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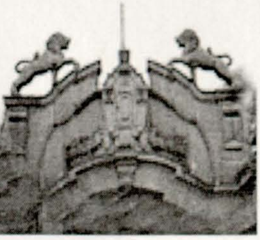

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Compact Linear Operators on Hilbert Space and their Application to Integral Equations

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Declaration

I declare that this project work has been composed by me and that no part of the project work has formed the basis for the award of any Degree, Diploma, Associate ship, Fellow ship, or any other similar title to me.

Anteneh Getachew

Permission

This is to certify that this project is compiled by Anteneh Getachew in the department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluations by examiners and eventual defense.

Seid Mohammed (PhD) ______________________________

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Abstract

Perhaps the most theory developed about eigenvalue, eigenvector and existence of a solution of a certain operator equation is for matrix operators, but the first parts to be considered apart from matrices are integral operators, since the classical formulation from physics, chemistry, engineering, statistics are of this nature and the study of an integral operators gives birth to the modern functional analysis. And the most common problems in applied mathematics are differential operators which are fruitful sources of integral equations. Due to various advantages of having integral equation rather than differential equation usually we would like to convert and formulate differential equation in to integral equation. Many of integral operators encountered in application are bounded operators and many of them are, in fact, in special classes of bounded operators called compact operators and again the most important classes of compact are the Hilbert-Schmidt operators.

Goal of the project: In applied mathematics we usually encounter with a linear operators $T: H \to H$ on a Hilbert space H. Such an operators gives to the rise of an equation $I - \lambda T$ where I is identity operator from H into H and λ is any constant complex number. Under such condition when T is an integral operator the resulting equation $I - \lambda T$ is called an integral equation and integral operators (equation) involve a function space H which is often infinite dimensional.

Due to this fact the main goal of this project is:

1. To explore and deal with eigenvalue, eigenfunction properties of self adjoint and compact linear operator $T: H \to H$ emphasizing on an infinite dimensional Hilbert space H.
2. To determine the solvability of an equation $(I - \lambda T)u = f$; with special emphasis given to $I - \lambda T$ is Fredholm integral equation.

Table of Content

Chapter 1: Preliminary Concepts

Notations

Enumeration of Theorems, Lemmas, Corollaries, Propositions and Definitions: Theorems, Lemmas, Corollaries, Propositions and Definitions are not enumerated together, rather they are enumerated separately. The reference to the Theorem or Lemma or Corollary or Proposition or Definition is analogues to what is said above. For instance Theorem 1.2.6 is cited as Theorem 6 in section 1.2 of chapter 1.

Symbol or abbreviation	**Description**
$\mathbb{C}$	Complex numbers
$\mathbb{Z}$	Integer numbers
$\mathbb{N}$	Natural numbers
$\mathbb{R}$	Real numbers
K	Complex numbers or ~~Integer~~ Real numbers
T	Operator
$T(V)$	The image of V under an operator T
$D(T)$	Domain of an operator T
$N(T)$	The null space or kernel of T
$T^\star$	The adjoint operator of the operator T
$B(V,W)$	The set of bounded linear operators from V in to W
$B(V)$	The set of bounded linear operators from V in to V
δ_{ij}	Kronecker symbol, $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$
$\mathrm{Span}\{\dots\}$	Space spanned by $\{\dots\}$
$\overline{f(t)}$	The complex conjugate of $f(t)$
$k^\star(s,t)$	Conjugate kernel $(= \overline{k(s,t)})$

Introduction

In this project we will use the theory developed so far to introduce the concept of eigenvalue and eigenvector for a linear operators in a Hilbert space. From linear algebra we know how self adjoint matrix can be diagonalized; this means there is an orthonormal basis $\{e_i\}$ the (finite dimensional) vector space such that the linear mapping $T: \mathbb{C}^n \to \mathbb{C}^n$ corresponding to the matrix can be expressed as,

$$Tx = \sum_{j=1}^{n} \lambda_j \langle x, e_j \rangle e_j \quad .$$

Here we recognize that $\langle x, e_j \rangle = \alpha_j , j = 1, \ldots, n$ as the coordinate of x, and λ_j , $j = 1, \ldots, n$ are eigenvalues with corresponding eigenvectors e_j , $j = 1, \ldots, n$ and there are a non trivial solution to the equation

$$Tx = \lambda x \quad .$$

In order to study eigenvalue, eigenfunction properties of integral operator $T: H \to H$ we will extend the finite dimensional results to the self adjoint and compact linear operators $T: H \to H$ in an infinite dimensional Hilbert space H for which we have a celebrated spectral theorem which is shared by large class of interesting integral equations and differential equation.

Differential equations are fruitful sources of integral equations and usually differential equations are preferred to be changed in to integral equations in application. One obvious reason for using the integral equation rather than differential equation is that all of the conditions specifying the initial value problems or boundary value problems for differential equation can be condensed in to single integral equation. And some of the advantages of replacing differentiation with integration is, integration is smooth, a features which has a significant implications when approximate solution are sought.

Let H be an infinite dimensional space and let $T: H \to H$ be an integral operator given by $Tu = f$, such an equation is either linear or non linear integral equation. In this project the main emphasis and discussion is about linear integral operators and equations;

that is an equation involving unknown function under one or more integrals and which satisfies $T(\alpha u_1+\beta u_2)=\alpha Tu_1+\beta Tu_2$ for arbitrary constants α and β. As integral operators play an important role in applications, this project aimed to explore for an answer to the following two questions; what are eigenvalue, eigenfunctions properties of compact linear and self adjoint operator $T: H \to H$ in a Hilbert space H? When dose the Fredholm integral equation $(I-\lambda T)u=f$ is solvable? Since many of integral operators encountered in application are bounded operators and many of them are, in fact, in special classes of bounded operators called compact operators, hence in this project in order to study eigenvalue, eigenfunction properties of integral operators or equation we consider a more general compact operator which are motivated by matrix operators and several spectral theory properties are discussed and investigated for self adjoint and compact linear operators $T: H \to H$ in a Hilbert space H. Moreover, we will make heavy use of the application of Fredholm Alternative Theorem to determine and study the solvability of Fredholm integral equation $(I-\lambda T)u=f$ where $T: H \to H$ is compact linear operator (integral operator) .

This project contains four Chapters, each being divided in to several sections. The topics included have been selected, not only for their scientific importance, but also because they allow a logical flow for the development of idea to deal with property of eigenvalue, eigenfunction of compact linear operator in a Hilbert space as well as for the development of Fredholm Alternative Theorem for an integral equation $(I-\lambda T)u=f$, where $T: H \to H$ is compact linear operator.

Chapter 1 of this project provides the necessary backgrounds that are useful for the desired discussion. In this chapter the basic equalities and inequalities in Hilbert space, the most common integral equations the so called Fredholm integral equation and connection of differential equations with integral equations are discussed.

Chapter 2 is devoted for bounded operators in Hilbert space and existence of an operator $T^\star$ satisfying $\langle Tx,y\rangle=\langle x,T^\star y\rangle$ for all $x,y\in H$.This two properties of an operator that is boundedness of an operator and existence of an operator $T^\star$ satisfying $\langle Tx, y\rangle = \langle x, T^\star y\rangle$ for all $x, y \in H$ are the most important concepts when dealing with spectral theory of compact linear operator and Fredholm Alternative Theorem.

Chapter 3 deals with special class of bounded operators called compact operators. Here this chapter covers some of the sufficient conditions satisfied by an integral operator T to be a compact operator. In addition the most common compact integral operators called Hilbert Schmidt operators are discussed.

The last chapter (chapter 4) is devoted to desired exploration and discussion for eigenvalue and eigenfunctions properties of compact linear operators $T: H \to H$ in a Hilbert space H when T is self adjoint operator and we will see Fredholm Alternative Theorem.

For the purpose of this project an integration theory that integrates a continuous function in the case of $\mathbb{R}$, over the bounded intervals, in the case of $\mathbb{R}^2$, over the bounded rectangles will normally suffice,

a. If f is integrable on $[a,b]$, then $|f|$ is integrable and

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|\, dt.$$

b. Fundamental Theorem of Calculus: If $f: [a,b] \to K$ is continuously differentiable, then $\int_a^b f'(t)dt = f(b) - f(a)$.

c. Fubini's Theorem: If $f: [a,b] \times [c,d] \to K$ is continuous

$\int_c^d \int_a^b f(s,t)dsdt$, $\int_a^b \int_c^d f(s,t)dtds$ exists and are equal.

d. Leibniz integral rule: If $\alpha: \mathbb{R} \to \mathbb{R}$, $\beta: \mathbb{R} \to \mathbb{R}$ continuously differentiable and if $f: \mathbb{R}^2 \to \mathbb{R}$ and $\frac{\partial f}{\partial s}$ are continuous, then

$$\frac{d}{ds}\left(\int_{\alpha(s)}^{\beta(s)} f(s,t)\, dt\right) = \int_{\alpha(s)}^{\beta(s)} \frac{\partial f}{\partial s}(s,t)dt + f(\beta(s), s)\frac{d\beta}{ds} - f(\alpha(s), s)\frac{d\alpha}{ds}$$

Chapter 1

Preliminary Concepts

1.1 Normed Vector Space

***Definition 1.1.1*:-** Let V be a vector space. A norm on V is a function $\|.\|: V \to [0, \infty)$ satisfying for all $x, y \in V$ and $\alpha \in K$:

i. $\|x + y\| \leq \|x\| + \|y\|$
ii. $\|\alpha x\| = |\alpha| \|x\|$
iii. $\|x\| \geq 0 \; and \; \|x\| = 0 \; iff \; x = 0$

If V is a vector space and $\|.\|$ is a norm on V then, $(V, \|.\|)$ is called normed vector space. Any normed vector space is a metric space when we use the norm induced metric $d(x, y) = \|x - y\|$.

***Definition 1.1.2*:-** A normed vector space that is complete in a metric induced by the norm is called Banach space.

1.2 Inner Product Space

***Definition 1.1.3*:-** Let V be a vector space. An inner product is a mapping $\langle .,. \rangle : V \times V \to K$ satisfying,

i. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
ii. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \; for \, all \; x, y \in V and \; \alpha . \beta \in K$
iii. $\langle x, x \rangle \geq 0 \; and \; \langle x, x \rangle = 0 \; iff \; x = 0$

Notice that the conditions imply that $\langle z, \alpha x + \beta y \rangle = \overline{\alpha} \langle z, x \rangle + \overline{\beta} \langle z, y \rangle$. So the inner product is linear in the first argument and conjugate linear in the second. The vector V with an inner product $\langle .,. \rangle$ is called inner product space.

Definition 1.1.4:- Let V be an inner product space.

a. We say x *and* y in V are orthogonal if $\langle x, y\rangle = 0$ and we denote $x \perp y$.
b. The set of vectors $\{x_i\}$ in V is said to be orthonormal set if $\langle x_n, x_m\rangle = 0 \,, for\; n \neq m$ and $\langle x_{n,}\, x_m\rangle = 1 \,, for\; n = m.$
c. The induced norm on V is defined by $\|x\| = \langle x, x\rangle^{1/2}.$

Remark 1.1.1:- Let V be an inner product space with an inner product $\langle .,.\rangle$ on V. Then,

i. ***Cauchy Schwarz inequality***

$$|\langle x, y\rangle| \leq \|x\|\,\|y\| \;, for\; all \quad x, y \in V$$

ii. ***Parallelogram law***

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \;, for\; all\;\; x, y \in V$$

iii. **If $x \perp y$, then**

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

1.3 Infinite Dimensional Spaces

Let V be a vector space. If V contains a finite set of n linearly independent vectors $x_{1,}\ldots, x_n$ and moreover the set $n+1$ vectors is linearly dependent we say that V is finite dimensional vector space and the dimension of V is n. It follows that any vector x is a linear combination $x = \alpha_1\, x_1 + \cdots + \alpha_n\, x_n$ of at most n linearly independent vectors. If there is no such, we say V is infinite dimensional.

Examples of infinite dimensional spaces:-

i. The sequence space $l^p\ (1 \leq p < \infty)$, that is the set of sequence of numbers $\{x_n\}_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} |x_n|^p < \infty$.

   $l^p\ (1 \leq p < \infty)$, is normed vector space with the norm induced on l^p

$$\|x\| = \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{\frac{1}{p}} \quad , \text{for } x = \{x_n\}_{n=1}^{\infty} \in l^p.$$

ii. Function spaces such as,

a. The set of all continuous functions defined on a closed interval $[a, b]$ denoted by $C[a, b]$ is an infinite dimensional vector space.

b. L^p Space.

Consider the real or complex valued function $f(x)$ of real variable x defined in an interval $[a, b]$.

- Thus, for $1 \leq p < \infty$, $L^p[a, b]$ denotes the set of all measurable function defined on $[a, b]$ such that

  $$\int_a^b |f(x)|^p \, dx < \infty$$

  where, the integral is taken in the sense of Lebesgue.
- L^p Space norm is defined by $\|f\| = (\int_a^b |f(x)|^p \, dx)^{\frac{1}{p}}$.
- If $f, g \in L^p[a, b]$, we define an inner product $\langle f, g \rangle$ on $L^p[a, b]$ by the equation $\langle f, g \rangle = \int_a^b f(x) \, \overline{g(x)} dx$ where the bar denotes the complex conjugate.

1.4 Hilbert Space

Definition 1.4.1:- A vector space with an inner product that is a Banach space with respect to the induced norm is called a Hilbert space.

i.e. a complete inner product space is called a Hilbert space (where the norm is the natural norm).

Throughout the rest of our discussion we will make heavy use of Hilbert spaces $L^2[a, b]$, and we will use l^2 space to illustrate some concepts.

***Example* 1:−** $L^2[a, b]$ and l^2 are an infinite dimensional Hilbert spaces.

Remark 1.4.1:- Let H be a Hilbert space.

i. **Bessel's inequality**:- If $\{e_n\}$ is an orthonormal sequence in H, then

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|^2 \quad \text{, for all } x \in H.$$

ii. Parsavel equation:- $\{e_n\}$ is an orthonormal basis if an only if

$$\sum_{n=1}^{\infty}|(x,e_n)|^2 = \|x\|^2 \text{ , for all } x \in H.$$

1.5 Metric space and Continuous Mapping

Definition 1.5.1: A metric space is a pair (X,d), where X is a set and d is a metric (distance function) on X, that is, $d(.,.): X \times X \to [0,\infty)$ such that for all $x,y,z \in X$ we have:

i. d is real valued, finite and nonnegative.
ii. $d(x,y) = 0$ if and only if $x = y$.
iii. $d(x,y) = d(y,x)$ (symmetry)
iv. $d(x,y) \le d(x,z) + d(z,y)$ (triangle inequality)

Notice that any normed vector space is a metric space when we use the norm induced metric $d(x,y) = \|x-y\|$. And since $|\|x\| - \|y\|| \le \|x-y\|$ any norm is continuous.

Definition1.5.2: Let $X = (X,d_1)$ and $Y = (Y,d_2)$ be metric spaces. A mapping $T: X \to Y$ is said to be continous at a point $x_o \in X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$d_2(Tx,Tx_o) < \varepsilon \quad \text{for all } x \text{ satisfying} \quad d_1(x,x_o) < \delta.$$

Theorem 1.5.1: A mapping $T: X \to Y$ of a metric space (X,d_1) into a metric space (Y,d_2) is continuous at a point $x_o \in X$ if and only if

$$x_n \to x_o \text{ implies } Tx_n \to Tx_o.$$

Proof: Assume T to be continuous at $x_o \in X$. Thus by the definition given $\varepsilon > 0$ there is $\delta > 0$ such that

$$d_1(x,x_o) < \delta \quad \text{implies} \quad d_2(Tx,Tx_o) < \varepsilon.$$

Let $x_n \to x_o$. Then there is an N such that for all $n > N$ we have

$$d_1(x_n,x_o) < \delta.$$

Hence for all $n > N$,

$$d_2(Tx_n,Tx_o) < \varepsilon.$$

By definition this means $Tx_n \to Tx_o$.
Conversely, we assume that

$$x_n \to x_o \text{ implies } Tx_n \to Tx_o$$

and prove that then T is continuous at x_o. Suppose this is false. Then there is an $\varepsilon > 0$ such that for every $\delta > 0$ there is an $x \neq x_o$ satisfying

$$d_1(x, x_o) < \delta \text{ but } d_2(Tx, Tx_o) \geq \varepsilon.$$

In particular, for $\delta = \frac{1}{n}$ there is an x_n satisfying

$$d_1(x_n, x_o) < \frac{1}{n} \text{ but } d_2(Tx_n, Tx_o) \geq \varepsilon.$$

Clearly $x_n \to x_o$ but $\{Tx_n\}$ does not converge to Tx_o.This contradicts $Tx_n \to Tx_o$ and proves the theorem.■
This Theorem is very useful, and we shall need it in proving other facts.

1.6 Integral Equations

Definition 1.6.1:- An equation in which an unknown function appears under one or more signs of integration is called an integral equation.

The equations

$$f(s) = \int_a^b k(s,t)u(t)dt \quad ,(a \leq s \leq b) \tag{1}$$

$$u(s) = f(s) + \int_a^b k(s,t)u(t)dt \quad ,(a \leq s \leq b) \tag{2}$$

$$u(s) = \int_a^b k(s,t)(u(t))^2 \, dt \quad ,(a \leq s \leq b) \tag{3}$$

$$u(s) = \int_a^b \int_a^b k(s,t,z)u(t)u(z)dtdz \quad ,(a \leq s \leq b) \tag{4}$$

in each of which *u(s)* is the unknown function, and all other functions are regard as given, are integral equations.

The equations (1) and (2) can be written in the form $(Tu)(s) = f(s)$, where the expression Tu containing the unknown function $u(s)$ is linear in the sense that

$$T(\alpha u_1(s) + \beta u_2(s)) = \alpha(Tu_1)(s) + \beta(Tu_2)(s) \text{ , for any constants } \alpha \text{ and } \beta.$$

Thus, for equation (1) , $(Tu)(s) = \int_a^b k(s,t)u(t)dt$

and for equation (2) , $(Tu)(s) = u(s) - \int_a^b k(s,t)u(t)dt$

Definition 1.6.2:- The equation of the type $(Tu)(s) = f(s)$, where $(Tu)(s)$ is a linear expression in which $u(s)$ appears under one or more signs of integration are called linear integral equations.

For instance, the equation of the form (1) and (2) are linear integral equations, but (3) and (4) are not.

Linear integral equations of the form $f(s) = \int_a^b k(s,t)u(t)dt$, $(a \leq s \leq b)$, where the unknown function appears under a sign of integration and nowhere else in the equation are called integral equation of first kind. And linear integral equations of the form $u(s) = f(s) + \int_a^b k(s,t)u(t)dt$, $(a \leq s \leq b)$, where the unknown function appears under a sign of integration and nowhere else in the equation are called integral equation of second kind.

In both first and second kind equations the function $k(s,t)$ which is defined on $[a,b] \times [a,b]$ of the (s,t) plane is called **kernel** of the equation.

If we take $f(s) = 0$ in equation (2) we obtain homogenous equation of second kind

$$u(s) = \int_a^b k(s,t)u(t)dt \quad ,(a \leq s \leq b) \quad .$$

It is often convent to introduce a parameter λ in to the equation of second kind which then assumes a form $u(s) = f(s) + \lambda \int_a^b k(s,t)u(t)dt$, $(a \leq s \leq b)$ .We can then obtain useful information by studying what happens when λ is allowed to vary in $\mathbb{R}$. The most commonly occurring integral equations in many of applications are Fredholm integral equations and Voltera integral equations. Next we will see the difference between these two integral equations and note that in our discussion we will emphasize on Fredholm integral equations that are frequently occurring than Voltera integral equation.

Definition 1.6.3:- An integral equation of the form

i. $f(s) = \int_a^b k(s,t)u(t)dt \quad ,(a \leq s \leq b)$ where *u(s)* is the unknown function and all other functions are regard as given is called **Fredholm integral equation** of first kind.

ii. $u(s) = f(s) + \int_a^b k(s,t)u(t)dt \quad ,(a \leq s \leq b)$ where *u(s)* is the unknown function and all other functions are regard as given is called **Fredholm integral equation** of second kind.

***Example* 2**: $-$ The integral equation $f(s) = \int_1^3 ln\left|\frac{t^s}{s^t}\right| u(t)dt$ is Fredholm integral equation of first kind with kernel involving logarithmic function.

