Addis Ababa University
College of Natural and Computational Sciences
Department of Mathematics

COEFFICIENTS ESTIMATE FOR LOG-HARMONIC MAPPINGS

A Thesis Submitted to the Department of Mathematics of Addis Ababa University
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By:Namomsa Tafasa Gurmesa

Advisor: Hunduma Legese (Ph.D)

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Addis Ababa University, Department of Mathematics

We, the undersigned, hereby certify that we have read and examined this thesis on **COEFFICIENTS ESTIMATE FOR LOG-HARMONIC MAPPINGS**, which is done by **Namomsa Tafasa** in partial fulfillment of the requirements for the degree of master of science and recommend to the school of graduate studies for acceptance of the thesis.

Advisor: **Dr. Hunduma Legese**
Signature: ______________________
Date: ______________________

Examiner 1: ______________________
Signature: ______________________
Date: ______________________

Examiner 2: ______________________
Signature: ______________________
Date: ______________________

Abstract

This thesis explores coefficient estimates for close-to-starlike log-harmonic mappings, On the unit disk, a subclass of univalent log-harmonic functions is defined..The results contribute to the understanding of log-harmonic mappings and propose a Log-Harmonic Coefficient Conjecture, which parallels classical conjectures in complex analysis. This work lays the foundation for future research aimed at further exploring and validating these conjectures and their broader implications. Topics discussed include log-harmonic polynomials, subclasses of log-harmonic mappings, we first give a general understanding of how to construct log-harmonic Koebe mappings. additionally, For some special subclasses of log-harmonic mappings, growth and distortion theorems are examined.

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Contents

Notations and Abbreviations

- $\mathbb{D}$: Unit disk in the complex plane.
- $\mathcal{A}$: Linear space of all analytic functions defined on the unit disk $\mathbb{D}$.
- $\mathcal{B}$: Set of all functions $\mu \in \mathcal{A}$ such that $|\mu(z)| < 1$ for all $z \in \mathbb{D}$.
- f: twice continuously differentiable function.
- $\Delta f = 0$: Harmonic function.
- $\log f$: Log harmonic mapping.
- $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$: Partial derivatives.
- Δ: Laplacian operator.
- $f_{\bar{z}}(z), f_z(z)$: Partial derivatives with respect to $\bar{z}$ and z.
- μ: Second complex dilatation of f, $\mu \in \mathcal{B}$.
- S_{Lh}:The class of univalent log-harmonic mapping.
- CST_{Lh}: The set of all close-to-starlike log-harmonic mappings.
- S^*_{Lh}: The set of all starlike univalent log-harmonic mappings.

Chapter 1

Introduction

Harmonic mappings play a significant role across various disciplines, including engineering and aerodynamics, due to their extensive applications in solving complex mathematical problems. We investigate coefficient estimates for log-harmonic close-to-starlike mappings in this thesis, with particular attention to a subset of univalent log-harmonic mappings formed on the unit disk.

Mostly in the last 2011, Abdulhadi, Bshouty, and Hengartner have been studied logharmonic mappings. And [1-9] developed the fundamental theory of logharmonic mappings. Coefficient estimates for the set S^*_{Lh} of all starlike univalent log harmonic mapping was studied [2]. Therefore, the aim of this thesis is to estimate coefficients for the set of all close-to-starlike logharmonic mappings CST_{Lh}.The set S^*_{Lh} of all starlike univalent logharmonic mappings is a subset of CST_{Lh}.

This thesis is organized as follows: In chapter 2, we introduces essential concepts and theorems relevant to Univalent Logharmonic Mappings in a Simply Connected Domain, including basic properties and representations. And also discusses about Logharmonic Polynomials. In chapter 3, we revised subclasses of log-harmonic mappings including spirallike and close-to-starlike log-harmonic mappings. Additionally, it introduces the growth and distortion theorems that go along with building techniques for univalent log-harmonic mappings.

In chapter 4, The core findings of the thesis are presented here, focusing on the coefficient estimates for close-to-starlike log-harmonic mappings. Key theorems are provided to establish these estimates. And also we proposes future research directions, including the Log-Harmonic Coefficient Conjecture, which invites further investigation into the coefficient bounds and their implications.

Chapter 2

preliminaries

In this chapter, we will discuss essential concepts which are relevant to our topics. We present some benefcial defnitions, theorems, and lemmas.

Let $\mathcal{A}$ be the linear space of all analytic functions defined on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and Let $\mathcal{B}$ be the set of all functions $\mu \in \mathcal{A}$ such that $|\mu(z)| < 1$ for all $z \in \mathbb{D}$. Let f be twice continuously differentiable function, then f is harmonic if $\Delta f = 0$, and f is log harmonic mapping if $\log f$ is harmonic, where

$$\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}},$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

A log-harmonic mapping is a solution of the nonlinear elliptic partial differential equation

$$\overline{f_{\bar{z}}(z)} = \mu(z)\left(\frac{\overline{f(z)}}{f(z)}\right)(f_z(z) \tag{2.1}$$

where μ is the second complex dilatation of f and $\mu \in \mathcal{B}$.

Unlike the linear space $\mathcal{A}$ consisting of analytic functions, the translations in the image do not preserve logharmonicity and the inverse of a logharmonic function is not necessarily logharmonic. If f_1 and f_2 are two logharmonic functions with respect to $\mu \in \mathcal{B}$, then $f_1 \cdot f_2$ is logharmonic with respect to the same μ. If, in addition, $0 \notin f_2(D)$, then f_1/f_2 is also logharmonic. With respect to $\mu \circ \phi$, the composition $f \circ \phi$ of a logharmonic mapping f with a conformal premapping ϕ is also logharmonic. Conversely, the composition $\phi \circ f$ of a conformal postmapping ϕ and a logharmonic mapping f is usually not logharmonic. If f is a logharmonic mapping in $\mathbb{D}$, then it is continuous, open, and light since it is a nonconstant locally quasiregular mapping. As a result, f can be expressed as the product of two functions, $f = A \circ \chi$, where $A \in \chi(\mathcal{A})$ and χ is a locally quasiconformal homeomorphism in D. Consequently, for logharmonic mappings, the maximum principle, the identity principle, and the argument principle remain valid. Mostly in the last few years, Abdulhadi, Bshouty, and Hengartner have been studying logharmonic mappings. and the basic theory of logharmonic mappings was developed in [1-9].

Theorem 1. *([1,Theorem 1.1]). Let f in D be represent a logharmonic mapping with respect to $\mu \in B(D)$. Suppose that $f(z_0) = 0$ and $B(z_0, \rho) \setminus \{z_0\} \subset D \backslash Z(f)$, where $B(z_0, \rho) = \{z : |z - z_0| < \rho\}$ and $Z(f) = \{z \in D : f(z) = 0\}$. Then f admits the representation*

$$f(z) = (z - z_0)\,|z - z_0|^{2\beta n}\, h(z)\overline{g(z)}, \quad z \in B(z_0, \rho) \tag{2.2}$$

where $n \in \mathbb{N}, \beta = n\overline{\mu(z_0)}\,(1 + \mu(z_0)) / \left(1 - |\mu(z_0)|^2\right)$ and, therefore, $\mathrm{Re}(\beta) > -n/2$.

$\mathcal{A}(B(z_0, \rho))$ contains the functions h and g , where $h(z_0) \neq 0$ and $g(z_0) = 1$. We obtain the following global representation for logharmonic mappings as a direct consequence of Theorem 1.

corollary 1. *([1,Corollary 1.2]). Let D be a simply connected domain in $\mathbb{C}$ and f a logharmonic mapping in D. If f has exactly p zeros $\{z_k\}_{k=1}^{p}$ in D (counting multiplicities), then f admits a global representation given by*

$$f(z) = \left[\prod_{k=1}^{p} (z - z_k)\,|z - z_k|^{2\beta_k}\right] h(z)\overline{g(z)} \tag{1.3}$$

where $\beta_k = \overline{\mu(z_k)}\,(1 + \mu(z_k)) / \left(1 - |\mu(z_k)|^2\right)$ and, therefore, $\mathrm{Re}(\beta_k) > -1/2$. The functions h and g are in $\mathcal{A}(D)$, and $0 \notin h \cdot g(D)$.

For the converse, Abdulhadi and Hengartner [2] proved the following theorem.

Theorem 2. *([1,Theorem 1.3]). Suppose that $f(z) = h(z)\overline{g(z)}$ is defined in a domain D, where h and g are in $\mathcal{A}(D)$, such that $f(D)$ does not lie on a logarithmic spiral. Then either $f = \bar{g}$ or f is a solution of (2.1).*

Remark. *([1,Remark 1.4]). The converse of Theorem 1.3 does not hold. Indeed, consider the partial differential equation $\overline{f_{\bar{z}}} = (1/3)(\bar{f}/f)f_z$. Then $f_1(z) = z^6\bar{z}^2$ and $f_2(z) = z|z|$ are solutions of this equation. The function f_1 can be written in the form $h\bar{g}$ while f_2 could not.*

As a result, f's Jacobian J_f is provided by

$$J_f = |f_z|^2 - |f_{\bar{z}}|^2 = |f_z|^2 \left(1 - |\mu|^2\right),$$

This is positive; hence, all non-constant log-harmonic mappings in $\mathbb{D}$ are open and sense-preserving. The expression for

$$f(z) = h(z)\overline{g(z)}$$

where $h(z)$ and $g(z)$ are non-vanishing analytic functions in $\mathbb{D}$ can be used to represent f if it is a non-vanishing log-harmonic mapping in $\mathbb{D}$. Conversely, if f vanishes at $z = 0$ but is not exactly zero, then for any $z \in D$, such a mapping f admits the representation

$$f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$$

where
(a) $\beta = \overline{\mu(0)}(1 + \mu(0))/\left(1 - |\mu(0)|^2\right)$, and therefore $\mathrm{Re}(\beta) > -1/2$,
(b) h and g are in $\mathcal{A}$ satisfying $g(0) = 1$ and $0 \notin h \cdot g(D)$. f is hence locally quasiconformal. The analogue of Caratheodory's Kernel Theorem might not apply to univalent logharmonic mappings. Specifically, for every function

$$f_r(z) = \frac{z}{(1-z)^2} \exp\left(-2r\left(\mathrm{Re}\int_0^z \frac{(1+z)}{(1+rz)(1-z)}dz\right)\right), \quad 0 < r < 1 \tag{2.4}$$

the unit disc D is mapped onto the slit domain $\mathbb{C}\backslash\left(-\infty, -p_r\right)$, which is both univalent and logharmonic with respect to $\mu_r(z) = -rz$. As r changes from 0 to 1, the tip p_r of the omitted slit varies monotonically from $-1/4$ to -1.

