T \lambda g_u(u - u^*) dt \\
&\quad = - \int_0^T \lambda [g_* - g + g_y(y - y^*) + g_u(u - u^*)] dt \\
&= - \int_0^T \lambda [g(t, y, u) - g(t, y^*, u^*)] dt - \int_0^T \lambda [g_y(t, y_*, u_*)(y - y^*) + g_u(t, y_*, u_*)(u - u^*)] dt \\
&\quad \leq g(t, y, u) - g(t, y^*, u^*), \text{ by convexity of } g. \\
&\geq - \int_0^T \lambda [g(t, y^*, u^*) - g(t, y, u)] dt - \int_0^T \lambda [g(t, y, u) - g(t, y_*, u_*)] dt
\end{aligned}$$

We have that

$F(y, u) - F(y^*, u^*) \geq 0, \forall (y, u)$ . Hence  $(y^*, u^*)$  is minimizer of the Optimal Control problem. ■

### 1.3 The Bang- Bang principle

Consider the optimal control problem

$$\begin{aligned}
\max_u \{ &F(u) = \int_0^T [f_1(t, x) + f_2(t, x)u(t)] dt \\
\text{S.t : } &\dot{x} = g_1(t, x) + g_2(t, x)u(t), x(0) = x_0 \\
&a \leq u(t) \leq b, \forall t \in [0, T]
\end{aligned} \tag{1.4}$$

#### Definition

A control  $u \in U_{ad}$  is called bang- bang if for each time  $t \geq 0$  and each index  $i = 1, 2, \dots, m$ , we have  $|u_i(t)| = 1$  where  $u(t) = (u_1(t), u_2(t), \dots, u_m(t))$  if the control problem (1.3) is given, then the Hamiltonian will have the form:

$H = (f_1(t, x) + \lambda g_1(t, x)) + f_2(t, x) + \lambda g_2(t, x)u$ , and also linear with respect to  $u$ .

Then, the optimality condition in Pontryagin's max principle becomes:

$$\frac{\partial H}{\partial u} = f_2(t, x^*) + \lambda g_2(t, x^*), \text{ which does not explicitly depend on } u. \tag{1.5}$$

In particular, when  $\frac{\partial H}{\partial u} = 0$  ( $a < u(t) < b$ ).

- There is no characterization of  $u$ , that means, we can not find a solution  $u^*$  from this procedure

Hence  $u^*(t)$  is either  $a$  or  $b$ . There is a function  $\psi(t) = f_2(t, x) + \lambda g_2(t, x)$  is the Switching function. Therefore,  $u(t)$  takes values of the boundaries of  $U_{ad}$ , but the times  $t_i$ , that  $u(t)$  switches its values from one boundary to the other is given by the switching function  $\psi(t)$ . The switching times  $t_i$ , is found by setting

$$\psi(t_i) = 0 \Rightarrow f_2(t_i, x(t_i)) + \lambda(t_i)g_2(t_i, x(t_i)) = 0$$

## 1.4 Pontryagins Maximum (or Minimum) Principle

Pontryagins maximum (or minimum) principle is used in optimal control problem to find the best possible control for taking a system from one state to another especially in the presence of constraints for the state or input controls. The principle states informally that the Hamiltonian must be minimized (or maximized) over  $U$ , the set of all permissible controls. If  $u^* \in U$  is the optimal control for the problem then the principle states that:

$$H(t, x^*(t), u^*(t), \lambda^*(t)) \leq H(t, x(t), u(t), \lambda(t)), \forall u \in U, \forall t \in [0, T] \quad (1.6)$$

where  $x^* \in C^1[0, T]$  is the corresponding optimal state trajectory and  $\lambda^* \in [0, T]$  is the optimal costate trajectory.

Consider the optimal control problem in maximum form :

$$\begin{aligned} \max F(x, u) &= \int_0^T f(t, x(t), u(t))dt + S(x(T), T) \\ \text{Subject to : } \dot{x}(t) &= g(t, x(t), u(t)), x(0) = x_o \end{aligned} \quad (1.7)$$

where  $S(x(T), T)$  is known as the salvage functions. Suppose  $x(t)$  and  $u(t)$  represent the state trajectory and optimal control respectively. Then there exists an adjoint  $\lambda(t)$  such that together  $x(t)$ ,  $u(t)$  and  $\lambda(t)$  satisfies the following conditions.

1. Adjoint condition:

$$\dot{\lambda}^*(t) = -\frac{\partial}{\partial x} H(t, x^*(t), u^*(t), \lambda^*(t))$$

2. State equation:  $\dot{x}^*(t) = g(t, x^*, u^*)$ , with  $x(0) = x_o$
3. The transversalis condition:

$$\dot{\lambda}(T) = S_x(x(T), T)$$

4. The maximum conditions:  $H(t, x^*(t), u^*(t), \lambda^*(t)) \leq H(t, x(t), u(t), \lambda(t))$

### Bounded controls

Consider the optimal control problem

$$\begin{aligned} \max F(x, u) &= \int_0^T f(t, x(t), u(t))dt + S(x(T), T) \\ \text{Subject to : } \dot{y}(t) &= g(t, y(t), u(t)), y(0) = y_o \\ a &\leq u(t) \leq b, \forall t \in [0, T], a < b. \end{aligned} \quad (1.8)$$

Hence the conditions for optimality for bounded controls are given

1. State equation:  $\dot{y}(t) = g(t, y(t), u(t))$ , with  $y(0) = y_0$
2. Adjoint condition:

$$\dot{\lambda}(t) = -\frac{\partial}{\partial y} H(t, y(t), u(t), \lambda(t))$$

3. The transversality condition:

$$\lambda(T) = S_y(y(T), T)$$

4. The Optimality conditions:

$$i. u^*(t) = a, \text{ if } f_u + \lambda g_u \leq 0 \text{ at } t$$

$$ii. a < u^*(t) < b, \text{ if } f_u + \lambda g_u = 0 \text{ at } t$$

$$iii. u^*(t) = b, \text{ if } f_u + \lambda g_u \geq 0 \text{ at } t$$

Hence the last conditions (the maximizing the Hamiltonian) in the Pontryagin's maximum principle is replaced by

$$u^* = a, \text{ if } \frac{\partial H}{\partial u} < 0$$

$$a < u^* < b, \text{ if } \frac{\partial H}{\partial u} = 0$$

$$u^* = b, \text{ if } \frac{\partial H}{\partial u} > 0$$

### Example 1.3

$$\max \int_0^2 [2x(t) - 3u(t) - u^2(t)] dt$$

$$\text{Subject to : } \dot{x}(t) = x(t) + u(t), x(0) = 5$$

$$0 \leq u(t) \leq 2$$

### Solution

$$\text{The Hamiltonian is : } H = f + \lambda g = (2x - 3u - u^2) + \lambda(x + u)$$

i. The Adjoint Condition

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = 0, \lambda(2) = 0 \text{ (TR)}$$

$$\lambda' = -2 - \lambda$$

$$\lambda_\lambda(t) = Ke^t, \lambda_p(t) = A, \text{ where } A = -2.$$

$$\lambda^*(t) = \lambda_\lambda(t) + \lambda_p(t)$$

$$\lambda(t) = Ke^{-t} - 2$$

$$\lambda(2) = 0 \Rightarrow Ke^{-2} - 2 = 0$$

$$K = \frac{2}{e^{-2}} = 2e^2$$

$$\lambda^*(t) = 2e^{2-t} - 2 = 2(e^{2-t} - 1)$$

ii. State Condition

$$\dot{x} = x + u, x(0) = 5, \quad x^*(t) = Ke^t + x_p(t), x(0) = 5$$

iii. Optimality condition (maximum condition)

$$\frac{\partial H}{\partial u} = -3 - 2u + \lambda = -2u - 3 + 2(e^{2-t} - 1)$$

- If  $0 < u^* < 2$ , then

$$\frac{\partial H}{\partial u} = 0 \Rightarrow 2u = 2e^{2-t} - 5$$

$$\Rightarrow u^*(t) = e^{2-t} - \frac{5}{2}$$

- If  $u^* = 0$ , then

$$\begin{aligned}\frac{\partial H}{\partial u} < 0 &\Rightarrow 2e^{2-t} - 5 < 0 \\ \Rightarrow e^{2-t} < \ln \frac{5}{2} &\Rightarrow 2 - t = \ln \frac{5}{2} \\ 2 - \ln \frac{5}{2} < y &\leq 2 \\ \text{i.e. } t &\in (2 - \ln \frac{5}{2}, 2], u^* = 0\end{aligned}$$

- If  $u^* = 2$ , then

$$\begin{aligned}\frac{\partial H}{\partial u} > 0 &\Rightarrow -2 \times 2 + 2e^{2-t} - 5 > 0 \\ \Rightarrow e^{2-t} > \ln \frac{9}{2} &\Rightarrow 2 - t \geq \ln \frac{9}{2} \\ 2 - \ln \frac{9}{2} > t &\geq 0 \\ \text{i.e. } t &\in [0, 2 - \ln \frac{9}{2}), \text{ then } u^* = 2\end{aligned}$$

