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SEMINAR REPORT ON:

**ON PERIODIC SOLUTIONS OF FUNCTIONAL DIFFERENTIAL
EQUATIONS**

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Declaration

I, Fikremariam Shitiye declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

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Permission

This is to certify that this project is compiled by **Mr.Fikremariam Shitiye** in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

Advisor Dr.Tadess Abdi

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ABSTRACT

In this project we study the existence of some positive, periodic solutions of systems of functional differential equations. In the first topic we introduce the delay differential equations from the simple ordinary differential equation. In the second topic we apply a cone theoretic fixed point theorem and determining values for λ obtains conditions for the existence of positive periodic solutions of the system of functional differential equations.

In the third topic we see the existence of periodic solutions for a single species model. Exponential and logistic growth models are the most common. But in this topic we would like to study a class of differential equation models for single species that involve a time delay. In particular, we are interested to determining the existence of periodic solutions in periodic equations and in delayed lotka-volterra type equations for single species.

Lastly we mainly concern on the global existence of periodic solutions in delayed multi-species models due to delays and/or periodicity. We will present the global existence result of periodic solutions in a class of delayed autonomous Gause-type predator-prey systems. Secondly, we present an existence and uniqueness result on periodic solutions in a class of delayed periodic systems.

1. INTRODUCTION

The simplest model for the growth, or decay, of a population says that the growth rate, or decay rate, is proportional to the size of the population itself. Increasing or decreasing the size of the population results a proportional increase or decrease in the number of births and deaths. Mathematically, this is described by the initial value problem of ordinary differential equation.

$$\frac{dy}{dt} = ky, \quad y(0) = 1$$

from this IVP we get the exponential solution

$$y(t) = e^{kt}$$

From this solution the knowledge of the present (here: $y(0)=1$) allows us to predict the future at any time t . The past is not involved in this solution. We illustrate a delay differential equation (DDE) from an ordinary differential equation (ODE) by considering the above linear first order ordinary differential equation. But when we use a DDE, the past time exerts its influence on the present and, hence, on the future. The following DDE

$$\frac{dy}{dt} = ky(t-\tau), \quad y(t) = 1 \quad \text{when } -\tau \leq t < 0$$

exhibits a right hand side that depends on y at time $t-\tau$. τ is called the delay or time lag. For an ordinary differential system, a unique solution is determined by an initial point in Euclidean space at an initial time t_0 . For a delay differential system, one requires information on the entire interval $[t_0 - \tau, t_0]$. Moreover, the initial condition is now replaced by an initial function defined on a finite interval of time. The delay differential equations (DDEs) are a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times.

A general form of the time-delay differential equation for $x(t) \in R^n$ is

$$\dot{x}(t) = \frac{dx(t)}{dt} = f(t, x(t), x_t)$$

where $x_t = \{x(\tau) : \tau \leq t\}$ represents the trajectory of the solution in the past. In this equation, f is a functional operator from $R \times R^n \times C^1([-\tau, 0], R^n)$ to R^n .

We can see some examples of DDEs

1) Continuous distributed delay

$$\dot{x}(t) = f(t, x(t), \int_{-\infty}^0 x(t+\tau) d\mu(\tau))$$

2) Discrete constant delay

$$\dot{x}(t) = f(t, x(t), x(t-\tau_1), x(t-\tau_2), \dots, x(t-\tau_n)) \text{ for } \tau_1 > \dots > \tau_n \geq 0.$$

The most fundamental Functional Differential Equation (FDE) is the linear first order Delay Differential Equation,

$$\dot{x}(t) = a_1(t)x(t) + a_2(t)x(t-\tau), \text{ for } t \geq 0.$$

DDEs and FDEs are often used as modeling tools in several areas of applied mathematics, including the study of epidemics, age-structured population growth, automation, traffic flow and problems related to engineering of high-rise buildings for earth quake protection.

We consider the initial value problem

$$\begin{aligned} \dot{x}(t) &= f(t, x_t), \quad t \geq t_o \\ x_{t_o} &= x(t_o + \theta) = \phi(\theta) \end{aligned} \tag{1.1}$$

where $\phi(\theta) \in C([-\tau, 0], R^n)$ represents the initial point or initial data and $x_t = x(t + \theta)$, $\theta \in [-\tau, 0]$, is a function belongs to the Banach space $C = C^0([-\tau, 0], R^n)$ of continuous functions mapping the interval $[-\tau, 0]$ into R^n .

Equation (1.1), also called the volterra functional differential equation includes both distributed delay differential equations, where f depends on x computed on a continuum, possibly unbounded ($\tau = +\infty$), set of past values, and discrete delay differential equations, where only a finite number of past values of the state variable x are involved.

The initial value problem (1.1) will be expressed in a more friendly manner by

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), x(t-\tau_1), x(t-\tau_2), \dots, x(t-\tau_n)) \text{ for } t \geq t_o \\ x(t) &= \phi(t), \quad t \leq t_o \end{aligned} \quad (1.2)$$

Here, according to the complexity of the phenomenon, the delay (or lags) τ_i , which always are nonnegative, may just be constants (the constant delay case), or functions of t , $\tau_i = \tau_i(t)$ (the variable or time dependent delay case), or even functions of t and x itself, $\tau_i = \tau_i(t, x(t))$ (the state dependent delay case). In order to simplify the notation, the function $\phi(t)$ is understood to be defined in $[\rho, t_o]$, where $\rho = \min_{\substack{1 \leq i \leq n \\ t \geq t_o}} \{ \min(t - \tau_i) \}$.

In particular, for state dependent delays, the bound ρ cannot be determined a priori. An interesting and quite common case is given by $n=2$ and $\tau_1=0$ for which (1.2) takes the standard form

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), x(t-\tau)) \text{ for } t \geq t_o \\ x(t) &= \phi(t), \quad t \leq t_o \end{aligned} \quad (1.3)$$

2. POSITIVE PERIODIC SOLUTION OF SYSTEMS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

In this section we apply a cone theoretic fixed point theorem and obtains conditions for the existence of positive periodic solutions of the system of functional differential equations

$$\dot{x}(t) = A(t)x(t) + \lambda f(t, x(t - \tau(t))) \quad (2.1)$$

We are concerned with determining values for λ so that the system of functional differential equations (2.1) has a positive periodic solution.

The matrix

$$A(t) = \text{diag}[a_1(t), a_2(t), \dots, a_n(t)], \quad a_j \in C(\mathbb{R}, \mathbb{R}), \quad \tau : \mathbb{R} \rightarrow \mathbb{R},$$

are continuous and ω -periodic, $j = 1, 2, \dots, n$ with $\omega > 0$.

The function $f : \mathbb{R} \times \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ is continuous, where

$$\mathbb{R}^n = (x_1, x_2, \dots, x_n)^n \quad \text{and} \quad \mathbb{R}_+^n = \{(x_1, x_2, \dots, x_n)^n \in \mathbb{R}^n : x_j > 0, j = 1, 2, \dots, n\}.$$

We denote BC the normed vector space of bounded functions $\phi : \mathbb{R} \rightarrow \mathbb{R}^n$ with the norm

$$\|\phi\| = \sum_{i=1}^n \sup_{t \in \mathbb{R}} |\phi_i(t)| \quad \text{where} \quad \phi = (\phi_1, \phi_2, \dots, \phi_n)^T. \quad \text{For each } x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n, \text{ the norm of } x \text{ is}$$

defined as $\|x\|_o = \sum_{j=1}^n |x_j|$; where we say that x is “positive” whenever $x \in \mathbb{R}_+^n$.

Definition 2.1: Let X be a Banach space and K be a closed, non-empty subset of X . K is a cone if

- i) if $u \in K$ and $\alpha \geq 0$, then $\alpha u \in K$
- ii) if $u \in K$ and $-u \in K$ then $u = 0$.

Definition 2.2: A vector function $x : \mathbb{R} \rightarrow \mathbb{R}^n$ is called ω -periodic solution of the system (2.1) if it is absolutely continuous, periodic with the period ω , i.e.

$$x(t + \omega) = x(t),$$

and satisfy the system (2.1) almost everywhere on \mathbb{R} .

Theorem2.1[] (krasnosel'skii) Let B be a Banach space, and let P be a cone in B . Suppose Ω_1 and Ω_2 are bounded open subsets of B such that $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ and suppose that

$$T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow P$$

is completely continuous operator such that

- i) $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$; or
- ii) $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$,

Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Definition2.3: An operator $A : E \rightarrow E$, for a Banach space E , is said to be completely continuous if it maps an arbitrary bounded set into a compact set.

Definition2.4: Let X be a metric space. $M \subseteq X$ is said to be precompact if for every $\varepsilon > 0$, there is a finite subset J of M such that $M \subset \bigcup_{\alpha \in J} B(\alpha, \varepsilon)$.

We denote $f = (f_1, f_2, f_3, \dots, f_n)^T$ and assume

$$(H1) \quad \int_0^{\omega} a_j(s) ds < 0 \text{ for } j = 1, 2, \dots, n.$$

Define the set C_{ω} by

$$C_{\omega} = \{x \in C(R, R^n) : x(t + \omega) = x(t), t \in R\}.$$

Then it is clear that $C_{\omega} \subset BC$ when it is endowed with supremum norm

$$\|x\| = \sum_{j=1}^n \|x_j\|_o, \text{ where } \|x_j\|_o = \sup_{t \in [0, \omega]} |x_j(t)|.$$

Next, we consider the scalar differential equation

$$\dot{x}(t) = a(t)x(t) + \lambda f(t, x(t - t(\tau))), \quad (2.2)$$

where λ is constant, $a \in C(R, R)$, $\tau : R \rightarrow R$ are continuous and ω -periodic with $\omega > 0$. The function $f : R \times R \rightarrow R$ is continuous and ω -periodic in t .