The univalent and logharmonic limit function $\lim_{r\to 1} f_r(z) = f_1(z) = (z(1-\bar{z}))/(1-z)$ translates D onto D. The unit circle is the cluster set of f_1 at point 1, and for $0 < |t| \leq \pi$, it has the boundary value $f\left(e^{it}\right) = -1$. Let f be a univalent logharmonic function defined in D with regard to $\mu \in B(D)$, and let D be a suitably simply connected domain in $\mathbb{C}$. φ denotes a conformal mapping from the unit disk D onto D. In U, $f \circ \varphi$ is univalent logharmonic given $\mu^* = \mu \circ \varphi \in B(U)$. Thus, we can assume that $f(0) = 0$ and $D = U$.

As in the analytical situation, we write

$$S_{Lh} = \left\{f(z) = z|z|^{2\beta}h\bar{g} : \text{where} f \text{ is a univalent logharmonic mapping defined in } U \text{ with } h(0) = g(0) = 1\right\}. \tag{2.5}$$

S_{Lh} is not compact in terms of the topology of normal convergence at this point, since $1^{2\beta} = 1$. When $f_n(z) = z|z|^{(1-n)/n}$ is applied, the sequence does, in fact, converge uniformly to $f(z) = z|z|^{-1}$, which is not in S_{Lh}. Our next result deals with the subclass S^0_{Lh} of S_{Lh} defined by $S^0_{Lh} = \{f \in S_{Lh} : \mu(0) = 0 \text{ (resp., } \beta = 0)\}$. In [3], the following outcome was demonstrated.

where $\operatorname{Re}\beta > -1/2, h, g \in \mathcal{A}, h(0) \neq 0$ and $g(0) = 1$ (cf. [3]).

The class of all functions of this kind has been the subject of numerous investigations. For instance, see [13, 14].

We take the class S^0_{Lh} of univalent log-harmonic mappings f of the type $\beta = 0$ for simplicity's sake.

$$f(z) = zh(z)\overline{g(z)}$$

Since $h(0) = g(0) = 1$ is the normalization for $h, g \in \mathcal{A}$, and such that

$$h(z) = \exp\left(\sum_{n=1}^{\infty} a_n z^n\right) \quad \text{and} \quad g(z) = \exp\left(\sum_{n=1}^{\infty} b_n z^n\right) \tag{2.6}$$

2.1 Univalent Logharmonic Mappings in a Simply Connected Domain

Let S be a nonparametric minimum surface that lies over Ω, and Let Ω be a domain in $\mathbb{C}$, the complex plane. S is defined in the following sense as a univalent orientation-preserving harmonic mapping $w = f(z)$ from a suitable domain D of $\mathbb{C}$ onto Ω. Then, S can be represented by a function $s = G(u, v), w = u + iv \in \Omega$.

The system of nonlinear elliptic partial differential equation

$$\overline{f_{\bar{z}}(z)} = \mu(z)\left(\frac{\overline{f(z)}}{f(z)}\right)(f_z(z)$$

is solved by the mapping F, where $\mu(z)$ in $\mathcal{A}(D)$. As f preserves orientation, it follows that in D, $|\mu(z)| < 1$. The second dilatation of f is the function $\mu(z)$. When dz changes on the unit circle,the quotient of the maximum and minimum values of the differential $|Df(z)|$ is $\frac{(1+|\mu(z)|)}{(1-|\mu(z)|)}$. (see, e.g., [4, 5]). Which give an equations:

$$(1 - |\mu(z)|) \leq |Df(z)| \leq (1 + |\mu(z)|) \tag{2.8}$$

Three real-valued harmonic functions provide the representation of the minimum surface S (see, for example, [5, 6]),

$$u(z) = \operatorname{Re}(\mu(z)), \quad v(z) = \operatorname{Im}(f(z)), \quad s(z) = \operatorname{Im}\int^z \sqrt{\mu} f_z dz \tag{2.9}$$

Since $(s_z)^2 = -\overline{f_{\bar{z}}} f_z = -\mu(z)\,(f_z)^2$ in D, it follows that $\sqrt{\mu(z)}$ belongs to $\mathcal{A}(D)$. Specifically, every zero in $\mu(z)$ has an even order. Since $ds^2 = |f_z|^2\,(1 + |\mu(z)|)^2|dz|^2$ is the Riemannian metric of S, then $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ are isothermal parameters for S.

If φ is univalent and analytic and f is univalent and harmonic, then the composition $f \circ \varphi$ (whenever well defined) is a univalent harmonic mapping; however, $\varphi \circ f$ does not always have to be harmonic. Therefore, $f(\varphi)$ represents the same minimal surface but in different isothermal parameters if F represents a minimal surface over Ω (in the sense of relation (2.2)). Let us assume that Ω in $\mathbb{C}$ is a proper simply-connected domain. Then, any proper simply connected domain in $\mathbb{C}$ may be selected for D. Specifically, $D = U$ or $D = \Omega$ are suitable options.

Now let's consider the left half-plane $D = \{z : \mathrm{Re}(z) < 0\}$. Let f be a univalent harmonic that is defined in D and that preserves orientation in order to satisfy the relation.

$$f(z + \alpha i) = f(z) + \beta \quad \forall z \in D \tag{2.11}$$

where α and β are real constants.

$$f(z + 2\pi i) = f(z) + 2\pi i \quad \forall z \in D \tag{2.12}$$

can be obtained by applying the transformation $(2\pi/\beta)f(2\alpha/2\pi)$, assuming without losing generality that $\alpha = \beta = 2\pi$. $\mathrm{Re}(f(-\infty)) = c$ will be written whenever $\lim_{x\to-\infty} \mathrm{Re}(f(z)) = c$ for some$c \in [-\infty, \infty)$.

Similarly, $\mu(z)(-\infty) = c$ means that $\lim_{x\to-\infty} \mu(z) = c$.

Let UHP be the class of all orientation-preserving mappings of univalent harmonics defined on the left half-plane $D = \{z : \mathrm{Re}(z) < 0\}$.

$$\begin{gathered} f(z + 2\pi i) = f(z) + 2\pi i \quad \forall z \in D \\ \mathrm{Re}(f(-\infty)) = -\infty \end{gathered} \tag{2.13}$$

is satisfied.

It follows that the second dilatation function $\mu(z)$ is periodic, that is, $\mu(z + 2\pi i) = \mu(z) + 2\pi i$ in D, and therefore the Gauss map is also periodic. Observe that $\mu(-\infty)$ exists. Furthermore, it was shown in [6] that mappings in the UHP class admit the representation.

$$f(z) = z + \beta x + h(z) + \overline{g(z)} \tag{2.14}$$

where

(a) h and g are in $\mathcal{A}(D)$ such that

(i) $g(-\infty) = 0$ and $h(-\infty)$ exists and finite in $\mathbb{C}$,
(ii) $h(z + 2\pi i) = h(z)$ and $g(z + 2\pi i) = g(z)$ for all $z \in D$;
(b)

$$\left|\frac{g'(z) + \bar{\beta}}{1 + \beta + h'(z)}\right| < 1 \quad \text{on } D$$

$$\beta = \frac{\overline{\mu(z)(-\infty)}(1 - \mu(z)(-\infty))}{1 - |\mu(z)(-\infty)|^2}, \text{ and hence } \operatorname{Re}(\beta) > -1 \tag{2.15}$$

Define

$$f(z) = e^{f(\log(z))}, \quad z \in D \tag{2.16}$$

Then f is a univalent logharmonic mapping in D with respect to $\mu(z) = \phi(z)(\log(z))$ and hence μ in $B(D)$. Observe that the family of all univalent logharmonic and orientation-preserving mappings f defined in D satisfying $f(0) = 0$ is isomorphic to the class UHP. It was demonstrated in [7, 8] that even when the differential equation becomes nonlinear, working with logharmonic mappings is simpler.

2.2 Logharmonic Polynomials

An analytic polynomial of degree n is indicated by p_n. A function with the formula $f = p_n\overline{p_m}$ is known as a logharmonic polynomial. There are nonconstant logharmonic polynomials that are not p-valent for any $p > 0$ in contrast to the analytic situation. For instance, with respect to $\mu = -1$, the function $f(z) = z\bar{z}$ in $\mathbb{C}$ is a logharmonic polynomial. Furthermore, with respect to $\mu(z) = (z+1)/(z-1)$, the function $f(z) = (z-1)(\bar{z}+1)$ in $\mathbb{C}$ is a logharmonic polynomial. The two-valent polynomial leaves off the half-plane $\operatorname{Re}(w) < -1$. However, they inherit the analytic polynomials' $\lim_{z\to\infty} f(z) = \infty$ feature. Given that $|f| = |p_n\overline{p_m}| = |p_n p_m|$, this follows. The opposite is not true, either; logharmonic functions$f = h\bar{g}$ defined in $\mathbb{C}$ have the condition $\lim_{z\to\infty} f(z) = \infty$ and are not logharmonic polynomials. One example is the function $f(z) = ze^z e^{-\bar{z}}$. It should be noted that some harmonic polynomials, $p_n(z)+\overline{p_m(z)}$, do not meet the condition $\lim_{z\to\infty} f(z) = \infty$. But if we suppose that $\mu(\infty)$ exists and that $|\mu(\infty)| \neq 1$, we have the following deduction [12].