Therefore,

$$u^*(t) = \begin{cases} 2, & \text{when } 0 \leq t < 2 - \ln \frac{9}{2} \\ e^{2-t} - \frac{5}{2}, & \text{when } 2 - \ln \frac{9}{2} < t \leq 2 - \ln \frac{5}{2} \\ 0, & \text{when } 2 - \ln \frac{5}{2} < t \leq 2 \end{cases}$$

The corresponding state solution is

$$\begin{aligned}x^*(t) &= \begin{cases} K_1 e^t - 2, & t \in [0, 2 - \ln \frac{9}{2}) \\ K_2 e^t - \frac{1}{2} e^{2-t} + \frac{5}{2}, & t \in [2 - \ln \frac{9}{2}, 2 - \ln \frac{5}{2}] \\ K_3 e^t, & t \in (2 - \ln \frac{5}{2}, 2] \end{cases} \\ x^*(0) = 5 &\Rightarrow K_1 - 2 = 5 \Rightarrow K_1 = 7\end{aligned}$$

$x^*(t)$  is continuous.

a)

$$\begin{aligned}7e^{t_1} - 2 &= K_2 e^{t_1} - \frac{1}{2} e^{2-t_1} - \frac{5}{2} \\ 7e^{t_1} + \frac{1}{2} e^{2-t_1} - 2 + \frac{5}{2} &= K_2 e^{t_1}\end{aligned}$$

$$7e^{t_1} + \frac{1}{2}e^{2-t_1} + \frac{1}{2} = K_2e^{t_1}$$

$$K_2 = 7 + \frac{1}{2}e^{2-2t_1} + \frac{1}{2}e^{-t_1}$$

b)

$$K_2e^{t_2} - \frac{1}{2}e^{2-t_2} + \frac{5}{2} = K_3e^{t_2}$$

$$(7 + \frac{1}{2}e^{2-2t_2} + \frac{1}{2}e^{-t_2})e^{t_2} - \frac{1}{2}e^{2-t_2} - \frac{5}{2} = K_3e^{t_2}$$

$$K_3 = 7 + \frac{1}{2}e^{-t_2}$$

The solution is:

$$x^*(t) = \begin{cases} 7e^t - 2, & t \in [0, 2 - \ln \frac{9}{2}] \\ 7e^t + 3, & t \in [2 - \ln \frac{9}{2}, 2 - \ln \frac{5}{2}] \\ 7e^t + \frac{1}{2}, & t \in (2 - \ln \frac{5}{2}, 2] \end{cases}$$

# Chapter 2

## Analysis of Constrained Optimal control problems

### 2.1 Introduction

Optimal control problems with state variable inequality constraints are an important in different areas, especially in Mechanics, aerospace, management science and economics. These problems are not easy to solve and even the theory is not unambiguous, since there exist various forms of the necessary and sufficient optimality condition .more specifically, we deal with problems with pure and mixed state variable constraints. Pure constraints are inequality constraints expressed only in terms of the state variables and possibly time. Mixed constraints are constraints on control variable that may depend on the state variables and time [6]

### 2.2 Optimal control problem with control state constraints

state  $x(t) \in \mathfrak{R}$ , control  $u(t) \in \mathfrak{R}$ . all functions are assumed to be sufficiently smooth boundary reconditions

$$\dot{x}(t) = f(x(t), u(t)), \text{a.e. } t \in [0, t_f],$$

$$x(0) = x_0 \in \mathfrak{R}^n, \psi(x(t_f)) = 0 \in \mathfrak{R}^k,$$

$$(0 = \varphi(x(o), x(t_f)) \text{ mixed boundary conditions}$$

mixed control state constraints

$$\begin{aligned} \alpha \leq c(x(t), u(t)) \leq \beta, t \in [0, t_f], c : \mathfrak{R}^n \times \mathfrak{R}^m \rightarrow \mathfrak{R} \text{ control bounds} \\ \alpha \leq u(t) \leq \beta \text{ are included by } c(x, u) = u \text{ Hamiltonian} \\ H(x, \lambda, u) = \lambda_o f(x, u) + \lambda f(x, u) \lambda \in \mathfrak{R}^n (\text{row vector}) \end{aligned}$$

## 2.3 Problem with mixed inequality constraints

Optimal control problems in which the state and control are subject to mixed constraints. optimal control problems with state inequality constraint arise frequently in practical application. optimal control with the state  $x$  and control  $u$  are subject to joint ,or mixed constraints through the condition( $x(t), u(t)$  an element of  $S(t)$ ) Consider the problem to find a piecewise continuous control  $u^* \in C[0, T]$  with associated response  $x^* \in C^1[0, T]$  and a terminal time  $T^* \in [0, T]$  such that the following constraints are satisfied and the cost function takes on its maximum value:

$$\begin{aligned} \max F = \int_0^T f(t, x, u) dt \\ \text{Subject to : } \dot{x}(t) = g(t, x(t), u(t)), x(0) = x_o \text{ and } x(T) = x_T \\ h(t, x(t), u(t)) \leq 0 \end{aligned} \quad (2.1)$$

Assume that the components of  $h(t, x(t), u(t))$  depend explicitly on the control  $u$  and the following constraint qualification condition holds. The matrix

$$\left( \frac{\partial h}{\partial u} \text{diag}(h) \right) \quad (2.2)$$

is full rank. In other words, the gradients with respect to  $u$  of all the active constraint  $h(t, x(t), u(t)) \leq 0$  must be linearly independent. A possible way of attempting to solve optimal control problems with mixed inequality constraints are to form a Lagrangian function  $L$  by adjoining  $h(t, x(t), u(t))$  to the Hamiltonian function  $H$  with a Lagrange multiplier vector function  $\mu$  [2].

$$\begin{aligned} L(t, x, u, \lambda, \mu) = H(t, x, u, \lambda) + \mu h(t, x, u) \\ \text{where, } H(t, x, u, \lambda) = f(t, x(t), u(t)) + \lambda g(t, x(t), u(t)) \end{aligned}$$

### 2.3.1 Necessary conditions for optimality

Consider the optimal control problem [4 ]  
Necessary conditions of optimality are derived for optimal control problems with pathwise state constraints.

$$\max F = \int_0^T f(t, x, u) dt \quad (2.3)$$

$$\text{Subject to : } \dot{x}(t) = g(t, x(t), u(t)), x(0) = x_0, x(T) = x_T$$

$$h(t, x(t), u(t)) \leq 0$$

With fixed initial time and free terminal time and where  $f$ ,  $g$  and  $h$  are continuously differentiable with respect to  $(t, x, u)$  on  $[0, T] \times \mathbb{R}^m \times \mathbb{R}^n$ . Suppose that  $u^* \in C[0, T]$  is a maximizer for the problem and let  $x^*$  denote the optimal response. If the constraint qualification conditions are holds for every  $t \in [0, T]$  :

1. The function

$$H(t, x^*(t), u(t), \lambda^*(t)) \text{ attains its maximum on } U(x^*(t), t) \text{ at } u = u^*(t) \\ \text{for every } t \in [0, T]$$

$$H(t, x^*(t), u^*(t), \lambda(t)) \geq H(t, x^*(t), u(t), \lambda(t)) \quad \forall u \in U(x^*(t), t)$$

$$\text{where, } U(x(t), t) := \{u(t) \in \mathbb{R}^n : h(t, x(t), u(t)) \leq 0\}$$

2. The quadruple  $(t, x^*, u^*, \lambda^*, \mu^*)$  satisfies the equations

$$\dot{x}^*(t) = L_\lambda(t, x, u, \lambda, \mu)$$

$$\dot{\lambda}(t) = -L_x(t, x, u, \lambda, \mu), \text{ and}$$

$$0 = L_u(t, x, u, \lambda, \mu)$$

at each instant  $t$  of continuity of  $u^*$ .

3. The vector function  $\mu^*$  is continuous at each instant of continuity of  $u^*$  and satisfies:  $\mu(t)h(t, x(t), u(t)) = 0, \mu(t) \geq 0$

### 2.3.2 Extension to General state Terminal constraints

The maximum principle given in above conditions can be extended to the case where general terminal constraints are specified on the state variables as:

$$a(x(T), T) \geq 0 \\ b(x(T), T) = 0$$

and a terminal term is added to the cost functional as

$$\max F = \int_0^T f(t, x, u) dt + S(x(T), T) \quad (2.4)$$

where  $a, b$  and  $S$  are continuously differentiable with respect to  $(t, x)$  for all  $(t, x) \in [0, T] \times \mathfrak{R}^m$ . Suppose that the terminal constraints satisfy the constraint qualification conditions [7].

$$\begin{pmatrix} \frac{\partial}{\partial x} a & \text{diag}(a) \\ \frac{\partial}{\partial x} b & 0 \end{pmatrix} \quad (2.5)$$

is full rank . Then in addition to the necessary condition of optimality there exists Lagrangian multiplier vectors  $\alpha \in \mathfrak{R}^l$  and  $\beta \in \mathfrak{R}^l$  such that:

$$\lambda(T) = S_x(x(T), T) + \alpha a_x(x(T), T) + \beta b_x(x(T), T) \quad (2.6)$$

Where  $\alpha \geq 0, \alpha a(x(T), T) = 0$

**Example 2.1** Consider the problem [7]

$$\max F = \int_0^1 u dt \quad (2.7)$$

subject to:

$$\begin{aligned} \dot{x} &= u, x(0) = 1 \\ u &\geq 0 \text{ and } x - u \geq 0 \end{aligned}$$

Note that the constraints  $u \geq 0$  and  $x - u \geq 0$ , are mixed type and they can be rewritten as  $0 \leq u \leq x$ .

