Realistic Mathematics Education-based Mathematization, Epistemological Beliefs, Problem-Solving Skills, and Geometry Achievement of Pre-Service Mathematics Teachers

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ABSTRACT

Realistic mathematics education-based mathematization, epistemological beliefs, problem-solving skills, and geometry achievement of pre-service mathematics teachers

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This study explored the mathematization and mathematical concepts utilized in the community, designed an intervention based on these realitic-based mathematizations, and examined their influence on pre-service mathematics teachers' epistemological beliefs, problem-solving abilities, and achievements in learning geometry. To achieve this intended objective, an exploratory sequential mixed research design was employed, where the first phase explored the mathematization and mathematical concepts in the community, and the second phase underwent an intervention study in teacher education. Data were collected through semi-structured interviews, observation, a questionnaire, geometry problems, and geometry achievement tests. In phase I, purposive sampling and snowball sampling were utilized, whereas in phase II, purposive and random sampling were utilized. After collecting data, thematic analysis was employed for the qualitative data, and independent t-test, MANOVA, and Pearson correlations were employed for the quantitative data. Measuring, counting, designing, explaining, and locating were the common mathematizing processes; area of plane geometry, congruency and similarity of polygons, rotational and radial symmetry, sector, arc length, and volume of solid figures were some of the mathematical concepts embedded in the activities of the local community. Statistically significant mean differences were observed on pre-service mathematics teachers' problem-solving skills, geometry achievement, and mathematical epistemological beliefs, indicating that the group taught via RME-based mathematization outperformed those taught via the conventional approach. The MANOVA result also indicated that there were significant mean differences between the treatment groups and comparison group on a linear combination of source, stability, and structure of mathematical knowledge. Qualitative findings revealed that the instruction helped pre-service teachers to adopt both absolutist and fallibilist perspectives on mathematical knowledge. In conclusion, societal and cultural activities are rich in mathematical notions and forms of mathematization. Integrating them into RME-based approaches can benefit pre-service teachers by transforming their mathematical epistemological ideas and enhancing their geometry performance and problem-solving skills. The findings of this study underscore the importance of integrating RME-based mathematization that is rooted in culturally situated practices into teacher education programs to enhance pre-service teachers' problem-solving skills, geometry achievement, and epistemological beliefs. Since emphasis is given to localization and vocational competence in education, among other things, integrating local, societal, and cultural mathematizations into teacher education programs could benefit preservice teachers and the education system. This study also has an implication for the way teacher education curriculum should be designed, and how pre-service teachers should be trained to be competent to deliver locally relevant but meaningful mathematics, while also laying the foundation for further research.

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LIST OF ACRONYMS AND ABBREVIATIONS

CG: Comparison group

CTE- College of Teacher Education

DV: Dependent variable

EG: intervention group

GA: Geometry Achievement

GAT: Geometry Achievement Test

HM- Horizontal Mathematization

IV: Independent variable

MEB: Mathematical Epistemological Beliefs

MEBQ: Mathematical Epistemological Beliefs Questionnaire

MOE- Ministry of Education

NEAEA- National Educational Assessment and Examinations Agency

PSS- Problem-Solving Skills

RGP-Realistic geometry problems

RME: Realistic Mathematics Education

SoMK-Source of mathematical knowledge

StaMK-Stability of mathematical knowledge

StrMK-Structure of mathematical knowledge

VM-Vertical Mathematization

CHAPTER 1: INTRODUCTION

1.1 Background of the Study

The evolution of mathematics has a profound connection with human history, with its various branches—including arithmetic, algebra, geometry, calculus, analysis, and set theory—serving as essential tools for interpreting and transforming the world. From the earliest uses of arithmetic for counting and trade to the development of algebra for solving abstract problems and geometry useful for practical and artistic applications, mathematics has long supported the growth of organized societies (Masanja, 2002). The introduction of advanced fields such as calculus, analysis, and set theory further enabled significant developments in science, technology, philosophy, and logic (Youvan, 2024). These mathematical disciplines have not only reflected human progress but also actively driven it throughout history.

Geometry is one of the learning areas that occupies a vital place in human activity and mathematics education. It allows people to describe, comprehend, and interact with the space they inhabit. Geometry is visible in countless aspects of daily life, from constructing homes and making pottery to sewing, weaving, plowing, and crafting, to modern advancements in road, building, machinery, and computer technology. In the educational context, geometry enhances learners' reasoning, deductive proof skills, logical argumentation, conjecturing, and problem-solving abilities (Dimla, 2018). It fosters critical thinking and strengthens students' capacity to connect abstract mathematical concepts to real-life experiences (Zutaah et al., 2023). Moreover, geometry enhances spatial awareness, visualization and serves as a unifying component in mathematics, providing a strong visual foundation for algebraic, statistical, and arithmetic reasoning (Choudhury & Das, 2012). Its historical relevance and direct link to everyday practices make geometry a crucial subject for meaningful and contextualized mathematics education (Jablonski & Ludwig, 2023).

Despite its significance, the nature of mathematics itself has long been subject to philosophical debates, primarily between absolutist and fallibilist schools of thought. From the absolutist perspective, mathematics is considered value-free, acultural, and absolute—existing independently of human interest or cultural contexts (Aikenhead, 2017; Shoaib & Akhter, 2020). In contrast, fallibilists view mathematics as corrigible, socially constructed, value-laden, and open to revision. Aligning with this latter perspective, the concept of mathematization emerged as a framework to explain how societies model, understand, and address social, environmental, and technological challenges using mathematical reasoning (Tegmark, 2014).

In mathematics education, mathematization refers to the process of representing real-world or mathematical problems in mathematical terms, facilitating both understanding and problem-solving. This concept was developed and popularized by Freudenthal in the early 1960s through his theory of Realistic Mathematics Education (RME) (Jupri & Drijvers, 2016). According to Freudenthal, mathematization characterizes mathematical activity as a continuous process involving problem-solving, exploring structures, and organizing content, whether originating from reality or within mathematics itself. It encourages learners to develop mathematical concepts while dealing with real-life challenges, and promotes the guided reinvention of mathematical ideas.

Mathematization is typically categorized into two interrelated forms: horizontal and vertical mathematization (Van den Heuvel-Panhuizen, 2003). Horizontal mathematization involves translating problems from real-world contexts into mathematical terms using observation, experimentation, and inductive reasoning (Treffers, 1987; Barnes, 2005). This allows learners to apply mathematical concepts to concrete problems. Vertical mathematization, on the other hand, involves refining, formalizing, generalizing, and abstracting mathematical ideas within mathematics itself (Treffers, 1987; van den Heuvel-Panhuizen, 2010). This process enables learners to deepen their understanding by restructuring mathematical models, integrating different models, and moving towards formalized reasoning and abstraction (De Lange, 1987; Rasmussen et al., 2009; Upu et al., 2017). The horizontal and vertical mathematization are two interrelated aspects. The horizontal mathematization enables vertical mathematization because, from real-life situations, informal models are developed, and then these models are refined, generalized, and formalized during vertical mathematization.

Various theoretical perspectives have guided the application of horizontal and vertical mathematization in mathematics education. Treffers (1987) identified four primary perspectives: mechanistic, empiricist, structuralist, and realistic. The mechanistic approach rejects any reference to real-life contexts and focuses on rote memorization; it possesses neither horizontal nor vertical mathematization. In empiricist approaches, the emphasis is on experientially real contexts with little concern for structured cognitive processes, which can lead to learning being confined to the horizontal mathematization level without much development towards vertical level. The structuralist focuses primarily on mathematical structures rather than students' real-life experiences, thus presenting predominantly vertical mathematization without considering horizontal mathematization. In comparison with other approaches, it is significant to note that the realistic perspective possesses both horizontal and vertical characteristics through the reinvention of students' learning of mathematical concepts with reference to real-life phenomena and learning processes, thus gaining significant recognition from an educational perspective (Yuanita et al., 2018).

Several empirical studies have demonstrated that a realistic mathematics education approach enhances teaching and learning, improves problem-solving skills, fosters positive mathematical beliefs, and strengthens students’ scientific reasoning and academic performance (Fauzan & Syarif, 2019; Syarif et al., 2019; Yilmaz, 2020; Özkaya & Yetim, 2017; Şahin et al., 2019; Aksu & Colak, 2021; Laurens et al., 2017). However, some studies have reported no significant effect on students' mathematical achievement (Cengiz & Eğmir, 2022; Hough et al., 2016), indicating the need for continued investigation. Additionally, research on students’ mathematical epistemological beliefs and effective strategies to influence them remains limited (Depaepe et al., 2016; Pantziara et al., 2020).

While RME-based mathematization approaches are widely practiced in mathematics education systems in countries such as the Netherlands, Indonesia, Turkey, South Africa, and the United States, localized adaptations are essential for optimizing their benefits. In Ethiopia, mathematics education has long been criticized for its heavy reliance on foreign-imported curricula and pedagogical approaches, detached from the country's socio-cultural, political, and economic realities (Shishigu, 2015; Tesfamicael et al., 2021; Weldeana, 2016). Consequently, Ethiopian students often struggle to relate academic mathematics to their daily lives and fail to model real-world problems mathematically (Tesfamicael et al., 2021). This disconnect contributes to low mathematics performance, particularly in geometry (Dereje et al., 2023; Yeshanew et al., 2024).

Furthermore, the prevailing teaching methods, often characterized by mechanical, teacher-centered instruction, reinforce absolutist epistemological beliefs among teachers and inhibit students' ability to mathematize both horizontally and vertically (Tegegne, 2015; Arsaythamby & Zubainur, 2014; Van Zanten & Van den Heuvel-Panhuizen, 2021). This has negatively affected both the quality and relevance of mathematics education in Ethiopia (MOE, 2022). Valuable mathematical knowledge embedded in Ethiopia's indigenous practices—including weaving, pottery, stitching, threshing, religious rituals, and traditional games such as 'Gebeta'—remains largely unexplored in formal mathematics education (Tegegne, 2015; Tesfamicael, 2021; Weldeana, 2016). Incorporating these culturally-rooted mathematical experiences into classroom teaching holds significant potential for bridging the gap between formal mathematics and learners' lived experiences (Eleni, 1992; Tesfamicael et al., 2021).

However, there has been limited empirical research examining these practices and their integration into formal education. It is therefore critically important to uncover local mathematization experiences and explore their potential in enhancing mathematics teacher education in Ethiopia. Such efforts could significantly improve pre-service teachers' epistemological beliefs, geometry achievement, and problem-solving skills, contributing to more relevant, culturally grounded, and effective mathematics education. This study was thus conceived to address these gaps and offer new insights for improving mathematics education in Ethiopia and internationally.

1.2. Context of the Study

Ethiopia, covering approximately 1.24 million square kilometers, is a landlocked country situated in the Horn of Africa between 4° and 14° north latitude and 33° and 48° east longitude. The country geographically houses the Great Rift Valley, with Addis Ababa serving as the capital city and home to numerous diplomatic offices. According to recent estimates, the total population is about 123.4 million, of which more than 80% reside in rural areas with diverse sociocultural backgrounds. Languages spoken in the country are more than 80, while the major languages include include Afan Oromo, Amharic, Tigrigna, Af Somali, Sidama Afoo, and Afar Aff. Some of these languages use the Latin alphabet, while Semitic languages, particularly Amharic, Tigrigna, and Harari continue to employ the indigenous Geez script. These have a direct association with cultural values and mathematization.

Ethiopia's historical significance is globally acknowledged, being considered one of the world's earliest human civilization. Renowned archeological discoveries such as Lucy (Australopithecus afarensis, 3.2 million years old) unearthed in 1974, and Ardi (4.4 million years old) discovered in 1992, have solidified the country's position as a cradle of humanity. Ethiopia also boasts a rich cultural heritage, as evidenced by ancient sites like the rock-hewn churches of Lalibela, dating from the 12th to 13th centuries, and the ruins of Aksum, which houses tombs, obelisks, fortifications, and the historically significant Church of Our Lady Mary of Zion some of which date back to the time before Christ. Ethiopia's cultural assets include other UNESCO World Heritage Sites such as Fasil Ghebbi, Harar Jugol, the Konso Cultural Landscape, the Lower Valleys of the Awash and Omo, Tiya Megaliths, Melka Kunture and Balchit, Simien National Park, and Bale National Park, and many others. Additionally, Ethiopia has a distinctive thirteen-month calendar, with the final month containing five days, or six in a leap year, and operates a twelve-hour clock system counted from dawn to dusk and from dusk to dawn. The Grand Ethiopian Renaissance Dam (GERD), a modern engineering feat, is also a recent symbol of national unity and progress. The mathematical arithmetic related to the rotation of the Sun, which led to the development of "Bahre-Hasab" literally meaning "Ocean of ideas" is another milestone of the Ethiopians.

From an educational perspective, modern schooling in Ethiopia publicly began in 1908. Before 1944, the country lacked formal teacher education programs, and foreign nationals were hired to serve as teachers and school administrators (Eyasu et al., 2017). Formal teacher training commenced in the mid-1940s with the establishment of a primary school teacher-training program at Menelik II School in Addis Ababa (Hussein, 2007; Semela, 2014). Since then, Ethiopia's teacher education system has undergone reforms aimed at improving the quality and reach of education. However, scholars have argued that these reforms often resulted in poor content delivery, lack of coordination, competency mismatches, and ineffective integration of local culture and realities into education structures (Tirusew, 2006). Kebede (2022) further noted that recurrent changes in curriculum design, teacher training models, institutional structures, and governance systems have continued to pose challenges to education quality in the country.

Even though there are significant reforms that have characterized teacher education in Ethiopia in recent years, the meaningful inclusion of indigenous knowledge, culture, and learning experiences in mathematics teaching-learning remains limited. Currently, the teaching of mathematics, particularly geometry, remains highly inclined towards abstract teaching approaches, which are highly disconnected from the sociocultural learning realities of the learners. Shishigu (2015), Weldeana (2016), Tesfamicael et al. (2021), among other researchers, have consistently condemned the reform strategies in teaching-learning in the mathematics curriculum in Ethiopia for its over-reliance on imported teaching-learning materials. Even though recent national reforms in education in Ethiopia emphasize the inclusion of culture and indigenous knowledge in teaching-learning in the curriculum and in teacher education, this remains far from being implemented in most teaching-learning in the math teaching-learning course.

The RME approach has been globally and widely promoted as instruction that connects formal mathematics to the real-life and cultural experiences of learners. Empirical studies conducted in primary and secondary school settings from different countries-the Netherlands, Indonesia, Turkey, South Africa, England, and the United States-report positive effects of RME on students' conceptual understanding, problem-solving capabilities, and mathematical beliefs. However, existing evidence is not yet conclusive, since several studies report inconclusive or insignificant effects of RME on mathematics achievement. Such inconsistency suggests that the effectiveness of RME might be context-dependent and modulated by several factors, including educational level, content domain, and teacher beliefs.

More critically, there is a notable scarcity of research examining the application of RME in teacher education, particularly in relation to how RME-based mathematization can support the integration of cultural mathematics and transform pre-service teachers' mathematical epistemological beliefs. In the Ethiopian context, empirical studies investigating the use of RME in pre-service mathematics teacher education—especially in geometry—are extremely limited. Little is known about how culturally grounded RME approaches influence pre-service teachers' epistemological beliefs, problem-solving skills, and geometry achievement. Consequently, a clear research gap exists at the intersection of Realistic Mathematics Education, cultural mathematics integration, and pre-service teacher education, both globally and, more acutely, within Ethiopia.

1.3. Statement of the Problem

Recent national assessments and educational research have consistently revealed that Ethiopian students' performance in mathematics remains below expected standards at all grade levels. Results from a series of National Learning Assessments (NEAEA 2000, 2004, 2008, 2010, 2012, 2013, 2014, 2016, 2017, 2020) showed average achievement scores in mathematics were often well below 50%, with particularly poor outcomes in geometry (Atnafu, 2010). Tefera et al. (2021) highlighted that pre-service teachers also struggled with mathematical problem-solving and mathematization, often failing to apply mathematical reasoning to real-world situations.

The primary reason for this persistent underperformance is the longstanding detachment of Ethiopia's mathematics education from its cultural and social realities, and its perpetuation of value-free, absolutist conceptions of mathematics (Tegegne, 2015; Tesfamicael et al., 2021). Critics have argued that the dominant approach in Ethiopian mathematics classrooms remains heavily didactic and authoritarian, prioritizing rote memorization of formulas and procedures while discouraging explorative or contextual learning (Ahmed et al., 2020; Bekele, 2016; Dejene et al., 2018; Tuge & Garoma, 2016). Consequently, mathematics education in Ethiopia often neglects the daily experiences and cultural practices of learners, limiting their ability to engage in horizontal and vertical mathematization (Arsaythamby & Zubainur, 2014; Van Zanten & Van den Heuvel-Panhuizen, 2021) and leading to poor geometry performance and limited problem-solving skills (Dereje et al., 2023; Yeshanew et al., 2024).

Although multiple studies have documented the existence of absolutist epistemological beliefs among mathematics teachers in Ethiopia (Tegegne, 2015; Tuge & Garoma, 2016) and their adverse effects on students' mathematical beliefs and achievement, most of these studies have primarily surveyed existing conditions without proposing actionable interventions. For instance, Nigus (2012) found that pre-service mathematics teachers' beliefs about the nature of mathematics often contradicted contemporary educational philosophies. Internationally, research has shown that learners' epistemological beliefs influence their problem-solving ability, geometry achievement, and overall mathematical thinking (Özgen et al., 2019; Suliani et al., 2024; Schubert et al., 2023; Schommer-Aikins et al., 2000; Schommer-Aikins et al., 2005; Septriyana et al., 2019; Yuanita et al., 2018). Yet Shoaib et al. (2021) observed that preservice teachers' epistemological beliefs were only weakly associated with geometry achievement, suggesting a complex relationship requiring further investigation.

Although there has been considerable international research on Realistic Mathematics Education (RME) and the inclusion of culturally relevant mathematization activities in mathematics education, there is a lack of empirical studies that have implemented RME principles in the Ethiopian context. Ethiopia has a strong indigenous mathematical knowledge base in the form of Ethiopian Orthodox Church practices, indigenous games like Gebeta, and daily activities like weaving, pottery, and agricultural measurement systems (Tegegne, 2015; Tesfamicael et al., 2021). There have also been studies that revealed advanced mathematical thinking in traditional practices, such as the computation of extreme values in the process of crop threshing (Weldeana, 2016). Nevertheless, despite the potential, these mathematically rich cultural practices have not yet been included in formal mathematics education through the lens of RME principles, nor have their impacts on students' mathematics learning been investigated in the Ethiopian context (MOE, 2018; Tesfamicael, 2021).

In light of these circumstances, this study was conceived to explore mathematical concepts embedded within Ethiopian cultural practices, and to prepare and implement a Realistic Mathematics Education (RME)-based mathematization intervention within teacher education programs. The intervention aims to assess its effects on pre-service teachers' epistemological beliefs, problem-solving skills, and geometry achievement. In doing so, this research seeks to contribute to a more culturally grounded, contextually relevant, and effective mathematics education in Ethiopia.

1.4. Objectives of the Study

This study aimed to explore how mathematization is practiced in the community and how it can be applied effectively in mathematics teacher education. It further sought to examine the impact of this approach on pre-service teachers' mathematical epistemological beliefs, problem-solving skills, and geometry achievement. The specific objectives were:

1. To explore and understand the m concepts and mathematization behind cultural activities like weaving, basketry, pottery and house building can be applied in mathematics teacher education.
2. To examine differences in problem-solving skills, geometry achievement, and epistemological beliefs between pre-service teachers taught via an RME-based mathematization approach and those taught through conventional approach.
3. To examine the effect of RME-based mathematization approach on the linear combination of source, stability and structure of mathematical knowledge between pre-service teachers taught using an RME-based mathematization approach and those taught using conventional approach.
4. To explore how pre-service teachers mathematize when solving real-life geometry problems.
5. To assess the contribution of geometric problem-solving skills and mathematical epistemological beliefs to pre-service teachers' geometry achievement.
6. To explore the challenges and shortcomings that prohibit pre-service mathematics teachers from engaging successfully in mathematization within the context of RME-based mathematization instruction.

1.5. Basic Research Questions

The study sought to answer the following research questions:

1. What mathematization and mathematical concepts are embedded in weaving, pottery, basketry and house building that can be applied in mathematics teacher education?
2. Is there a significant difference in problem-solving skills, geometry achievement, and mathematical epistemological beliefs between groups taught by RME-based mathematization and conventional approaches?

3. Are there significant differences in the mean scores on the combined dimensions (source, stability, and structure) of epistemological beliefs between pre-service teachers taught using an RME-based mathematization approach and those taught using conventional instruction?
4. How do pre-service teachers mathematize while solving real-life-based mathematics problems?
5. Do problem-solving skills and epistemological beliefs significantly contribute to pre-service mathematics teachers' geometry achievement?
6. What challenges and shortcomings prohibit pre-service mathematics teachers from engaging successfully in mathematization within the context of RME-based mathematization instruction?

1.6. Significance of the Study

The curriculum framework for teacher education promotes active learning and the integration of indigenous knowledge into appropriate teaching-learning activities (MoE, 2023). In this context, a realistic pedagogical approach is essential for making geometry learning meaningful and enhancing achievement among pre-service teachers (Laurens et al., 2017). The study also contributes to theory by making connections between Alan J. Bishop's six universal mathematical activities and the mathematization principles of Realistic Mathematics Education (RME) to create a culturally-grounded framework for mathematics learning. The study illustrates how cultural mathematical practices may be used to create meaningful contexts for mathematization and for enhancing pre-service teachers' mathematical epistemological beliefs, problem-solving abilities, and geometry achievement. In so doing, it extends the theoretical connection between cultural mathematics and RME in mathematics teacher education.

This study offers valuable insights for curriculum developers, teacher educators, and pre-service teachers on addressing mathematical epistemological beliefs, problem-solving skills, and geometry achievement in Ethiopia and beyond. It proposes implementing RME-based mathematization to improve the effectiveness of geometry teaching in colleges and primary schools, providing practical teaching activities applicable to both geometry and general mathematics classrooms.

Moreover, the study documents mathematical knowledge embedded in cultural practices, offering teachers, curriculum designers, and policymakers access to local mathematical traditions that can enrich mathematics education. The findings may benefit various stakeholders—including curriculum developers, policymakers, textbook authors, and researchers—and pave the way for new pedagogical courses and themes in Ethiopian teacher preparation programs focused on RME-based instruction. By addressing gaps in transforming pre-service teachers' mathematical epistemological beliefs, this research contributes both theoretically and practically. Furthermore, theoretically, this work broadens RME by viewing mathematization as a culturally situated process founded in social activities. It may also offer insights to mathematics educators and primary school teachers on student experiences with realistic-based mathematization, expanding knowledge for future research and fostering new initiatives in the field. Although policymakers sometimes overlook research findings, disseminating these results on global platforms could maximize their political impact.

1.7. Delimitation of the Study

Experiencing mathematics by mathematizing everyday activities enables pre-service teachers to build meaningful knowledge. When teachers see the mathematics in daily life and surroundings, they develop concrete connections to how mathematics functions in the real world. This study focuses on exploring mathematization and mathematical concepts behind cultural activities such as pottery, weaving, basketry, and house construction, and implementing an RME-based mathematization intervention in teacher education to assess its effects on pre-service teachers' epistemological beliefs, problem-solving skills and their geometry achievement.

The research centers on the Euclidean Plane Geometry (Math-111) course, a major subject offered to second-year pre-service mathematics teachers at Colleges of Teacher Education (CTEs) in Oromia Regional State. Topics include points, lines, planes, triangles, quadrilaterals, circles, and related properties, aligned with the Oromia Education Bureau's curriculum. Since pre-service teachers are future educators, this study emphasizes their preparation and evaluates the role of RME-based mathematization in transforming their epistemological beliefs, problem-solving skills, and geometry achievement.

CTEs were purposively selected based on the presence of mathematics departments and their establishment timing, with colleges randomly assigned to experimental (RME-based mathematization) and comparison groups. For confidentiality, the colleges are referred to as CTE1 and CTE2. Consequently, CTE1 assigned for intervention group, CTE2 is for comparison group. The study limits its scope to mathematical epistemological beliefs, problem-solving skills in geometry, and geometry achievement, excluding other variables. Data collection methods include semi-structured interviews, open-ended mathematics questions, classroom observations, questionnaires, and achievement tests.

1.8.Organization of the Dissertation

This dissertation is organized into six chapters. Chapter one presents the background of the study, the statement of the problem, and the context of the study. Chapter two reviews related literature on the purpose and philosophy of mathematics education, approaches to mathematics education, realistic mathematics education (RME), mathematization, ethnomathematics, epistemological beliefs, and studies on the teaching and learning of geometry. In this chapter, the concepts of the RME-based mathematization approach, its empirical findings, and the identified research gaps are discussed. In addition, the theoretical and conceptual frameworks of the study are presented at the end of the chapter.

Chapter Three describes the research paradigm, research design, and data collection methods and procedures, including the research instruments, sample selection, and data analysis procedures. Chapter Four presents the analysis of Phase I data based on Bishop's mathematizing processes, while chapter five focuses on the analysis of Phase II, which examines the effectiveness of the RME-based mathematization approach. Finally, chapter six discusses the findings from both phases, presents the meta-inferences, and provides conclusions and recommendations for practice and future research.

1.9. Operational Definitions

Below are key operational definitions used in this study to clarify essential terms and concepts.

Conventional Instruction: This refers to the traditional way mathematics is taught—typically teacher-centered, focused heavily on textbook content, memorization, and procedural learning, often emphasizing teacher-led instruction. In this study, it serves as the comparison or control group against a class taught with RME.

Ethnomathematics: Ethnomathematics captures the mathematical knowledge naturally embedded in cultural activities such as pottery, weaving, sewing, and other crafts practiced in the Oromia region. It highlights how local traditions and daily life involve rich mathematical thinking that is often overlooked in formal education.

Problem-Solving Skills: This refers to a pre-service teacher's ability to analyze a mathematical problem, select effective strategies, and carry through to a complete, logical solution. It reflects both creativity and critical thinking in tackling real or abstract mathematical challenges.

Geometry Achievement: Geometry achievement refers to the performance of pre-service teachers on tests designed to measure their understanding and mastery of geometric concepts, including shapes, properties, theorems, and problem-solving related to Euclidean geometry.

Mathematical Epistemological Beliefs: These are beliefs about the nature of mathematics and how mathematical knowledge is formed and validated. This includes views on whether math is fixed or evolving (stability), how math concepts are structured and connected (structure), and the sources from which math knowledge originates—whether from authorities, discovery, or human invention (source).

Mathematization: Mathematization is the process of transforming real-world situations, problems, or observations into mathematical forms—such as translating everyday experiences into numbers, shapes, symbols, or equations. It is how learners "make math" from life.

Horizontal Mathematization: This is the initial stage where students connect a real-world problem to mathematics by identifying relevant concepts and representing the problem through informal means such as sketches, verbal descriptions, visualizations, and pattern recognition. It involves describing relationships in everyday language before formalizing them.

Vertical Mathematization: At this stage, students deepen their mathematical understanding by transforming informal ideas into formal mathematical language—using symbols, formulas, or models—to generalize, abstract, refine, and extend the problem. This includes creating new definitions, formulating generalizations, and connecting ideas at a higher conceptual level.

Realistic Mathematics Education (RME): RME is a teaching approach that roots mathematics learning in real-life contexts meaningful to students. It views mathematics as a human activity closely linked to learners' experiences, encouraging them to reinvent mathematical ideas through problem-solving grounded in reality.

RME-based Mathematization: This instructional approach actively engages students in transforming real-life situations into mathematical problems and encourages them to solve these problems using appropriate tools and reasoning. It involves the use of algebra, symbols, diagrams, or other mathematical tools to express and analyze everyday phenomena mathematically.

CHAPTER 2: LITERATURE REVIE

2.1. Introduction

This section will go over the literature related to mathematics teaching and learning. Theoretical framework are also discussed in this chapter. The purpose and philosophy of mathematics education, as well as approaches in mathematics education, realistic mathematics education, mathematization, an overview and mathematical epistemological ideas are all explained. This chapter also presents empirical studies on mathematical epistemological beliefs and mathematical problem-solving skills and geometry achievement, and RME-based Mathematization in an Ethiopian Context. Finally, conclusion with a summary and statement of the conceptual framework developed based on the theoretical foundations is presented.

2.2.Mathematics Education

Mathematics education lies at the intersection of two vital fields of inquiry: mathematics and education. It goes beyond the traditional focus of mathematics, which primarily deals with abstract concepts, theories, and problem-solving methods. Mathematics education, however, integrates the **"what"** (subject matter) with the **"how"** (pedagogical strategies) to ensure effective teaching and learning. It involves a deep understanding of both the content and the best practices for teaching that content, as well as recognizing the diversity of learners and their individual needs (Ernest, 1991). This intersection means that mathematics education is concerned not only with the transmission of knowledge but also with how that knowledge is conceptualized, structured, and applied in various contexts, including the classroom and beyond.

For mathematics to serve the needs of students, rather than just the advancement of the field itself, mathematics education plays a critical role in shaping the goals, frameworks, organization, and delivery of mathematical disciplines. It aims to make mathematics meaningful and accessible to learners, with the goal of developing critical thinkers and problem solvers who can tackle real-world challenges systematically. The philosophical foundation of mathematics education views mathematics as a form of inquiry that utilizes both deductive and inductive reasoning to solve problems. This aligns with the broader educational objective of fostering a generation of learners who are equipped to apply mathematical principles in everyday life, not merely as abstract knowledge but as tools for understanding and solving practical problems (Ernest, 1991).

Mathematics education is therefore not just about mastering content; it is a scientific discipline that studies how people learn and practice mathematics in various forms. This field examines the processes by which mathematical knowledge is acquired, how learners engage with mathematical concepts, and how teaching methods, tools, and media can influence this learning. The discipline also explores how different forms of representation, including symbolic, visual, and real-world contexts, can be used to enhance understanding. Importantly, mathematics education also recognizes the social and cultural dimensions of learning mathematics, acknowledging that learning is shaped by the environments, interactions, and social organization of mathematical activities (Barton, 1996).

This broader understanding of mathematics education situates it as a dynamic and interactive field. It goes beyond viewing mathematics as a collection of static truths and instead considers it as an evolving body of knowledge that interacts with human activity. Mathematics education studies how mathematical thinking and problem-solving are influenced by different teaching practices, how students construct meaning from mathematical tasks, and how real-life contexts and cultural practices can be integrated into mathematical learning (Barnes, 2005). For example, ethnomathematics, which explores the cultural aspects of mathematical practices, highlights how mathematics is embedded in everyday activities such as trade, architecture, and traditional crafts (Barton, 1996).

In this sense, mathematics education can be metaphorically described as the study of the relationship between mathematics and humanity. It focuses on how mathematical knowledge is constructed, communicated, and applied in various social and cultural settings, emphasizing the importance of connecting classroom learning to students' real-life experiences. This connection enriches the learning process, making mathematics more engaging and relevant to learners, and helps them to see its value beyond the classroom. By considering the diverse ways in which individuals and communities engage with mathematical concepts, mathematics education contributes to a deeper and more meaningful understanding of the subject (Bariso et al., 2021).

In conclusion, mathematics education is a multifaceted field that integrates the technical rigor of mathematics with the pedagogical and social aspects of learning. Its aim is not only to impart mathematical knowledge but also to develop learners' abilities to use mathematics as a tool for reasoning, problem-solving, and understanding the world around them. Through this integration of content, pedagogy, and cultural context, mathematics education fosters the development of students as critical thinkers and active participants in the mathematical community.

2.3. **Philosophy underlying Mathematics Education**

Philosophy is about systematic analysis and the critical examination of fundamental problems. It involves the exercise of the mind and intellect, including thought, enquiry, reasoning and its results: judgements, conclusions, beliefs and knowledge. Since mathematics education is the unification of mathematics and education, its philosophy is deep-rooted not only from the nature of mathematics (Bendegem & François, 2007) but also from education (Ernest, 2016) as well. Philosophy of mathematics education encompasses the aim of mathematics education, the analysis of the nature of mathematics learning (and teaching) and the view of mathematics (Zheng, 1994).

2.3.1. Philosophy of Mathematics

There are several philosophical positions on the nature of mathematics and some of them are discussed below.

Platonism: Platonism is one of the oldest and most lasting mathematical philosophies. In terms of mathematics, a fundamental concept of Platonist ontology, it proposes that mathematical connections and structures as well as mathematical objects such as numbers, forms, and equations exist in an ideal, abstract universe and independent of the human mind, our language or any other human activity (Ambrozy, 2019; Palmerino, 2016; Peck, 2018; Startup, 2024). These mathematical objects exist outside space and time and are neither physical nor mental (Peck, 2018; Startup, 2024). These objects are eternal and unchangeable, and mathematicians find them rather than make them (Dembiński, 2019). For example, the number 2 is assumed to exist in a metaphysical sense, regardless of whether anyone thinks about it. In Platonist view, the truth-value of mathematical entities such as "2 + 2 = 4," are objectively true regardless of one's beliefs or feelings. This viewpoint is reflected by the work of early philosophers such as the Pythagoreans, who had mystical views regarding numbers (Nwanegbo-Ben, 2017). This author stated that Pythagoreans saw numbers as the foundation of existence and believed that mathematics revealed the divine structure of the universe. According to Platonists, mathematics is a systematic, unchanging body of knowledge in which the teacher serves as an explainer, guiding learners toward conceptual comprehension. However, this theory was challenged when the Pythagoreans discovered irrational numbers, such as the square root of 2, that could not be stated as a ratio of whole numbers (Makonye, 2013). This revelation was so upsetting to their worldview that the member who revealed it was supposedly punished because it jeopardized their belief in numerical perfection (Ratner, 2009).

Platonism has long dominated mathematical theory, shaping how it is studied and taught. However, it has faced several "crises" in its history, such as the discovery of irrational numbers, which undermined the belief in the perfect, rational nature of numbers; the invention of complex numbers, which combine real and imaginary parts, challenged the idea of a simple, ideal mathematical world. In addition, the development of non-Euclidean geometries by Gauss, Lobachevsky, and Riemann demonstrated that mathematical truths could vary depending on the axioms chosen (Makonye, 2013).

Platonism presents distinct educational obstacles. Its concept of mathematics as pre-existing knowledge suggests that instruction should center on imparting known facts. However, this strategy may fail to interest students or address the many ways in which they understand mathematical concepts. Effective teaching would include recognizing and resolving the unique problems that students confront in discovering these "pre-existing" realities.

Logicism: Logicism, rooted in the philosophical doctrine of realism, was founded by philosophers such as Frege and Bertrand Russell and holds that mathematics is a form of logic (Ernest, 1991; Schiemer, 2021). The primary focus of logicism is on the basic, rigorous, and universal character of logic and mathematics (Lv, 2024). The premise of logicism contains two primary elements: that mathematical notions may be defined in terms of logical conceptions and that mathematical theorems can be obtained from logical axioms by solely logical deduction (Startup, 2024). From Logicists point of view, mathematical notions and mathematical truths may be derived from a set of logical principles and axioms (Ernest, 1991; Makonye, 2013; Schiemer, 2021). According to logicism, all mathematical truths can be proven using logical reasoning utilizing axioms of logic and inference procedures (Ernest, 1991; Woleński, 2022). For example, Euclid's postulate that "the shortest distance between two points is a straight line" might be interpreted as a logical statement. These mathematical truths are eternal and unchangeable; they are independent of intuition or tangible experience (Lv, 2024). Furthermore, Lv (2024) stated that the ultimate purpose of logicism is to provide a pure and perfect foundation for mathematics by establishing the unity and universality of mathematics through a codified system of logic.

Logicists think that by beginning with such axioms, all subsequent mathematical findings can be meticulously and rationally obtained from logic, utilizing inferences regulated by logical rules (Patton, 2017). This method stresses reason and structure, presenting mathematics as a logical extension of clear ideas. Russell (1919) maintained that rules, axioms, and postulates might be used to deduce all mathematical findings by careful and thorough reasoning. Russel stated that axioms and postulates are mathematical assertions that are considered obviously true and do not require proof.

Logicism paved the way for the formal development of mathematics. It demonstrated how abstract notions such as limits in calculus might be formally stated using logical reasoning. However, this philosophy is challenged with like inconsistencies in axioms, in spherical geometry, for example, the shortest path between two places runs in a great circle rather than a straight line; hence, Euclid's "shortest distance" postulate does not apply. The other challenge is Russell's paradox**:** attempts to reduce mathematics totally to logic revealed conflicts, such as those found in set theory, undermining the notion that mathematics was a perfectly coherent system. Despite these limitations, logicism continues to shape how we approach mathematical proofs and the logical structure of theories. Furthermore, Gödel's incompleteness theorem is also another challenge for logicism, resulting in a realization of mathematics' complexity and variety, spurring research of various schools of thought to better explain its nature and knowledge.

Formalism: Formalism, as advocated by David Hilbert, adopts a different approach. This school of thought arose from the rule-based approach to mathematics, in which we begin with axioms and symbolic rules that lead to valid propositions (Spindler, 2022). It regards mathematics as a collection of symbols and rules with no intrinsic meaning. Formalism holds that mathematics is a meaningless formal game played with marks on paper and following rules (Makonye, 2013; Spindler, 2022). Formalist mathematicians are more concerned with the language used to convey a theory than with the abstractions contained inside it.

For formalist, the main concern of mathematics is not on the specific interpretation or practical meaning of mathematical objects but rather on the formal system and symbolic operations. This school of thought believes that any mathematical issue can be solved using finite proof procedures, and as a result, mathematics emerged as a new field of study for the structure, grammar, mathematical reasoning, and proof. Mathematics, according to formalists, is like a game in which symbols are used to generate theorems using specified rules (Makonye, 2013). For example, in algebra, the equation ‘a+b=b+a’ is correct because it adheres to arithmetic principles, not because it has a deeper meaning. Formalists claim that the value of mathematics is based on its logical coherence rather than its relevance to the actual world. Formalists, unlike logicists and intuitionists, deal with language rather than abstract entities.

Formalism stresses accuracy and structure, making it useful in fields such as computer science and automated theorem proving. However, its disassociation from meaning presents difficulties for teachers since learning becomes purely procedural, focusing on memorizing rules without understanding why they work and this approach can alienate learners, as it neglects the real-world applications and contexts that make mathematics meaningful. Educators who concentrate on rote learning or "rules without understanding" unintentionally align with formalism, limiting pupils' capacity to handle real-world situations and acquire profound mathematical intuition.

Intuitionism: The other mathematical philosophy is intuitionism, introduced by the Dutch mathematician LEJ Brouwer (1881–1966), stresses the significance of human intuition in mathematics. For intuitionism, mathematical facts do not exist outside of the human mind; rather, they are mental constructs formed by personal thinking and understanding. This school of mathematical thought provided an alternative mathematical framework to classical mathematics, viewing mathematical entities as mental constructions created in the mind rather than Platonic objects that exist independently of the human mind or formulas written on a piece of paper (Bar-On, 2024).

A mathematical assertion is not true in the eyes of intuitionists unless it is proven (Katz & Katz, 2011; Makonye, 2013). Intuitionists deny the law of excluded middle, which stipulates that every assertion is either true or false, arguing that some mathematical propositions may be neither true nor false unless proven or disproven (McConaughey, 2020). Intuitionsim emphasizes the process of discovery and encourages students to believe in their own intuition. For example, before officially establishing the attributes of geometric forms, a student may construct conjectures.

Intuitionism is compatible with exploratory and inquiry-based learning, in which students actively interact with issues rather than passively absorb information (Makonye, 2013). However, it has its own challenge. One challenge is the rejection of unproven mathematical truths can slow progress and lead to skepticism about widely accepted results. The other challenge is overreliance on personal intuition that may hinder the acceptance of collaborative or historically established knowledge. Despite these limitations, intuitionism emphasizes the value of creativity and critical thinking in mathematics.

Fallibilism: Fallibilism is a contemporary philosophy that questions the absolutist perspective of mathematics. It contends that mathematical knowledge is a human product molded by culture, history, and trial-and-error methods (Makonye, 2013). As such, mathematics is not perfect and can change throughout time. For example, the Pythagoreans' discovery of irrational numbers and the creation of non-Euclidean geometries indicate that long-held mathematical ideas may be challenged. Gödel's incompleteness theorems demonstrated that no mathematical system could be perfectly complete and consistent (Batzoglou, 2024). The formula $V-E+F=2$, which Euler formulated and proved in 1750 using intuitive reasoning but lacked rigor, relates the vertices (V), edges (E), and faces (F) of a polyhedron. Augustin-Louis Cauchy rigorously proved it in 1811 using a method that closely resembles modern standards, illustrating the dynamic self-correcting nature of mathematics. As a result, this mindset encourages students to view their failures as chances for growth and discovery. By combining historical and cultural aspects, the fallibilist viewpoint promotes tolerance and appreciation for many approaches to mathematics (Ghosh, 2020; Dede, 2019).

However, RME is against Platonsit view that considers mathematics as a ready-made system. Like that of formalists, RME does not consider mathematics as a meaningless symbol game, but rather highly grounded in constructivist (social constructivist)philosophy that believes that through the process of mathematization, learners can construct mathematical concepts(knowledge). Since RME utilizes learners' intuitive abilities to improve understanding and problem-solving skills, intuitionist mathematical objects only exist when created by the human mind. Therefore, both RME and intuitionism as a philosophy of mathematics emphasize the mental construction of mathematical understanding, suggesting that within the RME framework, "intuition" can be a valuable tool. On the other hand, a fallibilist considers mathematics as a dynamic human activity constructed and reconstructed through the interaction with real contexts, yet does not consider it as a body of absolute, eternal truths to be transmitted; hence, RME is rooted in the fallibilism philosophy of mathematics education.

2.3.2. Philosophy of Education

Both education and philosophy are highly interrelated. Since they present different views of the same thing, they are considered as two faces of the same coin, in which philosophy is the contemplative side, whereas education is the dynamic and practical side of philosophy. The philosophy of education is a field of philosophy that addresses philosophical questions regarding aims, methods, nature, policy, pedagogy, learning theories and curriculum, (Öksüz & Şentürk, 2021). Alternatively, it is the process of applying a philosophical approach to resolving problems related to education and arriving at philosophical conclusions. Educational philosophies explain why and how teachers implement their methods, not what they wish to do in the classroom to promote learning.

In general, philosophy dictates the nature of one's education by answering questions such as, “What is the nature of education? What link exists between education and philosophy? Why should education be imparted? What is the necessity for education? What is the influence of philosophical ideas on education? How can it be excellent? For what purposes should education be provided? How do you theorize or philosophize about educational practices?” It can determine achievements, content arrangement, select delivery approach and select measurement-assessment tools teachers utilized (Danner & Ofuani, 2022). Within the epistemological framework that examines the nature of knowledge and how humans acquire it, perennialism, essentialism, progressivism, and reconstructionism are the four main educational philosophies, each linked to one or more broader philosophies such as realism, idealism, pragmatism, and existentialism. These educational philosophy concepts are employed in schools across the world.

Perennialism: It is the oldest and most conservative educational philosophy rooted in realism, which dominated American education from the colonial period to the early 1990s (Ornstein & Hunkins, 2018). It is a subject-centered philosophy in which the goal of the perennialist educator is to teach everlasting ideas, to seek enduring truths which are constant, resulting in a developing mind that think critically (Danner & Ofuani, 2022).

The main concern of perennialists is the content and development of reasoning skills which is developed sequentially. In this philosophy, the teaching materials, learning activities, and pedagogy are not dependent on the students’ interests, but on what are necessary to enhance their intellectual capacity. Subsequently, teachers are viewed as the primary performers on stage; they ought to instill in their students a work ethic, deference to authority, and discipline. Perennialists believe that people are rational beings whose brains require development, and thus, the priority is intellectual development for a meaningful education. The curriculum of Perennialist is subject-centered, logically ordered content emphasizing language, literature, mathematics, and science (Ornstein & Hunkins, 2018).

Essentialism: Beginning in the 1930s and reemerging with increased strength in the 1950s and 1980s, essentialism has criticized progressive educational practices that focus on how children learn rather than on what children learn. A kind of neo-perennialism rooted in both idealism and realism, essentialism maintains that the purpose of schools is both to preserve the knowledge and values of the past and to provide children with the skills essential to live successful and meaningful lives in the present society (Arends, et.al, 2001). It shares a number of common features with perennialism. Essentialists advocate that schools should conserve social traditions in which the teacher and subject-centered curriculum is determined by the traditions and heritage that the students need to master, which in turn neglects the interests of the students. In this context, the role of the teacher is to conduct lectures while students are provided with a worksheet and a hands-on project, followed by an assessment of learning on the materials covered during this process. If their assessment indicates sufficient competence, they are promoted to the next level to learn the next level of more difficult materials.

Progressivism: It is a product of the pragmatic school of thought, which emphasizes the importance of ongoing experimentation with ideas. According to progressivists, learning is built on inquiry that arises from observing the world. Progressive education theory was developed in America between the mid-1920s and the mid-1950s, with John Dewey as its most prominent advocate. Progressivists opposed authoritarian teaching, overreliance on textbook methods, memorization through constant drilling, static goals and materials that ignore a changing world, corporal punishment, and efforts to separate education from individual experiences and social realities (Ornstein & Hunkins, 2018). Their main focus is on students as a whole, rather than on the curriculum or teachers. The learner is seen as a problem solver and thinker who creates meaning through personal experiences within their physical and cultural context. The teacher functions as a facilitator, guiding students to reflect on their learning experiences as they engage with real-life situations and determine their own suitable role in society (Arends et al., 2001). The curriculum centers around students’ interests and questions, emphasizing teamwork, group activities, and problem-solving, because learning occurs through cooperative approaches. The emphasis is on the process—how learning happens. Progressivists believe that schools should promote collaboration and self-discipline while also transmitting society's values and culture.

Re-constructionism: Re-constructionism is an ideology that focuses on fixing social issues and aims to build a better society and global democracy. Its root is embedded in Pragmatism, which is based on socialistic and utopian notions from the late nineteenth and early twentieth centuries; nevertheless, the great depression gave it fresh life. During this time, progressivists played an important role in popularizing their educational movement in the United States. However, small groups of progressivists became disillusioned with the United States and impatient for reform, and they advocated for a greater emphasis on society-centered education that addressed the needs of all social classes, rather than student-centered education, which progressivists overemphasized (Tan, 2006; Ornstein & Hunkins, 2018). Because schools are viewed as social agencies rather than just academic establishments, their pragmatic roots are taken into consideration.

Together with progressivists, social reconstructivists think that students should be given the tools they need to address social and personal issues, and they criticize traditional schooling for its preset curriculum and instruction that upholds the status quo (Tan, Charlene, 2006). Their curriculum places more emphasis on social sciences like history, political science, economics, sociology, philosophy, and psychology than it does on pure science. This is done in order to address pressing issues like hunger, international terrorism, inflation, and inequality (Alsalem, 2018). People are liberated from the confines, limitations, and restraints of society by gaining individual self-realization and independence via cognitive and intellectual activity. According to Alsalem (2018), the job of teachers in reconstructivism is to provide a favorable environment that encourages students' awareness of societal problems and guides them in identifying acceptable answers.

The four fundamental educational philosophies—perennialism, essentialism, progressivism, and reconstructionism—can be placed within an epistemological framework that considers the nature of knowledge and how people acquire it. Perennialism and essentialism promote teacher-centered pedagogy, which views information as fixed, universal, and best transferred through direct instruction and decontextualized learning. In contrast, progressivism and reconstructionism promote student-centered inquiry, experiential learning, problem solving, critical thinking, and a strong link between knowledge, learners' lives, and society as a whole.

Perennialism and essentialism emphasize the transfer and mastery of preset knowledge, which is more consistent with traditional, teacher-directed techniques than with Realistic Mathematics Education (RME). Progressivism and reconstructionism, on the other hand, have a lot in common with RME. Progressivism promotes hands-on learning, contextualized problem solving, and the growth of understanding via experience, whereas reconstructionism emphasizes knowledge's social significance and transformational role in addressing real-world problems.

RME-based mathematization sees mathematics as a human activity inextricably linked to reality. It enables students to recreate mathematical concepts by modeling and applying them in actual, real-world situations. This perspective is highly aligned with progressivist and reconstructionist beliefs because it encourages inquiry, contextual participation, and the social purpose of learning. Within this paradigm, RME also encourages students to modify their epistemological ideas from absolutist—where mathematics is considered definite and unchanging—to fallibilist, where knowledge is viewed as developing, open to improvement, and impacted by human action.

Furthermore, RME improves problem-solving abilities by placing mathematics challenges in meaningful cultural and social contexts, allowing students to draw on their life experiences to generate solutions and insights. This interaction not only enhances their problem-solving ability, but it also leads to more conceptually grounded geometry achievement. Rather than depending on rote memorization, students gain comprehension through active mathematization, which connects mathematical knowledge to real-world problems.

