CHIRAL PERTURBATION THEORY WITH APPLICATIONS TO SOME HADRONIC PROCESS

By

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Abstract

This project mainly describe the introduction to chiral perturbation theory for strong interaction at low energy level and deals with quantum chromodynamics and its global symmetry in the concept of chiral limit.

This project also describe the commutator and anti commutator of charge operators of quarks in hadron spectrum. Then we will construct effective Lagrangian for both global and local symmetry and it show effective Lagrangian is invariant under $Su(3)_L \times Su(3)_R$ symmetry transformation.Then we calculate currents from effective Lagrangian.

Finally, by discussing effective Lagrangian and weingers power counting scheme we will see application of quantum chromodynamics,especially pion decay.

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INTRODUCTION

Quantum Chromodynamics has been a very successful theory and is today widely accepted as the fundamental gauge theory underlying the so called strong interactions. These interactions govern the behavior of quarks, generally believed to be fundamental particles building up the composite particles called hadrons. Among these hadrons are the familiar proton and neutron which build up the nuclei of all the atoms that make up the matter we see around us in everyday life.

Historically however, Quantum Chromodynamics(QCD) has had its problems in gaining general acceptance as the theory of the strong interactions. Mainly this is due to the fact that it seemed impossible to isolate and detect the quarks themselves. This was later solved with the idea of confinement, but for a long time the existence of the quarks was hotly debated. Until the early 1970s it was common to talk about fictitious constituents, allowing for a simplified classification of the hadron spectrum, which could, however, not be interpreted as dynamical degrees of freedom in the context of quantum field theory. There still exists no analytical method for the description of QCD at low energies. How the hadrons are created from QCD dynamics is still insufficiently understood, and is one of the reasons why many phenomenological, more or less QCD-inspired models of hadrons are used.However, even though we still do not know how to make low-energy predictions from QCD itself, techniques have been developed to systematically explore the low-energy dynamics, based on certain symmetry properties of QCD. One method is called Chiral Perturbation Theory(chpt) and describes the dynamics of Gold-stone bosons in the framework of an effective field theory.

Chapter 1

CHIRAL PERTURBATION THEORY

Chiral perturbation theory(CHPT) is the effective field theory of quantum chromodynamics(QCD) at low energies. So it provide for discussing QCD at low energy by means of effective field theory. The basis of CHPT is the global $Su(3)_L \times Su(3)_R$ symmetry of the QCD Lagrangian. This symmetry of the QCD Lagrangian can be broken down spontaneously to $Su(3)_V$ symmetry with giving rise to eight massless Gladstone boson.

We will first consider the indication for a spontaneous break down of chiral symmetry in QCD and then discuss the transformation property of Gold-stone under the symmetry group of the Lagrangian and the ground state respectively. After introducing the lowest order effective Lagrangian that is important to the spontaneous break down from $Su(3)_L \times Su(3)_R$ to $Su(3)_V$. So effective Lagrangian is expressed interms of hadronic degree of freedom at low energy. At low energy level there are number of pseudo scalar octet(π, κ, η) which are called Gladstone bosons.

1.1 Remarks on SU(3)

Chiral perturbation theory provides a systematic framework for investigating strong interaction process at low energies. So, the group SU(3) plays an important roles in the context of strong interaction,
because **SU(3)** is

- a gauge group of QCD
- flavor Su(3) is approximately realized as a global symmetry of hadron spectrum and
- direct product of $Su(3)_L \times Su(3)_R$ is the chiral symmetry group of QCD for non vanishing u,d and s quark masses,So, it is better to see some properties of SU(3) and its lie group.

Therefore,the basic properties of $SU(3)$ and it's lie algebra should be given as follows. The group $Su(3)$ is defined as a set of unitary matrices U, that is U^+U=1 and det(U)=1. In mathematical terms, $SU(3)$ is an eight parameter simply connected,compacted lie group. This implice that any group element can be parametrized by a set of eight independent real parameters as,
Θ= $(\Theta_1, \Theta_2$............$\Theta_8)$, so elements of $Su(3)$ is conveniently written interms of exponential representation as,

$$U(\Theta) = \exp(-i \sum_{a=1}^{8} \Theta_a \frac{\lambda_a}{2}). \tag{1.1.1}$$

Where Θ_a is a real numbers and λ_a eight linearly independent matrices that we called Gell-mann matrices that satisfy the following relation,

$$\lambda_a = \lambda_a^+ \tag{1.1.2}$$

$$tr(\lambda_a \lambda_b) = 2\delta_{ab}. \tag{1.1.3}$$

So, we can represent the Gell mann matrices as follows,

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The structure of the lie group is enclosed in the commutator relation of this gell-mann matrices,which is given as follows,

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = \frac{\lambda_a}{2}\frac{\lambda_b}{2} - \frac{\lambda_b}{2}\frac{\lambda_a}{2} = i f_{abc}\frac{\lambda^c}{2}. \tag{1.1.4}$$

And from equation(1.14), we get

$$f_{abc} = \frac{1}{4i} tr([\lambda_a, \lambda_b]\lambda_c). \tag{1.1.5}$$

So, the anti- commutator relation also given as follows,

$$\{\lambda_a, \lambda_b\} = \frac{4}{3}\delta_{ab} + 2d_{abc}\lambda_c \tag{1.1.6}$$

$$d_{abc} = \frac{1}{4} tr(\{\lambda_a, \lambda_b\}\lambda_c). \tag{1.1.7}$$

From here, we conclude the anti-commutator of two Gell-mann matrices is not necessary a Gell-mann matrix.

1.2 The QCD Lagrangian

The gauge principle has proven to be a extremely successful in elementary particle physics to generate interaction between matter fields through the exchange of massless bosons. The best known example is quantum electrodynamics(QED). In QED the interaction between charged particle is mediated by the exchange of neutral gauge bosons called photon. Because of the neutrality of the photon,there don't exist vertices where a photon interacts directly with another photon.

Therefore,in QED only single vertex is required,the coupling of the photon to a fermion. The coupling constant is e in QED that is related to the fine structure constant through $\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$ and because of the smallness of α,the theory can be successfully treated perturbatively.

So as we know Lagrangian in QED is given by,

L_{QED}=$\bar{\psi}(i\gamma^{\mu}D_{\mu} - m)\psi$ - $\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

where, $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ and the covariant derivative of ψ is given by,

$$D_{\mu}\psi = (\partial_{\mu} - ieA_{\mu})\psi.$$

Therefore, the remarkable success of QED leads quite naturally to anon abelian generalization involving a triplet of color-charges interacting through the exchange of color gauge bosons called gluon. This is the theory of QCD,which is the so called quarks which are spin $\frac{1}{2}$ fermions with six different flavors in addition to their three possible colors.

Let us now concentrate on the flavor sector,there are six quark flavors in the spectrum, which are often divided into two parts.

These are light quarks(u,d and s) and heavy quarks(c,b and t).

For quarks masses,

$m_u, m_d, m_s << 1Gev < m_c, m_b, m_t$, where the scale 1Gev called hadron scale, is the natural value which associated with masses of hadrons containing the lightest quarks.

Therefore, in low energy region only light quarks can be taken into account,

So the QCD Lagrangian can be given as,

$$L_{QCD} = \sum_{f=u,d,s} \bar{q}_f(i\gamma^{\mu}D_{\mu} - m_f)q_f - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}. \qquad (1.2.1)$$

Where, q_f,m_f and $G_{\mu\nu}$ are quark field, quark mass and the gluon field tensor,respectively. For each quark flavor f,the quark field q_f consists a color triplet (r, g, b) ,which transform under gauge transformation $g(x)$ described by a set of parameters, $\Theta(x)$=[$\Theta_1(x) + ...\Theta_8(x)$,so the quark field can be given as,

$$q_f = \begin{pmatrix} q_{f,r} \\ q_{f,g} \\ q_{f,b} \end{pmatrix}. \tag{1.2.2}$$

The quark field can be transformed as,

$$q_f \longrightarrow q'_f = exp\Big[-i\sum_{a=1}^{8}\Theta_a(x)\frac{\lambda_a^c}{2}\Big]q_f = U[g(x)]. \tag{1.2.3}$$

The covariant derivatives of q_f is given by,

$$D_\mu q_f = (\partial\mu - ig_3 A_U)q_f.$$

$$D_\mu \begin{pmatrix} q_{f,r} \\ q_{f,g} \\ q_{f,b} \end{pmatrix} = \partial\mu \begin{pmatrix} q_{f,r} \\ q_{f,g} \\ q_{f,b} \end{pmatrix} - ig\sum_{a=1}^{8}\frac{\lambda_a^c}{2}A_{\mu,a}\begin{pmatrix} q_{f,r} \\ q_{f,g} \\ q_{f,b} \end{pmatrix}. \tag{1.2.4}$$