And the integral equation $f(s) = u(s) - \lambda\int_0^{2\pi} sin(s+t)u(t)dt$ is Fredholm integral equation of second kind with kernel involving trigonometric function.

Definition 1.6.4:- An integral equation of the form

i. $f(s) = \int_a^s k(s,t)u(t)dt$, where u is unkown function and all other functions are regard as given and f and k are functions defined on $[a,b]$ and on the triangle $a \leq t \leq s \leq b$ respectively is called **Voltera integral equation** of first kind.

ii. $u(s) = f(s) + \int_a^s k(s,t)u(t)dt$, where u is unkown function and all other functions are regard as given and f and k are functions defined on $[\mathrm{a},\mathrm{b}]$ and on the triangle $a \leq t \leq s \leq b$ respectively is called **Voltera integral equation** of second kind.

***Example* 3**: The integral equation $f(s) = u(s) + \alpha\int_1^s (s-t)e^{\beta(s-t)}\,dt$ is called Voltera integral equation of second kind.

- If $a = \pm\infty$ or $b = \pm\infty$, or if $\int_a^b \int_a^b |k(s,t)|^2 \, dtds = \infty$ then the integral equation of the form $u(s) = f(s) + \int_a^b k(s,t)u(t)dt$ is called singular integral equation.

Definition 1.6.5:- The integral equation of the form

$$f(s) = \int_a^s \frac{u(t)}{(S-t)^{\alpha}} dt \quad , \quad 0 < \alpha < 1 \ ,$$

where $f(S)$ is given function and $u(t)$ is unknown function is called Abel's integral equation.

1.7 Connection of Integral Equations with Differential Equations

The theories of ordinary and partial differential equations are fruitful sources of integral equations. That is integral equations can arise from ordinary differential equations. We shall sketch here some of the ways in which integral equations can arise from ordinary differential equation.

***Example* 4 :** Consider the boundary value problem

$$\frac{d^2 f}{ds^2} = u(s) \ , f(0) = 0 \ , \ f(1) = 0.$$

$$\frac{d^2 f}{ds^2} = \frac{d}{ds}\left(\frac{df}{ds}\right) = \frac{d}{ds}(f^{'}(s)) = u(s)$$

Then integrating both side of the equation from 0 to s we have

$f^{'}(s) = \int_0^s u(t)dt + c_1$ where c_1 is constant.

Then integrating both sides of the equation $f^{'}(s) = \int_0^s u(t)dt + c_1$ from *0* to *s*, we have, $f(s) = \int_0^s \left(\int_0^r u(t)dt + c_1\right) dr + c_2$, where c_2 is constant.

Now Appling integration by parts for an integral $\int_0^s \left(\int_0^r u(t)dt\right) dr$,

Let $\quad g = \int_0^r u(t)dt \quad and \quad dh = dr$

$\quad dg = u(r)dr \quad and \quad h = \mathrm{r}$

Thus, $\int_0^s \left(\int_0^r u(t)dt\right) dr = gh - \int_0^s hdg$

$$= s\int_0^s u(t)\,dt - \int_0^s tu(t)dt$$

$$= \int_0^s (s-t)\,u(t)dt$$

Hence, $f(s) = \int_0^s (s-t)u(t)dt + c_1 s + c_2$

Then using the boundary conditions $f(0) = 0$ and $f(1) = 0$ we have,

$$c_2 = 0 \quad and \quad c_1 = -\int_0^1 (1-t)u(t)dt$$

Thus, $$f(s) = \int_0^s (s-t)u(t)dt - s\int_0^1 (1-t)u(t)dt$$

But, $$\int_0^1 (1-t)u(t)dt = \int_0^s (1-t)u(t)dt + \int_s^1 (1-t)u(t)dt$$

Hence, $$f(s) = \int_0^s (s-t)u(t)dt - s\int_0^s (1-t)u(t)dt - s\int_s^1 (1-t)u(t)dt$$

$$= \int_0^s \big((s-t) - s(1-t)\big)\,u(t)dt - s\int_s^1 (1-t)\,u(t)dt$$

$$= \int_0^s t(s-1)u(t)\,dt + \int_s^1 s(t-1)\,u(t)dt$$

Hence, $f(s) = \int_0^1 k(s,t)\,u(t)dt$ where, $k(s,t) = \begin{cases} t(s-1)\,, & 0 \le t < s \le 1 \\ s(t-1)\,, & 0 \le s < t \le 1 \end{cases}$

is an equivalent integral representation of the boundary value problem.

Let see another method that converts initial value problem to an integral equation. Before outlining the method needed we wish to recall a useful transformation formula

$$\int_0^x \int_0^{x_1} \ldots \int_0^{x_{n-1}} f(x_n)\,dx_n \ldots dx_1 = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)\,dt$$

that converts a multiple integral in to single integral which appears in calculus. This is an essential and useful formula that will be employed in the method that will be used in conversion technique.

For practical consideration the formulas

$$\int_0^s \int_0^s f(t)dtdt = \int_0^s (s-t)f(t)dt \tag{5}$$

$$\int_0^s \int_0^s \int_0^s f(t)dtdtdt = \frac{1}{2!}\int_0^s (s-t)^2 f(t)dt \tag{6}$$

are the two special cases of the formula given above, and mostly used formulas that will transform double and triple integrals respectively to single integral.

For simplicity reasons, we prove (5) that convert double integral in to single integral. Since the right side of the (5) is a function of s allows us to set the equation

$$G(s) = \int_0^s (s-t)\, f(t)dt \tag{7}$$

Differentiating both side of (5) and using Leibnitz rule we obtain

$$G'(s) = \int_0^s f(t)dt \tag{8}$$

Integrating both side of (8) from 0 to s noting that $G(0) = 0$ from (7), we find

$$G(s) = \int_0^s \int_0^s f(t)dtdt$$

Hence, $\int_0^s \int_0^s f(t)dtdt = \int_0^s (s-t)f(t)dt$

***Example* 5**: Converting the initial value problem $y''' - 3y'' - 6y' + 5y = 0$ subjected to the initial conditions $y(0) = y'(0) = y''(0) = 1$ to its equivalent integral equation.

Now set $y'''(s) = u(s)$ (9)

then integrating both side from 0 to s and using initial condition $y''(0) = 1$ we get,

$$y''(s) = 1 + \int_0^s u(t)dt \tag{10}$$

Then integrating (10) from 0 to s and using initial condition $y'(0) = 1$ we get,

$$y'(s) = 1 + s + \int_0^s \int_0^s u(t)dtdt \tag{11}$$

Then integrating (11) from 0 to s and using initial condition $y(0) = 1$ we get,

$$y(s) = 1 + s + \frac{1}{2}s^2 + \int_0^s \int_0^s \int_0^s u(t)dtdtdt \tag{12}$$

Transforming the double and triple integrals (11) and (12) in to single integral using the formula we will obtain,

$$y'(s) = 1 + s + \int_0^s (s - t)u(t)dt \tag{13}$$

$$y(s) = 1 + s + \frac{1}{2}s^2 + \frac{1}{2}\int_0^s (s - t)^2\, u(t)dt \tag{14}$$

Substituting (9), (13) and (14) in to the given differential equation, we will obtain

$$u(s) = 4 + s - \frac{5}{2}s^2 + \int_0^s \left(3 + 6(s - t) - \frac{5}{2}(s - t)^2\right)u(t)dt$$

that is the equivalent Voltera integral equation form.

Chapter 2

Bounded Linear Operators in Hilbert Space and Fredholm Alternative

2.1 Bounded Linear Operators in Hilbert Space

Definition 2.1.1:- A mapping from a normed space V in to anther normed space W is called an operator.

Definition 2.1.2:- Let H be a Hilbert space and let $T: H \rightarrow H$ be a mapping. Then

i. T is linear if $T(\alpha f + \beta g) = \alpha Tf + \beta Tg$ for all $f, g \in H$, and $\alpha, \beta \in K$.
ii. T is bounded if there exist $M>0$ such that $\|Tf\| \leq M\|f\|$ for all $f \in H$.

An operator T which satisfies both i and ii is called bounded linear operator. I.e. an operator which is both bounded and linear is called bounded linear operator. The norm of abounded operator T is $\|T\| = sup_{f\neq 0}\frac{\|Tf\|}{\|f\|}$. That $\|T\|$ is the smallest constant M such that $\|Tf\| \leq M\|f\|$ for all $f \in H$.

Example **6**: Let $T: L^2[0,1] \rightarrow L^2[0,1]$ is an operator given by $Tf = f$.

For $f, g \in L^2[0,1]$ *and* $\alpha, \beta \in \mathbb{R}$, we have,

$$T(\alpha f + \beta g) = \alpha f + \beta g = \alpha Tf + \beta Tg$$ hence, T is linear.

And, $\|Tf\| = \int_0^1 |Tf|^2 \, dt = \left(\int_0^1 |f(t)|^2 dt\right)^{\frac{1}{2}} = \|f\|$.

That is, $\|Tf\| \leq M\|f\| \; for \; all \; f \in L^2[0,1]$ and $M = 1$ hence, T is bounded.

Therefore, the identity operator $T: L^2[0,1] \rightarrow L^2[0,1]$ is bounded linear operator.

Example **7**: Let $T: L^2[0,1] \rightarrow L^2[0,1]$ is a mapping given by $Tf = \int_0^1 f(t) \, dt$. We need to show that T is bounded linear operator.

Let $f, g \in L^2[0,1]$ and $\alpha, \beta \in \mathbb{R}$. Then

$T(\alpha f + \beta g) = \int_0^1 (\alpha f + \beta g)(t)\, dt = \int_0^1 \big(\alpha f(t) + \beta g(t)\big)\, dt$

$$\begin{aligned}\int_0^1 \big(\alpha f(t) + \beta g(t)\big)\, dt &= \int_0^1 \alpha f(t)\, dt + \int_0^1 \beta g(t) dt \\ &= \alpha \int_0^1 f(t)\, dt + \beta \int_0^1 g(t) dt \\ &= \alpha T f + \beta T g\text{ , hence } T \text{ is linear.}\end{aligned}$$

And $\|Tf\|^2 = \int_0^1 \left|\int_0^1 f(t)\, dt\right|^2 ds \leq \int_0^1 \left(\int_0^1 |f(t)|\, dt\right)^2 ds$

But, $\int_0^1 |f(t)|\, dt = \int_0^1 |f(t).1|\, dt \leq \|f\| \|1\|$ *by Hölders inequality*

Thus, $\|Tf\|^2 \leq \int_0^1 \|f\|^2 ds = \|f\|^2$

Hence, for $M = 1, \|Tf\| \leq M\|f\| \; for\ all\ f \in L^2[0,1]$

Hence, T is bounded linear operator.

***Example* 8**: Let $T: L^2[0,2\pi] \to L^2[0,2\pi]$ be an operator given by $Tf = \frac{df}{dt}$.

Let $f, g \in L^2[0,2\pi]$ and $\alpha, \beta \in \mathbb{R}$, then $T(\alpha f + \beta g) = \frac{d(\alpha f+\beta g)}{dt} = \frac{d(\alpha f)}{dt} + \frac{d(\beta g)}{dt}$

$$= \alpha \frac{df}{dt} + \beta \frac{dg}{dt} = \alpha T f + \beta T g$$

Hence, T is linear. If we take $f(t) = sinnt$, for $t \in [0,2\pi]$ Clearly $f \in L^2[0,2\pi]$.

Thus,

$$\|Tf\|^2 = \int_0^{2\pi} |Tf|^2\, dt = \int_0^{2\pi} \left|\frac{df}{dt}\right|^2 dt = \int_0^{2\pi} |ncosnt|^2 dt$$

$$= n^2 \int_0^{2\pi} (cos^2 nt) dt = n^2 \pi$$

However

$$\|f\|^2 = \int_0^{2\pi} |sinnt|^2\, dt = \int_0^{2\pi} sin^2 nt\, dt = \pi$$

Thus, $\|Tf\|^2 = n^2 \|f\|^2$ for all $f \in L^2[0,2\pi]$

That is $\|Tf\| = n\|f\|$ for all $f \in L^2[0,2\pi]$.

Hence, there doesn't exist a constant $M > 0$ which is bounded yet bigger than every integer such that $\|Tf\| \leq M\|f\|$ for all $f \in L^2[0,2\pi]$. Hence, T is linear operator but not bounded.

Proposition 2.1.1: The integral operator $T: L^2[a,b] \to L^2[a,b]$ given by

$$(Tu)(s) = \int_a^b k(s,t)u(t)dt$$

is linear and bounded in $L^2[a,b]$ if $\int_a^b \int_a^b |k(s,t)|^2 dtds < \infty$.

Proof:- Let $\int_a^b \int_a^b |k(s,t)|^2 dtds = M < \infty$, thus for any $u \in L^2[a,b]$ we have $\|Tu\|^2 = \int_a^b |(Tu)(s)|^2 \, ds = \int_a^b \left| \int_a^b k(s,t)u(t)dt \right|^2 ds$

$$= \int_a^b \left(\int_a^b |k(s,t)u(t)| \, dt \right)^2 ds$$

But by Hölders inequality , $\int_a^b |k(s,t)u(t)| \, dt \leq \left(\int_a^b |k(s,t)|^2 dt \right)^{\frac{1}{2}} \|u\|$

Thus, $\|Tu\|^2 \leq \|u\|^2 \int_a^b \left(\left(\int_a^b |k(s,t)|^2 dt \right)^{\frac{1}{2}} \|u\| \right)^2 ds = \|u\|^2 \int_a^b \int_a^b |k(s,t)|^2 dt$

$$= M\|u\|^2 \ for\ all\ f \in L^2[a,b]$$

That is , $\|Tu\| \leq \sqrt{M}\|u\| \ for\ all\ f \in L^2[a,b]$

Hence, T is bounded linear operator. ■

Linear operators in Hilbert space H are the natural generalization of matrices. We define domain, range and null space (kernel) of an operator $T: H \to H$ as,

$D(T) = domain\ of\ T = \{f \in H: Tf\ is\ defind\}$

$T(H) = range\ of\ T = \{f \in H: Tg = f,\ for\ some\ \ g \in H\}$

$N(T) = null\ space\ of\ T = \{f \in H: Tf = 0\}$

Notation:- Let H be a Hilbert space, $B(H)$ denotes the set of all bounded linear operators from H in to H.

$I.e.\ B(H) = \{T{:}\,T\ is\ bounded\ linear\ operator\ from\ \ H\ in\ to\ H\}.$

$$T \in B(H) \Leftrightarrow T{:}\,H \to H\ is\ bounded\ linear\ operator.$$

2.2 Closed Linear Manifold

Definition 2.2.1:-(i). A linear manifold (subspace) M is a non empty subset of a Hilbert space H which is closed under vector addition and scalar multiplication.

(ii). A subspace M of a Hilbert space H is closed if every sequence in M which is convergent in H is also convergent in M. That is M itself is a Hilbert space.

Lemma 2.2.1:- Let H be a Hilbert space then the null space of a bounded linear operator on H is closed.

Proof: Let $N(T) = \{f \in H{:}\,Tf = 0\}$ and let $\{f_n\}$ be a sequence of elements in the null space of T (in $N(T)$) which converges to $g \in H$. Then,

$$lim_{n\to\infty}\|Tf_n - Tg\| = lim_{n\to\infty}\|T(f_n - g)\|$$

but, T is bounded, thus there is $M > 0$ such that $\|Tf\| \le M\|f\|$ for all $f \in H$.

Thus, $$\|T(f_n - g)\| \le M\|(f_n - g)\|$$

hence , $lim_{n\to\infty}\|Tf_n - Tg\| \le lim_{n\to\infty} M\|(f_n - g)\| = M\, lim_{n\to\infty}\|(f_n - g)\|$.

But, $\{f_n\}$ is a sequence of elements in $N(T) \subset H$ which converges to g implies $\|f_n - g\| \to 0$. Thus, $\lim\limits_{n\to\infty}\|Tf_n - Tg\| \le M \lim\limits_{n\to\infty}\|f_n - g\| = 0$ this implies the sequence $\{Tf_n\}$ converges to Tg.

I.e. $\lim\limits_{n\to\infty} Tf_n = Tg$ but, since $f_n^{'}s$ are from $N(T)$ we have $Tf_n = 0$ thus,

$\lim\limits_{n\to\infty} Tf_n = 0 = Tg$ this implies $g \in N(T)$. Hence, the sequence $\{f_n\}$ of elements in $N(T)$ which converges to H also converges in H.

Therefore, the null space of a bounded linear operator is closed subspace of H. ∎

Definition 2.2.2:- Let M be a subset of a Hilbert space H. Then, we define the orthogonal complement of, denoted by $M^{\perp}$, by

$$M^{\perp} = \{f \in H : \langle g, f \rangle = 0 \; for \; all \; g \in M\} .$$

Lemma 2.2.2: Let M be a subset of a Hilbert space H. Then, $M^{\perp}$ is a closed subspace of H.

Proof: Let $\{f_n\}$ be a sequence in $M^{\perp}$ which converges to $g \in H$.

Thus, for every $h \in M$, we have,

$$\langle g, h \rangle = \langle lim_{n\to\infty} f_n , h \rangle = lim_{n\to\infty} \langle f_n, h \rangle = lim_{n\to\infty} 0 = 0$$ here the interchanging of the limit is valid since $\langle f, h \rangle$ is continuous for a fixed $h \; in \; M$.

Thus $\langle g, h \rangle = 0 \quad for \; all \; \; h \in M$

That is $\langle g, h \rangle = \overline{\langle h, g \rangle} = 0$ for all $h \in M, \langle h, g \rangle = 0$ for all $h \in M$. *Implies* $g \in M^{\perp}$. ■

Proposition 2.2.1: Let M be a closed convex subset of a Hilbert space H .For any $x_0 \in H$, there is a unique $y_0 \in M$ such that

$$\|x_0 - y_0\| \leq \|x_0 - y\| , \quad for \; all \; y \in M.$$

Proof:- Let $\delta = inf \, \{\|x_0 - y\| : y \in M\}$, then there is a sequence $\{y_i\}$ in M such that $\|x_0 - y_i\| \to 0$. Now by parallelogram law we have that

$$\|y_n - y_m\|^2 = \|(y_n - x_0) + (x_0 - y_m)\|^2$$

$$= 2(\|y_n - x_0\|^2 - \|x_0 - y_m\|^2) - \|(y_n - x_0) - (x_0 - y_m)\|^2$$

$$= 2(\|y_n - x_0\|^2 - \|x_0 - y_m\|^2) - 4 \left\|\frac{1}{2}(y_n + y_m) - x_0\right\|^2 .$$

But, since $\delta = inf \, \{\|x_0 - y\| : y \in M\}$ and $\frac{1}{2}(y_n + y_m) \in M$ (due to the convexity assumption), we have $\delta^2 \leq \left\|\frac{1}{2}(y_n + y_m) - x_0\right\|^2$.

Thus, $\quad \|y_n - y_m\|^2 \leq 2(\|y_n - x_0\|^2 - \|x_0 - y_m\|^2) - 4\delta^2$

But, $2(\|y_n - x_0\|^2 - \|x_0 - y_m\|^2) - 4\delta^2$ converges to *0* for $m, n \to \infty$. Hence, $\{y_i\}$

is a Cauchy sequence. Thus, since we are in a Hilbert space H, the sequence $\{y_i\}$ converges to y_0 in H, and since M is closed $y_0 \in M$ (the limit point y_0 belongs to M) and $\delta = \|x_0 - y_0\|$.