Theorem 3. *([1, Theorem 4.1]) Let* $\lim_{z\to\infty} f(z) = \infty$ *be a logharmonic function in* $\mathbb{C}$ *with* $f = h\bar{g}$. f *is a polynomial if* $\lim_{z\to\infty} \mu(z) = \mu(\infty)$ *exists and if* $|\mu(\infty)| \neq 1$.

Indicate $Z(f - w, D)$'s cardinality, or the number of zeros in $f - w$ in D, multiplicity not included, by denoting $NZ(f - w, D)$. $NZ(f - 1, \mathbb{C}) = \infty$ is a property of the polynomial $f(z) = |z|^2$. Nevertheless, a univalent logharmonic mapping in $\mathbb{C}$ is inherently a polynomial of the type $f(z) =$ const $\cdot(z-$ a) $\overline{(z - a)^m}$ or of its conjugate, where $const \neq 0, a \in \mathbb{C}$, and $m = 0, 1, 2, \ldots$, according to Theorem 2.3. Although they are not polynomials, some functions of the form $f = h\bar{g}$ have the property that $NZ(f - w, \mathbb{C})$ is finite and uniformly bounded for all $w \in \mathbb{C}$.

For every fixed $w \in \mathbb{C}$, the function $f(z) = ze^z e^{\bar{z}} - w$, for instance, has a maximum of two zeros. The outcome displayed below was found in [12].

Theorem 4. *([1, Theorem 4.2.]) If* $f = h\bar{g}$ *is a logharmonic function in* $\mathbb{C}$, *then* $\lim_{z\to\infty} \mu(z) = \mu(\infty)$ *exists with* $|\mu(\infty)| \neq 1$, *and* $NZ(f - w, G)$ *is finite for at least two distinct values of* w. *Consequently,* f *is a polynomial..*

It is easy to derive an upper bound on a logharmonic polynomial's w-point count by applying Bezout's theorem [39].

Theorem 5. *([1, Theorem 4.3]). Let* $p(x, y)$ *and* $q(x, y)$ *be polynomials with real coefficients in the real variables* x *and* y. *If* p *and* q *have infinitely many zeros, or if* $\deg(p) = n$ *and* $\deg(q) = m$, *then at most* nm *common zeros exist between them.*

Bezout's theorem provides a strong upper constraint for the number of zeros in a harmonic polynomial, and therefore for polyanalytic polynomials, as demonstrated by Wilmshurst [15] (see, e.g., [16]).

This is not the case with logharmonic polynomials, though. Let $f = p_n\overline{p_m}$ be a polynomial of degree $n + m$ that is logharmonic. $f(z) - w = \sum_{k=0}^{n} \sum_{j=0}^{m} a_{kj} z^k \bar{z}^j$ in that case. The real-valued polynomials in x and y, $P(z) = \mathrm{Re}(f(z))$ and $Q(z) = \mathrm{Im}(f(z))$, have degrees of $n + m$.. Applying Bezout's theorem, we conclude with the following estimate.

Theorem 6. *([1, Theorem 4.4]). Assume that the logharmonic polynomial* $f = p_n\overline{p_m}$ *is defined in* $\mathbb{C}$. *Then, for any* w *in* $\mathbb{C}$, *either* $f - w$ *has an infinite number of zeros or* $f - w$ *has at most* $(n + m)^2$ *zeros.*

It's not the best possible bound. Quadratic polynomials, in fact, have the following forms: $f(z) = p_2(z)$, $\overline{f(z)} = p_2(z)$, or $f(z) = a(z + b)\overline{(z + c)}$. $f - w$ has at most two zeros or infinitely many zeros in each of the three scenarios.

Note that $|\mu(\infty)| \neq 1$ and that the logharmonic polynomial $f(z) = (z-1)/(\bar{z}+1))$ is a two-valent polynomial that does not include the half-plane $\mathrm{Re}(w) < -1$. If $|\mu| \not\equiv 1$, on the other hand, the circumstances alter, and the outcome is as follows [2].

Theorem 7. *([1,Theorem 4.5]). Assume that $n > m$ and that $f = p_n\overline{p_m}$ is a logharmonic polynomial defined in $\mathbb{C}$. Assume that$Z(f-w, \mathbb{C}) \cap (\partial D \cup S_E(D))$ is empty for all $w \in \mathbb{C}$. Subsequently, the valency $V(f, \mathbb{C})$ of f in $\mathbb{C}$, as well as the number of zeros $VZ(f- \ w, S_E(\mathbb{C}))$ of $f-w$, are at least $n-m$. The bound is as close to ideal as it gets.*

One immediate consequence of Theorem 7 is the following result.

corollary 2. *([1, Corollary 4.6]). Let $f = p_n\overline{p_m}$ be a logharmonic polynomial defined in $\mathbb{C}$, and suppose that $n > m$. Then*

(i) $f(\mathbb{C}) = \mathbb{C}$,

(ii) for almost all $w \in \mathbb{C}$, the function $f - w$ has at least $n - m$ disjoint zeros.

The next result characterizes polynomials of finite valency [12].

Theorem 8. *Theorem 4.7. Let $f = p_n\overline{p_m}$ be a logharmonic polynomial with $p_n \not\equiv$ const $\cdot p_m$, defined in $\mathbb{C}$. Then, for all $w \in \mathbb{C}$, the cardinality $NZ(f-w, \mathbb{C})$ of the zero set $Z(f-w, \mathbb{C})$ is finite (and hence, uniformly constrained by Bezout's theorem).*

Remark. *The image is on a straight line if $p_n \equiv$ const $\cdot p_m$.*

Chapter 3

Subclasses of Logharmonic Mappings

3.1 Spirallike Logharmonic Mappings

If the origin is in $\mathbb{C}$, let Ω be a simply connected domain. If $w \in \Omega$ implies that $w \exp(-te^{i\alpha}) \in \Omega$ for every $t \geq 0$, then Ω is α-spirallike, $-\pi/2 < \alpha < \pi/2$. The domain Ω is considered starlike (relative to the origin) if $\alpha = 0$. The following notations will be used.
(a) The set S^{α}_{Lh} comprises any univalent logharmonic mappings f in D that meet the conditions $f(0) = 0$, $h(0) = g(0) = 1$, and $f(D)$ is a domain that resembles an α-spirallike.

(b) $S^{\alpha} = \{f \in S^{\alpha}_{Lh}$ and $f \in \mathcal{A}\}$.

(c) For which $f(D)$ is starlike (with regard to the origin), $S^{*}_{Lh} = S^{0}_{Lh}$ and $S^{*} = S^{0}$.

We attach the analytic function $\psi(z) = zh(z)/g(z)^{e^{i\alpha}}, \psi(0) = 0$, to each $f(z) = z|z|^{2\beta}h(z)\overline{g(z)} \in S^{\alpha}_{Lh'}$.

A representation theorem was provided by Abdulhadi and Hengartner [8] for mappings in the class S^{α}_{Lh}.

Theorem 9. *([1, Theorem 5.1.])*

(a) If $f \in S^{\alpha}_{Lh'}$ then $\psi \in S^{\alpha}$.

(b) For every given $\psi \in S^{\alpha}$ and $\mu \in \mathcal{B}$, there are h and g in $\mathcal{A}$ uniquely defined such that

(i) $0 \notin h \cdot g(D), h(0) = g(0) = 1$,

(ii) $\psi(z) = zh(z)/g(z)^{e^{i\alpha}}$,

(iii) $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ is a solution of (1.1) in $S^{\alpha}_{Lh'}$ where $\beta = (\overline{\mu(0)}(1 + \mu(0)))/(1- |\mu(0)|^2)$

Remark. *([1, Remark 5.2]). Theorem 9 has no equivalent for the class of all logharmonic mappings that are convex and univalent. While* $f(z) = z/\left|1-z^4\right|^{1/2}$ *is not a convex mapping,*$\psi(z) = z$ *is, and* $\mu(z) = z^4 \in \mathcal{B}$.

Remark. *([1, Remark 5.3]). According to Theorem 9,* $S^{\alpha} \times \mathcal{B})$ *is isomorphic to the class*$S^{\alpha}_{Lh'}\alpha$ *fixed in* $(-\pi/2, \pi/2)$.

Theorem 9 immediately leads to the following outcome.

corollary 3. *([1,Corollary 5.4]).For every* $r \in (0,1)$, $f(rz)/r \in S^{\alpha}_{Lh}$ *if* $f \in S^{\alpha}_{Lh'}$. *Stated differently, level sets inherit the* α*-spirallike nature.*

An integral representation for $f \in S^{\alpha}_{Lh}$[8] is the following result.