And the field tensor strength transform as,

$$G_{\mu\nu} = G_{a\mu\nu}\frac{\lambda_a}{2} = \partial\mu A_\nu - \partial\nu A_\mu + ig_3[A_\mu, A_\nu]. \tag{1.2.5}$$

We can express $G_{a\mu\nu}$ as follows,

$$G_{a\mu\nu}\frac{\lambda_a}{2} = \partial\mu A_\nu - \partial\nu A_\mu + ig_3\Big[A_\mu, A_\nu\Big]$$
$$= \partial\mu A_{a\nu}\frac{\lambda_a}{2} - \partial\nu A_{a\mu}\frac{\lambda_a}{2} + ig_3\Big[A_{b\mu}\frac{\lambda_b}{2}, A_{c\nu}\frac{\lambda_c}{2}\Big]$$
$$= \partial\mu A_{a\nu}\frac{\lambda_a}{2} - \partial\nu A_{a\mu}\frac{\lambda_a}{2} + ig_3 A_{b\mu}A_{c\nu}\Big[\frac{\lambda_b}{2}, \frac{\lambda_c}{2}\Big]$$
$$= (\partial\mu A_{a\nu} - \partial\nu A_{a\mu})\frac{\lambda_a}{2} + ig_3 A_{b\mu}A_{c\nu}if_{abc}\frac{\lambda_a}{2}$$
$$G_{a\mu\nu}\frac{\lambda_a}{2} = \Big[\partial\mu A_{a\nu} - \partial\nu A_{a\mu} + ig_3 A_{b\mu}A_{c\nu}if_{abc}\Big]\frac{\lambda_a}{2}$$
$$G_{a\mu\nu} = \Big[\partial\mu A_{a\nu} - \partial\nu A_{a\mu} - g_3 f_{abc}A_{b\mu}A_{c\nu}\Big]. \tag{1.2.6}$$

So we can put the field strength tensors as follows,

$$G_{\mu\nu} = \left[\partial\mu A_{a\nu} - \partial\nu A_{a\mu} - g_3 f_{abc} A_{b\mu} A_{c\nu}\right]\frac{\lambda_a}{2}. \tag{1.2.7}$$

where f_{abc} is the $su(3)$ structure constants and A_μ is the gluon field.

we will approximate the full QCD Lagrangian in equation (1.2.1), by considering the the concept of chiral limit, so the Lagrangian L^0_{QCD} is given by,

$$L^0_{QCD} = \sum_{l=u,d,s} \bar{q}_l i\gamma^\mu D_\mu q_l - \frac{1}{4} G^{\mu\nu} G^{\mu\nu}. \tag{1.2.8}$$

because under chiral limit m_u,m_d and $m_s \longrightarrow 0$, this implice that $m_f = 0$.

1.3 Left and right handed quark fields

In order to fully exhibit the global symmetry of the above QCD Lagrangian, we should consider the chirality matrix,

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5^+. \tag{1.3.1}$$

$$\{\gamma^\mu, \gamma_5\} = 0, \gamma_5^2 = 1. \tag{1.3.2}$$

We should introduce projection operators as follows,

$$P_L = \frac{1}{2}(1 - \gamma_5) = P_L^+. \tag{1.3.3}$$

$$P_R = \frac{1}{2}(1 + \gamma_5) = P_R^+. \tag{1.3.4}$$

The properties of the projection operators can be given as,

$$P_L + P_R = \frac{1}{2}(1 - \gamma_5) + \frac{1}{2}(1 + \gamma_5) = \frac{1}{2} + \frac{1}{2} = 1. \tag{1.3.5}$$

$$P_L^2 = [\frac{1}{2}(1 - \gamma_5)]^2 = \frac{1}{4}(1 - 2\gamma_5 + \gamma_5^2) = \frac{1}{2}(1 - \gamma_5) = P_L. \tag{1.3.6}$$

Similarly,

$$P_R^2 = P_R \tag{1.3.7}$$

And

$$P_L P_R = P_R P_L = \frac{1}{2}(1-\gamma_5)\frac{1}{2}(1-\gamma_5) = \frac{1}{4}(1-\gamma_5^2) = 0. \tag{1.3.8}$$

So, the left and right-handed quark fields can be expressed as,

$$q_L = P_L q. \tag{1.3.9}$$

$$q_R = P_R q. \tag{1.3.10}$$

And its $\bar{q_R}$ and $\bar{q_L}$ will be

$$\bar{q_R} = q_R^+\gamma^0 = (P_R q)^+\gamma^0 = q^+P_R^+\gamma^0 = q^+P_R\gamma^0 = q^+\gamma^0 P_L = \bar{q}P_L. \tag{1.3.11}$$

$$\bar{q_L} = q_L^+\gamma^0 = (P_L q)^+\gamma^0 = q^+P_L^+\gamma^0 = q^+P_L\gamma^0 = q^+\gamma^0 P_R = \bar{q}P_R. \tag{1.3.12}$$

Therefore, the QCD Lagrangian in the chiral limit,

$$L_{QCD}^0 = \sum_{l=u,d,s} (\bar{q}_{L,l} i\gamma^\mu D_\mu q_{L,l} + \bar{q}_{R,l} i\gamma^\mu D_\mu q_{R,l}) - \frac{1}{4}G^{\mu\nu}G^{\mu\nu}. \tag{1.3.13}$$

QCD Lagrangian in chiral limit is invariant under(covariant derivatives of flavor independent),

$$\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \to U_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} = \exp(-i\sum_{a=1}^{8}\Theta_a^L\frac{\lambda_a^f}{2})e^{-i\Theta^L}\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \tag{1.3.14}$$

$$\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \to U_R \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} = \exp(-i\sum_{a=1}^{8}\Theta_a^R\frac{\lambda_a^f}{2})e^{-i\Theta^R}\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \tag{1.3.15}$$

where U_L and U_R are independent unitary 3×3 matrices and the superscript f denotes Gell-mann matrices acting in the flavor space.

1.4 Global symmetry of currents of light quark sector

consider infinitesimal, local transformation as follows,from equation 1.3.14,

$$q_L \rightarrow \left[1 - \exp(-i\sum_{a=1}^{8}\Theta_a^L\frac{\lambda_a}{2} - i\Theta_a^L\right]q_L \tag{1.4.1}$$

and similarly,

$$q_R \rightarrow \left[1 - \exp(-i\sum_{a=1}^{8}\Theta_a^R\frac{\lambda_a}{2} - i\Theta_a^R\right]q_R. \tag{1.4.2}$$

Then by using the above two equation,we obtain

$$\delta L_{QCD}^0 = \bar{q_L}\left[\sum_{a=1}^{8}\partial_\mu\Theta_a^L\frac{\lambda_a}{2} + \partial_\mu\Theta_a^L\right]\gamma^\mu q_L + \bar{q_R}\left[\sum_{a=1}^{8}\partial_\mu\Theta_a^R\frac{\lambda_a}{2} + \partial_\mu\Theta_a^R\right]\gamma^\mu q_R. \tag{1.4.3}$$

From this equation we can calculate the currents as follows,

$$L_a^U = \frac{\partial(\delta L_{QCD}^0)}{\partial(\partial_\mu\Theta_a^L)} = \bar{q_L}\gamma^\mu\frac{\lambda_a}{2}q_L \tag{1.4.4}$$

$$\partial_\mu L_a^u = \frac{\partial(\delta L_{QCD}^0)}{\partial(\Theta_a^L)} = 0, \tag{1.4.5}$$

and

$$L^u = \frac{\partial(\delta L_{QCD}^0)}{\partial(\partial_\mu\Theta^L)} = \bar{q_L}\gamma^\mu q_L \tag{1.4.6}$$

$$\partial_\mu L^u = \frac{\partial(\delta L_{QCD}^0)}{\partial(\Theta^L)} = 0. \tag{1.4.7}$$

By using analogous expression for R_a^u and R^u,we obtain,

$$R_a^U = \bar{q_R}\gamma^\mu\frac{\lambda_a}{2}q_R. \tag{1.4.8}$$

$$R^u = \bar{q_R}\gamma^\mu q_R. \tag{1.4.9}$$

So the linear combination of L_a^U and R_a^U, lead us the vector and axial vector currents (from transformation of all left-handed and right-handed quark fields by the same phase) is given by

$$V_a^u = R_a^U + L_a^u = \bar{q}\gamma^\mu\frac{\lambda_a}{2}q. \tag{1.4.10}$$

Axial-vector current (from transformation of all left-handed quark fields with one phase and all right-handed with the opposite phase) is given by

$$A_a^u = R_a^U - L_a^u = \bar{q}\gamma^\mu\gamma_5\frac{\lambda_a}{2}q. \tag{1.4.11}$$

since

$$P_L\gamma^\mu P_R \pm P_R\gamma^\mu P_L = \gamma^\mu(P_R^2 \pm P_L^2) = \gamma^\mu(P_R \pm P_L) = \begin{cases} \gamma^\mu. \\ \gamma^\mu\gamma_5. \end{cases} \tag{1.4.12}$$

In the limit of massless quarks, the currents L_a^u and R_a^u or, alternatively, V_a^u and A_a^u are conserved.