Overall, progressivism and reconstructionism offer solid philosophical justification for RME-based mathematization, whereas perennialism and essentialism diverge from the RME tenets because of their teacher-centered focus. When taken as a whole, they demonstrate how RME can increase students' problem-solving abilities, mold their epistemological views, and promote significant success in mathematics education.

2.4. **Approaches in Mathematics Education**

In mathematics education, Treffers (1987) categorized different classroom scenarios according to the balance between teaching and learning as follows: mechanistic, structuralist, empiristic, and realistic. Each style is shaped by a different focus on vertical and horizontal mathematization which is described as follows.

Mechanistic or traditional instruction emphasizes memorizing and applying already known procedures and it lacks both horizontal as well vertical mathematization (Makonye, 2014; Wahyudi et al., 2017). In this instructional approach, students hardly need to make sense of any real problems and structure/build their mathematical ideas (Barnes, 2005). It treats human beings like a computer or machine.

A structuralist: The structuralist approach is a new math approach that is based on set theory, flowcharts, and games, all of which are examples of horizontal mathematization, but they are stated from an 'ad hoc' manufactured universe that has nothing in common with the learner's living world (Zulkardi, 2010). This approach concentrates on symbolic manipulation and the formation of mathematical concepts that characterize vertical mathematization (Menon, 2013). In a structuralist approach, mathematization involves abstracting and organizing mathematical concepts into a coherent structure. Learners are expected to abstract and discover principles for specific cases, engaging in deductive reasoning and generalization (Menon, 2013).

Empiricist: this approach focuses on solving many problems using informal strategies (horizontal mathematizing element) but lacks models and schema and therefore does not encourage the extended situation to come up with a formula or a model and this in turn doesn't enable students to develop mathematical concepts (Barnes, 2005).

Meanwhile, the realistic instructional approach focuses on both horizontal and vertical mathematization since the guiding principle of realistic mathematics education is guided by reinvention through progressive mathematization by a phenomenological exploration (Gravemeijer, 1994). In this approach, real-life situations or context-based problems are taken as the starting point of learning mathematics (i.e., students organize the problem, try to identify the mathematical concepts embedded in the real-life problems, discover regularities and relations (Dickinson &Hough, 2012; Makonye, 2014). Taking these informal strategies as a basis students make meaning and develop new mathematical concepts with the help of vertical mathematization.

2.5.**Realistic Mathematics Education (RME)**

Traditionally, in Dutch, primary school mathematics as a subject was termed as "rekenen," meaning arithmetic. In the 1960s this arithmetic was taught in a mechanistic approach characterized by the fixed procedure in a step-by-step and context-based problems that were used only for the application of previously learned calculation procedures and they hardily paid due attention to develop insight into the mathematics in the procedures (Van Den Heuval-Panhuizen& Drijvers, 2020). As a result, the country developed a project called Wiskobas against the notion that 'teacher is the source of knowledge; students are the passive listener and the classroom is the place where teacher transfer knowledge.

In this project an approach called realistic mathematics education (RME) was developed which considered mathematics classroom as the place in which students get an ample opportunity to reinvent mathematical ideas and concepts by exploring real-world problems under the guidance and supervision of a teacher (Wahyudi et al., 2017). RME was developed based on daily life problems to make mathematics learning more authentic, enthusiastic, and meaningful (Payadnya et al., 2023). Freudenthal, the chief proponent of RME, felt that the abstract nature of problems used in mathematics education was a weakness as ‘it is wasted on individuals who are unable to avail themselves of this flexibility (Freudenthal, 1968). The theory sticks with the notion that ‘learning of mathematics should begin with real problem situations which initiate students to develop mathematical concepts, tools and procedures and through the process, students can apply their mathematical knowledge, which then gradually has become more formal and general, and less context-specific (Jupri et al., 2014). Freudenthal believed that for mathematics to be of human value, it must be connected with reality, stay close to children, and should be relevant to society. He also stressed the social and cultural role of mathematics and its valuable place in society.

RME was developed to answer what a mathematics is human activity and how students learned mathematics and teachers taught it (should be learned and taught contextually). The ideas of situated learning theory also emphasize learning as a matter of creating meaning from the real activities of daily living (Stein, 1998), indicating the importance of learning that occurs relative to the teaching environment. A similar theory called situated cognition which was proposed by Brown et al. (1989) proposes the social nature of learning, and they argued that learning cannot be abstracted from the situations in which it is learned and cannot merely be transferred or applied in school. According to Herrera (2020), both theories view that knowledge is situated as the product of the activity, context, and culture in which it is authentically developed and used. The culture of conventional schooling influences what is learned, and decontextualized instruction does not lead to effective learning. These indicate the need for content-based RME.

The philosophy underpinning RME is that for students to resolve real problems, learning of mathematics should begin with real problem situations and students should be allowed to reinvent (rediscover) mathematical ideas and concepts by progressively mathematizing the real-world situations with the close assistance of their teacher (i.e., mathematics is a human activity) (Heuvel-Panhuizen & Zanten, 2020). RME is an art of teaching through which students develop mathematical objects, commencing from real-world problems and ending with a reflection on the solution of real-world problems (De Lange, 1987). Thus, in RME, we take things from the real context, make mathematics an inspiring subject by solving the real problem using informal as well as formal strategies with the guidance of the teacher, and finally bring them back to the real world.

Here in RME, even though the meaning of the word ‘realistic’ in real-world situations has a broader connotation, it is confined to problem situations that students can imagine (Heuvel-Panhuizen & Drijvers, 2014). Therefore, in RME, problems presented to students can come from the real world and the fantasy world of fairy tales, or the formal world of mathematics, as long as the problems are experientially real in the student’s mind. This domain-specific instructional theory for mathematics was governed by three fundamental principles: guided reinvention through progressive mathematization, didactical phenomenology, and emergent models.

Reinvention through progressive mathematization. According to the reinvention principle, students should be allowed to participate in a process comparable to the one that led to the invention of mathematics. The reinvention principle suggests that instructional activities should provide students with experientially realistic situations, and by facilitating informal solution strategies, students should have an opportunity to invent more formal mathematical practices (Freudenthal, 1973). Thus, the developer can look at the history of mathematics as a source of inspiration and at informal solution strategies of students who are solving experientially real problems for which they do not know the standard solution procedures yet (Gravemeijer, 1994) as starting points. Then the developer formulates a tentative learning sequence by a process of progressive mathematization.

Didactical phenomenology: Situations in which a given mathematical topic is used necessitated examination to disclose the types of applications that should be expected for education, as well as their usefulness as sites of impact for a progressive mathematization process. The objectives of phenomenological research are to identify issue circumstances in which situation-specific methodologies can be explored and generalized, as well as identifying situations that can elicit paradigmatic solution procedures that can be used as the foundation for vertical mathematization (Freudenthal, 1983).

Emergent models. Emergent models are essential for bridging the gap between informal and formal levels. Students created their models. Initially, the model was any model of a circumstance that the pupils were familiar with like sketches, pictures, diagrams, or longways of recording a calculation. By generalizing and formalizing the model, it becomes its entity. The models in the formal level take the forms of abstract mathematical tools such as tables, symbols for representation, formulae, or more generalizable strategies. This allowed this model to be used as a basis for mathematical reasoning (Gravemeijer, 1994).

These three principles are interlinked in a way of fulfilling one supports the other, and when three of them are integrated, they influence the instructional design, and the teaching-learning process. To put these theoretical principles into reality in the classroom, Realistic Mathematics Education articulates a set of instructional qualities that influence the design and implementation of teaching and learning activities. While the three principles serve as the conceptual and philosophical foundation of RME, the characteristics represent their practical application in instructional contexts. They explain how RME is used in mathematics classrooms and how such applications are designed to help students with progressive mathematization, conceptual understanding, and active building of mathematical knowledge. For these to happen, it is worth mentioning the key characteristics through which the RME can be implemented, and influence learning.

Characteristics of RME

Following the principles discussed above, the design and implementation of RME-based learning aim to achieve each of the following characteristics of RME. These RME characteristics serve as instructional guidelines, directing us on how to create an activity, classroom interaction, and the flow of learning from informal reasoning to formal mathematical comprehension, hence identifying the instructional focus and learning aim of RME. RME is distinguished by five characteristics. These are utilization of contexts, models, students' productions and constructions, interactivity, and the intertwining together of diverse learning strands.

1. **Real context:** In the RME context serves as a starting point for learning, which is not limited to something concrete or real rather, it includes meaningful situations like students' everyday environments that can be imagined in their minds. In this regard, real-life situations serve as a phenomenological starting point for learning, allowing learners to identify appropriate scenarios in which mathematical ideas and processes can be created through the mathematization process.

2. **Use of model:** models are situational models that are familiar with the students (i.e., concrete models like manipulatives, real objects, etc.) or mathematical models (abstract models like equations, formulas, graphs, etc.) developed by students themselves. These models serve as a bridge between informal mathematical activities and formal mathematical reasoning, which helps students in learning mathematics at a different level of abstraction. These models gradually grow from context-bound representations to formal mathematical tools as they are generalized and formalized, implementing the emergent model principle and facilitating the move from informal methods to abstract mathematical thinking.

3. **The use of student work alone or strategies:** as a result of their "doing mathematics"; the learners are expected to make a significant contribution to the

teaching and learning process, which takes them away from their informal approaches and toward more formal or raw methods. This feature exemplifies the principle of guided reinvention by emphasizing the strategies and productions of the learners as necessary starting points for the development of formal mathematical concepts under the close assistance of the teacher.

4. **Interactivity:** RME relies heavily on student-to-student and student-to-teacher interaction (De Lange, 1996; Gravenmeijer, 1994). Explicit negotiation, intervention, debate, cooperation, and evaluation are all important components of a constructive learning process in which the student's informal approaches are used as a lever to achieve the formal ones. Explaining, justifying, agreeing and disagreeing, challenging alternatives, and reflecting are all activities that students participate in during this interactive lesson. Such engagement is the critical to guided reinvention since it allows learners to articulate, justify compare and develop their informal techniques using social negotiation and instructional scaffolding.

5. **The intertwining of various learning strands or units:** The integration of mathematical strands or units is necessary for RME (De Lange, 1996; Gravenmeijer, 1994). It's also known as the holistic method because it includes applications and implies that learning strands can't be treated as independent entities; instead, in problem resolution, an intertwining of learning strands is used. One of the reasons is that applying mathematics is extremely difficult when it is taught "vertically," that is when different disciplines are taught individually without regard for the links between them. More than just algebra or geometry is frequently required in applications. This intertwining reflects the phenomenological aspect of real-world situations in which many mathematical concepts are used concurrently, allowing for meaningful application and the transfer of mathematical knowledge across contexts.

These characteristics show how the fundamental ideas of RME can be applied in classroom practice. RME promotes progressive mathematization and the development of deep, transferable mathematical understanding by utilizing meaningful contexts, supporting student-generated models and strategies, stimulating interaction, and integrating diverse mathematical strands. As a result, the use of RME is explicitly geared toward allowing students to experience mathematics as a meaningful human activity rather than a collection of isolated procedures.

2.6. **Theoretical overview of Mathematization**

Freudenthal argued that there is no mathematics without mathematizing, no abstraction without abstracting, no algorithms without algorithmizing, and no language without verbalizing, and no schemes without schematizing (Freudenthal 1991). This indicates that learners are not expected to learn mathematics from the usual view of presenting mathematics as a ready-made system with general applicability, and this led to the emergence of Realistic Mathematics Education in the Freudenthal Institute in collaboration with other universities and institutions in the Netherlands**.** In Mathematics Education, the notion of mathematization originated from the theory of Realistic Mathematics Education (Jupri & Drijvers, 2016). Mathematization has a different meaning among different scholars. Some consider it as the process of representing a real-life problem mathematically, solving and interpreting it into a real-life context (Amin et al., 2018; Jupri & Drijvers, 2016). It can also be considered as the process of applying mathematical techniques, concepts, principles, definitions, procedures, theorems, and the like developed in mathematics to real-life problems or other fields of science, and even in mathematics as well (Araújo & de Lima, 2020). It is the various ways of organizing activities to exhibit characteristics of mathematics, such as generality, certainty, exactness, and brevity (Loc & Hao, 2016).

Others like Berry and Davies (1996) define mathematization as the formulation of a real-life situation necessary for a mathematical model to be constructed and for mathematical operations to take place. Fosnot (2005) states that mathematization is a constructive process that includes observing patterns in special cases, analyzing the reasons for something happening, stating the findings in some form of generalization, and seeking flexibility in strategy making or proof. It can also be considered a process of designing pedagogical activities to progressively transform concrete perceptual mathematical experiences, from less precise to more precise, into generalized mathematical concepts reified by symbols and signs, making mathematization accessible to students (Lestariningsih et al., 2018).

In mathematization, horizontal and vertical mathematization is distinguished (Treffers, 1987; Van den Heuvel-Panhuizen, 2003) in which the former one deals with transforming a real problem into a symbolic mathematical problem via observation, experimentation, and inductive reasoning (Treffers, 1987). It is the process through which students use their informal strategies to describe and solve a contextual problem and interpret the mathematical solution with the respect to the given real-life problem. This is characterized by the identification of mathematical concepts from the given real-life problems, identifying, schematization, visualizing and describing a problem in different ways, and discovering relations, interpreting the mathematical solution with the respect to the given real-life problem (De Lange, 1987). It refers to modeling a problem situation into mathematics and vice versa.

Vertical mathematization is the process of translating the problem into mathematical language using symbols based on informal strategies to reach a higher level of mathematical abstraction. It is characterized by representing a relation in a formula, providing regularities, refining and adjusting models, using different models, combining and integrating models, formulating a new mathematical concept, and generalizing (De Lange, 1987).

Mathematizing in learning mathematics is a process of improvement and development of mathematical ideas (Freudenthal, 1987), which involves a series of progressive analyses and interpretations from one level to another and is termed progressive mathematization.

Progressive mathematization means the development of mathematical sophistication over time. It is a process that occurs gradually when the process of vertical mathematization forms the formalization process of mathematical concepts of mathematical models derived from horizontal mathematization. This process of progressive mathematization provides a trajectory through which learning may occur.

As stated by De Lange (2006) the process of mathematization is cyclic. (1) Given the real-life problem, students start their journey by understanding the given problem(s) and identifying the relevant mathematical concepts embedded in the real-life problem. (2) Based on the identified mathematical concepts, the students remove unnecessary elements that exist in reality by formulating a mathematical model. (3) At the third stage, students try to solve the mathematical problem included in the model and reflect on the solution process. (4) At last, they can interpret the mathematical solution about the original, realistic situation. In this cyclic process, the first two steps together deal with changing real problems into a mathematical symbol, which is termed horizontal mathematization. The third step is taken on the vertical mathematization, and the fourth step, which deals with changing the mathematical solution of mathematical symbols to realistic solutions, is a process of horizontal mathematization.

When Treffers' and De Lange's conceptualizations of mathematization are compared, they reveal a common theoretical underpinning as well as a difference in focus. Treffers (1987) views mathematization largely as a development from horizontal to vertical processes, emphasizing the transition from informal, context-bound techniques to formal mathematical abstraction. De Lange (2006), on the other hand, broadens this viewpoint by expressly describing mathematization as a cyclical process in which pupils alternate between real-world events and mathematical representations. While Treffers emphasizes the structural distinction between horizontal and vertical mathematization, De Lange emphasizes the iterative aspect of mathematization, emphasizing reflection, interpretation, and validation as necessary components of mathematical learning. This cyclic nature of mathematization is illustrated in the diagram below.

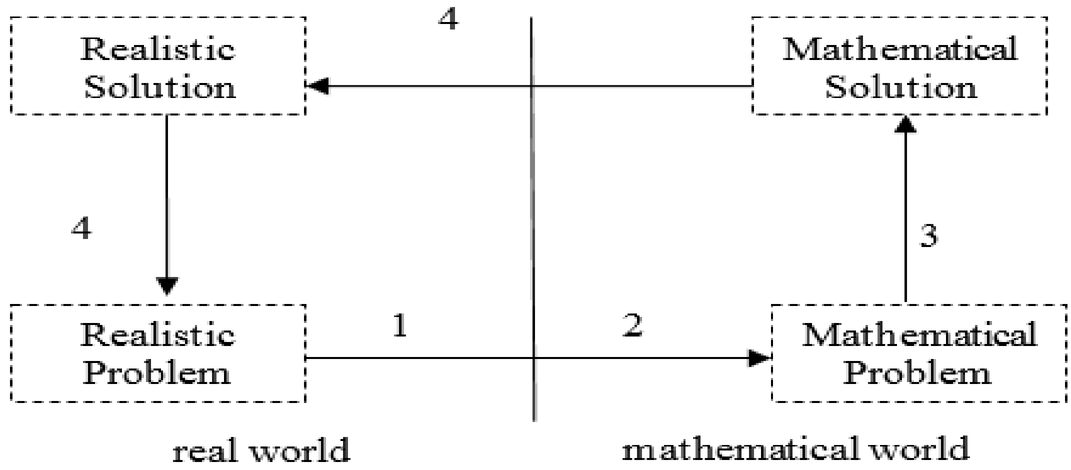

Figure 2.1 Cyclical characters mathematization (based on De Lange, 2006, p.17)

The above illustration mirrors De Lange's cyclical interpretation of mathematization while keeping in accordance with Treffers' distinction between horizontal and vertical mathematization, combining both perspectives into a unified understanding of mathematical learning.

Then, mathematization in this study is the activity of transforming a problem expressed in a real-life context into a mathematical model or representation, and then the completion of a mathematical model or representation is interpreted into a real-life context (Lestariningsih et al., 2018).

According to Lynch (2011), there are five steps in the process of mathematization:

1. Considering the real-world problem
2. Organizing based on mathematical contexts,
3. Paring down to the most important aspects,
4. Using mathematical skills and solving problems, and
5. Considering the solution in terms of the real situation.

2.7. **Ethno-mathematics**

Mathematics is extremely important in many areas of human activity, including trading, farming, cooking, tailoring, medical, and engineering, and it is found in all human cultures (Umoru & Abah, 2021). Surprisingly, every culture, regardless of its level of development, develops a certain type of mathematics to aid its citizens in solving daily societal problems. Even in the absence of formal schooling, people have exploited basic mathematical knowledge to solve a variety of problems (Abiam, 2006). This indicates that every cultures around the world have mathematics and mathematical education in their cultural history.

Ethiopia, as a country with a long human history, has indigenous resources that could be used to teach mathematics. However, there is a paucity of literature that has conceived, concretized, and indigenized mathematical ideas (Weldeana, 2016). As a result, indigenous mathematical concepts are not well-researched and documented. Consequently, indigenous mathematical knowledge is not incorporated into mathematical training at all levels of education. Instead of giving learners prepared knowledge, providing embedded education advice, and creating a framework for them to construct their knowledge as they learn.

The primary father of ethnomathematics, D'Ambrosio (2006), described ethnomathematics as a synthesis of three major constructs: Ethno; mathema; and tics that formed ethnomathematics, in which the prefix ethno denotes "all of the materials that go into a group's cultural identity," mathema denotes "doing/knowing" mathematics, and tics denote "methods/techniques" that involve ways and styles. Different scholars define ethnomathematics in different ways. For example, Rosa and Orey (2010) describe it as a study of mathematical ideas found in any culture. Bush (2005) on the other hand stated that ethnomathematics as a study of mathematics’ connection to culture and culture’s effect on mathematics teaching and learning. For Weldeana (2015) ethnomathematics is the way different cultural community mathematize (e.g., measure, count, compare, relate, sort, infer, hypothesize, problem pose, generalize, communicate, collect data and process, predict, analyze, record, evaluate, verify, and construct).

Learning mathematics through ethnomathematics allows for the abstraction, idealization, and generalization of mathematical notions and principles. When examined through the lens of mathematics features, mathematics is a human activity that involves the use of the learners' understanding of reality and environment to speed up the process of learning mathematics and attain a higher level of mathematics education (Gravemeijer, 2008). Mathematics considers the quantitative, relational, and cultural aspects of society that are integrated with the concrete things that learners can observe or understand through the process of mathematization, as mathematics is a reflection of the real world through an empirical abstraction process (Gravemeijer, 2008). Ethnomathematics can be a part of mathematics education if it uses the surrounding culture as a beginning point for learning. As a result, ethnomathematics can be heavily integrated into the mathematics curriculum (Rosa & Orey, 2010). Ethnomathematics-based learning can enhance students' enthusiasm, curiosity, and confidence while also increasing their activities, and it ensures relevance.

It recognizes that different cultures negotiate their mathematical practices in different ways by considering the appropriation of academic mathematical knowledge developed by various sectors of society as well as different modes in which different cultures negotiate their mathematical practices (D'Ambrosio, 2001). Such negotiated mathematical practices have been shown to have some activities that are universally practiced. Bishop has identified six activities that are universal, as described below.

By studying different cultures around the world, Bishop thematized the cultural activities into counting, locating, designing, playing, measuring, and explaining. In his first theme he identified mathematical concepts like numbers, number patterns, numbers system, place value, operations on numbers; combinatory; accuracy; approximation; errors; fractions; infinitely large and small numbers; limit; powers; number relationships; arrow diagrams; algebraic representation; events; probabilities; and frequency representations.

Within the locating category, he identified mathematical concepts like distance; straight and curved lines, angle as turning rotations, systems of location: polar coordinates, 2D/3D coordinates mapping, latitude/longitude, loci, linkages, circle, ellipse, vector, spiral.

Under the theme of designing, concepts like design; abstraction; shape; form; aesthetics; objects compared by properties of form; large, small; similarity; congruence; properties of shapes; common geometric shapes, figures, and solids; nets; surfaces; tessellations; symmetry; proportion; ratio; scale-model enlargements; rigidity of shapes were identified (Bishop, 1991).

Mathematical concepts like accuracy of units, estimation, length; area; volume; time; temperature; weight; conventional units; standard units; system of units (metric); money; compound units were identified under measuring category (Bishop, 1991). Mathematical concepts identified under playing are games; fun, puzzles, paradoxes, modelling, imagined reality, hypothetical reasoning, procedures, cooperative games, competitive games, chance, and prediction.

Under the explaining theme, he identified similarities, classifications, conventions, hierarchical classifying of objects, story explanation, logical connectives, linguistic explanations: logical arguments, proofs, symbolic explanations: graphs, diagrams, charts, matrices, mathematical modeling, criteria: internal validity, external generalizability (Bishop, 1991).

On the other hand, Barton (1996) recognized four ethno-mathematical activities as of descriptive activities (describing the activities studied), archeological activities (discovering the mathematical aspects of the activities), mathematizing activities (connecting the mathematical aspects uncovered to the existing mathematical concepts), and analytic activities (explaining why the practices are the way they are). All of Barton's ethno-mathematical activities, according to Weldeana (2016), were derived from Bishop's six universal activities.

Aikenhead (2017) conceptualized counting, locating, measuring, designing, playing, and explaining as fundamental or universal mathematizing processes (cultural practices) in the mathematical systems developed by numerous significant cultures throughout history. These fundamental procedures show that the local indigenous community engages in a variety of mathematical activities (Aikenhead, 2017). D'ambrosio and Rosa (2017) stated that ethno-mathematics centers on the mathematizing of frequent activities in specific communities. Such a stance highlights mathematics as a cultural practice that exists in human activity.

Several ethno-mathematics researchers used the six mathematizing processes (cultural practices) of Bishop as a framework for exploring the mathematization and mathematics embedded in various cultural activities. For example, some of the researchers who used this framework are (Aikenhead, 2017; Gebre et al., 2021; Graham, 2020; Tegegne, 2015; and Tesfamicael et al., 2021), and many others.

Thus, one ultimate goal of this study is exploring the mathematization of North Showa, Arsi, and Bale cultural community and therefore, the researcher selects Bishop's six-mathematizing process as framework for exploring the way North Showa, Arsi and Bale local community mathematize while practicing their cultural activities: pottery, weaving, basketary, and traditional house building.

2.8. **Ethno-mathematics in Ethiopia**

There has been little or no awareness of the role of cultural mathematics in mathematics education among teachers and curriculum experts for several years, even though around 80% of Ethiopians live in rural areas, in which the cultural background of the students could have contributed to a meaningful teaching-learning process (Weldeana, 2016). However, since recent years, researchers have given due attention to this area of study. Weldeana (2015) conducted research on grade 11 students in the Tigray region, Ethiopia, and collected data using semi-structured interviews followed by Mathematical questions.

The goal of the study was twofold: to identify the strategies they designed, and the reasoning they provided and to probe the depth of their understanding of the procedures while producing the local materials. The other purpose was to measure the capacity they have in transforming local mathematical knowledge into its equivalent global. The findings showed that some local materials are male-oriented and males performed better as compared to females, and some others are female-oriented and females outperformed as compared to males. It is also found that both males and females equally demonstrated and enhanced in transforming the local mathematics to the corresponding global one.

Tegegne (2015), for example, studied everyday mathematics among the Khimra people of Ethiopia. The study's target was to investigate the everyday mathematical practice and the issue of bridging the gap between in-school and out-of-school mathematics. His study was focused mainly on pottery, weaving, sewing craft, farming, traditional brewing, shop, shoeshine, house construction, and the two games gebeta and tirga, conducted around two schools. He collected his data through classroom and workplace observation, interviews, field notes, and document analysis. He found that in-school and out-of-school mathematical practices can work together to improve the process and means of reaching mutual goals.

By blending two of the Barton's (1996) models of ethno-mathematics research (i.e., the archeological and mathematizing), Weldeana (2016) found mathematical concepts like the maximum area from the local threshing plot. Furthermore he found concepts of volume and surface area of cones, cylinder, and frustum from local materials such as 'Gotera' and seat of the coffee pot, the concept of shortest path postulates and triangle inequalities from 'Aquarach'.

Gebre et al. (2021) also conducted an ethnographic case study on ethno-mathematics in Kafa people in Ethiopia to investigate the indigenous number and number sense in the local community and explore its level of integration in the school curriculum. The authors used focus group discussion, interview, document analysis, and observation for data collection and their findings showed that in Kafa's agricultural setting, fractions, base six (maqoo) and base sixty (uddoo) counting are embedded. Their results also showed that even though revision of the curriculum is needed, the local indigenous mathematical knowledge was incorporated into the existing curriculum partially.

Another ethno-mathematics research was conducted by Tesfamicael and his colleagues in 2021 in different parts of Ethiopia, like the Afar region (eastern part), South Omo (Southern part), Gambella (Western part), Gondar (Northwest), and Addis Ababa (central part). They collected data using unstructured interviews and observation and analyzed the data based on Bishop's mathematical activities framework. Under the Bishop's counting category, they found that there were different number systems like the Geez number system, the Opo numeral system with the Opo community, and the Ari number system in the Ari community. Under locating category, they found that the local communities described their spatial environment by exchanging information while they exchanged their greeting and some other navigate in their environment with their own expression for up/down, left/right, forward/backward, north, south, west and south, curve, and ecology. Under the measuring category, they found that the comparison concepts as well as the concept of estimation in measuring different kinds of stuff. Under the designing category, they found geometric contents: patterns, symmetry, diamond, triangle, rectangle, polygon, circle, parallel lines, symmetry, transformation, reflection, rotation, and tessellation.

The other recent study on ethnomathematics is that of Bariso et al. (2021) which targeted examining the ethno-medical concepts and games in the Guji Oromo community. Framing based on Bishop's mathematical activities, they found mathematical concepts like grouping, clustering, and making networks under counting and locating; planning, building, and pattern making under measuring; puzzles, paradoxes, games, and hypothetical reasoning under designing; classification, conventions, generalization, and symbolic explanations under explaining.

The above studies mainly focus on exploiting mathematical concepts embedded in different Ethiopian cultural activities in different parts of the country and try to see their existence in the teaching-learning of mathematics, yet the ways those local communities mathematize the activities identified seek further study. There is also the need to study integration or inclusion of those local mathematical activities in the actual mathematics classroom and examine their effects on learners' mathematics achievement problem problem-solving skills , and on the overall meaningfulness of mathematics learning, including associated changes around the epistemological beliefs of learners and teachers.

2.9. **Epistemological beliefs**

This section deals with different concepts like unidimensional and multidimensional epistemological beliefs, general and domain specific epistemological beliefs, Mathematical epistemological beliefs, how to change epistemological beliefs and empirical research on epistemological beliefs.

2.9.1. Unidimensional and multidimensional epistemological beliefs

Epistemology is a philosophical field that addresses questions including what the nature of knowledge is and how people come to know it, and epistemological beliefs address questions like how people learn, what theories and ideas they have about knowing, and how these epistemological premises relate to and impact cognitive processes like reasoning and thinking (Hofer & Pintrich, 1997). Piaget's introduction of genetic epistemology in the 1950s and Perry's research in the 1960s led to a significant increase in the area of epistemological beliefs, which is now expanding quickly. Piaget's theory describes the phases of intellectual growth and knowledge acquisition, demonstrating the fusion of psychology and philosophy (Hofer & Pintrich, 1997). By examining Harvard University undergraduate students' epistemic growth over their four-year college experience using his own theoretical framework, he classified them into nine positions depending on their understanding of knowledge and the world around them. He then further divided the nine positions into four developmental groups: dualism, multiplicity, relativism, and commitment (Grove & Bretz, 2010). He discovered that many first-year students assume that omniscient authority dictates basic, immutable facts, and by their senior year, students believe that complex, tentative knowledge comes from reason and empirical inquiry (Schommer-Aikins, 2004). He proposed nine developmental stances that help students transition from dualistic to committed relativistic thinking during their four-year college experience as illustrated below.

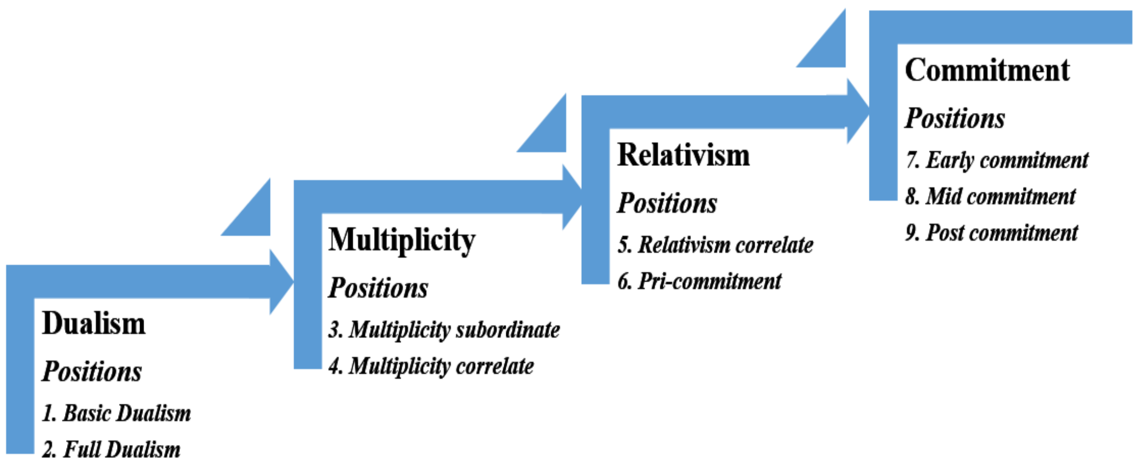

Figure 2.2 Perry's Intellectual Development Model

Dualism

Perry's model identifies the initial stage as dualism or naïve epistemology, characterized by binary thinking without ambiguity recognition (Perry, 1970). At this stage, students perceive knowledge as right or wrong, with authority figures being solely correct and the truth value is absolute (Black & Allen, 2017; Gallagher, 2019). Dualism fails to acknowledge subjectivity and diverse perspectives and dismisses alternative opinions

(Perry, 1970). It reflects a simplistic good vs. bad mentality and ignores diverse perspectives.

Position 1: Basic Dualism: In the initial dualistic stage, students perceive knowledge as absolute right or wrong answers from authority figures, with their thinking operating on discrete categories of true and false (Perry, 1970).

Position 2: Full Dualism: Late dualism is a transitional phase in students' thinking, where they recognize knowledge is not always definitive and that answers may depend on perspective rather than authority (Kunwar et al., 2024). These scholars stated that this stage denotes a shifting viewpoint and a more sophisticated comprehension of knowledge.

Multiplicity

Individuals at the multiplicity stage understand many viewpoints and challenge authority, yet they continue to rely on external sources without critical scrutiny (Saleem et al., 2017). They acknowledge subjectivity, uncertainties in knowledge, and differing opinions in place of absolute truth (Brownlee, 2004). This transitional phase marks the initiation of subjective thinking and conflicts between competing views, involving early and late phases (Black & Allen, 2017; Brownlee, 2004; Gallagher, 2019; Kunwar et al., 2024). Since everyone has the right to their own opinion, students are sucked into a maze of ambiguity where "truth" is determined by subjective opinion rather than judgment and conflicting viewpoints appear to be equally valid (Gallagher, 2019).

Position 3: Early Multiplicity/Subordinate: Early multiplicity is a critical stage in learning where students begin to recognize knowledge uncertainties and diverse perspectives, leading to confusion and relativism (Kunwar et al., 2024). They lack evaluation skills and doubt authority figures, causing anxiety and confusion. As certainty fades, critical reasoning emerges, highlighting the turbulent transition from prior beliefs to a new framework.

Position 4: Late Multiplicity/Correlate: Late multiplicity refers to a period of increased acceptance of diverse knowledge views and their uncertainty. Students recognize the importance of actively evaluating claims and developing defensible judgments. Thought incorporates context and relativism, but ambiguity between opinions can leave students feeling loose. Late multiplicity is distinguished by an increasing ability for relativistic thinking, although it still struggles to resolve doubts and mature critical abilities.

Relativism

Relativism is a critical movement in students' understanding of knowledge, moving away from contextual awareness and toward various interpretations alongside alternatives (Gallagher, 2019; Kunwar et al., 2024). It involves analyzing concepts critically, appreciating the significance of interpretation, and comprehending the process by which opinions are formed (Black & Allen, 2017; Brownlee, 2004; Saleem et al., 2017). Relativism encompasses two stages of growth and incorporates interpretation, contextual judgment, and pluralism into thought (Kunwar et al., 2024).

Position 5: Contextual Relativism/Correlate: Transitioning to relativistic thinking involves students understanding that knowledge is context-dependent and subject to interpretation, with authorities offering opinions based on evidence. This approach encourages critical evaluations and replaces dualistic right and wrong with skilled judgment within pluralism.

Position 6: Development of Relativistic Reasoning/Pre-commitment: Relativism is a learning strategy in which students recognize that knowledge is impacted by context and utilize sophisticated reasoning to assess situations. This shift from absolute to interpretive knowledge comprehension, fostering coordination, critical assessment skills, and contextual thinking, is a crucial phase in pre-commitment thinking.

Commitment

Commitment, the final stage of cognitive growth, entails forming personal values and beliefs, making informed decisions, actively participating intellectually, and matching ideals with disciplinary standards for decision-making (Black & Allen, 2017; Gallagher, 2019; Kunwar et al., 2024). While not all development is linear, Perry's model provides a useful framework for studying cognitive growth, which consists of three developmental phases (Kunwar et al., 2024).

Position 7: Early Commitment: At the commitment stage, students make personal commitments, recognizing the importance of committing to values and beliefs, even in ambiguity. This stage involves a transformative shift, altering perception and being, and ultimately leading to decision-making based on reason and personal knowledge.

Position 8: Mid Commitment: During this stage, the student learns about commitment and its implications, focusing on responsibility and decision-making. They expand their capacity to make commitments in various areas, evaluate their consequences, and resolve conflicts. This stage focuses on expanding one's capacity to make commitments in various domains.

Position 9: Post Commitment: Students learn that commitment is continuous and developing, necessitating a careful balance of numerous obligations and devotion. They integrate ambiguities from relativism into daily life, embracing them as part of personal growth. Ambiguities become integral to student identity, and students no longer resist uncertainty but embrace it naturally with equanimity, grace, and peace of mind.

Similarly, Baxter Magolda (1993a) extended this line of research with a longitudinal study of more than 100 male and female college students, categorizing female college students' epistemological views into four developmental phases (Brownlee, 2004). These include absolute knowledge, which is comparable to Perry's dualism; transitional knowing, which is similar to Perry's multiplicity; independent knowing, which is similar to Perry's relativism; and contextual knowing, which is similar to Perry's commitment. In Perry’s and Baxter Magolda’s categorization of epistemological beliefs, college students must pass through these four developmental stages, indicating that their model of classification was linear and unidimensional.

Schommer (1990), however, challenged this linear view and argued against Perry's proposed unidimensional and developmental approach to epistemic beliefs because of the complexity of the nature of knowledge. Instead, Schommer proposed a multidimensional framework with five dimensions: certainty of knowledge, source of knowledge, structure of knowledge, control of knowledge acquisition, and speed of knowledge acquisition. These five dimensions fall into two main categories that underpin individuals' epistemological theories: beliefs about the nature of knowledge and beliefs about the nature of knowing. The certainty and structure of knowledge reflect the nature of knowing, whereas the source, control, and speed of knowledge acquisition reveal the nature of knowing.

Schommer classified the five components of belief as simple knowledge, certain knowledge, omniscient authority, quick learning, and innate ability. Simple knowledge is the belief in knowledge's structure, which varies from discrete components to integrated ideas; particular authority is the belief in knowledge's dependability, which ranges from fixed to changing. Omniscient authority refers to beliefs in the legitimacy of a source of knowledge, which might be inherited from superiors or based on facts and reason. Innate ability refers to an individual's view that their learning capacity is set at birth or prone to change over time, whereas quick learning refers to the idea that the speed of learning varies from quick or not at all to slow.

According to Schommer-Aikins (2008), learners can display varying levels of complexity across several belief dimensions, highlighting the dynamic and multifaceted nature of knowledge. This showed that learners with sophisticated beliefs view knowledge as a complex, interrelated, and complete structure, whereas naïve learners see it as simple and discrete pieces (Mardiha & Alibakhshi, 2020). Naive learners believe that knowledge is passed down through authority that learning is permanent and hereditary, and that experts are the source of knowledge. Sophisticated learners, on the other hand, think that knowledge is gained not just via professionals but also from lifelong experiences and observations (Schommer-Aikins, 2008). According to Önal and Kırmızıgül (2021), sophisticated epistemological views suggest that enhancing learning requires significant effort.

Based on their significance and a well-cited review of six pioneering epistemological researches (i.e. King & Kitchener, 1994; Baxter Magolda, 1992; Kuhn, 1991; Schommer, 1990; Belenky et al., 1986 and Perry, 1970), Hofer and Pintrich (1997) developed a revised framework of multidimensional epistemological beliefs. By excluding dimensions related to learning or educational experience (e.g., the role of the instructor in Baxter Magolda's model or quick learning in Schommer's model), as well as fixed ability dimensions that were only present in Schommer's model, they identified two dimensions of epistemological beliefs. The nature or process of knowing and the nature of knowledge are both two-dimensional, in which the former consists of certainty of knowledge and simplicity of knowledge, whereas the latter consists of the source of knowledge and justification for knowing.

In Hofer's model, Certainty of knowledge spans from being perfect truth within certainty, which is less complicated, to being tentative in nature, which is more sophisticated. Simplicity of Knowledge spans from tangible, fragmented, and knowable truth (less sophisticated) to relative, contextual, and contingent (more complicated). Knowledge can come from the outside self and reside in external authority, or the knower can develop it through social interaction. The Justification for knowing dimension assesses how students judge knowledge based on observation or authority (less sophisticated) or how they utilize inquiry norms to individually investigate and integrate expert opinions (more advanced).

2.9.2. Domain General and Domain Specific Epistemological Beliefs

As indicated earlier, Epistemological beliefs refer to individuals' beliefs about the nature of knowledge and the process of knowing. These beliefs influence how people approach learning, problem-solving, and knowledge acquisition. The topic of whether epistemic beliefs are domain-general or domain-specific has long served as a guide for study. Researchers distinguish between domain general and domain specific epistemological beliefs to explore whether these beliefs apply uniformly across all knowledge domains or are influenced by the characteristics of specific disciplines. A domain-general approach to epistemological beliefs was common in the early days of study (Perry, 1970; Schommer, 1990), but this perspective has changed significantly over time (Urhahne & Kremer, 2023). Based on philosophical and empirical considerations of epistemic similarities and differences across domains, Muis et al. (2006) provide evidence that suggests beliefs are both domain general and domain specific. Schommer-Aikins & Duell (2013) also stated that epistemological beliefs are both domain general and domain specific. Merk et al. (2018) further argued that epistemological beliefs cover not only domain-general and domain-specific but also topic-specific.

Domain General Epistemological Beliefs: Studies found that epistemological beliefs in the past were treated as if they were domain-general (Baxter Magolda, 1988; Jehng et al., 1993; Perry, 1968). Domain-general epistemological beliefs are overarching principles about knowledge and learning that apply across various fields and contexts. These beliefs suggest that individuals hold overarching beliefs about knowledge that are consistent across different domains. For example, individuals who believe that knowledge is dynamic and evolves overtime might apply this perspective universally, irrespective of whether they are studying mathematics, art or social science (Schommer-Aikins et al., 2003). Schommer-Aikins and Duell (2013) highlight how such beliefs influence general academic behavior, emphasizing that learners with more sophisticated general epistemological beliefs tend to engage in deeper learning strategies.

According to Buehl, Alexander, and Murphy (2002), domain general beliefs provide a foundational framework for understanding knowledge but might lack the nuance necessary for specific fields of study. The generality of these beliefs allows individuals to navigate unfamiliar domains but may fail to capture the complexity of domain-specific practices.

Domain Specific Epistemological Beliefs: Domain-specific epistemological beliefs, on the other hand, are tailored to the characteristics and knowledge structures of specific domains, such as mathematics, physics, or psychology. These beliefs recognize that different fields vary in their methods of inquiry, levels of certainty, and processes of validation. Domain-specific epistemic beliefs are necessary preconditions for comprehending the knowledge and practices in academic fields including history (Stoel et al., 2022; VanSledright & Maggioni, 2016), mathematics (Depaepe et al., 2016), and science (Elby et al., 2016).

Domain-specific beliefs are proposed to have a greater influence on learning processes and outcomes than generic epistemic beliefs (Bråten et al. (2008); Buehl & Alexander, 2006; Muis et al., 2006). This hypothesis proposes that domain-specific beliefs emerge through interaction with domain-specific information and challenges. Berding et al. (2017) also found that domain-specific epistemological beliefs have a stronger impact on teaching, learning processes, and success than domain-general epistemological beliefs. These authors offered evidence that domain-specific epistemological beliefs lay a greater emphasis on the concrete topic knowledge to be acquired and the accompanying learning processes.

In a study of 19 empirical research, Muis et al. (2006) concluded that epistemic beliefs should consider both domain specific and domain universal. These authors found that every study they analyzed supported the idea that personal epistemology is somewhat domain specific. They stated in their theory of integrated domains in epistemology (TIDE) that, domain-specific epistemological beliefs of learners begin to develop when they enter the educational system. Other previous research has also demonstrated that learners' epistemological views are growing more specific and changing along certain dimensions because of their experiences in various fields (Bendixen & Rule, 2004). Urhahne and Kremer (2023) found that learners' exposure to domain-specific knowledge systems alters their understanding of knowledge in that context, which is reflected in the distinctiveness of these beliefs. Their finding also indicated that pre-service teachers' epistemic beliefs across four school topic domains produce a unique profile of epistemic beliefs (Urhahne & Kremer, 2023). These authors also disclosed that as students' views are formed over the course of a few years of education, they learn about the parallels and differences between fields, which implies that by the time they graduate, they should have different epistemic viewpoints in every topic.

2.9.3. Mathematical Epistemological Beliefs

Despite scholarly debate about the domain generality and domain specificity of epistemological beliefs, recent empirical evidence (e.g., Depaepe et al., 2017; Schommer-Aikins and Duell, 2013; Löfström and Pursiainen, 2015; Trakulphadetkrai, 2022) indicates that epistemological beliefs are more likely discipline-specific. Furthermore, domain-specific epistemological beliefs influence learners' academic performance, like mathematics performance, more than domain-general epistemological beliefs. For these domain specific mathematical epistemological beliefs, different scholars proposed distinct views on the nature of mathematical knowledge and process of knowing it. For example, Ernest (1989) proposed three distinct views: an instrumentalist, a Platonist and a problem-solving view. Advocators of instrumentalist view, perceive mathematics as a collection of independent facts, skills, and rules, Platonist view adhere mathematics as a stable body of highly structured and connected knowledge that has to be discovered and those who advocate problem-solving perspectives, consider mathematics as a dynamic, relative and creative human invention.

Blömeke et al. (2008) on the other hand categorized the mathematical epistemological beliefs of teachers into four distinct views: a scheme-related perspective, a formalist perspective, a process-related perspective and an application perspective. According to these authors, a scheme-related perspective considers mathematics as a collection of concepts, rules, and formulas (for example, "Mathematics is a collection of procedures and rules which precisely determine how a task is solved."). A formalist perspective adheres mathematics as an exact discipline with an axiomatic foundation that is formed by deduction (for example, "Mathematical thinking is determined by abstraction and logic."). A process-related perspective considers mathematics as a discipline that involves problem solving and the finding of structure and regularities (For example, it is frequently possible to uncover new connections, rules, and concepts by grasping mathematical issues). Individuals with application related orientation consider mathematics viewed as a science that is useful to society and life (For example, "Mathematics helps to solve daily tasks and problems.").

In 2014, Ernest modified how mathematical knowledge and how it is acquired into two categories by combining instrumentalist and platonist viewpoints into one and calling it absolutist and problem-solving viewpoints fallibilist. Absolutists believe that mathematics exists nearly independently of humanity, governing nature and connecting everything in existence together via its unchanging uniformity throughout time and space (Ernest, 2024). Advocates of the Absolutist perceive mathematical knowledge as absolutely secure, unchanging essence, unquestionable and objective (Youvan, 2023). They present mathematics as a culture- and value-free set of timeless, isolated facts (Ernest, 2024). Individuals with an absolutist worldview are more likely to encourage the execution of routine activities and procedures; they utilize memory to acquire mathematical information (Depaepe et al., 2017). There is limited room for mathematical creativity and innovation; instead, they profit on discoveries (Horton & Panasuk, 2011; Trakulphadetkrai, 2022).

Fallibilists oppose absolutists and believe that mathematics is linked to social practice sets, each with its own history, personalities, organizations, and social contexts, as well as symbolic forms, goals, and power dynamics, with mathematical truth being subject to modification and correction (Depaepe et al., 2017; Ernest, 2014). They no longer think of mathematics as a discipline differentiated by an abstract body of knowledge that exists in an objective, superhuman world (Ernest, 2014). Fallibilists focus on doing mathematics – within the social conventions in a particular context– and its human side, rather than emphasizing the acquisition of a fixed set of mathematical concepts and procedures (Depaepe et al., 2017).

Because mathematical epistemological beliefs are views that people have about the nature of mathematical knowledge and ways of knowing which are primarily represented by source of mathematical knowledge and knowledge justification, the stability of mathematical knowledge, and the structure of mathematical knowledge (Hofer & Pintrich, 1997). However, Trakulphadetkrai (2022) claimed that the two sub-constructs of the process of knowing (sources of mathematical knowledge and knowledge justification) had a substantial overlap. Then, he proposed three sub-constructs concerning mathematical epistemological beliefs. Stability and structure of mathematical knowledge are investigated under the nature of mathematical knowledge, and the source of mathematical knowledge is explored under the process of knowing.

Furthermore, based on Schommer's dimensions, Buehl and Fives (2009) proposed the three unique sub-constructs of mathematical epistemological beliefs described above. Accordingly, mathematics teacher educators' views on the nature of mathematical knowledge and ways of acquiring it in terms of its source, stability, and structure were taken into consideration in this study as mathematical epistemological beliefs. Many scholars have previously characterized epistemological beliefs as sophisticated (knowledge as interwoven, dynamic evolving, reason as source of knowledge) or naïve (knowledge as compartmentalized, ever changing, and authority as source of knowledge). In this study, we substituted naïve with absolutist and sophisticated with falliblist. As a result, this study utilizes philosophical labels to define different mathematical epistemological views concerning the source, stability, and structure of mathematical knowledge, as explained below.

Source of mathematical knowledge: Beliefs regarding the source of mathematical knowledge are critical for learners' learning outcomes and instructional effectiveness (Buehl & Fives, 2009). This belief is concerned with how one acquired knowledge or the origin of the knowledge. From an absolutist perspective, mathematical knowledge is naturally learned and conveyed by authoritative figures viewing students as passive recipients of information, accepting textbooks and professors without question or interaction in real-world situations (Luitel, 2009; Rai & Acharya, 2021). Absolutists maintained that mathematical knowledge existed independently of human or societal cognition, governing nature and holding the universe together by its everlasting uniformity throughout space and time (Hall, 2002). In other words, mathematical knowledge has always been there, just waiting to be discovered. According to this viewpoint, the sum of the squares of the two sides of a right-angle triangle equals the square of the third side, or the ratio of circumference to the diameter of the circle is equal to pi, which existed long before humans existed. Humans are just uncovering mathematical knowledge that has always existed.