Lemma2.1: $x(t) \in C_\omega$ is a solution of (2.2) iff

$$x(t) = \lambda \int_t^{t+\omega} \frac{\exp(\int_a^t a(u) du)}{\exp(-\int_0^s a(u) du) - 1} f(s, x(s - \tau(s))) ds \quad (2.3)$$

Now, we define the cone K and the Green's function $G(t,s)$ for equation (2.1).

For $(t, s) \in R^2$, $j = 1, 2, \dots, n$, we define

$$\sigma := \min\{\exp(-2\int_0^\omega |a_j(s)| ds), j = 1, 2, \dots, n\}, \quad (2.4)$$

$$G_j(t, s) = \frac{\exp(\int_a^t a_j(v) dv)}{\exp(-\int_0^s a_j(v) dv) - 1} \quad (2.5)$$

We also define

$$G(t, s) = \text{diag}[G_1(t, s), G_2(t, s), \dots, G_n(t, s)].$$

From (2.5), we have

$$\begin{aligned} G_j(t + \omega, s + \omega) &= \frac{\exp(\int_a^{t+\omega} a_j(v) dv)}{\exp(-\int_0^{s+\omega} a_j(v) dv) - 1} \\ &= \frac{\exp(\int_a^t a_j(v + \omega) dv)}{\exp(-\int_0^\omega a_j(v) dv) - 1} \\ &= \frac{\exp(\int_a^t a_j(v) dv)}{\exp(-\int_0^\omega a_j(v) dv) - 1} = G_j(t, s) \end{aligned}$$

It is clear that $G(t,s) = G(t+\omega, s+\omega)$ for all $(t,s) \in \mathbb{R}^2$ and, by (H1) and the assumption on f we have,

$$G_j(t,s) > 0, \quad f_j(u, \phi(u - \tau(u))) > 0$$

for $(t,s) \in \mathbb{R}^2$ and $(u, \phi) \in \mathbb{R} \times BC(\mathbb{R}, \mathbb{R}_+^n)$. Let K be the set defined by

$$K = \{x \in C_\omega : x_j(t) \geq \sigma \|x_j\|, t \in [0, \omega], x = (x_1, x_2, \dots, x_n)^T\}.$$

To verify that K is a cone, firstly we assume that $x \in K$ and $\alpha \geq 0$ we have $x_j(t) \geq \sigma \|x_j\|$, then

$$\alpha x_j(t) \geq \alpha \sigma \|x_j\| = \sigma \|\alpha x_j\| \quad \text{or} \quad \alpha x_j(t) \geq \sigma \|\alpha x_j\|$$

Secondly we assume that $x \in K$ and $-x \in K$, then $x_j(t) \geq \sigma \|x_j\|$ and $-x_j(t) \geq \sigma \|x_j\|$ respectively. But these two inequality are true when $x_j = 0$. This implies $x = 0$. Thus K is a cone.

Now, we are in a position to define an operator $\psi : K \rightarrow K$ as

$$(\psi x)(t) = \int_t^{t+\omega} G(t,s) f(s, x(s - \tau(s))) ds \quad (2.6)$$

for $x \in K, t \in \mathbb{R}$, where $G(t,s)$ is defined following (2.5). We denote $(\psi x) = (\psi_1 x, \psi_2 x, \dots, \psi_n x)^T$.

Before we proceed any further we state the followings:

$$A_j = \frac{e^{-\int_0^\omega |a_j(u)| du}}{e^{-\int_0^\omega |a_j(u)| du} - 1} \quad (2.7)$$

$$B_j = \frac{e^{\int_0^\omega |a_j(u)| du}}{e^{-\int_0^\omega |a_j(u)| du} - 1} \quad (2.8)$$

for $j = 1, 2, \dots, n$. It is easy to see that for $j = 1, 2, \dots, n$,

$$A_j \leq G_j(t,s) \leq B_j$$

for all $s \in [t, t + \omega]$.

If we set $A = \min_{1 \leq j \leq n} A_j$ and $B = \max_{1 \leq j \leq n} B_j$, then $A \leq G_j(t, s) \leq B$ for $j = 1, 2, \dots, n$.

Lemma2.2: If $(\psi x)(t)$ is given by (2.6), then $\psi : K \rightarrow K$ is completely continuous.

Proof: For each $x \in K$, since $f(t, x(t - \tau(t)))$ is a continuous function of t, we have $(\psi x)(t)$ is continuous in t and

$$\begin{aligned} (\psi x)(t + \omega) &= \int_{t+\omega}^{t+2\omega} G(t + \omega, s) f(s, x(s - \tau(s))) ds \\ &= \int_t^{t+\omega} G(t + \omega, s + \omega) f(s + \omega, x(s + \omega - \tau(s + \omega))) ds \\ &= \int_t^{t+\omega} G(t, s) f(s, x(s - \tau(s))) ds = (\psi x)(t) . \end{aligned}$$

Thus, $(\psi x) \in C_\omega$. Next we show that (ψx) is continuous. For $\theta, \mathfrak{G} \in C_\omega$, $\|\theta - \mathfrak{G}\| < \delta$ imply

$$\sup_{0 \leq s \leq \omega} |f_j(s, \theta(s - \tau(s))) - f_j(s, \mathfrak{G}(s - \tau(s)))| < \frac{\varepsilon}{\lambda n B_j \omega} .$$

If $x, y \in K$ with $\|x - y\| < \delta$, then

$$\begin{aligned} |(\psi_j x)(t) - (\psi_j y)(t)| &\leq \lambda \int_t^{t+\omega} |G_j(t, s)| |f_j(s, x(s - \tau(s))) - f_j(s, y(s - \tau(s)))| ds \\ &\leq \lambda B_j \omega \sup_{0 \leq t \leq \omega} |f_j(t, \theta(t - \tau(t))) - f_j(t, \mathfrak{G}(t - \tau(t)))| \\ &\leq \frac{\varepsilon}{n} \end{aligned}$$

for all $t \in [0, \omega]$. These yields

$$\|(\psi_j x)(t) - (\psi_j y)(t)\|_o < \frac{\varepsilon}{n}$$

Thus, $\|(\psi x)(t) - (\psi y)(t)\| < \varepsilon$.

Hence, ψ is continuous. For $x \in K$, let

$$(\psi_j x)(t) = \lambda \int_0^\omega |G_j(t, s) f_j(s, x(s - \tau(s)))| ds.$$

Then,

$$(\psi_j x)(t) \leq \lambda B_j \int_0^\omega |f_j(s, x(s - \tau(s)))| ds \quad \text{and}$$

$$\begin{aligned} (\psi_j x)(t) &\geq \lambda A_j \int_0^\omega |f_j(s, x(s - \tau(s)))| ds \\ &\geq \frac{A_j}{B_j} \|\psi_j x\|_o = \sigma \|\psi_j x\|_o, \quad j = 1, 2, \dots, n \end{aligned}$$

Therefore, $(\psi x) \in K$.

Next, we see that f maps bounded sets into bounded sets. Indeed, let $\varepsilon = 1$. For any $\mu > 0$, there exist $\delta > 0$ such that $x, y \in BC$, $\|x\| \leq \mu, \|y\| \leq \mu, \|x - y\| < \delta$ imply

$$|f(s, x(s - \tau(s))) - f(s, y(s - \tau(s)))| < 1$$

for $s \in [0, \omega]$. This leads

$$|f(s, x(s - \tau(s)))| \leq M, \quad \text{for } M > 0.$$

It follows from $(\psi_j x)(t) \leq \lambda B_j \int_0^\omega |f_j(s, x(s - \tau(s)))| ds$ that for $t \in [0, \omega]$,

$$\|\psi x\| = \sup_{0 \leq t \leq \omega} \sum_{j=1}^n |(\psi_j x)(t)| \leq \sum_{j=1}^n \lambda B_j \int_0^\omega |f_j(s, x(s - \tau(s)))| ds \leq \lambda B \omega M$$

Finally, for $t \in R$ we have

$$\begin{aligned} \frac{d}{dt}(\psi x)(t) &= G(t, t + \omega) f(t + \omega, x(t + \omega - \tau(t + \omega))) - G(t, t) f(t, x(t - \tau(t))) \\ &= [G(t, t + \omega) - G(t, t)] f(t, x(t - \tau(t))) \\ &= f(t, x(t - \tau(t))). \end{aligned}$$

Then we obtain

$$\left| \frac{d}{dt}(\psi x)(t) \right| < M$$

Hence, $\{(\psi x) : x \in K, \|x\| \leq \mu\}$ is a family of uniformly bounded and equicontinuous function on $[0, \omega]$. Thus, the function ψ is completely continuous.