**Solution:** The Hamiltonian  $H = u + \lambda u, H = (1 + \lambda)u$

So that the optimal control has the form  $u^* = \text{bang}[0, x; 1 + \lambda]$ .

To get adjoint equation and the multiplier associated with constraint

$u \geq 0, x - u \geq 0$  we form the Lagrangian

$$L = H + \mu_1 u + \mu_2(xu) = \mu_2 x + (1 + \lambda + \mu_1 - \mu_2)u.$$

From this we get the adjoint equation

$$\dot{\lambda} = -L_x(t, x, u, \lambda, \mu) = -\mu_2, \lambda(1) = 0$$

and also note that the optimal control must satisfy:

$$\frac{\partial L}{\partial u} = 1 + \lambda + \mu_1 - \mu_2 = 0$$

Where  $\mu_1$  and  $\mu_2$  must satisfy the complimentary slackness conditions

$$\begin{aligned}\mu_1 &\geq 0, \mu_1 u = 0 \\ \mu_2 &\geq 0, \mu_2(x - u) = 0.\end{aligned}$$

The mixed state constraint  $x(t) - u(t) \geq 0$  being active for each  $0 \leq t \leq 1$ , then we have  $u^*(t) = x^*(t)$ . Since  $x(0) = 1$  the control  $u^* = x$  gives  $x(t) = e^t$  as the solutions  $x(t) = e^t$  it follows that  $u^* = x > 0$  thus  $\mu_1 = 0$ .

Since  $\frac{\partial}{\partial u} L = 1 + \lambda + \mu_1 - \mu_2 = 0$ . Then  $1 + \lambda - \mu_2 = 0$  since  $\mu_1 = 0$

$$\mu_2 = 1 + \lambda, \dot{\lambda} = -\mu_2 \text{ and } \lambda(1) = 0, \text{ then } \dot{\lambda} = -1 - \lambda$$

$$\Rightarrow \dot{\lambda} + \lambda = -1, \lambda(t) = c_1 e^{-t} - 1 = 0,$$

$$c_1 e^{-1} - 1 = 0 \Rightarrow c_1 = e$$

$$\lambda(t) = e^{1-t} - 1$$

$$\mu_2(t) = e^{1-t} \geq 0 \text{ and } x - u^* = 0$$

## 2.4 Problems with pure state inequality constraints

Consider the function:

$k(t, x)$  where  $k : [0, T] \times \mathfrak{R}^n$ . Then the pure state constraints  $k(t, x) \geq 0$  are generally, more difficult to deal with since  $k(t, x)$  does not explicitly depend on  $u$ , and  $x$  can be controlled only indirectly. It is therefore, convenient to differentiate  $k(t, x)$  with respect to time  $t$  as many times as required until it contains a control variable. Let us for the moment define  $k^i(t, x), i = 1, 2, \dots, p$ ; recursively as follows

begin (center)  $k^0(t, x, u) = k(t, x)$

end( center)

$$k^1(t, x, u) = \frac{d}{dt} k = k_x(t, x)g(t, x, u) + k_t(t, x)$$

$$k^2(t, x, u) = \frac{d}{dt} k^1 = k_x^1(t, x)g(t, x, u) + k_t^1(t, x)$$

$\vdots$

$$k^p(t, x, u) = \frac{d}{dt} k^{p-1} = k_x^{p-1}(t, x, u)g(t, x, u) + k_t^{p-1}(t, x)$$

where subscripts denote partial derivatives. Depending on the context we also use a subscript such as  $i$  to denote the  $i^{th}$  component of a vector.

If

$$k_u^i(t, x, u) = 0, \text{ for } 0 \leq i \leq p - 1, k_u^p(t, x, u) \neq 0 \quad (2.8)$$

Then the state constraint  $k(t, x) \geq 0$  is of order  $p$ . In more general case of  $k(t, x)$ , the corresponding order  $p_i$  for each component  $k_i(t, x)$  of  $k(t, x)$  is obtained from equation of 2.8 and 2.9. If the state constraints will be order of  $p = 1$ , then it is easier to treat than the higher order cases [6]. With respect to the  $i^{th}$  constraint  $k_i(t, x) \geq 0$ , a subinterval  $(\tau_1, \tau_2) \in C[0, T]$  with  $\tau_1 < \tau_2$  is called an interior interval of a trajectory if  $k_i(x(t), t) > 0$  for all  $t \in (\tau_1, \tau_2)$  an interval  $[\tau_1, \tau_2]$  with  $\tau_1 < \tau_2$  is called a boundary interval if  $k_i(x(t), t) = 0$  for  $t \in [\tau_1, \tau_2]$ . An instant  $\tau_1$  is called an entry time.

If there is an interval ending at  $t = \tau_1$  and a boundary interval starting at  $\tau_1$ ; correspondingly,  $\tau_2$  is called an exist time if a boundary interval ends at  $\tau_2$  and an interior interval starts at  $\tau_2$ . If the trajectory  $x$  just touches the boundary at time  $\tau$ , i.e;  $k(\tau, x(\tau_c)) = 0$  and if the trajectory  $x$  is in the interior just before and after  $\tau$ , then  $\tau$  is called a contact time. Taken together entry exit and contact times are called junction times. Assume that the following full rank conditions on any boundary interval  $[\tau_1, \tau_2]$ ;

$$\begin{pmatrix} \frac{\partial}{\partial u} k_1^{p_1} \\ \cdot \\ \cdot \\ \frac{\partial}{\partial u} K_s^{p_s'} \end{pmatrix}$$

is full rank for all  $t \in (\tau_1, \tau_2)$ , where  $k_i^*(t) = 0$ , for  $i = 1, 2, \dots, s' \leq s$  and  $k_i^*(t, x) > 0$ , for  $i = s' + 1 \dots s$  for  $t \in (\tau_1, \tau_2)$  and  $p_i$  is the order of constraint  $k_i(t, x) \geq 0$  i.e the gradients of  $k_i^{p_i}(t, x)$  with respect to  $u$  of the active constraints  $k_i(t, x) = 0$ ,  $i = 1, 2, \dots, s'$  must be linearly independent along an optimal trajectory [2].

## 2.5 Direct adjoint approach

In this approach, the Hamiltonian  $H$  and Lagrangian  $L$  are defined as follows:

$$H(t, x, u, \lambda) = f(t, x, u) + \lambda g(t, x, u)$$

$$L(t, x, u, \lambda, \mu, \nu) = H(t, x, u, \lambda) + \mu h(t, x, u) + \nu k(t, x),$$

Where the vector  $\lambda \in \mathfrak{R}^n[t]$  is the adjoint function and  $\mu \in \mathfrak{R}^s[t]$  and  $\nu \in \mathfrak{R}^q[t]$  are multipliers. This method derives its name from the fact that the mixed constraints  $h(t, x, u) \geq 0$  as well as the pure state constraints  $k(t, x) \geq 0$  are directly adjoined to the Hamiltonian in order to form the Lagrangian.

**Theorem 2.1:**

Let  $(x^*, u^*)$  be an optimal pair for optimal control problem over a fixed interval  $[0, T]$ , such that  $u^*(\cdot)$  is right continuous with left hand limits and the constraint qualification condition of equation 2.1 holds for every triple  $(t, x^*, u^*), t \in [0, T]$  with  $u \in U(t, x^*(t))$ . Assume that  $x^*(t)$  has only finitely many junction times where  $\lambda(\cdot)$  are discontinuous at junction time. Then there exist a constant  $\lambda_0(t) > 0$ , a piecewise absolutely continuous costate trajectory  $\lambda(\cdot)$  mapping  $[0, T]$  in to  $\mathfrak{R}^n$ , piecewise continuous multiplier function  $\mu(\cdot)$  and  $\nu(\cdot)$ . Mapping  $[0, T]$  in to  $\mathfrak{R}^s$  and  $\mathfrak{R}^q$  respectively, a vector  $\eta(\tau_i) \in \mathfrak{R}^q$  for each point  $\tau_i$  of discontinuity of  $\lambda(\cdot)$  and  $\alpha \in \mathfrak{R}^l, \beta \in \mathfrak{R}^l$  and  $\gamma \in \mathfrak{R}^q$  such that  $(\lambda_0, \lambda(t), \mu, \nu, \alpha, \beta, \eta(\tau_1), \dots, \eta(\tau_i)) \neq 0$  for every  $t$  and the following conditions hold almost everywhere[6]:

$$u^*(t) = \operatorname{argmax}_{u \in U(t, x^*(t))} H(t, x^*, u, \lambda_0, \lambda(\cdot))$$

$$\operatorname{max}_{u \in U(t, x^*(t))} H(t, x^*, u, \lambda_0, \lambda(\cdot))$$

$$L_u^*[t] = H_u^*[t] + \mu h_u[t] = 0$$

$$\dot{\lambda} = -L_x^*[t]$$

$$\mu(t) \geq 0, \mu h^*(t, x, u) = 0$$

$$\nu \geq 0, \nu k^*(t, x) = 0$$

At the terminal time  $T$ , the following transversality conditions hold:

$$\lambda(T^-) = \lambda_0 S_x^*[T] + \alpha a_x[T] + \beta b_x[T] + \gamma k_x^*[T]$$

$$\alpha \geq 0, \gamma \geq 0, \alpha a[T] = \gamma k^*[T] = 0$$

For any time  $\tau$  in a boundary interval and for any contact time  $\tau$ , the co-state trajectory  $\lambda$  may have a discontinuity given by the following jump conditions:

$$\lambda(\tau^-) = \lambda(\tau^+) + \eta(\tau) k_x^*[\tau]$$

$$H^*(\tau^-) = H^*(\tau^+) - \eta(\tau) k_t^*[\tau]$$

$$\eta(\tau) \geq 0, \eta(\tau) k^*[\tau] = 0$$

Where  $\tau^+$  and  $\tau^-$  denotes the left hand and the right hand side limits respectively.