Chapter 2

The Hadron Spectrum

QCD Lagrangian posses a $Su(3)_L \times Su(3)_R$ symmetry group under chiral limit. Chiral limit is nothing, but the light quark mass should be vanishing. We have argued that strong interaction exhibit a global chiral symmetry $Su(N_f) \times Su(N_f)_R$. However, this symmetry cannot be realized in the usual manner, for then the spectrum should also exhibit this symmetry. Consider the Hamiltonian H_o symmetric under the chiral group. This Hamiltonian in particular commutes with the generators of the axial transformation,

$$[\hat{H}_0, q_A^a] = 0. \tag{2.0.1}$$

The linear combination of left and right handed charges can be related as follows,

$$q_V^a = q_R^a + q_L^a \tag{2.0.2}$$

$$q_A^a = q_R^a - q_L^a. \tag{2.0.3}$$

Where charge operators define as space integral of charge density which is given by,

$$q_L(t) = \int d^3x q_L^+(t,x) \frac{\lambda_a}{2} q_L(t,x). \tag{2.0.4}$$

$$q_R(t) = \int d^3x q_R^+(t,x) \frac{\lambda_a}{2} q_R(t,x). \tag{2.0.5}$$

$$q_V(t) = \int d^3x [q_L^+(t,x) \frac{\lambda_a}{2} q_L(t,x) + q_R^+(t,x) \frac{\lambda_a}{2} q_R(t,x)]. \tag{2.0.6}$$

What are the commutation relations among the charges?

The commutation relation of charges can be expressed as,

$$[q_L^a, q_L^b] = i f_{abc} q_L^c. \tag{2.0.7}$$

$$[q_R^a, q_R^b] = i f_{abc} q_R^c. \tag{2.0.8}$$

$$[q_L^a, q_R^b] = [q_L^a, q_V] = [q_R^a, q_V] = 0. \tag{2.0.9}$$

Let us verify these commutation relations as follows by using the anti-commutation of fermion fields

$$\{q(t,x), q'^{+}(t,y)\} = \delta^3(x-y). \tag{2.0.10}$$

$$\{q(t,x), q'(t,y)\} = 0. \tag{2.0.11}$$

$$\{q^{+}(t,x)\}, q'^{+}(t,y)\} = 0. \tag{2.0.12}$$

By using this expression ,we can show the above commutation relation as follows,

$$[q_L^a, q_L^b] = \int d^3x d^3y \Big[q^{+}(t,x) P_L^{+} \frac{\lambda_a}{2} P_L q(t,x), q^{+}(t,y) P_L^{+} \frac{\lambda_a}{2} P_L q(t,y)\Big]. \tag{2.0.13}$$

We can simplify the above commutation by using the following relation,

$[ab, cd]$ $=a\{b,c\}d$-$ac\{b,d\}$+ $\{a,c\}db$ - $c\{a,d\}b$.

Therefore,

$$\Big[q^{+}(t,x) P_L^{+} \frac{\lambda_a}{2} P_L q(t,x), q^{+}(t,y) P_L^{+} \frac{\lambda_a}{2} P_L q(t,y)\Big] =$$
$$\int d^3x d^3y \delta^3(x-y) q^{+}(t,x) P_L^{+} P_L P_L^{+} P_L \frac{\lambda_a}{2}\frac{\lambda_b}{2} q(t,y) -$$
$$\int d^3x d^3y \delta^3(x-y) q^{+}(t,y) P_L^{+} P_L P_L^{+} P_L \frac{\lambda_b}{2}\frac{\lambda_a}{2} q(t,x) =$$
$$\int d^3x q^{+}(t,x) P_L (\frac{\lambda_a}{2}\frac{\lambda_b}{2} - \frac{\lambda_b}{2}\frac{\lambda_a}{2}) q(t,x) = i f_{abc} \int d^3 q^{+}(t,x) P_L \frac{\lambda_c}{2} q(t,x)$$

finally we get,

$$[q_L^a, q_L^b] = i f_{abc} \int d^3 q^{+}(t,x) P_L \frac{\lambda_c}{2} q(t,x) = i f_{abc} q_L^c. \tag{2.0.14}$$

When we show equation (2.0.14), we use the following relation,

$q^{+}(t,x) q^{+}(t,y) \{q(t,x), q(t,y)\}$=0

$\{q^{+}(t,x), q^{+}(t,y)\}q(t,y)q(t,x)$=0

$P_L^{+}P_LP_L^{+}P_L = P_L$

$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = \frac{\lambda_a}{2}\frac{\lambda_b}{2} - \frac{\lambda_b}{2}\frac{\lambda_a}{2} = if_{abc}\frac{\lambda^c}{2}.$

Similarly we can show the commutation relation of equation $[q_R^a, q_R^b] = if_{abc}q_R^c$ as follows,

$$[q_R^a, q_R^b] = \int d^3xd^3y\Big[q^{+}(t,x)P_R^{+}\frac{\lambda_a}{2}P_Rq(t,x), q^{+}(t,y)P_R^{+}\frac{\lambda_a}{2}P_Rq(t,y)\Big].$$

We can simplify the above commutation by using the following relation,

$[ab, cd]$ =$a\{b,c\}d$-$ac\{b,d\}$+ $\{a,c\}db$ - $c\{a,d\}b$.

Therefore,

$$\Big[q^{+}(t,x)P_R^{+}\frac{\lambda_a}{2}P_Rq(t,x), q^{+}(t,y)P_R^{+}\frac{\lambda_a}{2}P_Rq(t,y)\Big] =$$

$$\int d^3xd^3y\delta^3(x-y)q^{+}(t,x)P_R^{+}P_RP_R^{+}P_R\frac{\lambda_a}{2}\frac{\lambda_b}{2}q(t,y)-$$

$$\int d^3xd^3y\delta^3(x-y)q^{+}(t,y)P_R^{+}P_RP_R^{+}P_R\frac{\lambda_b}{2}\frac{\lambda_a}{2}q(t,x) =$$

$$\int d^3xq^{+}(t,x)P_R(\frac{\lambda_a}{2}\frac{\lambda_b}{2} - \frac{\lambda_b}{2}\frac{\lambda_a}{2})q(t,x) = if_{abc}\int d^3q^{+}(t,x)P_R\frac{\lambda_c}{2}q(t,x)$$

finally we get,

$$[q_R^a, q_R^b] = if_{abc}\int d^3q^{+}(t,x)P_R\frac{\lambda_c}{2}q(t,x) = if_{abc}q_R^c. \quad (2.0.15)$$

In the above equation we used,

$q^{+}(t,x)q^{+}(t,y)\{q(t,x), q(t,y)\}$=0

$\{q^{+}(t,x), q^{+}(t,y)\}q(t,y)q(t,x)$=0

$P_L^{+}P_LP_L^{+}P_L = P_L$

$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = \frac{\lambda_a}{2}\frac{\lambda_b}{2} - \frac{\lambda_b}{2}\frac{\lambda_a}{2} = if_{abc}\frac{\lambda^c}{2}.$

Now let us consider the vector charges q_V^a=$q_R^a + q_L^a$, that satisfy the commutation relation of $Su(3)$ lie algebra,

$$[q_V^a, q_V^b] = [q_R^a + q_L^a, q_R^b + q_L^b] = [q_R^a, q_R^b] + [q_L^a, q_L^b] =$$

$$if_{abc}q_R^c + if_{abc}q_L^c = if_{abc}(q_R^c + q_L^c) = if_{abc}q_V^c$$

therefore,

$$[q_V^a, q_V^b] = i f_{abc} q_V^c. \tag{2.0.16}$$

Similarly ,we can do for the axial vector given by q_A^a=$q_R^a - q_L^a$,that satisfy commutation relation as follow,

$$[q_A^a, q_A^b] = [q_R^a - q_L^a, q_R^b - q_L^b] = [q_R^a, q_R^b] + [q_L^a, q_L^b] =$$

$$i f_{abc} q_R^c + i f_{abc} q_L^c = i f_{abc}(q_R^c + q_L^c) = i f_{abc} q_V^c$$

because $q_R^c + q_L^c$=q_V^c, in short we can put

$$[q_A^a, q_A^b] = i f_{abc} q_V^c. \tag{2.0.17}$$

And we can evaluate the commutation of the axial vector charge and the vector charge as follows,

$$[q_V^a, q_A^b] = [q_R^a + q_L^a, q_R^b - q_L^b] = [q_R^a, q_R^b] - [q_L^a, q_l^b] =$$

$$i f_{abc} q_R^c - i f_{abc} q_L^c = i f_{abc}(q_R^c - q_L^c) = i f_{abc} q_A^c$$

because $q_R^c - q_L^c$=q_A^c, in short we can put

$$[q_V^a, q_A^b] = i f_{abc} q_A^c. \tag{2.0.18}$$

From the above we conclude that the charge operators do not form the closed algebra,that is the commutator of two axial charge operator is not again an axial charge operator as we have seen in equation(2.0.17).