Now we are ready to state and proof our results. Before we proceed we state the following:

$$(L1) \quad \lim_{x_j \rightarrow 0^+} \frac{f_j(s, x(s - \tau(s)))}{x_j} = l_j, \text{ uniformly in } s \text{ with } 0 < l_j < \infty, \text{ and}$$

$$(L2) \quad \lim_{x_j \rightarrow \infty} \frac{f_j(s, x(s - \tau(s)))}{x_j} = L_j, \text{ uniformly in } s \text{ with } 0 < L_j < \infty$$

for $x \in R^n$. For notational convenience, we let

$$L_M = \max_{1 \leq j \leq n} L_j, \quad L_m = \min_{1 \leq j \leq n} L_j, \quad l_M = \max_{1 \leq j \leq n} l_j, \quad l_m = \min_{1 \leq j \leq n} l_j \quad \text{and} \quad |G(t, s)| = \max_{1 \leq j \leq n} |G_j(t, s)|.$$

Theorem2.2[3]: Assume that (H1), (L1), and (L2) hold. Then, for each λ satisfying

$$\frac{1}{\omega \sigma A L_m} < \lambda < \frac{1}{\omega B l_M} \tag{2.9}$$

(2.1) has at least one positive periodic solution.

Proof: We construct the sets Ω_1 and Ω_2 in order to apply Theorem2.1. Let λ be defined by (2.9), and choose $\varepsilon > 0$ such that

$$\frac{1}{\omega \sigma A (L_m - \varepsilon)} \leq \lambda \leq \frac{1}{\omega B (l_M + \varepsilon)}.$$

By condition (L1), there exists $H_1 > 0$ such that $f_j(t, y) \leq (l_j + \varepsilon)y_j \leq (l_M + \varepsilon)y_j$, for $0 < y_j \leq H_1$.

Define $\Omega_1 = \{x \in K : \|x_j\|_o < H_1, j = 1, 2, \dots, n\}$ and assume $x \in K \cap \partial\Omega_1$. Then

$$(\psi_j x)(t) \leq \lambda B \int_0^{\omega} f_j(s, x(s - \tau(s))) ds$$

$$\begin{aligned}
&\leq \lambda B \omega (l_j + \varepsilon) \int_0^\omega x_j(s, x(s - \tau(s))) ds \\
&\leq \lambda B \omega (l_j + \varepsilon) \|x_j\|_o \\
&\leq \lambda B \omega (l_M + \varepsilon) \|x_j\|_o \leq \|x_j\|_o
\end{aligned}$$

In particular, $\|\psi_j x\|_o \leq \|x_j\|_o$, and

$$\|\psi x\| = \sum_{j=1}^n \|\psi_j x\|_o \leq \sum_{j=1}^n \|x_j\|_o = \|x\| \text{ for all } x \in K \cap \partial\Omega_1. \quad (2.10)$$

Next we construct the set Ω_2 . Considering (L2) there exists \bar{H}_2 such that

$$f_j(t, y) \geq (L_j - \varepsilon)y_j \geq (L_m - \varepsilon)y_j, \text{ for all } y_j \geq \bar{H}_2.$$

Let $H_2 = \max\{2H_1, \frac{\bar{H}_2}{\sigma}\}$ and set $\Omega_2 = \{x \in K : \|x_j\|_o < H_2, j = 1, 2, \dots, n\}$.

If $x \in K$ with $\|x\| \geq H_2$, then $x_j \geq \sigma \|x_j\| \geq \bar{H}_2$.

Thus

$$\begin{aligned}
(\psi_j x)(t) &\geq \lambda A \int_0^\omega f_j(s, x(s - \tau(s))) ds \\
&\geq \lambda A \omega \sigma (L_m - \varepsilon) \|x_j\|_o.
\end{aligned}$$

Hence,

$$\|\psi_j x\|_o \geq \|x_j\|_o, \text{ and}$$

$$\|\psi x\| = \sum_{j=1}^n \|\psi_j x\|_o \geq \sum_{j=1}^n \|x_j\|_o = \|x\| \text{ for all } x \in K \cap \partial\Omega_2 \quad (2.11)$$

Applying (i) of Theorem 2.1 to (2.10) and (2.11) yields that ψ has a fixed point $x \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. This completes the proof.

Theorem 2.3[3]: Assume (H1), (L1), and (L2) hold. Then, for each λ satisfying

$$\frac{1}{\omega\sigma A l_m} < \lambda < \frac{1}{\omega B L_M} \quad (2.12)$$

(2.1) has at least one positive periodic solution.

Proof: We construct the sets Ω_1 and Ω_2 in order to apply Theorem 2.1. Let λ be given as in (2.12), and choose $\varepsilon > 0$ such that

$$\frac{1}{\sigma\omega A(l_m - \varepsilon)} \leq \lambda \leq \frac{1}{\omega B(L_M + \varepsilon)}.$$

By condition (L1), there exists $H_1 > 0$ such that $f_j(t, y) \geq (l_j - \varepsilon)y_j \geq (l_m - \varepsilon)y_j$, for $0 < y_j \leq H_1$.

Define $\Omega_1 = \{x \in K : \|x_j\|_o < H_1, j = 1, 2, \dots, n\}$ and assume $x \in K \cap \partial\Omega_1$. Then

$$\begin{aligned} (\psi_j x)(t) &\geq \lambda A \int_0^{\omega} f_j(s, x(s - \tau(s))) ds \\ &\geq \lambda A \omega (l_m - \varepsilon) x_j(t - \tau(s)) \\ &\geq \lambda A \sigma \omega (l_m - \varepsilon) \|x_j\|_o \\ &\geq \|x_j\|_o. \end{aligned}$$

In particular, $\|\psi_j x\|_o \geq \|x_j\|_o$, and

$$\|\psi x\| = \sum_{j=1}^n \|\psi_j x\|_o \geq \sum_{j=1}^n \|x_j\|_o = \|x\| \text{ for all } x \in K \cap \partial\Omega_1. \quad (2.13)$$

Next we construct the set Ω_2 . Considering (L2) there exists \bar{H}_2 such that

$$f_j(t, y) \leq (L_j + \varepsilon)y_j \leq (L_M + \varepsilon)y_j, \text{ for all } y_j \geq \bar{H}_2.$$

We consider two cases; $f_j(t, y)$ is bounded and $f_j(t, y)$ is unbounded. The case where $f_j(t, y)$ is bounded is straight forward. If $f_j(t, y)$ is bounded by $\theta > 0$, set

$$H_2 = \max\{2H_1, \omega\lambda\theta B\}.$$

Then if $x \in K$ and $\|x\|_o = H_2$, we have

$$\begin{aligned}
(\psi_j x)(t) &\leq \lambda B \int_0^{\omega} f_j(s, x(s - \tau(s))) ds \\
&\leq \omega \lambda B \theta \leq \|x_j\|_o.
\end{aligned}$$

Consequently, $\|\psi_j x\|_o \leq \|x_j\|_o$, and hence $\|\psi x\| \leq \|x\|$.

So, if we set $\Omega_2 = \{y \in K : \|y_j\|_o < H_2, j = 1, 2, \dots, n\}$, then

$$\|\psi x\| \leq \|x\|, \text{ for } x \in K \cap \partial\Omega_2 \quad (2.14)$$

When f is unbounded, we let $H_2 > \max\{2H_1, \bar{H}_2\}$ be such that $f_j(t, y) \leq f_j(t, H_2)$, for $0 < y_j \leq H_2$. For $x \in K$ with $\|x_j\|_o = H_2$,

$$\begin{aligned}
(\psi_j x)(t) &\leq \lambda B \int_0^{\omega} f_j(s, x(s - \tau(s))) ds \\
&\leq \lambda B \int_0^{\omega} f_j(s, H_2) ds \\
&\leq \lambda B \int_0^{\omega} (L_j + \varepsilon) H_2 ds \\
&\leq \lambda B \omega (L_M + \varepsilon) \|x_j\|_o \leq \|x_j\|_o.
\end{aligned}$$

Consequently, $\|\psi_j x\|_o \leq \|x_j\|_o$, which implies that

$$\|\psi x\| = \sum_{j=1}^n \|\psi_j x\|_o \leq \sum_{j=1}^n \|x_j\|_o = \|x\|$$

So, if we set $\Omega_2 = \{x \in K : \|x_j\|_o < H_2, j = 1, 2, \dots, n\}$,

then,

$$\|\psi x\| \leq \|x\|, \text{ for } x \in K \cap \partial\Omega_2 \quad (2.15)$$

Applying (ii) of Theorem 2.1 to (2.14) and (2.15) yields that ψ has a fixed point $x \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. Also, applying (ii) of Theorem 2.1 to (2.13) and (2.15) yields that ψ has a fixed point $x \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. This completes the proof.

3. PERIODIC SOLUTIONS IN SINGLE SPECIES MODELS

A single-species population growth model is considered, where the growth rate response to changes in its density has a periodic delay. It is shown that if the self-inhibition rate is sufficiently large compared to the reproduction rate, then the model equation has a globally asymptotically stable positive periodic solution. In this topic, we discuss about the global existence of periodic solutions in periodic equations and in delayed lotka-volterra type equations for single species.