Now, to show the uniqueness of y_0 assume that also $\delta = \|x_0 - z_0\|$ for some $z_0 \in M$. Then, $\|y_0 - z_0\|^2 = \|(y_0 - x_0) + (x_0 - z_0)\|^2$ again using parallelogram law we have,

$$\begin{aligned}\|(y_0 - x_0) + (x_0 - z_0)\|^2 \\ &= 2(\|y_0 - x_0\|^2 + \|x_0 - z_0\|^2) - \|(y_0 - x_0) - (x_0 - z_0)\|^2 \\ &= 2(\|y_0 - x_0\|^2 + \|x_0 - z_0\|^2) - \|y_0 + z_0 - 2x_0\|^2 \\ &= 2(\|y_0 - x_0\|^2 + \|x_0 - z_0\|^2) - 4\left\|\tfrac{1}{2}(y_0 + z_0) - x_0\right\|^2\end{aligned}$$

Thus, $\|y_0 - z_0\|^2 = 2(\|y_0 - x_0\|^2 + \|x_0 - z_0\|^2) - 4\left\|\frac{1}{2}(y_0 + z_0) - x_0\right\|^2$

$$\leq 2(\delta + \delta) - 4\delta = 0$$

So $y_0 = z_0$. Hence, for each $x_0 \in M$ there is a unique $y_0 \in H$ such that $\|x_0 - y_0\| \leq \|x_0 - y\|$, for all $y \in H$. ■

Definition 2.2.3:- Let M and N be closed subspaces of a Hilbert space H, with $N \perp M$. We define the orthogonal sum of N and M , denoted $N \oplus M$, by

$$N \oplus M = \{h \in H : h = x + y \ , \ x \in N \ , \ y \in M\}.$$

It is obvious that the representation $h = x + y$ is unique and $N \oplus M$ is a subspace of H. Now showing $N \oplus M$ is a closed subspace: let $\{h_n\}$ be a sequence in $N \oplus M$ that converges to h_0. Let $h_n = x_n + y_n$, $x_n \in N$, $y_n \in M$. *I.e.* $\{x_n\}$ is a sequence in N and $\{y_n\}$ is a sequence in M. But, since $N \perp M$,

$$\|x_n - x_m\|^2 + \|y_n - y_m\|^2 = \|h_n - h_m\|^2 \qquad \text{By Remark 1.1.1 (iii)}$$

Thus, we see that both $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences in N and M respectively. Hence, both $\{x_n\}$ and $\{y_n\}$ both are convergent say to x_0 and y_0

respectively. But, since both N and M are closed subspaces $x_0 \in N$ and $y_0 \in M$.

So, $h_0 = x_0 + y_0 \in N \oplus M$.

Hence, $N \oplus M$ is closed subspace.

Theorem 2.2.1:- If M is a closed subspace of a Hilbert space H, then $H = M \oplus M^{\perp}$.

Proof: Let $h \in H$. Then, by proposition 2.2.1, there is a unique $x \in H$ such that $\|h - x\| \leq \|h - y\|$, for all $y \in M$. Now, let $z = h - x$, thus we need only to show $z \in M^{\perp}$. For $\alpha \in \mathbb{K}$ and $y \in M$ we have,

$$\|z\|^2 = \|h - x\|^2 \leq \|h - (x + \alpha y)\|^2 \leq \|h - x - \alpha y\|^2$$

$$\begin{aligned} \|h - x - \alpha y\|^2 &= \|z - \alpha y\|^2 \\ &= \langle z - \alpha y, z - \alpha y \rangle \\ &= \|z\|^2 - \bar{\alpha}\langle z, y \rangle - \alpha \langle y, z \rangle + |\alpha|^2 \|y\|^2 \end{aligned}$$

Taking, $\alpha = \langle z, y \rangle$ and $\|y\|^2 = 1$ we have $\|z\|^2 \leq \|z\|^2 - |\alpha|^2$.

Thus, $\|z\|^2 \leq \|z\|^2 - |\alpha|^2$ shows that necessarily $0 = \alpha = \langle z, y \rangle$ for all $y \in M$.

Hence, $z \in M^{\perp}$. Hence, $h \in H$ is uniquely written as $h = z + x$, $z \in M^{\perp}$, $x \in M$.

i.e. any $h \in H$ has a unique decomposition as $h = z + x$ where, $z \in M^{\perp}$, $x \in M$.

Hence, $H = M \oplus M^{\perp}$. ■

Definition 2.2.4:- Let H be a Hilbert space. Then,

i. A linear operator T on H whose range is real or complex number is called a linear functional. i.e. a linear functional $T: H \to K$ is a linear mapping H in to real or complex number satisfying

$$T(\alpha f + \beta g) = \alpha T f + \beta T g \ , \ f, g \in H \ \ and \ \alpha, \beta \in K.$$

ii. A linear functional $T: H \to K$ is bounded if there is a $M > 0$ such that $|Tu| \leq M\|u\|$, for all $u \in H$.

2.3 Adjoint Operator

The Hilbert spaces are Banach spaces and all the results we have on bounded operators for normed as well as Banach space works in the Hilbert space also, but due to the simple nature of the dual of the Hilbert space that the Riesz representation theorem has revealed, we establish a reach operator theory.

Theorem 2.3.1 (Riesz representation theorem)

If φ is a continuous linear functional on a Hilbert space H.Then there is a unique $g \in H$ such that $\varphi(f) = \langle f, g \rangle$, for all $f \in H$.

Proof: Let $N = \{f \in H : \varphi(f) = 0\}$.

Since φ is continuous N is closed subspace of H by Lemma 2.2.1.

If $N = H$, we take $g = 0$, other wise that is if $N \subseteq H$ using the Proposition 2.2.1 to the fact that N is closed subspace of H we have $N \oplus N^{\perp}$ by Theorem 2.2.1.

Then, take $g_0 \in N^{\perp}$ such that $\|g_0\| = 1$.

Consider the vector $h = \big(\varphi(f)\big)g_0 - \big(\varphi(g_0)\big)f$, for any $f \in H$.

$$\varphi(h) = \varphi\Big(\big(\varphi(f)\big)g_0 - \big(\varphi(g_0)\big)f\Big) = \varphi(f)\varphi(g_0) - \varphi(g_0)\varphi(f) = 0$$

This implies, $\varphi(h) = \langle h, g \rangle = 0$ for all $f \in H$, thus $h \in N$.

Then,

$$\begin{aligned} 0 = \langle h, g_0 \rangle = \langle \big(\varphi(f)\big)g_0 - \big(\varphi(g_0)\big)f, g_0 \rangle &= \langle \big(\varphi(f)\big)g_0, g_0 \rangle - \langle \big(\varphi(g_0)\big)f, g_0 \rangle \\ &= \varphi(f)\langle g_0, g_0 \rangle - \varphi(g_0)\langle f, g_0 \rangle \\ &= \varphi(f) - \varphi(g_0)\langle f, g_0 \rangle \end{aligned}$$

Hence, $\varphi(f) = \varphi(g_0)\langle f, g_0 \rangle$ take $g = \overline{\varphi(g_0)}\, g_0$.

Therefore, $\varphi(f) = \langle f, g \rangle$.

To show uniqueness assume $g_1, g_2 \in H$ and we have two representations

$$\varphi(f) = \langle f, g_1 \rangle \text{ and } \varphi(f) = \langle f, g_2 \rangle \text{ for all } f \in H.$$

Thus, $\langle f, g_1 \rangle = \langle f, g_2 \rangle$. Implies , $\langle f, g_1 \rangle - \langle f, g_2 \rangle = 0$,for all $f \in H$.

$$\langle f, g_1 - g_2\rangle = 0 \quad \text{, for all } f \in H.$$

Taking $f = g_1 - g_2$ we have,

$\langle f, g_1 - g_2\rangle = \langle g_1 - g_2, g_1 - g_2\rangle = \|g_1 - g_2\|^2 = 0$

Thus, $g_1 - g_2 = 0$ *i.e* $g_1 = g_2$

Hence there is a unique $g \in H$ such that $\varphi(f) = \langle f, g\rangle$, for all $f \in H$. ■

As first application of Riesz representation Theorem, we will define adjoint of a bounded operator on Hilbert space. In finite dimension, where the bounded linear operators are represented by matrices, this is just the conjugate transpose. But, in infinite dimension, the situation is again much more delicate.

Definition 2.3.1:-Let H be a Hilbert space and let $T: H \to H$ be an operator. Then, the adjoint operator of T is the operator $T^\star: H \to H$ (if exists) such that

$$\langle Tf, g\rangle = \langle f, T^\star g\rangle \quad \text{for all } f, g \in H.$$

Theorem 2.3.2:- Let H be a Hilbert space and let $T \in B(H)$. Then, there is a unique adjoint operator $T^\star$ of T such that $T^\star \in B(H)$ and $\|T\| = \|T^\star\|$.

Proof: Take a $g \in H$ and define $\varphi_g: H \to K$ by $\varphi_g(f) = \langle Tf, g\rangle$ for all $f \in H$.

Thus, $|\varphi_g(f)| = |\langle Tf, g\rangle| \leq \|Tf\|\|g\|$ by Cauchy Schwarz inequality.

But since T is bounded we have $\|Tf\| \leq \|T\|\|f\|$ for all $f \in H$.

Thus, $|\varphi_g(f)| \leq \|T\|\|f\|\|g\|$ for all $f \in H$. Hence, φ_g is bounded.

Let $f_1, f_2 \in H$ and $\alpha, \beta \in K$, then

$$\begin{aligned}
\varphi_g(\alpha f_1 + \beta f_2) = \langle T(\alpha f_1 + \beta f_2), g\rangle &= \langle T(\alpha f_1) + T(\beta f_2), g\rangle \\
&= \langle \alpha T f_1 + \beta T f_2, g\rangle \quad \text{Since } T \text{ is linear.} \\
&= \langle \alpha T f_1, g\rangle + \langle \beta T f_2, g\rangle = \alpha\langle T f_1, g\rangle + \beta\langle T f_2, g\rangle \\
&= \alpha\varphi_g(f_1) + \beta\varphi_g(f_2)
\end{aligned}$$

Hence, φ_g is linear functional. Hence, φ_g is bounded linear functional.

Now by Riesz representation theorem there is a unique $g_0 \in H$ such that

$$\varphi_g(f) = \langle f, g_0 \rangle \quad \text{for all } f \in H.$$

Since g_0 depends on g , we have in this way an operator $T^\star g = g_0$ satisfying

$$\langle Tf, g \rangle = \langle f, T^\star g \rangle \quad \text{for all } f, g \in H.$$

To show $T^\star$ is linear let $g_1, g_2 \in H$ and $\alpha, \beta \in K$ then for all $f \in H$,

$$\begin{aligned}\langle f, T^\star(\alpha g_1 + \beta g_2)\rangle = \langle Tf, \alpha g_1 + \beta g_2 \rangle = \langle Tf, \alpha g_1 \rangle + \langle Tf, \beta g_2 \rangle \\ = \bar{\alpha}\langle Tf, g_1 \rangle + \bar{\beta}\langle Tf, g_2 \rangle \\ = \overline{\alpha}\langle f, T^\star g_1 \rangle + \bar{\beta}\langle f, T^\star g_2 \rangle \\ = \langle f, \alpha T^\star g_1 \rangle + \langle f, \beta T^\star g_2 \rangle \\ = \langle f, \alpha T^\star g_1 + \beta T^\star g_2 \rangle\end{aligned}$$

For which we conclude that $T^\star(\alpha g_1 + \beta g_2) = \alpha T^\star g_1 + \beta T^\star g_2$ hence, $T^\star$ is linear.

To show $T^\star$ is bounded,

$$\begin{aligned}\| T^\star g\|^2 = |\langle T^\star g, T^\star g \rangle| \quad \text{for all } g \in H \\ = |\langle T\, T^\star g, g \rangle| \leq \|T\, T^\star g\| \|g\| \quad \text{by Cauchy Schwarz inequality} \\ \leq \|T\| \| T^\star g\| \|g\| \quad \text{since T is bounded.}\end{aligned}$$

Thus, $\| T^\star g\| \leq \|T\| \|g\|$ for all $g \in H$. (15)

Hence $T^\star$ is bounded linear operator$\big(T^\star \in B(H)\big)$.

Now from (15) we have, $\| T^\star\| \leq \|T\|$.

$$\begin{aligned}\text{And,} \quad \|Tg\|^2 = |\langle Tg, Tg \rangle| \quad \text{for all } g \in H \\ = |\langle g, T^\star T g \rangle| \leq \|T^\star T g\| \|g\| \quad \text{by Cauchy Schwarz inequality} \\ \leq \|T^\star\| \|Tg\| \|g\| \quad \text{for all } g \in H\end{aligned}$$

Implies, $\|Tg\| \leq \|T^\star\| \|g\|$, for all $g \in H$ and gives $\|T\| \leq \| T^\star\|$.

Hence $\|T\| = \|T^\star\|$. ■

***Example* 9**: Let $k \in C([0,1],[0,1])$ and let $T: L^2[0,1] \to L^2[0,1]$ is given by

$$(Tf)(s) = \int_0^1 k(s,t)\,f(t)dt\,.$$

For all $f \in L^2[0,1]$, we have

$$\|Tf\|^2 = \langle Tf, Tf\rangle = \int_0^1 (Tf)(s)\overline{(Tf)(s)}\,ds = \int_0^1 |(Tf)(s)|^2\,ds$$

$$= \int_0^1 \left|\int_0^1 k(s,t)\,f(t)dt\right|^2 ds \le \int_0^1 \left(\int_0^1 |k(s,t)f(t)|\,dt\right)^2 ds$$

But, since $k \in C([0,1],[0,1])$ we have $|k(s,t)| \le max_{0\le s,t\le 1}\ |k(s,t)| = \|k\|_\infty$

Thus, $\|Tf\|^2 \le \|k\|_\infty{}^2 \int_0^1 \left(\int_0^1 |f(t)|\,dt\right)^2 ds$

But, $\int_0^1 |f(t)|\,dt = \int_0^1 |f(t).1|\,dt \le \|f\|\|1\| = \|f\|$ by Hölders inequality

Thus, $\|Tf\|^2 \le \|k\|_\infty{}^2\ \|f\|^2$ for all $f \in L^2[0,1]$.

That is $\|Tf\| \le \|k\|_\infty \|f\|$ for all $f \in L^2[0,1]$, hence T is bounded operator.

Let $f, g \in L^2[0,1]$ and $\alpha, \beta \in K$, then

$$T(\alpha f + \beta g) = \int_0^1 k(s,t)(\alpha f + \beta g)(t)dt$$

$$= \int_0^1 \big(k(s,t)\alpha f(t) + k(s,t)\beta g(t)\big)\,dt$$

$$= \int_0^1 \alpha\, k(s,t)f(t)dt + \int_0^1 \beta\, k(s,t)g(t)\,dt$$

$$= \alpha(Tf)(s) + \beta(Tg)(s)$$ hence, T is linear.

Hence, $T \in B(L^2[0,1])$. Therefore, by Theorem 2.3.2, $T^\star$ exists and is unique.

Now determining the adjoint operator $T^\star$ of T.

Consider $f, g \in L^2[0,1]$,

$$\langle Tf, g\rangle = \int_0^1 (Tf)(s)\overline{g(s)}\ ds = \int_0^1 \left(\int_0^1 k(s,t)\,f(t)dt\right)\overline{g(s)}\ ds$$

$$= \int_0^1\int_0^1 k(s,t)f(t)\overline{g(s)}\ dtds$$

$$= \int_0^1\int_0^1 k(s,t)f(t)\overline{g(s)}\ dsdt$$ By Fubini's Theorem.

$$= \int_0^1 f(t)\left(\int_0^1 k(s,t)\overline{g(s)}\, ds\right)dt = \int_0^1 f(t)\overline{\left(\int_0^1 \overline{k(s,t)}g(s)\, ds\right)}\,dt = \langle f, T^\star g\rangle$$

Where, $(T^\star g)(t) = \int_0^1 \overline{k(s,t)}g(s)\, ds$ is the adjoint operator of T.

Thus, $T^\star$of T is the integral operator with conjugate transpose kernel $k^\star(s,t) = \overline{k(t,s)}$.

Proposition 2.3.1:- Let H be a Hilbert space and $T \in B(H)$. Then,

i. $N(T^\star) = \left(T(H)\right)^\perp$.

ii. $\left(T(H)\right)^\perp$ is a closed subspace of H.

Proof:

i. Let $y \in \left(T(H)\right)^\perp$ then, $\langle Tx, y\rangle = 0$ for all $x \in H$.

Thus, $\langle Tx, y\rangle = \langle x, T^\star y\rangle = 0$ for all $x \in H$.

This implies $T^\star y = 0$ hence, $y \in N(T^\star)$.

Let $y \in N(T^\star)$, then $0 = \langle x, T^\star y\rangle$ for all $x \in H$.

Thus, $0 = \langle x, T^\star y\rangle = \langle Tx, y\rangle$ for all $x \in H. I.e.$ $\langle Tx, y\rangle = 0$ for all $x \in H.$ This implies, $\langle z, y\rangle = 0$ $for\ all\ z \in T(H)$ hence, $y \in \left(T(H)\right)^\perp$.

Hence, $N(T^\star) = \left(T(H)\right)^\perp$.■

ii. Since, $T \in B(H)$there exists a unique $T^\star \in B(H)$ by Theorem 2.3.2. And by Lemma 2.2.1. $N(T^\star)$ is a closed subspace of H. By (i) we have $N(T^\star) = \left(T(H)\right)^\perp$ hence, $\left(T(H)\right)^\perp$ is a closed subspace of H.■

Definition 2.3.2: Let H be a Hilbert space $T \in B(H)$. If $T = T^\star$ we say T is self adjoint.

***Example* 10**: −Let $T: L^2[0,1] \to L^2[0,1]$ be an operator given by $Tf = sf(s)$.

Now finding the adjoint operator $T^\star$, let $f, g \in L^2[0,1]$ and $\alpha, \beta \in K$.

$T(\alpha f + \beta g) = s(\alpha f + \beta g)(s) = \alpha sf(s) + \beta sg(s) = \alpha Tf + \beta Tg$

Hence T is linear.

$$\langle Tf,g\rangle = \int_0^1 tf(t)\overline{g(t)}\,dt = \int_0^1 f(t)\,\overline{tg(t)}dt = \langle f,tg(t)\rangle = \langle f,T^\star g\rangle$$

where $T^\star g = sg(s)$ i.e. $T = T^\star$ hence T is self adjoint operator.

Lemma 2.3.1:- Let T be bounded self adjoint operator on Hilbert space H.Then, $\langle Tf,f\rangle$ is real for all $f \in$ H.

Proof: $\langle Tf,f\rangle = \langle f,Tf\rangle = \overline{\langle Tf,f\rangle}$ for all $f \in H$. I.e. $\langle Tf,f\rangle = \overline{\langle Tf,f\rangle}$ for all $f \in H$. Hence, $\langle Tf,f\rangle$ is real for all $f \in H$.■

Proposition 2.3.2: Let T be bounded self adjoint operator on a Hilbert space H. Then,

$$\|T\| = \sup_{\|x\|=1}|\langle Tx,x\rangle|$$

Proof: Let $\beta = \sup_{\|x\|=1}|\langle Tx,x\rangle|$ thus, for $\|x\| = 1$, by Cauchy Schwarz inequality we have that $|\langle Tx,x\rangle| \leq \|Tx\|\|x\| \leq \|T\|\|x\|\|x\| = \|T\|$ so that taking the supermom over norm 1 of x we have $\beta \leq \|T\|$.

On the other hand

$$\langle T(x+y),x+y\rangle - \langle T(x-y),x-y\rangle = 2(\langle Tx,y\rangle + \langle Ty,x\rangle)$$

But since T is bounded self adjoint $\langle T(x+y),x+y\rangle - \langle T(x-y),x-y\rangle$ is real by Lemma 2.3.1. So from definition of β we have that

$$\langle T(x+y),x+y\rangle \leq \beta\|x+y\|^2 \quad \text{and}$$
$$-\langle T(x-y),x-y\rangle \leq \beta\|x-y\|^2$$

Using parallelogram law $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ we have ,

$$2(\langle Tx,y\rangle + \langle Ty,x\rangle) \leq \beta(\|x+y\|^2 + \|x-y\|^2) \leq 2\beta(\|x\|^2 + \|y\|^2)$$

Thus, $$\langle Tx,y\rangle + \langle Ty,x\rangle \leq \beta(\|x\|^2 + \|y\|^2) \qquad (16)$$

Now , if $Tx = 0$ it is clear that $\|Tx\| \leq \beta\|x\|$ that will lead us to $\|T\| \leq \beta$, so assume that $Tx \neq 0$ and define $y = \frac{\|x\|}{\|Tx\|}Tx$.