Fallibilists, on the other hand, argue that mathematics is a social construct created by people and constantly evolving, with individual learners actively constructing knowledge based on their experiences and reasoning, as per various studies (Ernest, 2014).

Stability of mathematical knowledge: Beliefs about the stability of mathematical knowledge contrast the absolutist and fallibilist perspectives. Absolutists view mathematics as a domain of unchanging truths that transcend time and culture, often emphasizing the need for accuracy and consistency (Lerman, 1990; Ernest, 1995). Absolutists used deductive arguments to describe the feature of their mathematics, including its public image, as value-free, non-ideological, culture-free, objective, certain, and accurate in describing reality (Ernest, 2018; Kennedy, 2018). This view underpins instructional practices focused on transmitting fixed facts and canonical methods, prioritizing precision over exploration (Hall, 2002; Penn, 2012).

The **fallibilist perspective**, on the other hand, sees mathematics as an evolving field subject to refinement and reinterpretation (Ernest, 1995; Peck, 2018). Falliblist also acknowledge the work-in-progress and incomplete nature of mathematical knowledge. For instance, in his paper, De Toffoli defended his point by using the following instances of mathematical proof modifications: In 1993, Wiles attempted to establish the Diophantus' theorem for no three non-zero integers a, b, and c that meet the equation $a^n + b^n = c^n$ for any $n > 2$. However, his approach was flawed and had substantial gaps. He resolved to revise his proof, and after extensive effort with his old student Richard Taylor, they arrived at the proper proof in 1995. Similarly, same author noted that the proof offered by D'Alembert in 1746 for the Fundamental Theorem of Algebra had numerous gaps, however the proof presented by Gauss in 1997 after many years had no such gaps, suggesting that mathematical proofs are amenable to modification.

Teachers with this view encourage their students to engage critically with mathematical concepts, exploring alternative methods and questioning assumptions (Rott & Leuders, 2017). This dynamic view aligns with contemporary philosophies of mathematics education, which value creativity, adaptability, and collaboration (Berding et al., 2017).

Structure of mathematical knowledge: The structure of mathematical knowledge belief examines how mathematical concepts and skills are organized. This structure of mathematical knowledge is often viewed along a spectrum ranging from isolated facts to interconnected networks (Trakulphadetkrai, 2022). The isolationist viewpoint, alongside the absolutist viewpoint, understands mathematics as a collection of discrete principles that need to be remembered and applied separately (Ernest, 1989). Teachers who hold such ideas tend to prioritize procedural mastery, compartmentalizing subjects with little regard for their interrelations (Lamichhane, 2017; Maphutha et al., 2023; Viholainen et al., 2017). Such approaches often fail to foster deep understanding, as they minimize opportunities for students to see the broader relationships among topics and connect mathematical concepts with the real world and with itself (Depaepe, et al., 2016; Shoaib & Akhter, 2020).

On the other hand, the connectionist viewpoint aligned with fallibilist asserts that mathematical knowledge is intrinsically linked; comprehension of one field enhances understanding of others (Askew, 2002). For instance, geometry and algebra are not independent domains but mutually reinforcing fields, as championed in constructivist pedagogies (Ernest, 2014). Studies revealed that connectionist instructors are more likely to employ inquiry-based and problem-solving techniques, which promote deeper conceptual comprehension and student involvement (Trakulphadetkrai, 2022).

Students' epistemological beliefs about the nature of knowledge and ways of knowing are linked to their adoption of surface or deep learning approaches (Hofer & Pintrich, 1997; Schommer. 1990). Those who view knowledge as simple, absolute, objective, certain, incorrigible isolated facts, and with or without accompanying understanding prefer in their learning a surface and an outcome oriented approach that neglect the involvement of students in the process of experimenting, inventing, creating, conjecturing, and meaning-making (Tuge, 2008*)*. Hence, they are unable to mathematize the real situations and mathematics itself and as a consequence, they are less likely to use higher-level cognitive strategies and perform poorly (Cano, 2005).

But those who view mathematical knowledge as dynamically constructed and context-dependent prefer deep learning and a process-oriented approach (Tuge, 2008), and therefore with the assistance of their teacher, students can reinvent mathematical concepts via mathematizing real situations and making mathematics.

However, individuals who believe on the co-existence of both perspectives hold that some knowledge is more tentative than others and that some knowledge will always be certain (Buehl & Fives, 2009). According to these authors, people with these perspectives thought that knowledge originated with and was transmitted by authoritative figures, but on the other end of the spectrum, knowledge is seen as actively being constructed by the individual pre-service teacher based on his or her particular experience and reasoning. In light of this, Godino et al. (2015) provide a synthesis of mathematics education instructional strategies that place an emphasis on students' personal inquiry and knowledge production as well as those of other crucial participants in the knowledge transfer process. In their study, these authors claim that the mixed model (i.e., objectivist model and constructivist models) encourages the acquisition of mathematical knowledge and the teacher's control of the student's developing knowledge.

According to Muis (2004), changing students' math-related epistemological beliefs employed a social-constructivist approach, primarily based on Yackel and Cobb's (1996) theory, which understands the development of these beliefs in terms of socio-mathematical norms and practices. These authors argue that if the classroom norms and practices are a major determinant of the development of beliefs, significantly altering those environments will enhance positive mathematics-related beliefs. The majority of Muis's review involved a quantitative assessment of changes in students' beliefs, based on a typical pretest-posttest design, with very few studies providing qualitative analyses of changes in beliefs. As stated in her review, it was found that implementing constructivist-oriented approaches to teaching mathematics resulted in more availing epistemological beliefs. Even though there were some studies on how to change students' math-related epistemological beliefs, the association between epistemological beliefs and instructional approaches is mainly theoretical, rather than based on extensive empirical evidence.

To measure epistemological beliefs, researchers used closed-ended questionnaires (Baydar, 2020; Tezci et al., 2016), interviews (Rott, 2021; Shoaib & Akhter, 2020), and observation (Shoaib & Akhter, 2020). However, as recommended by Depaepe et al. (2016), triangulation of data through the combination of different data collection techniques might be a fruitful approach in the examination of epistemological beliefs. Thus, in this study, RME-based mathematization rooted in RME theory is used as an instructional approach in changing pre-service teachers' mathematics-related beliefs since it focuses on learning mathematics as a meaningful, human activity rather than as a set of abstract rules. In this approach, pre-service teachers will get an opportunity to engage in real-life and culture-based situations in which they represent, model, and refine their informal reasoning into formal mathematical ideas.

We utilized multiple ways of data collection, like questionnaires to get a holistic picture of the whole class, semi-structured interviews to provide depth (rich explanation of why participants think that way), and classroom observation to get practice-based evidence (enacted practice). Therefore, to get a valid, reliable, and holistic understanding of epistemological beliefs. It is better to utilize all these three together than utilizing any one of them.

2.10. **Geometry Achievement**

In mathematics education research, geometry learning is frequently explored through two complementary lenses: geometric thinking, which represents the learning process, and geometry success, which reflects learning results. Geometric thinking focuses on how learners reason, envision, and create geometric understanding, whereas geometry success measures how well learners apply this information through problem solving, formal reasoning, and applications. Taking into account both parts provides a more complete picture of geometry learning, as learning processes have a direct impact on learning results. Geometry is a branch of mathematics that allows individuals to understand the universe by comparing shapes, sizes, relative positions, properties of space, and objects and their connections (Bassarear, 2012; Gunhan, 2014). It does not only provide a thorough understanding of the universe but also play a magnificent role in comprehending other concepts in mathematics and other subjects (Hamidah, et al., 2025). Geometry achievement is not limited to the person's capability and understanding to perform well in geometry but includes the ability to apply them in solving real-life problems. The quality of learners' geometric thinking processes, such as how they envision, reason, and make sense of geometric relationships during learning, has a significant impact on their accomplishment. It involves developing understanding of geometric relationship, reasoning logically, applying this knowledge in real life and abstract concepts (Gonza´lez & Herbst, 2006). It is closely related to geometric thinking skills because learners with such skills can observe things, build definitions based on the characteristics inherent in objects, recognize relationships between objects, and apply them to solve geometry problems (Ersoy et al., 2019).

These geometric thinking skills represent the cognitive processes through which learners construct geometric knowledge. As part of the learning process, they enable students to move from visual recognition to logical reasoning and abstraction, thereby forming the foundation upon which geometry achievement is built.

One of the key aspects of geometry achievement is spatial reasoning that involves understanding how objects fit and move in space, which is fundamental for learning and applying geometric principles (Putri et al., 2025). Research findings indicated that learners with excellent spatial reasoning abilities perform better in geometry because they can cognitively manipulate geometric objects and visualize intricate shapes (Laski et al., 2013). This skill is essential for understanding geometric transformations such as rotations, reflections, and dilations (Yuliardi & Rosjanuardi, 2021). Spatial reasoning is thus a critical component of geometric thinking that directly contributes to enhanced geometry accomplishment by allowing students to effectively interpret, modify, and apply geometric notions. The other aspects of geometry achievement involve application and problem solving as since geometry achievement is beyond memorizing facts and formulas (Putri et al., 2025). The capability to comprehend, apply, and prove geometric theorems is another aspect of geometry achievement. Learners develop a deeper understanding of geometry concepts and strengthen their thinking skills through the process of proving geometric theorems (Peligro et al., 2018).

Collectively, geometry achievement should be seen as the result of well-developed geometric thinking processes. Instruction and assessment can encourage meaningful learning processes and robust learning outcomes in geometry by stressing both how students think about geometry and how they demonstrate their understanding.

2.11. **Problem Solving Skills in Geometry**

Problem solving is a key aspect of mathematics learning, particularly geometry (Suarsana et al., 2019); allowing learners to practice and integrate previously learned concepts, theorems, and skills (Tambunan, 2019). Problem-solving skills in solving geometry problems are the learners' ability to thoroughly comprehend the problem, assess the available information, and use appropriate techniques in synthesizing the solution to the geometry problem. These problem-solving skills require visualization, reasoning, and spatial comprehension to solve problems involving shapes, sizes, and their attributes (Aziz et al., 2020). Nevertheless, such proficiency in solving geometry problems is beyond memorization of formula and needs the development of problem solving strategies, creativity and mathematical thinking, and the ability to apply abstract concepts in practical scenarios (Rejeki et al., 2021). Lestariningsih et al. (2018) stated the steps in solving mathematical problems as follows:

(1) Formulating a problem of real-world context into a mathematical problem,

(2) Using the facts, concepts, procedures, and mathematical reasoning to obtain the mathematical solution of the problem mathematically,

(3) Interpreting the mathematical solution to the real-world context to the initial problem, and

(4) Evaluating a solution to real-world context to the problem.

2.12. **Research on RME based Mathematization and Epistemological Beliefs**

In a study aiming at modifying pre-service primary school teachers' epistemological beliefs in mathematics, Gill et al. (2004) found that the treatment group receiving the instructional intervention (refutational text) displayed a stronger change in implicit epistemological beliefs than the control group. In their longitudinal study over two years, Charalambous et al. (2009) used the history of mathematics as an intervention to change the mathematical epistemological beliefs of pre-service teachers. The authors used questionnaires as well as semi-structured interviews as instruments of data collection on the three components of mathematical epistemological beliefs: instrumentalist view, platonic view, problem-solving view proposed by Ernest (Ernest, 1989).

To acquire more information about changes in epistemological beliefs, a follow-up semi-structured interview with six participants was undertaken. According to the findings, pre-service teachers held strong Platonic ideas about mathematics, perceiving it as a fixed set of truths. Exposure to mathematics history only partially induced the expected changes in pre-service teachers' epistemological beliefs. After two years of teacher training, pre-service teachers had weaker Platonic epistemological beliefs, but their problem-solving beliefs also declined.

Yuanita, et al. (2018) used RME-based mathematization as an intervention strategy to change secondary school students’ mathematical beliefs and their findings showed that the use of RME-based mathematization has a positive effect in changing students’ mathematical belief as compared to the traditional approach. In line with the above study, Yuanita and Zakaria (2016) also conducted quasi-experimental research on 645 students at a Junior High School on the effect of RME-based mathematization on students’ mathematical beliefs, and their findings indicated that there are significant differences among students in their mathematics beliefs. Furthermore, the finding of Dickinson and Hough (2012) found that RME based approach plays a significant role in changing teachers’ mathematical epistemological beliefs.

Based on the studies discussed in this part, RME-based mathematization is conceived as an instructional strategy that frames mathematics as a meaningful, problem-solving, and changing human activity rather than a set body of information. The use of RME-based mathematization—through contextual problems, guided reinvention, and student-centered exploration—has been shown to positively influence learners' and teachers' epistemological beliefs by weakening instrumentalist and Platonic views and promoting problem-solving-oriented beliefs at various educational levels. These findings indicate that

RME-based mathematization has an important role in altering how mathematics is viewed and comprehended in instructional practice.

2.13. **Research on RME-based Mathematization and Problem-solving Skills in Geometry**

Mathematical problem solving is the process of mathematically interpreting a situation that entails articulating, testing, and revising the mathematical interpretation as well as grouping, integrating, changing, revising, or refining clusters of mathematical concepts from many disciplines both within and outside of mathematics. It needs high-level thinking process that combines intellectual aptitude and significant cognitive processes (Bahar & Maker, 2015; Jupri & Drijvers, 2016). It refers to students' capacity to comprehend problems, plan problem-solving strategies, implement chosen strategies of completion, and re-examine problems to subsequently come up with new solutions or develop existing ones (Kuzle, 2013). Students can make use of previously acquired mathematical knowledge as a foundation to solve new problems. It is one of the twenty-first-century skills that students should develop through learning mathematics. RME encourages students to use self-developed models to solve and interpret problems that are experientially real in the setting. The teacher guides the student informally throughout class interactions (Dawkins, 2015). It is an approach to learning mathematics that uses contexts that experientially real to students as a mechanism to help them better understand mathematical problems.

Taking this into consideration Manurung, et al. (2020) conducted a study on the effectiveness of RME-based mathematization to enhance students' mathematical problem-solving skills and their result showed that group treated with RME-based mathematization approach outperform the conventional one indicating that RME-based mathematization boosts students problem-solving skills. Pratiwi and Widjajanti (2020) in their review study indicated that realistic mathematics education based mathematization increases students mathematical problem-solving skills. In addition to these, Yuanita, and Zakaria (2016) conducted a quasi-experimental study to determine the effectiveness of RME-based mathematization approach to mathematical problem-solving skills, and their result showed that groups treated with RME-based mathematization significantly outperformed the conventional one.

Overall, the studies examined in this section show that RME-based mathematization is consistently used as a problem-centered instructional approach in which students actively engage with experientially real contexts, develop informal strategies, and gradually refine these strategies into formal mathematical solutions. The findings suggest that such implementations of RME-based mathematization greatly improve students' mathematical problem-solving abilities by encouraging deeper knowledge, strategic flexibility, and reflective thinking. This shows that RME-based mathematization can successfully enhance the development of high-level cognitive processes required for geometry problem solving.

2.14. Researches on RME-based Mathematization and Geometry Achievement

The teaching process in the RME-based mathematization approach begins with real-world problems, and students arrive to the information required through the process of problem-solving. Previous research (Abubakar et al 2021; Aksu & Colak, 2021; Hakim & Sitepu, 2024) commonly note the beneficial impacts of the RME-based mathematization strategy on student geometry academic achievement.

In addition to the above mentioned, Laurens, et al. (2017) conducted quasi-experimental study aimed to investigate the difference in students’ mathematics cognitive achievement after implementing RME and conventional learning. Their result revealed that students who engaged in RME-based mathematization achieved better result than those who taught with the conventional one.

Furthermore, Aksu and Colak (2021) also conducted a study on 8th grade students to answer if there is a reasonable difference in the geometry achievement of students who were taught with the traditional and RME-based mathematization approaches. It was shown that there was significant mean difference between groups taught with RME-based mathematization and those taught with traditional group. This shows that learning activities prepared based on RME-based mathematization approach are much more effective than learning activities prepared based the traditional approach on students' academic success in geometry.

The studies reviewed in this section, taken together, suggest that the implementation of RME-based mathematization is done through instructional design that starts from real-world problems and leads students to formal geometric understanding through progressive mathematization. The common finding of the studies reviewed in this section is that students who underwent RME-based mathematization outperformed students who were instructed using traditional methods in geometry achievement. This finding indicates that the use of RME-based mathematization not only leads to procedural understanding but also to meaningful application of geometric knowledge.

The body of research on RME-based mathematization and mathematical epistemological beliefs, problem-solving skills, and geometry achievement as a whole suggests that RME-based mathematization is consistently conceptualized as an instructional strategy that combines real-world applications, student-centered problem-solving, and progressive mathematization to facilitate effective mathematics learning. In the body of research that has been reviewed, RME-based mathematization has been found to have a positive impact on epistemological beliefs, improve mathematical problem-solving skills, and geometry achievement. These findings suggest that RME-based mathematization has a dual role in facilitating both learning processes and outcomes.

Despite the growing international evidence for the effectiveness of RME-based mathematization, very little research has been conducted on its application in the Ethiopian school context. The teaching of mathematics in Ethiopia has been dominated by teacher-centered learning, which focuses on procedural skills and examination results. In this regard, the principles and practices of RME-based mathematization provide a refreshing alternative that promotes contextual learning and understanding. Nevertheless, empirical evidence on the impact of RME-based mathematization on the epistemological beliefs, problem-solving abilities, and geometry achievement of Ethiopian students is very limited.

2.15. RME-based Mathematization in the Ethiopian School Context

In many Ethiopian classrooms, students are taught mathematics using the traditional lecture method, which only prepares students for exams (Dejene, et al., 2018; Girma & Anagaw, 2020). This approach uses little to no real-life problem as a source of mathematical activity, pays little attention to applications, and focuses on rote learning. In this approach, students are not accustomed to using real/concrete models, manipulatives, images, figures, tables, graphs, and symbols to solve mathematical problems. Knowledge is transferred from the teacher and students are passive listeners. Since there is no proper interaction among students, and student and the outsid world, they face difficulty in developing mathematical problem-solving skills. As a result, they are challenged with oral communication, visual communication, textual communication, and symbolic communication. In this approach mathematical knowledge is considered as absolute and incorrigible; teachers and books are the main sources of mathematical knowledge.

The mechanistic metaphor treats the teacher as the sower, knowledge as a seed, and student as a field or considers the teacher as a mother who fetches water from spring, knowledge as water, and student as an empty jar. This mechanistic approach prevents pupils from solving mathematical problems using their own prior experience and knowledge; unable to make students formalize, axiomatize, generalize, and construct mathematical concepts on their own, resulting in a horizontal and vertical mathematization deficiency.

Tuge and Garoma, (2016) researched 172 mathematics teachers of grades 5-8 to examine their mathematical beliefs and the way they view problem-solving approaches in the mathematics classroom. They used questionnaires, observations, and interviews as instruments of data collection and their finding indicated that most of the teachers who took part in the survey view mathematics as absolute and certain (70%), independent of human culture (67%), and (67%) of the respondents consider teachers and textbooks as authoritative sources of mathematical knowledge in the classroom.

Tuge and Garoma (2016) stated what is commonly done in mathematics classroom from primary to tertiary level is that majority of Ethiopian mathematics teachers demonstrate a formula, an algorithm, or a technique, then show a few examples of how to solve problems using formulas and algorithms. And then, based on the algorithm/rules, students are given similar problems in the form of worksheets/exercises/ and solve the given problems by plugging numbers to formula and procedures. At the tertiary level, the prepared worksheets are typically designed to assist students in applying the algorithms they've learned, and they may also include some context problems that need their application. In this structuralist approach, students are instantly placed in the more formal vertical mathematization process in actual mathematics teaching, they do not have an opportunity to go through horizontal mathematization process like understanding the problem and identifying mathematical concepts in the given problem, and solving it informally based on their prior knowledge. The only option students have is memorizing the formula or algorithm they were taught and applying it accordingly. This is comparable to someone being shown and told what is on the other side of a river and expected to use it for their profit. They are not, however, given or shown the bridge that aids in crossing to the other side and making proper use of what is available. This link is provided by the horizontal mathematization bridge.

Even though the study of Tuge and Garoma (2016) focused on the epistemological beliefs of mathematics teachers and their finding indicated that the views of the majority of the teachers regarding the nature of mathematics were absolutist, the study did not tell us the ways to change mathematics teachers' epistemological beliefs. Thus, one of the focuses of this study is on changing pre-service teachers' views on the nature of mathematics through RME-based mathematization.

2.16. **Research Gaps**

From the review, particularly from the Ethiopian perspective, it was obtained that, even though the majority of the Ethiopian people live in the rural areas, cultural mathematics was not given due attention, and it is not practiced in the school, and teachers were observed facing difficulty in practicing these in their classrooms. Hence researchers recommended that pre-service, as well as in-service teachers, should practice during their stay in the teacher education program (Tegegne, 2015; Tesfamicael et al., 2021). In the Ethiopian context, almost all researchers have attempted to unearth the indigenous mathematical knowledge through cultural activities and have attempted to determine how much of this knowledge is incorporated into the curriculum. However, to the best of the researcher's knowledge, there has been a dearth of attempts to implement the explored indigenous mathematical knowledge in actual teacher education classroom settings and evaluate their efficacy. From the review, it was also found that there are contrasting results where RME-based mathematization has no significant effect on students' mathematical achievement (Cengiz and Eğmir, 2022; Hough et al., 2017). Their findings contradicts the finding of (Aksu & Colak, 2021; Laurens et al., 2017) who stood to have findings in favor of significant effect of RME-based mathematization on students' mathematics achievement. Thus, this study is intended to extract the way cultural people mathematize their cultural activities, connect it to pre-service teacher education, and examine its effect on the pre-service teachers' mathematical epistemological beliefs.

2.17. **Theoretical Framework**

Mathematics is not an independent and culturally neutral discipline but is rooted in the social and cultural practices of human life (Bishop, 1993; D'Ambrosio, 1990). Marketing, building houses, crafting, weaving, pottery, designing, and other social interactions are all involved in the development of mathematical knowledge. Culture and social interactions are basic and fundamental in the development of mathematical knowledge and understanding (d'Entremont, 2015). Mathematics is not only seen as a cognitive and individualistic pursuit but is also rooted in culture and is influenced by culture (Berch et al., 2018). Thus, mathematical knowledge, concepts, and techniques are seen as inventions of humans that emerged out of the need to engage with culture and cultural practices (Pradhan, 2020). However, traditional mathematics education systems, especially in Ethiopia, have neglected culturally embedded mathematical knowledge, creating a disconnection between students' realities and mathematics education. The theoretical framework is trying to address this disconnection and link mathematics with sociocultural and contextual realities. Within this framework, the exploration phase is theoretically grounded in *ethno-mathematics* and aims to systematically identify mathematization processes and mathematical concepts embedded in cultural activities, which serve as the foundational basis for the subsequent RME-based mathematization instruction.

In this sociocultural perspective on mathematics, various cultural groups have developed unique forms of mathematization of their experiences of living. People from different cultures use classification, quantification, measurement, explanation, comparison, inference, generalization, and evaluation in various ways (D'Ambrosio, 2006). These processes, which include counting, measuring, locating, playing, and designing, are essential parts of living and can be considered cultural products and sources of mathematical development (Bishop, 1997; Grattan-Guinness, 2004; Pradhan, 2020).

The term *ethno-mathematics*, first introduced by D'Ambrosio (1985), is considered a theory that helps us understand the concept of mathematization and how people from different cultural groups understand and express mathematical concepts meaningfully. Orey and Rosa (2010) describe *ethno-mathematics* as a program of study that examines how individuals conceptualize, communicate, and solve mathematical problems within their cultural environments. In this way, *ethno-mathematics* contests the hegemonic paradigms of traditional mathematics education by highlighting the production of mathematical knowledge as a cultural process (Rosa & Orey, 2011; Zito & Albanese, 2025).

Bishop (1988, 1997) expanded the idea of mathematics as a cultural product and described six universal mathematical practices that exist in all cultures: counting, measuring, locating, designing, playing, and explaining. These practices form a basic framework for understanding the development of mathematical ideas from cultural problem-solving activities.

In this study, Bishop's six universal mathematizing processes serve not only as analytical tools for examining the mathematical concepts and mathematization processes embedded in cultural activities but also as a conceptual bridge between cultural activities and instructional design, turning informal societal mathematization into experientially real starting points for RME-based classroom activities. These processes are examples of informal involvement with mathematics based on experience and thus represent horizontal mathematization, in which the meaning of mathematics is rooted in the learners' experience, not in the mathematics of the classroom.

Although *ethno-mathematics* offers a perspective on the cultural origins and meanings of mathematical activity, it does not answer the question of how this knowledge can be developed systematically in the classroom. This is the role of Realistic Mathematics Education (RME). RME shares *ethno-mathematics'* emphasis on meaningful contexts but extends it by offering an instructional framework for transforming informal, culturally grounded mathematical activity into formal mathematical knowledge.

RME is a domain-specific theory developed by Freudenthal and grounded in socio-constructivist principles. It treats mathematics as an activity that is acquired by organizing and mathematizing real-life situations (Freudenthal, 1973; Gravemeijer, 1994). Unlike other approaches that treat mathematics as an abstract subject, RME enables learners to acquire mathematics by constructing their own understanding through real-life problem situations that are experientially real. Learners acquire informal strategies that are later refined through reinvention to form more formal mathematics (Gravemeijer & Doorman, 1999). In this way, RME operationalizes *ethno-mathematical* insights within classroom instruction.

RME emphasizes both horizontal and vertical mathematization. Horizontal mathematization involves learners' interpretation and structuring of culturally meaningful situations, while vertical mathematization involves the progressive formalization, generalization, and abstraction of these informal strategies. Thus, RME provides the instructional mechanism through which culturally embedded mathematical practices identified by *ethno-mathematics* are transformed into structured mathematical knowledge. This interconnection supports the study's intervention aimed at enhancing pre-service teachers' geometry achievement and problem-solving skills.

By emphasizing the cultural origins and human production of mathematical ideas during both exploration and instruction, the framework implicitly challenges absolutist views of mathematics and lays the groundwork for epistemic transformation during the intervention stage. Beyond cognitive outcomes, the framework also considers how engagement in RME-based mathematization influences mathematical epistemological beliefs, that is, beliefs about the nature of mathematical knowledge and how it is acquired. Such beliefs are known to influence learning processes and instructional practices (Eichler & Erens, 2015; Feucht & Bendixen, 2010). Ernest (1989) initially described three views of mathematics—the Platonist, instrumentalist, and problem-solving views—which he later synthesized into two overarching epistemological perspectives: absolutist and fallibilist (Ernest, 2014). The absolutist perspective views mathematics as objective, fixed, and culture-free, whereas the fallibilist perspective views mathematics as socially constructed, evolving, and open to revision.

Other typologies, such as those proposed by Blömeke et al. (2008), similarly distinguish between absolutist-oriented views (scheme-related and formalist) and fallibilist-oriented views (process-related and application-oriented). In this study, epistemological beliefs are examined through three components: the source of mathematical knowledge, the stability of mathematical knowledge, and the structure of mathematical knowledge.

In the Ethiopian context, mathematics instruction has predominantly reflected an absolutist epistemological orientation, emphasizing fixed knowledge transmitted by authoritative sources such as textbooks and teachers (Tesfamicael, 2021; Tuge & Garoma, 2016; Weldeana, 2016). This perspective has led to rote memorization and minimal participation in meaningful, context-based mathematics learning (Ahmed et al., 2020; Bekele, 2016; Dejene et al., 2018).

This approach regards RME-based mathematization, which originated in *ethno-mathematical* contexts, as an epistemic transformation tool. By engaging pre-service teachers in culturally meaningful mathematical activities and guiding them through horizontal and vertical mathematization, RME emphasizes mathematics as a human creation (source), reveals its dynamic and revisable nature (stability), and promotes interconnected conceptual understanding. Consequently, the framework assumes that sustained engagement in RME-based mathematization supports a shift from absolutist toward fallibilist epistemological beliefs.

Overall, this theoretical framework integrates *ethno-mathematics*, realistic mathematics education, and mathematical epistemological beliefs into a coherent structure. *Ethno-mathematics*, particularly Bishop's six universal mathematizing processes, provides the conceptual lens for identifying mathematical concepts and informal mathematization embedded in cultural activities. These culturally grounded practices constitute the basis for horizontal mathematization. Realistic Mathematics Education builds on this foundation by offering a pedagogical pathway through which informal, culturally situated mathematical activity is transformed into formal mathematical knowledge via guided reinvention and vertical mathematization. As a result, the exploration phase findings guide the design of the RME-based intervention, ensuring that vertical mathematization in the classroom originates from culturally grounded horizontal mathematization rather than decontextualized or externally imposed circumstances. Engagement in this RME-based mathematization process is expected not only to enhance pre-service teachers' geometry achievement and problem-solving skills but also to influence their mathematical epistemological beliefs by foregrounding mathematics as humanly constructed (source), evolving (stability), and conceptually interconnected (structure). In this way, the framework positions RME-based mathematization rooted in ethno-mathematical contexts as the central mechanism linking cultural mathematical practices, instructional processes, epistemological transformation, and learning outcomes.

2.18. Conceptual Framework

The conceptual framework of this study is grounded in the view that mathematics is a human activity situated in a social and cultural context, and that meaningful mathematical learning emerges when instruction is rooted in learners' lived experiences. From this perspective, mathematical knowledge is not merely transmitted but actively constructed through engagement with meaningful contexts. Accordingly, the framework conceptualizes integration as a systematic and theoretically informed process through which mathematical knowledge and mathematization practices embedded in Ethiopian cultural activities are identified, interpreted, and transformed into instructional starting points within pre-service mathematics teacher education through a Realistic Mathematics Education (RME)–based mathematization approach. The framework is consistent with the mixed-method exploratory sequential research design used in the study. In this design, an initial qualitative investigation of cultural mathematization informs the subsequent instructional intervention and quantitative assessment of learning results, establishing coherence between theory, methodology, and practice.

In the first phase of the research, mathematical notions and mathematization processes contained in certain Ethiopian cultural activities—such as home construction, basketry, weaving, and pottery—are thoroughly explored. These activities involve rich mathematical engagement arising naturally through measuring, designing, shaping, arranging, and optimizing materials within real-life sociocultural contexts. Plane geometry notions such as points, lines, angles, triangles, quadrilaterals, polygons, circles, parallel lines, perimeter, and area are integrated in these activities, as are chosen parts of solid geometry such as surface area and volume. The activities also incorporate patterns, geometric progression, and transformational geometry, which includes translation, rotation, reflection, and dilation. Within the paradigm, these types of engagement are conceptualized as informal and experience-based mathematization, in which mathematical meaning is generated through interaction with cultural practices and the physical environment.

The mathematical ideas identified in the cultural activities are then examined in relation to the formal geometry curriculum of pre-service mathematics teacher education programs. The topics included in the geometry curriculum are points, lines, planes, triangles, congruent and similar figures, quadrilaterals, properties of quadrilaterals, polygons, circles, perimeter, area, and some topics of solid geometry. Though solid geometry is also included in both cultural practices, as well as in the curriculum, the main emphasis of the present study is on plane geometry. This alignment demonstrates that cultural mathematization is not external to academic mathematics but constitutes a meaningful and legitimate foundation for formal curricular content when appropriately interpreted and integrated.

Building on the findings of the exploratory phase, the second phase of the study involves designing and implementing an RME-based mathematization instructional intervention. In this intervention, culturally grounded problem situations derived from societal practices are used as instructional starting points. These problems allow pre-service teachers to engage in mathematization processes that progressively relate informal, context-bound reasoning to formal mathematical concepts, representations, and language. Pre-service teachers are helped to reinvent geometric notions through purposeful engagement rather than passive reception, guided by the basic principles of RME—guided reinvention, didactical phenomenology, and emergent models.

Within this instructional process, horizontal and vertical mathematization function as key learning mechanisms. Horizontal mathematization occurs as pre-service teachers translate realistic, culturally meaningful situations into mathematical representations using informal strategies. Vertical mathematization follows as these representations are refined, generalized, and formalized, leading to the development of more structured and interconnected geometric knowledge. Emergent models serve as cognitive bridges between these two forms of mathematization, supporting conceptual understanding and progressive formalization.

Engagement in RME-based mathematization is expected to influence pre-service teachers' geometric problem-solving skills by fostering active reasoning, strategy development, modeling, and justification. Solving culturally relevant geometry problems enables students to interpret events, construct images, and consider solution approaches, making them better problem solvers for both contextualized and formal geometric problems. Meanwhile, it is assumed that regular exposure to this teaching strategy also impacts pre-service teachers' mathematical epistemological beliefs, including beliefs about the nature, source, and certainty of mathematical knowledge. In that sense, pre-service teachers, as they experience mathematics as something that can be constructed, explored, and made relevant to their real-world experiences, are assumed to transform their beliefs from viewing mathematics as an authority-driven body of rules to viewing it as a coherent, evolving, and meaningful human endeavor.

Geometric problem-solving skills and mathematical epistemological beliefs are characterized in the framework not just as outputs of RME-based mathematization, but also as interconnected mediators of geometry achievement. Improved problem-solving abilities enhance deeper conceptual comprehension and flexible application of geometric concepts, whereas more nuanced epistemological views encourage persistence, sense-making, and introspective involvement with mathematical problems. In this way, geometry achievement is viewed as a function of the combined and synergistic effects of RME-based mathematization education, problem-solving skills, and mathematical epistemological beliefs.

The conceptual framework illustrates the ways in which cultural practices, mathematization, pedagogical strategies, epistemology, and learning outcomes are intricately interconnected in pre-service mathematics teacher education in Ethiopia. It offers a comprehensive and coherent theory base for examining the possibilities of integrating informal sociocultural mathematics and formal academic learning in a more meaningful and effective way in mathematics teacher education.

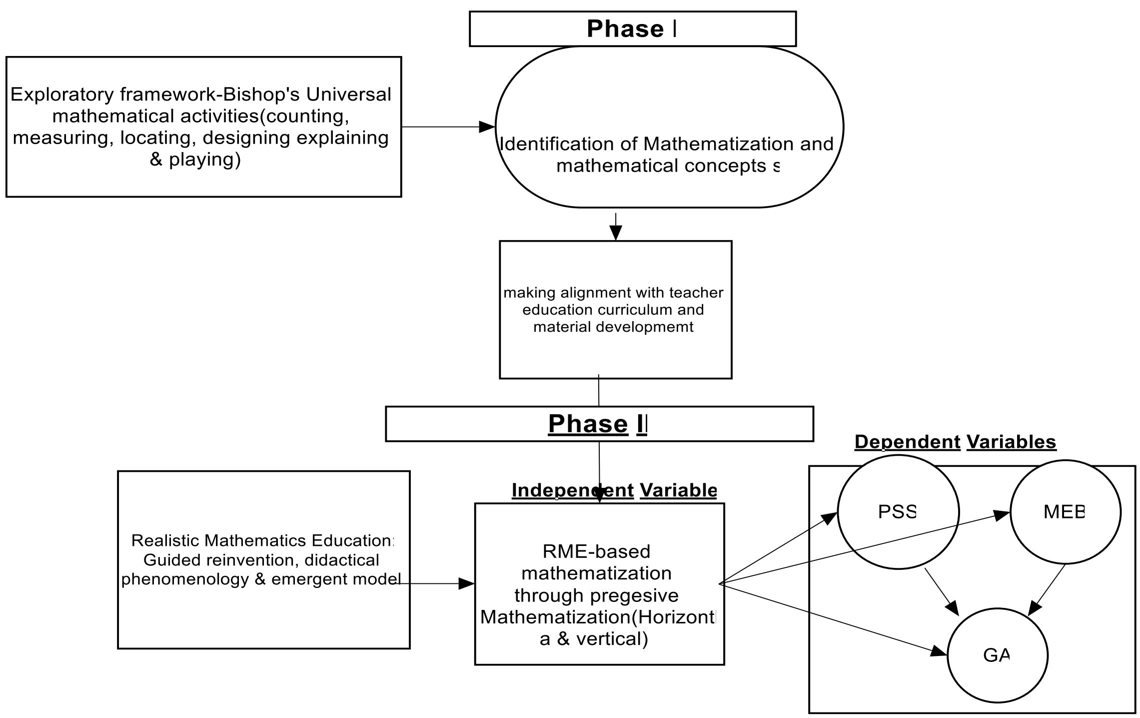

Figure 2.3 Conceptual frameworks for the study

CHAPTER 3: RESEARCH METHODOLOGY

3.1. Introduction

This chapter begins by conceptualizing research methodology as a systematic and scientific way to address research problems, defining the nature of researchable problems, testable hypotheses, and appropriate research designs and data collection methods (Tashakkori & Creswell, 2007; Teddlie & Tashakkori, 2011). It explains not only the practical steps involved in the research process but also the philosophical underpinnings that shape choices about data gathering, analysis, and interpretation (Kothahari, 2004). Methodology is therefore understood as a comprehensive strategy that governs every stage of the inquiry, from framing the research approach and paradigm to choosing data collection techniques and analysis methods.

This conceptualization of methodology aligns closely with the study's theoretical and conceptual frameworks. The theoretical framework, based on Freudenthal's Realistic Mathematics Education (RME) and Bishop's six universal mathematizing activities, informs the study's dual focus: firstly, exploring mathematization embedded in cultural activities, and secondly, designing and testing an RME-based intervention in teacher education. Similarly, the conceptual framework structures the study into two phases: a qualitative ethnomathematical exploration (Phase I) and a quantitative intervention informed by the first phase's findings (Phase II). Thus, the methodology provides a cohesive roadmap that integrates philosophical assumptions with practical research steps, enabling an in-depth exploration of cultural mathematization and its pedagogical application.

3.2. **Study Area and Sources of Data**

The study was conducted in Oromia Regional State, focusing on North Showa, Arsi, and Bale Zones, which were chosen due to the prevalence of cultural practices such as pottery, weaving, basketary, and traditional house building activities central to understanding mathematization as a culturally embedded process (Bishop, 1988b). Data related to these traditional pursuits were collected from community members residing in Fitche town administration, Sherero, Wucale, Degem woredas (North Showa), Arsi Robe town and Hetosa (Arsi Zone), and Robe town and Goba (Bale Zone).

The researcher's familiarity with these areas and their cultural practices further supported a rich and contextually grounded qualitative exploration consistent with the study's first phase of the conceptual framework. For Phase II, pre-service mathematics teachers from CTE1 and CTE2 provided data. These institutions were selected because they are the only regional colleges offering mathematics degree programs and draw students from diverse rural backgrounds with similar socioeconomic statuses and educational histories.

This demographic homogeneity provides a meaningful context to test the RME-based mathematization intervention in teacher education, addressing documented limitations in Ethiopia's teacher-centered, rote learning system that often neglects cultural relevance (Ahmed et al., 2020; Bekele, 2016). The linguistic context Afaan Oromoo as the medium in early grades and teacher education, and English in grades 9 to 12 also underscores the necessity of culturally grounded pedagogy such as RME. This strategy and setting enables an in-depth understanding and effective testing of culturally relevant mathematization in educational contexts, directly connecting with the conceptual framework's two-phase approach.

3.3. **Research Paradigm**

Research paradigms encompass the philosophical worldviews that guide how researchers understand reality (ontology), acquire knowledge (epistemology), conduct research (methodology), and assign value (axiology) (Mackenzie & Knipe, 2006; Kivunja & Kuyini, 2017; Patton, 2002). These paradigms influence decisions on research design, data collection, analysis, and interpretation (Guba & Lincoln, 1994). Given the dual nature of this study, the research paradigm integrates multiple philosophical stances.

The first phase, which explores mathematization in local cultural activities, aligns with the interpretivist paradigms. This paradigm recognizes knowledge as socially constructed, context-dependent, and culturally mediated (Lincoln & Guba, 1985; Denzin & Lincoln, 2018). This perspective supports qualitative methods such as interviews and observations and reflects the ethnomathematical emphasis of the theoretical framework, which views mathematical knowledge as embedded in cultural practices. The researcher engages as an insider, co-constructing knowledge with participants, a stance vital for understanding cultural mathematization authentically.

The second phase fits within the pragmatist paradigm, as a problem-centered philosophical stance in which "what works" refers to methods and explanations that meaningfully address the research questions and have practical (Creswell & Plano Clark, 2018). This paradigm prioritizes the selection of methods depending on their effectiveness in resolving the research challenge over adherence to a single ontological or epistemological perspective. This study presents a coherent framework for combining quantitative and qualitative methodologies to investigate the outcomes and processes of RME-based mathematization. This aligns with Freudenthal's RME theoretical emphasis on mathematization as a teachable human activity and the conceptual framework's intervention design aiming to modify preservice teachers' epistemological beliefs, problem-solving skills, and achievement in geometry.

Recognizing the complexity of integrating these distinct paradigms, the study adopts dialectical pluralism, which fosters dialogue between interpretivist and pragmatist perspectives (Creswell & Clark, 2018; Cronenberg, 2020). This pluralistic worldview is necessary to adequately address the layered research questions and methodologies involved, reflecting the study's commitment to both rich qualitative insight and rigorous quantitative analysis.

3.4. **Research Approach and Design**

A mixed methods approach underpins this study, combining qualitative and quantitative strategies to provide a more comprehensive understanding than either method alone (Creswell & Plano Clark, 2007). Specifically, an exploratory sequential design was employed, where qualitative data from Phase I inform the quantitative intervention and analysis in Phase II. This design is well-suited for the research problem, as it begins by exploring the understudied mathematization embedded in society's cultural activities through qualitative case study methods such as interviews, observations, and field notes.

This exploration is grounded in Bishop's six mathematizing activities, which frame the data analysis and provide culturally sensitive insights to identify mathematical concepts relevant for instructional design. Building on this foundation, the second phase implemented a quasi-experimental, nonequivalent comparison group method with embedded qualitative components (Kanga et al., 2015). Using simple lottery method, CTE1 randomly assigned as the intervention group, receiving RME-based mathematization instruction, while CTE2 assigned as the comparison group, taught via conventional methods.

Efforts were made to control for pre-existing group differences to strengthen validity. Data collection for this phase included a geometry achievement test, a mathematical epistemological beliefs questionnaire (based on Ernest's absolutist and fallibilist framework), open-ended realistic geometry problems, classroom observations, and semi-structured interviews. Quantitative data were analyzed using inferential statistics to assess intervention effectiveness, while qualitative data were thematically analyzed to capture participant experiences and inform instructional processes.

This mixed design effectively integrates the theoretical framework's dual focus on culturally grounded mathematization and RME-based pedagogy, allowing the conceptual framework's two-phase structure to be fully operationalized and evaluated.

3.5. **Sample and Sampling Techniques**

In alignment with the exploratory sequential mixed methods design outlined in the preceding sections, appropriate sampling techniques were carefully used to ensure the collection of rich, contextually relevant, and generalizable data. Considering factors such as availability, proximity, and accessibility, the researcher purposively and conveniently selected North Shoa, Arsi, and Bale Zones as research sites for Phase I. These areas were identified for their rich cultural traditions and practical activities involving pottery making, weaving, basketry, and traditional house building activities central to understanding culturally embedded mathematization processes as framed by Bishop's (1988b) six universal mathematizing activities and Freudenthal's Realistic Mathematics Education (RME).

Study participants within these cultural communities were initially identified through purposive sampling, based on their knowledge, experience, and involvement in traditional activities that entail implicit mathematical reasoning. As participants were invited to assist in identifying further suitable informants, a snowball sampling strategy was subsequently employed. This approach allowed the researcher to access additional participants with rich cultural knowledge and experience, thereby enhancing the depth and contextual validity of the qualitative data.

In Phase II, CTE1 and CTE2 were purposively selected as research sites. These two institutions were identified as the only colleges in Oromia Regional State offering degree programs in mathematics education.

The two colleges taking part in the research had similar characteristics in terms of institutional characteristics. In terms of institutional resources used for the research, it is evident that the colleges were similar in their well-equipped resources for the implementation of the teaching plans. In addition, the year the two colleges were established was similar.

In terms of human resources, all instructors who were involved in teaching different geometric-related courses at both institutions were found to possess at least a Master of Science (MSc) in Mathematics or Mathematics Education. In addition to this, the years of service between instructors were found to be relatively comparable between both institutions. This means that there were no significant differences between instructors at different institutions. All instructors from both institutions were also found to have undertaken the Higher Diploma Program.

Furthermore, the two colleges used the same national curriculum, similar course modules, and exact entry criteria for pre-service teachers entering the program. Among other factors, implying that there was reasonable assurance that the two colleges were similar in institutional context, instructional capacity, and student composition, and thus the variation in learning outcomes can be safely attributed to the intervention rather than institutional differences.

This made them ideal contexts for implementing and assessing an RME-based intervention aimed at transforming preservice teachers' epistemological beliefs, problem-solving strategies, and geometry achievement, as envisaged in the conceptual framework.

From these two teacher education colleges, one was randomly assigned to serve as the intervention group and the other as the comparison group. This random allocation was intended to minimize potential biases and ensure the comparability of groups prior to the intervention, addressing methodological rigor within the quasi-experimental method adopted in Phase II.

Beyond the first phase of the research that used explorative approach, qualitative methods were used with in the second phase to generate meaning and build deeper insights on the mathematization, problem solving skills and epistemological beliefs endowing that of the RME-based intervention. To this effect, the qualitative component within the second phase used seven preservice mathematics teachers from the three groups of lower, medium, and higher academic achievers who were purposively selected for in-depth interviews. Furthermore, the teacher educator who implemented the intervention was also interviewed, particularly regarding the mathematization process and how to assess it, as well as reflections on its implementation.

This sampling strategy — comprising purposive, snowball, convenience, and random techniques — was applied across two phases, effectively supporting the study's twofold objectives: to explore mathematization in cultural practices and to examine the pedagogical potential of culturally grounded RME-based instruction in teacher education. The integration of these sampling approaches is consistent with the study's theoretical commitment to interpretivist and pragmatist paradigms, and operationalizes the conceptual framework's two-phase research structure.

3.6. **Variables of the Study**

As identified in previous sections, this study examined relationships between instructional interventions and cognitive outcomes within a teacher education context. According to Best and Kahn (2006), variables are characteristics or qualities of persons, objects, or situations that the researcher can manipulate, measure, or categorize. Leech et al. (2014)

further classify variables into independent, dependent, and extraneous categories based on their functional roles within research studies.

The independent variable in this study is the RME-based mathematization instructional strategy. As the presumed cause or predictor, this variable was deliberately manipulated by the researcher to assess its influence on pre-service mathematics teachers' cognitive and epistemological outcomes. The intervention strategy operationalized Freudenthal's theory of realistic mathematization as a human activity contextualized within culturally relevant learning environments.

The dependent variables are pre-service mathematics teachers' epistemological beliefs, geometry achievement, and geometry problem-solving skills. These outcome variables were measured before and after the intervention to determine the instructional effects. Changes in these variables were expected to reflect the effectiveness of the RME-based mathematization approach in fostering a more fallibilist epistemology, enhancing conceptual understanding, and improving problem-solving abilities.

Extraneous variables are factors other than the independent variable that might influence the dependent variables. In educational interventions, these may include factors such as prior achievement levels, socio-economic background, language proficiency, or instructional resources. Recognizing their potential impact, the study sought to control extraneous variables through random assignment of intervention and comparison groups, the use of comparable pre-intervention tests, and careful monitoring of instructional practices across settings. Controlling these variables was essential to safeguard internal validity and ensure that observed differences could be attributed with greater confidence to the instructional intervention itself.

3.7. **Instruments for Data Collection**

In accordance with the exploratory sequential mixed methods research design previously outlined, this study employed multiple data collection instruments to address its research questions comprehensively. As Plano-Clark and Ivankova (2016) emphasize, integrating multiple data collection techniques allows researchers to capitalize on the strengths and offset the limitations inherent in any single source of data. Consistent with this methodological stance, the researcher collected data through a combination of semi-structured interviews, participant observation, field notes, questionnaires, open-ended geometry problems, achievement tests, and classroom observations.

In the first phase of the study, qualitative data was gathered through the use of cultural sites and community members to examine the mathematization process and the mathematical ideas inherent in the cultural practices. In this study, semi-structured interviews were used, and this was by design, given the approach's capacity to facilitate systematic research while remaining sensitive to the participants' culturally grounded knowledge.

The semi-structured interviews were designed in such a manner that they aligned with the study's theoretical and methodological approach, which focuses on the identification of the ways in which mathematical ideas are embedded, enacted, and communicated in the context of the cultural practices. Each of the questions in the semi-structured interviews was designed in such a manner that it would allow the researcher to obtain the participants' descriptions, interpretations, and explanations of the culturally grounded practices that involve the use of mathematical ideas, such as counting, measuring, spatial reasoning, pattern recognition, estimating, and proportional reasoning, to name a few.