### Proof

1. Now to formulate the maximum principle for the problem defined by the state equation, objective function, mixed inequality constraint and pure state variable inequality constraints, we form the Lagrangian as follows :

$$L(t, x, u, \lambda, \mu, \nu) = H(t, x, u, \lambda) + \lambda g(t, x, u) + \mu h(t, x, u) + \nu k(t, x),$$

Where the Hamiltonian is:

$$H(t, x, u, \lambda) = f(t, x, u) + \lambda g(t, x, u)$$

Now we can show that the maximum principle from Hamiltonian equation the only difference is that the expression of  $F(x^*, u^*) - F(x, u) \geq 0$

$$\int_0^T [f(t, x^*(t), u^*(t)) - f(t, x(t), u(t))] dt \geq 0$$

This equations are proved in theorem (1.1) then it follows that a necessary conditions for  $u^* = u$  to be maximizing control is that  $F(x^*, u^*) \geq 0$  for all admissible. This implies that  $H(t, x^*(t), u^*(t), \lambda^*(t)) \geq H(t, x^*(t), u(t), \lambda^*(t))$  for all admissible  $u$  and all  $t \in [0, T]$ . This state that  $u^*$  maximize the Hamiltonian.

Therefore,  $u^*(t) = \operatorname{argmax}_{u \in U(t, x(t))} H(t, x^*, u, \lambda_0, \lambda(\cdot))$

2. From equation of Euler Lagrangian equation derivative in chapter one we can get:

$$\begin{aligned} L_u^*[t] &= H_u^*[t] + \mu h_u[t] = 0 \\ \dot{\lambda} &= -L_x^*[t] \end{aligned}$$

Since the vector function  $\mu(t)$  and  $\nu(t)$  is piecewise continuous at each time  $t$  of continuity of  $u$ , then it is satisfies the following condition :

$$\begin{aligned} \mu(t) &\geq 0, \mu h^*(t, x, u) = 0 \\ \nu &\geq 0, \nu k^*(t, x) = 0 \end{aligned}$$

3. Suppose that the point  $\tau_i$  at which the control switches between the maximum and minimum time. Therefore, at this time the jump conditions form for the adjoint variable and the Hamiltonian function. Then we would have that  $\lambda(\cdot)$  and  $H(\cdot)$  are discontinuous at this time. Therefore, at any entry/contact time  $\tau_i$ , the adjoint function and Hamiltonian function may have discontinuities of the form:

$$\begin{aligned} \lambda(\tau^-) &= \lambda(\tau^+) + \eta(\tau) k_x^*[\tau] \\ H^*(\tau^-) &= H^*(\tau^+) - \eta(\tau) k_t^*[\tau] \end{aligned}$$

Where the Lagrange multiplier vector  $\eta(\tau)$  satisfies the condition:  $\eta(\tau) k_x^*[\tau] = 0, \eta(\tau) \lambda \geq 0$ .

**Proposition 2.1.** The adjoint function  $\lambda$  is continuous at a junction time

$\tau$ , i.e  $\eta(\tau) = 0$  if either (1) or (2) below holds:

1. The control  $u^*$  is continuous at  $\tau$  and the matrix.

$$\begin{pmatrix} \frac{\partial}{\partial u} h^*[\tau] & \text{diag}(h^*[\tau]) & 0 \\ \frac{\partial}{\partial u} k^{1*}[\tau] & 0 & \text{diag}(k^{1*}[\tau]) \end{pmatrix} \quad (2.9)$$

is full rank where  $k^1(t, x, u)$  is defined in equation 2.9.

2. The entry or exist is nontangential i.e  $k^{1*}(T^-) < 0$  or  $k^{1*}(T^+) > 0$ , then  $\lambda(t)$  is continuous at time  $t = \tau$  [6].

**Definition 2.1.**

The Hamiltonian is said to be regular if along a given,  $x(t)$ ,  $\lambda(t)$ ,  $\eta(t)$  and  $H(x(t), u(t), \lambda(t), \eta(t))$  has a unique maximum in  $u$  for all  $t \in [0, T]$ .

**Proposition 2.2.**

If the Hamiltonian is regular, which in this context means that the maximization of  $H$  with respect to  $u$  is unique, then  $u^*$  is continuous everywhere including the points on the boundary .

**Proof.**

Suppose Hamiltonian is regular then we want to show that  $u(t^-) = u(t^+)$ . Let  $(x, u)$  be an admissible pair for optimal control problem; therefore, Hamiltonian look as follows:

$$H(t, x(t), u(t)) = f(t, x(t), u(t)) + \lambda(t)g(t, x(t), u(t))$$

Then on the subinterval  $[0, t^-]$  the control functions are :

$$H_u(t, x(t), u(t)) = 0 \Rightarrow f_u(t, x(t), u(t)) + \lambda(t)g_u(t, x(t), u(t)) = 0, \text{ where } t \in [0, t^-]$$

and also on the subinterval  $[t^+, T]$  the control functions are:

$$H_u(t, x(t), u(t)) = 0 \Rightarrow f_u(t, x(t), u(t)) + \lambda(t)g_u(t, x(t), u(t)) = 0 \text{ where } t \in [t^+, T].$$

Hence the control functions are if the  $\lim_{t \rightarrow t^-} u(t) = \lim_{t \rightarrow t^+} u(t)$ .

$$\begin{aligned} H_u(t^-, x(t^-), u(t^-)) &= f_u(t^-, x(t^-), u(t^-)) + \lambda(t^-)g_u(t^-, x(t^-), u(t^-)) \\ &= H_u(t^+, x(t^+), u(t^+)) = f_u(t^+, x(t^+), u(t^+)) + \lambda(t^+)g_u(t^+, x(t^+), u(t^+)) = 0 \end{aligned}$$

Therefore,

$$\begin{aligned} f_u(t^-, x(t^-), u(t^-)) + \lambda(t^-)g_u(t^-, x(t^-), u(t^-)) &= 0 \\ f_u(t^+, x(t^+), u(t^+)) + \lambda(t^+)g_u(t^+, x(t^+), u(t^+)) &= 0 \end{aligned}$$

make both sides limit

$$\begin{aligned} \lim_{t \rightarrow t^-} (f_u(t^-, x(t^-), u(t^-)) + \lambda(t^-)g_u(t^-, x(t^-), u(t^-))) &= 0 \\ \lim_{t \rightarrow t^+} (f_u(t^+, x(t^+), u(t^+)) + \lambda(t^+)g_u(t^+, x(t^+), u(t^+))) &= 0 \end{aligned}$$

Therefore,  $u(t)$  is continuous everywhere including the points on the boundary.

## 2.6 The indirect adjoining approach with complementary slackness:(first order constraints)

The idea behind this approach is the following. if the trajectory hits the boundary at time  $\tau_1$  i.e  $k(x(\tau_1), \tau_1) = 0$ , then for it to remain on the boundary up to time  $\tau_2$  requires.

$$k^1(t, x^*(t), u^*(t)) = 0, \text{ for } t \in (\tau_1, \tau_2)$$

Where  $k^1(t, x, u)$  may or may not depend explicitly on the control variables. This asserts that the phase velocity of a point moving along the trajectory is tangential to the boundary at time  $t$ . At the exit point  $\tau_2$  we must have  $k^{1*}(\tau_2^+) \geq 0$ . Thus, one could formally impose the constraint  $k^1(t, x, u) \geq 0$  whenever  $k(t, x) = 0$  in order to prevent the trajectory from violating the constraint  $k(t, x) \geq 0$ . Then the Hamiltonian and Lagrangian can be defined as follows:

$$\begin{aligned} H^1(t, x, u, \lambda_0, \lambda^1) &= \lambda_0 f(t, x, u) + \lambda^1 g(t, x, u) \\ L^1(t, x, u, \lambda_0, \lambda^1, \mu, \nu^1) &= H^1(t, x, u, \lambda_0, \lambda^1) + \mu h(t, x, u) + \nu^1 k^1(t, x, u) \end{aligned}$$

Because the derivative  $k^1(t, x, u)$  of  $k(t, x)$  rather than  $k(t, x)$  itself is adjoined to  $H$  in forming the Lagrangian, this approach is known as the indirect adjoining approach.

The control region:

$$U^1(t, x) = \{u \in \mathfrak{R} \mid h(t, x, u) \geq 0, k^1(t, x, u) \geq 0, \text{ if } k(t, x) = 0\}$$

The necessary conditions of optimality that are used as a procedure while applying the indirect adjoining approach are now stated as follows.