Then $\|y\| = \frac{\|x\|}{\|Tx\|}\|Tx\| = \|x\|$ and $2\frac{\|x\|}{\|Tx\|}\|Tx\|^2 = \frac{\|x\|}{\|Tx\|}(\|Tx\|^2 + \|Tx\|^2)$

$$\frac{\|x\|}{\|Tx\|}(\|Tx\|^2+\|Tx\|^2)=\frac{\|x\|}{\|Tx\|}(\langle Tx,Tx\rangle+\langle Tx,Tx\rangle)$$
$$=\frac{\|x\|}{\|Tx\|}(\langle Tx,Tx\rangle+\langle TTx,x\rangle)$$
$$=\langle Tx,\frac{\|x\|}{\|Tx\|}Tx\rangle+\langle T\left(\frac{\|x\|}{\|Tx\|}Tx\right),x\rangle$$
$$=\langle Tx,y\rangle+\langle Ty,x\rangle$$
$$\leq \beta(\|x\|^2+\|y\|^2) \text{ By using (16)}$$

But above we have that $\|x\|=\|y\|$. Thus, $2\frac{\|x\|}{\|Tx\|}\|Tx\|^2\leq 2\beta\|x\|^2$.

Hence this will lead us to $\|Tx\|\leq\beta\|x\|$ for all $x\in H$ then taking the supermom over norm 1 of x we have $\|T\|\leq\beta$.

Therefore, $\|T\|=\sup_{\|x\|=1}|\langle Tx,x\rangle|$. ■

In the proof of existence of an adjoint operator for a bounded operator, the Riesz - representation Theorem was essential ingredient for all $x,y\in H$. When T is unbounded we can of course define a functional $\varphi_y(x)=\langle Tx,y\rangle$ for $x\in D(T)\subseteq H$, but it is not bounded on H and it is not clear if we can find $z\in H$ such that $\langle Tx,y\rangle=\langle x,z\rangle$ for all $x\in D(T)$.

***Example* 11**: – Let $T\colon L^2[0,1]\to L^2[0,1]$ be an operator given by $Tf=\frac{df}{dt}$.

This differential operator is unbounded (see Example 8) and it has no adjoint operator.

If we try, $\langle Tf,g\rangle=\int_0^1\frac{df(t)}{dt}\overline{g(t)}\,dt=\int_0^1\frac{d\left(f(t)\overline{g(t)}\right)}{dt}dt-\int_0^1 f(t)\frac{d\left(\overline{g(t)}\right)}{dt}dt$

$$=f(t)\overline{g(t)}\;|_0^1-\int_0^1 f(t)\frac{d\left(\overline{g(t)}\right)}{dt}dt\;=f(t)\overline{g(t)}\;|_0^1-\int_0^1 f(t)\overline{\frac{d(g(t))}{dt}}dt$$
$$=f(1)g(1)-f(0)g(0)+\langle f,-\frac{d(g(t))}{dt}\rangle$$

If we attempted to conclude that $T^\star g=-\frac{dg}{dt}$ but, the definition of adjoint doesn't hold for all choice of f and g unless $f(1)g(1)-f(0)g(0)=0$

Hence the differential operator $Tf=\frac{df}{dt}$ doesn't have an adjoint operator.

2.4 Fredholm Alternative Theorem

Theorem 2.4.1 :- (Fredholm Alternative Theorem) Let H be a Hilbert space and $T \in B(H)$ with closed range then the equation $Tf = g$ has a solution if and only if $\langle g, h\rangle = 0$ for every $h \in N(T^\star)$.

Proof: Suppose $Tf = g$ has a solution f_0. Since $T \in B(H)$ we have a unique adjoint operator $T^\star \in B(H)$ of T. Thus, $\langle Tu, v\rangle = \langle u, T^\star v\rangle$ for all $u, v \in H$.

Hence for $h \in N(T^\star)$ we have $\langle g, h\rangle = \langle T f_0, h\rangle = \langle f_0, T^\star h\rangle$

But $T^\star h = 0$ since $h \in N(T^\star)$. Thus $\langle g, h\rangle = \langle f_0, 0\rangle = 0$

Hence, $\langle g, h\rangle = 0$ for all $h \in N(T^\star)$.

Suppose $\langle g, h\rangle = 0$ for every $h \in N(T^\star)$.

Assume $Tf = g$ has no solution. This implies $g \notin T(H)$.

Since $T(H)$ is closed we can have $H = T(H) \oplus \left(T(H)\right)^\perp$ by Theorem 2.2.1.

Now g can be written uniquely $g = g_1 + g_2$ where $g_1 \in T(H)$ and $g_2 \in \left(T(H)\right)^\perp$ such that $\langle g_1, g_2\rangle = 0$. But we have for each $y \in \left(T(H)\right)^\perp$, $\langle x, y\rangle = 0$ for all $x \in T(H)$ that is $\langle Tf, y\rangle = 0$ for all $f \in H$. Hence since $g_2 \in \left(T(H)\right)^\perp$ we have $\langle Tf, g_2\rangle = 0$ for all $f \in H$.

Thus, $\langle Tf, g_2\rangle = \langle f, T^\star g_2\rangle = 0$ for all $f \in H$ now since $T^\star g_2 \in H$ taking $f = T^\star g_2$ we will have $\langle T^\star g_2, T^\star g_2\rangle = 0$ and this implies $T^\star g_2 = 0$ hence $g_2 \in N(T^\star)$.

Thus by the assumption $0 = \langle g, g_2\rangle = \langle g_1 + g_2, g_2\rangle = \langle g_1, g_2\rangle + \langle g_2, g_2\rangle$

But since $\langle g_1, g_2\rangle = 0$ hence $\langle g_2, g_2\rangle = 0$ implying that $g_2 = 0$.

Thus, $g = g_1 + g_2 = g_1 \in T(H)$ hence this contradicts to the assumption that $g \notin T(H)$. Hence it must be the case that $Tf = g$ has a solution.■

***Example* 12**: –Let $T: L^2[0,1] \to L^2[0,1]$ is given by $(Tu)(s) = \int_0^1 tu(t)\,dt$.

We need to find the cases in which the integral equation $(I + \lambda T)u = f$ has a solution.

Now first let us find the adjoint operator of the integral equation $I + \lambda T$.

For $u, v \in L^2[0,1]$

$$\langle (I + \lambda T)u, v\rangle = \int_0^1 \left[u(s) + \lambda \int_0^1 tu(t)\, dt\right] \overline{v(s)}\, ds$$

$$= \int_0^1 u(s)\, \overline{v(s)} ds + \int_0^1 \int_0^1 \lambda tu(t)\overline{v(s)} dtds$$

$$= \int_0^1 u(s)\, \overline{v(s)} ds + \int_0^1 \int_0^1 \lambda tu(t)\overline{v(s)} dsdt \quad \text{By Fubine's}$$

$$= \int_0^1 u(s)\, \overline{v(s)} ds + \int_0^1 u(t) \left(\lambda t \int_0^1 \overline{v(s)}\, ds\right) dt$$

$$= \int_0^1 u(s)\, \overline{v(s)} ds + \int_0^1 u(t) \left(\int_0^1 \overline{\lambda t v(s)}\, ds\right) dt$$

$$= \int_0^1 u(s)\, \overline{v(s)} ds + \int_0^1 u(t) \overline{\left(\lambda t \int_0^1 v(s)\, ds\right)}\, dt$$

$$= \langle u, v\rangle + \langle u, \lambda T^\star v\rangle \quad \text{where } (T^\star v)(s) = s \int_0^1 v(t)\, dt\, .$$

$$= \langle u, (I + T^\star)v\rangle = \langle u, (I + \lambda T)^\star v\rangle$$

Hence the adjoint operator of $I + \lambda T$ is

$$(I + \lambda T)^\star v = v(s) + \lambda s \int_0^1 v(t)\, dt$$

Now we need to find the null space of the adjoint operator $(I + \lambda T)^\star$. Thus,

$$v(s) + \lambda s \int_0^1 v(t)\, dt = 0$$

$$\Rightarrow v(s) = -\lambda s \int_0^1 v(t)\, dt$$

$$\Rightarrow v(s) = -\lambda s\alpha \quad \text{where } \alpha = \int_0^1 v(t)\, dt$$

$$\Rightarrow v(s) = -\lambda s\alpha \quad \text{multiplying both side by } s$$

$$\Rightarrow \int_0^1 v(s) ds = \int_0^1 -\lambda s\alpha ds = -\lambda\, \alpha \int_0^1 sds$$

$$\Rightarrow \alpha = \frac{-\lambda\alpha}{2}$$

$$\Rightarrow \left(1 + \frac{\lambda}{2}\right)\alpha = 0$$

Hence ,

i. If $\lambda = -2$ then, null space of $(I + \lambda T)^\star$ is spanned by $v(s) = s$ that is $N((I + \lambda T)^\star)$ =span$\{s\}$.

Hence, if $\lambda = -2$,then $(I + \lambda T)u = f$ has a (non unique) solution if and only if $\int_0^1 f(t)\ tdt = 0$.

That is $(I + \lambda T)u = f$ has a (non unique) solution for each $f \in L^2[0,1]$

ii. If $\lambda \neq -2$ then, null space of $(I + \lambda T)^\star$ is $\{0\}$ that is $N((I + \lambda T)^\star) = \{0\}$.

That is $(I + \lambda T)u = f$ has a unique solution for each $f \in L^2[0,1]$.

Notice that we cannot be sure that the Fredholm Alternative Theorem applies here since we have not yet shown that the range of the integral operator $I + \lambda T$ is closed. In fact the operator (equation) $I + \lambda T$ has a closed range and in our chapter 4 we will see when an operator T is in a special class of bounded operators the so called compact operators the operator equation $I + \lambda T$ will have a closed range.

Chapter 3

Compact Operators- Hilbert Schmidt Kernel

We will now introduce a special class of bounded operators that occur in many applications, in particular equations or as inverse of unbounded operator. The important property of an integral equation is that they are compact, which is stronger statement than boundedness. The compact operators are sometimes called completely continuous operators for the reason that will become clear in our discussion.

3.1 Compact and Relatively Compact Sets

Now let us recall the definition of compact set.

Definition 3.1.1: A subset S of a normed space H is relatively compact if any sequence $\{x_n\}$ in S has convergent subsequence.

Definition 3.1.2: A subset of a normed space is relatively compact if it's closer is compact.

Note that in finite dimensional spaces, the compact sets are precisely the closed and bounded ones. But this is not true for an infinite dimension.

***Example* 13**: $-$ Take a closed unit ball $S = \{x \in H: \|x\| \leq 1\}$ in an infinite dimensional Hilbert spaceH. Then any orthonormal sequence $\{e_n\}$ will belong to it. Thus, for $n \neq m$,

$$\|e_n - e_m\|^2 = \langle e_n - e_m, e_n - e_m\rangle = \langle e_n, e_n\rangle - \langle e_n, e_m\rangle - \langle e_m, e_n\rangle - \langle e_m, e_m\rangle$$

$$= \|e_n\| + \|e_m\| \qquad \text{Since for } n \neq m, \langle e_n, e_m\rangle = 0$$

$$= 2 \quad \text{Implies no subsequence can converge.}$$

In particular the sequence $\{sinnt\}_{n=1}^{\infty}$ which consists a mutually orthogonal element in $L^2[0, \pi]$ which is an infinite dimensional space cannot have a convergent subsequence since the distance between any two elements is$\sqrt{2}$.

Proposition 3.1.1:- A compact set is closed and bounded.

Proof: Assume S is compact set and $\{x_n\}$ is a sequence in S.

Assume S is not bounded. Thus, we can have for every $n \in \mathbb{N}$ take $x_n \in S$ such that $\|x_n\| > n$. Thus, for every subsequence $\{y_n\}$ of $\{x_n\}$ we have that $\|y_n\| > n$. Hence $\|y_n\| \to \infty$ $(diverges)$ as $n \to \infty$. So $\{y_n\}$ is not convergent and this contradicts with the fact that S is compact. Hence, it must be the case that S is bounded. Assume S is not closed. This implies there exists $x \in \bar{S} - S$ and there is a sequence $\{x_n\}$ in S that converges to x.But then every subsequence $\{y_n\}$ of $\{x_n\}$ will converge to $x \notin S$. Thus it contradicts with the fact that S is compact. Hence it must be the case that S is closed. ■

Note that form our topology point of view every finite dimensional subspace of a normed space is closed subset of the space and a bounded sequence in a finite dimensional normed space has a convergent subsequence.

3.2 Compact Operators

Let $T: H \to H$ (H is Hilbert) is bounded operator. Then, let A be any bounded subset of H, i.e there is $M_1 > 0$ such that $\|x\| \leq M_1$ for all $x \in A$. Since T is bounded operator there exist $M_2 > 0$ such that $\|Tx\| \leq M_2\|x\|$ for all $x \in A$.

That is, $\|Tx\| \leq M$ for all $x \in A$ where $M = M_1 M_2$ hence $T(A)$ is also bounded set. Therefore if $T: H \to H$ (H is Hilbert) is bounded operator then it maps bounded sets in to bounded set. Compact operators have even a stronger continuity property.

Definition 3.2.1:- Let X and Y be two normed spaces. An operator $T: X \to Y$ is said to be compact if for each bounded sequence $\{x_n\}$ in X the sequence $\{Tx_n\}$ in Y has a convergent subsequence.

Equivalently, let X and Y be normed spaces. An operator $T: X \to Y$ is said to be compact if for bounded set A in X , $T(A)$ is relatively compact in Y ($\overline{T(A)}$ is compact in Y) . That is a compact operator is an operator which maps bounded sets in to relatively compact sets.

Throughout our discussion we will see compact operators $T: H \to H$ where H is Hilbert space (infinite dimensional). Let us see the following example to illustrate compactness of an operator.

***Example* 14**: − Let $T: l^2 \to l^2$ be a mapping given by

$$Tx = \left(\frac{\mu_1}{2}, \frac{\mu_2}{4}, \frac{\mu_3}{8}, \ldots, \frac{\mu_m}{2^m}, 0,0,0, \ldots\right)$$

where $x = (\mu_1, \mu_2, \mu_3, \ldots) \in l^2$ and $m \in \mathbb{N}$

Now let $x = (\mu_1, \mu_2, \mu_3, \ldots) \in l^2$ then,

$$\|Tx\|^2 = \sum_{i=1}^{m} \left|\frac{\mu_i}{2^i}\right|^2 \le \sum_{i=1}^{\infty} \left|\frac{\mu_i}{2^i}\right|^2 \le \sum_{i=1}^{\infty} \left|\frac{\mu_i}{2}\right|^2 = \frac{1}{4}\sum_{i=1}^{\infty} |\mu_i|^2$$

$$= \frac{1}{4}\|x\|^2 \text{for all } x \in l^2. \qquad (17)$$

Hence, T is a bounded operator.

Let $\{x_n\}$ be a bounded sequence in l^2, and let $x_n = (\mu_1^n, \mu_2^n, \mu_3^n, \mu_4^n, \ldots)$.

Now, $\{x_n\}$ be a bounded sequence in l^2, i.e there is $M > 0$ such that $\|x_n\| \le M$.

Thus, from (17) we have

$$\|Tx_n\|^2 = \sum_{i=1}^{m} \left|\frac{\mu_i^n}{2^i}\right|^2 \le \sum_{i=1}^{\infty} \left|\frac{\mu_i^n}{2^i}\right|^2 \le \sum_{i=1}^{\infty} \left|\frac{\mu_i^n}{2}\right|^2 = \frac{1}{4}\sum_{i=1}^{\infty} |\mu_i^n|^2 = \frac{1}{4}\|x_n\|^2 \le \frac{M^2}{4}$$

for all n.That is $\|Tx_n\| \le \frac{M}{2}$ hence the sequence $\{Tx_n\}$ is abounded sequence in a finite dimensional range ($\dim T(H) = m < \infty$). But using the fact that a bounded sequence in a finite dimensional normed space has a convergent subsequence: thus $\{Tx_n\}$ in $T(H)$ possesses a convergent subsequence. Hence T is compact operator.

Proposition 3.2.1:-Let H be a Hilbert space and $T: H \to H$ be a mapping. Then if T is compact then it is bounded.

Proof: Let $A = \{x \in H : \|x\| = 1\}$. Clearly A is a bounded set.

But since T is compact operator, $\overline{T(H)}$ is compact set in H. Implying that $\overline{T(H)}$ closed and bounded subset of H ; $T(H)$ is bounded set in H.

That is there is $M > 0$ such that $\|Tx\| \leq M$ for all $x \in H$ with $\|x\| = 1$.

$$\sup_{\|x\|=1} \|Tx\| < \infty .$$

Hence T is bounded operator.■

Definition 3.2.2:- Let T be an integral operator in $L^2[a,b]$. Then, the kernel k of the form $k(s,t) = \sum_{i=1}^{n} p_i(s) q_i(t)$,where $\{p_1 , p_2 , \ . \ . \ . \ , p_n\}$ and $\{q_1 , q_2 , . \ . \ ., q_n\}$ are subsets of $L^2[a,b]$, is called degenerate or separable kernel, and the integral operator T involving degenerate kernel is called degenerate integral operator.

Now the following two theorems are sufficient conditions for an operator $T: H \to H$ is said to be compact in an infinite dimensional Hilbert space H.

Theorem 3.2.1:- Let H be a Hilbert space and $T: H \to H$ be a linear operator. If T is bounded and finite dimensional operator, then T is compact.

Proof: Let A be a bounded set in H. Since T is bounded operator it maps bounded subset $A \subseteq H$ in to a bounded subset $T(A) \subseteq T(H) \subseteq H$.Also the closure $\overline{T(A)}$ is contained in $T(H)$, since $T(H)$ is finite dimensional and hence $T(H)$ is a closed linear subspace in H. Consequently, $\overline{T(A)}$ is bounded and closed in a finite dimensional vector space $T(H)$ and therefore T is compact.■

***Example* 15**: Let $T: L^2[a,b] \to L^2[a,b]$ is an integral operator given by

$$(Tf)(s) = \int_a^b k(s,t) f(t)\, dt$$

where $k(s,t) = \sum_{i=1}^{n} p_i(s) q_i(t)$ with $p_i, q_i \in L^2[a,b]$ (k is degenerate kernel) .

Now, $(Tf)(s) = \int_a^b k(s,t) f(t)\, dt = \int_a^b \sum_{i=1}^{n} p_i(s) q_i(t)\, f(t) dt$

$$= \sum_{i=1}^{n} \int_a^b p_i(s) q_i(t) f(t) dt$$

$$= \sum_{i=1}^{n} p_i(s) \int_a^b q_i(t) f(t) dt = \sum_{i=1}^{n} \langle f , \overline{q_i} \rangle\, p_i$$

Hence T is finite dimensional.

$$\|Tf\|^2 = \int_a^b \left|\int_a^b k(s,t)f(t)\,dt\right|^2 ds \le \int_a^b \left(\int_a^b |k(s,t)f(t)|dt\right)^2 dt$$
$$= \int_a^b \left(\int_a^b \left|\textstyle\sum_{i=1}^n p_i(s)q_i(t)f(t)\right|dt\right)^2 dt$$
$$= \int_a^b \left(\textstyle\sum_{i=1}^n \int_a^b |p_i(s)q_i(t)f(t)|dt\right)^2 ds$$
$$= \int_a^b \left(\textstyle\sum_{i=1}^n |p_i(s)| \int_a^b |q_i(t)f(t)|\,dt\right)^2 ds$$

But, $\int_a^b |q_i(t)f(t)|\,dt \le \|q_i\|\|f\|$ by Hölders inequality.

Thus,

$$\int_a^b \left(\textstyle\sum_{i=1}^n |p_i(s)| \int_a^b |q_i(t)f(t)|\,dt\right)^2 ds = \int_a^b \left(\textstyle\sum_{i=1}^n |p_i(s)|\,\|q_i\|\|f\|\right)^2 ds$$
$$= \int_a^b \|f\|^2 \left(\textstyle\sum_{i=1}^n |p_i(s)|\,\|q_i\|\right)^2 ds$$
$$= \int_a^b \left(\textstyle\sum_{i=1}^n |p_i(s)|\,\|q_i\|\right)^2 ds\ \|f\|^2$$

Hence , $\|Tf\| \le M\|f\|$ for all $f \in L^2[a,b]$ where

$M = \left(\int_a^b \left(\sum_{i=1}^n |p_i(s)|\,\|q_i\|\right)^2 ds\right)^{\frac{1}{2}}$ Hence, T is bounded and $\dim T(H) = n < \infty$.

Therefore, by Theorem 3.2.1 T is compact.