To establish content validity, the interview protocol was informed by established literature in the areas of ethno-mathematics, culturally responsive mathematics education, and qualitative research methodology. The use of this literature ensured that the questions were representative of the conceptual domain of interest and captured relevant dimensions of mathematization that were established in the literature. The use of this theoretical underpinning ensured that the interpretive validity of the findings could be established by connecting the data with established conceptual frameworks.

In addition, the methodological rigor of this research was ensured by having experts in mathematics education, qualitative research methodology, and cultural studies evaluate the interview protocol. The experts were asked to evaluate the questions in terms of their clarity and conceptual and cultural appropriateness. The feedback ensured that potential researcher bias was minimized in the questions and that the credibility and trustworthiness of the research were established.

The semi-structured format helped ensure rigor by finding a balance between consistency and flexibility. A series of specific but open-ended questions created a common structure for the study, which helped ensure the data gathered could be made comparable, but it also allowed the respondents the opportunity to voice their own understanding of their experiences. This process is similar to Seidman's (2006) emphasis on the importance of open-ended interviewing as a tool to allow respondents to voice their own unique worldview.

Probing questions were used to ensure that the respondents provided detailed and rich descriptions of the cultural activities and mathematical concepts that underpinned those activities. All of the interviews were audio-recorded with the consent of the respondents, which helped ensure that the data could be analyzed. In addition to interviews, participant observation was utilized to allow the researcher to immerse within specific cultural contexts and gain a nuanced understanding of community members' mathematizing processes. Blessing and Chakrabarti (2002) note that observational methods require researchers to record events either manually or using recording devices. In this study, observations captured how community members engaged in practical, culturally embedded mathematization during traditional activities such as pottery making, weaving, house building, and basketary. These observations provided firsthand, objective data that complemented and triangulated the interview findings, following Dawadi et al. (2021).

Field notes were used as a complementary source of data for collecting information related to observational data gathered during Phase I of the study. These were recorded both during and after observation. During observation, brief and rapid jotting down of key words, phrases, symbols, and sketches were recorded. These included significant actions, interactions, tools, spatial arrangements, and mathematical ideas that were unfolding. These were recorded with minimal disturbance of the natural setting. Shortly after each observation, these were expanded into detailed field notes, which included rich descriptions of events, sequences of activities, and features of the setting, along with analytic reflections related to processes of mathematization and mathematical concepts.

In this respect, the field notes followed the recommendations by Roberts (1998), which emphasized the incorporation of both descriptive and reflective elements. While the descriptive component sought to provide an accurate and detailed account of the observed phenomena, the reflective component was used to represent the views and questions of the researcher. Consequently, this approach ensured the incorporation of an “emic” and “etic” perspective. To ensure methodological rigor and standardization in the collection of data using this approach, the same format was used for all the field notes. This format included details such as the date, time, location, and activity, as well as sections for descriptive and reflective notes.

Phase II of the study, which focused on testing the RME-based mathematization intervention, employed a range of quantitative and qualitative instruments. Data were collected using a Mathematical Epistemological Beliefs Questionnaire, semi-structured interviews, open-ended geometry problems, a geometry achievement test, and structured classroom observations.

The Mathematical Epistemological Beliefs Questionnaire, adapted from the works of Buehl and Fives (2009), Kul (2013), Penn (2012), and Trakulphadetkrai (2012), consisted of 32 items designed to assess preservice mathematics teachers' beliefs regarding the nature and acquisition of mathematical knowledge. Before its administration, subject experts, colleagues, and supervisors for contextual appropriateness reviewed the instrument. Ultimately, 26 items were piloted to assess reliability. The instrument consists of 26 items measuring mathematical epistemological beliefs. Of these, ten items assess beliefs about the sources of mathematical knowledge, with five reflecting an absolutist view and five reflecting a fallibilist view. Eight items measure beliefs about the stability of mathematical knowledge, with four representing an absolutist perspective and four representing a fallibilist perspective. The remaining eight items assess beliefs about the structure of mathematical knowledge, with four items reflecting an absolutist view and four reflecting a fallibilist view.

To capture qualitative insights in Phase II, semi-structured interviews were conducted with preservice mathematics teachers to explore their experiences, beliefs, and challenges in applying the RME-based mathematization strategy. As described by Harrell and Bradley (2009), semi-structured interviews combine a predefined set of open-ended questions with opportunities for interviewers to explore emergent themes. This flexible, conversational approach enabled a deep exploration of participants' beliefs and instructional experiences. All interview instruments were adapted from relevant literature and reviewed by experts for accuracy and relevance before use.

The interviews in both phases were conducted in either Amharic or Afan Oromo, based on participants' language preferences. All interviews were audio-recorded, transcribed verbatim, and translated into English for classification and thematic analysis. These interviews were essential for eliciting detailed accounts of how mathematical ideas and problem-solving strategies were understood, experienced, and transformed through culturally situated practices and the intervention program.

Observations were also systematically conducted throughout both phases. In Phase I, observations focused on cultural activities such as pottery making, farming, weaving, and traditional house construction, with particular attention to the embedded mathematizing practices. In Phase II, classroom observations examined preservice mathematics teachers' engagement with RME-based mathematization tasks, their interaction patterns, and the epistemological beliefs they demonstrated during geometry instruction. These observations were shaped using an interpretative protocol for qualitative observations that focused on the importance of reflective engagement with the physical setting, participant interactions, and routines related to the process of mathematizing practices. In addition to observer comments and field notes, casual interviews and informal dialogues with preservice teachers followed each observation session to clarify observed behaviors and gather supplementary information.

To assess the effect of the intervention quantitatively, a geometry achievement test was administered to both the intervention and comparison groups. The test was designed to measure preservice mathematics teachers' knowledge and skills in geometry after completing a semester-long intervention program. It consisted of multiple-choice questions constructed using a table of specifications to ensure proper coverage of the mandated content areas and cognitive levels and 28 multiple choice items were developed.

Although the instructional approach was founded on Realistic Mathematics Education (RME), the test items did not directly represent RME principles such as contextual modeling, openness, or mathematization processes. Instead, the test was used as a summative assessment of pre-service teachers' overall geometry achievement after receiving instruction, allowing for comparison of learning outcomes rather than examination of learning processes.

The test items were carefully developed to address key content areas within the geometry module and were reviewed for content validity by mathematics education experts. Out of 28 multiple-choice items, 24 were chosen for the study based on their difficulty index and discrimination power, which were within an acceptable range. The items included triangles, similarity and congruence, quadrilateral properties (including areas and perimeters), polygons, and circles (including properties, area, and circumference). Although the test questions did not explicitly include RME-based contexts or mathematization procedures, they measured conceptual mastery of the geometry concepts covered in the RME-based educational intervention.

Although the instructional intervention was based on Realistic Mathematics Education (RME), the test items were not explicitly designed to reflect RME principles such as contextual modeling, openness, or mathematization processes. Instead, the test functioned as a summative assessment of pre-service teachers' overall geometry achievement following instruction, allowing for comparison of learning outcomes rather than examination of learning processes.

In relation to geometry problems, ten context-based open-ended geometry problems were designed to investigate and inform the problem-solving abilities of pre-service mathematics teachers, all in line with the various contexts presented in this module. First and foremost, it is imperative to note that these problems were targeted to assess and ensure alignment with the various elements presented in the theory of Realistic Mathematics Education. Consequently, the problems were designed to incorporate horizontal and vertical mathematization within them and scored using a rubric scale adapted from Patac et al. (2021) and Yuanita et al. (2018), focusing on aspects such as problem interpretation, strategy formulation, justification, and solution accuracy.

This comprehensive suite of data collection instruments was carefully aligned with the study's conceptual framework and theoretical grounding in RME and ethnomathematics. The triangulation of qualitative and quantitative data sources enhanced the validity and depth of the findings, providing a holistic understanding of the role of culturally embedded mathematization in transforming preservice teachers' epistemological beliefs, geometry achievement, and problem-solving skills.

3.8. **Data Collection Procedures**

Following the research design and methodological framework established in the preceding sections, data for this study were collected in two distinct but interconnected phases. Each phase was carefully planned and executed in alignment with the mixed methods exploratory sequential approach, ensuring that qualitative insights from the first phase meaningfully informed the intervention and subsequent data collection in the second phase.

In Phase I, the researcher traveled to the designated research sites, equipped with all necessary materials, including participant consent forms and formal letters of support. These letters, secured from the then Department of Science and Mathematics Education, and now Department of teacher Education, and the respective district administration offices, officially authorized the researcher to conduct the study within the selected zones. Beyond fulfilling ethical and legal obligations, these documents facilitated access and cooperation from local authorities and participants. However, the most crucial aspect of this phase involved establishing rapport and negotiating consent with potential participants. The researcher personally introduced the study's objectives and sought voluntary participation, adhering to ethical research practices and ensuring participants' informed consent.

Data were collected using audio and video recordings of interviews and observations. The researcher transcribed these materials verbatim and translated them into English to prepare them for qualitative analysis. Observations were documented through detailed field notes capturing both descriptive accounts of events and reflective commentary on the researcher's experiences and insights. These procedures ensured a rich, contextualized understanding of the mathematizing practices embedded in local cultural activities.

Phase II began with the administration of pre-tests to both the intervention and comparison groups. These instruments included open-ended geometry problems, a geometry achievement test, and a mathematical epistemological beliefs inventory. These assessments established baseline data on participants' geometry knowledge, problem-solving abilities, and beliefs about mathematics, which would later be compared with post-test results to determine the effects of the intervention.

Following the pre-tests, the intervention group engaged in a geometry course delivered through a RME-based mathematization approach that utilized REACT(R-Relate, E-experience, A-apply, C-cooperate and T-transfer) strategy to implement it. Rooted in Freudenthal's (1968) theoretical framework, this instructional method views mathematics as a human activity grounded in learners' everyday experiences. The teaching and learning process emphasized progressive mathematization, where real-life problems formed the foundation for developing informal strategies, which were then systematically refined into formal mathematical concepts through guided reinvention (Prahmana et al., 2021; Risdiyanti & Prahmana, 2020).

The intervention was structured around six core principles of RME: the reality principle, activity principle, guidance principle, level principle, interactivity principle, and intertwinement principle. These principles guided the design and delivery of geometry lessons, ensuring that learning activities remained meaningful, interactive, and connected to learners' cultural contexts. Realistic geometry problems were introduced as entry points for classroom activities. Pre-service mathematics teachers were encouraged to identify relevant mathematical concepts within these problems, visualize and schematize them, and collaboratively develop informal or formal solution strategies.

Throughout this instructional process, classroom observations were conducted, capturing participants' engagement, mathematization behaviors, and instructional interactions. Observer comments, field notes, and informal dialogues complemented these observations, providing additional qualitative insights into the learning environment and participants' experiences.

The model of instruction for the comparison group was teacher-centered, as is typical in a geometry class for pre-service teachers of mathematics. Instruction in this group adhered to a deductive model, where each day of instruction began with the explicit stating of formal axioms, definitions, postulates, and theorems relevant to the geometry topic being studied, providing the main entry point for instruction on that topic, rather than arising out of problem situations or activities.

After the presentation of the definition and theorems, the instructor explaining the definition and illustrated and teaching the standard procedures for the solution of the subject matter presented lectures. The instructor presented examples on how the aforementioned principles of the subject matter could be applied to any of the problems presented in textbooks. The examples presented were of an abstract and symbolic nature and were not presented within any contextual situation.

Learning activities developed for the comparison group were mainly routine exercises based on the material from the course module. These exercises mainly focused on procedural skills and the proper application of formulas/theorems, as opposed to conceptual understanding, exploration, or multiple solutions. To what extent these activities were context-based or problem-led was very limited, as were the opportunities for the construction of mathematical knowledge by the student.

Adoption of teaching style: In the comparison group, the teaching style adopted was instructor-centered. The teacher was the key actor in explaining the content matter, asking some oral questions to test comprehension or recall sometimes, and providing feedback on the responses. Students' involvement was only focused on answering the teacher's questions or completing tasks individually. Other involvement of students like discussion with peers, problem-solving with peers, and mathematical argumentation were not incorporated.

In the case of representations, it relied on symbolic and formal mathematical representations, such as an algebraic expression and symbols or diagrams given by the instructor. Multiple representations, on the other hand, were used on a minimal basis and only for the clarification of defined concepts.

The nature of the assessment was mainly summative in nature in comparison group schools. Though oral questions were asked at certain points in teaching, these were not used consistently as a form of formative assessment to identify students' thinking and improve instruction. The assessment strategy mainly relied on gradational work such as quizzes, assignments, mid-term exams, and finally, exams. These assessments were mainly for grading, making decisions, and not necessarily for improving students' learning processes.

Overall, the teaching in the comparison group was marked by an oriented instruction type where mathematical knowledge was presented as a set of factual information that should be applied. Obviously, this type of educational structure is quite different from that of the RME-based type of mathematization of the mathematical knowledge of the students of the intervention group. This is because the intervention was clearly distinguished from conventional instruction by its emphasis on contextual problem situations, student-generated solution strategies, and the gradual transition from informal reasoning to formal mathematical representations.

Unlike traditional instruction, which usually starts with teacher explanation and abstract procedures, the progressive mathematization approach emphasized guided reinvention, mathematical modeling, and classroom discussion as important features of instruction. Furthermore, to ensure the progressive mathematization intervention as intended, there was careful design and implementation. The intervention was implemented through pre-designed lesson plans and worksheet that were carefully aligned with the principles of RME, specifically the stages of progressive mathematization and guided reinvention. Classroom implementation of the intervention was done in accordance with the lesson plans, and adherence to the intervention was maintained through classroom observations to ensure that the implementation of the instruction was consistent with the intended philosophy of instruction. Although it is impossible to ensure complete fidelity of implementation in an interactive classroom setting, especially in a paradigm like RME, which emphasizes adaptive teaching and student participation, these measures ensured that the intervention was implemented in a manner consistent with the principles of RME-based mathematization.

At the conclusion of the intervention period, post-tests were administered to both groups. These assessments mirrored the pre-tests in structure and content, comprising open-ended geometry problems, a geometry achievement test, and the mathematical epistemological beliefs inventory. This enabled a systematic comparison of pre- and post-intervention data to evaluate the effectiveness of the RME-based mathematization approach.

In addition to quantitative assessments, qualitative data collection continued within the intervention group. Field notes and informal dialogues captured participants' reflections and reactions during classroom activities. Furthermore, semi-structured interviews were conducted with seven purposively selected pre-service mathematics teachers from the intervention group. These interviews explored participants' overall experiences with the RME-based instruction, the challenges they encountered, and the perceived effects of the intervention on their problem-solving approaches, geometry achievement, and epistemological beliefs. Interviews were conducted in Afan Oromo, audio-recorded, transcribed, and translated into English for analysis.

This structured, two-phase data collection procedure ensured a coherent alignment between the study's theoretical foundations, research design, and methodological execution. The triangulation of quantitative and qualitative data sources strengthened the study's validity and provided a comprehensive understanding of how culturally contextualized mathematization processes influence mathematics education outcomes in the Ethiopian teacher education context.

3.9. **Trustworthiness**

Building on the research design and data collection methods described in the previous sections, ensuring the trustworthiness of qualitative findings is essential to guarantee the study's rigor and credibility. Neuman (2006) emphasizes that while both qualitative and quantitative researchers seek valid and reliable measurements, each approach interprets these concepts differently, despite sharing fundamental principles.

Many qualitative researchers acknowledge the importance of validity and reliability but often avoid these terms due to their close association with quantitative measurement. Instead, qualitative research focuses on credibility, transferability, dependability, and confirmability to establish trustworthiness (Lincoln & Guba , 1989). According to Pilot and Beck (2014), trustworthiness refers to the confidence in the data, interpretations, and procedures that ensure the quality of the study. It serves as a mechanism for the researcher to convince both themselves and the readers that the findings are meaningful (Lincoln & Guba, 1985).

A critical aspect of trustworthiness is credibility, which examines the extent to which findings reflect reality. Researchers must ask: Are the findings consistent with actual experiences? Do the results represent what truly exists? Such considerations are central to the internal validity of qualitative research (Shenton, 2004). In addition, transferability addresses how well the findings can be applied to other contexts, theories, or future studies (Lincoln & Guba, 1985). Dependability pertains to the stability of findings over time and conditions—whether replicating the study with the same context and methods yields similar results (Shenton, 2004). Lastly, confirmability relates to objectivity, ensuring findings are free from researcher bias or subjective influence.

To enhance trustworthiness in this study, multiple data sources—including interviews, observations, and field notes—were triangulated to develop a thorough rationale for coding and theme generation. To minimize bias during data analysis, participants were given access to review transcripts to verify the accuracy of interpretations. Furthermore, the research process, methodology, and context were fully described to provide transparency. Direct quotes from participants were incorporated to allow readers to interpret and assess the findings independently.

The researcher and his colleague did the coding of the qualitative data of phases I and phase II. First, the researchers independently coded the qualitative data. Then, the coding was compared and discussed together, with a focus on the points of disagreement, until agreement was reached on all codes. The final codes and themes were determined through a process of discussion and reflection. Rather than focusing on the statistical measures of inter-coder reliability, this process added rigor to the analysis and helped ensure that the themes were grounded in multiple perspectives.

An additional review by a PhD-qualified colleague ensured that interpretations and conclusions were well-grounded in the raw data. To improve transferability, the findings were analytically generalized to broader theoretical frameworks by cross-referencing relevant literature (Yin, 2003). This careful attention to trustworthiness strengthens the confidence in the study's qualitative results and prepares the foundation for the subsequent assessment of instrument validity and reliability.

3.10. **Validity and Reliability**

Following the establishment of trustworthiness for qualitative data, attention must also be given to the validity and reliability of quantitative measurement instruments used in this study. These concepts are crucial to ensure that the data collected are accurate, consistent, and reproducible.

Kimberlin and Winterstein (2008) highlight that validity and reliability is key to maintaining the integrity and quality of any measurement instrument. Without proper focus on these qualities, research risks losing rigor and becoming invalid or irrelevant (Bhattacharyya et al., 2017). Reliable and valid instruments ensure meaningful interpretation of scores derived from questionnaires, tests, and observational tools commonly used in education and other fields (Cook & Beckman, 2006).

Validity assesses the soundness of inferences made from the data, confirming that measurements represent the constructs under investigation (Kashif et al., 2023). It refers to the extent to which the collected data covers the actual area of inquiry (Ghauri & Gronhaug, 2005) and how reasonable and meaningful the conclusions are (Polit & Beck, 2006). Messick (1989) frames validity as the judgment of whether interpretations and subsequent actions based on test results are appropriate.

Among various types of validity, this study focused on face validity and content validity. Face validity concerns whether the instrument appears to measure what it is intended to, based on expert review and respondent perception (Taherdoost, 2016). Experts in mathematics education, along with supervisors and instructors, reviewed the mathematical epistemological beliefs questionnaire, geometry achievement test, and problem items to assess clarity, relevance, and comprehensibility. Necessary adjustments were made based on their feedback.

Content validity evaluates whether the instrument's items adequately represent the entire domain of the construct (Cohen & Swerdlik, 2018; Slaney, 2017). Experts verified the content of the geometry achievement test and problems through a table of specifications, ensuring the items properly covered all relevant aspects. They also assessed the mathematical epistemological beliefs scale on criteria such as relevance, clarity, simplicity, and ambiguity (Yaghmale, 2003).

Table 3.1 Criteria for measuring content validity of mathematical epistemological beliefs

Relevance	Clarity	Simplicity	Ambiguity
1= not relevant	1= not clear	1 = not simple	1 = doubtful
2= item need some revision	2= item need some revision	2 = item need some revision	2 = item need some revision
3= relevant but need minor revision	3 = clear but need minor revision	3 =simple but need minor revision	3 = no doubt but need minor revision
4= very relevant	4 = very clear	4 = very simple	4 = meaning is clear

Items with a content validity index (CVI) above 0.75 were retained, while others were removed.

Reliability indicates the consistency and stability of measurements (Chakrabartty, 2013; Haradhan, 2017). The four main types of reliability include test-retest, parallel forms, inter-rater, and internal consistency.

In this study, both inter-rater reliability and internal consistency were employed. Because geometry achievement test items were scored dichotomously, Kuder-Richardson Formula 20 (KR-20) was used to assess internal consistency. For the Likert-scale items measuring mathematical epistemological beliefs, Cronbach's alpha was calculated. Both measures showed acceptable reliability coefficients of 0.780 and 0.777 for the geometry achievement test and mathematical epistemological beliefs, respectively. Furthermore, the reliability coefficients for the components of mathematical epistemological beliefs (source, stability, and structure of mathematical knowledge) were computed and found to be 0.70, 0.72, and 0.71, respectively.

For the open-ended geometry problems, scoring was done using a rubric. The modified Pearson correlation coefficient (Fisher, 1954) was used to assess reliability due to the interval nature of the data. Additionally, the intraclass correlation coefficient (ICC) was calculated to assess inter-rater reliability, which is more suitable for interval scales than Cohen's kappa (Portney & Watkins, 2015).

Table 3.2 Results of ICC realistic geometry problems in SPSS using average measures, consistency, 2-way mixed -effects Model

	Intraclass Correlation	95% Confidence Interval		F Test with True Value 0			
		Lower Bound	Upper Bound	Value	df1	df2	Sig
Single Measures	.867	.771	.925	14.038	44	44	.000
Average Measures	.929	.870	.961	14.038	44	44	.000

In this study, the researcher used average measures, consistency, and a 2-way mixed-effects model to calculate ICC estimates. The 95% confidence intervals were then computed using IBM SPSS statistical package version 27, based on a mean rating (k=2) and a 2-way mixed-effects model. The ICC for inter-rater reliability, ICC (3, 2) = 0.929, is excellent, showing that open-ended geometry items were rated consistently between raters.

3.11. **Difficulty and Discrimination Indices**

Following the establishment of validity, reliability, and trustworthiness of the instruments, it was crucial to evaluate how effectively the test items measure students' learning outcomes. This evaluation is achieved through item analysis, a systematic process that involves reviewing and refining test questions to ensure their quality and relevance. Item analysis helps in identifying and modifying or eliminating test items that fail to meet acceptable standards, thus improving the overall effectiveness of the test (Moses, 2017; Rezigalla et al., 2024). Specifically, this process ensures that the test items align with the learning objectives and accurately reflect students' performance levels.

Classical Test Theory (CTT) serves as the foundation for item analysis by providing two essential statistical measures: the difficulty index and the discrimination index. The difficulty index reflects the proportion of students who answer an item correctly, with values ranging from 0.0 (very difficult) to 1.0 (very easy). Ideal test items tend to have difficulty indices between 0.3 and 0.7, balancing between too easy and too difficult to effectively gauge student ability (Boopathiraj & Chellamani, 2013; Ul Islam & Usmani, 2017). In this study, 24 of 28 geometry test items had difficulty indices ranging from 0.25 to 0.71, indicating that these 24 items represent a balanced range of difficulty levels, whereas the remaining four items were removed.

Complementing the difficulty index, the discrimination index assesses how well each item differentiates between high-achieving and low-achieving students. An effective test item should be answered correctly more frequently by higher achievers, thus demonstrating its discriminatory power. The discrimination index ranges from -1.0 to +1.0, where values above 0.4 are considered good, values between 0.3 and 0.39 reasonable, 0.2 to 0.29 marginal, and below 0.19 poor, warranting revision or removal (Marie & Sreekala, 2015).

Based on this, the discrimination index of the geometry achievement exam was intermediate, 0.39- 0.71, indicating that the geometry test items are successful in discriminating between high and low achievers. Together, these indices ensure that the geometry achievement test not only measures knowledge accurately but also distinguishes clearly between different levels of student proficiency. This rigorous evaluation of item quality is an essential extension of the previous focus on validity, reliability, and trustworthiness, forming a comprehensive approach to instrument quality assurance.

3.12. **Methods of Data Analysis**

Building upon the established rigor of measurement instruments, data analysis was the next crucial step that transforms raw data into meaningful findings, allowing the research questions to be answered with precision and depth. Central to this process was the identification of the unit of analysis, which determines the main entity about which data is collected and analyzed (Kumar, 2018). The unit of analysis is inherently linked to the research questions and defines whether the focus is on individuals, groups, organizations, or broader communities (Ellinger & McWhorter, 2016). In this study, the units of analysis are threefold: the local cultural communities engaged in daily mathematization activities, the mathematical concepts embedded within those activities, and the pre-service mathematics teachers enrolled in the two teacher education colleges in Oromia, Ethiopia.

Qualitative data obtained from interviews and observations were transcribed, verified by participants for accuracy, and analyzed thematically following Braun and Clarke's (2006) six-phase framework. This iterative process involved familiarization with data, coding, theme development, review, and interpretation, supported by Bishop's (1988) mathematical activity framework to contextualize findings within relevant theoretical perspectives. For the quantitative data, descriptive statistics such as means and standard deviations were calculated alongside inferential statistics including independent t-tests, one-way MANOVA, and multiple regression analyses. These tests examined group differences and predictive relationships among epistemological beliefs, achievement, and problem-solving skills. The qualitative data from the second phase further enriched the understanding of how instructional interventions influenced teacher candidates' learning experiences. Together, these analytic methods ensure a comprehensive and connected approach, building on the foundations of measurement validity, reliability, and trustworthiness established earlier.

3.13. **Scoring Criteria**

The 26-item mathematical epistemological beliefs scale assessed pre-service mathematics teachers' perspectives on the nature of mathematical knowledge using a 4-point Likert scale. The scoring system was based on research conducted by Penn (2012) and Tuge and Garoma (2016). For items describing fallibilist ideas, "strongly agree" was given a value of four, while "strongly disagree" was given a value of one. For items expressing absolutist ideas, "strongly agree" received a value of one, while "strongly disagree" was assigned a value of four.

Mean scores were calculated to assess pre-service teachers' epistemological beliefs, with scores above 2.5 suggesting fallibilist beliefs and scores below 2.5 indicating absolutist beliefs, according to Aydin and Tasci's (2005) classification. The small difference between scores of 2.49 and 2.51 may not have a significant influence on actions or beliefs in the real world, so a classification based on scores above or below 2.5 may not necessarily indicate meaning. Nevertheless, it is helpful for statistical analysis and research purposes because the averages in either group may represent the perspectives on the epistemological beliefs. In reality, epistemological views are more complex than a single score, and people may have a mix of fallibilist and absolutist views in different areas of knowledge.

Mathematical epistemological beliefs about where mathematical knowledge comes from, how stable it is, and how it is structured are on a continuum spectrum ranging from absolutist to fallibilist (Depaepe et al., 2017; Lerman, 1990; López, 2021; Schommer-Aikins & Duell, 2013; Trakulphadetkrai, 2022). This classification based on an arbitrary score below or above 2.5 is just for establishing categories for statistical analysis or reporting purposes. The little difference in a pre-service teacher's score between absolutist and fallibilist perspectives may not have a significant impact on their daily activities or learning strategy choices. Epistemological beliefs are inherently nuanced, and individuals may simultaneously hold absolutist and fallibilist beliefs across different dimensions of knowledge. Accordingly, epistemological beliefs about the source, stability, and structure of mathematical knowledge are conceptualized as existing along a continuum rather than as discrete categories (Depaepe et al., 2017; Schommer-Aikins & Duell, 2013; Trakulphadetkrai, 2022).

3.14. **Research Ethics**

Ethical considerations were paramount throughout the research process, underpinning the integrity and social responsibility of the study. In line with Creswell's (2007) guidelines, this research adhered strictly to ethical principles involving informed consent, confidentiality, voluntary participation, and the minimization of risk to participants. The study ensured that all participants—ranging from local artisans such as potters, weavers, and craftsmen to academic staff and pre-service mathematics teachers were fully informed about the research aims, procedures, and their rights before data collection commenced. This transparency was maintained both orally and through written consent forms, which participants signed to document their voluntary agreement to participate.

To protect participants' privacy, pseudonyms were used throughout the study unless individuals explicitly chose to use their real names. Participants were assured that the information gathered would be exclusively utilized for academic purposes and treated with strict confidentiality. The researcher also highlighted participants' right to withdraw at any point without any negative consequences, ensuring ethical autonomy. Furthermore, the research protocol was rigorously reviewed and approved by the institutional review committee (IRC) of the Department of Science and Mathematics Education and approved by the Institutional Review Board (IRB) of the College of Education and Behavioral Studies of Addis Ababa University, confirming that all ethical standards were met before the study began. These ethical safeguards reinforce the credibility and trustworthiness of the research findings by ensuring respectful, responsible, and transparent engagement with all individuals involved, seamlessly complementing the rigorous methodological processes detailed in the preceding sections.

CHAPTER 4: EXPLORATION OF MATHEMATIZATION AND MATHEMATICAL CONCEPTS EMBEDDED IN THE CULTURAL ACTIVITIES

This chapter looked at mathematization and mathematical principles ingrained in society's cultural activities. The exploration mainly focuses on basketry, weaving, pottery, and home construction. The following question served as the foundation for our investigation, in line with the use of mathematization and mathematical notions incorporated in cultural activities:

1. What mathematization and mathematical concepts are found in weaving, pottery, basketry making, and house building that can be utilized in mathematics teacher education?

4.1. Basketry

Basketry is the craft of weaving three-dimensional items from dry, split *migira (baqaaqsaa*); *alela-baqaaqsaa,* inked with different colors used to create various designs; and suta, all types of straw. These crafts are very common in rural areas of Arsi, Selale, Bale, and other parts of Ethiopia. Basket weaving is a traditionally female-dominated activity that is often passed down from mother to daughter. Most of the time, in the countryside, these are the common presents that the bride gathers from her family, relatives, neighbors, and colleagues and presents to her spouse. However, instead of *alela*, females increasingly use threads or medabera—a type of thread with flattened, sharp edges of different colors. These baskets are different in type based on their purpose. For example, *Mesob*—a cylindrical basket with a small leg and a cone-shaped lid—is used to store injera and bread (the lemat for bread does not have legs).

Figure 4.1 Mesob for stroring injera and bread

Lemat, which is similar to *Mesob* but with taller legs, and another type of *lemat* that lacks a cylindrical wall, having only a circular or cone-shaped top to hold injera and wot during communal meals, can also be used for decoration in a salon. Agalgilii serves multiple purposes, such as carrying injera and wot food during teamwork at the plowing field or storing unspoiled food when men go livestock herding.

Figure 4.2 Lemat of different shapes

Kuna is a frustum of a cone-shaped indigenous material made of straw that comes in a variety of sizes for different uses. The smaller one is known locally as a kuna segn, a portable traditional container used by farmers to store wheat specifically for sowing. It is also used for grain Winnowing. Designed for comfort, compact enough to be carried in one hand while planting with the other, trampling plowed land, and spreading grains by hand, the practice, deeply ingrained in regional culture, is an essence of the sustained relationship between people, the planet, and acquired methods of farming across generations. Besides the kuna segn, there is also a larger type of the kuna used for storage of grains such as wheat, corn, barley and others. Beyond storage, the greater kuna serves as a local unit of measurement when farmers lend or borrow grain from one another. This dual role emphasizes its cultural and practical relevance in the community's agrarian existence, serving as a container as well as a symbol of social exchange and mutual assistance.

Figure 4.3 Kuna of different sizes

Gindoo/Sefed -is a circular hand-woven basket used to remove the bidena/injera from the hot elee/mitad. It is also used for grain winnowing and decorative purposes, and is often placed on the walls of houses.

Figure 4.4 Gindoo or sefed

Elemitu, also known as qabee ananii, is another kind of hand-woven basket. It is a cultural object produced from straw and is used as a container both during and after cow breeding.

Figure 4.5 Elemtu that serves as a milk container

Shinote- is a gorgeous, hand-woven cultural item used in Oromo wedding ceremonies and other traditional occasions. It resembles a mesob but is smaller, making it portable and easier to transport—particularly by young ladies during bridal processions. Shinote is distinguished by its exquisite decoration, which adds to its visual appeal. It is constructed from alela, a local natural fiber that is painstakingly separated and dyed in various vibrant colors. The vibrant strands are artfully woven to form shapes and symbols that are aesthetically pleasing and culturally significant.

Figure 4.6 Shinote in the wedding ceremony for welcoming the bride and groom in Selale

Females covered the body of the shitote with dantel. Then they tied wet grass and sweet candy called Keremela of various colors all over their bodies, which not only looks attractive but also shows the community's inventiveness and identity. The shinote is held in both hands by young girls singing cultural songs in honor and praise of the groom when the groom leaves his house and return or when the bride is ready to leave her house. It is both a celebratory and symbolic act that unites artistry, tradition, and communal celebration. The shinote is not just a ritual object, but it is indicative of workmanship, pride in heritage, and the close identification of the Oromo with their heritage.

4.1.1. **Mathematization and mathematical concepts embedded in basketray**

Weavers utilize numerous varieties of alela to create distinct designs on the baskets they make. As one respondent noted in her interview response, they decorate the basket's body with various motifs to make it more appealing and intriguing. The design activity is expressed in the construction of numerous geometrical shapes, such as circles, different triangles of non-Euclidean geometry, spiral shapes, and zig-zag lines, different quadrilaterals of non-Euclidean geometry as shown below. For example, if you look at the irbo's left side, its legs are decorated with two white spirals, two red ones, two green ones, and two blue spirals. There is also a large spiral adorned with different triangles, followed by two more red spirals, two green ones, and two blue ones. It ends with a big spiral and is finally completed with two red spirals, and which indicates the existence of counting activity in producing different baskets.

Furthermore, when creating different motifs, the number of yellow *alela* used to form the back of the snake and *fuuloo fardaa* is determined by bringing back some yellow *alela* and bringing forward the remaining ones, resulting in two triangles—one with yellow alela and the other with either red or blue alleles. As noted during the interview, the number of cycles in weaving—especially in mesob weaving—is crucial for marketing; therefore, directly counting these cycles correlates with perceived quality and market value.

Basket makers frequently utilize non-standard measurements, like finger-span, to measure the inner surfaces of traditional products like Injera, bread mesob, kuna, and sefed. As explained in the interviews, they calculate the size of these goods by measuring the circumference of the circular base. For example, an Injera mesob is typically three to four finger-spans wide. This precise measurement is more than practical; it is the precision and expertise that exist in the basket making tradition in which size and proportion are critical both to utility and to beauty. This indicated that the mathematizing process of measuring is reflected in basketry.

One *mesob* maker demonstrated how she uses alela to create various designs. One of these is known as the "back of the snake." This design is created by arranging three yellow *chokorsa* and *suta*. To begin, she brings the two yellow *chokorsa* at each end backward and pulls the central one forward with *suta* before weaving it. She then reverses the movement, bringing the two yellow *chokorsa* in the corners forward with *suta* and moving the central one backward, followed by weaving. She keeps alternating between these positions, bringing the two corner *chokorsa* back while pushing the central one forward, and then bringing the middle one back while moving the corner ones forward. The "back of the snake" design emerges gradually from this repeated sequence.

Figure 4.7 Back of the snake design formation

One of the most valued patterns in mesob creation is *fuuloo fardaa*, which exudes both precision and beauty. It is woven with an abundance of yellow *chokorsa*, their warm color glowing like ripe sorghum kernels, and other *alela* to enhance the pattern. Unlike simple weaving, *fuuloo fardaa* is made using double weaving, a method that requires patience, talent, and a steady hand. While making, she also creates a motif that resembles a zig-zag line called *ficaan sangaa* in the local name, meaning the urine of an ox.

Figure 4.8 The process of making a fuuloo fardaa motifs and presentation of other motifs

The procedure begins with three full cycles woven in the traditional manner, utilizing white *migira* and *suta* to form a solid foundation. The weaver creates approximately fourteen yellow *chokorsa* for this motif. She begins by bringing the second *chokorsa* from the corner backward, while the first and remaining twelve stand in front like a line of attentive dancers, and then bind them in place with the *suta*. In the next step, she pulls the first and third *chokorsa* backward, allowing the second and all others to travel forward and be secured in the weave.

Each *chokorsa* is performed in a meticulously learned pattern, alternating between forward (BF) and backward (BB) positions. This dynamic dance continues row after row, with each turn of the suta tightening the strands and infusing new life into the pattern. For more information the following table depicts the patterns.

Table 4.1 Pattern formation for creating different triangles that make the fuuloo fardaa motifs

Arrangement of Chokorsa	Steps in weaving												
	1st	2nd	3rd	4th	5nd	6th	7th	8th	9th	10th	11th	12th	13th
1	BF	BB	BF	BB	BF	BB	BF	BB	BF	BB	BF	BB	BF
2	BB	BF	BB	BF	BB	BF	BB	BF	BB	BF	BB	BF	BB
3	BF	BB	BF	BB	BF	BB	BF	BB	BF	BB	BF	BB	BF
4	BF	BF	BB	BF	BB	BF	BB	BF	BB	BF	BB	BF	BB
5	BF	BF	BF	BB	BF	BB	BF	BB	BF	BB	BF	BB	BF
6	BF	BF	BF	BF	BB	BF	BB	BF	BB	BF	BB	BF	BB
7	BF	BF	BF	BF	BF	BB	BF	BB	BF	BB	BF	BB	BF
8	BF	BF	BF	BF	BF	BF	BB	BF	BB	BF	BB	BF	BB
9	BF	BF	BF	BF	BF	BF	BF	BB	BF	BB	BF	BB	BF
10	BF	BF	BF	BF	BF	BF	BF	BF	BB	BF	BB	BF	BB
11	BF	BF	BF	BF	BF	BF	BF	BF	BF	BB	BF	BB	BF
12	BF	BF	BF	BF	BF	BF	BF	BF	BF	BF	BB	BF	BB
13	BF	BF	BF	BF	BF	BF	BF	BF	BF	BF	BF	BB	BF
14	BF	BF	BF	BF	BF	BF	BF	BF	BF	BF	BF	BF	BB

While making this such a design, they utilized different types of alela and several bundles of suta to complete the mother body and its lid and and this indicated that basket makers utilze the counting and designing process of mathematizing process while making different types of baskets.

The weaver explained during the interview that the legs and upper portion of the *mesob* are woven from the outside to the inside, while the inner section is woven from the inside out. This technique, they claimed, improves beauty, attracts customers, and simplifies the weaving process—particularly when constructing a *mesob*. They also critiqued the habit of weaving the legs and lid from the inside outward, calling it impractical and detrimental to the cultural product's visual appeal. In their comments, they also mentioned the *mesob's* various applications, such as market sales, food storage for brides, *Shinnote* rituals, and decoration. This mirrors the mathematizing process of explanation that basket makers use in their daily work.

Finding starting locations, such as the inner circular portion, as well as bending points and directional changes—whether weaving from the inside out or the outside in—are all critical components of spatial orientation in weaving. The placement of design components is equally crucial; to improve visual appeal and draw in buyers, patterns and colors are carefully placed on the most noticeable areas, including the lid, upper region and the outside of the leg. Understanding part-to-whole linkages is also necessary for the procedure, which sees the mesob as consisting of three interrelated components, each with a unique size and orientation: the upper portion, the leg in the shape of a cone's frustum, and the inner circular base including the lid of the *mesob*. Thus, in basketry, spatial orientation in weaving, positioning of design elements, and part-to-whole relationships all together reflect the mathematizing process of locating.

Figure 4.9 Formation of different design by means of alela, madebera and threads

Basketry weaving is a rich source of mathematical notions that organically develop from traditional artisanship. Basketry is typically viewed as purely an artistic or utilitarian technique, although it requires advanced mathematical understanding. For example, the production of containers such as the Elemitu, which is used to store milk, necessitates a knowledge of volume—weavers must carefully estimate and control the basket's capacity to ensure it satisfies practical requirements. Similarly, the Kuna, which is meant to carry a variety of grains including wheat, barley, and corn, emphasizes the necessity of volume measurement and spatial reasoning in the weaving process.

Beyond volume, basketry displays a variety of geometric notions incorporated in its forms and structures. Weavers work with shapes like triangles, squares, and circles, and three-dimensional ones like cones, frustums of a cone, and cylinders. Symmetry, tessellation, and spatial transformation are demonstrated by the extremely accurate positioning of designs that include zigzag patterns, angles, and repetitive motifs. Not only are they pleasing to the eye but essential to the strength and usability of the basket.

Furthermore, measurement is crucial in the weaving process. Mesob and kuna basket sizes are well thought out and constructed, demonstrating a practical sense of ratio, proportion, and scale. The weaver's expertise at visualizing and executing these measurements makes the resultant product perform the way it should but remain structurally intact. Generally, basketry weaving is an excellent illustration of how traditional cultural practices may serve as natural environments for mathematizing, or using mathematical ideas to handle everyday concerns.

Figure 4.10 Mathematical modeling of a sample basketry

4.2. **Making Gingilcha**

Gingilcha is a cultural material made up of dry strings of straws. Its main purpose is to separate different finer particles of a mixture of flour or others like teff from coarser (i.e., the flour passes through with the hub as it is). While preparing this material, there are different steps. The first step is preparing strings from draw straws. In preparing the "*Gingilcha*," the women need to prepare the following materials: '*Suta*a' dry straws of one bundle and *migira*, separable straws of 3 to 4 bundles, '*mutaa*' water, material with a circular shape, and a dish for holding water.

To prepare the strings from straw, the collected wet straw is equally divided into two equal parts. These dividing the wet straw equally into two parts was continued until women produced as many straws as possible during autumn season. After dividing the wet straw, they made dry the straws by placing it on the roof of the house. The dry straws would be immersed for a while just to make a string by their hands on their leg. The reason why they made a string from the dry straw is for sake of strength of the 'Gingilcha'.

Figure 4.11 Steps in gingilcha making

After they completed making many strings, they started making the net of '*Gingilcha*' with two pairs of strings, one pair horizontal and one pair vertical, and added a pair of strings at a time until the desired size was met. Then '*tatta'uu*' or making the net continued by intertwining the vertical strings with a pair of horizontal string. According to their interview response, one horizontal thread is at the top and the other is at the bottom when the net is being built. In the next stage, the top one will be at the bottom and the bottom one at the top. This top, bottom, top, bottom pattern will continue until the required distance. In doing so, they completed a rectangular net with the desired size.

They started weaving the net with one vertical string from the second row to the last, then the leftover string from the first pair is combined with the one in the second pair. Then, the leftover of the second pair is entwined with the one in the third pair, and the leftover of the third one is entwined with the one in the fourth pair. This is done until the last pair but ends with one string of the last pair. The next row will start with the first pair, then the second, third, fourth, and last pair. For clarification, the following is the patters they followed while making the net. R stands for row whereas S stands for a string.

R2: S1, S1S2, S2S3, S3S4, S4S5… S47S48, S48,
R3: S1S1, S2S2, S3S3, S4S4… S48S48.
R4: S1, S1S2, S2S3, S3S4, S4S5… S47S48, S48
R5: S1S1, S2S2, S3S3, S4S4… S48S48.
R6: S1, S1S2, S2S3, S3S4, S4S5… S47S48, S48
. ………
. ………
. ………

Figure 4.12 Pattern formation in making the net of gingilcha

After completing the net, they stretched the net on circular material and tied it as indicated in the next picture. After completing tying on the circular material, they started weaving just as indicated in the fourth picture.

Figure 4.13 The second phase of gingilcha

The weaving process in the form of a spiral shape is continued up to eight to 10 rounds from above and two to three rounds from below, as indicated below. In their response, they stated that the most common form of '*Gingicha*' is circular, and this is due to the fact that the *suta* in weaving is dry, and they were easily bent. Otherwise, if the shape is rectangular or in another form, the materials (*suta*) they used for weaving will not be easily bent but rather broken. The upper weaving is for holding the floor, whereas the lower weaving is to keep the net from dust or water.

Figure 4.14 The lower, the net and the upper part of gingilcha

4.2.1. **Mathematization process and mathematical concepts embedded in making Gingilcha**

There are different mathematizing process and mathematical concepts embedded in making '*Gingilcha*'. Among the mathematizing process is counting. To start making the net of '*Gingilcha*,' the respondents need four strings: one pair horizontal and one pair vertical. In their interview response, to make counting manageable, they add a pair of vertical strings at a time and continue adding a pair of vertical strings until they arrived at the desired size. As stated in her interview, one respondent counted the vertical strings as 2, 4, 6…96 in the first row since she added a pair of strings at a time until the desired size will met. The other respondent mentioned in her interview response that she treated a pair of strings as a single unit and continued adding a pair of vertical strings, finally leading to 50 pairs (100 strings) for the net of the material.

The other mathematizing process in *Gingilcha* making is locating, which entails figuring out where to put materials (strings) to form the required structure. The placement of vertical strings allows the net of the material to form a mesh-like structure. The respondent arranges a pair of vertical strings and ties them with a pair of horizontal strings. Each string is arranged in a particular vertical and horizontal arrangement to form a grid-like structure. Furthermore, based on the size of *Gingilcha*, the respondent decides where vertical strings ought to be placed. Since the sizes of the *Gingilcha* vary from small to large depending on the customer's need, this variation in size necessitates modifying the vertical string location. These size preferences can affect the precise amount of vertical columns.

Gingilcha makers use measurement to determine its size, which is another mathematizing process utilized in making the desired material. They used measurements to ascertain the dimensions, lengths, and spacing of the constituents that went into making the substance, as they indicated in their response. According to the replies, keeping spaces between neighboring rows is essential to *Gingilcha's* proper operation, unless it is difficult to filter well. To make sure the gaps are the proper size—neither too big nor too small—they applied estimate approaches they had acquired via experience. The respondent infers the length of the strings since they are arranged according to the dimensions of the substance being made. Finding the number of pairings that fit inside the *Gingilcha's* specified dimensions yields the number of vertical strings. This process is in line with the *Gingilcha's* overall size measurement. The respondents in their interview response verify that the

Gingilcha functions as needed using both conceptual (such as the spaces between rows) and physical (such as the length of the string) methods.

Designing is another mathematizing process in the process of making *gingilcha*. Designing refers to the intentional arrangement and planning of the *gingilcha's* structure, which directly impacts its functionality. The *gingilcha* net is designed with alternating vertical and horizontal threads. During their interview, the respondents mentioned how the net is composed of horizontal strings that tie vertical strings. Stability and a consistent structure throughout the Gingilcha are guaranteed by the carefully crafted pattern, which alternates the horizontal strings in a top-bottom pattern. Additionally, the spacing between rows of vertical strings is fundamental in the design for flour to be separated from the coarser grains. The design is adapted to guarantee that the gap size is appropriate for filtering, which is essential to the 'Gingilcha's' efficiency. The foundation of Gingilcha design is guaranteeing both structural soundness and utilization. What the Gingilcha does in terms of separating flour from hubs depends on the design choices made at this stage.

Furthermore, explaining is another mathematizing process that deals with articulating and justifying the choices made during the process, as well as communicating the underlying principles of the design. Commencing from string preparation until finalizing the material, the respondents provided in-depth explanations for their activities. For example, they provide justification by stating that in order to make the gaps smaller and more efficient for filtration, the strings must be intertwined. They also explain why vertical strings need to be arranged in pairs to ensure stability and proper function. According to their explanation, the horizontal string pairs serve two functions: they tie the vertical threads together and keep the space between them open for efficient filtration. It is further noted that the *Gingilcha* is stabilized and made to be durable by the alternating pattern of horizontal strings. Their responses emphasize that the shape of the dish influences the strength and flexibility of the *Gingilcha* and explain that they require a circular dish for the purpose of stretching the net prior to sewing. They also explained that the strings are used at the start of the sewing process to keep the net from coming loose from the sewing portion, emphasizing how important design is to the sieve's longevity and performance.

Sharing the information and logic behind the decisions taken during the sieve-making process is essential to making sure that others can comprehend and replicate the work.

The researcher noticed that the respondents began with one vertical string when creating the net of *Gingilcha*, starting from the second row. They then tied the remaining first string to one of the second strings, the remaining second string to one of the third strings, and so on until the last string was tied alone. In the third row, the participants knot the first two strings together, the second two together, and the third two together and so on to the end. Starting with one vertical string in the fourth row, they tied the first string to a second string, the second string to a third string, and the third string to a fourth string. The respondents repeated these processes until the final string was left untied. From the second row to the last row of the net, the processes continue, creating a few triangles between two adjacent rows, suggesting that the Gingilcha net is composed of numerous tiny triangles.

Because there was a change from the first row to the second, the second to the third, the third to the fourth, and so on until the last row, the respondents unintentionally or intentionally created mathematical concepts such as string addition and pattern development during this process. The respondents created parallel lines while creating the net, which led to a row formation in tying pairs of vertical strings. These parallel lines then developed into parallel chords in the final *Ginglcha* construction. Furthermore, the other mathematical concept embedded in the *Gingilcha* construction is the symmetry principle, since for the *Gingilcha* to be both practical and long lasting; its structure relies on geometric and symmetrical principles. The strings are placed in a symmetrical manner within the construction of *Gingilcha*. The construction of each row is similar, and the arrangement of the horizontal and vertical strings is consistent. For *Gingilcha* to filter well, the spaces between the vertical strings must be uniformly distributed, which is ensured by this symmetry. Following the net's stretching on a circular dish, the respondents began sewing in a spiral pattern to create a cone-shaped frustum for storing flour, demonstrating the existence of the concepts of volume and surface area.