Theorem 10. *([1,Theorem 5.5]). A function* $f \in S^{\alpha}_{Lh}$ *if and only if there are two probability measures* u *and* ν *on the Borel sets of* ∂U *and an* $\mu(0) \in U$ *such that*

$$f(z) = z|z|^{2\beta} \cdot \exp\left\{\int_{\partial U \times \partial U} K(z,\zeta,\xi;\mu(0))du(\zeta)dv(\xi)\right\} \tag{3.1}$$

where

$$\beta = \frac{\overline{\mu(0)}(1+\mu(0))}{1-|\mu(0)|^2},$$

$$K(z,\zeta,\xi;\mu(0)) = -2\cos(\alpha)\cdot e^{i\alpha}\cdot\log(1-\zeta z) + 2e^{i\alpha}\operatorname{Re}\left\{e^{i\alpha}\log(1-\zeta z)\right\} + T(z,\zeta,\xi;\mu(0))$$

$$T(z,\zeta,\xi;\mu(0)) = 2e^{i\alpha}\operatorname{Re}\frac{e^{i\alpha}(1+\mu(0))\left(1-\overline{\mu(0)}e^{-2i\alpha}\right)\zeta + e^{-i\alpha}(1+\overline{\mu(0)})\left(1-\mu(0)e^{2i\alpha}\right)\xi}{(\zeta-\xi)\left|1-\mu(0)e^{2i\alpha}\right|^2} \times \log\frac{1-\xi z}{1-\zeta z} \tag{3.2}$$

if $|\zeta| = |\xi| = 1, \zeta \neq \xi$, and

$$T(z,\zeta,\zeta;\mu(0)) = 4\cos(\alpha)\cdot e^{i\alpha}\cdot\operatorname{Re}\frac{\zeta z}{(1-\zeta z)}\frac{1-|\mu(0)|^2}{|1-\mu(0)e^{2i\alpha}|^2} \tag{3.3}$$

Observe that S^{α}_{Lh} is not compact, but Theorem 5.5 can be used to solve extremal problems over the class of mappings in S^{α}_{Lh} with a given $\mu(0) = 0$.

We have seen in Corollary 5.4 that if f is a univalent logharmonic mapping in $U, f(0) = 0$, and if $f(U)$ is starlike, then $f(|z| < r)$ is starlike (with respect to the origin) for all $r \in (0,1)$. The next result proved in [8] shows that this property may fail whenever $f(0) \neq 0$.

Theorem 11. *([1, Theorem 5.6]) For each $z_0 \in D\backslash\{0\}$, there are univalent logharmonic mappings f such that $f(z_0) = 0, f(D)$ is starlike (with respect to the origin), but no level set $f(|z| < r), |z_0| = \rho < r < 1$, is starlike.*

3.2 Close-to-Starlike Logharmonic Mappings

3.2.1 Logharmonic Mappings with Positive Real Part

Let P_{Lh} be the set of all logharmonic mappings R in D which are of the form $R = H\bar{G}$, where H and G are in $\mathcal{A}$, $H(0) = G(0) = 1$, such that $\mathrm{Re}(R(z)) > 0$ for all $z \in D$. In particular, the set P of all analytic functions p in D with $p(0) = 1$ and $\mathrm{Re}(p(z)) > 0$ in D is a subset of P_{Lh} .

The next result [9] describes the connection between P_{Lh} and P.

Theorem 12. *([Theorem 5.7]). A function $R = H\bar{G} \in P_{Lh}$ if and only if $p = H/G \in P$.*

As a consequence of Theorem 12, it follows that R admits the representation

$$R(z) = p(z) \exp 2 \,\mathrm{Re} \int_0^z \frac{\mu(s)}{1-\mu(s)} \frac{p'(s)}{p(s)} ds \tag{3.4}$$

where μ in $\mathcal{B}(D)$ and p is an analytic function with positive real part normalized by $p(0) = 1$.

3.2.2 Close-to-Starlike Logharmonic Mappings

Let $F(z) = z|z|^{2\beta} h\bar{g}$ be a logharmonic mapping. If F is a product of a starlike logharmonic mapping $f(z) = z|z|^{2\beta} h^* \overline{g^*} \in S^*_{Lh}$ with respect to μ in $\mathcal{A}$ and a logharmonic mapping with positive real part $R \in P_{\mathrm{Lh}}$ with the same second dilatation function μ, then F is close to starlike.

A close-to-starlike logharmonic mapping can be understood geometrically as one in which the radius vector of the image of $|z| = r < 1$ never reverses by more than π.

Denote by CST_{Lh} the set of all close-to-starlike logharmonic mappings. It contains in particular the set CST of all analytic close-to-starlike functions which was introduced by Reade [10] in 1955. Also, the set S^*_{Lh} of all starlike univalent logharmonic mappings is a subset of CST_{Lh} (take $R(z) \equiv 1$ in the product). Furthermore, if $F(z) = z|z|^{2\beta} h\bar{g}$ is a logharmonic mapping with respect to μ in $\mathcal{A}$ satisfying $h(0) = g(0) = 1$ and $\mathrm{Re}\, F(z)/z|z|^{2\beta} > 0$, then F is a close-to-starlike logharmonic mapping, where

$$f(z)=z|z|^{2\beta}\left|\exp\left(\int_0^z \frac{\mu(s)/s}{1-\mu(s)}ds\right)\right|^2 \tag{3.5}$$

On the other hand, a mapping $F \in \mathrm{CST}_{Lh}$ need not necessarily be univalent. For example, take $F(z)=z(1+z)$, where $z \in S^*$ and $1+z \in P$.

A representation theorem for the class CST_{Lh}, demonstrated in [9], is the next outcome we have.

Theorem 13. *([9, Theorem 5.9]). (a) Let $F=z|z|^{2\beta}h\bar{g}$ be in CST_{Lh}. Then $\psi=zh/g \in CST$.*

(b) Given any $\psi \in CST$ and μ in $\mathcal{A}$, there are h and g in $\mathcal{A}$ uniquely determined such that

(i) $0 \notin h \cdot g(D), h(0)=g(0)=1$,

(ii) $\psi(z)=zh/g$,

(iii) $F=z|z|^{2\beta}h\bar{g}$ is in $CST\ _{Lh}$ which is a solution of (2.1) with respect to the given μ.

corollary 4. *([Corollary 5.10]). $F \in CST_{Lh}$ if and only if $F(rz)/r \in \mathrm{CST}_{Lh}$ for all $r \in (0,1)$.*

A function $f:\Omega \to \mathbb{C}$ with complex values is classified as being in the $C^1(\Omega)$ class (resp. $C^2(\Omega)$), if the continuous first order (resp. second order) partial derivatives of $\operatorname{Re} f$ and $\operatorname{Im} f$ exist in Ω. Examine the complex linear differential operator Df defined on $C^1(\Omega)$ for $f \in C^1(\Omega)$ by

$$Df=zf_z-\bar{z}f_{\bar{z}}$$

Considering the scenario when $\Omega=\mathbb{D}$ is sufficient in light of the Riemann mapping theorem.

3.2.3 Starlike Logharmonic Mappings of Order α

Definition 1. *Assume that the mapping $f=z|z|^{2\beta}h\bar{g}$ is univalent logharmonic. If f is a starlike logharmonic mapping of order α, then we claim that*

$$\frac{\partial}{\partial\theta}\left(\arg f\left(re^{i\theta}\right)\right)=\operatorname{Re}\frac{zf_z-\bar{z}f_{\bar{z}}}{f}>\alpha, \quad 0\le\alpha<1$$

for all $z=re^{i\theta} \in \mathbb{D}\backslash\{0\}$.

The set of all starlike harmonic functions of order α and starlike logharmonic functions of order α are denoted by $\mathcal{S}_H^*(\alpha)$ and $\mathcal{S}_{Lh}^*(\alpha)$, respectively.

These classes are simply denoted by $\mathcal{S}_H^*$ and $\mathcal{S}_{Lh'}^*$, respectively, if $\alpha = 0$.The set of all starlike harmonic functions is denoted by $\mathcal{S}_H^*$ and the set of starlike log-harmonic functions by $\mathcal{S}_{Lh}^*$.Several authors look into these classes in great detail. Consider the case of $[1, 3]$.

The class of all normalised analytic starlike functions of order α are denoted by $\mathcal{S}^*(\alpha)$.

The classes $\mathcal{S}_{Lh}^*(\alpha)$ and $\mathcal{S}^*(\alpha)$ are related by the following theorem.

Theorem 14. *([11, Theorem 2.1]) Consider a log-harmonic mapping on $\mathbb{D}$, $0 \notin (hg)(\mathbb{D})$, such that $f(z) = zh(z)\overline{g(z)}$. If and only if $\varphi(z) = zh(z)/g(z) \in \mathcal{S}^*(\alpha)$, then $f \in \mathcal{S}^*_{Lh}(\alpha)$.*

Proof. Let $f(z) = zh(z)\overline{g(z)} \in ST^*_{Lh}(\alpha)$, then it follows that

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\arg f\left(re^{i\theta}\right)\right) &= \operatorname{Re}\frac{zf_z - \bar{z}f_{\bar{z}}}{f}\\ &= \operatorname{Re}\left(1 + \frac{zh'}{h} - \frac{\bar{z}\overline{g'}}{\bar{g}}\right)\\ &= \operatorname{Re}\left(1 + \frac{zh'}{h} - \frac{zg'}{g}\right) > \alpha\end{aligned}$$

Setting

$$\varphi(z) = \frac{zh(z)}{g(z)}$$

we obtain

$$\operatorname{Re}\frac{zf_z - \bar{z}f_{\bar{z}}}{f} = \operatorname{Re}\frac{z\varphi'}{\varphi} > \alpha$$

Given that f is univalent, $0 \notin f_z(D)$, as we can see. Additionally,

$$\varphi \circ f^{-1}(w) = q_1(w) = w\left|g \circ f^{-1}(w)\right|^{-2}$$

is locally univalent on $f(D)$. It is true that for any $z \in D$, we get $\frac{z\varphi' z)}{\varphi(z)} = (1 - \mu(z))\frac{zf_z}{f} \neq 0$. We deduce that φ is univalent on D from Lemma 2.3 in [4]. $\varphi \in ST(\alpha)$, thus.