Theorem 3.2.2:-Let H be a Hilbert space and let $\{T_n\}$ is sequence of compact linear operators converging to an operator T. Then, T is compact.

Proof: let $\{T_n\}$ is sequence of compact linear operators converging to an operator T. We need to show that T is compact. Let $\{y_n\}$ be a bounded sequence in $T(H)$.

Then there exist possibly unique bounded sequence $\{x_n\}$ in H with $Tx_n = y_n$.

We are now going to Show that it is possible to extract a subsequence $\{x_{n_k}\}$ such that $\{Tx_{n_k}\}$ that is a subsequence of $\{y_n\}$ is convergent.

Since T_1 is compact, $\{x_n\}$ has a subsequence $\{x_n^1\}$ such that $\{T_1x_n^1\}$ is convergent.

Since T_2 is compact, $\{x_n^1\}$ has a subsequence $\{x_n^2\}$ and $\{T_2x_n^2\}$ is convergent.

Continuing this way, for every k , $\{x_n^{k-1}\}$ has a subsequence $\{x_n^k\}$ such that $\{T_k x_n^k\}$. Consider now the diagonal sequence $\{x_n^n\}$. It is obvious that $\{T_k x_n^n\}$ is convergent for every k , and we will now show that $\{T x_n^n\}$ is a Cauchy sequence in H.

Since $\{x_n\}$ is bounded there is $\alpha > 0$ such that $\|x_n^n\| \le \alpha$ for every n. And since $T_n \to T$, let $\varepsilon > 0$ be given we can find a fixed number k such that

$$\|T - T_k\| \le \frac{\varepsilon}{3\alpha}\,.$$

Since $\{T_k x_n^n\}$ is a cauch sequence for every k there is N such that

$$\|T_k x_n^n - T_k x_m^m\| \le \frac{\varepsilon}{3} \text{ for all } n, m > N$$

Now for all $n, m > N$ we have

$$\|T x_n^n - T x_m^m\| \le \|T x_n^n - T_k x_n^n\| + \|T_k x_n^n - T_k x_m^m\| + \|T_k x_m^m - T x_m^m\|$$

$$\le \|T - T_k\|\|x_n^n\| + \frac{\varepsilon}{3} + \|T_k - T\|\|x_m^m\| \le \frac{\varepsilon}{3\alpha}\alpha + \frac{\varepsilon}{3} + \frac{\varepsilon}{3\alpha}\alpha = \varepsilon$$

Hence, $\{T x_n^n\}$ is a Cauchy sequence in H , thus it converges in H.

Hence, since the diagonal sequence $\{x_n^n\}$ is a subsequence of $\{x_n\}$ such that $\{T x_n\}$ has a convergent subsequence $\{T x_n^n\}$ that converges.

Therefore, T is compact.■

***Example* 16**: – Let $T: l^2 \to l^2$ defined by

$Tx = \left(\frac{\mu_1}{2}, \frac{\mu_2}{4}, \frac{\mu_3}{8}, \ldots\right)$ where , $x = (\mu_1, \mu_2, \mu_3, \ldots) \in l^2$.

For $n \in \mathbb{N}$ define $T_n: l^2 \to l^2$ by $T_n x = \left(\frac{\mu_1}{2}, \frac{\mu_2}{4}, \ldots, \frac{\mu_n}{2^n}, 0, 0, 0, \ldots\right)$.

Let $x = (\mu_1, \mu_2, \mu_3, \ldots)$, $y = (\delta_1, \delta_2, \delta_3, \ldots)$ and $x, y \in l^2$ and α, β be constants.

Then, $T_n(\alpha x + \beta y) = \left(\frac{\alpha\mu_1 + \beta\delta_1}{2}, \frac{\alpha\mu_2 + \beta\delta_2}{4}, \ldots, \frac{\alpha\mu_n + \beta\delta_n}{2^n}, 0, 0, 0, \ldots\right)$

$$= \left(\frac{\alpha\mu_1}{2}, \frac{\alpha\mu_2}{4}, \ldots, \frac{\alpha\mu_n}{2^n}, 0, 0, 0, \ldots\right) + \left(\frac{\beta\delta_1}{2}, \frac{\beta\delta_2}{4}, \ldots, \frac{\beta\delta_n}{2^n}, 0, 0, 0, \ldots\right)$$

$$= \alpha T_n x + \beta T_n y .$$

Hence, T_n is linear operator for all $n \in \mathbb{N}$.

And, $\|T_n x\|^2 = \sum_{i=1}^{n} \left|\frac{\mu_i}{2^i}\right|^2 \leq \sum_{i=1}^{\infty} \left|\frac{\mu_i}{2^i}\right|^2 \leq \sum_{i=1}^{\infty} \left|\frac{\mu_i}{2}\right|^2 = \frac{1}{4}\sum_{i=1}^{\infty}|\mu_i|^2 = \frac{1}{4}\|x\|^2$ for all $x \in l^2$. That is, $\|T_n x\| \leq \frac{\|x\|}{2}$ for all $x \in l^2$. Hence, T_n is bounded operator.

Since T_n is bounded linear operator and $\dim T_n(l^2) = n < \infty$

Hence, T_n is compact operator by Theorem 3.2.1.

That is $\{T_n\}$ is a sequence of compact linear operator on l^2.

$$\|(T - T_n)x\|^2 = \|Tx - T_n x\|^2 = \sum_{i=n+1}^{\infty} \left|\frac{\mu_i}{2^i}\right|^2 \leq \sum_{i=n+1}^{\infty} \left|\frac{\mu_i}{2^{n+1}}\right|^2$$

$$= \frac{1}{2^{2(n+1)}} \sum_{i=n+1}^{\infty} |\mu_i|^2$$

$$\leq \frac{1}{2^{2(n+1)}} \sum_{i=1}^{\infty} |\mu_i|^2$$

$$= \frac{1}{2^{2(n+1)}} \|x\|^2 \text{ for all } x \in l^2.$$

Thus, $\|Tx - T_n x\| \leq \frac{1}{2^{n+1}} \|x\|$ for all $x \in l^2$ now taking the supermom over all x of norm 1 we have, $\|T - T_n\| \leq \frac{1}{2^{n+1}}$

Hence, $T_n \to T$, hence T is compact by Theorem 3.2.2.

Definition 3.2.3:-A sequence $\{x_n\}$ in a Hilbert space H is weakly convergent with weak limit x if, for all $y \in H$, the sequence $\langle x_n, y\rangle$ converges to $\langle x, y\rangle$ in the usual sense, and we write $x_n \rightharpoonup x$ in this case.

Proposition 3.2.2: A weakly convergent sequence $\{x_n\}$ in a Hilbert space H has a unique weak limit.

Proof: Let $\{x_n\}$ a sequence in a Hilbert space H such that $x_n \rightharpoonup z_1$ and $x_n \rightharpoonup z_2$. That is

$$\langle x_n, y\rangle \to \langle z_1, y\rangle \quad \text{for all } y \in H \quad \text{and} \quad \langle x_n, y\rangle \to \langle z_2, y\rangle \quad \text{for all } y \in H.$$

But $\langle x_n, y\rangle$ is a sequence of numbers its limit is unique, implying that

$\langle z_1, y\rangle = \langle z_2, y\rangle$ for all $y \in H$.

Thus,

$0 = \langle z_1, y\rangle - \langle z_2, y\rangle = \langle z_1 - z_2, y\rangle$ for all $y \in H$ i.e $\langle z_1 - z_2, y\rangle = 0$ for all $y \in H$

Hence, $z_1 - z_2 = 0$ which implies $z_1 = z_2$ and weak limit is unique. ■

***Example* 17**: – Let $\{e_n\}$ be an orthonormal sequence in the Hilbert space H.

For $x \in H$, we have from Bessel's inequality that

$$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \leq \|x\|^2.$$

The terms of the left hand side goes to 0 for $n \to \infty$.

So $\langle e_n, x\rangle \to \langle 0, x\rangle = 0$ for all $x \in H$, showing that e_n converges weakly to 0.

Proposition 3.2.3: Let $\{x_n\}$ be a sequence in H, and assume that $x_n \to x$. Then $x_n \rightharpoonup x$.

Proof: Assume $x_n \to x$, that is $\|x_n - x\| \longrightarrow 0$.

Thus, $|\langle x_n, y\rangle - \langle x, y\rangle| = |\langle x_n - x, y\rangle|$

$\leq \|x_n - x\|\|y\|$ By Cauchy Schwarz inequality

Implying that $|\langle x_n, y\rangle - \langle x, y\rangle| \to 0$ for all $y \in H$ for $n \to \infty$.

Hence $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in H$. Therefore, $x_n \rightharpoonup x$.■

Let us recall one of the central theorems of functional analysis which will help us to verify that a weakly convergent sequence is bounded.

Uniform Boundedness Theorem: Let $\{T_\alpha\}_{\alpha \in I}$ be the set of bounded linear operators from a Banach space X in to a normed linear space Y such that

$\|T_\alpha(x)\| \leq M_x$ for all $\alpha \in I$, ($\{T_\alpha\}_{\alpha \in I}$ is a family pointwise bounded linear operators) then there is a constant M independent of x such that $\|T_\alpha\| \leq M$ for all $\alpha \in I$.

Proposition 3.2.4: A weakly convergent sequence in a Hilbert space H is bounded.

Proof: Let $\{x_n\}$ be a weakly convergent sequence in H with weak limit x. Now for each x_n define a mapping $T_n: H \to \mathbb{C}$ by $T_n(y) = \langle y, x_n \rangle$ for $y \in H$. Clearly T_n is linear. By Cauchy Schwarz inequality $\|T_n(y)\| = |\langle y, x_n \rangle| \leq M\|y\|$ for all $y \in H$ where $\|x_n\| = M$. Hence, $\{T_n\}$ is the set of bounded linear operators from a Hilbert space H in to a normed linear space $\mathbb{C}$.

But for each $y \in H$, $\langle x_n, y \rangle \to \langle x, y \rangle$, implying that for each $y \in H$ the sequence $\{\langle x_n, y \rangle\}$ bounded , this implies $|T_n(y)| = |\langle x_n, y \rangle| \leq M_y$ for all $n \in \mathbb{N}$ that is, M_y is constant depending on $y \in H$ but not on n. Hence, the family $\{T_n\}$ is pointwise bounded linear operators.

Therefore, by uniform boundedness Theorem it is uniformly bounded, i.e there is some constant $M > 0$ independent of y such that $\|T_n\| \leq M$ for all $n \in \mathbb{N}$.

Thus, from $$\|x_n\|^2 = T_n(x_n) \leq \|T_n\|\|x_n\| \leq M\|x_n\|$$

we have $\|x_n\| \leq M$ for all $n \in \mathbb{N}$. Hence, a weekly convergent sequence in a Hilbert space H is bounded.■

The difference between strong and weak convergence is a genuine infinite dimensional phenomenon; if the Hilbert space is finite dimensional weak and strong convergence are the same. But we have strong convergence implies weak convergence and any weakly convergent sequence is bounded. One importance of compact linear operator $T: H \to H$ for Hilbert space H is that it maps weak convergence in to strong convergence. Now let's see the following Theorem to verify this.

Theorem 3.2.5:-Let H be a Hilbert space and let $\{x_n\}$ be a weakly convergent sequence, with weak limit x .If $T \in B(H)$ is compact, then $\{Tx_n\}$ converges in the norm to Tx.

Proof: Since $x_n \rightharpoonup x$ we have for all $y \in H$ that

$\langle Tx_n, y\rangle = \langle x_n, T^{\star}y\rangle \to \langle x, T^{\star}y\rangle = \langle Tx, y\rangle$ i.e. $\langle Tx_n, y\rangle \to \langle Tx, y\rangle$ for all $y \in H$.

Hence $Tx_n \rightharpoonup Tx$, since by proposition 3.2.3 strong convergence implies weak convergence, we see that Tx is the only possible limit. So assume that $\{Tx_n\}$ does not converge to Tx . Then it possible to extract a subsequence $\{Ty_n\}$ of $\{Tx_n\}$ such that

$$\|Ty_n - Tx\| > \delta \text{ for all } n \in \mathbb{N} \text{ and for some } \delta > 0 .$$

But , since by proposition 3.2.4 $\{x_n\}$ is bounded and T is compact and we can have a subsequence $\{Ty_{n_k}\}$ of $\{Ty_n\}$ that is strongly convergence to y in H hence, by proposition 3.2.3 $Ty_{n_k} \rightharpoonup y$. But since $Ty_{n_k} \rightharpoonup Tx$ we must have that by proposition 3.2.2 $y = Tx$, which is not possible according to the inequality above.

Hence $Tx_n \to Tx$ ■

Corollary 3.2.1:-Let H be a Hilbert space and $T \in B(H)$ is compact .Then if $\{e_n\}$ is an orthonormal sequence in the Hilbert space H then, $Te_n \to 0$.

Proof:- Since $e_n \rightharpoonup 0$, hence $Te_n \to 0$ by Theorem 3.2.3 .■

Proposition 3.2.5:- Let H be an infinite dimensional Hilbert space. If $T \in B(H)$ is compact and T^{-1} exists, then T^{-1} is unbounded.

Proof: Let $\{e_n\}$ be an orthonormal sequence in H. Then, $Te_n \to 0$ from Corollary 3.2.1, but since T^{-1} exists, $\|T^{-1}(Te_n)\| = \|e_n\| = 1$ for all, so T^{-1} is not continuous. ■

Theorem 3.2.4: Let H be a Hilbert space and let T_1 and $T_2 \in B(H)$. Then the product $T_2T_1: H \to H$ is compact if one of the two operators T_1 or T_2 is compact.

Proof: Let $\{x_n\}$ be a bounded sequence in H.

If T_1 is compact there exist a subsequence $\{x_{n_k}\}$ such that $T_1x_{n_k} \to x \in H$,as $k \to \infty$.

Since T_2 is bounded therefore continuous, we have

$$T_2(T_1x_{n_k}) = (T_2T_1)x_{n_k} \to T_2x \text{ in } H \text{ , as } k \to \infty.$$

Hence, T_2T_1 is compact.

If T_1 is bounded and T_2 is compact, since bounded operators map bounded sets in to bounded sets, the sequence $\{T_1 x_n\}$ is bounded in H. Therefore, there exists a subsequence $\{x_{n_k}\}$ such that

$$T_2(T_1 x_{n_k}) = (T_2 T_1)x_{n_k} \to x \text{ in } H \text{ , as } k \to \infty.$$

Hence again $T_2 T_1$ is compact. ■

Theorem 3.2.5: Let H be a Hilbert space and $T \in B(H)$. If T is compact then $T^\star$ is also compact.

Proof: Let $\{x_n\}$ be a bounded sequence in H. I.e $\|x_n\| \le M$ for some $M > 0$.

By Theorem 3.2.4 above $TT^\star: H \to H$ is compact. Hence there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{TT^\star x_{n_k}\}$ converges in H. But then from

$$\left\|T^\star\left(x_{n_k} - x_{n_j}\right)\right\|^2 = \langle T^\star\left(x_{n_k} - x_{n_j}\right), T^\star\left(x_{n_k} - x_{n_j}\right)\rangle$$

$$= \langle TT^\star\left(x_{n_k} - x_{n_j}\right), x_{n_k} - x_{n_j}\rangle \le 2M\left\|TT^\star\left(x_{n_k} - x_{n_j}\right)\right\|$$

We observe that $\{T^\star x_{n_k}\}$ is a Cauchy sequence, and therefore it converges in the Hilbert space H .■

3.3 Hilbert-Schmidt Operator

Definition 3.3.1:- If the kernel $k(s,t)$ satisfies $\int_a^b \int_a^b |k(s,t)|^2 dsdt < \infty$, then the kernel $k(s,t)$ is called Hilbert Schmidt kernel and the corresponding integral operator is called Hilbert Schmidt operator.

That is, an integral operator on $L^2[a,b]$ is called Hilbert Schmidt operator if the kernel k is in $L^2([a,b],[a,b])$, that is if $\|k\|^2 = \int_a^b \int_a^b |k(s,t)|^2 dsdt < \infty$.

Proposition 3.3.1:- Hilbert Schmidt operator on $L^2[a,b]$ is bounded and $\|T\| \le \|k\|$.

Proof: Suppose T is Hilbert Schmidt operator with kernel $k \in L^2([a,b],[a,b])$ that is $\int_a^b \int_a^b |k(s,t)|^2 dsdt < \infty$ then, using L^2 norm

$\|Tf\|^2 = \int_a^b \left|\int_a^b k(s,t)f(t)\,dt\right|^2 ds \le \int_a^b \left(\int_a^b |k(s,t)f(t)|\,dt\right)^2 ds$

But, $\int_a^b |k(s,t)f(t)|\,dt \le \|k(s,.)\|\|f\|$ By *Hölders inequality*

Hence, $\|Tf\|^2 = \int_a^b \left|\int_a^b k(s,t)f(t)\,dt\right|^2 ds \le \int_a^b \left(\int_a^b |k(s,t)f(t)|\,dt\right)^2 ds$

$$\le \int_a^b (\|k(s,t)\|\|f\|)^2\,ds = \|f\|^2 \int_a^b |k(s,t)|^2\,ds = \|f\|^2\|k\|^2$$

Thus, $\|Tf\| \le \|k\|\|f\|$ for all $f \in L^2[a,b]$ hence, T is bounded. Now taking supermom over all f of norm 1 we have, $\|T\| \le \|k\|$ ■

Definition 3.3.2: An orthonormal set $\{e_n\}_{n=1}^{\infty}$ in Hilbert space H is said to be complete if $\langle e_n, f\rangle = 0$, for all $n \in \mathbb{N} \Rightarrow f = 0$.

Equivalently, an orthonormal set $\{e_n\}_{n=1}^{\infty}$ in Hilbert space H is said to be complete if

$$g = \sum_{n=1}^{\infty} \langle g, e_n\rangle\, e_n \quad \text{, for all } g \in H.$$

That is to mean that there is no function $\varphi \neq 0$ in H for which the set $\{e_n\}_{n=1}^{\infty} \cup \{\varphi\}$ forms an orthonormal set in H.

In order to prove Hilbert Schmidt operator is compact, we must construct a complete orthonormal set for $L^2([a,b],[a,b])$ from complete orthonormal sets of $L^2[a,b]$. This is done by the so-called tensor product, which is defined in the following manner; let $f, g \in L^2[a,b]$ then we define the function $f \otimes g$ by

$$f \otimes g(s,t) = f(s)g(t) \quad , \ (s,t) \in [a,b] \times [a,b]\,.$$

It is obvious that $f \otimes g \in L^2([a,b],[a,b])$ since

$$\int_a^b \int_a^b |f \otimes g(s,t)|^2\,dsdt = \int_a^b \int_a^b |f(s)g(t)|^2\,dsdt$$
$$= \int_a^b |f(s)|^2 ds \int_a^b |g(t)|^2 dt < \infty$$

Theorem 3.3.1:- If the set $\{e_i\}_{i=1}^{\infty}$ is a complete orthonormal set in $L^2[a,b]$, then the set of all products $\{e_i e_j\}_{i,j=1}^{\infty}$ is complete orthonormal set in $L^2([a,b],[a,b])$.

Proof: To prove $\{e_ie_j\}_{i,j=1}^{\infty}$ is orthonormal set.

$$\langle e_ne_m, e_ie_j\rangle = \int_a^b\int_a^b e_n(s)e_m(t)\overline{e_i(s)e_j(t)}dsdt$$

$$= \int_a^b e_m(t)\overline{e_j(t)}\,dt \int_a^b e_n(s)\overline{e_i(s)}\,ds = \langle e_m, e_j\rangle\langle e_n, e_i\rangle = \delta_{mj}\delta_{ni}$$

So that $\{e_ie_j\}_{i,j=1}^{\infty}$ is an orthonormal set in $L^2([a,b],[a,b])$.