The process of creating *gingilcha* requires optimization and problem-solving. The manufacturers of *gingilcha* modify their methods in accordance with the intended result and the constraints of the material at hand. When creating *gingilcha*, the primary optimization challenge is balancing the spaces between the strings to guarantee appropriate filtration. *Gingilcha* cannot separate flour from coarser (hubs) if there is an excessive gap between two consecutive rows. *Gingilcha* producers continually adjust and improve the spacing sizes to meet the functional requirements of *Gingilcha*.

A range of mathematical ideas, from fundamental counting and measurement to more complex geometry and optimization concepts and integral calculud are encapsulated in the process of creating *Gingilcha* out of straw strings. *Gingilcha*-maker illustrates how arithmetic plays a crucial role in craftsmanship through the thoughtful placement of strings, the application of patterns and sequences, and the balancing of gaps for efficient filtration. The *gingilcha*-maker is able to create a useful item that satisfies the community's practical demands by using mathematical reasoning in their art. This study emphasizes the close relationship between mathematics and culture, demonstrating how mathematical concepts and mathematizing processes are frequently incorporated into daily life and handicrafts.

4.3. **Hat making**

Different materials, such as straws—*chokorsa, migira, and kula* in the local dialect—are used to make the traditional hat. When cattle are being herded, shepherds typically build the hat. Male shepherd usually prepares it. Shepherds make hats to protect their heads from sun and rain and even from heavy winds. The shepherd collects a large quantity of wet chokorsa and migira from various locations, then cuts off the unwanted portion and chews it to make it flexible enough to form many strings from it. Before commencing the hat making, they have to produce as many strings as possible.

Figure 4.15 The process of hat making

The process of hat making begins with two strings tied together with chokorsa at the center. After tying the strings together, four spaces between them will be created, and weaving with a pair of chokorsa will begin. The sewing process continued, and as the size of hat increased in size, so did the space between adjacent strings, necessitating the addition of new strings. Since there were four spaces in between the strings, the shepherd has inserted four strings. In his interview, the hat maker described two processes for making a hat. The first method begins with a flattened piece about the size of a hand, which is then molded into a cylinder that wraps around the ears and flattens to four to six or seven fingers (palm). The second approach starts with a cone-shaped structure that wraps around the ears and is then twisted and flattened to the size of five to six fingers (palm).

The weaving technique in hat making is the same as net forming in *Gingilcha* construction, with the exception that in hat making, a single vertical string must be tied with a pair of horizontal strings, whereas Gingilcha makers tie a pair of vertical strings with horizontal strings. Another distinction is that hats require more strings made from wet chokorsa and migira than Gingilcha nets made up of dry strings.

4.3.1. **Mathematization process and mathematical concepts embedded in hat making**

As stated by Bishop (1998), there are different universal mathematical activities termed as mathematizing process in Aikenhead (2017) and mathematical concepts embedded in different cultures. Thus, in this case the researcher tried to explore the mathematizing process and mathematical concepts embedded in hat making. In his interview report, a hat maker stated that he began the hat with two strings tied together at the middle, resulting in four strings, and that weaving would follow. After weaving a few steps, the distance between adjacent strings widened, and he would insert four strings before continuing to weaving, creating eight strings and eight spaces between adjacent strings. The hat maker then added 8 strings, bringing the total to 16, with a distance of sixteen units between each pair of adjacent strings. The process continued until its completion. As the size of the hat grew, he added another 16 strings, totaling 32 strings with 32 spaces between them. Using the same process, the hat maker eventually completed the hat with 256 strings. Throughout this process, he kept track of both the number of strings and the spaces between each pair.

The hat maker counts the number of strings required to complete the work and monitors the formation of the structure of the hat by iterative counting (i.e., the addition of strings when the distance between adjacent strings widens). Starting with the knotting of two strings, the numbers recorded were 2, 4, 8, 16, 32, 64, 128, and 256. This is evidence of the cultural relevance of counting in the hat-making process, and Aikenhead would argue that this activity of "mathematizing" bridges cultural practices and numerical comprehension and pattern finding.

Measuring is the process of determining quantities such as length, weight, or size in comparison to a particular standard. According to the interview report, the hat maker claimed that the size of the hat varies depending on the individual (i.e., the size of the hat for the child is not the same as for an adult). When the flattened part at the top becomes one hand, the hat maker bends and begins sewing by creating a cylindrical form. As a result, he measures the flattened part with his hand, as is customary. Furthermore, he continues to sew, making a cylindrical structure that wraps around his two ears. Then he curved it and began sewing, making a flat surface equal to the size of four, six, or seven fingers. In making the hat, the hatter takes the measure of different portions of the hat using his hand and four fingers.

The participant measures in a non-standard manner, such as with his fingers and hand. By varying the spacing between the strings while keeping a continuous, expanding framework, the hat maker exhibits an intuitive understanding of scale, growth, and spatial relationships. These "informal measurements" are closely related to a mathematizing process in which the participant uses practical knowledge to calculate the size and fit of the hat. This method is consistent with Aikenhead's notion of how mathematics is incorporated in cultural practices through practical measurement rather than reliance on formal units of measurement.

Designing is the other mathematizing process embedded in hat-making. This designing, according to Bishop's perspective, is the process of using mathematical reasoning to create an object or find a solution to a problem. Understanding how to weave the strings into spiral shapes is necessary for the hat's design. As one hat maker described in his interview, the process of hat making involves sewing in a spiral form, starting from tying the two strings together until the completion of the hat.

The hat makers refer to this spiral construction as maramaa, and it exhibits a thoughtful geometric design. They are using trial and error along with mathematical pattern recognition to create a practical and aesthetically pleasing product. Strings are added recursively in a geometric progression that clearly illustrates a mathematical design process. Conversely, in place of chokorsa, hat makers utilize wool to tangle up the strings to make the hat more classy and gorgeous. They utilize two or more than two various colors, like white, green, red, pink, etc., simultaneously to build a pleasing pattern and design on the hat to make it more inviting and pretty. Consequently, they remodeled the hat's part.

The hat makers' iterative, methodical application of cultural knowledge in conjunction with abstract mathematical concepts like geometric progression and spatial symmetry to build the hat's structure reflects Aikenhead's mathematizing process. A functional hat is the end result of cultural design that is grounded in mathematical principles.

Explaining, which entails expressing the process or logic behind a solution or action, is another mathematizing process used in the hat-making process. The hat maker describes how the strings are sewn into the hat, detailing how each chokorsa is positioned in relation to the strings: "One chokorsa will be above a given string, and the other chokorsa will be below it," before going on to explain how this method creates a spiral. Additionally, the hat maker explains how the process is recursive, with the number of strings doubling and the gap between them gradually growing. The hat makers deduce that they will be able to give the hat the required shape and structure by progressively increasing the amount of strings. This provides a clear view of the hat's development pattern as it spirals.

On the other hand, the hat makers use their experiential knowledge to predict how many strings will be required and how the pattern will evolve, which is an inference process. This indicates a mathematizing process in which the participant makes predictions based on observed patterns, such as how the strings will intertwine with wool and chokorsa as they are added and how the structure will grow. This technique is strongly related to pattern recognition, which is an important aspect of Aikenhead's mathematizing process.

The hat makers made numerous strings of chokorsa and migira in various diameters. They used very long strings at the start, medium-sized ones in the middle, and shorter ones as they got closer to finishing the hat. Starting with tying the two strings with chokorsa, the hat makers added the strings in a geometric progression at various stages as the spacing between neighboring strings widened. This procedure involves the addition of strings in a geometrically progressive manner. Other mathematical ideas are embedded in the hat-making process, such as cone formation, cylinder formation, concentric circles, and other patterns generated by using different colors of wool.

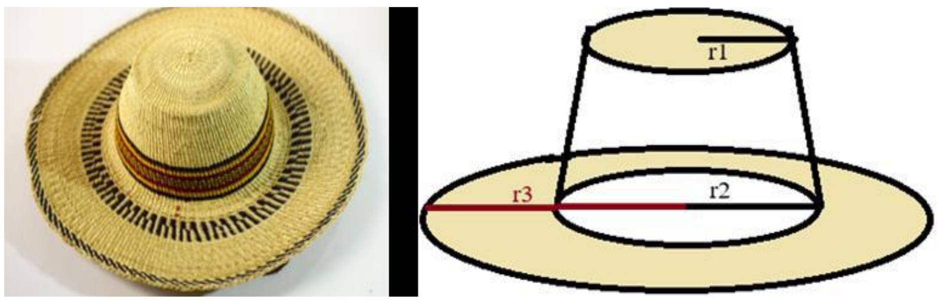

Figure 4.16 One sample of mathematical model of hat making

Basketry, which includes hat and gingilcha making, offers a rich cultural foundation for RME-based mathematization, in which mathematical concepts develop via observation of authentic actions rather than formal symbolism. Observing basketry operations exposes learners to counting, measuring, proportional reasoning, spacing, and pattern development as naturally ingrained processes. These activities highlight that mathematics emerges from purposeful human action and cultural demands, assisting pre-service teachers in understanding mathematics as a living practice directly linked to everyday life rather than an abstract and detached discipline.

Basketry activities demonstrate a wide range of mathematical principles stressed in the teacher education mathematics curriculum, notably geometry. These include circles, cylinders, symmetry, radius, diameter, area, perimeter, congruence, resemblance, and recurring patterns. The ordered arrangement of materials, consistent spacing, and balanced forms seen in basketry, hat making, and gingilcha making reflect spatial reasoning and geometric thinking gained through experience. The explanations and reasons provided by artisans demonstrate how reasoning, generalization, and logical thinking occur naturally within cultural activities, even when formal mathematical terminology is not explicitly utilized.

Using basketry as a context for mathematics instruction allows pre-service teachers to grasp mathematical topics, particularly geometry, in a relevant and culturally grounded manner. Considering basketry activities as a context allows to apply classroom mathematics in real-world circumstances, create mathematical ideas based on earlier experiences, and integrating mathematical knowledge to their cultural surroundings. This technique promotes problem-solving abilities, increases conceptual knowledge, and aids in the transformation of mathematical epistemological views by revealing mathematics to be relevant, contextual, and socially based. Finally, basketry provides pre-service teachers with a compelling way to connect geometry learning to everyday life and culture, improving both mathematical understanding and professional development.

4.4. **Weaving**

Weaving is a traditional method for making garments such as ***Netela***—a two-layer Ethiopian textile with attractive bands of multi-color design at the edges, usually made with silver or gold metallic threads and worn by women during ceremonial occasions such as funerals, holidays, and church services. As mentioned in their interview, netela differs from other cultural materials like *Gabi* and *Bulluko* because weaving is done by a single magii. However, with *Gabi*, weaving involves a pair of magii. Additionally, *Netela* is two-layered, while gabi and *Bulluko* are four-layered.

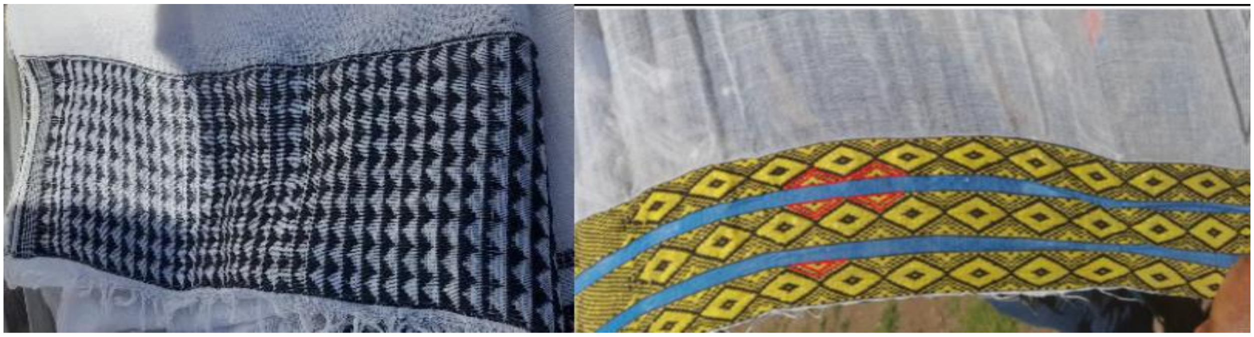

Figure 4.17 Different Netela with different designs

Gabi- a thick four-layer textile often worn by men during holidays, at church, funerals, and weddings. *Dirii* (Warps) and a pair of *magii* (weft) are used to make this cultural product; however, they use cotton-based *fo'aa* instead of the *magii*; thus, they use one *fo'aa* rather than a pair of *magii* (weft).

Figure 4.18 Gabi and Bullukko with different design

Bullokko- another well-known cultural product of the Oromo people, traditionally given by the groom to the bride's father during a marriage ceremony. *Bullukko* is similar to *Gabi* but the only difference is that the length of gabi is around 7m to 8m where as the the length of Bullukko is around 12m.

Degemo is a local cloth made from *kuula* (tilet) that protects against the cold during the night and morning. It is commonly woven by the Selale people, named after Degem, a town near Fitche. As mentioned during the interview, what distinguishes degemo from other cultural products, such as gabi and bullukko, is the material it is made from. It is made from threads of different colors. There are different types of degemo: woven with two manii, four manii, six manii, and seven or eight manii. As the number of manii increases, the design becomes more complex. For example, the picture below shows a degemo woven with six manii. The degemo maker explained during the interview that about 600 threads (warps) are tied on six manii, with 100 warps tied to each many. Since there are six *mergecha,* which corresponds to the number of manii, the weaver presses each *mergecha* one by one with his feet during weaving. When he presses with his right foot, the first mergecha tied to the first *manii* is pressed down, causing the warps attached to it to go down, while the remaining *manii* are pressed up, creating space for the *derbeto* to pass through horizontally. Next, he presses the 6th *mergecha* with his left foot, pressing down the *manii* tied to it along with its warps, while the other manii are pressed up, forming a gap for the *derbeto to weave*. In the third step, he presses the 2nd *mergecha* with his right foot, pressing down the *manii* tied to it and leaving the others pressed up to allow the *derbeto* to pass through. Then, he presses the 5th *mergecha* with his left foot, pressing down the *manii* tied to it along with its warps, while the remaining manii are pressed up to create another hole for the *derbeto*. The pattern then repeats with 1st, 6th, 2nd, 5th, 3rd, and 4th, continuing throughout the entire degemo-making process. This process appears in making other cultural products like *gabi, bullukko*, and *netela* only when the weaver wants to create different motifs. This distinguishes *degemo,* especially those woven by four *manii* or more *manii*.

Figure 4.19 Degemo woven with six manii, two manii and woven with guurtuu

The most important raw materials in weaving are *dirii, maagii (fo'aa),* and *kuula*. To prepare spun yarn and cotton yarn, the weaver spins the cotton fibers using the inzirtii, which is the spindle. Warping—the process of preparing warp threads (dirii) by stretching yarns that will be used for weaving—follows. The warp thread is then starched with wheat flour to strengthen it (*dirii*). However, during their interview, they revealed that they usually bought the starched thread and then wrapped it.

Figure 4.20 Warping and tying threads on looms (manii) and design making

After warping the warp threads (dirii), the weaver starts the weaving process by tying the dirii to the manii (loom), which takes about a day. The primary resources necessary for weaving are magii, which can be factory-produced or fo'aa (hand-spun by women from cotton), as well as dirii (warp) and kula (pre-dyed colored threads) utilized to create numerous designs and patterns on the final product.

Weavers usually employ a pair of manii (looms) and tie the warp threads to them. Responses to interviews indicate that they usually tie 500–600 warp threads in parallel to each manii (loom). This configuration allows the weaver to transfer the magii (fo'aa, or weft) transversely by raising or lowering certain warp threads to make an opening while moving the two feet alternately. The weft is then placed via this aperture using a tool called the darbatto (shuttle).

4.4.1. **Mathematization and Mathematical concepts embedded in weaving process**

During the weaving process, weavers perform counting while warping. As mentioned in their interview, depending on customer needs, weavers use 500 to 600 dirii (warps), warping 50 dirii at a time. One weaver also said he warped the threads for six gabs at once because warping takes time. As shown in the figure below, there are six pegs; he stretched the threads on each peg one by one until each contained 500 to 600 threads. Weavers also count while tying the warps on the loom. During the interview, they mentioned that they utilized non-standard units like ulee (stick) and their cubit to measure the length of gabi, bullukko, nexala, or degemo. For example, as mentioned by one respondent the length of one gabi is about 8 *ulee (stick)* or 32 cubits where as the length of Bullukko is around 16 *ulee* or 64 cubits. However, the length as well as the width of the material they prepare may vary depending on the customers' needs and orders. Measuring is another mathematizing process performed by weavers while weaving different cultural products. As mentioned earlier, they utilized non-standard units like cubit, fingers, yards, and sticks to measure the length as well as the width of the material they produce. This indicated that the mathematizing process of counting and measuring is commonly practiced in weaving.

Figure 4.21 Counting performed during warping and tying warps on looms

Designing is another mathematizing process performed by weavers. Weavers create different designs using threads of different colors on the edge of different materials, gabi, bullukko, netela and but in degemo the design is throughout the entire material. In designing the create different geometrical figures like congruent and similar triangles, congruent and similar rhombuses, different parallel lines, zigzag lines, and many other geometrical shapes. The participants reported two culturally significant techniques for generating woven designs. The first approach uses guurtuu, whereas the second employs four, six, seven, or even eight manii.

According to one reply, using *guurtuu* is a classic technique for design formation. In this method, the weaver physically handles the warps using a small stick called *guurtuu*, picking up two, dropping three, picking up five, dropping three again, then picking up two, and so forth, based on the pattern desired and position within the fabric. This method allows the weaver to add designs at will and spontaneously during the weaving process.

Figure 4.22 The process of making design using guurtuu

The second approach, regarded as a more recent innovation as stated by one respondent, includes preparing the pattern while tying the warps to four, six, or more *manii*. In this situation, the weaver does not modify the warps while weaving. Instead, the required pattern develops simply by alternating the usage of the feet and adhering to the predetermined warp settings.

For instance, as one respondent mentioned his design formation as follows.

> *"I tied 600 warps to four manii. The sequence had a repeating pattern: the first warp was attached to the first manii, the second to the fourth, the third to the second, and the fourth to the third manii. The 5th warp was then attached to the 1st manii, the 6th to the 4th, the 7th to the 2nd, and the 8th to the 3rd—and so on until the 600th warp was knotted. The keremela design is created by weaving a sequence of foot alternations with the tool known as derbeto. In the first step, the first and fourth manii are lowered, and the second and third are lifted. The weaver will next weave twice with the derbeto. In the second step, lower the first and third manii and elevate the second and fourth, followed by two weaves with derbeto. The final step involves lowering the second and third manii while raising the first and fourth and weaving twice. The fourth stage involves lowering the second and fourth manii and raising the first and third, followed by two rounds of weaving. In the fifth stage, the first and fourth are lowered again, while the second and third are raised, repeating the two-weave procedure.*

In the sixth step, the second and fourth manii are dropped while the first and third are raised, followed by two weaves. In the seventh step, lower the second and third and raise the first and fourth, followed by two weaves. In the ninth phase, the first and third manii are lowered and the second and fourth lifted, followed by two additional weaves''.

This indicated that the mathematizing process of designing is a common activity practiced by weavers in the weaving activity. The following are some examples of designs found in netala and gabi.

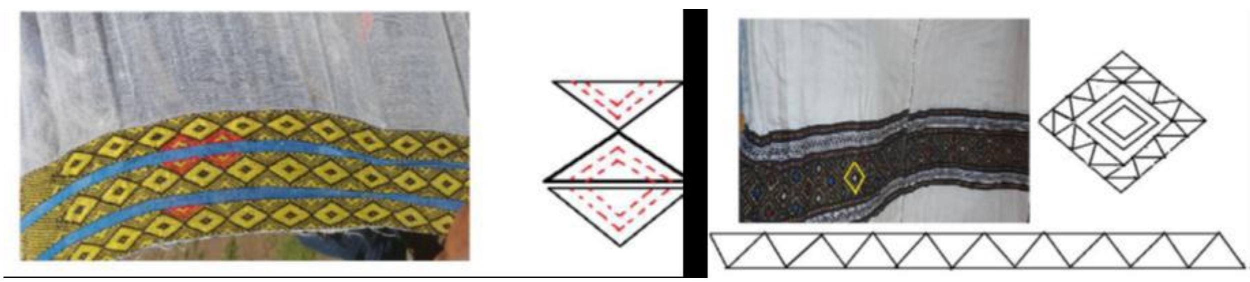

Figure 4.23 Different designs in netela and gabi making.

In addition to the above mathematizing processes like counting, measuring, and designing, geometric concepts are present in the shapes and motifs of woven fabrics. Plane shapes, such as rhombus and similarity, are evident in the weaving process. Furthermore, geometric rigid transformations such as translation, reflection, and rotation, as well as non-rigid transformations such as dilation, are mathematical notions incorporated in society's weaving practices. Thus, weaving activities and woven fabrics can be used as both background and media for learning mathematics, particularly geometry. Thus, weaving activities and woven fabrics can be utilized as a background and medium to teach mathematics. Woven fabrics can be used to teach about various plane figures, their similarities, and the congruence of plane shapes and geometric transformations. This allows students to investigate and discover the geometric concepts under study. Woven fabrics can also aid in distinguishing plane shape elements like sides, vertices, angles, and so on.

Weaving provides a meaningful cultural context for RME-based mathematization, in which mathematical concepts emerge spontaneously from ordinary practice rather than formal instruction. Pre-service teachers observe weaving activities and learn that measuring, counting, spacing, and pattern generation are all part of the process. These techniques highlight how mathematics is integrated in human existence and cultural activity, assisting pre-service teachers in seeing mathematics as a practical and socially placed subject rather than an abstract body of information disconnected from reality.

Weaving exemplifies a variety of geometric ideas covered in the teacher education mathematics curriculum. These include parallel and perpendicular lines, angles, congruence, similarity, tessellations, and transformational geometry like reflection and translation. Weaving produces repetitive patterns and organized designs that demand spatial reasoning, proportionate thinking, and logical sequencing. Weavers' explanations and justifications reflect experience-based reasoning, demonstrating that abstract geometric notions are embedded in culturally learned activities.

Weaving as a context for studying mathematics helps pre-service teachers understand mathematical topics, particularly geometry, in a meaningful way. Observing weaving allows kids to apply classroom mathematics in real-life settings, build mathematical concepts based on prior experience and their surroundings, and recognize mathematics' cultural importance. This technique improves problem-solving abilities, fosters deeper conceptual comprehension, and aids in the change of mathematical epistemological beliefs. Finally, weaving helps pre-service teachers see mathematics as relevant to everyday life and culture, resulting in better geometry comprehension and overall mathematical achievement.

4.5. **Pottery**

Making pottery starts with digging a waterlogged area about 1.5 m to 2.5 m or 3 m deep, as the needed clay soil is found at this depth. As the Goba participants noticed during there are two types of clay soil: ‘hache,’ which is sandy and rough, and ‘lotica,’ which is red, very smooth, and used to strengthen the pottery. After collecting the soil, they expose it to the sun for one or two days to dry. Because both lotica and hache are important in making pottery, they mix an equal amount of both or in proportion. After drying, the next step is beating the dry soil with a stick to make the soil fine-grained, and then sieving it using a gingilcha to separate the fine from the coarse material. Next, they mix it with water and knead it with their hands until it becomes extremely strong. As a result, most potters do not start making pottery immediately; instead, they wait one or two days.

In addition to soil, pottery making needs plastic material for shelter to cover from rain, a *gingilcha* for separating fine-grained soil from the coarse-grained one, torn clothes for making shapes and for protecting from the heat of the sun, and steel for carving the place where it is most abundant. Furthermore, the potter utilizes shoe soles to smooth or level the pottery surfaces; small stones to refine the rough texture; plastic materials for a sleek finish; and wood, straw, and fire to strengthen the pottery and increase its durability by removing moisture.

Figure 4.24 Process of pottery making from its source of clay to draying and firing in Goba

There are different types of pottery. *Shekla Dist (Dist)* is a type of pottery made from clay soil. This material is used for cooking the so-called *wot*, especially in rural areas. As respondents stated in their interviews, the *wot* cooked using a dist has more flavor and is tastier compared to the one cooked in a steel dish. This dist is used to cook *doro wot,* which is popular in Ethiopian society, at numerous rituals and holidays. As mentioned by the dist maker, it consists of three parts: the bottom, the wall, and the lid. They began by building the cone-shaped frustum wall, and after approximately a day, when it was strong enough, they began working on the bottom. Finally, they prepared the upper part, the cone-shaped lid made to fit the size of the frustum of cone-shaped wall.

Figure 4.25 Dist of different size

The *Mitad (Eelee)* is a circular object that has a diameter ranging from 2 to 4 finger spans. The one with a smaller diameter is used for cooking *Ambasha*-the local bread that meticulously divided into separate sectors, while the larger one is for cooking *Enjera (budenaa).* This local material is raised at the corner and gradually slopes down toward the center. The other type of mitad has a circular base and a cylindrical shape at the corner, and this type of mitad is used for cooking *difo dabo* (local bread).

Figure 4.26 Mitad of different sizes and designs

The **pot** is a cultural artifact made of clay that local people highly value. It serves many functions, including boiling water, cooking cabbage, cooking *shummo (nufro)*, and preparing *markaa (genfo)*—the most delicious and healthy cuisine popular among the Arsi community.

Figure 4.27 Markaa local food common in Arsi

Making a clay pot involves a creative and time-consuming method that has been used for ages. First, fine clay, preferably red clay, is carefully selected and then kneaded and honed to the desired texture. The potter shapes the clay by hand or with minimal tools to eventually make the pot. After finishing the pot, the pot maker placed it in the sun to dry it. Finally, they burned it to make it dry and sturdy enough to withstand the heat required for cooking.

This pot comes in a variety of sizes to meet the needs of each family. The smaller ones are usually used for everyday meals like markaa, while the larger pots are reserved for big families and festive occasions, such as when building a house with teamwork known as jig and for communal gatherings like weddings and burial ceremonies, where food. The smaller ones are usually used for everyday meals like markaa, while the larger pots are reserved for large families and festive occasions, such as building a house with teamwork known as jig and communal gatherings like weddings and burial ceremonies, where food is cooked in bulk.

Figure 4.28 Pots of different sizes

The jar is another local material made from clay soil and has many purposes in the daily activities of society. This local material is mainly made by women, and it is utilized to fetch water from rivers and springs. It is also used for storing water, *Tella,* and milk. This makes the jar more than just a container; it symbolizes the essential link between domestic life and natural resources, ensuring that families have access to clean water throughout the day.

As stated by the respondents, nowadays, it is used in the distillation process to make a locally distilled alcoholic beverage called Areqe, and the local community also utilizes it for churning milk to produce butter. The jar comes in different sizes, with the larger one, termed "gan," used for preparing Ethiopian beer made from barley called *Tella*.

Figure 4.29 Jars of different sizes

A ***jebena*** is a cultural artifact constructed of clay with a spherical base, a narrow neck of a cone-shaped frustum, and a spout. It is a traditional Ethiopian coffee pot that holds great cultural significance in the country's coffee ceremonies.

Figure 4.30 Jebena of different designs and sizes in Fitche

Gulilat is another cultural artifact made of clay that is placed at the tip of a straw house's roof to cover the top and has both practical and symbolic purposes. It is made up of three to four frustums of cones of varied sizes, in which their diameter varies and decreases from bottom to top. This is meticulously made to fit perfectly at the peak of the thatched roof, preventing rainfall from entering the home. It protects the home and strengthens the structure by directing rain away, especially during strong rains.

Figure 4.31 Gulilat that placed at the tip of a straw house

4.5.1. **Mathematization and mathematical concepts embedded in pottery making**

Measuring is one of the mathematizing processes that pottery makers utilize when determining the amount of hache and lotica soil needed for making mitad. According to an interview, a 34-year-old experienced participant who learned from her friend in grade 8 mixes lotica and hache proportionally. In contrast, other pottery makers mix ¼ hache with 100 kg of lotica. Measuring activity is a task performed to determine the amount of clay soil needed and the size of the mitad customers require. They use the yard as a measurement unit to gauge their materials' size. For example, during their interview, they mentioned that a small mitad has a diameter of two yards, a medium size has a diameter of three yards, and a large mitad has a diameter of four yards.

The other mathematizing process involved in pottery making is explaining, which is characterized by the justification they made. In her interview report, Goba I stated that mixing in equal ratios of hache and lotica is not only for preserving the strength of the pottery (mitad) but also to shorten the drying time of the mitad. In her interview, she also addressed that she learned from experience that reducing the hache lengthens the mitad's drying period in addition to weakening it and causing cracking. Goba II’s response indicated that the reason why the shape of mitad is circular rather than rectangular or triangular is to prevent pouring the batter on the ground. If the mitad we prepared has a corner, when pouring the batter, changing from one direction to another could cause the batter to spill onto the ground. Therefore, the shape of the mitad should be circular to avoid spilling the batter at the junction. In her response, Fitche I supports the issue raised by Goba II, and she mentioned that the batter for Enjera is not as solid as that of bread, which makes it easier to manage and change the shape of Enjera. According to Goba I's interview, women start pouring batter of Enjera at the perimeter of the circular surface and finish at the center because the surface is purposefully slanted in the traditional mitad-making process, rising at the margins and gradually slanting down toward the center.

Another mathematizing process in pottery making is designing, which is characterized by shaping the surface of the mitad. While making the mitad, the pottery maker intentionally controls the gradient by creating a circular shape with a slightly sloping surface from the edge toward the center to control the flow of the enjera batter inward, thereby controlling the thickness of the injera.

Furthermore, the design is expressed by shaping the mitad circular to guarantee consistent cooking and prevent batter from overflowing, highlighting both geometric and functional choices. Bishop's design process involves planning, creating, and sculpting objects and spaces with specific goals in mind, which usually reflecting aesthetic, functional, or symbolic purposes.

This design process is also seen in the design of the coffee pot called Jebena. The maker of the jebena does not design it haphazardly but with both practical and cultural meaning in mind, for example, the wide, rounded bottom, which permits stability and easy passage of heat in an even distribution throughout the coffee, and the narrow neck, which slows down the escape of steam and helps maintain pressure and heat constant. So that they might pour the coffee that had been brewed without disturbing the coffee grinds that had settled at the bottom, they positioned the spout low on the body of the jebena. These are instances of thoughtful spatial design evolved out of centuries of actual use and cultural custom. The interview response provided by Fitche II supports this justification. Her interview is as follows:

> *The narrow neck of the Jebena is used to control the coffee's flavor as it evaporates. The taste of coffee boiled in a teapot differs from that of Jebena coffee because the flavor evaporates more easily when boiled in a teapot. We positioned the spout low on the Jebena's body to pour brewed coffee without disturbing the grounds that have settled at the bottom.*

The respondent in Goba supports the justification raised by the respondent in Fitche by stating that the narrow neck helps the evaporation of the coffee not to escape easily and hence the secure the flavor of the coffe.

The counting activity is also another mathematizing process embedded in pottery making which is reflected through quantifiers of materials, sizing by diameter, and the number of mitads they prepared every day. In their interview responses, Goba Mitad producers stated that they employed 100 kilos of lotica and 100 kilograms of hache, both from sandy and clay soils. Similarly, a pottery maker from Fitche reported that seven mitads were created from a combination of seven kuna of clay soil and seven kuna of a black and sandy soil blend (mixture of 4 kuna of black soil and 3 kuna of sandy soil or mixture of 5 kuna of black soil and 2 kuna of sandy soil). To quantify "kuna" (local unit of measurement) as units of clay and mixtures of earth, such as 7 kuna of clay and 4 kuna of black earth and 3 kuna of sandy earth, discrete units must be enumerated. The proportional change (e.g., 4:3 or 5:2 proportions of soils) demands other mathematical operations such as addition and subtraction. Moreover, mitad makers in Fitche and Goba indicated that mitads are graded small, medium, or large based on their diameter, showing how numerical sizes impact mitad sizes. For example, mitads with diameters two, three, and four yards are graded small, medium, and large, respectively. Pottery production is highly reliant on numbering for material preparation, size, and scheduling. Potters use Bishop's universals—quantifiers, number operations, fractions, ratios, and measurement—through culturally rooted practices (e.g., kuna units and seasonal adjustments).

Figure 4.32 Different pottery and their making process

Along with the mathematizing process, pottery making also incorporates various mathematical concepts. For example, figure 4.23 illustrates the concept of volume, which is one of the mathematical concepts embedded in pottery making. Geometrically, there are various shapes, such as hemispherical, elliptical, and cone-shaped frustums. In addition, pottery is rich in other mathematical concepts such as the concept of circle, radial and rotational symmetry, concept of area, number operations, fractions, ratios, concept of slope (the concept of gradient), surface area, and others.

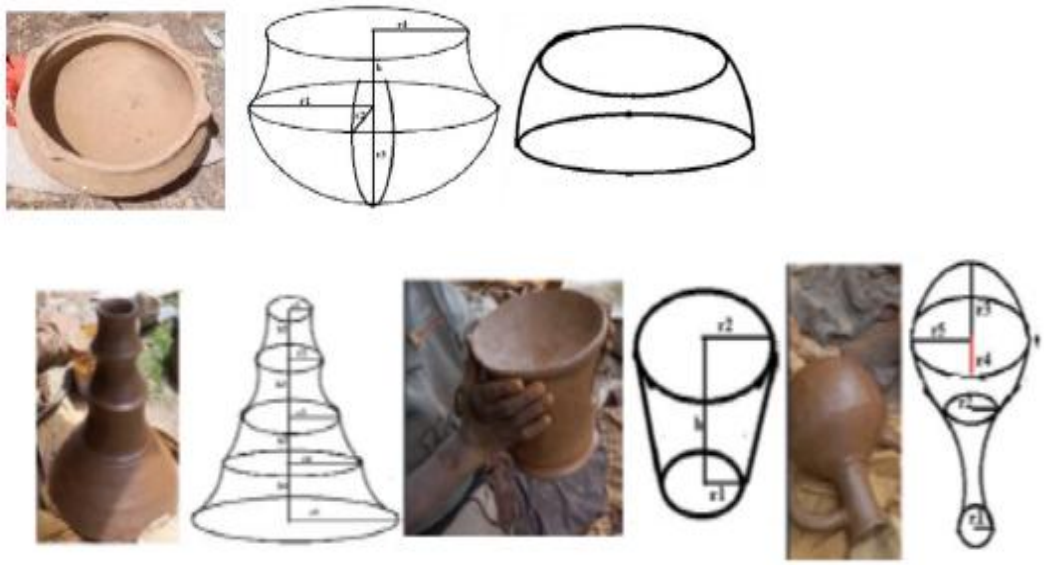

Figure 4.33 Some geometrical concepts embedded in pottery making

Pottery making, particularly the making of mitads and Jebenas, provides an important cultural context for RME-based mathematization, in which mathematical concepts emerge spontaneously from everyday practice. Pottery producers measure materials, mix soils in predetermined proportions, count units, and determine sizes according on customer specifications. These activities demonstrate an intuitive grasp of number operations, ratios, fractions, and measuring. Observing such actions helps pre-service teachers understand that mathematics is embedded in human activity and everyday life, rather than existing solely as abstract rules and formulae.

Geometric reasoning may be shown in pottery manufacture with the round shape of mitads, the hemispherical and frustum forms of Jebenas, and the deliberate control of slope and surface area. Circle, area, volume, symmetry, gradient, and proportionality—all fundamental subjects in the mathematics curriculum for teachers—are used to ensure optimal cooking, heat distribution, and functional design. Pottery makers' explanations and arguments illustrate experiential reasoning and abstraction, demonstrating how formal geometric notions are grounded in practical cultural knowledge.

Connecting mathematics concepts in the teacher education curriculum to cultural practices like pottery production helps pre-service teachers internalize mathematical concepts, especially geometry. Pre-service teachers can relate classroom mathematics to real-world circumstances by observing pottery manufacturing, build mathematical concepts based on prior experiences and observations, and gain a deeper understanding of how mathematics is related to human life. This technique improves problem-solving abilities, fosters deeper conceptual comprehension, and aids in the transformation of mathematical epistemological views by portraying mathematics as relevant, helpful, and culturally grounded. Finally, employing pottery as a learning setting improves geometry performance and strengthens the link between mathematics, culture, and everyday practice.

4.6. **Building a house**

Straw houses are very common houses in Ethiopia in rural areas. This straw house is made up of straw, wood, and mud. It has two parts: the wall and the roof, in which the former is made up of wood, mud, and straw, whereas the latter is commonly made up of wood and straw. The wall of the house is either cylindrical with a cone-shaped roof or a rectangular prism, or a hexagonal shape of wall with a pyramid roof. This straw house is built by teamwork called *jigi*. The house, which has a cylindrical-shaped wall and a cone-shaped roof, has a pole called utuba in the center that supports the roof. The owner positions the pole of the house and doubles its radius to determine the diameter of the house ahead of building its wall. Based on the owner's needs, they begin measuring the house's radius by starting from the pole and working their way down using their foot and then making a circle with the help of a rope and a small stick. A house with a radius of at least 16 feet may be constructed if they choose a wide home. As illustrated below, the home's owner might require two cylindrical walls by partitioning the inner one into four (bed area, fire area, gate, and shelf area). The inner cylinder wall, with a radius of 10 feet, is approximately 5 to 6 feet taller than the outer cylindrical wall, which has a radius of 16 feet but is about 10 to 12 feet shorter than the length of the pole. By establishing four equal spaces, the inner wall is separated into four equal sections. This is accomplished by using the pole as a reference frame to measure the same distance up and down, as well as right and left. To build the bed and fire area, for example, they plant one peg four feet up from the pole and one peg four feet down from it. The rope is connected to the inner wall's two ends in both directions by attaching pegs in two locations. Similarly, they partition the gate and the shelf area by measuring 4 feet to the right and 4 feet to the left, finally connecting the inner wall's end to its opposite end wall using ropes. The bed area serves for sleeping, the fire area for cooking, the shelf area is for storing different materials, and the gate is for entering and exiting purposes.

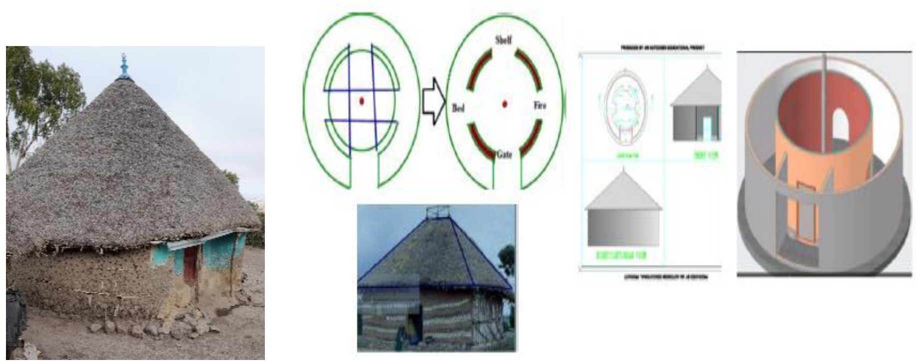

Figure 4.34 Straw house

4.6.1. **Mathematization process and mathematical concepts embedded in the house building**

The core Bishop's mathematical activities, referred to as the mathematizing process in Aikenhead (2017), were discovered in the construction of a straw house in a rural area of Oromia, Ethiopia. One of the mathematizing processes in this straw house construction is counting. Respondents reported that they use the principle of counting to quantify construction materials and structural elements. For example, to determine the quantity of straw needed to roof a house, they used a variable number of bundles (donkeys) depending on the size of the house (roughly 10 donkeys for a small house, 15 donkeys for a medium-sized house, and between 20 and 25 donkeys for a large house). The interview reveals that counting is interwoven in the division of the buildings into four divisions, each of which has a specific purpose. Furthermore, as indicated by the interview, counting is integrated in the quantification of the lengths of the house's radius, as well as the distance between the pole and the pegs used to form partitions.

Furthermore, counting is employed for both enumeration and arranging goods and labor. This is evident when the carpenter creates straw or steel dwellings of various shapes, such as four- or six-sided structures. He demonstrates proportional reasoning and the capacity to effectively detect differences, especially when splitting the diagonals of a four-sided home into two equal portions. This suggests that counting is used to structure both physical activity and mathematical relationships.

Locating is another mathematizing activity that occurs throughout the straw house construction process, from planning to completion, as it involves establishing spatial positions or orientations. In straw house construction, positioning and spatial orientation play significant roles in the planning and building stages. The first task involves determining where to plant the central pole of the house, as this position serves as a reference frame for the entire structure. Before planting the pole, they put pegs and then made a circle with the required radius for the inner and outer walls as well. The interview revealed that to partition the inner wall evenly, pegs were placed at precise distances in all four directions. One respondent described in his interview as follows:

> *To divide the inner circular wall into four sections, we measure four feet up from the center and we measure four feet down to the opposite side, put the pegs, four feet to the left and four feet to the right of the center, and put the pegs.*

They also employ locating for relational purposes, such as sleeping in front of the kitchen or shelving in front of the gate.

The carpenter explained that by tying the ropes to the four pegs, he caused the two diagonals to intersect at a point. The carpenter's operations included a sense of symmetry, alignment, and geometric positioning, identifying the location at which he needed to plant the roof pole, proving the existence of spatial reasoning in straw or steel house construction.

Another mathematizing process utilized in home construction is measuring, which is used to measure length, area, and volume. Measuring is dominant in the entire house construction process. The community utilized local units such as an arm to measure the depth of the dug foundation for planting wood to build a wall, a foot to measure the circle's radius, and for partitioning, and a bundle (donkey) to determine how much straw to use to cover the house's roof. One respondent stated in his interview response as follows:

> *The amount of straw depends on the size of the roof of the house. For the small house, we need around ten bundles of straw (10 donkeys), for medium, it needs around 15 donkeys, and for the big house, we need around 20 to 25 bundles (20 to 25 donkeys).*

The steepness of the house's roof, which is influenced by the size of the pole, the inner and outer walls, and the waleta, further demonstrates how local measuring units determine both the utility and the design.

To ensure accuracy while building a four-sided house, the carpenter used nonstandard units of measurement, such as arm, finger, and rope, as well as the standard unit of measurement, the meter. The process of equalizing the two diagonals exemplifies geometric precision rooted in practical necessity. During the interview, he claimed that, "If the difference is 10 cm, we shift one corner 5 cm inward and the other 5 cm outward." This demonstrates relational reasoning—using measurement to maintain balance and proportion. This demonstrates the contextual mathematization of geometry using bodily measuring activities. Measurement is central to his work. He explains, “We need a very long rope, around 50 meters or 100 arms,” using both standard and non-standard units to ensure correctness. His attention to “making the two diagonals equal” reflects geometric precision grounded in practical necessity. When he states, “If the difference is 10 cm, we shift one corner 5 cm inward and the other 5 cm outward,” he demonstrates relational reasoning—

using measurement to maintain balance and proportion. This reveals a contextual mathematization of geometry through embodied measuring practices.

It is impossible to construct a house without a design; thus, people try to design it before creating their dream home. The process of designing an architectural plan involves both planning and construction, both of which call for geometric and spatial principles. The local community uses the mathematizing process of designing while building their homes. The creation of a circle with instruments such as pegs, rope, and a sickle depicts geometric construction. The use of radial symmetry, which provides equal partitioning and visual balance, is another example of design. Furthermore, the design process is expressed by the construction of a cylindrical wall and a canonical or frustum-shaped roof. The pole, two walls, and a walta, a frustum-shaped object, are used to determine the steepness of the roof of the house.

On the other hand, design is mirrored in the in-house creation of diverse shapes, such as a quadrilateral with trapezoidal and triangular roofs or a hexagon with a triangular roof. The carpenter remarked that the clients' interests should be considered while developing the form of the house. In his interview response, he states, "If my customer is interested in a house with a quadrilateral base and a roof shaped like a pyramid with four triangular faces, I start designing using ropes and pegs." While prioritizing his clients' demands, the carpenter converts their practical requirements into geometrical models that offer geometric construction and proportionate division, notably the division of sides into equal halves and thirds. Designing, therefore, connects conceptual geometry with practical architecture, providing a genuine setting for RME-based mathematization.

The other mathematizing process involved in house construction is explaining, which articulates the reasons and justification for the design decisions. In their interview responses, the respondents indicated that they created a partition to make their home more beautiful and smart. Explanation is also evident in the practical use of functional domains. Furthermore, design and measurement decisions are founded on local logic and lived experiences. On the other side, the carpenter clarified why geometrical features are important. According to his interview response, if the two diagonals are not the same length, there is no ‘squadra’, which impacts the usage of steel for the roof and ceramics for the floor. In this way, the carpenter links geometric principles (diagonal equality) to practical results (material efficiency and aesthetic appeal).

The process of building a house involves many different types of mathematical ideas. The design incorporates a variety of mathematical aspects, including canonical and frustum roof shapes, congruent arcs, arc lengths, circular foundation structures, and radial symmetry in arrangement. Other mathematical concepts include measurement, such as lengths measured in feet or traditional measures like arm's length; material estimation, such as straw bundles; and proportional reasoning. This explanation demonstrates how material requirements, such as the amount of straw required, rise in proportion to the size of the house and roof area.

Additional mathematical notions are also required for house construction. The arrangement of pegs reflects coordinate principles, which use distances calculated from a central point in four directions, implying the usage of axes—up/down and left/right—to guide spatial organization. Volume is used to determine how much straw is required for the roof. Angle and roof steepness are assessed via the angle produced between the central pole and the cone's base, revealing an implicit understanding of trigonometric connections that govern the construction process.

House construction offers an excellent scenario for RME-based mathematization, in which mathematical concepts emerge spontaneously from culturally significant and practical actions. When planning and building houses, builders measure, estimate, count, and compare dimensions, frequently utilizing regionally based procedures and instruments. These activities instill ideas like area, perimeter, volume, diameter, radius, and proportional relationships in everyday life. This demonstrates that mathematics, particularly geometry, is actively employed to solve real-world problems in the community rather than being limited to abstract classroom knowledge.

Using house construction as a learning context reveals various geometric concepts taught in the teacher education mathematics curriculum. These include polygons like hexagonal and octagonal shapes, cylinders in structural elements, arcs and arc lengths in curved designs, and circles and concentric circles in floor plans. Wall alignment, roof construction, and spatial organization all use symmetry, spatial relationships, and triangular similarity and congruence. Even in the absence of formal mathematical symbolism, these reasoning, explanation, and decision-making exercises show how abstract geometric concepts are incorporated into culturally grounded construction processes.

Using house construction as a context for mathematics learning can assist pre-service teachers get a deeper and more meaningful understanding of mathematics, particularly geometry. Observing how geometric principles work in real-world building allows pre-service teachers to connect classroom mathematics to their daily lives and cultural context. This strategy improves problem-solving abilities, facilitates the building of mathematical concepts based on prior experience, and aids in the transformation of mathematical epistemological views by demonstrating mathematics' relevance, practicality, and connection to everyday life. Finally, this leads to higher geometry performance and a better understanding of the link between mathematics, culture, and real-world application.

CHAPTER 5: EFFECT OF RME-BASED MATHEMATIZATION ON THE EPISTEMOLOGICAL BELIEFS, GEOMETRY ACHIEVEMENT, AND PROBLEM-SOLVING SKILLS OF PRE-SERVICE TEACHERS

The previous chapter, which explored the mathematization and mathematical concepts embedded in the cultural activities, was used as a basis for designing RME-based mathematization that was used as an intervention to investigate the effect of RME-based mathematization on the epistemological beliefs, geometry achievement, and problem-solving skills of pre-service teachers. This chapter begins with discussing how cultural mathematics is integrated teacher education curriculum particularly plane geometry through RME approach that utilized REACT strategy to implement the intervention and presents the findings obtained from it.

5.1. Integrating Cultural Mathematical Knowledge into Teacher Education Curriculum through RME-Based Mathematization Approach Supported with REACT Strategy

In contemporary mathematics education, however, formal mathematical knowledge cannot be seen as developing in isolation but rather through collective and cultural human activities. For effective teaching of future mathematics teachers, they must not only be well-versed in the abstract discipline of geometry but also value and incorporate mathematical knowledge rooted in local cultural activities; this approach makes the process of teaching and learning geometry more meaningful to them.

The cultural activities of the Ethiopian society, especially weaving, basketry, pottery, and traditional house construction, are rich in mathematical ideas, particularly geometry. The daily activities of the society reflect profound intuitions about geometrical shapes, proportion, spatial reasoning, and patterns, which are fundamental to plane geometry. Integrating cultural mathematics into teacher education curricula provides a means to bridge learners’ lived experiences and formal mathematics (Gbormittah et al., 2025).