3.1. GLOBAL EXISTANCE OF PERIODIC SOLUTIONS IN

$$\dot{X}(t) = f(x(t-1)) - g(x(t))$$

In this section, we establish the global existence of periodic solutions of

$$\dot{x}(t) = f(x(t-\tau)) - g(x(t)) \quad (3.1)$$

which is a special case of the general non-linear delay equations

$$\dot{x}(t) = f\left(\int_{-r}^{-\sigma} x(t+s)d\mu(s)\right) - g(x(t)).$$

Our approach can be easily modified to cover the equation

$$\dot{x}(t) = x(t)[f(x(t-1)) - g(x(t))]. \quad (3.2)$$

H1a) $f(x)$ is strictly decreasing, $f(0) > 0$, $\lim_{x \rightarrow \infty} f(x) = 0$

H1b) $f(0) = 0$; there is an $x_M > 0$, such that $f(\cdot)$ is strictly increasing in $[0, x_M]$ and strictly decreasing in $[x_M, +\infty]$; $\lim_{x \rightarrow \infty} f(x) \geq 0$.

A1) $f(x)$ satisfies (H1a) or (H1b) with $x_M < x^\circ = 1$.

For convenience, we denote $x_M = 0$ when f satisfies (H1a). In the following, we also assume

A2) There is an $x^* \in [x_M, 1]$ such that $f(x^*) > g(x_*)$, where $x^* = g^{-1}(f(x_*))$.

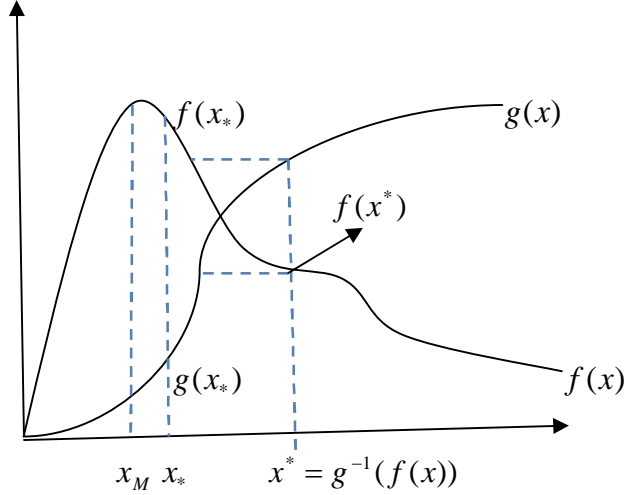


Fig3.1. $f(g^{-1}(f(x_*)))$ implies that $[x_*, x^*]$ is invariant.

Lemma3.1: Assume that $f(x)$ in (3.1) satisfies (A1) and (A2), and $x(s) \in [x_*, x^*]$ for $s \in [t - \tau, t_0]$. Then $x(t) \in [x_*, x^*]$ for $t \geq t_0$.

Proof: Let $0 \leq x_o < x_*$ such that, for $x \in [x_o, x_*]$, $f(g^{-1}(f(x))) > g(x)$. If the lemma is false, then there are two cases we consider:

- i) There is $t^* > t_0$, $x_o < x(t^*) < x_*$, $x(t^*) < x(t) \leq x^*$ for $t \in (t_0 - \tau, t^*)$, and $x'(t^*) < 0$;
- ii) There is $t^* > t_0$, $x(t^*) > x^*$, $x_* < x(t) \leq x(t^*)$ for $t \in (t_0 - \tau, t^*)$, and $x'(t^*) > 0$.

We consider the first case (i). Since $x(t^*) < x_*$, $x'(t^*) < 0$, we have

$$f(x(t^* - \tau)) < g(x(t^*)) \quad (3.3)$$

Since $f(x) \geq g(x)$ for $x \in [0, 1]$, and $g'(x) > 0$, (3.3) implies that $x(t^* - \tau) > 1$. If $1 < x(t^* - \tau) \leq x^*$, the monotonicity of $f(x)$ for $x \geq x_M$ together with (A2) imply

$$f(x(t^* - \tau)) \geq f(x^*) > g(x_*)$$

And the monotonicity of $g(x)$ gives as

$$f(x(t^* - \tau)) > g(x_*) > g(x(t^*))$$

a contradiction to (3.3).

Assume now the second case (ii) holds. We have $x'(t^*) > 0$ implies that

$$f(x(t^* - \tau)) > g(x(t^*))$$

The monotonicity of $g(x)$ implies that

$$f(x(t^* - \tau)) > g(x^*) = g(g^{-1}(f(x_*))) = f(x_*)$$

Now the monotonicity of $f(x)$ for $x > x_M$ clearly indicates that

$$x(t^* - \tau) < x_*,$$

Which contradicts our assumption $x_* \leq x(t) < x(t^*)$ for $t_o - \tau < t < t^*$. This completes the proof of lemma3.1.

In the rest of this section we assume that the initial condition for (3.1) satisfies

$$x(s) = \phi(s), \quad s \in [-\tau, 0], \quad x_* \leq \phi(s) \leq x^*$$

and $\phi(s) \in C([-\tau, 0], R)$. Lemma3.1 thus implies that $x_* \leq x(t) \leq x^*$ for $t \geq 0$. Clearly, in $[x_*, x^*]$, both $-f(x)$ and $g(x)$ are strictly increasing. Without loss of generality, may assume in the following that $f(x)$ satisfies (H1a). Also, for convenience, we assume that $\tau = 1$. Equation (3.1) thus reduces to

$$\dot{x}(t) = f(x(t-1)) - g(x(t)) \tag{3.4}$$

We denote $\alpha = -f'(1)$, $\beta = g'(1)$. we first linearized the functions $f(x(t-1))$ and $g(x(t))$ by Taylor series expansion at $x=1$ we get

$$f(x(t-1)) = f(1) + f'(1)(x(t-1) - 1) = f(1) + f'(1)x(t-1) - f'(1) = f'(1)x(t-1)$$

$$g(x(t)) = g(1) + g'(1)(x(t) - 1) = g(1) + g'(1)x(t) - g'(1) = g'(1)x(t)$$

Then, the linearized equation of (3.4) at $x=1$, takes the form

$$\dot{x}(t) = -\alpha x(t-1) - \beta x(t) \quad (3.5)$$

It has the characteristic equation

$$\lambda + \beta + \alpha e^{-\lambda} = 0 \quad (3.6)$$

We note that $\alpha > 0$, $\beta > 0$.

Lemma3.2: Let $\alpha > \alpha_\beta$, where α_β is the smallest positive solution of

$$\beta + \alpha \cos \sqrt{\alpha^2 + \beta^2} = 0 \quad (3.7)$$

Then (3.6) has a solution λ with $\operatorname{Re} \lambda > 0$, $\frac{\pi}{2} < \operatorname{Im} \lambda < \pi$.

In order to present our main results, we need the following notations

$$V = \max\{g'(x) : x \in [x_*, x^*]\} \quad (3.8)$$

$$K = \{\phi(s) : \phi \in C, \phi(-1) = 1, x_* \leq \phi(s) \leq x^*, \frac{d}{ds}((\phi(s)-1)e^{vs}) > 0 \text{ for } s \in [-1, 0]\} \quad (3.9)$$

$$K_1 = K \setminus \{1\}, \quad (3.10)$$

where 1 is the function $\phi(s) \equiv 1$, $s \in [-1, 0]$. It is easy to see K is a closed, bounded, and convex subset of the Banach space $C([-1, 0], R)$ with the standard supremum norm $\| \cdot \|$. This set is crucial importance in the proof of the following theorem, which is the main result of this section.

Theorem3.1[1]: Assume (A1) and (A2) hold, and $\alpha > \max\{1, \alpha_\beta\}$. Then equation (3.4) has a non constant periodic solution $x(t)$ with period greater than 2, and that satisfies $x_* \leq x(t) \leq x^*$.

Roughly, our approach involves showing that the initial conditions near 1 in K are taken away from it, and those away from it tend in some sense to approach it (with the help of lemma3.1). This leads to the existence of non constant fixed points of an operator A of the form

$A\phi = x_\sigma(\cdot, \phi)$, where $\sigma = \sigma(\phi)$ is a non negative number (to be defined), $\phi \in K$, and $x_\sigma(s, \phi) = x(\sigma + s, \phi)$, $s \in [-1, 0]$.

The primary tool available for proving the existence of periodic solutions is the theorem below from nonlinear functional analysis. Before stating the theorem, we need to define what it means for a fixed point of a map to be ejective.

Definition3.1[4]: Suppose B is a Banach space, $U \subseteq B$, and x is a given point in U . Given a map $A: U \setminus \{x\} \rightarrow B$, the point $x \in U$ is said to be an ejective point of A if there is an open neighborhood $G \subseteq B$ of x such that for every $y \in G \cap U$, $y \neq x$, there is an integer $m = m(y) > 0$ such that $A^m y \notin G \cap U$.

Intuitively, a point is ejective if it is surrounded by a neighborhood of points, which the map will sent outside the neighborhood eventually. We now state the theorem we apply in this topic and the next topic.

Theorem3.2[4]: If K is a closed, bounded, convex, and infinite dimensional set in a Banach space X , and $A: K \setminus \{x_o\} \rightarrow K$ is completely continuous, and $x_o \in K$ is ejective, then there is a fixed point of A in $K \setminus \{x_o\}$.

Theorem3.3[1]: suppose the following conditions are satisfied:

- i) There is a characteristic root λ of Eq(3.6) satisfying $\text{Re } \lambda > 0$
- ii) There is a completely continuous function $\tau: K \setminus \{x_o\} \rightarrow [\alpha, \infty)$, $\alpha \geq 0$, such that the map A defined by

$$A\phi = x_{\tau(\phi)}(\phi), \quad \phi \in K \setminus \{x_o\},$$

Takes $k \setminus \{x_o\}$ into K and is completely continuous.