Chapter 3

Lowest-order Effective Lagrangian

3.1 Equation of Lowest-order Effective Lagrangian

When we construct an effective low energy theory with a global symmetry of QCD the following condition be satisfied,

•The effective Lagrangian should be invariant under global $Su(3)_L \times Su(3)_R$.

•Assume spontaneous chiral symmetry braking such that the diagonal sub-group $Su(3)_V$ is unbroken, so the broken generators q_A^a imply the existence of eight Gold stone boson.

If the matrix U contains the Gold-stone boson field, so it is matrix representation will be

$$U(x) = e^{i\frac{\phi(x)}{f_0}} \tag{3.1.1}$$

where

$$\phi(x) = \sum_{a=1}^{8} \lambda_a \phi_a(x) = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}} & \sqrt{2}\pi^+ & \sqrt{2}\kappa^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\kappa^0 \\ \sqrt{2}\kappa^- & \sqrt{2}\kappa^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix} \tag{3.1.2}$$

and f_o is pion decay constant.

So, effective Lagrangian L_{eff} is given by,

$$L_{eff} = \frac{F_0^2}{4}(\partial_\mu U \partial_\mu U^+). \tag{3.1.3}$$

So we can show the the effective Lagrangian is invariant under the global $Su(3)_L \times Su(3)_R$ transformation as follows,

$$U \longrightarrow RUL^{+} \tag{3.1.4}$$

$$\partial_\mu U \longrightarrow \partial_\mu (RUL^{+}) = \partial_\mu RUL^{+} + RU\partial_\mu L^{+} + R\partial_\mu UL^{+}$$

but, $\partial_\mu R$ and $\partial_\mu L^{+} = 0$,finally we get

$$\partial_\mu U \longrightarrow R\partial_\mu UL^{+} \tag{3.1.5}$$

$$U^{+} \longrightarrow (RUL^{+})^{+} = LU^{+}R^{+} \tag{3.1.6}$$

$$\partial_\mu U^{+} = \partial_\mu (LU^{+}R^{+}) = \partial_\mu LU^{+}R^{+} + L\partial_\mu U^{+}R^{+} + LU^{+}\partial_\mu R^{+} = L\partial_\mu U^{+}R^{+}$$

but, $\partial_\mu L$ and $\partial_\mu R^{+} = 0$,finally we get

$$\partial_\mu U^{+} = L\partial_\mu U^{+}R^{+}U \longrightarrow RUL^{+} \tag{3.1.7}$$

so, we can show L_{eff} is invariant under the global $Su(3)_L \times Su(3)_R$ transformation,

$$L_{eff} \longrightarrow \frac{F_0^2}{4} tr(R\partial_\mu UL^{+}L\partial_\mu U^{+}R^{+}) = \frac{F_0^2}{4} tr(RR^{+}\partial_\mu U\partial_\mu U^{+}) = \frac{F_0^2}{4} tr(\partial_\mu U\partial_\mu U^{+}) = L_{eff}. \tag{3.1.8}$$

Since trace property Tr(AB) = Tr(BA). So, the Lagrangian is invariant under the global $Su(3)_L \times Su(3)_R$ transformation.

We can discuss the vector and axial vector currents associated with the global $Su(3)_L \times Su(3)_R$ symmetry of the effective Lagrangian as follows,we can parametrize infinitesimal transformation as,

$$L = 1 - i\Theta_a^L \frac{\lambda_a}{2} \tag{3.1.9}$$

$$R = 1 - i\Theta_a^R \frac{\lambda_a}{2} \tag{3.1.10}$$

To construct the left current (J^{μ}_{L}), set $\Theta^R_a = 0$ and choose $\Theta^L_a(x)$,

$$U \longrightarrow U' = RUL^+ = (1 - i\Theta^R_a \frac{\lambda_a}{2})U(1 - i\Theta^L_a \frac{\lambda_a}{2}) = U(1 + i\Theta^L_a \frac{\lambda_a}{2}) \tag{3.1.11}$$

$$U^+ \longrightarrow U'^+ = (1 - i\Theta^L_a \frac{\lambda_a}{2})U^+ \tag{3.1.12}$$

$$\partial_\mu U \longrightarrow \partial_\mu U' = \partial_\mu U(1 + i\Theta^L_a \frac{\lambda_a}{2}) + Ui\partial_\mu \Theta^L_a \frac{\lambda_a}{2} \tag{3.1.13}$$

$$\partial_\mu U^+ \longrightarrow \partial_\mu U'^+ = (1 - i\Theta^L_a \frac{\lambda_a}{2})\partial_\mu U^+ - i\partial_\mu \Theta^L_a \frac{\lambda_a}{2} U^+ \tag{3.1.14}$$

from the above we obtain for δL_{eff};

$$\delta L_{eff} = \frac{F_0^2}{4} tr\left[Ui\partial_\mu \Theta^L_a \frac{\lambda_a}{2}\partial_\mu U^+ + \partial_\mu U(-i\partial_\mu \Theta^L_a \frac{\lambda_a}{2} U^+)\right] = \frac{F_0^2}{4} i\partial_\mu \Theta^L_a tr\left[\frac{\lambda_a}{2}(\partial_\mu U^+ U - U^+ \partial_\mu U)\right]$$

$$\delta L_{eff} = \frac{F_0^2}{4} i\partial_\mu \Theta^L_a tr(\lambda_a \partial_\mu U^+ U). \tag{3.1.15}$$

Since,$U^+U = 1 \longrightarrow \partial_\mu(U^+U) = \partial_\mu(1) = 0 \longrightarrow \partial_\mu U^+ U = -U^+ \partial_\mu U$

Then we can calculate the left, right,vector and axial vector currents as follows,

The left is current given by,

$$J^U_L = \frac{\partial \delta L_{eff}}{\partial(\partial_\mu \Theta^L_a)} = i\frac{F_0^2}{4} tr(\lambda_a \partial_\mu U^+ U). \tag{3.1.16}$$

To construct the right current (J^{μ}_{R}), set $\Theta^L_a = 0$ and choose $\Theta^R_a(x)$,

$$U \longrightarrow U' = RUL^+ = (1 - i\Theta^R_a \frac{\lambda_a}{2})U(1 + i\Theta^L_a \frac{\lambda_a}{2}) = (1 - i\Theta^R_a \frac{\lambda_a}{2})U \tag{3.1.17}$$

$$U^+ \longrightarrow U'^+ = U^+(1 + i\Theta^R_a \frac{\lambda_a}{2}) \tag{3.1.18}$$

$$\partial_\mu U \longrightarrow \partial_\mu U' = (1 - i\Theta^R_a \frac{\lambda_a}{2})\partial_\mu U - iU\partial_\mu \Theta^R_a \frac{\lambda_a}{2} \tag{3.1.19}$$

$$\partial_\mu U^+ \longrightarrow \partial_\mu U'^+ = \partial_\mu U^+(1 + i\Theta_a^R \frac{\lambda_a}{2}) + iU^+\partial_\mu \Theta_a^R \frac{\lambda_a}{2}. \tag{3.1.20}$$

From the above we obtain for δL_{eff};

$$\delta L_{eff} = \frac{F_0^2}{4} tr[\partial_\mu U \partial\mu U^+] = \frac{F_0^2}{4} tr\Bigg[- Ui\partial_\mu \Theta_a^R \frac{\lambda_a}{2} \partial_\mu U^+ + \partial_\mu U(iU^+\partial_\mu \Theta_a^R \frac{\lambda_a}{2})\Bigg]$$

$$= \frac{F_0^2}{4} i\partial_\mu \Theta_a^R tr\Bigg[- U\frac{\lambda_a}{2}\partial_\mu U^+ + \partial_\mu U U^+ \frac{\lambda_a}{2}\Bigg]$$

$$= \frac{F_0^2}{4} i\partial_\mu \Theta_a^R tr\Bigg[\frac{\lambda_a}{2}(-U\partial_\mu U^+ + \partial_\mu U U^+)\Bigg]$$

$$= \frac{F_0^2}{4} i\partial_\mu \Theta_a^R tr\Bigg[2\frac{\lambda_a}{2}(\partial_\mu U U^+)\Bigg]$$

$$\delta L_{eff} = \frac{F_0^2}{4} i\partial_\mu \Theta_a^L tr(\lambda_a \partial_\mu U^+ U) = -i\frac{F_0^2}{4}\partial_\mu \Theta_a^L tr(\lambda_a U \partial_\mu U^+). \tag{3.1.21}$$