### **Theorem 2.2.**

Let  $(x^*(.), u^*(.))$  be an optimal pair for optimal control problem such that  $x^*(.)$  has only finitely many junction times and the strong constraint qualification condition of equation 2.10 holds. then there exists a constant  $\lambda_0 \geq 0$

a piecewise absolutely continuous costate trajectory  $\lambda^1(\cdot)$  mapping  $[0, T]$  in to  $\mathfrak{R}^n$ , piecewise continuous multiplier function  $\mu(\cdot)$  and  $\nu^1(\cdot)$  mapping  $[0, T]$  in to  $\mathfrak{R}^s$  and  $\mathfrak{R}^q$  respectively, a vector  $\eta^1(\tau_i) \in \mathfrak{R}^q$  for each point  $\tau_i$  of discontinuity of  $\lambda^1(\cdot)$ ,  $\alpha \in \mathfrak{R}^l$  and  $\beta \in \mathfrak{R}^l$ , not all zero, such that the following conditions hold almost everywhere[6]:

$$\begin{aligned} u^*(t) &= \operatorname{argmax}_{u \in U(t, x(t))} H^1(t, x^*, u, \lambda_0, \lambda^1(\cdot)) \\ \dot{\lambda}^1 &= -L_x^{1*}[t] \\ L_u^{1*}[t] &= 0 \\ \mu(t) &\geq 0, \mu h^*(t, x, u) = 0 \end{aligned}$$

$\nu_i^1$  is non increasing on boundary intervals of  $k_i(t, x)$ ,  $i=1, 2, \dots, q$ .

with  $\nu^1(t) \geq 0$ ,  $\dot{\nu}^1 \leq 0$ ,  $\nu^1 k^{1*}(t, x, u) = 0$

and  $\frac{d}{dt} H^{*1}[t] = \frac{d}{dt} L^{*1}[t] = L_{t^1}^*[t]$

Whenever these derivatives exist. at the terminal time  $T$  the transversality conditions

$$\begin{aligned} \lambda^1(T^-) &= \lambda_0 S_x^*[T] + \alpha a_x[T] + \beta b_x[T] + \gamma k_x^*[T] \\ \alpha &\geq 0, \gamma \geq 0, \text{ then } \alpha a[T] = \gamma k^*[T] = 0 \end{aligned}$$

holds. At each entry or contact time, the costate trajectory  $\lambda^1$  may have a discontinuity of the form

$$\begin{aligned} \lambda^1(\tau^-) &= \lambda^1(\tau^+) + \eta^1(\tau) k_x^*[\tau] \\ H^{*1}(\tau^-) &= H^{*1}(\tau^+) - \eta^1(\tau) k_x^*[\tau] \\ \eta^1(\tau) &\geq 0, \eta^1(\tau) k^*[\tau] = 0 \end{aligned}$$

### Proof

Suppose that the constraint  $k(t, x) \geq 0$  is called a constraint of first order since first derivative of  $k(t, x)$  the first time at a term in control  $u$  appears in the expression by putting  $g(t, x, u)$  for  $\dot{x}$ . Then in the case of first order constraints, we need to define  $k^1(t, x, u)$  as follows:

$$k^1(t, x, u) = \frac{d}{dt} k(t, x) = \frac{\partial}{\partial x} k g(t, x, u) + \frac{d}{dt} k(t, x)$$

Then using first order conditions we can form Lagrangian function as follows:

$$L^1(t, x, u, \lambda_0, \lambda^1, \mu, \nu^1) = H^1(t, x, u, \lambda_0, \lambda^1) + \mu h(t, x, u) + \nu^1 k^1(t, x, u)$$

where the Hamiltonian is

$$H^1(t, x, u, \lambda_0, \lambda^1) = \lambda_0 f(t, x, u) + \lambda^1 g(t, x, u)$$

Then from above Lagrangian equation we can derivative the maximum principle states that the necessary conditions for  $u^*$  with the state trajectory  $x^*$  to be an optimal control for the problems. Then there exist adjoint variable  $\lambda$  and Langrangian multipliers  $\mu, \nu, \alpha, \beta$  and the jump parameter  $\eta$  which satisfies the conditions, then to find adjoint equation and optimality condition using partial derivative interims of  $x$  and  $u$  and the proof of the Hamiltonian maximizing condition is similar to Theorem (2.1) as follows:

$$H^1(t, x^*(t), u^*(t), \lambda^1(t)) \geq H^1(t, x^*(t), u, \lambda^1(t))$$

at each  $t \in [0, T]$  for all  $u$  satisfying the following conditions  $g(x^*(t), u, t) \geq 0$ , and  $k^1(x^*(t), u, t) \geq 0$ , whenever  $k(x^*(t), t) = 0$ . Therefore,

$$\begin{aligned} u^*(t) &= \operatorname{argmax}_{u \in U(t, x(t))} H^1(t, x^*, u, \lambda_0, \lambda^1(\cdot)) \\ \dot{\lambda}^1 &= -L_x^{1*}[t] \\ L_u^{1*}[t] &= 0 \\ \mu(t) &\geq 0, \mu h^*(t, x, u) = 0 \end{aligned}$$

Since the derivative of  $\nu$  less than zero then  $\nu$  is non increasing on the bounder interval of  $k_i(t, x)$ . At any entry or contact  $\tau$  the control is switches between maximum and minimum. Therefore, at this time the adjoint function and Hamiltonian function have discontinuities, then it is the form of:

$$\begin{aligned} \lambda^1(\tau^-) &= \lambda^1(\tau^+) + \eta^1(\tau) k_x^*[\tau] \\ H^{*1}(\tau^-) &= H^{*1}(\tau^+) - \eta^1(\tau) k_t^*[\tau] \\ \eta^1(\tau) &\geq 0, \eta^1(\tau) k^*[\tau] = 0 \end{aligned}$$

## 2.7 The indirect adjoining approach for higher order constraints

In this section, we shall consider constraints of higher order ,i.e  $p \geq 2$ . This means if  $p = 1$  and  $k^1(t, x, u)$  does not depend on the control variable  $u$ , then we differentiate  $k(t, x)$  with respect to time  $t$  as many time as required until it contains a control variable  $u$ . Then such type of condition are said to be indirect adjoint approach for higher order constraints. The Hamiltonian and

Lagrangian of the indirect adjoining approach for the state constraint of order  $p$  are now:

$$H^p(t, x, u, \lambda_0, \lambda^p) = \lambda_0 f(t, x, u) + \lambda^p g(t, x, u)$$

$$L^p(t, x, u, \lambda_0, \lambda^p, \mu, \nu^p) = H^p(t, x, u, \lambda_0, \lambda^p) + \mu h(t, x, u) + \nu^p k^p(t, x, u)$$

With  $k^p$  defined in equation of 2.5. Then the control region  $U^p(t, x)$  is defined as follows:

$$U^p(t, x) = \{u \in \mathfrak{R}^n | h(t, x, u) \geq 0, k^p(t, x, u) \geq 0, \text{ if } k(t, x) = 0\}$$

### Theorem 2.3

Let  $(x^*(\cdot), u^*(\cdot))$  be an optimal pair for optimal control problem with  $x^*(\cdot)$  having only finitely many junction times, and where constraint  $k(t, x)$  is of order  $p$ , let the constraint qualification condition, of equation 2.7, holds. Then there exist a constant  $\lambda_0 \geq 0$ , a piecewise absolutely continuous costate trajectory  $\lambda^p(\cdot)$  mapping  $[0, T]$  in to  $\mathfrak{R}$ , piecewise continuous multiplier function  $\mu(\cdot)$  and  $\nu(\cdot)$  mapping  $[0, T]$  in to  $\mathfrak{R}^s$  and  $\mathfrak{R}^q$ , respectively, vectors  $\eta^1(\tau_i), \eta^p(\tau_i) \in \mathfrak{R}^q$  for each point  $\tau_i$  of discontinuity of  $\lambda^p(\cdot)$ ,  $\alpha \in \mathfrak{R}^l$  and  $\beta \in \mathfrak{R}^l$  not all zero, such that the following conditions hold almost everywhere [6].

$$u^*(t) = \operatorname{argmax}_{u \in U(t, x(t))} H^p(t, x^*, u, \lambda_0, \lambda^p(\cdot))$$

$$\dot{\lambda}^p = -L_x^{*p}[t]$$

$$L_u^{*p}[t] = 0$$

$$\mu(t) \geq 0, \mu h^*(t, x, u) = 0$$

The multiplier function  $\nu^p$  is  $p-1$  times differentiable and  $(\nu^p)^{p-1}$  is of bounded variation

$$(-1)^r (\nu^p)^r(t) \geq 0, \quad r = 0, 1, \dots, p. \quad \nu^p k^{p*}(t, x, u) = 0$$

$$\frac{d}{dt} H^{*p}[t] = \frac{d}{dt} L^{*p}[t] = L_t^{*p}[t]$$

At the terminal time  $T$ , the transversality conditions with  $\lambda^1(\cdot)$  replaced by  $\lambda^p(\cdot)$ . At entry times, the costate trajectory  $\lambda^p$  may have a discontinuity of the form:

$$\lambda^p(\tau^-) = \lambda^p(\tau^+) + \sum_{r=1}^p \eta^r(\tau) (k^{r-1})_x^*[\tau]$$

$$H^p(\tau^-) = H^p(\tau^+) + \sum_{r=1}^p \eta^r(\tau)(k^{r-1})_t^*[\tau] \quad (2.10)$$

$$\eta^r(\tau) \geq 0, \eta^r(\tau)k_t^*[\tau] = 0, r = 1, 2, \dots, p$$

**Proof**

The condition of the maximum principle states that the necessary conditions for  $u^*$  with the state trajectory  $x^*$  to be an optimal control for the problem is the same approach as first order state constraints (indirect adjoining approach). Suppose that the constraint  $k(t, x)$  is derivative  $p$  times until it contains a control variable  $u$ . In the case of  $p$  order constraints, we need to define  $k^p(t, x, u)$  as defined in the equations of (2.6). Then using  $p$  order conditions we can form Lagrangian functions as follows.