Then x_o is an ejective point of A .

Lemma3.3: Assume that all conditions of Theorem3.1 are satisfied and $x(t) = x(t, \phi)$, $\phi \in K_1$, is a solution of (3.4). Then the following hold:

- 1) There is a sequence $\{z_i\}_{i=1}^{\infty}$, $0 < z_1 < z_2 < \dots$ such that $x(z_i) = 1$, $z_{i+1} > z_i$, $i=1,2,\dots$
- 2) $\dot{x}(z_{2k-1}) < 0$, $\dot{x}(z_{2k}) > 0$, for $k = 1, 2, 3, \dots$
- 3) The function $e^{vt}(x(t) - 1)$ is non-increasing on each of the intervals $(z_{2k-1}, z_{2k-1} + 1)$ and non-decreasing on each the intervals $(z_{2k}, z_{2k} + 1)$, for $k = 1, 2, 3, \dots$
- 4) There is a constant $q > 0$ such that, for $\phi \in K$, $z_2 \leq q$.

We are now ready to define the operator $A : K \rightarrow K$ as

$$A(\phi(s)) = x(z_2(\phi) + 1 + s, \phi), \quad s \in [-1, 0], \quad \phi \in K_1, \quad A(1) = 1 \quad (3.11)$$

Lemma3.4: The mappings $\phi \rightarrow z_2(\phi)$ of K_1 into $(1, +\infty)$ and $A : K \rightarrow K$ are completely continuous.

Proof:- The continuous dependence on the initial data together with the fact that $\dot{x}(z_1(\phi), \phi) < 0$, $\dot{x}(z_2(\phi), \phi) > 0$ clearly indicates that if $\|\varphi - \phi\|$ is very small, then the function $x(t, \varphi)$ has two zeros \bar{z}_1, \bar{z}_2 very close to $z_1(\phi), z_2(\phi)$, and $\dot{x}(\bar{z}_1, \varphi) < 0$, $\dot{x}(\bar{z}_2, \varphi) > 0$, and cannot have any other zeros for $t \leq \bar{z}_2$. The complete continuity follows from the fact that $z_2(\phi) \leq q$ for $\phi \in K_1$. The continuity of A follows from the continuity of $z_2(\phi)$ and again the continuous dependence on the initial data.

Since $z_2 : K_1 \rightarrow (1, +\infty)$ is completely continuous, we see that, for any bounded $B \subset K_1$, $A(B)$ is bounded and equicontinuous (since $z_2 > 1$) and, thus, compact. Therefore, A is completely continuous.

Lemma3.5: Let $\bar{\alpha} = \max\{1, \alpha_\beta\}$, $\delta = \min\{x^* - 1, 1 - x_*\}$, and J be a compact set of $(\bar{\alpha}, \infty)$. Then

$$\mu = \inf\{\pi_{\lambda(\alpha)} \phi : \phi \in K, \|\phi - 1\| = \delta, \alpha \in J\} > 0 \quad (3.12)$$

Finally, we are ready to state the proof of Theorem3.1.

Proof (of Theorem3.1). From the definition of K , we see that it is a closed, bounded and convex set of infinite dimension in the Banach space C . A as defined in (3.11) is completely continuous by lemma3.4 and 1 is an ejective fixed point of A by Theorem3.3 and lemma3.5. Therefore, by Theorem3.2, we conclude that A has a fixed point ϕ in K_1 , which clearly corresponds to a non constant periodic solution $x(t, \phi)$ of period greater than 2. This completes the proof.

3.2. PERIODIC SOLUTIONS IN DELAYED PERIODIC LOTKA-VOLTERRA TYPE EQUATIONS

Frequently, we observe that populations in the real world tend to fluctuate. There are three typical approaches for modeling such behavior:

- i) Introduce more species into the model, and consider the higher dimensional systems (like predator-prey interaction);
- ii) Assume that the per capita growth rate is time dependent;
- iii) Take into account the time delay effect in the population dynamics.

Generally speaking, approach (i) is rather artificial, while (ii) and (iii) emphasize only one aspect of reality. Naturally, more realistic model of single species growth should take into account both the changing environment and the effects of time delays. Therefore, it is important to study the following general nonlinear non autonomous delayed Lotka-Volterra type equation for single species:

$$\dot{x}(t) = x(t)[b(t) - \sum_{i=1}^n \int_{t-r(t)}^t f_i(t, x(s)) d\mu_i(t, s)], \quad x(0) > 0 \quad (3.13)$$

When (3.13) has a positive steady state, it reduces to the following general delayed nonlinear nonautonomous logistic equation:

$$\dot{x}(t) = -(1 + x(t)) \sum_{i=1}^n \int_{t-r(t)}^t f_i(t, x(s)) d\mu_i(t, s),$$

where $f_i(t, 0) = 0$, $r(t) > 0$, $\mu_i(t, s)$, $i = 1, 2, 3, \dots, n$ are non-decreasing.

We show that, under reasonable conditions, Eq(3.1) has only one asymptotic state, in the sense that if $x_1(t)$ and $x_2(t)$ are two solutions of Eq(3.1), then we have

$$\lim_{t \rightarrow +\infty} (x_1(t) - x_2(t)) = 0.$$

In equation (3.13), we always assume that $f_i(t, x)$ and $r(t)$ are continuous with respect to their arguments, and $\mu_i(t, s)$ continuous with respect to t , non-decreasing with respect to s , and are defined for all $(t, s) \in \mathcal{R}^2$. In addition, we always assume the following:

H1) For $x > 0$, $xf_i(t, x) \geq 0$, $f_i(t, x)$ is non-decreasing with respect to x , and $\sum_{i=1}^n f_i(t, x)$ is strictly increasing with respect to x ;

H2) $r(t) > 0$, $t - r(t)$ is non-decreasing, and $\lim_{t \rightarrow +\infty} (t - r(t)) = +\infty$;

H3) $\mu_i(t, t) > \mu_i(t, t - r(t))$;

H4) for any $c \neq 0$, there exist $a_i(t) \geq 0$, $b_i(|c|) \geq 0$, $b_i(|c|) = 0$ if and only if $c = 0$, and $b_i' \geq 0$ are such that

$$|f_i(t, c_1) - f_i(t, c_2)| \geq a_i(t)b_i(|c_1 - c_2|) \quad \text{and} \quad \lim_{t \rightarrow \infty} \int_0^t \left(\sum_{i=0}^n \int_{\tau-r(\tau)}^{\tau} a_i(\tau) d\mu_i(\tau, s) \right) d\tau = +\infty;$$

H5) There exist $K_1 > 0$ and $K_2 > 0$ with $K_1 \leq K_2$ such that

$$b(t) - \sum_{i=1}^n \int_{t-r(t)}^t f_i(t, K_1) d\mu_i(t, s) \geq 0 \quad \text{and}$$

$$b(t) - \sum_{i=1}^n \int_{t-r(t)}^t f_i(t, K_2) d\mu_i(t, s) \leq 0$$

Let $r = r(0)$; the initial value problem for (3.13) is assumed to take the form $x(\theta) = \phi(\theta) \geq 0$, $\theta \in [-r, 0]$, $\phi \in C$.

Theorem 3.4: In (3.13), assume that there exists $M > 0$ such that

$$\int_{t-r(t)}^t b(\tau) d\tau \leq M < +\infty. \quad (3.14)$$

Then $x(\phi)(t)$ exists for all $t \geq 0$. Also, for large t ,

$$x(\phi)(t) \leq K_2 e^M. \quad (3.15)$$

If, in addition, there is $N > 0$ and $\bar{M} > 0$ such that, for $0 < x < K_2 e^M$, $f_i(t, x) < \bar{M}$ for all $t \geq 0$, $i = 1, 2, 3, \dots, n$, and

$$\int_{t-r(t)}^t \left(\sum_{i=1}^n [\mu_i(\tau, \tau) - \mu_i(\tau, \tau - r(\tau))] \right) d\tau \leq N, \quad (3.16)$$

then, for large t ,

$$x(\phi)(t) \geq K_1 e^{-\bar{M}N} \quad (3.17)$$

Proof: Let $x(\phi)(t)$ be the solution of (3.13) with the maximal interval of existence $[0, \tau_{\max})$.

Then, we claim that $x(\phi)(t) > 0$ for $t \in [0, \tau_{\max})$. If not, there is $t_o > 0$ such that $x(\phi)(t_o) = 0$.

Without loss of generality, taking $t_o = \min\{t \in [0, \tau_{\max}) : x(\phi)(t) = 0\}$, we have

$$\ln x(t) = \ln \phi(0) + \int_0^t b(\tau) d\tau - \int_0^t \left(\sum_{i=1}^n \int_{\tau-r(\tau)}^{\tau} f_i(\tau, x(s)) d\mu_i(\tau, s) \right) d\tau.$$

Letting $t \rightarrow t_o$, we get the contradiction: The left side goes to $-\infty$, and the right side is bounded.

From Eq(3.15) that is for large $t \geq 0$, $x(\phi)(t) \leq K_2 e^M$. Thus $x(\phi)(t)$ exists for all $t \geq 0$.