Conversely, let $\varphi \in ST(\alpha)$ and μ in $\mathcal{A}$ such that $|\mu(z)| < 1$ for all $z \in D$ be given. We consider

$$g(z) = \exp\left(\int_0^z \frac{\mu(s)\varphi'(s)}{\varphi(s)(1 - \mu(s))}ds\right) \tag{3.6}$$

where $\frac{z\varphi' z)}{\varphi(z)} = (1 - \alpha)p(z) + \alpha$, and p in $\mathcal{A}$ such that $p(0) = 1$ and $\operatorname{Re}(p) > 0$.

Also, let

$$h(z) = \frac{\varphi(z)g(z)}{z}$$

and

$$f(z) = zh(z)\overline{g(z)} = \varphi(z)|g(z)|^2 \tag{3.7}$$

Then, f is a solution of (2.1) with regard to μ, and h and g are non-vanishing analytic functions defined on U, normalized by $h(0) = g(0) = 1$.

Simple calculations give that

$$\frac{\partial}{\partial\theta}\left(\arg f\left(re^{i\theta}\right)\right) = \operatorname{Re}\frac{zf_z - \bar{z}f_{\bar{z}}}{f} = \operatorname{Re}\frac{z\varphi' z)}{\varphi(z)} > \alpha$$

Using the same argument we conclude that

$$f \circ \varphi^{-1}(w) = q_2(w) = w\left|g \circ \varphi^{-1}(w)\right|^2$$

is locally univalent on $\varphi(D)$ and that f is univalent from Lemma 2.3 in [4]. It follows that $f \in ST_{Lh}(\alpha)$, which completes the proof □

Theorem 15. *Since $\varphi \in C^1(\mathbb{D})$ is starlike (not necessarily harmonic) in $\mathbb{D}$, let $f(z) = \varphi(z)|z|^{2(p-1)}(p \geq 1)$. In $\mathbb{D}$, $f \in C^1(\mathbb{D})$ is hence univalent and starlike.*

We present the following straightforward conclusion, which applies to both Theorems 14 and 15.

proposition 1. *([2, proposition 1.2]) Let $f(z) = \varphi(z)|g(z)|^2$ be a complex-valued function on $\mathbb{D}$, where $\varphi, g \in \mathcal{A}$ such that φ and g are non-vanishing in $\mathbb{D}\backslash\{0\}$. Then $f \in \mathcal{FS}^*(\alpha)$ if and only if $\varphi \in \mathcal{FS}^*(\alpha)$.*

Proof. A straightforward computation demonstrates that

$$zf_z(z) = z\varphi_z(z)|g(z)|^2 + \varphi(z)zg'(z)\overline{g(z)} = \left(\frac{z\varphi_z(z)}{\varphi(z)} + \frac{zg'(z)}{g(z)}\right)f(z)$$

and similarly

$$\bar{z}f_{\bar{z}}(z) = \left(\frac{\bar{z}\varphi_{\bar{z}}(z)}{\varphi(z)} + \overline{\left(\frac{zg'(z)}{g(z)}\right)}\right)f(z)$$

which clearly implies that

$$\operatorname{Re}\left(\frac{zf_z(z) - \bar{z}f_{\bar{z}}(z)}{f(z)}\right) = \operatorname{Re}\left(\frac{z\varphi_z(z) - \bar{z}\varphi_{\bar{z}}(z)}{\varphi(z)}\right)$$

The desired conclusion follows. □

Definition 2. *([2, Definition 1.3]) Let $\alpha \in [0,1)$. A function $f \in C^2(\mathbb{D})$ with $f(0)=0$ and $\frac{\partial}{\partial\theta} f\left(re^{i\theta}\right) \neq 0, 0<r<1$, is called a fully convex of order α, denoted by $\mathcal{FC}(\alpha)$, if*

$$\frac{\partial}{\partial\theta}\left(\arg \frac{\partial}{\partial\theta} f\left(re^{i\theta}\right)\right) = \operatorname{Re}\left(\frac{D^2 f(z)}{Df(z)}\right) > \alpha$$

for $z = re^{i\theta} \in \mathbb{D}\backslash\{0\}$ (see also [9, 12] in order to distinguish convexity in the analytic and the harmonic cases), where

$$D^2 f = z(Df)_z - \bar{z}(Df)_{\bar{z}}$$

We see that

$$D^2 f = z f_z(z) + \bar{z} f_{\bar{z}}(z) - 2|z|^2 f_{z\bar{z}}(z) + z^2 f_{zz}(z) + \bar{z}^2 f_{\bar{z}\bar{z}}(z)$$

Assign $\mathcal{FC}(0) =: \mathcal{FC}$ to $\mathbb{D}$, the fully convex (univalent) class. The class $C(\alpha)$ of convex functions of order α coincides with $\mathcal{FC}(\alpha)$ in the analytic case.

Specifically, we designate the set of all fully convex harmonic functions of order α and fully convex log-harmonic functions of order α, respectively, as $\mathcal{FC}_H^0(\alpha)$ and $\mathcal{FC}_{Lh}(\alpha)$.

These classes are denoted, if $\alpha = 0$, by the sets of all fully convex harmonic functions and fully convex log-harmonic functions, respectively, $\mathcal{FC}_H^0$ and $\mathcal{FC}_{Lh}$. Although it is not the case in a formal sense, we treat fully starlike (or convex) mappings as starlike (or convex) mappings throughout the study.

Even if $\varphi(z)$ is a complexvalued convex mapping in $\mathbb{D}$, we note that the function $f(z) = \varphi(z)|g(z)|^2$ is not always convex in $\mathbb{D}$. For instance, think about

$$\varphi(z) = \frac{z}{1-z}, \quad g(z) = \exp\left(\frac{1}{4}\log\left(\frac{1-z}{1+z}\right)\right)\exp\left(\frac{z}{2(1-z)}\right)$$

and therefore, by (2.1), f defined by

$$f(z) = \varphi(z)|g(z)|^2 = \frac{z}{1-z}\left|\frac{1-z}{1+z}\right|^{1/2}\exp\left(\operatorname{Re}\left(\frac{z}{1-z}\right)\right)$$

is a univalent log-harmonic mapping in $\mathbb{D}$ but is not convex in $\mathbb{D}$ although $\varphi(z) = \frac{z}{1-z}$ is convex in $\mathbb{D}$. see ([2,definition 1.3]).

Theorem 16 (Theorem 1.4)**.** *Since $p \geq 1$), let $f(z) = \varphi(z)|z|^{2(p-1)}$. If and only if $\varphi \in \mathcal{FC}(\alpha)$, then $f \in \mathcal{FC}(\alpha)$.*

Proof. We calculate

$$zf_z(z) = |z|^{2(p-1)}\left[z\varphi_z(z) + (p-1)\varphi(z)\right]$$

and similarly,

$$\bar{z}f_{\bar{z}}(z) = |z|^{2(p-1)}\left[\bar{z}\varphi_{\bar{z}}(z) + (p-1)\varphi(z)\right]$$

Thus, we have

$$Df = zf_z - \bar{z}f_{\bar{z}} = |z|^{2(p-1)}\left(z\varphi_z - \bar{z}\varphi_{\bar{z}}\right)$$

This is easily expressed as $F(z) = |z|^{2(p-1)}\Phi(z)$. Thus, we first determine that in order to compute $D^2 f = D(Df) = DF$.

$$z\Phi_z(z) - \bar{z}\Phi_{\bar{z}}(z) = z\left(\varphi_z(z) + z\varphi_{zz}(z) - \bar{z}\varphi_{\bar{z}z}(z)\right) - \bar{z}\left(z\varphi_{z\bar{z}}(z) - \varphi_z(z) - \bar{z}\varphi_{\bar{z}\bar{z}}(z)\right)$$

so that

$$D^2 f = |z|^{2(p-1)}\left(z\Phi_z - \bar{z}\Phi_{\bar{z}}\right) = |z|^{2(p-1)}\left(z\varphi_z + \bar{z}\varphi_{\bar{z}} - 2|z|^2\varphi_{z\bar{z}} + z^2\varphi_{zz} + \bar{z}^2\varphi_{\bar{z}\bar{z}}\right)$$

Finally, we see that

$$\mathrm{Re}\left(\frac{D^2 f}{Df}\right) = \mathrm{Re}\left(\frac{z\varphi_z + \bar{z}\varphi_{\bar{z}} - 2|z|^2\varphi_{z\bar{z}} + z^2\varphi_{zz} + \bar{z}^2\varphi_{\bar{z}\bar{z}}}{z\varphi_z - \bar{z}\varphi_{\bar{z}}}\right) = \mathrm{Re}\left(\frac{D^2\varphi}{D\varphi}\right)$$

and this completes the proof. □

Example 1. *([2, Example 1.5]) With* $0 < |\lambda| \leq 1/2$, *set* $\varphi(z) = z - \lambda|z|^2$. *The log-harmonic mapping of* $\varphi(z)$ *in* $\mathbb{D}$ *is easily observed as a solution of (2.1) with the dilatation as*

$$\mu(z) = \frac{\bar{\lambda}z}{1 + \bar{\lambda}z}$$

which, since $0 < |\lambda| \leq 1/2$, *is analytic in* $\mathbb{D}$ *and* $|\mu(z)| < 1$ *in* $\mathbb{D}$.

3.3 Construction of univalent log-harmonic mappings

Clunie and Sheil-Small [2] proposed a method for generating a harmonic mapping onto a domain convex in one direction by "shearing" a given conformal mapping convex in the same direction.In [12,] a method was provided for creating univalent log-harmonic mappings from the unit disk onto a strictly starlike domain Ω. $f(z) = zh(z)\overline{g(z)} \in \mathcal{S}_{Lh}$ Following the method of shearing-construction by Clunie and Shell-Small, we now present a method of log-harmonic mappings with prescribed dilatation $\mu(z)$ with $\mu(0) = 0$.