In this study, the researchers utilized a RME-based mathematization approach supported by the REACT strategy to integrate cultural mathematics into the teacher education curriculum. This is because the foundational RME lies in the view of mathematics as a human activity instead of considering it as a static body of knowledge. Freudenthal stated that mathematics education should allow students to recreate mathematics by partaking in the process of mathematizing both reality and mathematics (Freudenthal, 1983). That is, students should move from contextual problem situations into mathematical abstractions, rather than beginning with ready-made formal abstractions (Freudenthal, 1973).

In the RME-based mathematization approach, the main concerns are context, activity, and reinvention. Besides Hans Freudenthal, scholars like Koeno Gravemeijer and Marja van den Heuvel-Panhuizen have expanded the heuristics of progressive mathematization and emergent modeling in RME, in which situational, reference, general, and formal are the four levels that bridge learners' informal knowledge to more formal mathematical understanding (Gravemeijer, 1994; van den Heuvel-Panhuizen, 2001; Cobb, Zhao, & Visnovska, 2008).

RME has six interrelated pedagogical principles. The first principle, the reality principle, addresses how mathematics should be taught and how the scenario should be presented. According to this principle, learning should begin with circumstances that are meaningful to the learners and taken from real-world contexts or experiences. The activity principle, on the other hand, governs how students should be involved in mathematical learning and doing it. In this principle, learners should be fully involved in solving mathematical problems, modeling situations, and reflecting on what they obtained (Cobb et al., 2008).

The level principle is about how learners move from informal, contextualized reasoning (horizontal mathematizing) to increasingly abstract and formal mathematical reasoning (vertical mathematizing) (van den Heuvel-Panhuizen, 2001). The other pedagogical principle is the intertwinement principle that discusses how the different mathematical strands, like geometry, measurement, algebra, and other problem-solving, are interwoven rather than treated in isolation (Gravemeijer & Stephan, 2002).

The one concerned with how learning occurs via social interaction—dialogue, negotiation, and reflection—with peers and the instructor is the interaction principle (Cobb et al., 2008). The last one is the guiding principle that states how teachers provide scaffolding, questions, and prompts that guide the process of reinvention to let the learners construct their own mathematical knowledge (Gravemeijer, 1994).

As illustrated in the diagram 5.1, the mathematization process followed sequential steps beginning with the contextual problem (reality principle), progressing through horizontal mathematization (informal modeling of real-world problems), and culminating in vertical mathematization (formal reasoning, generalization, and abstraction). The instructor functioned as a facilitator, guide, and evaluator, supporting students as they navigated these stages. Pre-service mathematics teachers worked individually and in groups to solve problems, share and discuss their strategies, and collectively refine their solutions during whole-class discussions. This process aimed to foster guided reinvention, enabling learners to construct mathematical knowledge from their lived experiences.

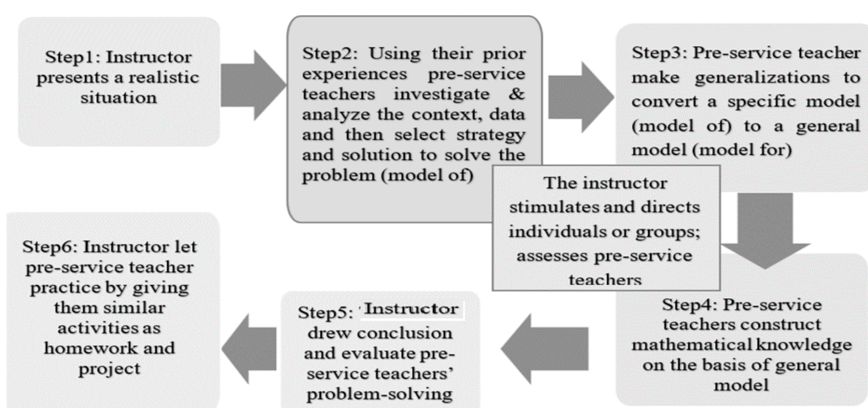

Figure 5.1 Steps in mathematizing geometry problems in the teaching learning process

Thus, in this study, the researcher selected RME-based mathematizatio approach to integrating cultural mathematical knowledge into teacher education, focusing on plane geometry for several key reasons. Firstly, the approach emphasizes contextual rootedness by engaging learners with their cultural practices, such as weaving and pottery, making geometry more meaningful and accessible. This connection legitimizes indigenous mathematical thinking and contextualizes learning. Secondly, RME bridges informal and formal knowledge, encouraging learners to translate cultural activities into mathematical representations and refining these into formal structures, aligning with teacher education objectives.

Besides, the RME model encourages active learning and agency, which enable pre-service teachers to research and model their cultural contexts. Thus, pedagogical awareness is attained. Interconnectivity between geometry and cultural artefacts is established; geometric objects occur as real-world artefacts. In other words, geometry is not isolated but interwoven with cultural practices. More importantly, the guidance principle leads lecturers to guide pre-service teachers in reflecting on the cultural derivation of mathematical ideas and the content of the formal geometry curriculum in order for them to develop culturally responsive teaching tasks.

In general, RME offers a theoretically sound and practically consistent framework for mathematics educators to connect with cultural mathematical knowledge, integrate geometry instruction in context, and assist pre-service teachers in building both subject and pedagogical expertise.

RME-based mathematization informed the theoretical and pedagogical direction of this study, whereas the REACT strategy (Relating, Experiencing, Applying, Cooperating, and Transferring) was incorporated into the instructional design to transform RME's concepts into actual teaching practices. In fact, each component of the REACT strategy is inherently consistent with the six RME principles. The REACT technique was viewed as an operational structure that facilitates the implementation of RME-based mathematization in real-world learning situations, enabling potential instructors to experience and reflect on mathematics as a cultural, social, and human activity. It was included as a useful teaching resource for adopting RME concepts in the classroom. Each REACT component is inherently compatible with the six RME principles.

In RME, relating conforms to the reality principle, because students begin studying geometry by connecting geometric principles to cultural experiences such as home construction and weaving design. The experiencing is consistent with the activity principle, in which pre-service teachers use materials, build models, and investigate geometric patterns contained in cultural practices. The applying reinforces progressive mathematization, in which pre-service teachers move from contextual understanding to applying concepts in structured geometrical problems, whereas the cooperating promotes the interaction principle, which encourages collaboration and shared mathematical knowledge construction. The transferring embodies the level and guiding principle since pre-service teachers were allowed to deduce or generalize geometric concepts from cultural activities to new, formal mathematical situations.

RME-based mathematization guaranteed that the incorporation of cultural mathematics into the plane geometry curriculum was founded on contextual meaning, guided mathematization, and gradual abstraction. Meanwhile, REACT translated these ideas into organized, participatory, and reflective classroom experiences. Thus, by including REACT into RME-based mathematization, the intervention guaranteed that learning was both contextually grounded and pedagogically organized, thereby enhancing pre-service teachers' conceptual understanding and instructional competency. Furthermore, the RME-based mathematization approach, which utilized the REACT strategy to implement it, provided pre-service mathematics teachers with the opportunity to learn geometry in culturally appropriate settings while also experiencing a transformational teaching model that they could apply to their future classrooms. As a sample, the following sample lesson plans were utilized during intervention. The following is the diagrammatic representation of the process of integration

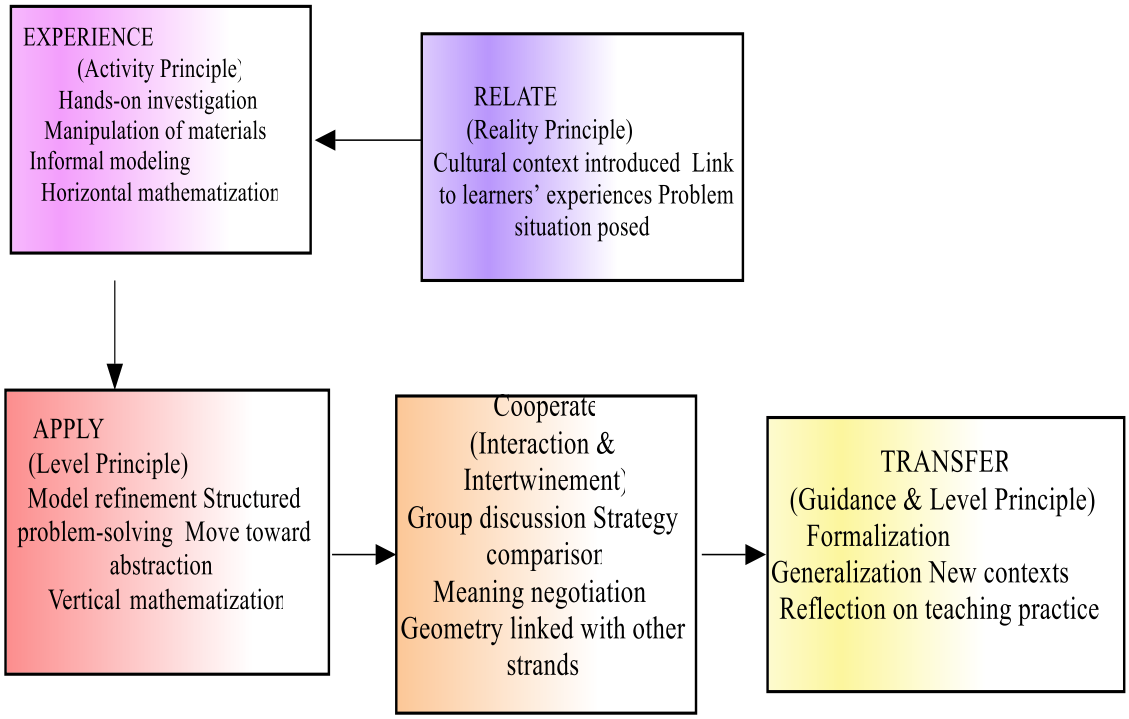

Figure 5.2. The process of integrating of cultural mathematics teacher education curriculum using RME principles and REACT strategy

Game of Triangle Congruence (Roof Construction Context)

Duration:100 minutes

Level : Pre-service Mathematics Teachers

Approach: RME-Based Mathematization

Strategy: REACT

Competency

After completing the lesson, pre-service teachers will be capable of:

- Determine and explain the sufficient and necessary requirements for triangle congruency.
- Engage collaboratively in a contextualized problem-solving activity reflecting cultural mathematical practices.
- Appreciate the role of cultural and practical knowledge in understanding geometric concepts.

Present a Realistic Situation (Relate) – 10 minutes: At this stage, the instructor offers the scenario*:* A carpenter constructing an inclined roof must make several triangular trusses that must be exactly congruent for roof stability. He can also provide the roof trusses and ask questions like "How do you think carpenters ensure the trusses are identical? The activity of the pre-service teacher here is to observe the roof trusses, share their prior experience about the roof structure of their community. They also discuss why identical triangles are needed for stability.

Investigation & Context Analysis (Experience) – 20 minutes: The SSS, AAS, ASA, SSA, and AAA measurement cards are now in the instructor's possession. The instructor should also give cardboard, scissors, rulers, and thread. Furthermore, the instructor divides the class into two groups: lead carpenters and trainee carpenters. Furthermore, the instructor divides the students into two groups: trainees and lead carpenters. He explains the challenge rules: one group builds a hidden triangle, while the other tries to replicate it using partial information. At this stage, pre-service teachers' role is to construct triangles using the measurements on the cardboards based on limited information (like one angle, two angles, etc.) using real or drawn materials. At this stage, the assessment mechanism is observation that mainly focuses on pre-service teachers' participation and ability to relate geometry to real-life contexts.

Generalization from Specific to General Model (Apply) – 20 minutes: The instructor's role is to provide step-by-step information (angle only, two angles, SAS, SSS, etc.) and guide learners to compare results after each round. He leads the discussion on findings and guides the comparison of triangle sets. Pre-service teachers are expected to construct triangles, compare them with other groups, and record results. At this stage, the assessments mainly focus on the accuracy of construction, reasoning about the sufficiency of data that can be assessed through peer and teacher evaluation of constructed models.

Cooperate / Abstraction and Reflection: At this stage the instructor is expected to facilitate group discussion on findings and raise questions like "Which information ensured identical triangles?" "Why did some attempts fail?" "How do carpenters avoid errors?" the role of the pre-service teachers is discuss in groups, share results, and collectively identify valid congruence conditions.

Transfer / Reflection and Application: At this stage, the role of the instructor is expected to assign practical homework/project and create a blueprint of a triangular roof truss using one congruence rule. He can guide formalization: lists and defines triangle congruency conditions (SSS, SAS, ASA, AAS, and RHS). Furthermore, links them back to carpentry practice: "How do these rules apply when building multiple roof trusses?" pre-service teachers are expected to summarize conditions, relate them to real-world carpentry, and record conclusions. At this stage, the assessment is mainly focuses on conceptual understanding, ability to generalize.

Lesson plan on Median and centroid of a triangle (Balancing point)

Topic: Centroid of a Triangle (Balancing Point)
Context: TV Dish Stand in a Small Rural Town
Duration: 100 minutes
Approach: RME-based Mathematization
Technique- REACT Strategy
Grade level: pre-service mathematics teachers
Materials: cardboard triangles, rulers and pencils, triangle diagram for group work

Learning competency

Up on the lesson, pre-service teachers will:

- Identify and create the medians of a triangular board using geometric tools.
- Determine the centroid of a triangle properly, both geometrically and using coordinates.
- Explain the link between a triangular slab's centroid and balancing point.
- Use the centroid principle to establish the proper placement of a single support post beneath the triangular board.

- Work with peers to solve a real-world geometry issue by measuring, building, and reasoning.
- Transfer the centroid concept to new contexts**,** such as other triangular objects or irregular boards.

Context -TV-Dish Problem: In a small rural town, Kebede and his family enjoy watching television, especially football games like the Premier League. Kebede wants to install a television dish using a triangular board supported by a single post. The post must be placed exactly at the board's balancing point so the dish remains stable. However, the slab builder has no idea where the point to secure the balance is, and he decided to consult you by asking, "At which exact point under this triangular board should I place the support post so the stand remains balanced?"

Reality (Relate stage) for 10 minutes: As instructor's activities, he can start the lesson by offering the TV-dish context as a worksheet and ask question like why stability is important. He expected to encourage pre-service teachers to share experiences with balancing objects by asking how one can find the exact balancing point of a triangular board. As pre-service teachers' activities, they receive the scenario and attempt to relate it to their daily life experiences. They share their idea about the reason why the placement of a single support post matters. Discuss the initial concepts of balance and stability.

Guided re-invention/ experiencing stage- for 25 minutes: The instructor expected to distribute cardboard triangles and direct them to try balancing the cardboard on a pen/pencil. Encourage them to make an approximate balancing point, and ask probing questions like “What do you notice about these balance points?” he also expected to distribute triangles of different colored papers. Pre-service teachers, in their return, try to experiment with balancing the triangle using a pen or pencil. They also try to fold the triangles made from different colored paper to find the distance from the vertex to the centroid and the distance from the midpoint of side and the centroid. They mark the balancing point; they can also compare the balancing point by considering different triangles. They can share their observation, such as it is somewhere inside, not on the edge.

Apply / Model Development- for 15minutes: Under model development, there is geometric construction. The instructor demonstrates how to find the midpoint of each side. Model drawing medians without yet naming the centroid. Ask a question where the triangle's medians meet. By associating with the scenario, introduce the term centroid once pre-service teachers discover the common point of the three medians. Explains the balancing property by returning to the scenario “where should the builder place the support post?” The pre-service teachers try to find the midpoints of the sides. They draw medians on the cardboard triangle and observe the points where they intersect and compare their balancing marks. Besides, can also determine how the centroid divides the triangles' medians into ratios. They also recognize the centroid as the junction of the medians, understand it corresponds to the actual balance mark, and infer that the post must be located at the centroid.

Cooperate / Abstraction and Reflection- 20 minutes: As an instructor's responsibility, he let the pre-service teachers sit in groups. He provides a triangular diagram of the actual TV-dish board (with side lengths) and asks groups to construct medians and locate the centroid accurately. Facilitates group discussions. Pre-service teachers, in their turn, work in groups to analyze the given triangular board. They discuss and agree on the point where the real support post should go and briefly present their reasoning.

Transfer / Reflection and Application: This stage involves generalization and extending new contexts. The instructor gives extension questions like how to find the balancing point of other triangles, like right-angle triangles and even for rectangles. The instructor can summarize the need for mathematization. Pre-service teachers apply the concept of centroid to new situations, such as a table with one leg and a coffee tray with one leg, and reflect on how real-world problems lead to mathematical concepts.

Assessment: Formative assessment activities includes observing, balancing an experiment, examining geometric construction, group reasoning and justification, dialogue, and oral questions, As home work, provide an exercise dealing with median and its centroid such as " compute and explain why this point solves the TV-dish support prolem for triangle ABC with coordinate A(x1, y1), B(x2,y2) and C(x3,y3)".

Topic: Exploring Circles and Area through Basketry (Mesob and Sefed)

Approach: RME-based mathematization
Strategy: REACT (Relate–Experience–Apply–Cooperate–Transfer)
Grade level: Pre-service Mathematics Teachers

Context: In the Selale community, females are preparing Mesob and Sefed baskets for sale in the market. Despite having little or no academic education, they strive to create beautiful, symmetrical, and consistent-sized baskets in order to attract purchasers. Despite the fact that they do not employ mathematical formulae, their art demonstrates a strong intuitive grasp of geometry, such as concentric circles, proportion, and spatial reasoning.

Reality (Relate stage): This stage is the time of relating pre-service teachers' experience with the context. For 10-15 minutes, the instructor starts the lesson with a greeting ahead of introducing the lesson topic. Then, he provided the image of mesob and sefed for pre-service teachers. He presented the scenario: Females are designing a basket for market sale and must make them consistent in size. Ask pre-service teachers questions like "What measurements or patterns might they use to achieve this?" At this stage, pre-service teachers are expected to share local experiences of mesob and gindo(sefed). They also discuss how the basketry determines the size or the proportion of the mesob and gindo/sefed. Furthermore, they are expected to identify mathematical concepts like circle, radius, diameter, symmetry, and different patterns.

Guided re-invention/experiencing stage: At this stage, the instructor offers a sample circular template or grid paper simulating mesobor sefed and lets them estimate, measure, and compare the radii, diameters. He may pose questions like "how can basket makers ensure equality of the size of the sefed"? At this stage, pre-service teachers are expected not only to measure and record values of different circular models, but also to draw and label radius and concentric circles. They also exchange their experience on how such measuring might occur in real weaving.

Apply / Model Development: At this stage, the classroom instructor may offer pre-service teachers an activity, like if the inner flat surface of the mesob and the sefed have a radius of 25cm and 24cm, calculate and compare their area. He further guides them to apply the formula for the area of the circle and its circumference. He can also facilitate reasoning about proportionality and scaling. At this stage, pre-service teachers are expected either to work in groups or individually to calculate area, circumference of circles, and ratios. They can also discuss how basket makers could plan materials like suta and migira, and even alela, they use or their price based on the size differences. They record their findings on their paper.

Cooperate / Abstraction and Reflection: This stage is a meaning-making stage, and the classroom instructor is expected to lead their group discussion to formal geometry concepts like circle and their radius, their area, and concentric circles. The responsibility of the pre-service teachers at this stage is to collaboratively justify how real basket weaving relates to geometry, particularly the concept of a circle.

Transfer / Reflection and Application: This stage is the stage where generalization and extension to the new context are made. At this stage, the instructor's role is to ask pre-service teachers to design their own basket pattern using geometric reasoning, i.e., to determine radii, the number of concentric circles, and proportions. He also further questions how they can use such cultural context to teach geometry meaningfully. On the other hand, pre-service teachers create individual basket designs that describe geometry concepts and present how they could use similar contexts in the mathematics classroom. They reflect on how to transfer learning to other cultural or real-world situations.

Assessment Methods: The assessment mechanisms is observing the pre-service teacher in their group discussion/dialogue, in their accuracy of mathematical reasoning and calculation, their creativity, and the cultural relevance of basket design and quality reflection on cultural mathematical integration. Furthermore, a summative posttest will be utilized.

Thus, based on the results obtained from the intervention, this section tries to answer the following research questions:

1. Is there a significant difference in problem-solving skills, geometry achievement, and mathematical epistemological beliefs between groups taught by RME-based mathematization and conventional approaches?
2. Are there significant differences in the mean scores on the combined dimensions (source, stability, and structure) of epistemological beliefs between pre-service teachers taught using an RME-based mathematization approach and those taught using conventional instruction?
3. How do pre-service teachers mathematize while solving real-life-based mathematics problems?
4. Do problem-solving skills and epistemological beliefs significantly contribute to pre-service mathematics teachers' geometry achievement?
5. What challenges and shortcomings prohibit pre-service mathematics teachers from engaging successfully in mathematization within the context of RME-based mathematization instruction?

Quantitative data analysis involves the systematic collection and evaluation of data that can be quantified and validated. As per Cohen et al. (2007), it includes a statistical tool for evaluating or examining numerical data. Putting hypothetical data situations into numerical form is the main objective of quantitative analysis. This means that the researcher needs to determine the degree of measurement related to the quantitative data before beginning the inquiry. Three continuous dependent variables (DV), one independent variable (IV) (treatment with two levels: RMEBM and CG), and two separate groups of participants in this study might be considered determinants for using the aforementioned statistical methods. The following table depicts the variables assessed and the tools used for this study.

Table 5.1The variable types and its measurement

Variables	Type	Tools for Measuring	Nature of data
Mathematical Epistemological beliefs	DV	MEBQ	Continuous
Geometry achievement	DV	GAT	Continuous
Problem solving skills	DV	RGP	Continuous
RME-based Mathematization instruction	IV		

DV: Dependent Variable, IV: Independent Variable

5.2. **Assumptions in Statistical Testing**

Assumptions about the data to be tested are the foundation of all statistical tests, whether they are parametric or non-parametric. The underlying population distribution is usually assumed rigidly in the parametric tests. Certain distributions, such as a normal distribution, serve as the foundation for parametric statistics (t-tests, ANOVA, ANCOVA, correlation, regression analysis, etc.) (Mishra et al., 2019; Orcan, 2020). Parametric tests are thought to be more effective in detecting group differences than non-parametric tests (Guglielmetti et al., 2021), because non-parametric tests such as chi-square, Mann-Whitney, and others are assumption-free statistics that do not make these types of assumptions (Nahm, 2016). Since non-parametric tests do not fulfill assumptions, the researchers may draw incorrect conclusions from the data and make poor decisions (Shatz, 2024). Nonetheless, certain parametric techniques like Pearson's correlation, regression, independent t-test, ANOVA, and others are "robust" against specific hypotheses while maintaining the validity of the data (Shatz, 2024; Wilcox, 2023). Some of the most common assumptions for parametric tests are normality, the absence of outliers, and homogeneity of variance, data independence, and linearity.

5.2.1.Sample and Independence of Observations

All inferential tests rely on fundamental assumptions, like the independence of observations (Guetterman, 2019). The independence of observations suggests that observations in a sample are independent of one another, which means that each sample subject's measurements are unaffected or linked to the measurements of other individuals (Pallant, 2016). The absence of effect or alteration on the importance of other observations is what characterizes the independence of observation. There is no connection between the different bits of observation (Palant, 2016). Non-independent observations introduce bias into our statistical test, resulting in an excessive amount of false positives (Type I error). One ultimate goal of this study was to examine the mean differences of two independent groups drawn at random from two teacher education colleges. Moreover, each pre-service mathematics teacher answered the test items independently, indicating that there was no violation of the assumption of independence of observations.

5.2.2. Normality test for the dependent Variables

According to Mishra et al. (2019), graphical or numerical techniques, such as statistical testing, can be used to verify if the provided data is normal. When it comes to numerical approaches, the Shapiro-Wilk test (for sample sizes under 50), the Kolmogorov-Smirnov test (for large sample sizes above 50), skewness, and kurtosis are the most often used techniques for normalizing data (Avram & Mărușteri, 2022; Mishra et al., 2019; Tomšik, 2019). In order to ensure that the data is normal, especially for small sample sizes, we can also analyze it using graphical techniques such as box plots, histograms, P-P plots, and Q-Q plots. This is because graphical representations have the advantage of enabling excellent judgment to determine normality in situations where numerical tests may be excessive or under sensitive (Avram & Mărușteri, 2022; Mishra et al., 2019).

On the other hand, skewness is a measure of symmetry, kurtosis is a measure of a distribution's peakedness, and a z-test is one in which the z score is the skewness or excess kurtosis value divided by their standard errors (Mishra et al., 2019). Other than the aforementioned methods, these three are techniques to confirm the normality of a dataset.

According to Mishra et al. (2019), the normality of a data distribution is maintained if the skewness or kurtosis of a given data ranges between -1 and +1 and if the z-value ranges between -1.96 and +1.96. In this research study, the normality of the data distribution regarding epistemological beliefs (pre-post), geometry achievement (pre-post), and problem-solving skills (pre-post) of pre-service mathematics teachers was verified using skewness, kurtosis, z-test.

From table 5.2, it was observed that the value of skewness (0.23 for the treatment group and $-.38$for the comparison group) and the value of kurtosis (-0.05 for the treatment group and -0.39 for the comparison group) of geometry achievement were within± 1. The z-values for the treatment group and comparison group, respectively, are 0.51 and -.88 in the case of skewness and -.06 and -0.47 in the case of kurtosis, indicating that they are within ±1.96. This indicates that the pre-geometry achievement data for the treatment group and the comparison group are normally distributed. As indicated in Table 5.2, the skewness value, kurtosis value, and z-score values for pre-problem-solving skills and the components of epistemological beliefs are within an acceptable range, showing that all the data on pre-problem-solving skills and the general and mathematical epistemological beliefs and its components for both comparison and treatment groups are normally distributed.

Table 5.2 Normal Distribution Analysis for pre-service mathematics teachers' geometry achievement, problem-solving skills, and epistemological beliefs (pre-test)

Dependent variable	Group type	N	Skewness	SE	z-value	Kurtosis	SE	z-value
	CG	30	-.38	.43	-.88	-.39	.83	-.47
GA	EG	27	.23	.45	.51	-.05	.87	-.06
	EX	27	.28	.45	.62	-.87	.87	-1
PSS	CO	30	-.50	.45	-1.1	-.64	.87	-.74
	EX	27	.32	.45	.71	-.27	.87	-.31
SoMK	CG	30	-.41	.43	-.95	.1	.83	0.12
	CG	30	.16	.43	.37	-.59	.85	-.69
StaMK	EG	27	-.58	.45	-1.28	.29	.87	.33
	CG	27	.09	.43	.21	.15	.83	.18
StrMK	EG	30	.37	.45	.82	-.45	.87	-.52
MEB	CG	30	-.24	.43	-.56	.71	.83	.86
	EG	27	.28	.45	.62	-.69	.87	-.79

For post-geometry achievement, the values of skewness (for the treatment group and for the comparison group) and kurtosis (for the treatment group and for the comparison group) all lie within the acceptance range of -1 and $+1$. It is also found that the values of z-score for treatment group and comparison group, respectively, are 1.36 and $.51$ for skewness and $-.09$ and -.61 for kurtosis, respectively, which are in an acceptable range. The skewness, kurtosis, and value of the z-score indicated that the data are normally distributed. According to table 5.3, for post-problem solving skills, mathematical epistemological beliefs and its components, the values of skewness, kurtosis, and z-score obtained from comparison and treatment group data are within the acceptable range, indicating that the data distribution in both groups is normal.

Table 5.3 Normal Distribution analysis for pre-service mathematics teachers' geometry achievement, problem-solving skills, and epistemological beliefs (post-test)

Dependent variable	Group type	N	Skewness	SE	z-value	Kurtosis	SE	z-value
	CG	30	.45	.43	−1.05	.21	.83	.25
GA	EG	27	.61	.45	1.36	−.08	.87	−.09
	CG	30	.22	.43	.51	-.61	.83	−.61
PSS	EG	27	.17	.45	.38	−.42	.87	−.48
SoMK	CG	30	−.14	.43	−.33	−.64	.83	−.77
	EG	27	.32	.83	.38	−.27	.87	−.31
StaMK	CG	30	−.15	.43	−.35	−.06	.83	−.07
	EG	27	.01	.45	.02	−.61	.87	−.70
StrMK	CG	30	−.65	.43	−1.51	1.1	.83	1.33
	EG	27	−.52	.45	−1.16	−.07	.87	−.08
MEB	CG	30	.03	.43	.07	−.22	.83	−.27
	EG	27	−.15	.45	−.33	−.32	.87	−.37

5.2.3. **Verification of the Dependent Variables for Outliers**

A researcher's initial step is to identify any outliers in the data collection. A dataset's "outlier" is a data point that deviates from the average (Agarwal & Gupta, 2021). These authors stated that the outliers provide an inaccurate depiction of the data and other relationships, skewing the statistical measurements. Erroneous input or the use of an excessive score during the data gathering process may be the source of this type of data. As part of the first data screening, box plots were utilized to look for outliers in terms of univariate normality (Mowbray et al., 2019). The study of the intervention groups did not reveal any univariate outliers. An approach to searching for outliers in multivariate normality is to compute the Mahalanobis distance. To find an outlier, it is necessary to comprehend the significant chi-square value. When all the dependent variable values are taken combined, the multivariate normality is the normal distribution of all those values. Thus, computing the Mahalanobis distance is sufficient in and of itself to confirm multivariate normality and outliers, as demonstrated by (Ghorbani, 2019). The greatest Mahalanobis distance must be less than the critical value in order for multivariate normality to be presumed (Pallant, 2016).

Table 5.4 Mahalanobis and Cook's distances for posttest data

Variable		Minimum	Maximum	Mean	Std. Deviation	N
Post test	Mahal. Distance	.21	16.75	3.93	3.36	57
	Cook's Distance	.00	.16	.07	.06	57

The table of critical values shows that the critical Mahalanobis value for four independent variables is 18.47. The Mahalanobis distance for the post-test data (PSS and the components of MEB) in this study was 16.75, which is less than the critical value of 18.47 and less than the Cook's distance of 0.16. These, therefore, suggest that both multivariate normality and the absence of outliers in the research samples are tenable.

5.2.4. **Homogeneity of Variance for Dependent Variables**

The parametric test assumes homogeneity of variances to verify that samples are from populations with similar variation. The equality of variances assumption assures that deviations from averages in subgroups are homogeneous, whereas the normality assumption ensures that the data is distributed symmetrically. Residuals account for differences in sample data owing to departures from normality. Numerous statistical tests, including Levene's, Box's M test, Hartley's Fmax, Cochran's, and Barlett's tests, are employed to confirm the homogeneity of variance assumption for the given data (Friendly & Sigal, 2018; Wang et al., 2017; Zhou et al., 2023). We typically employed Box's M test for multivariate analysis and Levene's test for univariate analysis. We evaluated the null hypothesis of equal group variance using the F-test in order to confirm variance homogeneity. A p-value less than.05 indicates unequal variance, whereas a p-value greater than.05 reflects equal variation in the data. The p-value reflects whether or not the variance homogeneity assumption has been violated.

The following table shows the results of Levene's test of homogeneity of variances of post geometry achievement (post-test), problem-solving skills (post-test), source, stability and structure of mathematical knowledge (post-test).

Table 5.5 Levene's Test of Equality of Error Variances

Variables	F	df1	df2	Sig.
GA	2.436	1	55	.124
PSS	13.462	1	55	.001
SoMK	2.034	1	55	.165
StaMK	4.526	1	55	.053
StrMK	.530	1	55	.648
MEB	.013	1	55	.909

From Table 5.5, $F(1, 55) = 2.436$, $p = .124$, for GA; $F(1, 55) = 2.034$, $p = .165$ for SoMK; $F(1, 55) = 4.526$, $p = .053$ for StaMK, and $F(1,55) = 0.530$, $p = .648$ for StrMK and for MEB; $F(1,55) = 0.013$, $p = .909$. Due to this assumption not being violated for the variables listed, it may be concluded that the homogeneity of variance for all of the variables listed except problem-solving skills is plausible. For PSS, $F(1, 55) = 13.462$, $p = .001$, showing that the assumption of equality of variance is violated. Thus, based on the recommendation of Delacre et al. (2017), instead of using an independent t-test, the researchers utilized Welch's t-test to examine the significant mean difference between the conventional group and the treatment group in relation to pre-service mathematics teachers' problem-solving skills.

5.2.5. **Homogeneity of Variance-Covariance Matrices for Dependent Variables**

In multivariate analysis, we used Box's M test to ensure the equivalence of numerous variance-covariance matrices. This test statistic is often used in multivariate and mixed models of variance to ensure that the variance-covariance assumption is homogeneous. If the variance-covariance matrices are equal across all the cells produced by between-subjects effects in the multivariate analysis, the vectors of the dependent variables are assumed to follow a multivariate normal distribution.

Since the Box's M test is very sensitive, unlike most other tests, the level of significance is usually .001 (Darcy et al., 2017; Hahs-Vaughn, 2016). If the statistic is not significant at p =.001, the variance-covariance matrices must be equal. One can utilize Wilk's lambda (λ) test statistic in multivariate analysis of variance whenever the homogeneity of the variance-covariance assumption is tenable. However, in cases of a violation of the homogeneity of variance-covariance in multivariate analysis, one may utilize Pillai's Trace multivariate test statistic to assess the multivariate results. The table below shows the homogeneity of variance-covariance matrices. Box's M test was employed for the homogeneity test, which verifies the assumption of equal variances. Based on the data provided, the box's M value for the epistemological beliefs component, computed by SPSS, is 12.21. This suggests that homogeneity of covariance matrices between groups is taken for granted F ((6, 21236.479) = 1.91, p =.08). According to Darcy et al.'s (2017) and Hahs-Vaughn's (2016) criteria (i.e., p <.05), the box's M value of 12.21 corresponded to a p value of 0.08, which was deemed non-significant. Consequently, the covariance matrices between the groups were taken to be equal for the MANOVA.

Table 5.6 Box's M Test for equality of covariance matrices

Variables	Box's M	F	df1	df2	Sig.
Components of MEB (source, stability& structure of mathematical knowledge)	12.21	1.91	6	21236.48	.08

5.2.6. **Multi-collinearity or Singularity among Dependent Variables**

Multi-collinearity can hinder interpretation and reduce the model's power to identify statistically significant independent variables (Ibrahim et al., 2022; Kim, 2019). In regression and multivariate analysis, multi-collinearity arises when explanatory variables (in the case of regression) and dependent variables (in the case of MANOVA) have a substantial correlation. There is a possibility of multi-collinearity if the correlation coefficient between variables is greater than or equal to 0.8 (Alabi et al., 2020; Ibrahim et al., 2022).

As stated by Shrestha (2020) and Ibrahim et al. (2022), MANOVA and regression analysis work best when there is a moderate correlation among variables. This study used Pearson correlations across the four variables to test the assumptions of MANOVA and multiple regression. The following table indicates a moderate correlation between the four variables, indicating an absence of multi-collinearity.

Table 5.7 Correlation between problem solving skills, source of mathematical knowledge, stability of mathematical knowledge and structure of mathematical knowledge

	PSS	SoMK	StaMK	StrMK
PSS				
SoMK	.302*			
StMK	.194	.447**		
StrMK	.166	.529**	.483*	

**. Correlation is significant at the 0.01 level (1-tailed).

*. Correlation is significant at the 0.05 level (1-tailed).

It is evident from Table 5.7 above that the correlation between the variables does not exceed the 0.8 criterion ($r < 0.8$). Thus, there is no violation of linearity (Pallant, 2016).

Furthermore, for multiple regression analysis, the tolerance that quantifies the impact of a single independent variable on every other independent variable and the variance inflation factor (the ratio of one to the tolerance) were also checked to see the existence of multi-collinearity. This variance inflation factor measures how much the predicted regression coefficient's variance is inflated when the independent variables are associated. According to Shrestha (2020) and Kim (2019), multi-collinearity arises when the variance inflation factor (VIF) is between 5 and 10 and the tolerance (T) is less than 0.1 to 0.2. For this study, the researcher used variance inflation factor and tolerance to check the existence of multi-collinearity among the independent variables. Table 5.8 depicts the tolerance and variance inflation factor.

Table 5.8 Collinearity Statistics

Variables	Collinearity Statistics	
	Tolerance	VIF
PSS	.71	1.4
SoMK	.61	1.64
StaMK	.66	1.52
StrMK	.61	1.64

As indicated in Table 5.8, the value of tolerance ranges from 0.61 to 0.79, and the value of the variance inflation factor is between 1.26 and 1.64. Hence, based on the guidelines of Shrestha (2020) and Kim (2019), the value of tolerance (T) and the value of variance inflation factor are in the acceptable range, indicating that there was no multi-collinearity among the independent variables listed above.

5.2.7. **Homogeneity of Regression Slopes**

The idea of homogeneity of regression slopes refers to the assumption that there is a consistent relationship between the dependent variable (post-test scores) and the covariate (pre-test scores) across comparisons and intervention groups. For covariance analysis, the homogeneity of regression slopes assumption is permissible if both the model differences and the interaction terms do not exhibit statistical significance. This means that the ANCOVA's homogeneity of regression slopes assumption is tenable if the p-value for the interaction term is larger than the pre-specified Type I error rate (e.g., $\alpha = 0.05$) (Johnson, 2016). Table 5.9 shows tests of between-subjects effects.

Table 5.9 Test for homogeneity of regression slopes using a general linear model

Dependent Variable: GA						
Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared
Corrected Model	188.671[a]	3	62.890	5.846	.002	.249
Intercept	560.875	1	560.875	52.133	.000	.496
Groups	42.345	1	42.345	3.936	.052	.069
Pre-G	.090	1	.090	.008	.927	.000
Groups*pre-GA	11.176	1	11.176	1.039	.313	.019

a. R Squared = .249 (Adjusted R Squared = .206)

Table 5.9 provides the summary table for ANCOVA with only the interaction term. The interaction term of pre-geometry achievement and the independent variable (Groups) was not significant (p =0.313), suggesting that the assumption of homogeneity of regression slopes is not violated.

5.3. **Analysis of Pre-test Scores**

This section deals with the analysis of pre-test results regarding the pre-service teachers' mathematical epistemological beliefs, problem-solving skills, and their geometry achievement at the baseline. The pre-test result may serve as a covariate for the post-test whenever there is a significant mean difference between groups in relation to the aforementioned dependent variables. Thus, the ultimate purpose of pretest (the baseline data) administration is to check if there was a significant mean difference between the comparison group and the treatment group before intervention. Thus, ANCOVA may account for baseline differences across groups by using the posttest score as the dependent variable, the pretest score as a covariate, and the pretest variance as the residual variation to confirm treatment effects, reduce bias, and ensure statistical independence between treatment and covariate (England et al., 2023; Miyazaki et al., 2022).

From the beginning, it was intended that the intervention groups used in this study would be comparable. Thus, prior to implementing the intervention, the researcher made an effort to assess the homogeneity of the intervention groups, adhering to the recommendations given by Wiersma and Jurs (2005). Based on the pre-administration of pre-epistemological beliefs, pre-problem-solving skills, and pre-geometry achievements in the treatment and comparison groups, the pretest mean scores were analyzed using an independent t-test.

Table 5.10 Comparison of intervention & comparison groups on pre-service teachers' pre-test geometry achievement, problem-solving skills, source, stability and structure of mathematical knowledge

variables	Group	n	Mean	SD	t	df	Sig
GA	CG	30	8.80	3.85			
	EG	27	10.52	4.48	2.45	55	0.018
PSS	CG	30	8.67	2.33			
	EG	27	10.85	2.97	1.98	55	0.053
SoMK	CG	30	2.24	0.26			
	EG	27	2.25	0.24	0.13	55	0. 90
StaMK	CG	30	2.2	0.28			
	EG	27	2.3	0.26	1.36	55	0.18
StrMK	CG	30	2.28	0.32			
	EG	27	2.22	0.26	-0.90	55	0.37
MEB	CG	30	2.25	0.19	0.50	55	
	EG	27	2.22	0.17			0.62

Prior to intervention, the researcher administered a pre-test to compare pre-service teachers' geometry achievement, problem-solving skills, and epistemological beliefs (source, stability, and structure of mathematical knowledge) in both the comparison and intervention groups. As indicated in Table 5.10 above, the mean score of the intervention group (M = 10.52, SD = 4.48) is higher than the mean score of the comparison group (M = 8.80, SD = 3.85) on geometry achievement. In relation to problem-solving skills, the mean score of the intervention group (M = 10.85, SD = 2.33) is somewhat higher than the comparison group (M = 8.67, SD = 2.33). The mean score of intervention group (M = 2.25, SD = 0.26) is somewhat higher than and the comparison group (M = 2.24, SD = 0.24) on source of mathematical knowledge. In relation to stability of mathematical knowledge, the mean score of the intervention group (M = 2.3, SD = 0.26) is a little bit higher than the comparison group (M = 2.2, SD = 2.28). The mean score of the intervention group (M = 2.22, SD = 0.26) is a little bit lower than the comparison group (M = 2.28, SD = 0.32) in relation to structure of mathematical knowledge. In relation to the general mathematical epistemological beliefs, the mean score of the intervention group (M = 2.22, SD = 0.17) is a little bit less than the comparison group (M = 2.25, SD = 0.19). The researcher used an independent t test to see if there was a significant difference in mean scores between the comparison and intervention groups.

The two groups had statistically significant mean difference t[55]=2.45, p=0.018 for geometry achievement, t[55]=1.98, p=0.053 a marginal value for problem-solving skills, t[55]=0.13, p=0.90 for source of mathematical knowledge, t[55]=1.36, p=0.18 for stability of mathematical knowledge and t[55]=-0.90, p=0.37. The result indicated that prior to intervention; the level of geometry achievement of the comparison group was not the same as the level of geometry achievement of the intervention group. Thus, in order to account for initial differences, the researcher utilized the pre-geometry achievement result as a covariate variable. To maintain group differences, the researcher wants to remove some of the pre-geometry achievement-test and pre-problem solving skills variability in the post-test using ANCOVA.

However, in relation to the mathematical epistemological beliefs and its three components (source, stability and structure of mathematical knowledge) pre-service mathematics teachers were equivalent before they exposed to the RME-based mathematization instruction and the conventional approach.

5.4. Quantitative Data Analysis of Group Differences on Posttest Scores

Prior to proceeding with the study of post-test findings, the analysis pre-test results were done. The analysis of pre-test result indicated that there was a statistically significant mean difference between comparison group and treatment group in relation to geometry achievement. This indicated that the two groups were not on the same level in geometry achievement. Thus, to control the effect of the covariate variable on the post geometry achievement, a one-way analysis of covariance was computed. However, in terms of pre-service teachers' pre-problem-solving skills and pre-epistemological beliefs (source, stability, and structure of mathematical knowledge), the independent t-test result showed no significant mean difference between the comparison and treatment groups. These results indicated that, in relation to pre-service teachers’ pre-problem-solving skills and epistemological beliefs, the two groups are approximately on the same level.

ANOVA and MANOVA are two inferential statistical analyses that compare the means of groups. The latter one is a generalization or extension of the former one and is used whenever there is an inquiry to analyze more than one dependent variable at a time, whereas the former one is applicable whenever there is an inquiry to analyze a single dependent variable. This way of analysis allows researchers to look at the association among dependent variables simultaneously instead of looking at each variable separately. Whenever a study has to examine the impact of one or more independent variables on a number of dependent variables, this statistical analysis is employed.

5.4.1. Treatment effects on problem-solving skills of pre-service teachers

The analysis of pre-test result on problem-solving skills indicated that the p value is 0.053, which is almost non-significant. Because of this, the researcher utilized ANCOVA insteady of independent t-test to control the effect of the covariate variable (pre-problem solving skills) on the post-test result (post-problem-solving skills). Prior to utilize ANCOVA, assumptions like outliers, and homogeneity of regression slopes (p=.264) were computed, and the result indicated that there was no violation

Table 5.11 Means and Variability for Post problem solving skills using pre-*problem solving skills* score as covariate Variable

Variable	Groups	n	Unadjusted		Adjusted	
			M	SD	M	SE
PSS	CG	30	8.10	5.33	9.39	1.59
	EG	27	22.48	10.92	21.50	1.66

As indicated in Table 5.11, after statistically controlling the effect of the covariates (pretest score), the mean score of intervention group is higher than the comparison group on the problem solving skills of post-test score. The analysis of covariance presented in Table 5.12 below further computed the significant difference between the intervention and comparison groups in problem solving skills post-test score.

Table 5.12 ANCOVA result for problem-solving skills of post-test score

Variable	Source	*df*	*MS*	*f*	*p*	η^2
PSS	Pre-PSS	1	480.18	7.39	.0.009	0.122
	Groups	1	568.84	8.75	0.005	.142
	Error	54	10.766			

The interaction between group and pretest score was not statistically significant, $F(1, 53) = 0.26$, $p = .609$, indicating that the assumption of homogeneity of regression slopes was satisfied. $F(1, 53) = 7.388$, $p = .009$, partial $\eta^2 = .122$, indicating that pretest scores significantly predict posttest problem solving skills, meaning that pre-service teachers' initial problem-solving skills matter and hence, ANCOVA appropriately adjusts posttest scores for baseline differences. Therefore, after controlling for pretest scores, ANCOVA revealed a significant effect of group on PSS, $F(1, 53) = 8.75$, $p = .005$, partial $\eta^2 = .142$. This supports your claim that the RME-based mathematization approach had a meaningful effect on problem-solving skills**.** According to Cohen (1988), this ETA value is considered to be a large but a medium effect size based on AlWahaibi et al. (2020). This finding suggested that, in the absence of covariate effects, the intervention might explain a 14.2% variation in problem-solving skills.

5.4.2. **Treatment effects on geometry achievement of pre-service teachers**

The researcher proposed to utilize an independent t-test to examine the significant effect of RME-based mathematization on pre-service teachers’ geometry achievement after checking all the assumptions of the independent t-test. However, the utilization of the independent t-test was determined based on the analysis result of the pre-test. Since the analysis of pre-geometry achievement result indicated that there was a significant mean difference between the comparison group and the treatment group (see Table 5.10). This shows that the two groups are not on the same level regarding geometry achievement. Thus, instead of computing an independent t-test, the researcher subjected to a preferred analysis of covariance to control the effect of the covariate variable (pre-geometry achievement) on the post-test result (post-geometry achievement).

The researcher computed a one-way ANCOVA to examine the effect of the RME-based mathematization approach on pre-service teachers’ geometry achievement. Before utilizing ANCOVA, assumptions of outliers, homogeneity of variance ($p= 0.115$), and homogeneity of regression slopes ($p=.313$) (see Table 5.9) were computed, and the result indicated that there was no violation.

Table 5.13 Means and Variability for Post geometry achievement using pre-geometry achievement score as covariate Variable

Variable	Groups	n	Unadjusted		Adjusted	
			M	SD	M	SE
GA	CG	30	11.20	3.72	11.16	.60
	EG	27	14.70	2.66	14.75	.64

As indicated in Table 5.13, after statistically controlling the effect of the covariates (pretest score), the mean score of intervention group is higher than the comparison group on the geometry achievement post-test score. The analysis of covariance presented in Table 5.14 below further computed the significant difference between the intervention and comparison groups in geometry achievement post-test score.

Table 5.14. ANCOVA result for geometry achievement post-test score

Variable	Source	*df*	*MS*	*f*	*p*	η^2
GA	Pre-GA	1	3.047	.283	.60	0.01
	Groups	1	176.956	16.44	0.00	.23
	Error	54	10.766			

The ANCOVA test indicated a significant mean difference between groups in geometry achievement, F (1, 56), p <.001, $\eta 2$ =.23. According to Cohen (1988), this ETA value is considered to be a large but a medium effect size based on AlWahaibi et al. (2020). This finding suggested that, in the absence of covariate effects, the intervention might explain a 23% variation in geometry achievement.

5.4.3. **Treatment effects on mathematical epistemological beliefs of pre-service teachers**

For the effectiveness of the RME-based mathematization, the difference in mean gain scores were computed for mathematical epistemological beliefs and its three components. For the source of mathematical knowledge, the EG exhibited a mean gain of 0.74, compared with a mean gain of 0.17 for the CG, indicating substantially greater epistemological development among participants exposed to the intervention. Furthermore, the difference in mean gain was .57, indicating that the improvement in the source of mathematical knowledge exceeded that of the CG by 0.57 points. In the baseline data, both CG and EG were virtually identical, indicating that the initial differences cannot be attributed to the source of mathematical knowledge. This suggests that conventional instruction produced minimal epistemological change regarding sources of mathematical knowledge. However, the intervention resulted in a significant shift in how pre-service teachers conceptualize the origin of mathematical knowledge, which was in line with teaching techniques that emphasized cultural mathematization and RME.

For mathematical knowledge stability, EG showed a mean gain of 0.35, while CG showed a mean gain of 0.15, indicating greater epistemic change among intervention participants. Furthermore, the difference in mean gain between the two groups is 0.2, showing that the EG improved perceptions regarding the stability of mathematical knowledge by 0.20 points more than the CG. Furthermore, the baseline result showed a slight difference between EG and CG and the independent t-test revealed no statistically significant mean difference between the two groups. Thus, the mean gain demonstrates that posttest differences are caused by the RME-based mathematization approach rather than the baseline status. Conventional methods produce little change in beliefs about the stability of mathematical knowledge, whereas the intervention promotes a greater shift toward viewing mathematical knowledge as evolving, constructed, and contextually grounded—consistent with culturally embedded mathematization and RME principles.