Now we prove that (3.15) is true. If there is some $t_o \geq 0$ such that, for all $t \geq t_o$,

$$x(\phi)(t) \geq K_2,$$

then,

$$\begin{aligned} \dot{x}(t) &= x(t) \left[b(t) - \sum_{i=1}^n \int_{t-r(t)}^t f_i(t, x(s)) d\mu_i(t, s) \right] \\ &\leq x(t) \left[b(t) - \sum_{i=1}^n \int_{t-r(t)}^t f_i(t, K_2) d\mu_i(t, s) \right] \\ &\leq 0 \end{aligned}$$

Hence, $\lim_{t \rightarrow \infty} x(t) = c \geq K_2$.

Suppose that $c > K_2$; then, for all large t , $x(t) > c$. By (H5) and (H4), we have

$$\begin{aligned}
\dot{x}(t) &\leq -x(t) \sum_{i=1}^n \int_{t-r(t)}^t (f_i(t, x(s)) - f_i(t, K_2)) d\mu_i(t, s) \\
&\leq -x(t) \sum_{i=1}^n \int_{t-r(t)}^t a_i(t) b_i(x(s) - K_2) d\mu_i(t, s) \\
&\leq -x(t) \sum_{i=1}^n \int_{t-r(t)}^t a_i(t) b_i(c - K_2) d\mu_i(t, s) \quad .
\end{aligned}$$

By (H4), this inequality yields a contradiction. Thus,

$$\lim_{t \rightarrow +\infty} x(t) = K_2 \quad (3.18)$$

Now, suppose that $x(t) - K_2$ has infinitely many zeros $\{t_n\}$ with $\lim_{n \rightarrow \infty} t_n = +\infty$.

If $x(t)$ assumes a local maximum at t^* with $x(t^*) \geq K_2$, then $\dot{x}(t^*) = 0$, and (H1) implies that there is a $t_o \in [t^* - r(t^*), t^*]$ such that $x(t_o) = K_2$.

Thus,

$$\begin{aligned}
\ln x(t^*) - \ln K_2 &= \int_{t_o}^{t^*} b(\tau) d\tau - \sum_{i=1}^n \int_{t_o}^{t^*} \int_{\tau-r(\tau)}^{\tau} f_i(\tau, x(s)) d\mu_i(\tau, s) d\tau \\
&\leq \int_{t^*-r(t^*)}^{t^*} b(\tau) d\tau \leq M
\end{aligned}$$

Hence, for large t ,

$$\ln x(t) - \ln K_2 \leq M$$

The above inequality gives us

$$x(t) \leq K_2 e^M .$$

Equation (3.18) and the preceding inequality complete the proof for (3.15). The proof of (3.17) is similar.

In order to establish the global asymptotic behavior of the solutions of (3.13), we need the following lemma.

Lemma3.6: In (3.13), suppose that the assumptions made in Theorem3.4 (conditions (3.14) and (3.16)) are true. Then, for any two solutions $x(t)$ and $y(t)$, if $x(t) > y(t)$ for all large t , then

$$\lim_{t \rightarrow +\infty} (x(t) - y(t)) = 0$$

Proof:- From (3.13), we have

$$\frac{d}{dt} \left(\frac{x(t)}{y(t)} \right) = \frac{\dot{x}(t)y(t) - x(t)\dot{y}(t)}{(y(t))^2} = -\frac{x(t)}{y(t)} \sum_{i=1}^n \int_{t-r(t)}^t [f_i(t, x(s)) - f_i(t, y(s))] d\mu_i(t, s) \leq 0.$$

Hence,

$$\lim_{t \rightarrow +\infty} \frac{x(t)}{y(t)} = c \geq 1$$

Suppose $c > 1$; then,

$$\lim_{t \rightarrow +\infty} \frac{x(t) - y(t)}{y(t)} = c - 1.$$

Theorem 3.4 implies that

$$x(t) - y(t) > \frac{c-1}{2} K_2 e^M$$

for all large t . Now, (H4) yields

$$\frac{d}{dt} \left(\frac{x(t)}{y(t)} \right) \leq -\frac{x(t)}{y(t)} \sum_{i=1}^n b_i \left(\frac{c-1}{2} K_2 e^M \right) \int_{t-r(t)}^t a_i(t) d\mu_i(t, s).$$

Also, by (H4), we get a contradiction. Thus,

$$\lim_{t \rightarrow +\infty} \frac{x(t)}{y(t)} = 1,$$

which, together with the fact that $y(t)$ is bounded between two positive constants for large t ,

$$\lim_{t \rightarrow +\infty} (x(t) - y(t)) = 0.$$

Theorem 3.5: In (3.13), in addition to the assumptions made in Theorem 3.4, assume further that there are continuous functions $\alpha_i(t)$ such that

$$|f_i(t, c_1) - f_i(t, c_2)| \leq \alpha_i(t) |c_1 - c_2|, \quad i = 1, 2, 3, \dots, n,$$

and for large t ,

$$\int_{t-r(t)}^t \left[\sum_{i=1}^n \int_{\tau-r(\tau)}^{\tau} \alpha_i(\tau) d\mu_i(t, s) \right] d\tau \leq K_2^{-1} e^{-M} \quad (3.19)$$

Then, for any two solutions $x(t)$ and $y(t)$ to (3.13), $\lim_{t \rightarrow +\infty} (x(t) - y(t)) = 0$.

Proof: Let $x(t)$ and $y(t)$ be two solutions of (3.13). Lemma 3.6 implies that $x(t) - y(t)$ has infinitely many zeros $\{t_n\}$ with $\lim_{n \rightarrow +\infty} t_n = +\infty$; otherwise, the proof is finished.

Let $z(t) = \frac{x(t)}{y(t)}$; then from (3.13), it follows that

$$\frac{dz(t)}{dt} = -z(t) \sum_{i=1}^n \int_{t-r(t)}^t [f_i(t, x(s)) - f_i(t, y(s))] d\mu_i(t, s) \quad (3.20)$$

In a similar argument to that in Theorem 3.4, we have

$$e^{-\bar{M}N} \leq z(t) \leq e^{\bar{M}N}, \quad (3.21)$$

for all large t .

Let $u = \lim_{t \rightarrow +\infty} \sup(z(t) - 1)$ and $v = \lim_{t \rightarrow +\infty} \sup(1 - z(t))$. Then,

$$0 \leq u \leq e^{\bar{M}N} - 1 \quad \text{and} \quad 0 \leq v \leq 1 - e^{-\bar{M}N}.$$

Choose $t_1 > 0$ such that (2.7) and (2.9) are true for all $t \geq t_1 - r(t_1)$. Also, for $t > t_1 - r(t_1)$,

$$\begin{aligned} -v - \varepsilon &< z(t) - 1 < u + \varepsilon; \\ \text{i.e., } -(v + \varepsilon)y(t) &< x(t) - y(t) < (u + \varepsilon)y(t). \end{aligned}$$

Hence,

$$-(v + \varepsilon)\alpha_i(t)y(s) < f_i(t, x(s)) - f_i(t, y(s)) < (u + \varepsilon)\alpha_i(t)y(s).$$

Assume $z(t^*)$ is a maximum or a minimum such that $t^* - r(t^*) \geq t_1$. Then, by (H1) and (H3), there is a $t_2 \in [t^* - r(t^*), t^*]$ such that $z(t_2) = 1$. Thus,

$$\ln z(t^*) = - \int_{t_2}^{t^*} \left(\sum_{i=1}^n \int_{\tau-r(\tau)}^{\tau} f_i(\tau, x(s)) d\mu_i(\tau, s) \right) d\tau$$

$$\begin{aligned}
&\leq (v + \varepsilon) \int_{t^* - r(t^*)}^{t^*} \sum_{i=1}^n \int_{\tau - r(\tau)}^{\tau} y(s) \alpha_i(\tau) d\mu_i(\tau, s) d\tau \\
&\leq (v + \varepsilon) K_2 e^M \int_{t^* - r(t^*)}^{t^*} \sum_{i=1}^n \int_{\tau - r(\tau)}^{\tau} \alpha_i(\tau) d\mu_i(\tau, s) d\tau \\
&\leq u + v,
\end{aligned}$$

And so

$$z(t^*) - 1 < e^{v+\varepsilon} - 1.$$

Similarly, we have $z(t^*) - 1 > e^{-(u+\varepsilon)} - 1$. The definitions of u and v and the preceding inequalities lead to $v - \varepsilon < 1 - e^{-(u+\varepsilon)}$, $u - \varepsilon < e^{v+\varepsilon} - 1$.

Since, we have

$$|f_i(t, c_1) - f_i(t, c_2)| \leq \alpha_i(t) |c_1 - c_2|$$

and from (2.7), then, $x(t) = x(\phi)(t)$ is oscillatory,

$$\lim_{t \rightarrow +\infty} x(\phi)(t) = 0.$$

Thus, $\lim_{t \rightarrow +\infty} (x(t) - y(t)) = 0$.

Theorem 3.6: For (3.13), in addition to the assumptions (H1)-(H4), assume further that there exists a solution $x_o(t)$ to (3.13) with $0 < l_2 \leq x_o(t) \leq l_1$ for large t , where l_1 and l_2 are constants.