Algorithm: Consider a sense-preserving log-harmonic mapping $f(z) = zh(z)\overline{g(z)}$, in which h and g are non-vanishing analytic functions in $\mathbb{D}$, normalized by $h(0) = g(0) = 1$. Then the dilatation μ, defined by

$$\mu(z) = \frac{\overline{f_{\bar{z}}(z)}}{f_z(z)} \cdot \frac{f(z)}{\overline{f(z)}} = \frac{zg'(z)/g(z)}{1 + zh'(z)/h(z)}, \tag{3.8}$$

$|\mu(z)| < 1$ in $\mathbb{D}$ makes it analytic. Construction of logharmonic mappings begins by allowing $zh(z)/g(z) = \varphi(z)$, where φ is analytic fulfilling $\varphi(0) = \varphi'(0) - 1 = 0$ and $\varphi(z) \neq 0$ for all $z \in \mathbb{D}\backslash\{0\}$. This is in accordance with the description of log-harmonic mapping. Thus, the two nonlinear differential equations are obtained.

$$\frac{zh(z)}{g(z)} = \varphi(z) \quad \text{and} \quad \frac{zg'(z)/g(z)}{1 + zh'(z)/h(z)} = \mu(z) \tag{3.9}$$

it might also be written as

$$z(\log h)'(z) - z(\log g)'(z) = \frac{z\varphi'(z)}{\varphi(z)} - 1 \quad \text{and} \quad z(\log g)'(z) = \mu(z)\left(1 + z(\log h)'(z)\right)$$

After resolving these two equations,

$$(\log g)'(z) = \frac{\mu(z)}{1 - \mu(z)} \cdot \frac{\varphi'(z)}{\varphi(z)}$$

Using the normalization $g(0) = 1$ to integrate, we get

$$g(z) = \exp\left(\int_0^z \left(\frac{\mu(s)}{1 - \mu(s)} \cdot \frac{\varphi'(s)}{\varphi(s)}\right) ds\right) \tag{3.10}$$

In this way, we obtain the log-harmonic mapping f defined by

$$f(z) = zh(z)\overline{g(z)} = \frac{zh(z)}{g(z)}|g(z)|^2 = \varphi(z) \exp\left(2\,\mathrm{Re}\int_0^z \left(\frac{\mu(s)}{1-\mu(s)} \cdot \frac{\varphi'(s)}{\varphi(s)}\right) ds\right) \tag{3.11}$$

The following procedures can be applied to generate the univalent log-harmonic mappings f of the type $f(z) = zh(z)\overline{g(z)}$:

1. Choose an arbitrary $\varphi \in \mathcal{S}^*$ and an arbitrary analytic function $\mu : \mathbb{D} \to \mathbb{D}$ with $\mu(0) = 0$.

2. Establish the pair of equations given by (3.9).

3. Solving for $(\log g)'(z)$, and then integrating with normalization $g(0) = 1$ yields (3.10).

4. The desired univalent log-harmonic mapping $f \in \mathcal{S}_{Lh}$ is then given by (3.11).

A variety of univalent logharmonic mappings can be obtained by selecting different values for the conformal mapping φ and the dilatation μ.

Example 2. *([2,Example 2.2]). Consider a log-harmonic mapping in $\mathbb{D}$ such that $f_\alpha(z) = zh(z)\overline{g(z)} \in \mathcal{S}_{Lh}$, which satisfies the two equations.*

$$\varphi_\alpha(z) = \frac{zh(z)}{g(z)} = \frac{z}{(1-z)^{2(1-\alpha)}} \quad \text{and} \quad \mu(z) = \frac{zg'(z)/g(z)}{1 + zh'(z)/h(z)} = z \quad (3.12)$$

where $0 \leq \alpha < 1$. The nonlinear system that is produced

$$\begin{cases} z(\log h)'(z) - z(\log g)'(z) = \frac{2(1-\alpha)z}{1-z} \\ (\log g)'(z) - z(\log h)'(z) = 1 \end{cases}$$

possesses the answer

$$(\log g)'(z) = \frac{1 - z + 2(1-\alpha)z}{(1-z)^2}$$

This, through integration, provides

$$\log g(z) = 2(1-\alpha)\frac{z}{1-z} + (1-2\alpha)\log(1-z)$$

So, we've got

$$\begin{cases} g(z) = (1-z)^{1-2\alpha} \exp\left(2(1-\alpha)\frac{z}{1-z}\right) \\ h(z) = \frac{g(z)}{(1-z)^{2(1-\alpha)}} = \frac{1}{1-z} \exp\left(2(1-\alpha)\frac{z}{1-z}\right) \end{cases} \quad (3.13)$$

Consequently, the form of $f_\alpha(z)$ is

$$\begin{aligned} f_\alpha(z) &= \varphi_a(z)|g(z)|^2 \\ &= \frac{z}{(1-z)^{2(1-\alpha)}} \left|(1-z)^{1-2\alpha} \exp\left(2(1-\alpha)\frac{z}{1-z}\right)\right|^2 \end{aligned} \quad (3.14)$$

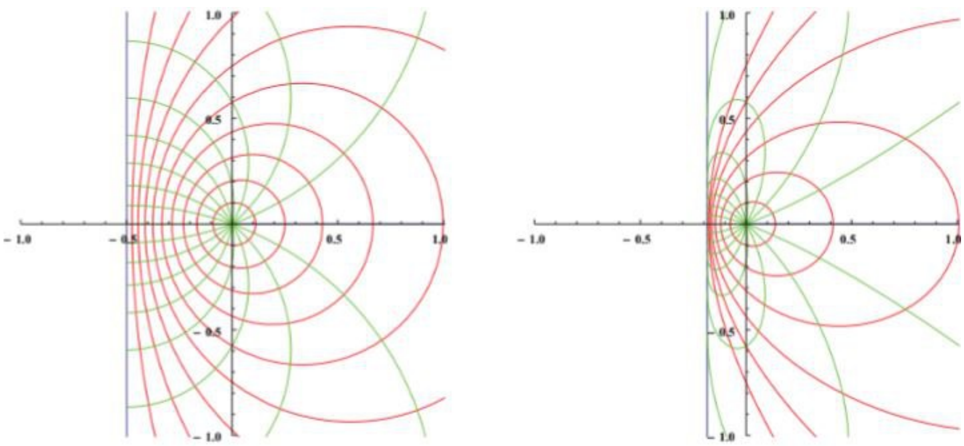
Observe that φ_α has an order of α stars. We also know that the log-harmonic mapping f_α in $\mathbb{D}$ is starlike of order α, according to Theorem 14. For more discussion see ([2, example 2.2]).

Example 3. *([2, Example 2.3]). (Log-harmonic two-slits mapping) We know that* $s(z) = z/\left(1 - z^2\right)$ *maps* $\mathbb{D}$ *onto* $\mathbb{C}\backslash\{u+iv : u = 0, |v| \geq 1/2\}$*. Let us now construct a log-harmonic two-slits mapping. Consider* $LS(z) = zh(z)\overline{g(z)} \in \mathcal{S}_{Lh}$ *with*

$$\varphi(z) = \frac{zh(z)}{g(z)} = s(z) = \frac{z}{1 - z^2} \qquad \text{and} \qquad \mu(z) = \frac{zg'(z)/g(z)}{1 + zh'(z)/h(z)} = z^2$$

Thus, as before, we have

$$z(\log h)'(z) - z(\log g)'(z) = \frac{2z^2}{1 - z^2} \qquad \text{and} \qquad (\log g)'(z) - z^2(\log h)'(z) = z$$



(a) The right half-plane mapping $\frac{z}{1-z}$ (b) The right half-plane log-harmonic mapping $f_{\frac{1}{2}}(z)$

Figure 4: Right half-plane mapping and log-harmonic right half-plane mapping images of $\mathbb{D}$

Finding the solution produces

$$(\log g)'(z) = \frac{z\left(1 + z^2\right)}{\left(1 - z^2\right)^2}$$

Integrating with the normalization $h(0) = g(0) = 1$, we arrive at

$$\begin{cases} g(z) = \sqrt{1 - z^2}\exp\left(\frac{z^2}{1-z^2}\right) = \exp\left(\sum_{n=1}^{\infty}\left(1 - \frac{1}{2n}\right)z^{2n}\right) \\ h(z) = \frac{g(z)}{1-z^2} = \frac{1}{\sqrt{1-z^2}}\exp\left(\frac{z^2}{1-z^2}\right) = \exp\left(\sum_{n=1}^{\infty}\left(1 + \frac{1}{2n}\right)z^{2n}\right) \end{cases}$$

and thus,

$$\mathrm{LS}(z)=\varphi(z)|g(z)|^{2}=\frac{z}{1-z^{2}}\left|1-z^{2}\right| \exp \left(\operatorname{Re}\left(\frac{2 z^{2}}{1-z^{2}}\right)\right) \tag{3.18}$$

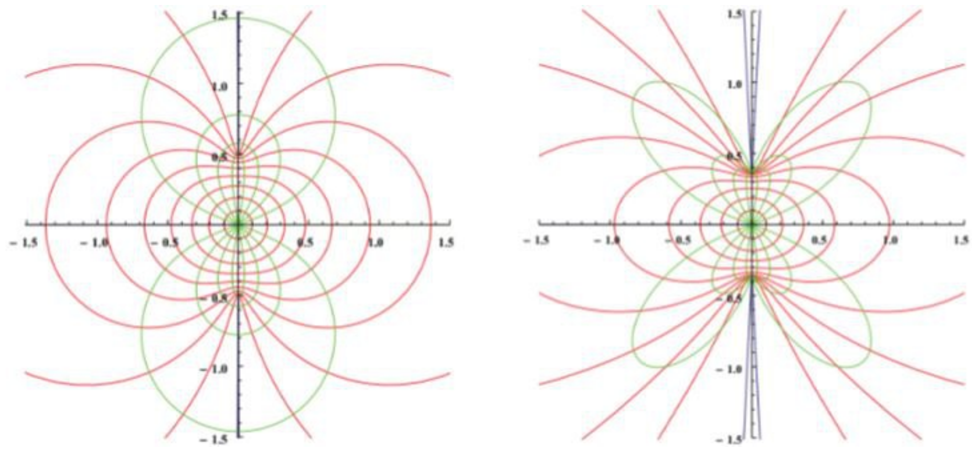
We know that $LS(z)$ is univalent and starlike in $\mathbb{D}$ according to Theorem 14. Specifically, we assert that $LS(z)$ maps $\mathbb{D}$ onto the two-slits plane $\mathbb{C} \backslash\{u+i v:|v| \geq 1 / e\}$. Figures 5(a) and 5(b) depict the pictures of $\mathbb{D}$ under $s(z)=z /\left(1-z^{2}\right)$ and the log-harmonic two-slits mapping $LS(z)$, respectively.