To see $\{e_ie_j\}_{i,j=1}^{\infty}$ is also complete, assume $f \in L^2([a,b],[a,b])$ and

$\langle e_ie_j, f\rangle = 0$ for all $n, m \in \mathbb{N}$.

$$\langle e_i\varphi_j, f\rangle = \int_a^b\int_a^b e_i(s)e_j(t)\overline{f(s,t)}\,dtds = 0 \quad \text{for all } i,j \in \mathbb{N}.$$

$$= \int_a^b e_i(s)\left(\int_a^b e_j(t)\overline{f(s,t)}\,dt\right)ds = 0 \quad \text{for all } i,j \in \mathbb{N}$$

since $\{e_i\}_{i=1}^{\infty}$ is a complete set in $L^2[a,b]$ we have $\int_a^b e_j(t)\overline{f(s,t)}\,dt = 0$ for all $j \in \mathbb{N}$ in $L^2[a,b]$. And again since $\{e_j\}_{j=1}^{\infty}$ is complete set in $L^2[a,b]$,

$\int_a^b e_j(t)\overline{f(s,t)}\,dt = 0$ for all $j \in \mathbb{N}$ implies that $f = 0$ in $L^2([a,b],[a,b])$.

Hence, $\{e_ie_j\}_{i,j=1}^{\infty}$ is complete set.

Hence, $\{e_ie_j\}_{i,j=1}^{\infty}$ is complete orthonormal set in $L^2([a,b],[a,b])$.■

Corollary 3.3.1:- Hilbert Schmidt operator is compact.

Proof: Let $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal set in $L^2[a,b]$, then by Theorem 3.3.1 $\{e_ie_j\}_{i,j=1}^{\infty}$ is a complete orthonormal set in $L^2([a,b],[a,b])$.

Then since k is in $L^2([a,b],[a,b])$ we can represent the kernel $k(s,t)$ as

$k(s,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\beta_{ij}e_i(s)\,e_j(t)$ where, $\beta_{ij} = \langle k, e_ie_j\rangle$ are Fourier coefficients for the kernel $k(s,t)$ and the operator is of the form $(Tu)(s) = \int_a^b k(s,t)u(t)\,dt$.

By Parseval's equality we have

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\beta_{ij}\right|^2 = \|k\|^2 = \int_a^b\int_a^b |k(s,t)|^2 dsdt < \infty .$$

Let us define the degenerate kernel $k_n(s,t)$ by $k_n(s,t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} e_i(s)\, e_j(t)$

which converges to $k(s,t)$ in $L^2([a,b],[a,b])$.

The operator $(T_n u)(s) = \int_a^b k_n(s,t) u(t)\, dt$ is compact by Theorem 3.2.1.

Now it suffice to show that T_n converges to T in the operator norm. To see this ,

$$
\begin{aligned}
&\|T_n u - Tu\|^2 \\
&= \int_a^b \left| \int_a^b \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} e_i(s)\, e_j(t) u(t) dt - \int_a^b \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \beta_{ij} e_i(s)\, e_j(t) u(t) \right|^2 ds \\
&= \int_a^b \left| \int_a^b \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} - \big(\sum_{i=1}^{n} \sum_{j=1}^{\infty} + \sum_{i=n+1}^{\infty} \sum_{j=1}^{\infty}\big)\Big) \beta_{ij} e_i(s) e_j(t) u(t) dt \right|^2 ds \\
&= \int_a^b \left| \int_a^b \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} - \big(\sum_{i=1}^{n} \sum_{j=1}^{n} + \sum_{i=1}^{n} \sum_{j=n+1}^{\infty} + \sum_{i=n+1}^{\infty} \sum_{j=1}^{\infty}\big)\Big) \beta_{ij} e_i(s) e_j(t) u(t) dt \right|^2 ds \\
&= \int_a^b \left| \int_a^b \Big(\sum_{i=1}^{n} \sum_{j=n+1}^{\infty} + \sum_{i=n+1}^{\infty} \sum_{j=1}^{\infty}\Big) \beta_{ij} e_i(s) e_j(t) u(t) dt \right|^2 ds \\
&\le \int_a^b \left(\Big(\sum_{i=1}^{n} \sum_{j=n+1}^{\infty} + \sum_{i=n+1}^{\infty} \sum_{j=1}^{\infty}\Big) |\beta_{ij}|^2 |e_i(s)|^2 \int_a^b |e_j(t)|^2 dt \int_a^b |u(t)|^2 dt \right) ds \\
&= \Big(\sum_{i=1}^{n} \sum_{j=n+1}^{\infty} + \sum_{i=n+1}^{\infty} \sum_{j=1}^{\infty}\Big) |\beta_{ij}|^2 \int_a^b |e_i(s)|^2 ds \int_a^b |e_j(t)|^2 dt \int_a^b |u(t)|^2 dt \\
&= \Big(\sum_{i=1}^{n} \sum_{j=n+1}^{\infty} + \sum_{i=n+1}^{\infty} \sum_{j=1}^{\infty}\Big) |\beta_{ij}|^2 \|e_i\|^2 \|e_j\|^2 \|u\|^2 \\
&= \Big(\sum_{i=1}^{n} \sum_{j=n+1}^{\infty} + \sum_{i=n+1}^{\infty} \sum_{j=1}^{\infty}\Big) |\beta_{ij}|^2 \|u\|^2
\end{aligned}
$$

Since $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\beta_{ij}|^2 < \infty$, for all $\varepsilon > 0$ there exists a positive integer n_0 such that

$$\left| \sum_{i=1}^{n} \sum_{j=1}^{n} |\beta_{ij}|^2 - \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\beta_{ij}|^2 \right| = \Big(\sum_{i=1}^{n} \sum_{j=n+1}^{\infty} + \sum_{i=n+1}^{\infty} \sum_{j=1}^{\infty}\Big) |\beta_{ij}|^2 < \varepsilon^2 ,$$

for all $n \ge n_0$ Then , we have

$$\|T_n u - Tu\|^2 \le \Big(\sum_{i=1}^{n} \sum_{j=n+1}^{\infty} + \sum_{i=n+1}^{\infty} \sum_{j=1}^{\infty}\Big) |\beta_{ij}|^2 \|u\|^2 < \varepsilon^2 \|u\|^2, \text{ for all } n \ge n_0$$

That is , $\|T_n u - Tu\| \le \varepsilon \|u\|$ for all $n \ge n_0$ and for all $u \in L^2[a,b]$.

But, $\|(T - T_n)u\| = \|T_n u - Tu\| \le \varepsilon \|u\|$ and taking supermom over all u of norm 1 we have $\|T - T_n\| \le \varepsilon$ for all $n \ge n_0$ implies T_n converges to T in the operator norm .

Hence T is compact by Theorem 3.2.2■

Chapter 4

Spectral Theory for Compact Linear Operators and Fredholm Alternative Theorem

4.1 Equivalence between Degenerate Operators and Matrix Operators'

Degenerate operators are equivalent to matrix operators. To see this suppose we wish to solve,

$$u(s) = f(s) + \lambda T_n u \tag{18}$$

where, T_n is the degenerate operator given by $(T_n u)(s) = \int_a^b \sum_{i=1}^n \phi_i(s)\varphi_i(t)\, u(t)dt$.

Thus,

$$\begin{aligned} f(s) &= u(s) - \lambda \int_a^b \sum_{i=1}^n \phi_i(s)\varphi_i(t)\, u(t)dt \\ &= u(s) - \lambda \sum_{i=1}^n \int_a^b \phi_i(s)\varphi_i(t)\, u(t)dt \\ &= u(s) - \lambda \sum_{i=1}^n \phi_i(s) \int_a^b \varphi_i(t)\, u(t)dt \end{aligned}$$

Hence, the equation (18) can be written as,

$$f(s) = u(s) - \lambda \sum_{i=1}^n \xi_i\, \phi_i(s)$$

where, $\xi_i = \int_a^b \varphi_i(t)\, u(t)dt$ are unknown constants.

Then taking the inner product of u with $\varphi_i(t)$, we obtain,

$$\begin{aligned} \xi_j = \langle u, \varphi_j\rangle &= \langle f + \lambda \sum_{i=1}^n \xi_i\, \phi_i, \varphi_j\rangle \\ &= \langle f, \varphi_j\rangle + \langle \lambda \sum_{i=1}^n \xi_i\, \phi_i, \varphi_j\rangle \\ &= \langle f, \varphi_j\rangle + \lambda \sum_{i=1}^n \xi_i\, \langle \phi_i, \varphi_j\rangle \end{aligned}$$

Now let $\eta_j = \langle f, \varphi_j\rangle \qquad j = 1,2,\dots,n$

$\alpha_{ij} = \langle \phi_j, \varphi_i\rangle \qquad i\,,j = 1,2,\dots,n$

$A = \left(\alpha_{ij}\right)_{i,j=1,2,\dots,n}$

$I = \left(\delta_{ij}\right)_{i,j=1,2,\dots,n}$, where $\delta_{ij} = \begin{cases} 1 \,, if\ i = j \\ 0 \,, if\ i \neq j \end{cases}$

$\eta = (\eta_1\,, \eta_2\,, \dots, \eta_n)^T$

$$\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$$

Then the equation $u(s) = f(s) + \lambda T_n u$ has an equivalent matrix representation form,

$$\left[\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}_{n\times n} - \lambda \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}\right] \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$$

It can be written compactly as,

$$(I - \lambda A)\xi = \eta \tag{19}$$

***Example* 18**: Let the Fredholm integral operator $T: L^2[0,1] \to L^2[0,1]$ is given by

$$(Tu)(s) = \int_0^1 ln\frac{t^s}{s^t}\, u(t)dt \quad , u(t) \in L^2[0,1] .$$

We wish to find the equivalent matrix operator of the integral equation of

$$s = u(s) - \lambda(Tu)(s).$$

Here, is a degenerate operator given as,

$$(T_2u)(s) = (Tu)(s) = \int_0^1 \textstyle\sum_{i=1}^2 \phi_i(s)\varphi_i(t)u(t)dt$$

where, $\phi_1(s) = s\ ,\ \phi_2(s) = -lns\quad ,\ \varphi_1(t) = lnt\ \ and\quad \varphi_2(t) = t$

Thus, $f(s) = u(s) - \int_0^1 \sum_{i=1}^2 \phi_i(s)\varphi_i(t)u(t)dt$ where, $f(s) = s$

Then, $\alpha_{11} = \langle \phi_1, \varphi_1 \rangle = \int_0^1 t\,\overline{lnt}\,dt = \int_0^1 t\,lntdt = \frac{1}{4}$

$\alpha_{12} = \langle \phi_2, \varphi_1 \rangle = -\int_0^1 lnt\,\bar{t}\ dt = \frac{1}{4}$

$\alpha_{21} = \langle \phi_1, \varphi_2 \rangle = \int_0^1 t^2 dt = \frac{1}{3}$

$\alpha_{22} = \langle \phi_2, \varphi_2 \rangle = -\int_0^1 tlnt\,dt = -\frac{1}{4}$

And, $\eta_1 = \langle f, \varphi_1 \rangle = \int_0^1 t\,lntdt = \frac{1}{4}$

$\eta_2 = \langle f, \varphi_2 \rangle = \int_0^1 t^2\,dt = \frac{1}{3}$

Hence, $A = \left(\alpha_{ij}\right)_{ij=1,2}$, $I = \left(\delta_{ij}\right)_{i,j=1,2}$ and $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$. Thus the integral equation $s = u(s) - \lambda(Tu)(s)$ has an equivalent matrix representation form,

$$\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \lambda\begin{pmatrix}1/4 & -1/4\\ 1/3 & 1/4\end{pmatrix}\right]\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}1/4\\ 1/3\end{pmatrix}$$

The main goal of this section is to determine how to generalize the spectral theory properties (Eigen pair properties) of matrices to an infinite dimensional compact linear operator. The reason this can be done for compact operator is that the compacting action of operator tends to minimize the effect of "most" of the possible independent directions in the vector space. As a result the difference between the action of finite dimensional operator and the action of operator is "small".

4.2 Spectral Properties of Compact Linear Operators on Hilbert Space

Definition 4.2.1: Let H be a Hilbert space and let $T: H \to H$ be a compact operator. An eigenvalue, eigenfuncton pair of T is a pair (λ, u) satisfying $Tu = \lambda u$ where, λ is scalar, $u \in H$, $u \neq 0$.

For Hilbert space H let $T: H \to H$ be a compact linear operator. The following are some properties of eigenvalues and eigenfunctions for the compact linear operator T.

1. The multiplicity of any eigenvalues $\lambda \neq 0$ is finite. That is the number of linearly independent eigenfuncton for any eigenvalue $\lambda \neq 0$ is finite.

   Proof: Suppose not! That is there is an infinite set of functions (vectors), $\{\phi_i\}_{i=1}^{\infty}$. Thus, $\{\phi_i\}_{i=1}^{\infty}$ can be made orthonormal set of vectors using Gram Schmidt for which $T\phi_i = \lambda\phi_i$. So that for $n \neq m$
   $\|T\phi_n - T\phi_m\| = \|\lambda\phi_n - \lambda\phi_m\| = |\lambda|\|\phi_n - \phi_m\| = \sqrt{2}\,|\lambda| > 0$, so $\{T\phi_n\}_{n=1}^{\infty}$ has no convergent subsequence and this contradicts with the fact that T is compact. Hence, it must be the case that the number of linearly independent eigenfunctions for any eigenvalue $\lambda \neq 0$ is finite. ■

2. The adjoint operator $T^\star$ exists and is bounded.

   Proof: A compact linear operator is bounded linear operator by Proposition 3.2.1. Thus, $T^\star$ exists and is bounded by Theorem 2.3.2. ■

3. If T is self adjoint operator, then T has at least one eigenvalue.

   Proof: Without loss of generality we may assume $T \neq 0$.

Let us consider the limits of the operator

$$m = \inf_{\|x\|=1} \langle Tx, x\rangle \quad , \quad M = \sup_{\|x\|=1} \langle Tx, x\rangle$$

From $\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$, we know that $\|T\| = max\{|m|, M\}$.

We shall prove that $\lambda_1 = \begin{cases} m & , if \ \|T\| = |m| \\ M & , if \ \|T\| = M \end{cases}$ is an eigenvalue of T.

Consider the case $\|T\| = M$. There exists a sequence of elements $\{x_n\}$ with $\|x_n\| = 1$ such that

$$\langle Tx_n , x_n\rangle \to M = \lambda_1 \text{ as } n \to \infty. \qquad (20)$$

Since T is compact and $\{x_n\}$ is bounded, we assume that the sequence $\{Tx_n\}$ has a convergent subsequence $\{Ty_n\}$, say converges to y_0 that is $Ty_n \to y_0$.

Let us regard,

$$\|Ty_n - \lambda_1 y_n\|^2 = \langle Ty_n - \lambda_1 y_n , Ty_n - \lambda_1 y_n \rangle$$

$$= \|Ty_n\|^2 - 2\lambda_1 \langle y_n , y_n \rangle + {\lambda_1}^2 \leq \|T\|^2 - 2\lambda_1 \langle y_n , y_n \rangle + {\lambda_1}^2$$

From the relation (20) and using the fact that ${\lambda_1}^2 = \|T\|^2$ we obtain

$$Ty_n - \lambda_1 y_n \to 0 \text{ as } n \to \infty$$

which implies in turn, that $y_n = \frac{1}{\lambda_1}\left(Ty_n - (Ty_n - \lambda_1 y_n)\right)$ converges.

Namely, $y_n \to y = \frac{1}{\lambda_1} y_0$

Indeed, $y_0 = Ty = \lambda_1 y$ because $Ty_n \to Ty = y_0$.

As $y \neq 0$, $(\|y\| = 1)$, then λ_1 is an eigenvalue . ■

4. If T is self adjoint operator, then all its eigenvalues are real.

   Proof: Let λ be an eigenvalue of T and $x \neq 0$ be the corresponding eigenfunctions of λ. That is $Tx = \lambda x$. Then,

   $\lambda\|x\|^2 = \lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle = \langle x, Tx\rangle$ since T is self adjoint.

   $= \langle x, \lambda x\rangle = \bar{\lambda}\langle x, x\rangle = \bar{\lambda}\|x\|^2$

   Since, $x \neq 0$, $\|x\| > 0$, hence $\lambda = \bar{\lambda}$. Hence λ is real. ■

5. If T is self adjoint operator, then eigenfunctons corresponding to different eigenvalues are orthogonal.

   Proof: Let λ, μ be distinct eigenvalues and $x, y \in H$, $x, y \neq 0$ be eigenfunctions corresponding to eigenvalues λ and μ respectively. That is $Tx = \lambda x$ and $Ty = \mu y$.

   Thus, $\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Tx, y\rangle = \langle x, Ty\rangle$ since T is self adjoint

   $= \langle x, \mu y\rangle = \bar{\mu}\langle x, y\rangle = \mu\langle x, y\rangle$ since μ is real

   But since $\lambda \neq \mu$, we must have $\langle x, y\rangle = 0$. Hence x and y are orthogonal. ■

6. If T is self adjoint operator, then the number of distinct eigenfuncton is either finite or $\lim_{n\to\infty} \lambda_n$.

   Proof: For each eigenvalue λ_n there are at most a finite number of eigenfunctions by (1). Let u_n with $\|u_n\| = 1$ be one of possibly many eigenfunctions of λ_n.

   The sequence $\{u_n\}$ is a orthonormal sequence since $\lambda_n \neq \lambda_m$, for $n \neq m$. Since T is compact if there are an infinite number λ_n ,

   $$0 = lim_{n\to\infty}\|Tu_n\|^2 = lim_{n\to\infty}\|\lambda_n u_n\|^2 = \lim_{n\to\infty} {\lambda_n}^2 = \lim_{n\to\infty} {\lambda_n}^2$$

   Hence, $\lim_{n\to\infty} \lambda_n = 0$. ■

7. (**Maximum principle**) Every non trivial self adjoint compact operator has a non trivial eigenpair (λ_1, ϕ_1) where $|\lambda_1| = \max_{\|u\|=1}|\langle Tu, u\rangle| = \|T\|$ and the maximum is attained by ϕ_1 (normalized).

Proof: Let $\beta = \sup_{\|u\| \neq 0} \frac{|\langle Tu,u\rangle|}{\|u\|^2}$.

Now, $|\langle Tu, u\rangle| \leq \|Tu\|\|u\|$ by Cauchy Schwarz inequality so that $|\langle Tu, u\rangle| \leq \|Tu\|\|u\| \leq \|T\|\|u\|\|u\| = \|T\|\|u\|^2$ and this gives us

$$\beta \leq \|T\|. \tag{21}$$

Now we wish to show $\beta \geq \|T\|$, we observe that

$$\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle = 2(\langle Tx, y\rangle + \langle Ty, x\rangle)$$

But, since T is compact self adjoint operator hence T is bounded self adjoint operator thus $\langle T(x+y), x+y\rangle$ and $\langle T(x-y), x-y\rangle$ are real by Lemma 2.3.1

So from the definition of β we have,

$$\langle T(x+y), x+y\rangle \leq \beta\|x+y\|^2 \quad \text{and}$$

$$-\langle T(x-y), x-y\rangle \leq \beta\|x-y\|^2$$

Thus,

$$2(\langle Tx, y\rangle + \langle Ty, x\rangle) \leq \beta(\|x+y\|^2 + \|x-y\|^2)$$

Now using parallelogram law, we have

$$2(\langle Tx, y\rangle + \langle Ty, x\rangle) \leq 2\,\beta(\|x\|^2 + \|y\|^2)$$

That is $\langle Tx, y\rangle + \langle Ty, x\rangle \leq \beta(\|x\|^2 + \|y\|^2)$ (22)

Now if $Tx = 0$ it is clear that $\|Tx\| \leq \beta\|x\|$ that will lead us to $\beta \geq \|T\|$. so assume that $Tx \neq 0$ and define $y = \frac{\|x\|}{\|Tx\|} Tx$. Then, $\|y\| = \frac{\|x\|}{\|Tx\|}\|Tx\| = \|x\|$

And $2\frac{\|x\|}{\|Tx\|}\|Tx\|^2 = \frac{\|x\|}{\|Tx\|}2\|Tx\|^2 = \frac{\|x\|}{\|Tx\|}(\|Tx\|^2 + \|Tx\|^2)$

$$= \frac{\|x\|}{\|Tx\|}(\langle Tx, Tx\rangle + \langle Tx, Tx\rangle)$$

$$= \frac{\|x\|}{\|Tx\|}(\langle Tx, Tx\rangle + \langle TTx, x\rangle) \text{ Since } T \text{ is self adjoint}$$

$$= \langle Tx, \frac{\|x\|}{\|Tx\|}Tx\rangle + \langle T\left(\frac{\|x\|}{\|Tx\|}Tx\right), x\rangle$$

$$= \langle Tx, y\rangle + \langle Ty, x\rangle$$

$$\le \beta(\|x\|^2 + \|y\|^2) \quad \text{By using equation (21)}$$

But from above we have that $\|y\| = \|x\|$. Thus, $2\frac{\|x\|}{\|Tx\|}\|Tx\|^2 \le 2\beta\|x\|^2$.