Also, suppose that there are continuous functions $\alpha_i(t)$ such that

$$|f_i(t, c_1) - f_i(t, c_2)| \leq \alpha_i(t) |c_1 - c_2|, \quad i=1, 2, 3, \dots, n$$

and for a large t ,

$$\int_{t-r(t)}^t \left[\sum_{i=1}^n \int_{\tau-r(\tau)}^{\tau} \alpha_i(\tau) d\mu_i(t, s) \right] d\tau \leq l_1^{-1} \quad (3.22)$$

Then, for any two solutions $x(t)$ and $y(t)$ to (3.13), $\lim_{t \rightarrow +\infty} (x(t) - y(t)) = 0$.

Theorem 3.7: Suppose $b(t+T) = b(t)$, $r(t+T) = r(t)$, $f_i(t+T, x) = f_i(t, x)$, and $\mu_i(t+T, s) = \mu_i(t, s)$ for some $T > 0$, and the assumptions made in Theorems 3.5 or 3.6 are true. Then Eq(3.13) has a unique, globally asymptotically stable periodic solution.

Proof: Let $x(t)$ be an arbitrary positive solution of Eq(3.13). Then $x_{nT} \in C$, $n = 1, 2, 3, \dots$. Since $x(t)$ is bounded from above and the right hand side of (3.13) is completely continuous, the sequence $\{x_{nT}\}_{n=1}^{\infty}$ is pre-compact. Assume that this subsequence x_{niT} converges to $x^* \in C$, that is,

$$\lim_{i \rightarrow \infty} x(niT + \theta) = x^*(\theta), \quad \theta \in [-r, 0] \quad (3.23)$$

Since Eq(3.13) is periodic, $y(t) = x(t+T)$ is also a solution of (3.13). $\lim_{t \rightarrow +\infty} (x(t) - y(t)) = 0$ implies that

$$\lim_{i \rightarrow \infty} x(niT + T + \theta) = x^*(\theta) \quad (3.24)$$

Let $x^*(t)$ be the solution of (3.13) with initial condition as $x^*(\in C)$; then, (3.23) and (3.24) imply that $x_T^* = x^*$, or equivalently,

$$x^*(T + t) = x^*(t), \quad \text{for all } t \in R.$$

Now, from Theorem 3.5 or 3.6 we can conclude that Eq(3.13) has a unique, globally asymptotically stable periodic solution.

4. PERIODIC SOLUTIONS IN MULTI-SPECIES MODELS

We mainly concern in this topic is the global existence of periodic solutions in delayed multi-species models due to delays and/or periodicity of environments. We present on the first section of this topic is a global existence result of periodic solutions in a class of delayed autonomous Gause-type predator-prey systems. Secondly, we present an existence and uniqueness result on periodic solutions in a class of delayed periodic systems.

4.1. PERIODIC SOLUTIONS IN DELAYED GAUSE-TYPE PREDATOR-PREY SYSTEM

One of the most universally recognized models in mathematics is the classic model for interaction of a single predator species and a single prey species. Our objective in this section is to establish sufficient conditions for the global existence of non constant periodic solutions in the following Gause-type predator-prey system:

$$\begin{aligned} \dot{x}(t) &= x(t)[g(x(t)) - p(x(t))y(t)], \\ \dot{y}(t) &= y(t)[-v + h(x(t-\tau))] \end{aligned} \tag{4.1}$$

where $x(t)$, $y(t)$ stated for the population density of prey and predator at time t , respectively. We always assume that

$$y(0) > 0, x(\theta) = \psi(\theta), \theta \in [-\tau, 0], \psi \in C([-\tau, 0], R^+), x(0) > 0 \tag{4.1}'$$

Moreover, we assume the following hold:

(A1) $g(x) \in C^2([0, +\infty), R)$; there exists $x_o > 0$ such that $g(x) > 0$ for $x \in [0, x_o)$, $g(x_o) = 0$, and $g''(x) \leq 0$ for $x \geq 0$.

(A2) $p(x) \in C^1([0, +\infty), R)$, and $p(x) > 0$ for $x \geq 0$; $p(x)$ is monotone non-increasing for $x \geq 0$.

(A3) $h(x) \in C^1([0, +\infty), R)$, and $h'(x) > 0$ for $x \geq 0$; $h(0) = 0$.

Theorem4.1: Let $(x(t), y(t))$ be the solution of (4.1) and (4.1)'. Then there is a constant $M > 0$, independent of initial data, such that

$$\max \{ \lim_{t \rightarrow +\infty} \sup x(t), \lim_{t \rightarrow +\infty} \sup y(t) \} \leq M.$$

Theorem4.2[1]: suppose system (4.1) has no positive steady state, that is, $v \geq h(x_o)$. Then

$$\lim_{t \rightarrow +\infty} (x(t), y(t)) = (x_o, 0).$$

Since we want to establish the existence of periodic solutions in (4.1), it is thus necessary to assume that it has a positive steady state. Without loss of generality, we assume in the following

$$\text{(H1)} \quad g(1) = p(1), v = h(1).$$

Also without loss of generality, we can assume (by time scaling) that

$$\text{(H2)} \quad \tau = 1$$

For convenience, we make the change of variables

$$u(t) = x(t) - 1, \quad v(t) = y(t) - 1 \tag{4.2}$$

This result in (4.1)

$$\begin{aligned} \dot{u}(t) &= (1 + u(t))[g(u(t) + 1) - p(u(t) + 1)(1 + v(t))] \\ \dot{v}(t) &= (1 + v(t))[-v + h(u(t) - 1)] \end{aligned} \tag{4.3}$$

The initial condition (4.1)' becomes

$$v(0) > -1, \quad u_o = \psi, \quad \psi(\theta) \geq -1, \quad u(0) > -1. \tag{4.3}'$$

The variational system of (4.3) at (0,0) takes the form

$$\begin{aligned} \dot{u}(t) &= -\alpha u(t) - \beta v(t), \\ \dot{v}(t) &= \omega u(t - 1). \end{aligned} \tag{4.4}$$

where $\alpha = p'(1) - g'(1)$, $\beta = p(1)$, $\omega = h'(1)$. In what follows, we always assume that

$$\text{(A4)} \quad p'(1) > g'(1).$$

The characteristic equation of (4.4) is

$$\lambda^2 + \alpha\lambda + \beta\omega e^{-\lambda} = 0 \tag{4.5}$$

$$\text{Denote } q(u) = \frac{g(u+1)}{p(u+1)}, \quad u \in (-\infty, \infty).$$

We assume further that

(A5) $q(\cdot)$ is concave, i.e. $q''(\cdot) \leq 0$.

Theorem4.3: In system (4.3), assume that (A1)-(A5) and (H1), (H2) hold, and

(B) $\omega\beta > K_o(\alpha)$, where $K_o(\alpha) = \alpha \left(\frac{\sigma_o(\alpha)}{\sin \sigma_o(\alpha)} \right)$, $\sigma_o(\alpha) \in (0, \frac{\Pi}{2})$.

Then the system has at least one non-constant positive periodic solution, with period $T > 2$.

Let

$$\sigma = \sigma(\phi) \equiv \sigma_2(\phi) + 1, U_\sigma(\phi) \equiv \text{col}(U_{\sigma_2+1}(\phi), V_{\sigma_2+1}(\phi)) \quad (4.6)$$

Define an operator F on K as follows:

$$F\phi = U_\sigma(\phi) \text{ for } \phi \in K \setminus \{0\}, \text{ and } F0 = 0 \quad (4.7)$$

Lemma4.1: F maps K into K, and $F : K \setminus \{0\} \rightarrow K$ is completely continuous with respect to the C topology. If $F\phi = \phi \in K \setminus \{0\}$, then $(U(t, \phi), V(t, \phi))$ is a non-constant periodic solution of period $\sigma(\phi) > 2$.

Lemma4.2: There exists $\varepsilon > 0$ (sufficiently small) such that $\sigma_2 : B_\varepsilon \cap K \setminus \{0\} \rightarrow (0, +\infty)$ is completely continuous, where B_ε denotes the closed ball in C with radius ε .

Lemma4.3: Let $\lambda = \mu + i\sigma$ be the simple root of (4.5) with $\mu > 0$, $\sigma \in \left(0, \frac{\Pi}{2}\right)$. Then $\inf\{\|\prod_\lambda \phi\| : \phi \in K, \|\phi\| = \delta > 0\} = v(\delta) > 0$.

Now we are ready to state the proof of Theorem4.3.

Proof of Theorem4.3: By the definition of K, we know that it is a closed, bounded, convex set of infinite dimension in the Banach space $C([-1, 0], R^2)$. F as defined in (4.7) is completely continuous by lemma4.1. By lemma4.2, Theorem(3.3), and lemma4.3, we see that 0 is an ejective fixed point of F. We conclude by Theorem(3.2) that F has a fixed point ϕ in $K \setminus \{0\}$, which by lemma4.1 corresponds to a non constant periodic solution of system (4.3) with period $\sigma(\phi) > 2$. This completes the proof.