Since,$UU^+ = 1 \longrightarrow \partial_\mu(UU^+) = \partial_\mu(1) = 0 \longrightarrow \partial_\mu U U^+ = -U\partial_\mu U^+$

$$J_R^U = \frac{\partial \delta L_{eff}}{\partial(\partial_\mu \Theta_a^R)} = -i\frac{F_0^2}{4} tr(\lambda_a U \partial_\mu U^+). \tag{3.1.22}$$

Then we calculate the vector and the axial vector currents as follows,

The vector current is given by,

$$J_V^U = J_R^U + J_L^U = -i\frac{F_0^2}{4}\Bigg[tr(\lambda_a U\partial_\mu U^+) - tr(\lambda_a \partial_\mu U^+ U)\Bigg].$$

$$J_V^U = -i\frac{F_0^2}{4}\Bigg[tr(\lambda_a(U\partial_\mu U^+ - \partial_\mu U^+ U))\Bigg] = -i\frac{F_0^2}{4} tr(\lambda_a [U, \partial_\mu U^+]). \tag{3.1.23}$$

since, $U\partial_\mu U^+ - \partial_\mu U^+ U = [U, \partial_\mu U^+]$

similarly, the axial vector current can be calculated,

$$J_A^U = J_R^U - J_L^U = -i\frac{F_0^2}{4}\Bigg[tr(\lambda_a U\partial_\mu U^+) + tr(\lambda_a \partial_\mu U^+ U)\Bigg]$$

$$J_A^U = -i\frac{F_0^2}{4} tr(\lambda_a(\{U, \partial_\mu U^+\})). \tag{3.1.24}$$

since, $U\partial_\mu U^+ + \partial_\mu U^+ U = \{U, \partial_\mu U^+\}$.

L_{eff} is invariant under $Su(3)_L \times SU(3)_R$ transformation. This implice that both left and right currents are conserved. This also implice that the vector and the axial vector currents are also conserved.

The quark mass components of the QCD Lagrangian can be written as,

$$L_M = -\bar{q_R} M_{qR} - \bar{q_L} M^+_{qR} \tag{3.1.25}$$

where $M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$
is a constant matrix that does not transform along with quark fields.
L_M of equation (3.1.25) would be invariant if M transform as,

$$M \longrightarrow RML^+ \tag{3.1.26}$$

let us construct the most general Lagrangian $L(U, M)$ that is invariant under $U \longrightarrow RUL^+$ and $M \longrightarrow RML^+$ and expand in powers of M,at lowest order in M,

$$L_{s.b} = \frac{F_0^2 B_0}{2} tr(MU^+ + UM^+). \tag{3.1.27}$$

In order to interprate the new parameter B_0,let us consider energy density of the ground state ($U = U_0 = 1$),

$$\langle H_{eff} \rangle_{min} = -F_0^2 B_0 (m_u + m_d + m_s) \tag{3.1.28}$$

we can verify the above equation by constructing the Hamiltonian density corresponding to equation $\frac{F_0^2 B_0}{2} tr(MU^+ + UM^+)$ and $\frac{F_0^2}{2} tr(\partial_\mu U \partial_\mu U^+)$, by defining dynamical field ϕ_a, so, the Hamiltonian density will be given by,

$$H = \pi_a \dot{\phi_a} - L \tag{3.1.29}$$

$$\pi_a = \frac{\partial L}{\partial \dot{\phi_a}} = \frac{F_0^2}{4} tr(\frac{\partial \dot{U}}{\partial \dot{\phi_a}} \dot{U^+} + \dot{U} \frac{\partial \dot{U^+}}{\partial \dot{\phi_a}})$$

Therefore,

$$\pi_a\dot{\phi}_a = \frac{F_0^2}{4} tr(\dot{\phi}_a \frac{\partial \dot{U}}{\partial \dot{\phi}_a} \dot{U^+} + \dot{U}\dot{\phi}_a \frac{\partial \dot{U^+}}{\partial \dot{\phi}_a}) \tag{3.1.30}$$

but as we mentioned,

$$U = 1 + \frac{i\phi}{F_0} - \frac{\phi^2}{2F_0^2} + \frac{i\phi^3}{6F_0^3} - \frac{\phi^4}{24F_0^4} + ...$$

$$U^+ = 1 - \frac{i\phi}{F_0} - \frac{\phi^2}{2F_0^2} - \frac{i\phi^3}{6F_0^3} - \frac{\phi^4}{24F_0^4} + ...$$

From U,we can evaluate the following,

$$\dot{U} = \frac{i\dot{\phi}}{F_0} - \frac{\dot{\phi}\phi + \phi\dot{\phi}}{2F_0^2} + ...$$

$$\frac{\partial \dot{U}}{\partial \dot{\phi}_a} = \frac{i\lambda_a}{F_0} - \frac{\lambda_a\phi + \phi\lambda_a}{2F_0^2} + ...$$

$$\dot{\phi}_a \frac{\partial \dot{U}}{\partial \dot{\phi}_a} = \frac{i\dot{\phi}}{F_0} - \frac{\dot{\phi}\phi + \phi\dot{\phi}}{2F_0^2} + ... = \dot{U}$$

$$\dot{\phi}_a \frac{\partial \dot{U^+}}{\partial \dot{\phi}_a} == \dot{U^+}$$

So,equation (3.1.30) can be write as,

$$\pi_a\dot{\phi}_a = \frac{F_0^2}{4} tr(\dot{U}\dot{U^+} + \dot{U}\dot{U^+}). \tag{3.1.31}$$

Then the Hamiltonian density will be,

$$H = \pi_a\dot{\phi}_a - L = \frac{F_0^2}{4} tr(\dot{U}\dot{U^+} + \dot{U}\dot{U^+}) - L \tag{3.1.32}$$

$$H = \frac{F_0^2}{4} tr(\dot{U}\dot{U^+}) + \frac{F_0^2}{4} tr(\nabla U.\nabla U^+) - \frac{F_0^2 B_0}{2} tr(MU^+ + UM). \tag{3.1.33}$$

Hamiltonian density is minimized by constant and uniform fields, determine the minimum of the last term as follows,

$$\frac{\partial V}{\partial \phi_a} = \frac{\partial}{\partial \phi_a}\Big[- \frac{F_0^2 B_0}{2} tr(MU^+ + UM)\Big] \tag{3.1.34}$$

$$But, tr(MU^+ + UM) = 2tr\Big[M(1 - \frac{\phi^2}{2F_0^2} + \frac{\phi^4}{24F_0^4} + ...\Big] \tag{3.1.35}$$

$$\frac{\partial}{\partial \phi_a} tr\Big[M(1-\frac{\phi^2}{2F_0^2}+\frac{\phi^4}{24F_0^4}+...)\Big] = tr\Big[M(0-\frac{\lambda_a\phi+\phi\lambda_a}{2F_0^2} + \frac{\lambda_a\phi^3+\phi\lambda_a\phi^2+\phi^2\lambda_a\phi+\phi^3\lambda_a}{24F_0^4}+..\Big] \quad (3.1.36)$$

We can parametrize ,

$$M = m_0\lambda_0 + m_3\lambda_3 + m_8\lambda_8,$$

where

$$m_0 = \frac{m_u+m_d+m_s}{\sqrt{6}},$$

$$m_3 = \frac{m_u-m_d}{2},$$

and

$$m_8 = \frac{(\frac{m_u+m_d}{2})-m_s}{\sqrt{3}}.$$

for final solution, let us use, $\phi = \phi_0 + \frac{\phi_2}{F_0^2} + \frac{\phi_4}{F_0^4} + ...$,organize in powers of $\frac{1}{F_0^2}$. write $\phi_0 = \lambda_b\phi_{0b}$,so terms proportional to $\frac{1}{F_0^2}$ will be

$$tr[M(\lambda_a\phi_0+\phi_0\lambda_a)] = tr[M(\lambda_a\lambda_b+\lambda_b\lambda_a)]\phi_{0b} = tr[(M\{\lambda_a,\lambda_b\}]\phi_{0b}$$

but,$\{\lambda_a,\lambda_b\} = \frac{4}{3}\delta_{ab} + 2d_{ab3}\lambda_c$. Therefore $tr[M(\lambda_a\phi_0+\phi_0\lambda_a)]$ can be given as follows,

$$tr[M(\lambda_a\phi_0+\phi_0\lambda_a)] = [\frac{4}{3}\delta_{ab}(m_u+m_d+m_s)+4m_3d_{ab3}+4m_8d_{ab8}]\phi_{0b} \quad (3.1.37)$$