$$L^p(t, x, u, \lambda_0, \lambda^p, \mu, \nu^p) = H^p(t, x, u, \lambda_0, \lambda^p) + \mu h(t, x, u) + \nu^p k^p(t, x, u)$$

Where Hamiltonian is

$$H^p(t, x, u, \lambda_0, \lambda^p) = \lambda_0 f(t, x, u) + \lambda^p g(t, x, u)$$

**Note:**  $P$  is indicate order.

Assume that the function  $g$  and  $k$  are continuously differentiable with respect to all their argument up to order  $p - 1$  and  $p$  respectively, the necessary condition of optimality as follows:

$$u^*(t) = \operatorname{argmax}_{u \in U(t, x(t))} H^p(t, x^*, u, \lambda_0, \lambda^p(\cdot))$$

$$\dot{\lambda}^p = -L_x^{*p}[t]$$

$$L_u^{*p}[t] = 0$$

$$\mu(t) \geq 0, \mu h^*(t, x, u) = 0.$$

If the switching function of order  $p$  the jump condition at entry times, the costate trajectory  $\lambda^p$  and Hamiltonian function may have a discontinuity of the form:

$$\lambda^p(\tau^-) = \lambda^p(\tau^+) + \sum_{r=1}^p \eta^r(\tau)(k^{r-1})_x^*[\tau]$$

$$H^p[\tau^-] = H^p[\tau^+] + \sum_{r=1}^p \eta^r(\tau)(k^{r-1})_t^*[\tau]$$

$$\eta^r(\tau) \geq 0, \eta^r(\tau)k_t^*[\tau] = 0, r = 1, 2, \dots, p.$$

## 2.8 The indirect adjoining approach with continuous adjoint functions

In this section, the adjoint function  $\check{\lambda}$  is continuous. The Hamiltonian  $H$  and the control region  $U$  respectively:

$$\begin{aligned}\tilde{H}(t, x, u, \lambda_0, \check{\lambda}, \mu, \check{\nu}) &= \lambda_0 f(t, x, u) + \check{\lambda} g(t, x, u) + \mu h(t, x, u) + \check{\nu} k^1(t, x, u) \\ \tilde{U}(t, x) &= u \in \mathfrak{R}^n | h(t, x, u) \geq 0, k^1(t, x, u) \geq 0, \text{ if } k(t, x) = 0 \\ &\cup (u \in \mathfrak{R}^n : h(t, x, u) \geq 0, k^1(t, x, u) \leq 0, \text{ if } k(t, x) = 0)\end{aligned}$$

### Theorem 2.4.

Let  $(x^*(.), u^*(.))$  be an optimal pair for optimal control problem such that the strong constraint qualification condition, of equation 2.4, holds. Then there exist a constant  $\lambda_0 \geq 0$ , a continuous and a piecewise continuously differentiable adjoint function  $\check{\lambda}(\cdot) : [0, T] \rightarrow \mathfrak{R}^n$ , and multiplier function  $\mu(\cdot) : [0, T] \rightarrow \mathfrak{R}^s$ , and  $\check{\nu} : [0, T] \rightarrow \mathfrak{R}^q$ , such that the following conditions are satisfied whenever  $u$  is continuous.

$$\begin{aligned}u^*(t) &= \operatorname{argmax}_{u \in U(t, x(t))} \tilde{H}(t, x^*, u, \lambda_0, \check{\lambda}(t), \mu(t), \check{\nu}(t)) \\ \check{\lambda} &= -\tilde{H}_x[t] \\ \tilde{H}_u[t] &= 0 \\ \frac{d}{dt} \tilde{H}[t] &= \tilde{H}_t[t]\end{aligned}$$

the multipliers  $\mu(\cdot)$  and  $\check{\nu}(\cdot)$  are continuous on intervals of continuity of  $u^*(\cdot)$ . Furthermore,  $\check{\nu}(\cdot)$  is non increasing on  $[0, T]$ , continuous when ever  $K_i^*(\cdot)$  is discontinuous ( i.e when entry to or exist from the corresponding state constraint is nontangential), and constant on intervals up on which  $K_i^*[\cdot] \geq 0$ . At the terminal time  $T$ , the following transversality conditions hold [6]:

$$\begin{aligned}\check{\lambda}(T) &= \lambda_0 S_x^*[T] + \alpha a_x[T] + \beta b_x[T] \\ \alpha &\geq 0, \alpha a[T] = 0\end{aligned}$$

### Proof.

Suppose  $u^*$  is continuous the Hamiltonian is regular along a given  $x(t)$ ,  $\lambda(t)$ ,  $\eta(t)$  and  $H(t, x(t), u, \lambda(t), \eta(t))$  has a unique maximum in  $u$  for all  $t \in [0, T]$  including the points on the boundary by proposition (2.2). Therefore, the necessary conditions are holds since maximum principle is unique and using partial derivative we can obtained optimality conditions and the

adjoint equation as follows.

$$\begin{aligned}
u^*(t) &= \operatorname{argmax}_{u \in U(t, x(t))} \tilde{H}(t, x^*, u, \lambda_0, \tilde{\lambda}(t), \mu(t), \tilde{\nu}(t)) \\
\dot{\tilde{\lambda}} &= -\tilde{H}_x[t] \\
\tilde{H}_u[t] &= 0 \\
\frac{d}{dt} \tilde{H}[t] &= \tilde{H}_t[t]
\end{aligned}$$

Since adjoint function is continuous then at the terminal time  $T$ , the following transversality conditions holds as follows:

$$\begin{aligned}
\tilde{\lambda}(T) &= \lambda_0 S_x^*[T] + \alpha a_x[T] + \beta b_x[T] \\
\alpha &\geq 0, \alpha a[T] = 0
\end{aligned}$$

## 2.9 Existence result

we can review several different sets of optimality conditions for optimal control problems since optimality conditions do not mean much in the absence of an optimal solution then we briefly provide some existence results for the problems our purpose here is not to make a review of existence result we choose to mention two characteristic:

- The first result uses strong assumptions such as boundedness of all admissible state and control paths.
- The second result uses growth conditions on the state and control variable [6].

The growth condition[2] if  $f$  and  $g$  satisfy the following conditions for every bounded subset  $X$  of  $\mathfrak{R}^n$ , then there exist a constant  $c$  and a summable function  $d$  such that, for almost every  $t$ , for every  $(x, y) \in \operatorname{dom} f(t, x, u)$  with  $x \in X$ , we have:

$$||g_x(t, x, u)|| \leq c|g(t, x, u)| + f(t, x, u) + d(t)$$

and for all  $\xi, \psi$

$$|\xi|(1 + ||g_u(t, x, u)||) \leq c\{|g_u(t, x, u)| + f(t, x, u)\} + d(t)$$

We define the (state dependent) control region:

$$U(t, x) = \{u \in \mathfrak{R}^n | h(t, x, u) \geq 0 \subset \mathfrak{R}^n$$

and the set

$$N(t, x) = \{(f(t, x, u) + \gamma, g(t, x, u))/\gamma \leq 0, u \in U(t, x) \subset \mathfrak{R}^{n+1}\}$$

**Lemma 2. 1.**

Let  $U(y)$  be the upper semicontinuous set-valued mapping  $\mathfrak{R}^m \rightarrow \mathfrak{R}^n$  with compact values. Then, on any compact (and hence on any bounded) set of  $y$ ; the values  $U(y)$  are uniformly bounded, i.e., for any compact set  $K \in \mathfrak{R}^m$  there exist a constant  $\delta$  such that the set  $U(y)$  is contained in the ball  $B(0, \delta)$  for any  $y \in K$  [2].

**Proof.**

Since  $U(y)$  is an upper semicontinuous mapping, for any  $y$  there exists a neighborhood  $O(y)$  of  $y$  such that the inclusion  $U(y^*) \subset U(y) + B_1(0, \delta)$  holds for any  $y^* \in O(y)$ . The union of these neighborhoods  $O(y)$  over all  $y \in K$  covers the entire compactum  $K$ ; and, by the definition of a compact set, a finite subcovering can be chosen from this covering. Namely, there exist finitely many points  $y_1, y_2, \dots, y_m \in k$ ; and their neighborhoods  $O(y_i)$ ; such that the set  $U(y^*)$  is contained in  $U(y_i) + B_1(0, \delta)$  for any  $y^* \in O(y_i)$ ; and these neighborhoods cover the entire compactum  $K$ . The union  $V$  of the bounded sets  $U(y_i) + B_1(0, \delta)$  over all  $i=1,2,\dots,m$  is also bounded, i.e., it is entirely contained in the ball  $B(0, \delta)$  for some  $\delta$ . Since for any  $y \in K$ ; there exists a number  $i$  such that  $y \in O(y_i)$ ; we have  $U(y) \subset U(y_i) + B_1(0, \delta) \subset V$ ; and hence  $U(y) \subset B(0, \delta)$ .