Hence this will lead us to $\|Tx\| \le \beta\|x\|$ for all $x \in H$.

Thus, taking the supermom over x of norm 1, we have $\|T\| \le \beta$. (23)

From (21) and (23) we have $\|T\| = \beta$.

Then,

$$\|T\| = \sup_{\|u\|\neq 0}\frac{|\langle Tu,u\rangle|}{\|u\|^2} = \sup_{\|u\|\neq 0}\langle T\frac{u}{\|u\|}, T\frac{u}{\|u\|}\rangle$$

thus, for any v , $v_0 = \frac{v}{\|v\|}$ we have $\sup_{\|v\|\neq 0}\langle Tv_0, Tv_0\rangle = \|T\|$.

Hence, there is a sequence of normalized function $\{z_n\}$, $\|z_n\| = 1$ for which

$\lim_{n\to\infty}|\langle Tz_n, z_n\rangle| = \|T\|$ which implies $\left|\lim_{n\to\infty}\langle Tz_n, z_n\rangle\right| = \|T\|$.

Then therefore a subsequence $\{x_n\}$ for which $\lim_{n\to\infty}\langle Tx_n, x_n\rangle = \pm\|T\|$ that is $\langle Tx_n, x_n\rangle$ converges to either $\|T\|$ or $-\|T\|$.

Let $\lambda_1 = \lim_{n\to\infty}\langle Tx_n, x_n\rangle$, then

$$\begin{aligned}\|Tx_n - \lambda_1 x_n\|^2 &= \|Tx_n\|^2 - 2\lambda_1\langle Tx_n, x_n\rangle + {\lambda_1}^2\|x_n\|^2 \\ &\le \|T\|^2\|x_n\|^2 - 2\lambda_1\langle Tx_n, x_n\rangle + {\lambda_1}^2\|x_n\|^2 \\ &= \|T\|^2 - 2\lambda_1\langle Tx_n, x_n\rangle + {\lambda_1}^2\end{aligned}$$

But $\|T\|^2 - 2\lambda_1\langle Tx_n, x_n\rangle + {\lambda_1}^2 \to \infty$ as $n \to \infty$. This means that

$$\lim_{n\to\infty}(Tx_n - \lambda_1 x_n) = 0$$

Since T is compact the sequence $\{Tx_n\}$ possesses a convergent subsequence $\{Ty_n\}$ which converges to some function $y_0 \in H$.

Since $Tx_n - \lambda_1 x_n \to 0$ we have also $Ty_n - \lambda_1 y_n \to 0$.

Thus, $\lim_{n\to\infty}(Ty_n - \lambda_1 y_n) = \lim_{n\to\infty} Ty_n - \lim_{n\to\infty}\lambda_1 y_n = 0$

Since T is continuous $\lim_{n\to\infty} Ty_n = T\lim_{n\to\infty} y_n$ and we have also

$\lim_{n\to\infty} \lambda_1 y_n = \lambda_1 \lim_{n\to\infty} y_n$

Implies $T\lim_{n\to\infty} y_n = \lambda_1 \lim_{n\to\infty} y_n$ (24)

Again, since $Ty_n - \lambda_1 y_n \to 0$, y_n converges to some function ϕ_1 that is

$$y_0 = \lim_{n\to\infty} Ty_n = \lambda_1 \lim_{n\to\infty} y_n \quad \Rightarrow \lim_{n\to\infty} y_n = \frac{y_0}{\lambda_1} = \phi_1$$

That is y_n converges to $\phi_1 = \frac{y_0}{\lambda_1}$ and

$$\|\phi_1\| = \left\|\frac{y_0}{\lambda_1}\right\| = \left\|\frac{\lim_{n\to\infty} Ty_n}{\lambda_1}\right\| = \left\|\frac{\lim_{n\to\infty} \lambda_1 y_n}{\lambda_1}\right\|$$

$$= \frac{\left\|\lim_{n\to\infty} \lambda_1 y_n\right\|}{|\lambda_1|} = \frac{\lim_{n\to\infty} \|\lambda_1 y_n\|}{|\lambda_1|} = \lim_{n\to\infty} \|y_n\| = 1$$

Hence, from equation (24) we have $T\phi_1 = \lambda_1\phi_1$. This implies that λ_1 is an eigenvalue. Hence, for non trivial self adjoint compact operator we have found a non trivial eigenpair (λ_1, ϕ_1) where $|\lambda_1| = \max_{\|u\|=1} |\langle Tu, u\rangle| = \|T\|$ and the maximum is attained by ϕ_1.■

8. A compact self adjoint operator T is either degenerate or else it has an infinite number of mutually orthogonal eigenfunctions and either $\|T\|$ or $-\|T\|$ is the largest among all the eigenvalues of T.

Proof: To see this we will use induction.

For $n = 1$ by property 6, we have a non trival eigenpair (λ_1, ϕ_1). Assume for $n - 1$ the eigenpairs (λ_i, ϕ_i) , $i = 1, \dots, n-1$ of T , ϕ_i are mutually orthogonal.

Now we define a new compact operator T_n by

$$T_n u = Tu - \sum_{i=1}^{n-1} \lambda_i \langle u, \phi_i\rangle\, \phi_i \ .$$

If T is an integral operator the kernel corresponding to T_n is

$$k_n(s,t) = k(s,t) - \sum_{i=1}^{n-1} \lambda_i\, \phi_i(s)\phi_i(t).$$

Now T_n have the following properties

i. Let $\langle u, \phi_i \rangle = 0$ for $i = 1, \dots, n-1$, then

$$T_n u = Tu - \sum_{i=1}^{n-1} \lambda_i(0)\, \phi_i = Tu.$$

Hence, if $\langle u, \phi_i \rangle = 0$ for $i = 1, \dots, n-1$, then $T_n u = Tu.$

ii. Let $u = \sum_{j=1}^{n-1} \alpha_j \phi_j$ then

$$\begin{aligned} Tu &= T\left(\textstyle\sum_{j=1}^{n-1} \alpha_j \phi_j\right) - \sum_{i=1}^{n-1} \lambda_i\, \phi_i \langle \textstyle\sum_{j=1}^{n-1} \alpha_j \phi_j\, , \rangle \\ &= \textstyle\sum_{j=1}^{n-1} \alpha_j T\phi_j - \sum_{i=1}^{n-1} \lambda_i\, \phi_i \sum_{j=1}^{n-1} \alpha_j \langle \phi_j, \phi_i \rangle \\ &= \textstyle\sum_{j=1}^{n-1} \alpha_j T\phi_j - \sum_{j=1}^{n-1} \alpha_j \sum_{i=1}^{n-1} \lambda_i\, \phi_i \langle \phi_j, \phi_i \rangle \\ &= \textstyle\sum_{j=1}^{n-1} \alpha_j T\phi_j - \sum_{j=1}^{n-1} \alpha_j\, \lambda_j \phi_j \\ &= \textstyle\sum_{j=1}^{n-1} \alpha_j T\phi_j - \sum_{j=1}^{n-1} \alpha_j T\phi_j \quad \text{Since } T\phi_j = \lambda_j \phi_j \text{ for} , j = 1, \dots, n-1 \\ &= 0 \end{aligned}$$

Hence, if $u = \sum_{j=1}^{n-1} \alpha_j \phi_j$, then $T_n u = 0$.

iii. Let $v \in T_n(H)$ then there is $u \in H$ such that $T_n u = v$.

Thus for $j = 1, \dots n-1$,

$$\begin{aligned} \langle v, \phi_j \rangle &= \langle\, T_n u, \phi_j \rangle \\ &= \langle Tu - \textstyle\sum_{i=1}^{n-1} \lambda_i \langle u, \phi_i \rangle\, \phi_i, \phi_j \rangle \\ &= \langle Tu, \phi_j \rangle - \textstyle\sum_{i=1}^{n-1} \lambda_i \langle u, \phi_i \rangle \langle \phi_i, \phi_j \rangle \end{aligned}$$

But since $\{\phi_i\}_{i=1}^{n-1}$ are orthonormal we have

$$\sum_{i=1}^{n-1} \lambda_i \langle u, \phi_i \rangle \langle \phi_i, \phi_j \rangle = \lambda_j \langle u, \phi_j \rangle \quad \text{thus,}$$

$$\begin{aligned} \langle v, \phi_j \rangle &= \langle Tu, \phi_j \rangle - \lambda_j \langle u, \phi_j \rangle \\ &= \langle u, T\phi_j \rangle - \lambda_j \langle u, \phi_j \rangle \quad \text{Since } T \text{ is self adjoint.} \\ &= \langle u, T\phi_j \rangle - \langle u, \lambda_j \phi_j \rangle \quad \text{Since } \lambda_j \text{ are real by property 2.} \end{aligned}$$

$$\begin{aligned} \langle v, \phi_j \rangle &= \langle u, T\phi_j \rangle - \langle u, T\phi_j \rangle \quad \text{Since } T\phi_j = \lambda_j \phi_j \text{ for} , \ j = 1, \dots, n-1. \\ &= 0 \end{aligned}$$

Hence, the range of T_n is orthogonal to span $\{\phi_1 , \dots, \phi_{n-1}\}$.

iv. For $\lambda \neq 0$, let $T_n\varphi = \lambda\varphi$, then $\frac{T_n\varphi}{\lambda} = \varphi$ this implies $T_n\left(\frac{\varphi}{\lambda}\right) = \varphi$.

Hence if $T_n\varphi = \lambda\varphi$ then, $\varphi \in T_n(H)$. Moreover, by (iii) φ is orthogonal to span$\{\phi_1, \dots, \phi_{n-1}\}$. And by (i) $T_n\varphi = T\varphi = \lambda\varphi$ so that if (λ, φ) is an eigenpair of T_n , then it is also an eigenpair of T.

Now we can apply property (6) to the reduced operator T_n to find another eigenpair.

If T_n is non trivial, it has a non trivial eigenpair (λ_n, ϕ_n) which is also an eigenpair of T by (iv) above, where

$$|\lambda_n| = \max_{\|u\| \neq 0} \frac{|\langle T_n u, u\rangle|}{\|u\|^2} = \| T_n \| \text{ and the maximum is}$$

attained by ϕ_n , which is in $T(H)$.

Furthermore by (iv) ϕ_n is orthogonal to ϕ_i for $i = 1, \dots, n-1$ (the previous eigenfunctions), and of course it can be normalized.

Now for $\langle u, \phi_i\rangle = 0$, $i = 1, \dots, n-1$, then $T_n u = Tu$ by (i).

Hence $$|\lambda_n| = \max_{\substack{u \neq 0 \\ \langle u, \phi_i\rangle = 0 \\ i=1,\dots,n-1}} \frac{|\langle T_n u, u\rangle|}{\|u\|^2} = \max_{\substack{u \neq 0 \\ \langle u, \phi_i\rangle = 0 \\ i=1,\dots,n-1}} \frac{|\langle Tu, u\rangle|}{\|u\|^2}$$

Let $U_m = \{\phi_1, \dots, \phi_m\}^{\perp} = \{u : \langle u, \phi_i\rangle = 0 \ \ , i = 1, \dots, m\}$.

Hence $\ . \ . \ . \subseteq U_m \subseteq U_{m-1} \subseteq U_{m-2} \subseteq \ . \ . \ . \subseteq U_2 \subseteq U_1 \subseteq H$

Thus,

$$\max_{\substack{u \neq 0 \\ \langle u, \phi_i\rangle = 0 \\ i=1,\dots,n-1}} \frac{|\langle Tu, u\rangle|}{\|u\|^2} \leq \max_{\substack{u \neq 0 \\ \langle u, \phi_i\rangle = 0 \\ i=1,\dots,n-2}} \frac{|\langle Tu, u\rangle|}{\|u\|^2}$$

That is

$$|\lambda_n| = \max_{\substack{u \neq 0 \\ \langle u, \phi_i\rangle = 0 \\ i=1,\dots,n-1}} \frac{|\langle Tu, u\rangle|}{\|u\|^2} \leq \max_{\substack{u \neq 0 \\ \langle u, \phi_i\rangle = 0 \\ i=1,\dots,n-2}} \frac{|\langle Tu, u\rangle|}{\|u\|^2} = |\lambda_{n-1}|$$

Hence, we have the following generalization.

(**The maximum principle for self adjoint compact operators**) The n^{th} eigenpair of T can be characterized by,

$$|\lambda_n| = \max_{\substack{u \neq 0 \\ \langle u,\phi_i\rangle = 0 \\ i=1,\dots,n-1}} \frac{|\langle Tu,u\rangle|}{\|u\|^2} \leq |\lambda_{n-1}|$$

where the maximum is attained by ϕ_n and $\langle \phi_n, \phi_i \rangle = 0$ for $i = 1, \dots, n-1$.

Therefore, we have two possibilities to occur,

a) If $T_n = 0$, $Tu = \sum_{i=1}^{n-1} \lambda_i \langle u, \phi_i \rangle \, \phi_i$ for some n , then T is degenerate.

b) If $T_n = 0$, $Tu = T_n u + \sum_{i=1}^{n-1} \lambda_i \langle u, \phi_i \rangle \, \phi_i$, then T has an infinite orthonormal set of eigenfunctions $\{\phi_n\}$ with corresponding eigenvalues $\{\lambda_n\} \subseteq \mathbb{R}$,

$$. \; . \; . \leq |\lambda_n| \leq |\lambda_{n-1}| \leq \; . \; . \; . \leq |\lambda_3| \leq |\lambda_2| \leq |\lambda_1| \text{ such that } \lambda_n \to 0.$$

Let λ_n is an eigenvalue and ϕ_n be the corresponding normalized eigenfunctions under T. Then, $|\lambda_n| = |\lambda_n| \langle \phi_n , \phi_n \rangle = |\langle T\phi_n , \phi_n \rangle| \leq \|T\| = |\lambda_1|$

Hence either $\|T\|$ or $-\|T\|$ is the largest among all the eigenvalues of T.■

Theorem 4.2.1: If T is compact self adjoint linear operator, then the set of $\{\phi_i\}_{i=1}^{\infty}$ of its orthonormal eigenfunctions is complete over the range of T.

That is every function in the range of T can be expressed in the Fourier series in terms of the sets $\{\phi_i\}_{i=1}^{\infty}$.

Proof: Suppose $u_n = f - \sum_{i=1}^{n-1} \alpha_i \phi_i$ for all $f \in H$, where $\alpha_i = \langle f, \phi_i \rangle$, $i = 1, \dots, n-1$

But, for $j = 1, \dots, n-1$,

$$\langle u_n, \phi_i \rangle = \langle f - \textstyle\sum_{i=1}^{n-1} \alpha_i \phi_i , \phi_j \rangle = \langle f, \phi_j \rangle - \sum_{i=1}^{n-1} \alpha_i \langle \phi_i, \phi_j \rangle$$

$$= \langle f, \phi_j \rangle - \alpha_j = \langle f, \phi_j \rangle - \langle f, \phi_j \rangle = 0$$

That is $\langle u_n, \phi_i \rangle = 0$, for $j = 1, \dots, n-1$.

Now, using (i) of property (7) we have $\|Tu_n\| = \|T_n u_n\| \leq \|T_n\| \|u_n\| \leq |\lambda_n| \|f\|$

But as $n \to \infty$, $|\lambda_n| \to 0$, hence $\|Tu_n\| \to 0$ as $n \to \infty$ i.e. $\lim_{n\to\infty} \|Tu_n\| = 0$.

Now let $g \in T(H)$, hence there is $f \in H$ such that $Tf = g$. Thus,

$$\begin{aligned}
0 = \lim_{n\to\infty}\|Tu_n\| &= \lim_{n\to\infty}\left\|T\left(f-\textstyle\sum_{i=1}^{n-1}\langle f,\phi_i\rangle\phi_i\right)\right\| \\
&= \lim_{n\to\infty}\left\|Tf-\textstyle\sum_{i=1}^{n-1}\langle f,\phi_i\rangle T\phi_i\right\| \\
&= \lim_{n\to\infty}\left\|Tf-\textstyle\sum_{i=1}^{n-1}\langle f,\phi_i\rangle \lambda_i\phi_i\right\| \quad \text{Since } T\phi=\lambda_i\phi_i\ ,\ i=1,\dots,n-1 \\
&= \lim_{n\to\infty}\left\|Tf-\textstyle\sum_{i=1}^{n-1}\langle f,\lambda_i\phi_i\rangle \phi_i\right\| \quad \text{Since } \lambda_i\text{'s are real by property(2)} \\
&= \lim_{n\to\infty}\left\|Tf-\textstyle\sum_{i=1}^{n-1}\langle f,T\phi_i\rangle \phi_i\right\| \quad \text{Since } T\phi=\lambda_i\phi_i\ ,\ i=1,\dots,n-1 \\
&= \lim_{n\to\infty}\left\|Tf-\textstyle\sum_{i=1}^{n-1}\langle Tf,\phi_i\rangle \phi_i\right\| \quad \text{Since } T \text{ is self adjoint} \\
&= \lim_{n\to\infty}\left\|g-\textstyle\sum_{i=1}^{n-1}\langle g,\phi_i\rangle \phi_i\right\| \\
&= \left\|g-\lim_{n\to\infty}\textstyle\sum_{i=1}^{n-1}\langle g,\phi_i\rangle \phi_i\right\| \quad \text{By continuity of norm } \|.\| \\
&= \left\|g-\textstyle\sum_{i=1}^{\infty}\langle g,\phi_i\rangle \phi_i\right\| \quad \text{which implies } g-\textstyle\sum_{i=1}^{\infty}\langle g,\phi_i\rangle\phi_i=0
\end{aligned}$$

Hence, $g=\sum_{i=1}^{\infty}\langle g,\phi_i\rangle\phi_i$ for all $g\in T(H)$.

Hence, is g represented as Fourier series of the sets $\{\phi_i\}_{i=1}^{\infty}$. ■

Let $T\in B(L^2[a,b])$ is an integral operator given by $(Tu)(s)=\int_a^b k(s,t)u(t)dt$ with continuous kernel k $(k\in C([a,b],[a,b]))$. For $u,v\in L^2[a,b]$

$$\begin{aligned}
\langle Tv,u\rangle &= \int_a^b\left(\int_a^b k(t,s)v(s)ds\right)\overline{u(t)}dt \\
&= \int_a^b\int_a^b k(t,s)v(s)\,\overline{u(t)}dsdt \\
&= \int_a^b\int_a^b k(t,s)v(s)\overline{u(t)}\ dtds \qquad \text{by Fubini's} \\
&= \int_a^b v(s)\left(\int_a^b k(t,s)\overline{u(t)}\ dt\right)ds \\
&= \int_a^b v(s)\overline{\left(\int_a^b \overline{k(t,s)}\,u(t)dt\right)}ds \\
&= \langle v,T^{\star}u\rangle \quad \text{where} \quad (T^{\star}u)(s)=\int_a^b\overline{k(t,s)}\,u(t)dt\ .
\end{aligned}$$

Therefore, if $T\in B(L^2[a,b])$ is an integral operator given by

$$(Tu)(s)=\int_a^b k(s,t)u(t)dt$$

with continuous kernel k ($k \in C([a,b],[a,b])$), then $T^{\star}$ of T existes and is given by

$(T^{\star}u)(s) = \int_a^b \overline{k(t,s)}\, u(t)dt.$

Observe that T is self adjoint operator (that is $T = T^{\star}$) if $k(s,t) = \overline{k(t,s)}$. A kernel k of an integral operator satisfying $k(s,t) = \overline{k(t,s)}$ is called hermitian kernel.

Hence, if $T \in B(L^2[a,b])$ is an integral operator given by

$$(Tu)(s) = \int_a^b k(s,t)u(t)dt$$

with continuous kernel k ($k \in C([a,b],[a,b])$), then T is self adjoin operator if the kernel k of T is hermitian kernel.