This equation gives eight equation ($a = 1.....8$) with eight unknown ϕ_{0b}.

for example if $a = 1$ the above equation give us the following result,

$$[\frac{4}{3}\delta_{ab}(m_u+m_d+m_s)+4m_3d_{ab3}+4m_8d_{ab8}]\phi_{0b} = [\frac{4}{3}\delta_{ab}(m_u+m_d+m_s)+4m_3d_{1b3}+4m_8d_{1b8}]\phi_{0b} = 0$$

but, $d_{1b3} = 0$ and $d_{1b8} = \frac{1}{\sqrt{3}}$, so the above equation can be written us,

$$[\frac{4}{3}\delta_{ab}(m_u+m_d+m_s)+4m_3d_{1b3}+4m_8d_{1b8}]\phi_{0b} = [\frac{4}{3}(m_u+m_d+m_s)+4\frac{(\frac{m_u+m_d}{2})-m_s}{\sqrt{3}}\frac{1}{\sqrt{3}}]\phi_{01} = 0$$

Therefore

$$tr[M(\lambda_a\phi_0+\phi_0\lambda_a)] = 2(m_u+m_d)\phi_{01} = 0 \Rightarrow \phi_{01} = 0 \quad (3.1.38)$$

Proceed analogously for remaining cases.⇒ For non-vanishing quark masses $\phi_{0b} = 0$,$b = 1...8$, Now consider $\frac{1}{F_0^4}$ terms,

$$tr\{M[(\lambda_a\phi_2 + \phi_2\lambda_a\}) - \frac{1}{12}(\lambda_a\phi_0^3 + \phi_0\lambda_a\phi_0^2 + \phi_0^2\lambda_a\phi_0 + \phi_0^3\lambda_a)]\} = 0$$

since $\phi_0 = 0$,

$$tr[M(\lambda_a\phi_2 + \phi_2\lambda_a)] = 0 \tag{3.1.39}$$

Calculation for ϕ_2 as for ϕ_0 above, ϕ_2= 0. And so on. In total we obtain $\phi = 0$ as the configuration minimizing H and thus Eq. (3.1.28). $\phi = 0$ is indeed minimum. Verified by taking second derivative of V as follows,

$$\frac{\partial^2 V}{\partial\phi_a\partial\phi_b}|_{\phi=0}\phi_a\phi_b \geqslant 0 \tag{3.1.40}$$

for all ϕ,where $V = \frac{f_0^2 B_0}{2}\ tr(MU^+ + UM^+)$

Compare derivative of Eq. (3.1.28) with respect to m_q with corresponding quantity in QCD

$$\frac{\partial\langle 0|H_{QCD}|0\rangle}{\partial m_q}\ |_{m_u=m_d=m_s} = \frac{1\langle 0|\bar{q}q|0\rangle}{3} \tag{3.1.41}$$

this implice that $3F_0^2 B_0 = -\langle\bar{q}q\rangle$

In order to determine the masses of the Gold stone bosons, we identify the terms of second order in the fields in $L_{s.b}$ as follows,

$$L_{s.b} = \frac{-B_0}{2}tr(\phi^2 M) + ... \tag{3.1.42}$$

we can express $tr(\phi^2 M)$ as follows,

$$tr(\phi^2 M) = 2(m_u + m_d +)\pi^+\pi^- + (m_u + m_s)\kappa^+\kappa^- + 2(m_d + m_s)\kappa^0\bar{\kappa}^0 +$$
$$(m_u + m_d)\pi^0\pi^0 + \frac{2}{\sqrt{3}}(m_u - m_d)\pi^0\eta + \frac{m_u + m_d + 4m_s}{3}\eta^2 \tag{3.1.43}$$

For the sake of simplicity we consider the isospin-symmetric limit $m_u = m_d = \check{m}$ so that the $\pi^0\eta$ term vanishes and there is no $\pi^0 - \eta$ mixing. We then obtain for the masses of the Gold stone bosons, to lowest order in the quark masses,we obtain

$$M_\pi^2 = 2B_0\check{m} \tag{3.1.44}$$

$$M_\kappa^2 = B_0(\check{m} + m_s) \tag{3.1.45}$$

$$M_\eta^2 = \frac{2}{3}B_0(\check{m} + 2m_s). \tag{3.1.46}$$

Gell-Mann-Okubo mass formula can be expressed as,

$$4M_\kappa^2 = 4B_O(\check{m} + m_s) = 2B_0(\check{m} + 2m_s) + 2B_0\check{m} = 3M_\eta^2 + M_\pi^2$$

$$4M_\kappa^2 = 3M_\eta^2 + M_\pi^2. \tag{3.1.47}$$

this equation is independent of B_0.

Finally Gell-Mann-Okubo mass formula is independent of B_0.

Chapter 4

Effective Lagrangian and Weinberg's power counting scheme

Perturbative calculation in effective field theory require two main ingredients;

• knowledge of the most general effective Lagrangian and

• consistent expansion scheme for observables

The effective Lagrangian can be written as,

$$L_{eff} = L_2 + L_4 + L_6 + ... \tag{4.0.1}$$

where, the subscripts refers to order in momentum and quark-mass expansion, for example index 2 refers two derivatives or one quark-mass term.

4.1 Construction of effective Lagrangian

We have derived the lowest order effective Lagrangian for global $Su(3)_L \times Su(3)_R$ symmetry. Now we can construct the effective Lagrangian for local $Su(3)_L \times Su(3)_R$ symmetry by taking the Goldstone boson in special unitary matrix,

$$U(x)e^{\frac{i\phi(x)}{F_0}}. \tag{4.1.1}$$

If U transform as $U \longrightarrow U'{=}V_R U V_R^+$, where V_R and V_L are independent space-time Dependant $Su(3)_L \times Su(3_R)$matrix.

The transformation behavior under $G = Su(3)_L \times Su(3_R)$ will be,

$$U \longrightarrow V_R U V_L^+. \tag{4.1.2}$$

Let an object A transform as $V_R A V_L^+$,this implice

$$A \xrightarrow{G} V_R A V_L^+. \tag{4.1.3}$$

And its covariant derivative transform as,

$$D_\mu A = \partial\mu A - ir_\mu A + iAl_\mu$$

$$= \partial\mu(V_R A V_L^+) - i(V_R r_\mu V_R^+ + iV_R \partial\mu V_R^+)(V_R A V_L^+) + iV_R A V_L^+(V_L l_\mu V^+ L + iV_L \partial\mu V_L^+)$$

$$= \partial\mu V_R A V_L^+ + V_R \partial\mu A V_L^+ + V_R A \partial\mu V_L^+ - iV_R r_\mu A V_L^+ - \partial\mu V_R A V_L^+ + IV_R A l\mu V_L^+ - V_R A \partial\mu V_L^+$$

$$D_\mu A = V_R(\partial\mu A - ir_\mu A + iAl_\mu)V_L^+ = V_R(D_\mu A)V_L^+. \tag{4.1.4}$$

So,we can say from here, covariant derivatives should transform as object it acts on. Since the effective Lagrangian contain high powers of derivatives, we should also need the field tensor $F^L_{\mu\nu}$ and $F_{\mu\nu}$ corresponding to the gauge fields,which is given by

$$F^R_{\mu\nu} = \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu] \tag{4.1.5}$$

and its transformation can be given as,

$$F^R_{\mu\nu} \xrightarrow{G} V_R \Big[\partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu]\Big] V_R^+ = V_R F^R_{\mu\nu} V_R^+ \tag{4.1.6}$$

similarly,

$$F^L_{\mu\nu} \xrightarrow{G} V_L \Big[\partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu]\Big] V_L^+ = V_R F^L_{\mu\nu} V_L^+. \tag{4.1.7}$$

The trace of the gauge field tensor can be given by,

$$tr(F^L_{\mu\nu}) = F^R_{\mu\nu}) = 0. \tag{4.1.8}$$

because,the trace of any commutator is zero and $tr(l_\mu) = tr(r_\mu)$=0.

The construction of the effective Lagrangian interms of $U, U^+, X, X^+, F^R_{\mu\nu}, F^L_{\mu\nu}$ and covariant derivatives of these objects proceed as follows.

Suppose we have matrices A,B.... all of which transform as,

$$A \xrightarrow{G} V_R A V_L^+ \tag{4.1.9}$$

$$B \xrightarrow{G} V_R B V_L^+ \tag{4.1.10}$$

we can form the above two equation invariant by multiplying in the following way,

$$tr(AB^+) = tr[V_R A V_L^+ (V + RBV_L^+)^+] = tr(V_R V_R^+ A V_L^+ V_L B^+ V_R^+)$$

but $V_R V_R^+ = V_L^+ V_L$=1, therefore,

$$tr(AB^+) = tr(AB^+). \tag{4.1.11}$$

So, the product of invariant trace is also invariant.