**Corollary 2.1.**

Suppose that  $U(t, x)$  is an upper semicontinuous mapping  $\mathfrak{R}^{m+1} \rightarrow \mathfrak{R}^n$  with compact values. Then, for any  $T > 0$  and any bounded set  $Q \subset \mathfrak{R}^m$ ; there is an  $R = R(T, Q)$  such that the inclusion  $U(t, x) \subset B(0, \delta)$  holds for any  $t \in [0, T]$  and any  $x \in Q$  [2].

**Proof.**

One should apply Lemma above to the mapping  $U(y)$ ; where  $y = (t, x)$ ; and to the compact set  $K = [0; T] \times Q$ .

**Theorem 2.5.**

Consider the optimal control problem where  $T$  is free to vary in the interval  $[0, T]$ . Assume that  $f, g, h, k, S, a$  and  $b$  are continuous in all their arguments at all points  $(t, x, u) \in [0, T] \times \mathfrak{R}^m \times \mathfrak{R}^n$ . Suppose that there exist an admissible solution pair and that the following conditions holds:

1.  $N(t, x)$  is convex for all  $(t, x) \in \mathfrak{R}^m \times [0, T]$ .

Suppose further that

2. There exists  $\delta > 0$  such that  $\|x(t)\| < \delta$  for all admissible pair  $(x(t), u(t))$  and  $t$ .
3. There exists  $\delta_1 > 0$  such that  $\|u\| < \delta_1$  for all  $u \in U(t, x)$  with  $\|x(t)\| < \delta$ . Then there exists an optimal triple  $(T^*, x^*, u^*)$  with  $u^*(\cdot)$  measurable [6].

**Proof.**

$$\text{let } (f_1(t, x, u) + \gamma_1, g_1(t, x, u)) \text{ and } (f_2(t, x, u) + \gamma_2, g_2(t, x, u))$$

be two value of  $N(t, x)$ . Then for all any  $0 \leq a \leq 1$ :

$$a((f_1(t, x, u) + \gamma_1, g_1(t, x, u)) + (1 - a)(f_2(t, x, u) + \gamma_2, g_2(t, x, u)))$$

$$af_1(t, x, u) + a\gamma_1, ag_1(t, x, u) + (f_2(t, x, u) + \gamma_2, g_2(t, x, u)) - a(f_2(t, x, u) + \gamma_2, g_2(t, x, u)),$$

collect like term together

$$(a((f_1(t, x, u) + \gamma_1) + (1 - a)(f_2(t, x, u) + \gamma_2)), ag_1(t, x, u) + (1 - a)g_2(t, x, u))$$

$$(a((f_1(t, x, u) - f_2(t, x, u) + f_2(t, x, u) + (a(\gamma_1 - \gamma_2) + \gamma_2)), ag_1(t, x, u) + (1 - a)g_2(t, x, u))$$

Therefore,  $N(x, t)$  is convex.

By Lemma and Corollary 2.1 above there exists  $\delta > 0$  such that  $\|x(t)\| < \delta$  for all admissible pair  $(x(t), u(t))$  and  $t$  and also there exists  $\delta_1 > 0$  such that  $\|u\| < \delta_1$  for all  $u \in U(t, x)$  with  $\|x(t)\| < \delta$ . Therefore, the triple  $(T^*, x^*, u^*)$  always belongs to the compact set  $[0, T] \times B(0, \delta) \times B(0, \delta_1)$ .

Thus, the set of solutions  $x(t)$  of optimal control problem is uniformly bounded and continuous, and the set of controls  $u(t)$  is uniformly bounded.

## 2.10 Sufficient conditions and uniqueness

### Theorem 2.6.

Let  $(x^*(\cdot), u^*(\cdot))$  be a feasible pair for the optimal control problem with a fixed horizon time  $T < \infty$ . If there exists a piecewise continuously differentiable function  $\lambda : [0, T] \rightarrow \mathfrak{R}^n$  such that for every other feasible pair  $(x(\cdot), u(\cdot))$  the following conditions holds [6]:

#### 1. The maximum Hamiltonian:

$$H(t, x^*(t), u^*(t), \lambda(t)) - H(t, x(t), u(t), \lambda(t)) \geq \dot{\lambda}(t)(x(t) - x^*(t)), \forall t \in [0, T].$$

**2. The jump conditions** This argues for allowing a jump in the adjoint variable  $\lambda(t)$ , which satisfies the maximum principle and the presence of state constraints allows for a jump  $\lambda(t)$  at a point in time when the state  $x(t)$  enters its constraints boundary. Moreover, the jump must satisfy certain conditions, which in the case of the state constraints are:

$$(\lambda(\tau^-) - \lambda(\tau^+))(x(\tau) - x^*(\tau)) \geq 0 .$$

$\forall t \in [0, T]$ , where  $\lambda$  is discontinuous and the transversality condition.

$$3. \lambda(T)(x(T) - x^*(T)) \geq S(x(T), T) - S(x^*(T), T).$$

Then  $(x^*, u^*)$  is optimal [4].

**Proof.**

1. Let  $u \in U(t, x)$  and  $x$  be an admissible trajectory generated by  $u$  then by definition of Hamiltonian we:

$$\begin{aligned} f(t, x^*(t), u^*(t)) + \lambda(t)g(t, x^*(t), u^*(t)) - (f(t, x(t), u(t)) + \\ \lambda(t)g(t, x(t), u(t))) - \dot{\lambda}(t)(x(t) - x^*(t)) \geq 0 \end{aligned}$$

Since  $\dot{x} = g(t, x(t), u(t))$ ,

Then  $f(t, x^*(t), u^*(t)) + \lambda(t)\dot{x}^*(t) - f(t, x(t), u(t)) - \lambda(t)\dot{x}(t) - \dot{\lambda}(t)(x(t) - x^*(t)) \geq 0$

Hence,  $f(t, x^*(t), u^*(t)) - f(t, x(t), u(t)) + \lambda(t)\dot{x}^*(t) - \lambda(t)\dot{x}(t) - \dot{\lambda}(t)(x(t) - x^*(t)) \geq 0$

Then we have by definition of derivative:

$$\begin{aligned} f(t, x^*(t), u^*(t)) - f(t, x(t), u(t)) - \frac{d}{dt}[\lambda(t)(x^*(t) - x(t))] \geq 0 \\ f(t, x^*(t), u^*(t)) - f(t, x(t), u(t)) \geq \frac{d}{dt}[\lambda(t)(x^*(t) - x(t))] \end{aligned}$$

So that make integration both side then

$$\begin{aligned} \int_0^T f(t, x^*(t), u^*(t)) - f(t, x(t), u(t)) dt \geq \int_0^T \left(\frac{d}{dt}\right)[\lambda(t)(x^*(t) - x(t))] dt \\ \int_0^T f(t, x^*(t), u^*(t)) dt - \int_0^T f(t, x(t), u(t)) dt \geq \lambda(t_0)(x^*(0) - x(0)) - \lambda(T)(x^*(T) - x(T)) \end{aligned}$$

But, by initial and transversality conditions  $x^*(t_0) = x_0, x(t_0) = x_0$  and  $\lambda(T) = 0$  such that:

$$\lambda(0)(x^*(0) - x(0)) - \lambda(T)(x^*(T) - x(T)) = 0$$

Hence  $\int_0^T f(t, x^*(t), u^*(t)) dt - \int_0^T f(t, x(t), u(t)) dt \geq 0$

2. The criterion for  $(x^*, u^*)$  to be optimal is the difference:

$$\int_0^T f(t, x^*(t), u^*(t)) dt - \int_0^T f(t, x(t), u(t)) dt \geq 0$$

For all admissible pairs  $(x, u)$ . Let use of definition of Hamiltonian and the fact that  $\dot{x} = g(t, x(t), u(t))$  is satisfied for all admissible pairs, we easily obtained from conditions one above:

$$H(t, x^*(t), u^*(t), \lambda(t)) - H(t, x(t), u(t), \lambda(t)) \geq \dot{\lambda}(t)(x(t) - x^*(t)), \forall t \in [0, T].$$

it follows that:

$$\geq \int_0^T \dot{\lambda}(t)(x(t) - x^*(t)) + \lambda(t)(\dot{x}(t) - \dot{x}^*(t)) dt = \int_0^T \left(\frac{d}{dt}\right)[\lambda(t)(x^*(t) - x(t))] dt$$

Now we can write this equations as follows:

$$\int_0^T \left(\frac{d}{dt}\right)[\lambda(t)(x^*(t) - x(t))] dt + \int_\tau^T \left(\frac{d}{dt}\right)[\lambda(t)(x^*(t) - x(t))] dt$$

Then we obtain:

$$\lambda(\tau^-)(x(\tau^-) - x^*(\tau^-)) - \lambda(0)(x(0) - x^*(0)) + \lambda(T)(x(T) - x^*(T)) - \lambda(\tau^+)(x(\tau^+) - x^*(\tau^+)).$$

Using initial condition and transversalies condition  $x^*(0) = x_0, x(0) = x_0$  and  $\lambda(T) = 0$  such that

$$\lambda(\tau^-)(x(\tau^-) - x^*(\tau^-)) - \lambda\tau^+(x(\tau^+) - x^*(\tau^+)) \geq 0$$

Since  $x(\tau^-) = x(\tau^+) = x(\tau)$  and  $x^*(\tau^-) = x^*(\tau^+) = x^*(\tau)$ , then we can write as follows:

$$\lambda(\tau^-)(x(\tau) - x^*(\tau)) - \lambda(\tau^+)(x(\tau) - x^*(\tau)) \geq 0$$

Hence,  $(\lambda(\tau^-) - \lambda(\tau^+))(x(\tau) - x^*(\tau)) \geq 0$  Since  $\lambda(t)(x(t) - x^*(t)) \geq 0, \forall t$ , so that  $(x^*, u^*)$  is optimal pair.