In terms of the structure of mathematical knowledge, the EG's mean gain is 0.34, which is three times the CG's mean gain of 0.11, indicating that participants exposed to the intervention had stronger epistemic progress. The difference in mean gain is 0.23, showing that the EG's improvement in beliefs about the strucrture of mathematical knowledge outperformed the control group by 0.23 points. The two groups were fairly close at baseline, and the independent t-test found no statistically significant mean difference between them. Thus, this result indicated that conventional instruction results in minimal restructuring of beliefs about how mathematical knowledge is organized, whereas the intervention promotes a more connected, coherent view of mathematical knowledge—consistent with RME and culturally grounded mathematization practices.

In terms of general mathematical epistemological views, the EG's major increase is .52, which is represented by a significantly bigger mean gain, while the CG's mean gain is .013, indicating a minor gain. The difference in mean gain also revealed that the EG improved total mathematical epistemological beliefs by 0.39 points more than the CG. At baseline, the two groups are essentially identical, implying that pre-existing belief differences are not substantial, and post-test variations indicate intervention-induced epistemic change. This suggests that the intervention group encourages a significant shift toward more sophisticated perspectives of mathematical knowledge, which is consistent with the theoretical foundations of mathematics as a human activity, cultural mathematization, and Realistic Mathematics Education (RME).

In addition to the result from the mean gain, independent t-test was also utilized to examine the effect of RME-based mathematization instruction on epistemological beliefs of pre-service mathematics teachers. Prior to computing the independent t-test, the researcher checked assumptions like normality and equal variance, and there were no violations. Thus, based on the components of epistemological beliefs results obtained through questionnaire, the analysis of mean, standard deviation, t-test, effect size(Cohen’s d) significant difference of the two groups were computed. As shown in Table 5.15, there was a mean difference between group treated with RME-based mathematization approach and the comparison group on the source, stability, and structure of mathematical knowledge. The group treated with mathematization-based approach has a higher mean score of 2.99 than the mean score of comparison group (2.41) on the source of mathematical knowledge. The table also indicated that in terms of stability of mathematical knowledge, the mean score (2.65) of the group treated with mathematization-based approach was higher than the mean score (2.34) of the group treated with the conventional one. It was also true that, with respect to the structure of mathematical knowledge, the group treated with mathematization-based approach has a higher mean score (2.56) than the mean score (2.39) of the group treated with conventional approaches. Furthermore, on the general mathematical epistemological beliefs, the mean score (2.74) of the group treated with the mathematization-based approach was higher than the mean score of the group treated with the conventional one (2.38).

Table 5.15 The comparison of treatment and comparison groups on pre-service mathematics teachers' epistemological beliefs

Variable	Group	n	M	SD	t	df	Sig.	*d*
SoMK	CO	30	2.41	.21				
	EX	27	2.99	.29	8.89	55	.000	2.32
StaMK	CO	30	2.35	.29		55	.000	
	EX	27	2.65	.21	4.59			1.15
StrMK	CO	30	2.39	.13		55	.000	
	EX	27	2.56	.14	4.67			1.31
	CO	30	2.38	.14	9.23	55	.000	2.45
MEB	EX	27	2.74	.15				

From Table 5.15, it can be shown that RME-based mathematization has a statistically significant effect on source, t (55) = 8.89; $p < .001$; $d = 2.32$, stability, t (55) = 4.59; $p < .001$; $d = 1.15$, structure, t (55) = 4.67; $p < .001$; d = 1.31) and for mathematical epistemological beliefs, t (55) = 9.23; $p < .001$; $d = 2.45$). Based on AlWahaibi et al. (2020), the d value for mathematical epistemological beliefs and its component source of mathematical knowledge is in the large category, whereas the d value for stability and structure of mathematical knowledge are in the medium range. However, according to Cohen (1988), the effect size d of mathematical epistemological beliefs and all of its components is considered to be large.

Mathematization-based instruction's effect size on the general mathematical epistemological beleifs and the source of mathematical knowledge indicates that the group receiving it outperformed the conventional technique by 2.45 and 2.32 standard deviation points respectively. Likewise, the impact size on the stability of mathematical knowledge showed that the group receiving mathematization-based instruction outperformed the conventional method group by 1.15 standard deviations. Additionally, the effect size on the structure of mathematical knowledge showed that the group treated with a mathematization-based teaching approach outperformed the group treated with a comparison instructional approach by 1.31 points of standard deviation.

In general, the effectiveness of the intervention was explored using differences in mean gain scores, independent samples t-tests, and the effect size of Cohen's d. The results showed statistically significant and large improvements in all aspects of mathematical epistemological beliefs of the intervention group over the comparison group, establishing the effectiveness of the RME-based mathematization strategy.

5.4.4.**Treatment Effects on Combined Dependent Variables**

The analysis of variance (ANOVA) is applicable to examine whether there is a significant mean difference in a single dependent variable between two or more groups; however, if we want to treat two or more dependent variables all together, we use the multivariate analysis of variance (Huang, 2020). Finch (2016) and Huang (2020) assert that the primary benefit of using MANOVA over univariate analysis is its capacity to be applied to the analysis of many dependent variables simultaneously. Additionally, a MANOVA can uncover differences that separate ANOVAs overlook (Finch, 2016; Huang, 2020). In addition, MANOVA can reduce Type I error since doing several univariate analyses results in a notably higher total Type I error rate (Warne, 2014).

This statistical analysis works if there is a moderate correlation (0.2−0.6) among the dependent variables under study (Lahti et al., 2019), but it is not suitable if there is a very high or very low correlation among the dependent variables (Finch & French, 2013). In this study, before running MANOVA, the researchers computed the correlation, and the obtained result indicated that there was a moderate relationship among the dependent variables, in addition to checking other important assumptions. Considering this, the researchers employed a one-way MANOVA to assess if there were significant mean differences between the intervention group and comparison group on a linear combination of dependent variables (source, stability, and structure of mathematical knowledge).

Table 5.16 Multivariate analysis for source, stability and structure of mathematical knowledge scores by group differences

Effect	Wilk's Lambda value	F	Sig	η^2
Wilks' Lambda	.361	31.266	.000	.639

Table 4.16 shows that there was a statistically significant mean difference between the comparison group and the intervention group on the linear combination of components of epistemological beliefs (source, stability, and structure of mathematical knowledge) because Wilk's Lambda =.361, F (2, 54) = 31.266, p =.000, and =.639. The multivariate analysis suggested that the group factor is responsible for 63.9% of the variation in the dependent variables.

5.4.5. The effect of problem-solving skills and epistemological beliefs on geometry achievement

In this section, we will see the effect of problem-solving skills and epistemological beliefs on geometry achievement via regression analysis. Multiple regression analysis is used to study the relationship between measured criterion variables and two or more continuous predictor variables (Lee, 2022). Furthermore, multiple linear regression allows one to calculate the model's overall fit as well as the relative contributions of each predictor to the total variance explained (Ray-Mukherjee et al., 2014). In this study, mathematical epistemological beliefs and problem solving skills are predictor variables and geometry achievement serve as dependent or measured criterion variable. To see the existing relationship between the dependent variable and the independent variables, the researcher computed correlation coefficients, p-values, standard Pearson's correlation coefficients and p-values, standardized coefficients, zero-order, and many others.

Prior to utilizing multiple regression analysis, the researcher checked the normality, autocorrelation, independence of independent variables, and independence of errors, linearity, and homoscedasticity. The skewness and kurtosis results indicated that the data for both the intervention group and the comparison group were normally distributed. In relation to autocorrelation, the Durbin-Watson value is 1.75, which lies in the acceptable range (1.5 to 2.5) (Azami et al., 2020), indicating there is no autocorrelation, which in turn showed that the assumption of independence of error is not violated in the case of comparison group. However, in the case of intervention group the Durbin-Watson value is 1.41which close to the commonly used 1.5 threshold indicating that that there is no serious violation of the independence of errors assumption.

The assumption of multi-collinearity in multiple regression analysis states that the independent variables should not be highly correlated with each other in order to provide reliable estimates for the regression coefficients. In this analysis, the values for the tolerance for all the independent variables (SoMKCG = 0.948, StaMKCG = 0.857, StrMKCG = 0.905, and PSSCG = 0.905) are significantly higher than the recommended minimum of 0.2. In addition to this, the Variance Inflation Factor (VIF) values for all the variables range from 1.055 to 1.167, which are significantly lower than the recommended maximum of 5. This analysis indicates that there is minimal correlation between the variables. Hence, multi-collinearity can be considered low for this regression analysis.

In carrying out multiple regression analysis, one of the most important assumptions is that the independent variables should not be too correlated with each other in order to avoid multi-collinearity, which can make the regression unstable. In the results, the tolerance level of all the independent variables (i.e., PSSEG = 0.889, SoMKEG = 0.907, StaMKEG = 0.880, and StrMKEG = 0.742) is greater than 0.2, which implies that there is low correlation between the independent variables. On the other hand, the Variance Inflation Factor (VIF) will vary between 1.103 and 1.347, which is less than 5 and implies that there is low multi-collinearity.

The above tests indicate that the independent variables in the model are sufficiently independent, and this will give us more confidence in carrying out the regression coefficient estimates in the model. Since there is low multi-collinearity in the model, we can be sure that the standard error is not inflated, and we can be confident in interpreting the contribution of each independent variable to the dependent variable GAEG.

From the analysis of the residuals and influence plots, it can be concluded that there are no significant outliers or influential points in the regression model. The values of the Mahalanobis distance range from 0.318 to 11.158. For 4 predictors (considering the independent variables to be PSSEG, SoMKEG, StaMKEG, and StrMKEG), with a sample size of 27, the critical chi-square value for $p = 0.001$ is approximately 18.47. Since all the values of the Mahalanobis distance are less than this critical value, there are no multivariate outliers in the data.

Other values like Cook's Distance and residuals for influential points are not available in the output. From the zero values of the variance and residual plots, it can be inferred that there are no extreme deviations in the dependent variable. In essence, it can be concluded that all the points in the data are consistent with the regression model. Hence, there are no points that have a significant impact on the regression coefficients.

The assumption of no significant outliers in multiple regression analysis can be checked with the help of Mahalanobis Distance and Cook's Distance values. In this analysis, the Mahalanobis Distance varies between 0.987 and 12.095. Since the sample size is 30 and the number of predictors is 4, the critical chi-square table at $p < 0.001$ with $df = 4$ is approximately 18.47. Since the maximum Mahalanobis Distance (12.095) is less than this threshold, there are no instances of multivariate outliers in the data. In the same vein, Cook's Distance varies between 0.000 and 0.125, which is less than the threshold limit of 1. This implies that there are no instances of outliers in the data.

The above analysis implies that the Mahalanobis and Cook's Distance values are within the threshold limits, and hence the assumption of no significant outliers or influential points in the data is satisfied, and the regression model is valid and stable.

For intervention group, the assumption of homoscedasticity was checked using a scatterplot of the standardized residuals plotted against the standardized predicted values. The residuals appeared to be randomly dispersed around the zero line, with constant variance over the range of the predicted values, thus indicating that the assumption of homoscedasticity had been satisfied.

Furthermore, for intervention group, the assumption of linearity for the GA model was tested with a scatterplot of the standardized residuals versus the standardized predicted values for the model. The residuals were randomly distributed around the zero line without any curvilinear pattern. Therefore, the assumption of linearity for the model holds as the relationship between the predictors and the GA is linear.

Form comparison group, homoscedasticity was checked using a scatterplot of the standardized residuals plotted against the standardized predicted values. It was observed that the residuals are randomly scattered around zero, indicating that there was no violation of the assumption of homoscedasticity.

Form comparison group, to check the assumption of linearity of the GA model, a scatterplot of the standardized residuals plotted against the standardized predicted values was used. It was observed that the residuals are randomly scattered above and below the zero line. This indicates that the relationship between the predictor variables and GA of CG is linear, and hence the assumption of linearity was met.

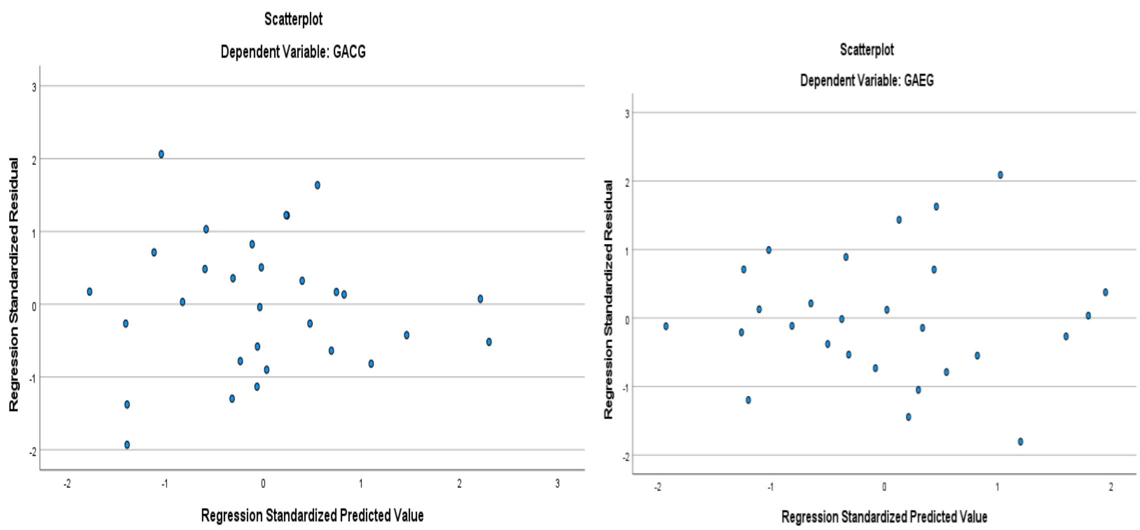

Figure 5.3 Scatter plot of geometry achievement and predictor variables score in both groups

Table 5.17 Prediction of geometry achievement from predictor variables

Multiple R = 0.673 , R^2=0.453, Adj R^2=0.354						
ANOVA Table						
	Sum of squares	df		Mean Square	F	Sig
Regression	83.25.	4		20.81	4.56	0.008
Residual	100.38	22		4.56		
Total	183.63	26				
Variables in the Equation						
Variable	**B**	SE	β	Zero-order(r)	t	Sig.
Constant	28.68	9.11			3.15	0.005
SoMK	.26	1.54	.03	-.15	.17	0.868
StaMK	0.18	2.17	.01	-.11	.08	0.939
StrMK	-6.93	3.38	-.38	-.51	-2.05	0.052
PSS	.11	.04	.47	.58		.011

As we can see from Table 5.17, F (2, 22) = 4.56, p = 0.008, indicating that the multiple regression results were found to be significant. This suggested that the combination of problem-solving skills and the components of mathematical epistemological beliefs significantly predicted geometry achievement. Unstandardized Coefficients B column provides us the value of the intercept (for the Constant row) and the slope of the regression line (from the problem-solving skills), and this gives the regression equation:

From Table 5.17 one can observe that the multiple correlation coefficient *(*R*),* using the four predictors simultaneously was 0.673 with r^2 =0.453) and the adjusted R-squared was 0.354, meaning that 35.4% of the variance in geometry achievement can be predicted from a combination of problem solving skills and components of mathematical epistemological beliefs. This suggested that other than the four predictor variables mentioned above, there are variables that account for about 64.6% of the variance in geometry achievement. The following equation may be used to calculate the cumulative percentage effect of each predictor variable, which are pre-service teachers' problem-solving abilities and source mathematical epistemological beliefs:

$$R^2 \times 100\% = (\beta PSS r_{PSS} + \beta SoMK r_{SoMK} + \beta StaMK r_{StaMK} + \beta StrMK r_{StrMK}) \times 100\%$$

$$(0.673)^2 100\% \approx (.47(.58) + .03(-.15) + .01(-.11) + (-.38)(-.51)) \times 100\%$$

$$45.3 \approx 27.26\% - .45\% - .11\% + 19.38\%$$

From four predictor variables, source and stability of mathematical knowledge predicted geometry achievement by 0.45% and 0.11% respectively, whereas problem-solving skills and structure of mathematical knowledge predicted the geometry achievement by 27.26% and 19.38% respectively.

In the intervention group, the regression results revealed that problem solving skills was the only significant predictor of geometry achievement ($\beta = .465$, $p = .011$). Although structure of mathematical knowledge ($\beta = -.375$, $p = .052$) showed a relatively strong standardized coefficient, it was not statistically significant at the 0.05 level. In contrast, source of mathematical knowledge ($\beta = .028$, $p = .868$) and stability of mathematical knowledge ($\beta = .014$, $p = .936$) were non-significantly predictor of geometry achievement.

Moreover, multiple regression analysis was also conducted on the comparison group. The overall regression analysis showed that the model was not statistically significant, $F(4, 25) = 1.41$, $p = .259$, indicating that the predictors, taken together, were not significant in explaining the variance in geometry achievement. The problem-solving skills and compoents of mathematical epistemological beliefs, taken together, explained only 5.4% of the variance in geometry achievement ($R^2 = .18$), with an adjusted R^2 of .054, which indicated very weak explanatory power. Problem-solving skills, taken independently, were positively but not significantly related to geometry achievement ($B = 0.224$, $\beta = .321$, $t = 1.69$, $p = .103$). Components of mathematical epistemological beliefs were also not significant predictors of geometry achievement. This showed that in the comparison group, problem-solving skills and components of mathematical epistemological beliefs were not significant predictors of geometry achievement, and the overall model did not provide a good explanation of the students' performance.

The findings suggested that PSS was a significant predictor of geometry achievement only in the intervention group, implying that the intervention may have strengthened PSS's significance in explaining students' geometry achievement. Although MEB explained a modest amount of the variation in geometry achievement in the intervention group, its predictive effect was not statistically significant.

Although mathematical epistemological beliefs differed significantly between group taught by RME-based mathematization approach and group taught conventional approach, they failed to appear as a significant predictor of geometry achievement in the intervention group. This could be explained by the short duration of the intervention. Epistemological beliefs are deeply rooted cognitive beliefs that might need a longer period of instructional exposure before influencing achievement outcomes. In addition, the small sample size of the intervention group might have impeded the ability to detect their unique contribution. It is also possible that the relationship between epistemological beliefs and achievement is mediated by other variables, such as problem-solving skills, which appeared as a significant predictor.

5.5. **Analysis of Qualitative Data**

Addressing the aforementioned study problems necessitates not only the use of quantitative data analysis but also qualitative data and analytical methodologies. Since comprehensive and in-depth insights about subjects and realities characterize qualitative research, the researcher seeks an in-depth understanding of the overall process of the second phase of the study in relation to epistemological beliefs, problem-solving skills, and geometry achievement. Therefore, in Phase II of this study, the researcher conducted an exploratory case study to investigate how RME-based mathematization instruction affected the thought process and mathematization process of pre-service teachers when solving geometry problems. Furthermore, the researcher used an exploratory case study technique to explore how the classroom instructor assessed the mathematization process of pre-service teachers. This is because an exploratory case study is a research method that aims to deepen understanding of a specific phenomenon or topic by conducting a detailed investigation and collecting and analyzing qualitative data through observations, interviews, and group discussion (Azungah, 2018; Ridder, 2017).

As stated by Creswell and Clark (2018), we can apply qualitative data collection to the experiment before, during, or after the intervention. Thus, the study utilized classroom observation, semi-structured interviews, and post-intervention interviews to gather qualitative data on participants' experiences and the long-term, sustained effects of the intervention.

For this study, because of its flexible nature, a semi-structured interview was used with seven pre-service mathematics teachers who were selected purposefully for the exploration of the mathematization process. For the exploration of the thought process of pre-service teachers because of intervention, the researcher selected three pre-service teachers from lower achievers, three from medium achievers, and three from higher achievers based on their mid-exam results, yet one from medium and one from higher achievers did not participate in the interview. Classroom instructors took part in the interview to learn more about assessment techniques for assessing pre-service teachers' mathematization processes.

In addition to a semi-structured interview, the researcher utilized classroom observation and field notes to collect the necessary data. The researcher also recorded video during the teaching learning process.

Thematic analysis is a qualitative research tool that allows researchers to organize and evaluate complicated data sets in a systematic manner. Researchers commonly utilize this method of qualitative data analysis due to its transparency, accessibility and conceptual flexibility, which enables it to be implemented in a variety of contexts to address research problems (Dawadi, 2020; Sovacool et al., 2023). Its ultimate purpose is to identify themes in the narratives that encapsulate the data sets. It requires studying and rereading the recorded material to uncover themes (Sovacool et al., 2023). Its thorough strategy can yield accurate and informative results (Naeem et al., 2023). These authors also stated that throughout the dataset, thematic analysis uses coding categories to uncover meaningful patterns.

In thematic analysis, there are two approaches: deductively and inductively, in which the former uses the existing concept as a basis to develop codes and meaningful themes, whereas the latter uses the data as a source for the emergence of codes and meaningful themes without taking account to themes in previous studies. As stated by Dawadi (2020), utilizing both deductive and inductive methods of thematic analysis maximizes the overall depth of the analysis. Taking this into consideration, the analysis of this study is not confined to either of the two approaches to thematic analysis but rather uses a combined deductive and inductive approach.

To maintain the requisite rigor in the analytical process, the researcher followed the six phases of thematic analysis outlined by Braun and Clarke. The researcher begins his deductive thematic data analysis process with themes derived from research questions or themes identified throughout the study's literature review. Using unexpected themes can lead to a greater understanding of the issue under study. Thus, a significant number of inductive codes may emerge throughout the data analysis process

Thus, the data gathered from classroom observations, field notes, and interviews was thematically analyzed to answer research questions. The data was analyzed by first transcribing audio and video-recorded data, after which the researcher gave the transcribed data to each participant for evaluation, giving participants the chance to check and maybe clarify their claims.

Following transcript verification, the researchers began data analysis using Braun and Clarke's six-phase theme analysis model. Theme development began with data transcription, which was then reviewed and reread in order to get acquaint with the data. We started the second part by doing some basic coding to reduce a big volume of data into manageable meaning bits. We looked over our data and added descriptive labels or codes that corresponded to our study questions. We categorized each transcript separately, finding portions that addressed our research question. Before proceeding to the remaining transcripts, we compared, debated, and changed our codes after finishing each one. During the third step, the researchers attempted to combine several codes based on their characteristics and study objectives. The fourth step is the review stage, in which the researchers critically evaluate the ideas developed in the third stage. The researchers meticulously examined the data to establish how much support each topic got before assessing the themes' relevance within the larger data set, as well as the fifth stage of defining and naming. Finally, we created the report using the data's extracted themes.

5.5.1. The influence of RME-based mathematization instruction pre-service teachers' thought process

This section looks at how pre-service teachers' epistemological beliefs—about the source, stability, and structure of mathematical knowledge—are affected by RME-based mathematization education. The findings of this study were derived from interviews and classroom observations.

Sources of Mathematical Knowledge

Learners as Active Constructors of Mathematical Knowledge

Reinvention of Mathematical Concepts through Experiential Contexts: In RME-based mathematization, learning mathematics begins by providing learners with experientially realistic scenarios that encourage them to reinvent (rediscover) mathematical concepts through progressive mathematization with close guidance from their teacher. This approach reflects the core RME principle that mathematics should be experienced as a human activity, where learners actively construct mathematical knowledge rather than passively receiving it. In this context, pre-service mathematics teachers developed different mathematical ideas, including triangle inequality, the sum of the degree measure of a triangle's inner angles, and determining the necessary and sufficient criteria for a triangle to be congruent. They proved the equivalence of the arc lengths created by two parallel chords and the triangle's point of concurrence. They were able to recognize specific quadrilaterals by folding the paper and develop an algorithm for establishing the relationship between the sides and the diagonals of a polygon.

These learning experiences show that the pre-service mathematics teachers were constructed as the main constructors of knowledge. This represents a shift from absolutist epistemological beliefs in which the teacher is solely responsible for knowledge transmission to a more constructivist and fallibilist approach to learning in which the learner participates in knowledge construction, fostering critical thinking and collaboration among students.

Teacher Guidance in the Process of Knowledge Construction: When developing algorithms to determine the link between a polygon's sides and diagonals, pre-service mathematics teachers encountered difficulties with mathematical induction. In this situation, the instructor stepped in to help them prove the mathematical induction. This illustrates the RME notion of guided reinvention, where the teacher does not simply transmit knowledge but supports learners in refining and formalizing their emerging mathematical ideas. In this way, the teacher plays the role of a facilitator who supports learners' mathematical reasoning while allowing them to remain actively engaged in constructing knowledge.

Multiple Sources of Mathematical Knowledge

Society and Everyday Activities as Sources of Mathematics: Furthermore, an interview with four pre-service mathematics teachers found that they accepted the coexistence of absolutist and fallibilist perspectives. This change can be associated with the RME-based mathematization strategy implemented during the intervention. Before this, they thought that the only source of knowledge in mathematics was the instructor, and the learner was just a receiver of the knowledge, and the only way they mastered the algorithms was by practicing them. However, after the intervention, they started realizing that knowledge in mathematics can be drawn from many sources, especially from their daily life experiences and societal activities, which is in line with the RME approach that knowledge in mathematics is drawn from real-life situations and human activities.

For instance, C022 stated:

> *Not only textbooks and videos, but society is the source of mathematical knowledge. In school, everyday activities are structured and taught as mathematics. To illustrate the source of mathematical knowledge, teachers bring practical mathematics practiced in society into the classroom. Therefore, a learner-centered teaching and learning approach that relies on real-life contexts should be promoted.*

This statement reflects the participants' growing awareness that mathematical knowledge practiced in society can be organized and formalized into school mathematics, which aligns with the RME process of mathematization where informal contextual knowledge is gradually transformed into formal mathematical concepts.

Teachers and Educational Resources as Supporting Sources

Another pre-service teacher, C026, reflected:

> *The source of mathematical knowledge is society. Additionally, teachers are also sources of mathematical knowledge since they are part of society. Students can construct mathematical knowledge using their own prior experiences and even*

learn from each other. The way in which the student understands the specific problem is different, and this is an indication that students are capable of creating their own mathematical knowledge.

This view illustrates an understanding of the student's epistemology, which is not limited to textbooks and instructors but is developed by interacting with different sources.

Learning Mathematics through Social Interaction and Experience

Prior Experiences and Mistakes as Sources of Mathematical Understanding: Some pre-service teachers believed mathematical knowledge originates from earlier experiences and mistakes. For example, C027 mentioned:

> *Individuals have different experiences. Teachers guide me to my destination. We learn a lot from our teachers and books. Students learn from each other since they come from different places, with different cultures and ways of living. Even buildings in different regions vary, reflecting diverse knowledge.*

This view highlights the role of diverse experiences and reflection on mistakes in developing mathematical understanding. Such beliefs align with social constructivist perspectives and with RME's view that learners refine their understanding by comparing different strategies, discussing ideas, and reflecting on errors during the mathematization process.

Social Interaction as a Basis for Mathematical Knowledge: However, three pre-service mathematics teachers viewed the development of mathematical knowledge as a result of social interaction. These pre-service teachers disregarded books, the internet, and other forms of media as sources of mathematical knowledge. One pre-service teacher supported this by stating that mathematical knowledge does not originate from books, videos, or the internet but rather from scholars who have prepared or written about it. These scholars, being members of society, construct their ideas of mathematics from the real world and then proceed to organize them into knowledge. This is a fallibilist approach in the understanding of mathematics as a construct of the human mind.

Dual Nature of Mathematical Knowledge: Discovery and Invention

Discovery of Mathematical Knowledge: In their interviews, pre-service mathematics teachers acknowledged both the discovery and invention of mathematics. Pre-service mathematics teachers recognized the discovery of mathematical knowledge by considering the relationship between the square of the sides of a right-angled triangle (Pythagoras' Theorem) and the ratio of the circumference of a circle to its diameter (π). Furthermore, the discovery of irrational numbers such as the square root of two and the formula for calculating the area of a triangle using its sides (Hero's formula) prompted them to acknowledge the discovery of mathematical knowledge.

Recognizing mathematics as discovered reflects elements of an absolutist epistemological belief, where mathematical truths are viewed as existing independently and waiting to be found.

Invention of Mathematical Knowledge: By taking into account negative numbers, coordinate geometry, calculus inventions, and many others, they also acknowledged the invention of the nature of mathematical knowledge. This understanding of mathematics as invented is a fallibilist approach that understands that the field of mathematics is a human-made endeavor to help us overcome problems. This is similar to the RME understanding that the origin of mathematics lies in human activity that gradually becomes more formalized through the process of mathematization.

Stability of Mathematical Knowledge

Another part of mathematical epistemology related to the nature of mathematical knowledge is its stability. In addition, findings from the interview revealed that pre-service teachers maintained two epistemological views on the nature of mathematical knowledge. They considered some mathematical knowledge to be eternal and unwavering, while some mathematical knowledge changes with time. This two-aspect view of mathematical knowledge was related to three main themes among the interview results: 1) Evolving and

Corrigible Nature of Mathematical Knowledge, 2) Permanent or Everlasting Nature of Mathematical Knowledge, and 3) Relationship Between Mathematical Knowledge, Human Activity, and Culture.

Evolving and Corrigible Nature of Mathematical Knowledge

Many participants emphasized that mathematical knowledge is dynamic, evolving, and corrigible rather than a finished product. C022 captured this viewpoint by stating:

> "*Mathematics from yesterday differs from mathematics today. The development of new mathematical knowledge is the outcome of civilized generations. Throughout the intervention period, we established mathematical concepts while attempting to infer various equations and build congruency requirements for triangles. Producing new mathematical knowledge necessitates extensive study, and the teaching-learning process must consider societal reality."*

Similarly, C026 highlighted the corrigible nature of mathematical knowledge, noting that even long-established truths may be revised in the future:

> "*Euclid's axioms are all true right now, but someone else may modify them later. As a result, one's understanding of mathematics is always evolving and open to revision."*

C029 further connected the evolving nature of mathematics to technological and societal changes:

> *"Today's technology is not the same as yesterday's, and future technology will be more sophisticated. Mathematical knowledge is a human activity, a continuous process of inquiry that is constantly susceptible to change*."

These responses demonstrate that mathematical knowledge is not absolute; it develops with new discoveries, societal needs, and technological advancements, reflecting its dependence on time and context.

Permanent or Everlasting Nature of Mathematical Knowledge

Recognizing the evolution of mathematics, the participants were also aware of the existence of permanent knowledge. The existence of both dynamic and permanent knowledge was stressed by C024:

> *"Even if mathematical knowledge changes throughout time, certain mathematical knowledge remains permanent. Conclusions on the fixed and permanent character of all mathematical knowledge are difficult to make since different scholars have produced contradictory results, demonstrating mathematical knowledge's corrigibility. New mathematical knowledge will emerge throughout time as mathematicians tackle a variety of problems."*

C023 also confirmed this duality, stating:

> *"There is mathematical knowledge that evolves over time and mathematical information that is permanent. I believe that the vast bulk of mathematical knowledge is correct. Our parents used to build circular houses, but now they have built houses in the shape of rectangles, squares, and other polygons. This revealed that they gained knowledge of how to construct their buildings in numerous ways, showing a shift in their knowledge of mathematics."*

These observations offer an insight that mathematical knowledge has lasting truths but also has the potential for refinement through human investigation and reasoning.

Relationship between Mathematical Knexaminerowledge, Human Activity, and Culture

Participants were also able to make connections between mathematics and human activities, daily lives, and society, as well as the understanding that knowledge stability and development are, to some extent, influenced by the human experiences and practices carried out within society. For one, C022, C026, and C029 made connections about mathematics as a human-oriented activity that evolves with civilization:

"Producing new mathematical knowledge necessitates extensive study, and the teaching-learning process must consider societal reality." C022"Mathematical knowledge is a human activity, a continuous process of inquiry that is constantly susceptible to change." – C029.

C023's example of changing house construction techniques illustrates how cultural practices and societal needs drive the development and application of mathematical knowledge.

This perspective emphasizes the fact that the stability of knowledge is not purely abstract but rather merges with human culture, practical needs, and the advancement of technology.

Structure of Mathematical Knowledge

The structure of mathematical knowledge, which concerns how mathematical ideas are organized and related, is a key component of mathematical epistemological belief examined in this study. Analysis of the interview data indicates that pre-service mathematics teachers generally positioned themselves toward the fallible end of the epistemological continuum, viewing mathematics as dynamic, interconnected, and evolving rather than fixed and absolute. Their responses reveal three overarching themes:

mathematics as a structured and hierarchical system, complexity and connectedness of mathematical knowledge, and multiple pathways to mathematical understanding.

Mathematics as a Structured and Hierarchical System

Hierarchical Organization of Knowledge: Several pre-service teachers perceived mathematical knowledge as being organized hierarchically, where earlier knowledge forms the foundation for later understanding. This view reflects a belief that mathematical ideas are systematically arranged and internally consistent. For instance, C022 emphasized the foundational role of prior knowledge in constructing new mathematical understanding:

> *"Mathematics has practical applications. This is the area of our current conversation. Through our teaching and learning experiences, we understood how intimately tied mathematics is to our environment. Prior experience may be a source of mathematical knowledge; new insights can be based on previous knowledge"* (Interview with C022).

This response suggests that mathematical knowledge builds upon itself, with previous experiences serving as the basis for acquiring new concepts. The participant's stance illustrates a coherent structure of knowledge in which earlier learning supports later conceptual development.

Progression from Simple to Complex: The hierarchical nature of mathematical knowledge was further highlighted through participants' recognition of progression from simple to more complex ideas. C023 explicitly acknowledged this structured progression, stating that:

> *"The structure of some mathematical knowledge is in hierarchical order."*

He elaborated that mathematical understanding develops through a process that moves from simple concepts to more complex ones. This belief was also evident when he discussed problem-solving, noting that while some problems may have single solutions, others—particularly open-ended problems—allow for multiple solutions and approaches. For example:

> *"An open-ended problem like $x + y = 9$ has multiple solutions, and students can approach it in a variety of ways."*

This response reinforces the idea that mathematical learning advances through stages, with increasing levels of abstraction and complexity.

Complexity and Connectedness of Mathematical Knowledge

Complexity of Mathematical Structure: Some participants emphasized the complex nature of mathematical knowledge, arguing that it cannot be reduced to a single, linear structure. C027, for instance, noted that although mathematics contains fundamental concepts, its overall organization is diverse and multifaceted:

> *"There are certain fundamental mathematical concepts, [but] the organization of mathematical knowledge is complex and varied."*

He further suggested that different problems require different rules and approaches, implying that mathematics is composed of multiple interrelated structures rather than one fixed framework. This view underscores the inherent complexity of mathematical knowledge.

Interconnectedness of Concepts: A strong sense of interconnectedness among mathematical concepts emerged across interviews. Participants frequently described how understanding one area of mathematics depends on knowledge from others. C029 highlighted this interconnectedness by explaining that comprehension of advanced mathematics relies on mastery of earlier concepts:

> *"Understanding current mathematics would be impossible unless I first understood the lower one."* (Interview with C029)

He illustrated this idea through a classroom problem involving the selection of a design for a farmer's house with a 48-foot perimeter. Solving the problem required the integration of multiple mathematical concepts:

> *"We used differentiation, the trigonometric principle, the circumference of a circle, the perimeter of a rectangle, and the perimeter of a regular hexagon."*

This example demonstrates the interconnected nature of geometry, calculus, and trigonometry, reinforcing the view that mathematical knowledge is both cumulative and relational. Similarly, C022 described a problem involving the relationship between the number of sides of a polygon and the number of diagonals. He observed that different solution methods revealed connections across mathematical domains:

> *"Some of the pre-service teachers employed combinatory concepts to solve this problem, while others used the rule of arithmetic sequence."*

He concluded that this problem demonstrated clear links between geometry, combinatorics, and arithmetic sequences, further evidencing the interconnected structure of mathematical knowledge.

Multiple Pathways to Mathematical Understanding

Existence of Multiple Solution Strategies: The existence of multiple approaches to solving mathematical problems was a recurring theme in participants' responses. C028 provided a clear example by discussing different methods for finding the area of a triangle:

> *"We can solve the area problem of a triangle in three ways: we can use half of the area of a rectangle; we can use half the product of the length of the two sides and the sine of an included angle…; or we can use Hero's formula."* (Interview with C028).

These approaches draw on concepts from trigonometry, geometry, arithmetic operations, and square roots, illustrating how a single mathematical problem can be understood through multiple conceptual pathways. Similarly, C023 emphasized that mathematical problems do not always have a single approach or solution:

> *"A given problem can be approached in several ways and may have one or more solutions; what matters is the problem itself."*

This belief reflects a flexible understanding of mathematics and aligns with a fallibilist perspective that values reasoning, exploration, and alternative strategies.

Mathematics as a Human and Everyday Activity

Beyond formal structures, participants also viewed mathematics as deeply embedded in everyday life and human activity. C022 conceptualized mathematics as a practical activity rooted in daily experience, while C023 described it as a cultural practice evident in community activities such as houses building, weaving, pottery, and basketry. C029 articulated this view vividly:

> *"Mathematics is the basis for everything. Mathematics is part of our everyday lives… Nothing is outside of mathematics."*

These perspectives further support the fallibilist view of mathematics as evolving, context-dependent, and socially constructed.

In general, the pre-service mathematics teachers—including C024 and C026—shared consistent views on the structure of mathematical knowledge. Their responses reflect beliefs aligned with the fallible end of the epistemological continuum, emphasizing hierarchical organization, progression from simple to complex ideas, the complexity and interconnectedness of mathematical concepts, and the existence of multiple solution strategies. Together, these themes portray mathematics as a structured yet flexible system that is both conceptually rich and deeply connected to human experience.

5.6. **Mathematization Process of Pre-Service Mathematics Teachers**

During RME-based instruction, the preservice teachers exhibited a variety of mathematizations. The mathematization process followed the conceptual framework shown in Figure 5.1, and the emergent mathematization was consistent with the curriculum areas taught by the pre-service teachers. The highlights of mathematization within each topic area are shown below. The mathematizations were explained by assigning activities to pre-service teachers and seeing how they mathematized while participating in the activities.

The activities were designed through the provision of a realistic situation following the reality principle. Pre-service teachers use the activity principle to guide their learning, and they use the guiding principle to facilitate their learning so they may make conjectures or generalizations. The interactivity principle, which encourages pre-service teachers to interact with one another and with the teacher educator through discussion-based activities, is followed by the intertwinement principle, which helps them integrate various mathematical concepts and improve their mathematization ability. The actions and accompanying outcomes are given below:

Mathematization on triangle's sides

One of the most notable successes witnessed was when pre-service instructors addressed the activity. The researcher observed the action in the classroom. Pre-service teachers were supposed to follow the guidelines provided by the instructor, who described the activity's scenario in full on a whiteboard. Here is what the activity is.

Activity: A farmer bought a 36-meter-long saaxara [read as sa't'ara in English], which is a hand-woven rectangular surface with varying width and a height of around 2.5 m. It is made of broken reed wood and used for fencing purposes. The farmer wishes to enclose a parcel of land in a triangular form, with the perimeter equaling the length of the saaxara. The farmer may cut the saaxaraa into 18 m, 12 m, and 6 m lengths for fencing, is it possible

to fence this triangle farm without leaving any gap or overlapping these stated dimensions? Likewise, given the same circumstances, how many triangles could be formed from the 36-meter-long saaxaraa, assuming that the sides of the triangles are natural numbers?
The majority of pre-service teachers first felt they could make a triangle with sides of 18m, 12m, and 6m. They immediately noticed their mistake when they attempted a real-world demonstration using a 36-meter-long rope instead of saaxaraa.

First, they lengthened the rope by 18 meters in one direction. They then bent the rope and stretched it for another 12 meters in a different direction. Finally, when they attempted to join the end of the remaining 6 m length to the beginning point, they discovered that it was impossible to make a triangle. This insight was clearly articulated during a post-exercise conversation with one of the pre-service teachers.

Researcher: Is it practical for a farmer to fence a triangle farm without leaving any room or overlapping any of its dimensions? Why not?

Respondent pre-service teacher: No, since he did not employ the proper triangle measurements.

Researcher: What exactly are you referring to? Earlier, you claimed that a triangle of such proportions could be built, but now you have retracted that claim. For what reason?

Respondent, pre-service teacher: Our trials yielded an open curve rather than a closed one. However, a triangle is a closed curve with three sides. We cannot construct a triangle with those measurements (18 m, 12 m, and 6 m).

Although the respondent pre-service teacher recognized that a triangle is a closed curve made up of three sides, he first assumed that any three lengths might form a triangle. However, hands-on testing highlighted the importance of choosing acceptable proportions. Without the proper proportions, one can produce a three-sided open curve but not necessarily a triangle, as seen in this excerpt from their study.

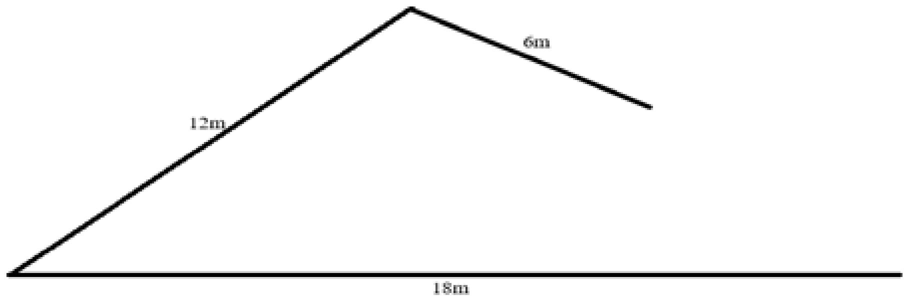

Figure 5.4 An open curve formed from 18m, 12 m, and 6 m

During the post-activity reflection, pre-service teachers sketched down on paper what they had done in the field. They drew an open curve measuring 12m, 4m, and 8m in length. Following that, conversation with another pre-service teacher proceeded as follows:

Researcher: After reviewing the farmer's proportions, you considered your own and created triangles. How are the farmer's measurements different from yours?

Preservice Mathematics Teacher: The primary distinction is that the farmer's measurements produced an open curve, whereas our specifications produced a closed curve or triangle.

Researcher: Did your experiments reveal any patterns in the relationship between the three sides?

Pre-service Mathematics Teacher: The three sides of the triangles we investigated have relationships.

Researcher: When you compared the farmer's chosen proportions with your own, what conclusions did you draw?

Preservice Mathematics Teachers: We learned that in our particular scenario, the sum of any two sides was always greater than the sum of the third. However, given the farmer's measurements, there were times when the sum of two sides matched the third and others when the sum of two sides exceeded the third.

Researcher: What conclusions did you draw from this activity and your field experiments? Will any three-line segment's lengths create the sides of a triangle?

Preservice Mathematics Teacher: Based on our findings, we determined that three given lengths a, b, and c can form the sides of a triangle if and only if: i) $a + b > c$, ii) $a + c > b$, and iii) $b + c = a$.

Pre-service teachers discovered via hands-on practice that the lengths of any three sides form a triangle if and only if the difference between any two sides is smaller than the third. Furthermore, if the lengths of any two sides do not add up to more than the third, there is no way to build a triangle. The description of these circumstances clarified the relationship between triangle sides and exposed a natural fact that the preservice teachers may utilize to refine their mathematizations.

Mathematization on triangles ‘congruence

Triangle congruence is a fundamental notion in geometry. Pre-service mathematics teachers attempted to develop criteria for determining the congruency of two or more triangles. The method they used was experiential, involving a group game.

Activity: Establishing necessary and sufficient criteria for triangle congruence.

Lead carpenters group secretly drew a triangle with three sides and angles as part of the exercise, while the apprentice carpenters group, which was not aware of the original triangle, was gradually given conditions to replicate it. Initially, the lead carpenters group only provided one angle's degree measurement to the other group participating in the activity. Despite numerous tries, drawing triangles such as scalene, isosceles, right-angled, obtuse, acute, and equilateral could not yield a congruent result. We discovered that triangle congruence is not ensured by a single congruent angle.

Despite offering the degree measure of a second angle, the opposing group was unable to replicate the original triangle. This led to the insight that even two congruent angles do not establish triangle congruency until a side is included.

They also attempted a third condition, a common side not covered by the previously revealed angles, and the second group was successful in replicating the original triangle. Contrary to their previous understanding, they found that congruency arises when two angles and the excluded side of one triangle are congruent to the equivalent portions of another triangle. As a result, the Angle-Angle-Side (AAS) criterion arose as a reliable requirement for triangle congruency. Expanding on this accomplishment, other groups discovered more congruency criteria that the preservice teachers were used to, including side-angle-Side (SAS), angle-side-Angle (ASA), and side-side-Side (SSS).

Mathematization linked to polygonal diagonals

When students go from a triangle notion to quadrilaterals and, eventually, generalized polygons, one of the obstacles they encounter is breaking a polygon into several triangles with which to handle various concepts such as angle measures and areas. It also highlights the relationship between a polygon's total number of diagonals and sides. Pre-service teachers were given an activity in this regard. On a whiteboard, the instructor described the activity scenario and guided the pre-service teachers based on what was expected of them. The activity is as follows:

Activity: Examining the link between a polygon's sides and its total number of diagonals. Is there a relationship between the total number of diagonals and the number of sides in a polygon? If this is the case, what is the relationship? How do we mathematically express this? Create a formula for the total number of diagonals in an n-sided polygon and demonstrate a proof for $n \geq 3$, where n comes from the natural numbers.

The course instructor gave pre-service math teachers the freedom to design their own strategies for solving geometry problems in the classroom. The pre-service teachers had ten to fifteen minutes to work on the task individually, rather than relying solely on pre-established approaches. It was evident from the class observations that they made a conscious attempt to apply horizontal mathematization while solving difficulties. This entailed making intentional attempts to recognize and communicate mathematical concepts as they completed normal problem-solving activities.

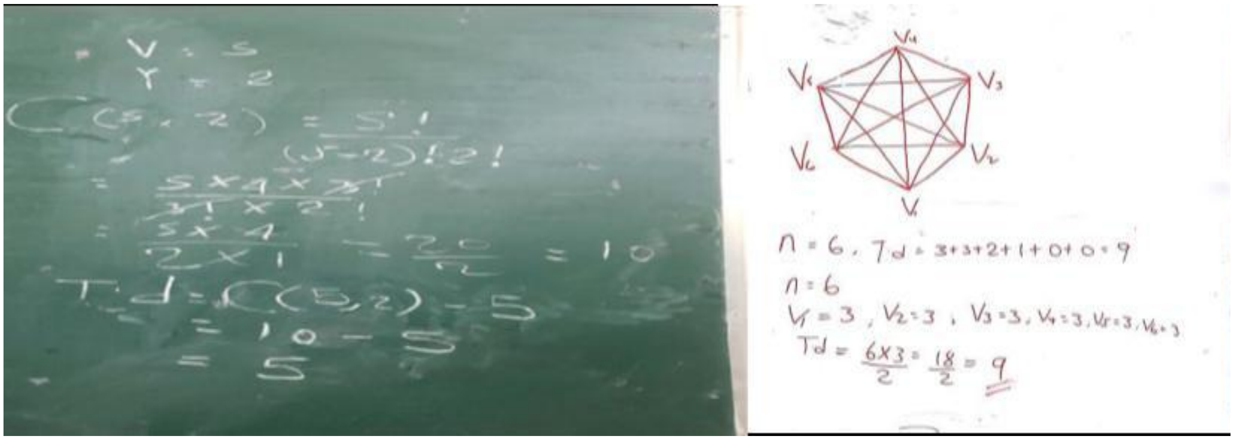

Figure 5.5 Mathematical ideas employed in formulating the number of diagonals of polygons.

Pre-service teachers identified various concepts, such as the four basic operations, counting, factorials, combinatorics, and the arithmetic sum of natural numbers, in the process of solving geometry problems. While solving the problems, they utilized different tools like picturing, schematizing, and expressing their answers, and they discovered a correlation between the number of diagonals and sides of a polygon by counting them once. Others utilized combinatorics to determine the total number of diagonals. Another group utilized the formula "n + the sum of the first n natural numbers," where n is the greatest number of diagonals from an initial vertex, to illustrate their mathematical knowledge of the problem. A hexagon, for example, begins with three diagonals from a vertex. The total of the diagonals is 3 + (3 + 2) + 1 = 9.

The sections below show how pre-service teachers arrived at their formulations of the relationship between the sides of a particular polygon and the total number of diagonals using inductive reasoning.