4.2. PERIODIC SOLTION IN PERIODIC SYSTEMS

We consider first the following non autonomous delay system

$$\dot{x}_i(t) = x_i(t)G_i(t, x_1(t), \dots, x_n(t), x_1(t - \tau(t)), \dots, x_n(t - \tau(t))) \quad (4.8)$$

where $x_i(t)$ is the population of the i^{th} species and $i = 1, 2, \dots, n$,

$$x = (x_1, x_2, \dots, x_n) \in R_+^n = \{x \in R^n : x_i \geq 0\}$$

Denote $IntR_+^n = \{x \in R_+^n : x_i > 0\}$. We assume that G_i is continuously differentiable and

$$\begin{aligned} \text{(H1)} \quad & \frac{\partial G_i(t, x_1, \dots, x_n, y_1, \dots, y_n)}{\partial x_j} > 0 \quad \text{for } j \neq i, \quad \text{and} \\ & \frac{\partial G_i(t, x_1, \dots, x_n, y_1, \dots, y_n)}{\partial y_k} > 0, \quad i, j, k = 1, 2, \dots, n. \end{aligned}$$

$\tau(t)$ is continuously differentiable, non negative, and bounded above by τ^* . For $x, y \in R^n$, $x \geq y$ means $x_i \geq y_i$, $i = 1, 2, \dots, n$, and $x > y$ means $x_i > y_i$, $i = 1, 2, \dots, n$.

We also assume that

$$\text{(H2)} \quad \text{there is a } p = (p_1, p_2, \dots, p_n) \in IntR_+^n \text{ such that, for } t \in R, i = 1, 2, \dots, n, \text{ and } G_i(t, p_1, \dots, p_n, p_1, \dots, p_n) < 0;$$

$$\text{(H3)} \quad G_i(t, \lambda x_1, \dots, \lambda x_n, \lambda y_1, \dots, \lambda y_n) \geq \lambda G_i(t, x_1, \dots, x_n, y_1, \dots, y_n) \text{ for } \lambda \in (0, 1] \text{ where } i = 1, 2, \dots, n;$$

$$\text{(H4)} \quad G_i(t, x_1, \dots, x_n, y_1, \dots, y_n) \text{ is uniformly continuous with respect to } (x_1, \dots, x_n, y_1, \dots, y_n) \text{ and } G_i(t + \omega, x_1, \dots, x_n, y_1, \dots, y_n) = G_i(t, x_1, \dots, x_n, y_1, \dots, y_n) \text{ for some } \omega > 0 \text{ and } i = 1, 2, \dots, n.$$

Lemma4.4: suppose that (4.8) satisfies (H1)-(H2). Then the following hold:

- i) For any $\eta \in IntR_+^n$, there exists $M(\eta) \in IntR_+^n$ such that, for any $\phi \in C$ with $0 \leq \phi \leq \eta$ on $[-\tau^*, 0]$, one has $0 \leq x(t, \phi) \leq M(\eta)$ for all $t \geq 0$;

- ii) There exists a $\Delta \in \text{Int}R_+^n$ such that for any $\alpha \in R_+^n$, there is a constant $T = T(\alpha) > 0$ such that, for any $\phi \in C$ with $0 \leq \phi \leq \alpha$ on $[-\tau^*, 0]$, one has $0 \leq x(t, \phi) \leq \Delta$ for all $t \geq T(\alpha)$.

If (H3) is replaced by the following assumption:

(H3)* $G_i(t, f_1(\lambda)x_1, \dots, f_n(\lambda)x_n, f_1(\lambda)y_1, \dots, f_n(\lambda)y_n) \geq g_i(\lambda) \times G_i(t, x_1, \dots, x_n, y_1, \dots, y_n)$ for $\lambda \in [0, 1]$ and $i = 1, 2, \dots, n$, where $f_i, g_i : [0, 1] \rightarrow [0, 1]$ satisfy $f_i(0) = 0, g_i(0) = 0, f_i(1) = g_i(1) = 1$, and f_i, g_i are non decreasing $i = 1, 2, \dots, n$, then lemma4.4 is still true.

Lemma4.5[1] (Horn's fixed point theorem). Let $S_0 \subset S_1 \subset S_2$ be convex subsets of the Banach space X , with S_0 and S_2 compact and S_1 open relative to S_2 . Let $P: S_2 \rightarrow X$ be continuous mapping such that, for some integer $m > 0$,

- a) $P^j(S_1) \subseteq S_2, 1 \leq j \leq m-1,$ and
- b) $P^j(S_1) \subseteq S_0, m \leq j \leq 2m-1,$

Then P has a fixed point in S_0 .

Consider now the periodic system

$$\dot{x}_i(t) = x_i(t)F_i(t, x_1(t), \dots, x_n(t), x_1(t-\tau(t)), \dots, x_n(t-\tau(t))) \quad (4.9)$$

where $x = (x_1, x_2, \dots, x_n) \in R_+^n$. We assume $F_i(t, x_1, \dots, x_n, y_1, \dots, y_n)$ is continuously differentiable in its variables, and there is an $\omega > 0$ such that

$$F_i(t + \omega, x_1, \dots, x_n, y_1, \dots, y_n) = F_i(t, x_1, \dots, x_n, y_1, \dots, y_n)$$

for $i = 1, 2, \dots, n$; $\tau(t)$ is also a continuously ω -periodic function, and $\tau(t) \geq 0$ for $t \in R$.

We denote $\tau^* = \max_{0 \leq t \leq \omega} \tau(t)$. Assume further that the following hold:

(A1) For any $\eta_1, \eta_2 \in R_+^n$ with $0 < \eta_1 \leq \eta_2$, there exists $\gamma(\eta_1, \eta_2) \in \text{Int}R_+^n$ such that, for any $\phi \in C$ with $\eta_1 \leq \phi(\theta) \leq \eta_2$ on $[-\tau^*, 0]$, one has $x(t, \phi) \geq \gamma(\eta_1, \eta_2)$ for all $t \geq 0$.

(A2) There exists $\delta \in \text{Int}R_+^n$ such that, for any $\phi(\theta) \in C$ with $\phi(\theta) \geq 0, \phi(0) > 0$,
 $\lim_{t \rightarrow +\infty} \inf x(t, \phi) \geq \delta$,

where $\lim_{t \rightarrow +\infty} \inf x(t, \phi) = (\lim_{t \rightarrow +\infty} \inf x_1(t, \phi), \dots, \lim_{t \rightarrow +\infty} \inf x_n(t, \phi))$;

(A3) For any $K > 0$, there exists an $L(K) > 0$ such that, for $\|\phi\| \leq K$,
 $|F(t, \phi)| = \sum_{i=1}^n |F_i(t, \phi)| \leq L(K)$ for $t \in R$;

(A4) There are $G_i(t, x_1(t), \dots, x_n(t), x_1(t - \tau(t)), \dots, x_n(t - \tau(t)))$ such that

$F_i(t, x_1(t), \dots, x_n(t), x_1(t - \tau(t)), \dots, x_n(t - \tau(t))) \leq G_i(t, x_1(t), \dots, x_n(t), x_1(t - \tau(t)), \dots, x_n(t - \tau(t)))$ for
 $i = 1, 2, \dots, n$, where $G_i(t, x_1(t), \dots, x_n(t), x_1(t - \tau(t)), \dots, x_n(t - \tau(t))) (i = 1, 2, \dots, n)$ satisfies the
assumptions (H1)-(H4).

Theorem 4.4[1]: Suppose that the system (4.9) satisfies (A1)-(A4) and has no positive steady state. Then the system (4.9) has a non constant positive ω – periodic solution.

The following assumption is less restrictive than (H2).

(H2)* There exists positive ω – periodic function $B_1(t), \dots, B_n(t)$ and $p = (p_1, p_2, \dots, p_n) \in \text{Int}R_+^n$

such that, for $t \in R$ and $i = 1, 2, \dots, n$, $G_i(t, p_1 B_1(t), \dots, p_n B_n(t), p_1 B_1(t - \tau(t)), \dots, p_n B_n(t - \tau(t))) + \frac{|\dot{B}_i(t)|}{B_i(t)} < 0$

Theorem 4.5[1]: Suppose that system (4.9) satisfies (A1)-(A4), and

$G_i(t, x_1(t), \dots, x_n(t), x_1(t - \tau(t)), \dots, x_n(t - \tau(t))) (i = 1, 2, \dots, n)$ satisfies the assumptions (H1), (H2)*,
(H3)*, and (H4). Then (4.9) has a non-constant positive ω – periodic solution provided that it has
no positive steady states.

5. CONCLUSION

The use of ordinary and partial differential equations to model in the real life has a long history. In the real life an initial value problem (IVP) for ordinary differential equations (ODEs) and partial differential equations (PDEs) can be modeled in physics, engineering, biology, medicine, etc. As these models are used in an attempt to better our understanding of more and more complicated phenomena, it is becoming clear that the simplest models cannot capture the rich variety of dynamics observed in natural systems. There are many possible approaches to dealing with these complexities. On one hand, one can construct larger systems of ordinary and partial differential equations, that is, systems with more differential equations. These systems can be quite good at approximating observed behavior, but they suffer from the downfall of containing many parameters, often signifying quantities which cannot be determined experimentally.

Another approach which is the inclusion of time delay terms in the differential equations. The delay differential equation models have the advantage of combining a simple, intuitive derivation with a wide variety of possible behavior regimes for a single system. Now a delay differential equation (DDE) models are becoming more common, appearing in many branches of real life modeling like in physics, engineering, biology, medicine, etc. Modeling using DDEs the past time and the present is included, and then we predict the future.

Moreover, the existence of positive periodic solutions of systems of functional differential equations would be an important subject for modeling in biology, ecology, medicine or other fields.

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