Set $z=e^{i \theta}(\theta \in(0, \pi) \cup(\pi, 2 \pi))$ into (3.18) to demonstrate that $LS(\mathbb{D})=\mathbb{C} \backslash\{u+i v:|v| \geq 1 / e, u=0\}$. We've got

$$LS\left(e^{i \theta}\right)=i \frac{\sin \theta}{|\sin \theta|} e^{-1}=\begin{cases} i / e & \text{for} \quad 0<\theta<\pi \\ -i / e & \text{for} \quad \pi<\theta<2 \pi\end{cases}$$

This demonstrates that, aside from the places where $z= \pm 1$, $LS(z)= \pm i / e$ on the unit circle. The desired assertion is stated in an argument similar to the analysis of Example 3, and the specifics are left out.

Remark. *Over the subclasses of $\mathcal{S}_{Lh}$, the two univalent log-harmonic mappings mentioned above serve as extremal functions for numerous extremal issues.*



(a) The two-slits function $\frac{z}{1-z^2}$ (b) The two-slits log-harmonic function $LS(z)$

Figure 5: $\mathbb{D}$ images underneath $LS(z)$ and $s(z)$

3.4 Growth and Distortion Theorem

In this section, we introduce the growth and distortion theorems for the classes of P_{Lh} and CST_{Lh} mappings.

Theorem 17. *([Theorem 5.8]). Let $R(z) = H(z)\overline{G(z)} \in P_{Lh}$, and suppose that $\mu(0) = 0$. Then for $z \in D$*
(i) $\exp(-2|z|/(1-|z|)) \le |R(z)| \le \exp(2|z|/(1-|z|))$,
(ii) $|R_z(z)| \le \left(2/(1-|z|)\left(1-z|^2\right)\right)\exp(2|z|/(1-|z|))$,
(iii) $|R_{\bar{z}}(z)| \le (2|z|/(1-|z|)(1-|z|^2))\exp(2|z|/(1-|z|))$.

For the right inequalities, equality is reached if R is a function of the type $R_0(\zeta z), |\zeta| = 1$, where

$$R_0(z) = \frac{1+z}{1-z}\left|\frac{1+z}{1-z}\right|\exp\left(\mathrm{Re}\,\frac{2z}{1-z}\right) \qquad (3.19)$$

and the left inequalities are equivalent if R has the following form

$$\frac{1}{R_0(\zeta z)}, \quad |\zeta| = 1 \qquad (3.20)$$

The radius of starlikeness and the radius of univalence for those mappings in the set CST_{Lh}[9] are found in the following result.

Theorem 18. *([Theorem 5.11]). In CST_{Lh}, let $F = z|z|^{2\beta}h\bar{g}$. The disc $|z| < R, where R \leq 2-\sqrt{3}$, is then mapped into a star-like domain by F. Best possible for all μ in $\mathcal{A}$ is the upper bound.*

Combining Theorems 19 and 20 with $\alpha = 0$, for the class CST_{Lh}, the distortion theorem that follows is obtained.

Theorem 19. *([Theorem 5.12]). Determine that $F = zh\bar{g} \in CST_{Lh}$. When $z \in D$, then,*
(a) $|z|\exp(-2|z|/(1-|z|) - 4|z|/(1+|z|)) \leq |F(z)| \leq |z|\exp(6|z|/(1-|z|))$,
(b) $|F_z(z)| \leq ((|z|^2+4|z|+1)/(1-|z|)^2(1+|z|))\exp(6|z|/(1-|z|))$,
(c) $|F_{\bar{z}}(z)| \leq (|z|(|z|^2+4|z|+1)/(1-|z|)^2(1+|z|))\exp(6|z|/(1-|z|))$.

Equality holds for the right inequalities if F is a function of the form

$$F_{\eta,\zeta}(z) = \frac{z(1-\overline{\eta z})}{(1-\eta z)}\frac{1+\zeta z}{1-\zeta z}\left|\frac{1-\zeta z}{1+\zeta z}\right|\exp\left(\mathrm{Re}\left[\frac{4\eta z}{1-\eta z}+\frac{2\zeta z}{1-\zeta z}\right]\right) \quad (3.21)$$

where F is a function of the kind $|\eta| = |\zeta| = 1$, and for the left inequalities

$$F_{\eta,\zeta}(z) = \frac{z(1-\overline{\eta z})}{(1-\eta z)}\frac{1+\zeta z}{1-\zeta z}\left|\frac{1-\zeta z}{1+\zeta z}\right|\exp\left(\mathrm{Re}\left[\frac{4\eta z}{1-\eta z}-\frac{2\zeta z}{1-\zeta z}\right]\right) \quad (3.22)$$

where $|\eta| = |\zeta| = 1$

Lemma 1. *([22, Corollary 3.6]) Given $p(0) = 1$, let $p(z)$ be analytic in $\mathbb{D}$. If and only if there is a probability measure δ on $\partial\mathbb{D}$ such that, then $\mathrm{Re}\, p(z) > 0$ in $\mathbb{D}$.*

$$p(z) = \int_{\partial\mathbb{D}} \frac{1+\eta z}{1-\eta z} d\delta(\eta), \quad z \in \mathbb{D}$$

given that every p has the form

$$p(z) = \frac{1+\mu(z)}{1-\mu(z)} = 1 + \frac{2\mu(z)}{1-\mu(z)}$$

regarding a $\mu \in \mathcal{B}$. $f(z) = zh(z)\overline{g(z)} \in CST_{LH}$ if and only if $\varphi = zh/g \in CST$, that is, as per Theorem 1.1.

$$p(z) = \frac{z\varphi'(z)}{\varphi(z)}$$

where p in $\mathbb{D}$ is analytic such that $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$ and $p(0) = 1$.
$s_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $s_2(z) = \sum_{n=0}^{\infty} b_n z^n$ are the variables to be considered. be functions that are analytic in $\mathbb{D}$. When $s_1(z) \prec s_2(z)$, or just $(s_1 \prec s_2)$, is written, we say that $s_1(z)$ is subordinate to $s_2(z)$ is subordinate to $s_2(z)$. if

$$s_1(z) = s_2(\omega(z))$$

given $\omega(0) = 0$, for some analytic function $\omega : \mathbb{D} \to \mathbb{D}$. According to the Schwarz lemma, $|\omega'(0)| \leq 1$ and $|\omega(z)| \leq |z|$ follow, ensuring that $|s_1'(0)| \leq |s_2'(0)|$.
For further information on subordination classes, see [16, p. 35] or [11, Chapter 6].
Assume a univalent mapping $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ log-harmonic. If f is a log-harmonic mapping that is close-to-starlike, then we say that

$$\frac{\partial \operatorname{Arg} f\left(re^{i\theta}\right)}{\partial \theta} = \operatorname{Re} \frac{zf_z - \bar{z}f_{\bar{z}}}{f} > 0 \tag{3.23}$$

for all $z \in \mathbb{D}$.
CST_{LH} represents the set of all close-to-starlike log-harmonic mappings.
Denote by S the family of functions $s(z) = z + c_2z^2 + c_3z^3 + \cdots$ regular in $\mathbb{D}$, such that $s(z)$ is in S if and only if

$$\operatorname{Re}\left(z\frac{s'(z)}{s(z)}\right) > 0 \Leftrightarrow z\frac{s'(z)}{s(z)} = p(z) \tag{3.24}$$

for some function $p(z) \in P$ and every $z \in \mathbb{D}$.
Let $s_1(z)$ and $s_2(z)$ be analytic functions in $\mathbb{D}$ with $s_1(0) = s_2(0)$. We say that $s_1(z)$ is subordinate to $s_2(z)$ and denote it by $s_1(z) \prec s_2(z)$ if $s_1(z) = s_2(\phi(z))$ for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$. If $s_2(z)$ is univalent, then $S_1(\mathbb{D}_r) \subset S_1(\mathbb{D}_r)$, $\mathbb{D}_r = z||z| < r, 0 < r < 1$ (subordination principle or Lindelof Principle [5]).