**Remark:** This theorem does not use any concavity or convexity assumption [4],[6].

**Example 2.2.**

In this example we illustrate the approaches in theorem 2.1, 2.2 and 2.4 by applying them to some illustrate examples. Consider the following example:[6]

$$\begin{aligned} & \max \int_0^3 -x dt \\ & \text{subject to : } \dot{x} = u, x(0) = 1 \end{aligned}$$

$$\begin{aligned}
u + 1 &\geq 0 \\
1 - u &\geq 0 \\
x &\geq 0 \\
x(3) &= 1
\end{aligned}$$

### Solution

The Hamiltonian is  $H = -x + \lambda u$  which implies the optimal control to be  $u^* = \text{bang}[-1, 1; \lambda]$ , when  $x \geq 0$

and which optimal control on the state constraint boundary is  $U^* = \text{bang}[-1, 1; \lambda]$ , when  $x = 0$ .

The boundary condition  $x(0) = x(3) = 1$ . Thus

$$u^*(t) = \begin{cases} -1, & \text{for } t \in [0, 1]; \\ 0, & \text{for } t \in [1, 2]; \\ 1, & \text{for } t \in (2, 3]. \end{cases}$$

and

$$x^*(t) = \begin{cases} 1 - t, & \text{for } t \in [0, 1]; \\ 0, & \text{for } t \in [1, 2]; \\ t - 2, & \text{for } t \in (2, 3]. \end{cases}$$

Now let first, we apply the direct adjoint approach and let form the Lagrangian  $L$  as

$$L = H + \mu_1(u + 1) + \mu_2(1 - u) + \nu x.$$

The necessary condition of theorem 2.1 are

$$\begin{aligned}
L_u &= \lambda + \mu_1 - \mu_2 = 0 \\
\dot{\lambda} &= -L_x = 1 - \nu \\
\text{but, } \mu_1 &\geq 0, \mu_1(u + 1) = 0 \\
\mu_2 &\geq 0, \mu_2(1 - u) = 0 \\
\nu &\geq 0, \nu x = 0, \dot{\nu} \leq 0 \\
\lambda(3) &= \beta, \text{ where } \beta \in \mathbb{R}
\end{aligned}$$

The enters of the boundary of  $x = 0$  in a nontangential way at time  $\tau_1 = 1$ , since  $k^1(1)^- \leq 0$  and also at time  $\tau_2 = 2$ . it leaves this boulder nontangential since  $k^1(2)^+ \geq 0$ . therefore, according to proposition (2.1),  $\lambda$  is continuous at time  $t = 1$  and  $t = 2$ , as well as in  $[0, 1]$ , and  $(2, 3]$  where the state constraint is not active. Now consider the boundary interval  $[1, 2]$ . here  $u = 0$  and implies that  $\mu_1 = \mu_2 = 0$ .

$$l_u = \lambda + \mu_1 + \mu_2 = 0, \lambda = 0$$

Thus,  $\lambda$  is also continuous in  $(1,2)$ . Furthermore, since  $\lambda = 0$  from equation:  $\dot{\lambda} = -L_x = 1 - \nu, \nu = 1$ .

Thus, all multipliers are uniquely determined in  $[1,2]$ . In  $[0,1)$  we have  $x > 0, \nu = 0$ , then  $\lambda = t - 1$ , because of  $\dot{\lambda} = 1$  and  $\lambda(1) = 0$ . Similarly, in  $(2,3]$  we have  $x > 0$  and  $\nu = 0$ . then  $\lambda = t - 2$  because of  $\dot{\lambda} = 1$  and  $\lambda(2) = 0$ . Now we can determined  $\mu_1$  and  $\mu_2$  from equation:

$$\begin{aligned} L_u &= \lambda + \mu_1 - \mu_2 = 0 \text{ and} \\ \mu_1 &\geq 0, \mu_1(u + 1) = 0 \\ , \mu_2 &\geq 0, \mu_2(1 - u) = 0 \end{aligned}$$

In  $[0,1)$  we have  $\lambda = t - 1$  and  $u = -1$  then  $\mu_2 = 0$ . in the indirect adjoining approach the Hamiltonian  $H_1$  and Lagrangian  $L_1$  are

$$\begin{aligned} H_1 &= -x + \lambda u \\ L^1 &= H^1 + \mu_1(u + 1) + \mu_2(1 - u) + \nu u \end{aligned}$$

the necessary conditions of theorem 2.2 are

$$\begin{aligned} L_u^1 &= \lambda + \mu_1 - \mu_2 + \nu = 0 \\ \dot{\lambda} &= -L_x = 1 \end{aligned}$$

where  $\mu_1, \mu_2$  and  $\nu^1$  satisfy the complementary conditions:

$$\begin{aligned} \mu_1 &\geq 0, \mu_1(u + 1) = 0 \\ \mu_2 &\geq 0, \mu_2(1 - u) = 0 \\ \nu^1 &\geq 0, \nu^1 x = 0, \nu^1 \leq 0 \\ \lambda(1^-) &= \lambda^{1+} + \eta^1(1), \eta^1(1) = 1 \geq 0. \end{aligned}$$

since  $x^*(t)$  enters the boundary zero at  $t = 1$  there are no jumps in interval  $(1,2]$  and the solutions for  $\lambda^1(t)$  is

$$\lambda_1(t) = t - 2, t \in (1, 2].$$

Hence  $\lambda_1(t) \leq 0$  and  $x^*(t) = 0$  on  $(1,2]$ , we have  $u^*(t) = 0$ . Now let us see what must happen at  $t=1$ . we now from equation:

$$\begin{aligned} \lambda_1(t) &= t - 2, t \in (1, 2], \lambda^1(1^+) = -1. \text{ then,} \\ H^1(1^+) &= -x^*(1^+) + \lambda_1(1^+)u^*(1^+) = 0 \\ H^1(1^-) &= -x^*(1^-) + \lambda_1(1^-)u^*(1^-) = -\lambda_1(1^-) \end{aligned}$$

By equation  $H(1^-)$  to  $H(1^+)$  we obtain  $\lambda(1^-) = 0$ , then the value of jump condition:  $\eta^1(1) = \lambda(1^-) - \lambda(1^+) = 1 \geq 0$ . in time interval  $[0,1), \mu_2 = 0$ . since  $u^* = -1$  and  $\nu^1 = 0$  because  $x > 0$ , for  $t \in [0, 1)$ . Therefore,

$$\begin{aligned} \frac{\partial}{\partial u} L &= \lambda + \mu_1 - \mu_2 + \nu = 0, \text{ then} \\ \lambda^1 + \mu_1 &= 0 \text{ since } \mu_2 \text{ and } \nu = 0, \text{ for } t \in [0, 1). \text{ hence} \\ \mu_1(t) &= -\lambda^1(t) = 2 - t \text{ for } t \in [0, 1) \text{ with } u = -1. \end{aligned}$$

At  $t = 1$  we have  $x(1)$  so that the optimal control  $u^* = 0$ . now assume that we continue to use the control  $u^*(t) = 0$  in the interval  $[1, 2]$  then  $x(t) = 0$  for  $t \in [1, 2]$ . since  $\lambda^1(t) \leq 0$  for  $t \in [1, 2]$ ,  $u^*(1) = 0$ , on the same interval then  $\mu_1$  and  $\mu_2 = 0$  for  $t \in [1, 2]$ . but we can obtain  $\nu^1(t) = -\lambda^1(t)$ ,  $t \in [1, 2]$ . therefore, the adjoint function  $\lambda$  is continuous every where,  $\nu$  is constant in  $[0, 1)$  and  $(2, 3]$  where the state constraint is not active and  $\nu$  is continuous at  $t = 1, 2$  where  $k^1(t, x, u) = \dot{x} = u$  is discontinuous. the adjoint function  $\lambda$  is continuous, since the entry to and the exit from the state constraint is non tangential.

# Summary

An optimal control is a set of differential equations describing the paths of optimal control can be derived using bang-bang principle and pontryagin's maximum or minimum principle or solving the Hamiltonian Jacobin equation. Optimal control problems with state inequality constraints . To solve the problem with pure state inequality constraint . we use different approaches with complementary slackness (first order and higher order constraints), the indirect adjoint approach with continuous adjoint functions.

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