In the chiral counting scheme the element count as,

$$U = O(q^0)$$

$$D_\mu U = O(q)$$

$$r_u, l_u = O(q)$$

$$f^{L/R}_{\mu\nu} = O(q^2)$$

Any additional covariant derivatives count as $O(q)$,that means each covariant derivatives produces a power of q,so the construction of chirally invariant expressions up to and including order of q^2 proceeds as follows.

At q^0 the only invariant term is a constant,which is

$$tr(UU^+) = tr(1) = constant. \tag{4.1.12}$$

and because of

$$tr(D\mu UU^{+}) = 0, \tag{4.1.13}$$

Therefore, at $O(q^2)$,we have the following,

$$tr(D_\mu D_\nu UU^{+}) = -tr[D_\nu U(D_\mu U)^{+}] \tag{4.1.14}$$

$$tr[D_\mu U(D_\nu U)^{+} = tr[U(D_\mu D_\nu U)^{+} = -tr[D_\mu U(D_\nu U)^{+} \tag{4.1.15}$$

$$tr(f^L_{\mu\nu}) = tr(f^R_{\mu\nu}) = 0. \tag{4.1.16}$$

We can show $tr(D_\mu UU^{+}) = 0$ as follows,

First we can write

$$D_\mu UU^{+} = -U(D_\mu U^{+})$$

$$tr(D_\mu UU^{+}) = tr[\partial_\mu UU^{+} + iUl_u U^{+} + iUl_u U^{+}]$$

since the covariant derivatives of$D_\mu A$

$$D_\mu A = \partial_\mu A - ir_u A + iAl_u,$$

similarly,the covariant derivatives of $D_\mu UU^{+}$ is given by,

$$D_\mu UU^{+} = [\partial_\mu UU^{+} - ir_u UU^{+} + iUl_u U^{+}]$$

then

$$tr(D_\mu UU^{+}) = tr[\partial_\mu UU^{+} - ir_u UU^{+} + iUl_u U^{+}]$$

$$= tr[\partial_\mu UU^{+}] - tr[ir_u UU^{+}] + tr[iUl_u U^{+}]$$

as we have mentioned before, $tr[\partial_\mu UU^{+}] = 0$ and $tr(r_u) = tr(l_u) = 0$,

Therefore,

$$tr(D_\mu UU^{+}) = 0. \tag{4.1.17}$$

Now we can write the lowest order effective Lagrangian as,

$$L_2 = \frac{f_0^2}{4} tr[D_\mu U(D_\mu U)^{+}] + \frac{f_0^2}{4} tr[XU^{+} + UX^{+}]. \tag{4.1.18}$$

4.2 Weinberg's power counting scheme

How do different diagrams compare?

The behavior of a given diagram analyze under;

• linear rescaling of all external momenta,

• quadratic rescaling of light quark mass and Goldstone mass.

Linear rescaling of all external momenta can be expressed as,

$$p_i \longrightarrow tp_i \tag{4.2.1}$$

and quadratic rescaling of light quark-mass and goldstone mass given as follows,respectively,

$$m^2 \longrightarrow t^2 m^2 \tag{4.2.2}$$

$$M^2 \longrightarrow t^2 M^2. \tag{4.2.3}$$

So, the chiral dimension D of a given diagram with amplitude $M(p_i, m_q)$is defined as

$$M(tp_i, t^2 m_q) = t^D M(p_i, mq). \tag{4.2.4}$$

The chiral dimension D can be given by

$$D = nN_L - 2N_i + \sum_{1}^{\infty} 2kN_{2k}$$

$$D = 2 + (n-2)N_L + \sum_{k=1}^{\infty} 2(k-1)N_{2k}. \tag{4.2.5}$$

Where, n, N_L, N_I and N_{2k} are number of space-time dimension, independent loops, internal goldstone boson line and vertices from L_{2k},respectively.

Perturbative scheme interms of external momenta and quark mass(meson mass), which are small compared to some scale, for example consider the following figure, if n=4 what will be the chiral dimension D?

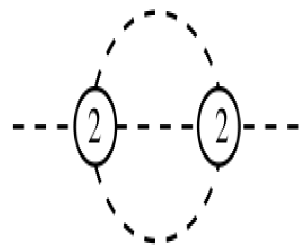

Figure 4.1: Loop Diagram for n=4

$$D = nN_L - 2N_I + \sum_1^{\infty} 2kN_{2k} = 4 \times 2 - 2 \times 3 + 2 \times 2 = 6 \tag{4.2.6}$$

or

$$D = 2 + (n-2)N_L + \sum_1^{\infty} 2(k-1)N_{2k} = 2 + 2 \times 2 + (2-2) \times 2 = 6. \tag{4.2.7}$$

similarly we can consider the following figure

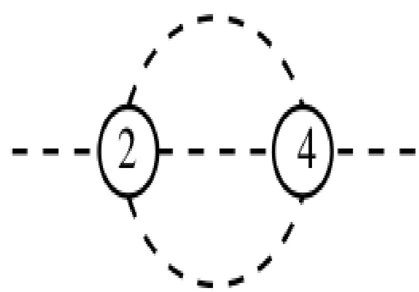

Figure 4.2: Loop Diagram for n=4

$$D = 4 \times 2 - 2 \times 3 + 1 \times 2 + 1 \times 4 = 8. \tag{4.2.8}$$

If we consider the following figure the chiral dimension D will be

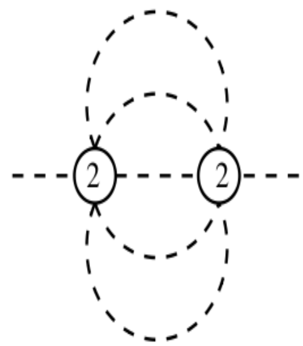

Figure 4.3: Loop Diagram for n=4

$$D = 4 \times 4 - 2 \times 5 + 2 \times 2 = 10. \tag{4.2.9}$$

Then the scaling behavior of the contribution to M of a given diagram ,its chiral dimension will be

$$D = 4 + 2N_I + \sum_{n=1}^{\infty} N_{2n}(2n-4). \tag{4.2.10}$$

The relation between number of independent loops,number of internal lines and total number of vertices are related by

$$N_L = N_I - (N_V - 1). \tag{4.2.11}$$

because of each of the N_V vertices generate delta function , after extracting one over all delta function,this yields $N_V - 1$ condition for the internal momenta, using

$$N_V = \sum_n N_{2n}. \tag{4.2.12}$$

we can put D as follows by using equation number (4.2.10),(4.2.11) and 4.2.12),

$$D = 4 + 2N_I + \sum_{n=1}^{\infty} N_{2n}(2n-4) = 4 + 2N_L + 2N_V - 2 + \sum_n N_{2n}(2n-4)$$
$$= 2 + 2N_L + \sum_{n=1}^{\infty} N_{2n}(2n-4+2) = 2 + 2N_L + \sum_{n=1}^{\infty} N_{2n}(2n-2)$$

Therefore,

$$D = 2 + 2N_L + \sum_{n=1}^{\infty} N_{2n}(2n-2). \tag{4.2.13}$$

.

4.3 Application at lowest order and pion decay

Let us consider two example at lowest order D=2 in equation
$D = 2 + 2N_L + \sum_{n=1}^{\infty} 2(n-1)N_{2n}$,we need only to consider a tree level diagram with vertices L_2.

Pion decay is deals with the weak decay $\pi^+ \longrightarrow \mu^+\nu_\mu$, which allow us to relate the free parameters F_0 of L_2 to the pion decay constant.