***Example* 19**: Let $T: L^2[0,\pi] \to L^2[0,\pi]$ given by $(Tu)(s) = \int_0^{\pi} k(s,t)\, u(t)dt$ with continuous kernel

$$k(s,t) = \begin{cases} \frac{(\pi-s)t}{\pi} & , \ 0 \le t < s \le \pi \\ \frac{(\pi-t)s}{\pi} & , \ 0 \le s < t \le \pi \end{cases}$$

This integral equation $f(s) = \int_0^{\pi} k(s,t)\, u(t)dt$ with the given continuous kernel is equivalent to the ordinary differential equation $f''(s) = -u(s) \quad , \quad 0 \le s \le \pi$ with the homogenous boundary condition $f(0) = f(\pi) = 0$.

To each function $u \in C[0,\pi]$ there exist a unique solution $f \in C^2[0,\pi]$ of boundery value problem which is given by

$$f(s) = \int_0^{\pi} k(s,t)\, u(t)dt \quad , \quad 0 \le s \le \pi.$$

The compact integral operator T with this so called triangular kernel is self adjoint, since its real valued kernel is symmetric $k(s,t) = k(t,s)$ hence $k(s,t) = \overline{k(t,s)}$ I,e k is hermitian kernel.

Let $u, v \in L^2[0,\pi]$, then

$$\langle Tu, v\rangle = \int_0^{\pi}\left(\int_0^{\pi} k(s,t)\, u(t)dt\right) \overline{v(s)}ds$$

$$= \int_0^\pi \int_0^\pi k(s,t)\, u(t)\overline{v(s)}dtds$$

$$= \int_0^\pi \int_0^\pi k(s,t)\, u(t)\overline{v(s)}dsdt \qquad \text{by Fubini's}$$

$$= \int_0^\pi u(t)\big(\int_0^\pi k(s,t)\overline{v(s)}ds\big)dt$$

$$= \int_0^\pi u(t)\overline{\big(\int_0^\pi k(s,t)v(s)ds\big)}dt \qquad \text{since } k(s,t) \text{ is real valued}$$

$$= \int_0^\pi u(t)\overline{\big(\int_0^\pi k(t,s)v(s)ds\big)}dt \qquad \text{since } k(s,t) = k(t,s)$$

$$= \langle u, T^{\star} v\rangle \quad \text{where } (Tv)(s) = \int_0^\pi k(s,t)v(t)\, dt$$

Hence, T is self adjoint.

Now the eigenvalue and eigenfunction equation $Tu = \lambda u$ is equivalent to the differential equation $\lambda u'' + u = 0$, $u(0) = u(\pi) = 0$. The non trivial solution to the boundary value problem are given by

$$\lambda_n = \frac{1}{n^2} \quad \text{and } \phi_n(s) = \sqrt{\frac{2}{\pi}}\, sinnt \text{ , for all } n \in \mathbb{N}.$$

Here observe that $\{\phi_n\}_{n=1}^{\infty}$ is the set orthonormal eigenfunctions with corresponding eigenvalues $\{\lambda_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ with

$$. \ . \ . \leq |\lambda_n| \leq |\lambda_{n-1}| \leq \ . \ . \ . \leq |\lambda_3| \leq |\lambda_2| \leq |\lambda_1| \ .$$

Hence, the set of eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ of T is complete over the range of T.

Implying,

$$(Tu)(s) = \int_0^\pi k(s,t)\, u(t)dt = \sum_{n=1}^{\infty}\langle Tu, \phi_n\rangle\phi_n$$

$$= \sum_{n=1}^{\infty}\langle u, T\phi_n\rangle\phi_n \qquad \text{since } T \text{ is self adjoint}$$

$$= \sum_{n=1}^{\infty}\langle u, \lambda_n\phi_n\rangle\phi_n \qquad \text{since } T\phi_n = \lambda_n\phi_n$$

$$= \sum_{n=1}^{\infty}\lambda_n\langle u, \phi_n\rangle\phi_n \qquad \text{since } \lambda_n \text{ is real}$$

$$= \sum_{n=1}^{\infty}\sqrt{\frac{2}{\pi}}\frac{sinns}{n^2}\int_0^\pi u(t)\sqrt{\frac{2}{\pi}}\, sinnt\, dt$$

Hence,

$$(Tu)(s) = \int_0^\pi k(s,t)\, u(t)dt = \sum_{n=1}^{\infty}\langle Tu, \phi_n\rangle\phi_n = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{sinns}{n^2}\int_0^\pi u(t)\, sinnt\, dt \ .$$

***Example* 20** : For any 2π – periodic function $k(t)$ in $L^2[0,2\pi]$,

let $T: L^2[0,2\pi] \to L^2[0,2\pi]$ given by $Tu = \int_0^{2\pi} k(s-t)\, u(t) dt$.

Take $\phi_n(s) = e^{ins}$ thus, $(T\phi_n)(s) = \int_0^{2\pi} k(s-t)\, e^{int} dt$.

Let $x = s - t$, *thus* $dx = -dt$ and $t = 0 \Rightarrow x = s$, $t = 2\pi \Rightarrow x = s - 2\pi$.

$$\text{Hence,} \quad (T\phi_n)(s) = -\int_s^{s-2\pi} k(x)\, e^{in(s-x)} dx = \int_{s-2\pi}^{s} k(x)\, e^{in(s-x)} dx$$

$$= \int_{s-2\pi}^{s} k(x)\, e^{in(s-x)} dx$$

$$= e^{ins} \int_{s-2\pi}^{s} k(x)\, e^{-int} dt = \lambda_n \phi_n$$

But, $\int_{s-2\pi}^{s} k(x)\, e^{-int} dt = \int_0^{2\pi} k(x)\, e^{-int} dt$ Since $k(x)$ is periodic function.

Hence, the eigenvectors are $\lambda_n = \int_0^{2\pi} k(x)\, e^{-int} dt$, $n \in \mathbb{Z}$ and the corresponding eigenfunctions are $\phi_n(s) = e^{ins}$, $n \in \mathbb{Z}$.

That is the eigenpairs are $\left(\int_0^{2\pi} k(x)\, e^{-int} dt \, , e^{ins} \right)$, $n \in \mathbb{Z}$.

Hence $\left\{ e^{ins} \right\}_{n=-\infty}^{\infty}$ are complete eigenfunctions in the range of T.

Let $u, v \in L^2[0,2\pi]$, then

$$\langle Tu, v \rangle = \int_0^{2\pi} \left(\int_0^{2\pi} k(s-t)\, u(t) dt \right) \overline{v(s)} ds = \int_0^{2\pi} \int_0^{2\pi} k(s-t)\, u(t) \overline{v(s)} dt ds$$

$$= \int_0^{2\pi} \int_0^{2\pi} k(s-t)\, u(t) \overline{v(s)} ds dt \qquad \text{By Fubini's}$$

$$= \int_0^{2\pi} u(t) \overline{\left(\int_0^{2\pi} k(s-t) v(s)\, ds \right)} dt = \langle u, T^\star v \rangle$$

where, $(T^\star v)(s) = \int_0^{2\pi} k(t-s)\, v(t) dt$.

We already know that the eigenfunctions $\left\{ e^{ins} \right\}_{n=-\infty}^{\infty}$ are complete for functions in $L^2[0,2\pi]$. It is interesting to note, however, that T is self adjoint only if $k(x) = k(-x)$ for all x , although its eigenfunctions are complete even when T is not self adjoint.

4.3 Fredholm Alternative Theorem for Compact Linear Operator

Let H be a Hilbert space and $T: H \to H$ is compact linear operator, with $D(T) \subseteq H$.

For $\lambda \in K$, we define an operator, $I - \lambda T$ and $D(I - \lambda T) \subseteq D(T)$ where I is identity on H.

Now, let $f \in H$ and consider the equation $(I - \lambda T)u = f$.
The aim of this section is to determine the solvability of equation $(I - \lambda T)u = f$ for a compact operator $T: H \to H$ (H Hilbert space) with the help of Fredholm Alternative Theorem. The following Fredholm Alternative Theorem tells us when the equation $(I - \lambda T)u = f$ is solvable.

Theorem 4.3.1: If T is compact linear operator, then $(I - \lambda T)u = f$ has a solution if and only if $\langle f, g \rangle = 0$ for all $g \in N((I - \lambda T)^{\star})$.

That is T is if is compact linear operator, then $(I - \lambda T)u = f$ has a solution if and only if f is orthogonal to the null space of $(I - \lambda T)^{\star}$.

Proof:- Recall that in a Fredholm Alternative Theorem if a bounded linear operator T has a closed range then, $H = N(T^{\star}) \oplus T(H)$, or equivalently, the equation $Tu = f$ has a solution if and only if $\langle f, g \rangle = 0$ for all $g \in N(T^{\star})$.

Hence to prove the theorem we only need to show that $I - \lambda T$ has a closed range.

Suppose not! Let $I - \lambda T$ does not have a closed range. Let $L = I - \lambda T$.

Then there is $g \in \overline{L(H)}$, $g \notin L(H)$ and a sequence $\{f_n\}$ in H such that

$$y_n = Lf_n \to g \tag{25}$$

Since $L(H)$ is a vector space (a subspace of H), $0 \in L(H)$.

But $g \notin L(H)$, so that $g \neq 0$. This implies $y_n \neq 0$ and $f_n \notin N(H)$ for all sufficiently large n. Without loss of generality we may assume that this holds for all n.

Since L compact it is bounded linear operator thus $N(L)$ is closed by Lemma 2.2.1. So that the distance δ_n from f_n to $N(L)$ is positive, that is

$$\delta_n = inf_{h\in N(L)}\|f_n - h\| > 0 \qquad (26)$$

By definition of infimum there is a sequence $\{h_n\}$ in $N(L)$ such that

$$a_n = \|f_n - h_n\| < 2\delta_n \qquad (27)$$

Now let us show that

$$a_n = \|f_n - h_n\| \to \infty \text{ for } n \to \infty \qquad (28)$$

Suppose (28) does not hold. Then $\{f_n - h_n\}$ has a bounded subsequence. Since T is compact, it follows that $\{T(f_n - h_n)\}$ has a convergent subsequence.

Now from $L = I - \lambda T$ we have $I = L + \lambda T$.

Thus

$$\begin{aligned} f_n - h_n = (L + \lambda T)(f_n - h_n) &= L(f_n - h_n) + \lambda T(f_n - h_n) \\ &= Lf_n + \lambda T(f_n - h_n) \quad \text{Since } h_n \in N(L), Lh_n = 0. \end{aligned}$$

Now, since $\{T(f_n - h_n)\}$ has a convergent subsequence and $\{Lf_n\}$ converges by (25), hence $\{f_n - h_n\}$ has a subsequence $\{f_{n_k} - h_{n_k}\}$ converges to some v in H.

That is $f_{n_k} - h_{n_k} \to v$.

Since T is compact, T is continuous so is L hence by Theorem 1.5.1 we have,

$$L\left(f_{n_k} - h_{n_k}\right) \to Lv.$$

Here $Lh_{n_k} = 0$ because $h_n \in N(L)$, so that by (25)

$$L(f_{n_k} - h_{n_k}) = Lf_{n_k} \to g$$

hence $Lv = g$ implying that $g \in L(H)$ which contradicts $g \notin L(H)$. So that it must be the case that $a_n = \|f_n - h_n\| \to \infty$ for $n \to \infty$.

Now setting

$$w_n = \frac{f_n - h_n}{a_n} \qquad (29)$$

we have $\|w_n\| = 1$.

Since $a_n \to \infty$ whereas $Lh_n = 0$ and $\{Lf_n\}$ converges to g from (25), it follows that

$$Lw_n = \frac{Lf_n}{a_n} \to 0. \qquad (30)$$

Using again $I = L + \lambda T$, we obtain

$$w_n = L\,w_n + \lambda T\,w_n \tag{31}$$

Since T is compact and $\{w_n\}$ is bounded, $\{Tw_n\}$ has a convergent subsequence. Furthermore, $\{Lw_n\}$ converges by (30). Hence (31) shows that $\{w_n\}$ has a convergent subsequence $\left\{ w_{n_i} \right\}$, say

$$w_{n_i} \to w \tag{32}$$

Comparison with (30) implies that $Lw = 0$ hence $w \in N(L)$. Since $h_n \in N(L)$, also

$$u_n = h_n + a_n w \in N(L)$$

Hence for the distance form f_n to u_n we must have

$$\delta_n \leq \|x_n - u_n\|$$

Writing u_n out and using (29) and (28), we thus obtain

$$\delta_n \leq \|x_n - (h_n + a_n w)\| = \|x_n - h_n - a_n w\| = \|a_n w_n - a_n w\|$$

$$= a_n\|w_n - w\| < 2\delta_n\|w_n - w\|$$

Dividing by $2\delta_n > 0$, we have $\|w_n - w\| > \frac{1}{2}$ and this contradicts (32).

Therefore, it must be the case that $I - \lambda T$ has a closed range.

Hence the Theorem is proved.■

***Example* 21**: Let $T: L^2[0,1] \to L^2[0,1]$ is given by $(Tu)(s) = \int_0^1 tu(t)\,dt$.

We need to find the cases in which the integral equation $(I + \lambda T)u = f$ has a solution.

Now $\quad Tu = \int_0^1 tu(t)\,dt = \langle u, \phi_1\rangle \phi_2(s) \quad , \quad \phi_1(t) = t \; and \; \phi_2(s) = 1$

Hence it is finite dimensional.

$$\|Tu\|^2 = \int_0^1 \left|\int_0^1 tu(t)\,dt\right|^2 ds \leq \int_0^1 \left(\int_0^1 |tu(t)|\,dt\right)^2 ds$$

But $\int_0^1 |tu(t)|\,dt \leq \left(\int_0^1 |t|^2 dt\right)^{\frac{1}{2}} \left(\int_0^1 |u(t)|^2 dt\right)^{\frac{1}{2}} = \frac{1}{\sqrt{3}}\|u\|$ by Hölders inequality

Thus, $\|Tu\|^2 \leq \int_0^1 \left(\frac{1}{\sqrt{3}}\|u\|\right)^2 ds = \frac{\|u\|^2}{3}$ hence $\|Tu\| \leq \frac{1}{\sqrt{3}}\|u\|$ for all $u \in L^2[0,1]$

Thus, T is bounded finite dimensional hence T is compact by Theorem 3.2.1.

Therefore, $I - \lambda T$ has a closed range. Hence, we can use Fredholm Alternative Theorem to study and determine the solvability of the equation $(I + \lambda T)u = f$.

Thus what is done in example 12 is true (see example 12). That is

i. If $\lambda = -2$ then, null space of $(I + \lambda T)^{\star}$ is spanned by $v(s) = s$ that is $N((I + \lambda T)^{\star})$ =span$\{s\}$.

   Hence, if $\lambda = -2$,then $(I + \lambda T)u = f$ has a solution if and only if

   $$\int_0^1 f(t)\, t dt = 0.$$

ii. If $\lambda \neq -2$ then, null space of $(I + \lambda T)^{\star}$ is $\{0\}$ that is $N((I + \lambda T)^{\star}) = \{0\}$.

   Hence, $(I + \lambda T)u = f$ has a unique solution for each $f \in L^2[0,1]$.

***Example* 22**: Let $T: L^2[0,1] \to L^2[0,1]$ is given by $(Tu)(s) = \int_0^1 stu(t)\, dt$.

We need to find the cases in which the integral equation $(I - \lambda T)u = f$ has a solution.

Now Observe that since T is linear and

$$(Tu)(s) = s\int_0^1 tu(t)\, dt = \langle u, \phi_1\rangle \phi_2(s)$$

where $\phi_1(t) = t \; and \; \phi_2(s) = s$ hence, T it is finite dimensional.

$$\|Tu\|^2 = \int_0^1 \left|\int_0^1 stu(t)\, dt\right|^2 ds \leq \int_0^1 \left(s\int_0^1 |tu(t)|\, dt\right)^2 ds$$

But $\int_0^1 |tu(t)|\, dt \leq \left(\int_0^1 |t|^2\right)^{\frac{1}{2}} dt \left(\int_0^1 |u(t)|^2 dt\right)^{\frac{1}{2}}$ by Hölders inequality

$$\left(\int_0^1 |t|^2\right)^{\frac{1}{2}} dt \left(\int_0^1 |u(t)|^2 dt\right)^{\frac{1}{2}} = \frac{1}{\sqrt{3}}\|u\|$$

Thus,

$\|Tu\|^2 \leq \left(\frac{1}{\sqrt{3}}\|u\|\right)^2 \int_0^1 s^2 ds = \frac{\|u\|^2}{9}$ hence $\|Tu\| \leq \frac{1}{3}\|u\|$ for all $u \in L^2[0,1]$

Thus, T is bounded linear and finite dimensional operator hence T is compact by Theorem 3.2.1.

OR the kernel k of T is $k(s,t) = st$ and

$$\int_0^1\int_0^1 |k(s,t)|^2\,dsdt = \int_0^1\int_0^1 |st|^2\,dsdt = \int_0^1 s^2\,ds\int_0^1 t^2\,dt = \frac{1}{9} < \infty$$

Hence , T is Hilbert Schmidt operator thus T is compact operator.

Now let us first find the adjoint operator of the integral equation $I - \lambda T$.

For $u, v \in L^2[0,1]$

$$\langle (I - \lambda T)u, v\rangle = \int_0^1 \left[u(s) - \lambda \int_0^1 stu(t)\,dt\right]\overline{v(s)}\,ds$$

$$= \int_0^1 u(s)\overline{v(s)}\,ds - \int_0^1\int_0^1 \lambda stu(t)\overline{v(s)}\,dtds$$

$$= \int_0^1 u(s)\overline{v(s)}\,ds - \int_0^1\int_0^1 \lambda stu(t)\,\overline{v(s)}\,dsdt$$ By Fubini's

$$= \int_0^1 u(s)\overline{v(s)}\,ds - \int_0^1 \lambda tu(t)\int_0^1 s\,\overline{v(s)}\,dsdt$$

$$= \int_0^1 u(s)\overline{v(s)}\,ds - \int_0^1 u(t)\left(\lambda t\int_0^1 s\,\overline{v(s)}\,ds\right)dt$$

$$= \int_0^1 u(s)\overline{v(s)}\,ds - \int_0^1 u(t)\overline{\left(\lambda t\int_0^1 s\,v(s)\,ds\right)}\,dt$$

$= \langle u, v\rangle - \langle u, \lambda T^\star v\rangle$ where $(T^\star v)(s) = s\int_0^1 t\,v(t)\,dt$

$= \langle u, v - \lambda T^\star v\rangle$ where $(v - \lambda T^\star v)(s) = v(s) - \lambda s\int_0^1 tv(t)\,dt$

$$= \langle u, \ (I - \lambda T^\star)v\rangle = \langle u, (I - \lambda T)^\star v\rangle$$

Thus, the adjoint operator of $I - \lambda T$ is $(I - \lambda T)^\star v = v(s) - \lambda s\int_0^1 tv(t)\,dt$

Hence $I - \lambda T$ is self adjoint operator.

Now we need to find the null space of the adjoint operator $(I - \lambda T)^\star$.

Thus, $v(s) - \lambda s\int_0^1 tv(t)\,dt = 0$

$\Rightarrow \ v(s) = \lambda s\int_0^1 tv(t)\,dt$

$\Rightarrow \ v(s) = \lambda s\alpha$ where $\alpha = \int_0^1 tv(t)\,dt$

$\Rightarrow \ sv(s) = \lambda s^2\alpha$ multiplying both side by s

$\Rightarrow \ \int_0^1 sv(s)ds = \int_0^1 \lambda s^2\alpha ds = \lambda\alpha\int_0^1 s^2 ds$

$\Rightarrow \ \alpha = \frac{\lambda\alpha}{3}$

$$\Rightarrow \quad \left(1-\frac{\lambda}{3}\right)\alpha = 0$$

Hence,

i. If $\lambda = 3$ then, null space of $(I-\lambda T)^{\star}$ is spanned by $v(s) = s$ that is $N((I-\lambda T)^{\star})$ =span$\{s\}$.

   Hence, if $\lambda = 3$,then $(I-\lambda T)u = f$ has a (non unique) solution if and only if $\int_0^1 f(t)\, t dt = 0$.

ii. If $\lambda \neq 3$ then, null space of $(I-\lambda T)^{\star}$ is $\{0\}$ that is $N((I-\lambda T)^{\star}) = \{0\}$. Hence $(I-\lambda T)u = f$ has unique solution for each $f \in L^2[0,1]$.

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