Lemma 2. *([6, Lemma 1.1]). In the unit disc $\mathbb{D}$, let $\phi(z)$ be regular, and let $\phi(0) = 0$. Therefore, for some $k \geq 1$, one has $z_1\phi'(z_1) = k\phi(z_1)$ if $|\phi(z)|$ reaches its highest value on the disc $|z| = r$ at the point z_1.*

Lemma 3. *([2, Lemma 3.1.]) or ([11, Theorem 6.4]) If* $s_1(z) \prec s_2(z)$*, where* $s_i(0) = 0$ *and* $s_i'(0) = 1 (i = 1, 2)$*, and*
(a) if $s_2 \in C$*, then* $|a_n| \leq 1$ *for* $n = 2, 3, \cdots$*;*
(b) if $s_2 \in \mathcal{S}^*$*, then* $|a_n| \leq n$ *for* $n = 2, 3, \cdots$*.*

Theorem 20. *([2, theorem 3.2]) Let* $f(z) = zh(z)\overline{g(z)}$ *belong to* $\mathcal{S}^*_{Lh}(\alpha)(0 \leq \alpha < 1)$*, where* $h(z)$ *and* $g(z)$ *are given by (2.1). Then for every* $n \geq 1$*,*

$$|a_n - b_n| \leq \frac{2(1-\alpha)}{n} \tag{3.25}$$

Moreover,
(a) $|a_n| \leq 2(1-\alpha) + \frac{1}{n}$;
(b) $|b_n| \leq 2(1-\alpha) + \frac{2\alpha-1}{n}$.
If $f(z) = f_\alpha(z)$ or one of its rotations, the equality is maintained; the value of $f_\alpha(z)$ is provided by (3.14).

Proof. $\mathcal{S}^*_{Lh}(\alpha)$ includes $f(z) = zh(z)\overline{g(z)}$. Next, we have

$$\alpha < \operatorname{Re}\left(\frac{zf_z(z) - \bar{z}f_{\bar{z}}(z)}{f(z)}\right) = \operatorname{Re}\left(1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}\right), \quad z \in \mathbb{D}$$

By (2.1) and Theorem 14, we obtain

$$1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} \prec \frac{1 + (1-2\alpha)z}{1-z}, \quad z \in \mathbb{D} \tag{3.26}$$

which is equivalent to

$$\frac{1}{2(1-\alpha)}\left(\frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}\right) = \sum_{n=1}^{\infty} \frac{n\,(a_n - b_n)}{2(1-\alpha)} z^n \prec \frac{z}{1-z'}, \quad z \in \mathbb{D}$$

Moreover, Lemma 1 states that $z/(1-z)$ is convex in $\mathbb{D}$.

$$\frac{n\,|a_n - b_n|}{2(1-\alpha)} \leq 1 \quad \text{for} \quad n \geq 1$$

and (3.24) follows.
[19, Theorem 2.6] provided the coefficient estimate inequalities (a) and (b) of Theorem 22. Ultimately, it becomes clear that the equalities are achieved through an appropriate rotation of $h(z)$ and $g(z)$, as indicated by (3.14). The proof is complete.

□

Theorem 21. *([2, theorem 3.3]) Let* $f(z) = zh(z)\overline{g(z)}$ *be a log-harmonic mapping in* $\mathbb{D}$*, where* $h(z)$ *and* $g(z)$ *are given by* (2.1)*, and satisfy the condition*

$$\sum_{n=1}^{\infty} n\,|a_n - b_n| \leq 1 - \alpha \tag{3.27}$$

for some $\alpha \in [0, 1)$. Then $f \in \mathcal{S}^*_{Lh}(\alpha)$.

Theorem 22. *([9, Theorem 3.7]) Let $F = z|z|^{2\beta} h\bar{g} \in CST_{Lh}$. Then we have*

$$\left|\arg \frac{F(z)}{z}\right| \leq 2 \arcsin |z| + \arcsin \frac{|z|}{(1 + |z|^2)} + 2\,\mathrm{Im}(\beta) \ln |z|.$$

There is equality if and only if

$$\psi(z) = \frac{zh}{g} = \frac{z(1 + \eta z)}{(1 - \eta z)^2}, |\eta| = 1$$

and

$$p(z) = \frac{1 + \zeta z}{1 - \zeta z}, |\zeta| = 1$$

Chapter 4

Main Results

4.1 Coefficients estimate for log-harmonic Close-to-Starlike Mappings

Our main results are stated in theorem 23 and theorem 24 as follows.

Theorem 23. *Let* $f(z) = zh(z)\overline{g(z)} \in \mathcal{CST}_{Lh}$, *then*

$$\left(1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right) \prec \frac{1+z}{1-z} \tag{4.1}$$

Proof. Let $f(z) = zh(z)\overline{g(z)} \in \mathcal{CST}_{Lh}$, then

$$0 < \mathrm{Re}\left(\frac{zf_z - \bar{z}f_{\bar{z}}}{f}\right) = \left(1 + z\frac{h'(z)}{h(z)} - \bar{z}\frac{\overline{g'(z)}}{g(z)}\right)$$

so, we have

$$0 < \mathrm{Re}\left(\frac{zf_z - \bar{z}f_{\bar{z}}}{f}\right) = \mathrm{Re}\left(1 + z\frac{h'(z)}{h(z)} - \bar{z}\frac{\overline{g'(z)}}{g(z)}\right) = \mathrm{Re}\left(1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right)$$
$$\Leftrightarrow 1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} = p(z) = \frac{1+\mu(z)}{1-\mu(z)} \Leftrightarrow 1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} = \frac{1+\mu(z)}{1-\mu(z)}$$
$$\Leftrightarrow 1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} \prec \frac{1+z}{1-z}$$

where $p(z) \in P, \mu(z) \in \Omega$. □

Theorem 24. *Let* $f(z) = zh(z)\overline{g(z)} \in \mathcal{CST}_{Lh}$*, where* $h(z)$ *and* $g(z)$ *are given by (2.1). Then for all* $n \geq 1$*,*

$$|a_n - b_n| \leq \frac{2}{n}.$$

Proof.

Let $f(z) \in \mathrm{CST}_{Lh}$. so that we have

$$0 < \mathrm{Re}\left(\frac{zf_z(z) - \bar{z}f_{\bar{z}}(z)}{f(z)}\right)$$
$$= \mathrm{Re}\left(1 + z\frac{h'(z)}{h(z} - z\frac{g'(z)}{g(z)}\right) \quad , z \in \mathbb{D}$$

By (2.6) and Theorem 23 we obtain

$$1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} \prec \frac{1+z}{1-z}, z \in \mathbb{D}$$
$$\Rightarrow 1 + z\frac{h'(z)}{h(z)} - \frac{zg'(z)}{g(z)} \prec 1 + \frac{2z}{1-z}, z \in \mathbb{D}$$
$$\Rightarrow z\frac{h'(z)}{h(z)} - \frac{zg'(z)}{g(z)} \prec \frac{2z}{1-z} \quad , z \in \mathbb{D}$$
$$\frac{1}{2}\left(\frac{zh'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right) \prec \frac{z}{1-z} \quad , z \in \mathbb{D} \tag{4.2}$$

from (2.6)

$$h(z) = \exp\left(\sum_{n=1}^{\infty} a_n z^n\right) \quad \text{and} \quad g(z) = \exp\left(\sum_{n=1}^{\infty} b_n z^n\right)$$

Then

$$z\frac{h'(z)}{h(z)} = z\frac{\sum_{n=1}^{\infty} na_n z^{n-1} \exp\left(\sum_{n=1}^{\infty} a_n z^n\right)}{\exp\left(\sum_{n=1}^{\infty} a_n z^n\right)}$$

$$= z\sum_{n=1}^{\infty} na_n z^{n-1}$$
$$= \sum_{n=1}^{\infty} na_n z^n \tag{4.3}$$

and

$$z\frac{g'(z)}{g(z)} = z\frac{\sum_{n=1}^{\infty} nb_n z^{n-1} \exp\left(\sum_{n=1}^{\dot\infty} b_n z^n\right)}{\exp\left(\sum_{n=1}^{\infty} b_n z^n\right)}$$

$$= z\sum_{n=1}^{\infty} nb_n z^{n-1}$$

$$= \sum_{n=1}^{\infty} nb_n z^n \tag{4.4}$$

Substitusting (4.3) and (4.4) in (4.2)

$$\Rightarrow \frac{1}{2}\left(\sum_{n=1}^{\infty} na_n z^n - \sum_{n=1}^{\infty} nb_n z^n\right) \prec \frac{z}{1-z}, z \in \mathbb{D}$$

$$\Rightarrow \frac{1}{2}\left(\sum_{n=1}^{\infty} n\left(a_n - b_n\right) z^n \prec \frac{z}{1-z}, z \in \mathbb{D}\right.$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n\left(a_n - b_n\right) z^n}{2} \prec \frac{z}{1-z}, z \in \mathbb{D}$$

Furthermore, by Lemma 3 we obtain that $\frac{z}{1-z}$ is convex in $\mathbb{D}$.

$$\frac{n\left|a_n - b_n\right|}{2} \leq 1 \text{ for all } n \geq 1$$

$$\Rightarrow n\left|a_n - b_n\right| \leq 2$$

$$\Rightarrow \left|a_n - b_n\right| \leq \frac{2}{n} \text{ for all } n \geq 1$$

□

4.2 Conclusion

This paper investigates the coefficients estimates for close-to-starlike log-harmonic mappings, a special class of univalent log-harmonic functions on the unit disk.The study presents key theorems that establish precise bounds on the coefficients of these mappings, offering a deeper understanding of their geometric properties and behavior.These results extend the existing knowledge in the field by providing new inequalities and laying the groundwork for further exploration of log-harmonic mappings. The findings underscore the importance of coefficient estimates in characterizing close-to-starlike log-harmonic mappings. The proposed Log-Harmonic Coefficient Conjecture suggests that the coefficients a_n and b_n should meet specific bounds, echoing the classical Bieberbach conjecture for this broader function class. Future research should focus on proving or refining these conjectures, exploring the full implications of these estimates, and extending them to other subclasses of log-harmonic mappings. This work opens avenues for deeper exploration into the structural properties and potential applications of log-harmonic functions in complex analysis.

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