Pion decay is described by the annihilation of a u quark and a $\bar{d}$ anti-quark,forming the π^+,into a W^+ boson,propagation of the intermediate W^+ and creation of the leptons μ^+ and ν_μ in the final state,considerfollowing figure the

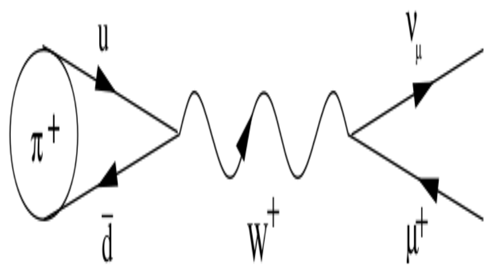

Figure 4.4: pion decay

From the above we observe that the interaction of massive charged weak boson W can coupling with leptons, then the coupling of W boson to the leptons given by,

$$L = -\frac{g}{2\sqrt{2}}\left[W_\mu^+ \bar{\nu_\mu}\gamma^\mu(1-\gamma_5)\mu + W_\mu^- \bar{\mu}\gamma^\mu(1-\gamma_5)\nu_\mu\right]. \tag{4.3.1}$$

Then the coupling of W boson to the Goldstone bosons is given by,

$$\frac{F_0^2}{4}tr[D_\mu(D_\mu U)^+ = \frac{F_0^2}{4}tr\left[(\partial_\mu U + iUl_u)(\partial_\mu U^+ - il_u U^+)\right]$$

$$= \frac{F_0^2}{4}tr\left[\partial_\mu U\partial_\mu U^+ - i\partial_\mu U l_u U^+ + iUl_u\partial_\mu U^+ + Ul_u l_u U^+\right]$$

$$= ......\frac{iF_0^2}{4}tr\left[Ul_u\partial_\mu U^+ - l_u U^+\partial_\mu U\right] + ...$$

$$= \frac{iF_0^2}{4}tr[l_u\partial_\mu U^+ U] + ...$$

since,

$$U^{+}\partial_{\mu}U = -\partial_{\mu}U^{+}U$$

but,

$$\partial_{\mu}U^{+} = \partial_{\mu}(e^{\frac{i\phi}{F_0}})^{+} = \partial_{\mu}(e^{\frac{-i\phi}{F_0}}) = \frac{-i\partial_{\mu}\phi}{F_0}e^{\frac{-i\phi}{F_0}}$$

$$\partial_{\mu}U^{+}U = -i\frac{\partial_{\mu}\phi}{F_0}e^{\frac{-i\phi}{F_0}}e^{\frac{i\phi}{F_0}} = \frac{-i\partial_{\mu}\phi}{F_0}$$

Therefore,the above equation becomes,

$$\frac{F_0^2}{4}tr[D_{\mu}(D_{\mu}U)^{+} = \frac{iF_0^2}{4}tr[l_u\partial_{\mu}U^{+}U] + ... = \frac{F_0}{2}tr[l_u\partial_{\mu}\phi]. \qquad (4.3.2)$$

We can express the QCD lagrangian with coupling to external field by setting,

$$l_u = -\frac{g}{\sqrt{2}}[W_{\mu}^{+}T_{+} + W_{\mu}^{-}T_{-}] \qquad (4.3.3)$$

$$r_{\mu} = 0 \qquad (4.3.4)$$

Where,

$$T_{+} = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (4.3.5)$$

$$T_{-} = \begin{pmatrix} 0 & 0 & 0 \\ V_{ud} & 0 & 0 \\ V_{us} & 0 & 0 \end{pmatrix} \qquad (4.3.6)$$

Therefore,the above equation becomes,

$$L_{W\phi} = -\frac{g}{\sqrt{2}}\frac{F_0}{2}tr\left[[W_{\mu}^{+}T_{+} + W_{\mu}^{-}T_{-}]\partial_{\mu}\phi\right] \qquad (4.3.7)$$

but,

$$\phi = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}} & \sqrt{2}\pi^{+} & \sqrt{2}\kappa^{+} \\ \sqrt{2}\pi^{-} & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\kappa^0 \\ \sqrt{2}\kappa^{-} & \sqrt{2}\kappa^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

then to simplify the above equation,first we have to calculate,

$$tr[T_+\partial_\mu\phi] = tr\Bigg[\begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \partial_\mu \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}} & \sqrt{2}\pi^+ & \sqrt{2}\kappa^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\kappa^0 \\ \sqrt{2}\kappa^- & \sqrt{2}\kappa^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}\Bigg]$$

$$tr[T_+\partial_\mu\phi] = V_{ud}\sqrt{2}\partial_\mu\pi^- + V_{us}\sqrt{2}\partial_\mu\kappa^-. \tag{4.3.8}$$

Similarly,we can calculate,

$$tr[T_-\partial_\mu\phi] = tr\Bigg[\begin{pmatrix} 0 & 0 & 0 \\ V_{ud} & 0 & 0 \\ V_{us} & 0 & 0 \end{pmatrix} \partial_\mu \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}} & \sqrt{2}\pi^+ & \sqrt{2}\kappa^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\kappa^0 \\ \sqrt{2}\kappa^- & \sqrt{2}\kappa^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

$$tr[T_-\partial_\mu\phi] = V_{ud}\sqrt{2}\partial_\mu\pi^+ + V_{us}\sqrt{2}\partial_\mu\kappa^+. \tag{4.3.9}$$

where V_{ud} and V_{us} is areal.

Therefore,the interaction Lagrangian can be given by,

$$L_{W\phi} = -g\frac{F_0}{2}\Bigg[W_\mu^+[V_{ud}\partial_\mu\pi^- + V_{us}\sqrt{2}\partial_\mu\kappa^-] + W_\mu^-[V_{ud}\sqrt{2}\partial_\mu\pi^+ + V_{us}\partial_\mu\kappa^+]\Bigg]. \tag{4.3.10}$$

The feynman rule for the invariant amplitude for the weak pion decay,

$$M = i\Bigg[-\frac{g}{2\sqrt{2}}\bar{U}_{\nu\mu}\gamma^\rho(1-\gamma^5)V_\mu^+\Bigg]i\frac{g_{\rho\sigma}}{M_W^2}i\Bigg[-g\frac{F_0}{2}V_{ud}(-ip^\rho)\Bigg]$$

$$= -G_F V_{ud}F_0\bar{u}_{\nu\mu}p(1-\gamma^5)V_\mu^+. \tag{4.3.11}$$

Where p denotes the four momentum of the pion and

$$G_F = \frac{g^2}{4\sqrt{2}M_W^2} \tag{4.3.12}$$

is the fermi constant.

The evaluation of the decay rate is given in this book[6],so here we only quote the final result,

$$\frac{1}{\tau} = \frac{G_F^2 \mid V_{ud}\mid^2}{4\pi}F_0^2 M_\pi M_\mu(1-\frac{M_\mu^2}{M_\pi^2})^2. \tag{4.3.13}$$

The constant F_0 is referred to as a pion decay constant in chiral limit.

Chapter 5

Conclusion

In this paper we have studied chiral perturbation theory is a cornerstone of our understanding of the strong interaction at low energies.
This implice that mesonic chiral perturbation theory has been extremely successful and may be considered as a full grown and mature area of the lowest energy physics.

Chiral perturbation theory(CHPT) is the effective field theory of quantum chromodynamics(QCD) at low energies.The basis of CHPT is the global $Su(3)_L \times Su(3)_R$ symmetry of the QCD Lagrangian. Chiral perturbation theory provides a systematic framework for investigating strong interaction process at low energies. Quarks are spin $\frac{1}{2}$ fermions with six different flavors in addition to their three possible colors. Left and right handed quark fields can be expressed interms of chiral matrices.

QCD Lagrangian posses a $Su(3)_L \times Su(3)_R$ symmetry group under chiral limit. The charge operators do not form the closed algebra,that is the commutator of two axial charge operator is not again an axial charge operators.The effective Lagrangian should be invariant under global $Su(3)_L \times Su(3)_R$. This implice that both left and right currents are conserved. This also implice that the vector and the axial vector currents are also conserved.

The behavior of a given diagram of Weinberg's power counting scheme analyze under,

- linear rescaling of all external momenta,
- quadratic rescaling of light quark mass and Goldstone mass.

Pion decay is described by the annihilation of a u quark and a $\bar{d}$ anti quark,forming the π^+,into a W^+ boson,propagation of the intermediate W^+ and creation of the leptons μ^+ and ν_μ.

Reference

[1] J S. Weinberg, Phys. Rev. 166, 1568 (1968)

[2] H. Fritzsch, M. Gell-Mann, and H. Leutwyler, Phys. Lett. B 47, 365(1973)

[3] S. Weinberg, Phys. Rev. Lett. 31, 494 (1973)

[4] H. F. Jones, Groups, Representations and Physics (Hilger,Bristol,1990)

[5] David Griffiths,Introduction to elementary particle physics

[6] Stephen Wofram,Introduction to the weak interaction,volume two.

[7] J. Gasser and H. Leutwyler, Nucl. Phys. B250, 465 (1985)

[8] S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. 177, 2239 (1969)

[9] H. Georgi, Weak Interactions and Modern Particle Theory (Ben-jamin/Cummings, Menlo Park, 1984)

[10]J. Gasser and H. Leutwyler, Chiral perturbation theory: expan- sions in the mass of the strange quark, Nucl. Phys. B250, 465 (1985)

[11] S. Weinberg, Nuclear forces from chiral Lagrangians, Phys. Lett.B 251, 288 (1990)

[12]H. Georgi, Weak interactions and modern particle theory (Ben- jamin/Cummings, Menlo Park, 1984).

[13] S. Weinberg, Physica A 96, 327 (1979).

Declaration

This project is my original work, has not been presented for a degree in any other University and that all the sources of material used for the project have been dully acknowledged.

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