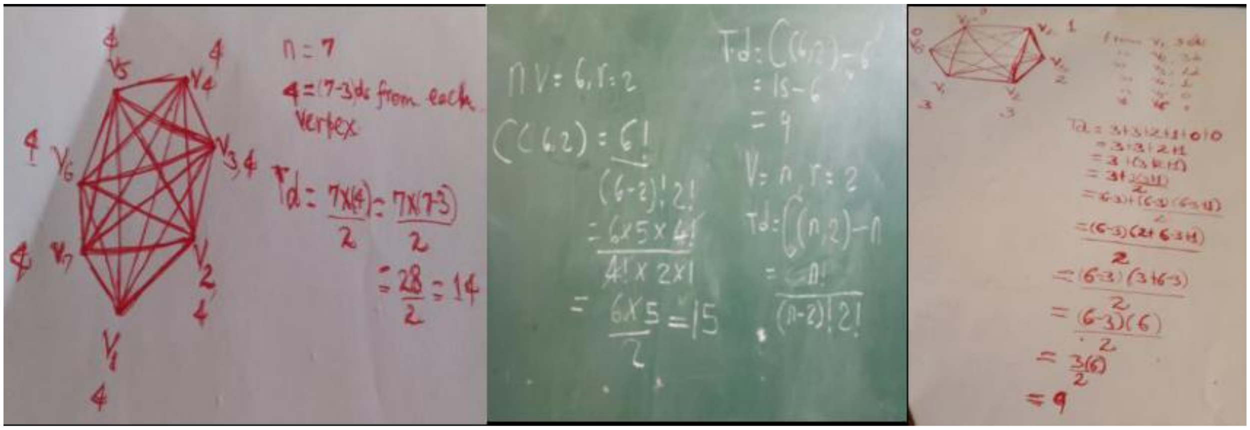

Figure 5.6 Illustration of strategies pre-service mathematics teachers apply to formulate the relationship between sides of a given polygon and the total number of diagonals

Mathematization on an equiangular quadrilateral

This contextualized quadrilateral activity offers insight into how pre-service teachers practice and develop their own mathematical reasoning, especially horizontal and vertical mathematization within the RME framework. Such translation, when teachers-in-training interpret a real-world scenario, such as a carpenter designing a four-sided house, into the geometrical properties of quadrilaterals, represents choices in how they connect practical contexts with formal mathematical structures. The activity of ensuring an equiangular quadrilateral through diagonal conditions provides a rich lens through which to explore how they model, represent, and progressively formalize geometric ideas. The instructor provided following activity.

Activity: A carpenter wants to design the ground before building a four-sided house. The shape of the house he wants to build is a four-sided polygon in which its pair of opposite sides are at equidistance from each other. The carpenter wants to use the roof sheets and the materials for the cornice efficiently because he has observed that this is true when the polygon is equiangular. Does making the diagonal of the quadrilateral equal an effective strategy to ensure the equiangular quadrilateral? How?

In solving the realistic quadrilateral design problem, the pre-service teachers demonstrated both horizontal and vertical mathematization as they transitioned from informal, context-based strategies to a formal geometric proof.

The pre-service teachers began to solve the problem by interacting with it using informal, context-connected techniques that helped them make sense of the real-life situation of a carpenter building a four-sided house. Their first attempts involved measuring each interior angle using a protractor that allowed them to realize whether the quadrilateral appeared equiangular. Their other approach was to use paper folding to overlap the quadrilateral's corners (vertices) using the principle that if corresponding angles coincide, they must be congruent. This was informed by their attention to the congruency of the diagonals. Some other pre-service teachers utilized A4-sized paper as a right-angle template, positioning the corner of the sheet on each vertex to visually confirm if the angles were 90 degrees. In addition, instead of using A4-sized paper, they utilized a 3 cm-4 cm-5 cm triangle, placing one on each of the four corners to ensure that each angle of the quadrilateral is 90 degrees. All of these informal approaches to the problem demonstrate how pre-service instructors transform real-world contexts into mathematical thinking using concrete actions, tools, and intuitive insights. Horizontal mathematization is distinguished by the use of measurement, folding, and physical alignment, all of which contribute to the transformation of a genuine situation into an initial mathematical model that students may study. Through these experiments, the pre-service teachers arrived at an informal conclusion: since all four corners seem congruent and their total must be 360°, then each angle should be 90°, implying that the quadrilateral is equiangular. This informal understanding helped them advance to reasoning that is more abstract.

Following their hands-on experiments, the pre-service teachers started formalizing their results. They meticulously justified their findings by using mathematical characteristics, logical structure, and deductive reasoning. This movement denotes vertical mathematization, in which informal models are restructured into formal mathematical connections. In their formal proof, they identified alternate internal angles by viewing the diagonal as a transversal line and the quadrilateral's parallel sides, and they determined triangle congruency using the SAS congruency criterion. Then, using the congruency of the two diagonals and the SAS congruency principle, they demonstrated that ABCD is a parallelogram. After researching the parallelogram's properties, pre-service teachers proved that the diagonals created isosceles triangles. Using additional adjacent angles and equal angles within these isosceles triangles, they discovered that each internal angle is 90°. In conclusion, the quadrilateral is a rectangle and so equiangular. Finally, they found that having the diagonals congruent is an effective method for ensuring that such a quadrilateral is equiangular.

By considering ABCD to be the four-sided polygon the carpenter wishes to design, and the pair of opposing sides to be at equidistance (parallel), and the diagonals of the quadrilateral to be congruent, their formal proof is as follows.

Table 5.18 Formal proof of the equi-angular of the quadrilateral that characterizes vertical mathematization

No	Premises	Reason
1	$\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$	Given
2	$\angle ABD \cong \angle CDB$,$\angle BAC \cong \angle DCA$, $\angle CBD \cong \angle ADB$ & $\angle CAB \cong \angle ACD$	Since $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$ and $\overline{AC}$ & $\overline{DB}$ are transversal lines, and alternate interior angles are congruent.
3	$\overline{AC} \cong \overline{DB}$	Assumption.
4	$\Delta ABC \cong \Delta CDA$	By SAS from 2 and 3.
5	$\overline{AB} \cong \overline{DC}$ & $\overline{BC} \cong \overline{DA}$	From 3
6	ABCD is a parallelogram	From 1 &5
7	ΔAED, ΔCEB, ΔDEC & ΔAEB are isosceles triangles.	From 3, 6, and the properties of a parallelogram.
8	$\angle EAD \cong \angle DAE \cong \angle$BCE $\cong \angle EBC$	From 2, 7, and the properties of an isosceles triangle.

9	$\angle DAB$ and $\angle ADC$ are supplementary.	Since adjacent angles of a parallelogram are supplementary
10	$u + x + n + r = 180^{o}$	From 9
11	$n + r + n + r = 180^{o} \Longrightarrow 2n + 2r = 180^{o}$	Since $u = n$ and $z = r$ and $z = x$ from 2 and 7
12	$n + r = 90^{o}$	From 11
13	Thus ABCD is a rectangle	From 1, 3, 12
14	Thus, ABCD is an equiangular polygon.	From 13.
15	Thus, making the diagonal equal is an effective strategy to ensure equiangular polygon.	

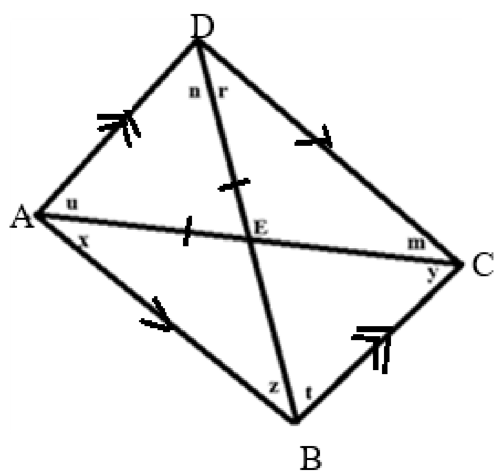

Figure 5.7 Quadrilateral with congruent diagonals

Mathematization on the Circle

The instructor offers an activity in which preservice teachers mathematize and develop a mathematical formulation based on RME-based mathematics for a house in a rural area with a concentric cylinder shape and concentric circles as a basis. The activity provided the following scenario. As seen in the figure below, the inner circular wall is separated into four sections: bed, shelf, fire, and gate. The owner of the home divides the inner cylindrical wall into four pieces by measuring two steps up, down, left, and right from the center of the circle and placing the pegs. Finally, using pegs, students stretch a thread from one end of the inner circle to the other to create open space in four directions. Do you believe the four arcs are congruent? If so, in what way?

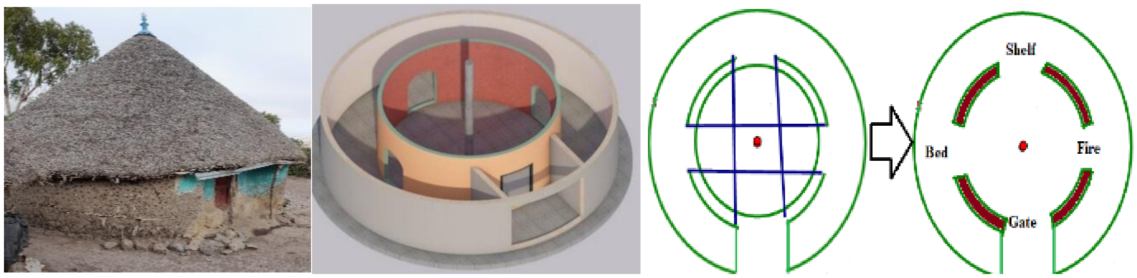

Figure 5.8 Straw house with concentric cylinder-shaped walls and partitioned into four parts

The instructor assigned the above activity. The activity said that the inner wall was separated into four sections. Furthermore, the inner cylindrical wall is separated into four sections (four independent walls). Do you believe these maintain the congruency of the four arcs generated by the four entries? How?

PST1: My response is yes. First, we analyzed one pair of opposing arcs at a time. We demonstrated in the group that a pair of opposite entries is equal, which was accomplished using folding. For example, we may check that the entry to bed and the fire are identical by folding the circle horizontally and seeing whether or not the two arcs' lengths overlap. Similarly, we folded the circle vertically to guarantee that the gate and shelving entry are equal. We used meter tape, a protractor, and ropes to measure each arc individually, and we validated that all four entries were equal. Researcher: PST2, how did you respond to this inquiry?

PST2: In our group, we addressed the congruency of these four entries by removing one sector from the circle and then overlaying it on the remaining portions, and we found that it fits perfectly, confirming that all four arcs (entry) are congruent.

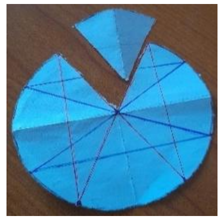

Figure 5.9 Utilization of the removed sector for confirming the congruency of arcs

Researcher: Are the four portions of the four walls equal? How?

PST1: To guarantee the four areas where the wall falls are congruent, we first folded the circle vertically, then horizontally. We checked that all four portions, in which all of the inner walls fell, were equivalent. We also used the rope to measure the four arcs where the four walls meet, and we checked that all the walls are congruent.

PST2: In our situation, we cut off one arc where one wall falls and then overlapped it with the other remaining portions where the walls fall; we made sure it was an identical replica of the other parts, demonstrating congruence.

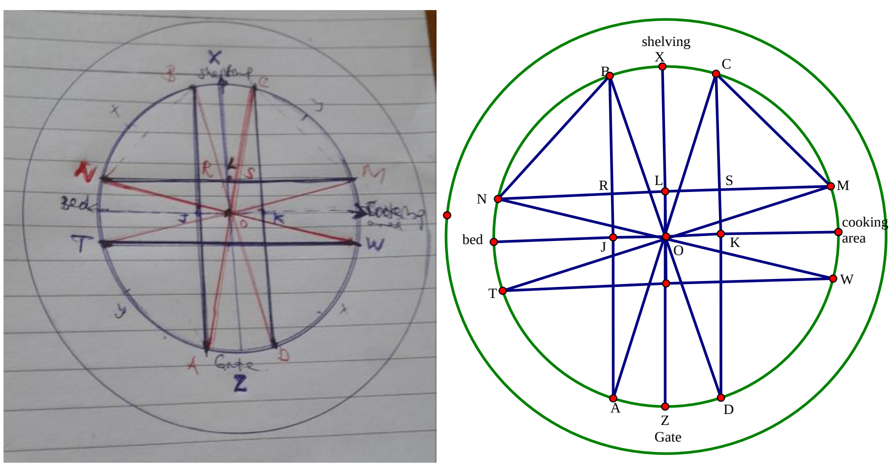

Figure 5.10 Diagrammatic representation of the proof drawn by the pre-service teachers and the right-hand side one is drawn by Sketchpad for clarification

Researcher: Is there any method to verify that the four portions of the four walls are congruent?

They designated each point as follows and continued their discussion.

PST1: Yes. Assume the letter 'O' represents the circle's center. The circle's diameter is $\overline{ZX}$

Researcher: How does $\overline{ZX}$ represent the circle's diameter?

PST3: PST3: Because it runs through the center of the circle and is the longest chord.

Researcher: Why are $\overline{AB}$ and $\overline{CD}$ parallel to $\overline{ZX}$?

PST1: Both are equidistant from $\overline{ZX}$ (two steps distance).

Researcher: I noticed that you used $\overline{NM} \perp \overline{AB}$ and $\overline{NM} \perp \overline{CD}$. Why?

PST3: If $\overline{AB}$ // $\overline{ZX}$ and $\overline{ZX} \perp \overline{NM}$ at point L, then $\overline{NM} \perp \overline{AB}$ at point R, ensuring the four parts' congruency. Furthermore, because $\overline{AB}$ // $\overline{ZX}$ and $\overline{ZX} \perp \overline{NM}$ at point L, then $\overline{NM} \perp \overline{CD}$ at point S.

Researcher: PST1, do you agree with PST3's statement? What prompted him to act this way?

PST1: Simply set m ($N\hat{R}B$)= 90^o & m(M$\hat{S}$C) = 90^o.

Researcher: How?

PST1: Because $\overline{ZX}$ is the diameter of the circle, and$\overline{ZX} \perp \overline{NM}$. $\overline{AB}$ & $\overline{CD}$ are parallel to $\overline{ZX},$ and both are perpendicular to $\overline{NM}$ at point R and point S, respectively.

Researcher: I observed that you made $\overline{NR} \equiv \overline{SM}$. PST3, please, would justify it?

PST3: Since $\overline{ZX}$ the bisector of $\overline{NM}$ at point L, and R and S are at equidistance from $\overline{ZX}$.

Researcher: How is $\overline{RB} \equiv \overline{SC}$? PST1, would you justify your reason?

PST1: Since $\Delta BJO \equiv \Delta CKO$ by the RHS congruency criteria and $\overline{RJ} \equiv \overline{SK}$.

Researcher: PST3, how do you see the relationship between ΔNRB & ΔMSC?

PST3: They are congruent.

Researcher: How do you justify it? PST1, would you clarify it more?

PST1: Sure. Since $\Delta NRB \equiv \Delta MSC$ by SAS, and this in turn shows the congruency of $\overline{NB}$ *and* $\overline{MC}$.

Researcher: Does the congruency of $\overline{NB}$ *and* $\overline{MC}$ tell us about the relationship between $\widehat{NB}$ and $\widehat{MC}$?

PST3: Of course. $\widehat{NB} \equiv \widehat{MC}$ since congruent chords subtend congruent arcs.

Researcher: PST1, Does arc NB have a relationship with DW? Does arc MC have a relationship with arc TA?

PST1: we know that both ($N\hat{O}B$) are (W$\hat{O}$D) are central angles, m ($N\hat{O}B$)= m($\widehat{NB}$) and m(M$\hat{O}$C)= m($\widehat{MC}$). Moreover, ($N\hat{O}B$) ≡ (W$\hat{O}$D) since they are vertically opposite angles. Furthermore, $T\hat{O}A$ ≡ M$\hat{O}$C since they are central angles and they are vertically opposite angles, showing that ($\widehat{TA}$) ≡ ($\widehat{MC}$) as m ($T\hat{O}A$)= m ($\widehat{TA}$) and m (M$\hat{O}$C)= m($\widehat{MC}$).

Researcher: PST3, do you want to say about the relationship among$\widehat{TA}$, $\widehat{MC}, \widehat{NB}, and$ $\widehat{DW}$?

PST3:Sure! All coincide since our major purpose was to assure their congruency, which we accomplished by the approach we have followed up to this point.

Researcher: What conclusions can you draw from your previous work on the link between the four arc length measurements?

PST1: Since the degree measure of the arc is equal to the degree measure of the central angle, the arc length is stated as $\ell = \frac{\pi r \theta^o}{180^o}$, where θ represents the center angle. Because all arcs have the same degree measure, their arc lengths must also be equal.

Researcher: PST3, do you agree with the idea of PST1?

PST3: Yes.

Researcher: Good job, thank you.

To show that the four arcs of the circle were congruent, the pre-service teachers used a variety of practical, context-dependent techniques that clearly define horizontal mathematization. Their thinking remained based on the actual representation of the circle, and they created meaning by rearranging and interpreting the figure with informal tools and techniques. Several pre-service teachers used the folding concept as an obvious approach to compare arcs. They folded the circle vertically through the center, then horizontally, so that the arcs' ends coincided. By physically overlaying the folded parts, they discovered that all four arcs matched perfectly. This hands-on manipulation of the figure demonstrates how learners turned the genuine scenario into a more comprehensible shape, ensuring horizontal mathematization.

Furthermore, some pre-service teachers cut out one of the arcs and placed it on top of the other three. The cutout piece's accurate alignment with each of the other arcs enabled them to conclude (informally but firmly) that all four arcs were congruent. This method indicates their desire to restructure the issue through direct comparison rather than formal geometric reasoning. Others utilized measurement-based procedures, such as using a protractor to compare central angles or a rope to measure arc lengths. They provided an empirical basis for determining equality by converting the arcs into measurable quantities. This also demonstrates horizontal mathematization because the preservice teachers were still working at the level of the physical or sketched figure while employing readily available tools to structure and reinterpret the issue circumstance.

These practical activities demonstrate how pre-service teachers engaged in horizontal mathematization by folding, cutting, comparing, and measuring to make sense of the situation. Their study reveals how learners informally examine and arrange the actual or visual surroundings, laying the groundwork for more formal mathematical reasoning to follow.

The pre-service teachers subsequently advanced from their informal explorations to developing a formal argument about the congruency of the four arcs—that is, the equality of the circular parts on which the four walls are positioned. During this vertical mathematization procedure, they discovered that the diameter of a circle is the perpendicular bisector of any chord running through it. They built on this by identifying pairs of congruent triangles using both the SAS and the RHS congruency criterion. They subsequently determined that the accompanying chords were congruent by examining the corresponding sides of these congruent triangles.

They used the fact that congruent chords subtend congruent arcs to conclude that the arcs corresponding to each wall must be identical. They also established that the four arcs covered by the four walls are congruent by using the congruency of vertically opposing angles and the concept linking central angles to arc measurements. This rigorous progress from informal thinking to the building of a justified, formal geometric argument exemplifies vertical mathematization, as pre-service mathematics teachers restructured and polished their thoughts into a cohesive mathematical framework.

5.6.1. **Assessing the Mathematization Process of pre-Service Mathematics Teachers**

The method of assessing pre-service mathematics teachers' mathematization processes is covered in this part. The collected data revealed that the classroom instructor utilized a variety of assessment techniques to assess the mathematization process of preservice teachers. These methods are informal talk, observation, probing during presentations, and argumentation.

Within this RME framework, these assessment activities allowed the instructor to investigate both horizontal and vertical mathematization processes. Horizontal mathematization evident when the pre-service teachers interpreted the task situations, game rules, and diagrammatic configurations into mathematical representations while vertical mathematization occurred in the refinement of these mathematical representations through formal reasoning, generalization, and application of established mathematical properties and theorems.

Informal dialogue

The observation results revealed that the teacher educator employed argumentation, observation, and interviews implicitly to measure how well pre-service mathematics teachers were mathematizing. This is an interview with a teacher educator who used these to help pre-service mathematics teachers establish what conditions were necessary and sufficient for triangles to be congruent.

Pre-service teachers, at an earlier stage in the problem-solving process, horizontally mathematized the game context by constructing physical and diagrammatic representations of triangles and testing given side lengths and angle measures. These activities involved the organization and structuring of the problem situation into mathematical objects such as sides, angles, and congruent figures.

To establish the congruency requirements, the instructor created a game for two groups and instructed them to play it. One group built a triangle without revealing it to the other group, while the other group was given one condition at a time to build a triangle, which was then compared to the one constructed initially by the first group. The instructor's duty was to supervise, observe, and analyze the progress of the pre-service mathematics teachers by interviewing them as they worked to build the congruency criteria. One group constructed a triangle and requested that their opponent group use SSA as one of the criteria for determining the congruence of two or more triangles. One group discovered that given two sides and a non-included angle, they could create congruent triangles. However, the other group did not produce congruent triangles under these conditions.

Herein, attempts to align the constructed triangles and the use of sketches and diagrams to compare figures represent cases of horizontal mathematization; pre-service teachers depended on visual and contextual reasoning when trying to make sense of the congruency conditions. At this point, there was dispute among the pre-service mathematics teachers in the classroom, and after consulting with their instructor; they concluded that Side-Side-Angle (SSA) might hold, although it is not necessarily true. This led to the realization that while some criteria may work for a specific mathematical problem, the same technique may not be applicable in general.

This shift of context-bound reasoning towards questioning the general validity of SSA marks the transition from horizontal to vertical mathematization. The discourse between the instructor and the pre-service mathematics teachers went as follows:

Instructor: There seemed to be discord among the groups, as I noticed. In your opinion, what is this problem?

PMT4: Yes, instructor. At the start of the game, the group gave us the length of one side of the triangle, which was 2 cm. Despite our best efforts, we were unable to replicate the triangle they created.

Instructor: What differences did you notice between the triangles your group and your opponent built?

PMT3: I realized that a single criterion or condition does not ensure the congruence of two triangles.

Instructor: So what did they give you the second time then?

PMT1: They gave the length of the second side of the triangle, which is 3 cm. We tried several times as well, but we were unable to obtain a perfect duplicate of what our rival group constructed.

Instructor: Can you make any generalizations based on your efforts? And why?

PMT2: Even if two sides of one triangle are congruent to the corresponding two sides of another triangle, the two triangles may not be congruent, suggesting that congruency of two sides of two triangles does not ensure their congruency. I believe the angle between the two sides matters.

Instructor: At the end, what did the other group give you?

PMT2: They gave us the degree measurement for the non-included angle, which was 40 degrees. We used the "sine law" to calculate the degree of an angle opposite the side with a length of 3 cm.

Instructor: Why did you wish to use the sine law?

PMT1: to obtain the angle.

Instructor: What results did you get from your calculations?

PMT2: The trigonometric table gave us x = 74 degrees, and we also found Sin(x) = 0.964.

Instructor: Is 74 degrees the possible value that angle x can have? Assume you are interacting with a triangle. What happens if the value of the degree measure of the angle is 106 degrees? You may recall the sine of the difference between two angles and its value. *Particularly Sin(*180^o-x^o*)*?

PMT4: Of course, yes. Suppose $y = 180^o - x^o$.Therefore, sin(180^o-x^o) = sin180^ocosx^o $- cos180^o sinx^o$ = 0-(-1)sinx^o= sinx^o. For x = 74 degrees, then siny^o= sin(180^o-74^o) $= sin106^o$.

PMT3: We calculated that the requisite angle was around 74 degrees. The remaining degree measurement of the third angle and length of the third side are 66 degrees and 2.85 cm, respectively. We were unable to replicate our opponent's triangle precisely. We attempted to align our ΔABD with ΔABC (the triangle our opponent group created), but ours did not match.

The first group constructed the first figure on the left, while the opposite group built the middle triangle, which measures 60 degrees, 2 cm, and 3 cm. The third picture depicts the triangles formed by the opposing group as they attempted to match their drawn triangles with the original triangle of the first group.

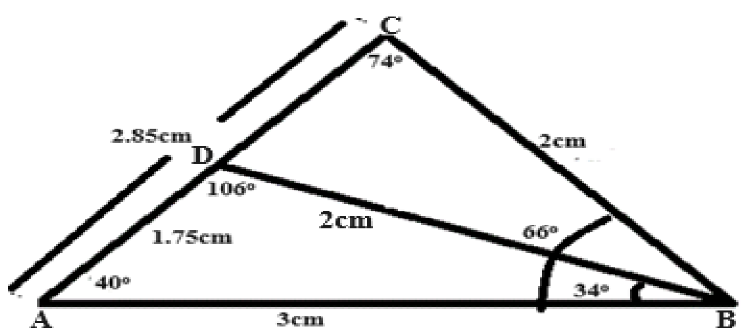

Figure5.11 Two triangles having two equal sides and one equal angle, but different in the remaining side and angles

Instructor: Why? What do you believe is causing the difference?

PMT4: I'm not sure, but it may be because we didn't meet our opponent's requirements for side length and angle degree measurement.

PMT2: Since Sin(x) = 0.964, the degree measure of x can be either 74 degrees or 106 degrees. These two parameters affect the third angle's degree measurement. If we take 74 degrees, the degree measure of the third angle is 66 degrees. However, if we take 106 degrees, the degree measure of the third angle is 34 degrees. This might be the explanation for the discrepancy.

Instructor**:** Can you give me each interior angle of a triangle if the degree measure of one angle is 106 degrees?

PMT2: Sure, Sir. A triangle's three angles measure 40 degrees, 106 degrees, and 34 degrees, respectively. This is what the other group had previously developed.

Instructor: What angle measurements did you use previously?

PMT3: 40, 74, and 66 degrees.

Instructor: Have you noticed a difference?

PMT2: Sure, sir.

Instructor: What conclusions can you draw from this?

PMT1: I believe that using the side-side-angle (SSA) criterion with random angles cannot ensure triangle congruence**.**

The conversation between the instructor and pre-service math teachers, as demonstrated above, not only suggests instructional and learning strategies to improve progressive mathematization. It also serve as a lens to identify the pre-service math teachers' conceptual maturity in determining the sufficient and necessary conditions for triangle congruency.

Specifically, vertical mathematization is manifested as the pre-service teachers' progress from empirical trial and error to the use of mathematical reasoning. Specifically, pre-service teachers were seen to use the sine rule, angle sum rule, and general properties of triangles to reason why SSA is not enough to ensure congruence between two triangles.

They can make conjectures about their outcomes since they are based on key mathematical procedures. The discussion has two functions: it is a teaching and learning tool, as well as a way of assessing pre-service teachers' mathematization processes.

Through informal discourse with the pre-service teachers, the instructor measured the horizontal mathematization by observing their ability to model and experiment with the task situation via diagrams and measurements, and the vertical mathematization by examining their ability to formalize, justify, and generalize their arguments via rules and symbols.

The approach encouraged the pre-service mathematics teachers to examine whether SSA was a necessary and sufficient condition for the congruency of triangles. Pre-service mathematics teachers used “Sine law” as well as the Angle sum theorem: “sum of the degree measure of interior angles of a triangle is 180 degrees”, resulting in two triangles having two sides and any random angle in common doesn’t indicate that the two triangles are congruent. To examine the way pre-service mathematics teachers mathematize in establishing congruency criteria for triangles, the instructor utilized probing and raising different questions as a means of assessment during the dialogue.

Observation

In the way of his observation, the classroom instructor observed strange concepts while a group of pre-service mathematics teachers was trying to construct a triangle with dimensions of 7cm and 5cm, and a non-included angle with degree measure of 60 degrees.

In the present study, a form of horizontal mathematization was noted among the pre-service teachers who tried to interpret the numerical data and create a concrete form of triangle construction. The instructor decided to discourse the issue with the group in the following way.

Instructor: You started the game by constructing a triangle of with dimension 5cm and 7cm and a non-included angle of 60 degrees that located opposite to the side of the triangle with side length 5cm. You provided triangle with dimension mentioned earlier to your opponent group. Your opponent group applied sine law to find the degree measure of an angle opposite to the side that was 7 cm long in their way of finding congruent triangle. Yet, they came up with strange result. Why did they come up with such an unusual result? Before providing the triangle with such dimension to your opponent group, did you check the appropriateness of the side lengths and the angle with such degree measure?

PSMT1: No, but it is possible to do now. Assume that x is the degree measure of an angle of a triangle opposite to the side with 7cm length. Since the degree measure of the angle opposite to the side with 5cm length is 60 degree, we can apply sine law to find the value of x. $\frac{\text{sinx}}{7\text{cm}} = \frac{\text{sin}60^{o}}{5\text{cm}}$, $\text{sinx} = \frac{7\text{cmsin}60^{o}}{5\text{cm}} = 1.212 \Longrightarrow \text{sinx} = 1.212.$

Instructor: The value of sinx you obtained was 1.212. How do you see such sine value for the angle of a triangle? Is it possible? What is the maximum value of 'sine' of an angle in a triangle? Does this value tell us something? Do not you think that there is some lesson that we get from this?

PSMT2: The value of the sine of an angle ranges between -1 and 1 and in a triangle the maximum value of 'sine' of the given angle is 1 whenever the degree measure the angle of a triangle is 90 degrees. This value indicated that there was either a problem with the lengths of the triangle's sides or a problem with the degree measure of the triangle's angle. We learned that while constructing a triangle we should care about the relationship between the sides of the triangles and the degree measure of each interior angle of the triangle.

Instructor: Suppose someone provide you two side lengths and the degree measure of the non-included angle. If you apply the sine law, when will you get a triangle? Why?

PSMT3: Let 'b' and 'a' be the lengths of two sides and let B and A be the angles opposite with dimensions 'b' and a, respectively. In applying the sine law, we will get the triangle if and only if $0 < \text{sinA} = \frac{\text{asinB}}{\text{b}} \leq 1$, $0 < \text{sinB} = \frac{\text{bsinA}}{\text{a}} \leq 1$ and the degree measure of the two angles need not be 90 degrees simultaneously.

The dialogue provided pre-service mathematics teachers an opportunity to realize their misconceptions. This was seen in the instances when pre-service teachers mathematized vertically by arguing the impossibility of the construction by appealing to the bounded nature of sine values and by establishing necessary and sufficient conditions for triangle formation. In both of these aspects, abstract thought detached from the original attempt of a physical construction was involved.

The classroom instructor provide feedback to pre-service mathematics teachers to through the above stated dialogue. Both pre-service mathematics teachers and their classroom instructor learned a lesson when and how to construct a triangle. During dialogue, pre-service mathematics teachers realized that a triangle is constructed if and only if the sine value of the provided angle is greater than zero and less than or equal to 1, provided that lengths of two sides and the degree measure of the non-included angle. Here, they are revealing the necessary and sufficient conditions under which, given two sides and a non-included angle, it is impossible to form a triangle with those two dimensions and a non-included angle degree measure that results in a sine value greater than 1.

Probing during presentation

The instructor asked a group to present the angle-angle (AA) or angle-angle-angle (AAA) criteria for congruency of triangles in another discussion. The presentation phase requested pre-service teachers to reorganize their previously produced task-based reasoning (horizontal mathematization) as structured arguments and formal justifications (vertical mathematization. One pre-service mathematics teacher attempted to explain what his group did in the discussion. The following was the conversation that the instructor and the presenter with his group.

Instructor: Is AA a necessary and sufficient condition for two or more triangles to be congruent?

P2: No, we have tried several times, yet we could not replicate the triangle our opponents constructed.

P1: The congruency of two or three corresponding angles of triangles is not a necessary and sufficient criterion for two or more triangles to be congruent.

Instructor: Why are AA and AAA not a necessary and sufficient condition for two or more triangles to be congruent?

P3: The main determining factors are the length of the sides of triangles. Suppose, we can consider two equilateral triangles with side length 2cm and 4cm respectively. Even though the degree measure each interior angles of the two triangles measures 60 degrees, their side lengths are different indicating that the two triangles are not congruent. The same is true in the case of AAA.

Instructor: What would you deduce from this then?

P1: From this, we can deduce that if two or three angles of one triangles are congruent to the corresponding angles of another triangle, the two triangles may not be congruent indicating that AA and AAA are not the necessary and sufficient conditions for two or more triangles to be congruent. Yet, if either the included or the non-included side of the two triangles are congruent, provided that two or three angles of one triangle are congruent to the corresponding angles of another triangle, then the triangles' congruency will be ensured.

P5: I agree with the idea of P1. Regardless of the sides' congruency of at least one corresponding sides, AA or AAA principle does not guaranty for the congruency of two triangles

The instructor let the pre-service mathematics teachers to present their discussion about AA or AAA principles. While presenting their discussion, there was probing whether AA and AAA are the necessary and sufficient condition for triangles congruency or not. Furthermore, the instructor probed pre-service mathematics teachers to justify their reasons why AA and AAA are/not the necessary and sufficient condition for two or more triangles to be congruent. Then pre-service mathematics teachers tried to present their justification whether AA and AAA are necessary and sufficient criteria for congruency of triangles or not. They supported their reasons by taking particular example whether AA and AAA principles ensure congruency criteria for two or more triangles. In this scenario, the instructor utilized presentation as a means of assessment to examine their reasons and justification while they were establishing the congruency conditions for two or more triangles.

Through the explanations, examples, and counterexamples they used, the instructor assessed the students' vertical mathematization, such as the pre-service teachers' capacity to formally describe conditions for congruent triangles.

Argumentation

From a theoretical perspective, argumentation captures important elements of both RME and Toulmin's model of argumentation, which conceptualize mathematical reasoning in terms of claims, data, and warrants. In this latter respect, argumentation functions as an analytic lens for assessing mathematization by showing how learners justify, generalize, and formalize ideas for mathematical phenomena.

Alternatively, the observation results indicated that classroom instructor utilized argumentation as a means of assessment strategy to assess the mathematization process of pre-service mathematics teachers. While establishing the necessary and sufficient conditions for triangles congruency, it was observed that pre-service mathematics teachers provided claim, data, and warrant.

The presence of claims, supporting data, and warrants is indicative of the engagement in vertical mathematization, whereby pre-service teachers moved beyond task-based experimentation into formal justification and logical reasoning about triangle congruency. As an example, the following figure depicted the data, warrant and claim that pre-service mathematics teachers provided while establishing congruency conditions for triangles.

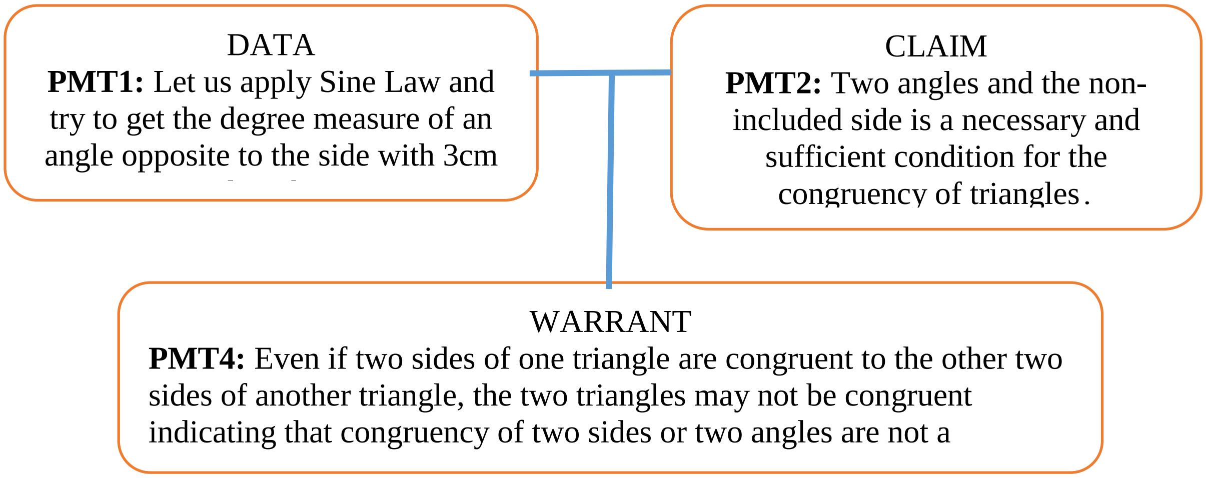

Figure 5.12 Toulmin's model of argumentation on congruency criteria of triangles

The claim, data, and warrant indicated in figure 4.2 that were provided by pre-service teachers during the establishment of triangle congruency conditions indicated that the argumentation the instructor utilized as a means assessment strategy helped pre-service mathematics teachers to justify their reasons.

Notably, the evidence was gathered from observed classroom activities and engaged arguments of students rather than just the instructor's self-reporting, thus enhancing the analytical support of the findings.

Furthermore, the researcher interview the classroom instructor and the interview response indicated that during teaching learning process particularly for the establishing the congruency criteria of triangles, he utilized interview, presentation, observation and argumentation. The instructor's interview answers will be employed for this evaluation to enlarge and support the observations within the classroom, but will not be a sole means of examining the processes of mathematization. The following is his response on the type of assessment for learning he employed to assess the mathematization process pre-service mathematics teachers.

To assess the mathematization process of pre-service teachers I used different assessment for learning strategies like interview (informal dialogue), presentation, observation, interview, and argumentation. Since they are dialogic in nature, argumentation, interview and presentation supported with probing were preferable in assessing the mathematization process of pre-service mathematics teachers. This due to the fact that this approach gave me a chance to assess pre-service teachers' thinking skills, reasoning skills, mathematical communication skills, their justification and argumentation skills and many other. Undeniably, I have tried to catch the thinking process of pre-service mathematics teachers thinking process through classroom observation supported by probing questions (TI1).

The classroom instructor favored assessment for learning strategies like interview/informal dialogue, argumentation and presentation in assessing pre-service mathematics teachers' thought processes These dialogic strategies are conceptually consistent with the theoretical underpinning of RME on interaction, reflection, and progressive mathematization because they provide scope for students to develop and refine their thinking and explanations. The reason why he preferred these the above mentioned strategies is that they helped him in exploring the thought process of pre-service mathematics such as argumentation skills, communication skills, reasoning skills and their justification skills that characterizes mathematization. As compared to classroom observation, the instructor believed mathematical discourses are much better at assessing the mathematization process of pre-service mathematics teachers. In his response, he stated that one could utilize observation as an input in identifying the gaps of the mathematization processes of pre-service mathematics teachers. However, to better understand pre-service mathematics teachers' level of thought processes, different assessment for learning strategies must be employed. He further explained his ideas regarding tests/ quizzes in assessing ones mathematization processes as follows:

In my teaching career, I have been commonly using tests, quizzes, and oral questions to assess the performance of my students. However, I was not certain that a single *test could assess all mathematical concepts. Even the nature of mathematical concepts matters. We cannot fully understand the thinking process of our students regarding some mathematical concepts using tests or quizzes. Therefore, it is better to use two or more assessments for learning strategies like mathematical discourse in addition to a test or quiz to assess the thinking process of our students. What I understood from this is that mathematical discourse like argumentation and interviews do not only serve as teaching and learning strategies, but also as assessments for learning to better understand learners' levels of thoughts (TI1).*

In his interview response, he asserted that to assess ones thought process it had better to utilize other different assessment for learning techniques rather than relying on the usual test or quiz. This is because a test or quiz alone does not guarantee to understand the thought process of the learners, particularly their mathematization process. He recognized the assessment for learning strategies that have dialogic nature since they provided ample opportunities to explore ones thought processes. He also recognized that mathematical discourse has dual purposes, serving as a teaching and learning strategy and assessing the mathematization process of pre-service mathematics teachers.

In his interview response, he also stated that the strategy he utilized helped him to provide feedback to pre-service mathematics teachers. This feedback process demonstrates how argumentation supported vertical mathematization, as pre-service teachers were encouraged to correct incorrect structures via formal mathematical relationships, such as the cosine law. Pre-service mathematics teachers mistakenly constructed a triangle without realizing the association between the sides of a triangle and the interior angles of a triangle in their journey of establishing the congruency criteria for triangles.

In his interview, the instructor mentioned that, the assessment strategy he applied gave him a chance to guide them to utilize the "Cosine Law" to see the relationship between the length of the sides and the degree measure of the interior angles. As he stated during interview, the assessment strategy helped pre-service mathematics teachers to realize their mistakes they made in constructing the triangle. Using cosine law, they tried to check the relationship between the lengths of the sides of the triangle and the degree measure of its interior angles and they came up with the mismatch between triangle's side lengths and the degree measure of interior angles.

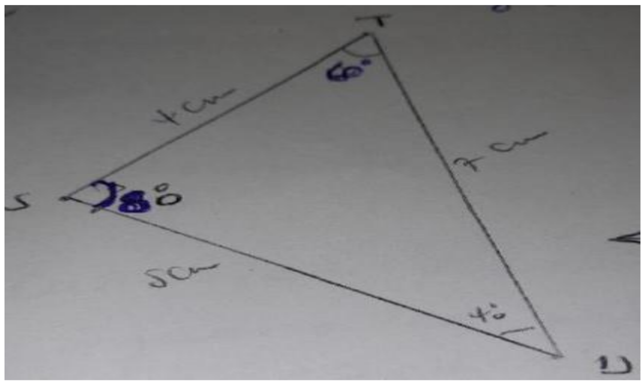

Figure 5.13 A triangle constructed by pre-service teachers having misalignment between length of sides of a triangle and its degree measure of interior angles

Application of the Cosine Law indicated:

$$(7m)^2 = (4cm)^2 + (5cm)^2 - 2 \times 5cm \times 4cm cos 80^{\circ}$$

$$\Rightarrow \quad 49cm^2 = 16cm^2 + 25cm^2 - 40 cos 80^{\circ}.$$

$$\Rightarrow \quad 49cm^2 = 41cm^2 - 40 \times .174cm^2.$$

$$\Rightarrow \quad 49cm^2 = 41cm^2 - 6.96cm^2$$

$$\Rightarrow \quad 49cm^2 = 27.04\ cm^2 \ \Rightarrow \Leftarrow (Contradiction).$$

5.7. Shortcomings limiting pre-service teachers' mathematization

Pre-service teachers frequently lack the capacity to perceive things from several angles since they have been predominantly exposed to experienced instructors and book-based mathematics instruction, according to the observational results. This sometimes causes learners to become dependent on formulas. One pre-service teacher described his view in the following way:

> *I used to study mathematics by memorizing different formulas without knowing where they came from and then applying them to various problems, much like our teacher did. As a result, I believed that we, the learners, were the recipients of mathematical knowledge, while the instructor was the source and transmitter. Therefore, what was the purpose of spending our time determining the formula? Instead, it is preferable to utilize the formula in our module.*

This pre-service teacher believes that his role is just to replicate what others have done. This, however, does not assist pre-service teachers in seeing things from other perspectives, nor does it allow them to produce new mathematical concepts or improve their mathematical knowledge. These also contribute to a misunderstanding of mathematization and its relationship to common workplace mathematics. Even though the sum of the degree measures of a triangle's interior angles is 180° and the principle governing the lengths of the sides is correct where the sum of the lengths of two sides is greater than the third side. For example, in achieving triangle congruency, some pre-service teachers were found to be unable to relate the degree measure of each interior angle of the triangle to the lengths of its sides.

The issue of spatial ability, or the difficulty of realizing or localizing mathematics, is another area in which some pre-service mathematics teachers fail in their mathematization attempts during RME-based mathematization instruction.

Challenges were observed from the works of the pre-service teachers at mathematizing during RME-based mathematization instruction. Some of the challenges are outlined below.

5.7.1. **Unresolved or Persistent Difficulties**

The researchers noted difficulties some pre-service teachers encountered while defining the necessary and sufficient conditions for the congruency of triangles. Particularly, they challenged in mathematically articulating the relationship between the degree measures of each interior angles of a triangle and the sides length of a given triangle. As an evidence, we can utilize the following illustration provided by pre-service teachers. We can apply cosine law to discern the relationship between the lengths of the sides of a triangle and the degree measures of the interior angles of a triangle, if properly measured. If we consider the first triangle pre-service teachers drew, we can noticed the difficulty they encountered since the relationship between triangle's sides length and the degree measure each interior angles fails to fulfil the cosine law. Applying the deductive cosine rule, we can determine that the square of a side's length is equal to the product of the squares of its lengths less twice the product of the cosine of the subtended angle and the lengths of its neighboring sides. In a triangle ABC, if AB and AC subtend$s\ \theta$, then $(BC)^2$ = $(AC)^2$ + $(AB)^2$– 2(AB)(AC) Cosθ.

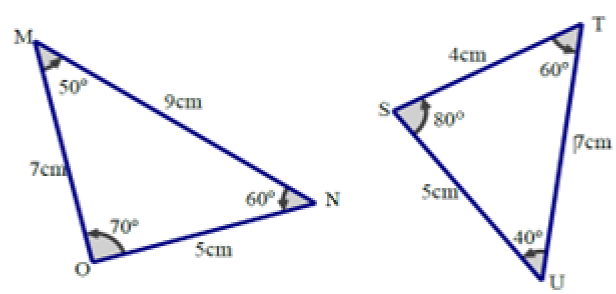

Figure 5.14 Mismatch of interior angles of the triangles and the length of their sides

From the drawings of the pre-service teachers, if we take the first one, we find out by the cosine law that:

$$(9cm)^2 = (7cm)^2 + (5cm)^2 - 2 \times 5cm \times 7cmcos70^{\circ}.$$

$$81cm^2 = 49cm^2 + 25cm^2 - 70cos70^{\circ}. \Rightarrow 81cm^2 = 74cm^2 - 70 \times .342cm^2.$$

$\Rightarrow 81cm^2 = 74cm^2 - 23.94cm^2 \Rightarrow\ 81cm^2 = 50.06cm^2$ and this is a contradiction.

In a similar manner, if we apply the cosine law to the remaining two angles, we will obtain similar contradictory result. This revealed that the triangles pre-service teachers modelled were not the right triangle since there is a mismatch between the lengths of sides of a triangle and the degree measure the interior angles of the triangle. This shows that pre-service teachers should continue to consider the potential relationship between side length and interior angle measurement.

5.7.2. **Inability to apply mathematical induction**

The other area that some pre-service teachers faced challenges was the utilization of mathematical induction in determining the relationship between sides of the polygon and the total numbers of diagonals of the polygon. These pre-service teachers formulated the relationship between sides of the polygon and the total number of the diagonals stemming from all vertices of an n-sided polygon as Td = (n(n-3))/2, where n signifies the number of sides of a specified polygon.

Furthermore, the interview results revealed that there are still additional challenges preventing pre-service mathematics teachers from fully mathematizing. Some of these challenges include time constraints, pre-service mathematics teachers' perspectives, instructor expertise and perspectives, and topic breadth.

While some pre-service teachers claim that time is not a significant problem in the mathematization process, one of the participants stated:

> *Mathematics, unquestionably, demands time. Even simple actions like eating require time. Mathematization refers to the practical application of mathematics. Without enough time, we are unable to construct mathematical conceptions autonomously; instead, we become passive listeners who must remember multiple formulae.*

Despite his claim that time should not be the major issue, the aforementioned reply recognized that mathematization did require it.

CHAPTER 6: DISCUSSION, CONCLUSION AND RECOMMENDATIONS

6.1. Discussion

In the preceding chapter, the researcher presented data analysis and interpretation of both quantitate data and qualitative data. This chapter includes discussions and triangulations in terms of mathematical ideas and mathematization rooted in cultural activities, as well as pre-service mathematics teachers' epistemological beliefs, problem-solving skills, and geometry achievement in response to study questions.

The purpose of this study is twofold: exploring the mathematization practiced in the community and finding out its effect on the pre-service mathematics teachers' mathematical epistemological beliefs, problem solving skills and achievement in geometry. Thus, the researcher employed exploratory sequential research design and utilized semi- structured interview, observation, epistemological beliefs questionnaire, realistic geometry problems and geometry achievement test as a means of data collections. In the preceding section, the researcher analyzed the first phase of the study by investigating the mathematization and mathematical concepts that underpin weaving, sewing craft, pottery, farming, and housebuilding. Under this section, the researcher presented, discussed, and triangulated the findings of the first phase on the mathematization and mathematical concepts embedded in the cultural activity of the society.

After presenting the first phase's discussion and triangulation, the researcher discussed, interpreted, and triangulated the second phase using the previous phase's results. Throughout both phases, the researcher discussed all of the qualitative data based on an inductive theme that emerged from the data or used pre-existing frameworks to discuss the data deductively.

6.1.1. Mathematization and Mathematical concepts embedded in cultural activities

In this study, the mathematization process and mathematical concepts embedded in cultural activities, such as hat and gigilcha making, weaving, straw house construction, dantel making, and sewing crafts, were explored. In phase I, the analysis was done based on the Bishop's six fundamental activities. In the above-stated cultural activities, the mathematization process, like counting, measuring, designing, locating, and explaining, was identified.

For instance, when creating hats, the hat maker used number patterns like 2, 4, 8, 16, 32, 64, 128, and 256. Gingilcha, on the other hand, was made using numerical patterns such as 2, 4, 6,..., 96, or pairs of strings as a single unit, which were then joined together until the necessary size—50 pairs of strings, or 100 strings—was reached. These designs demonstrate how hat and gingilcha producers utilized mathematizing counting. When building straw houses, local communities used a bundle of straw (equivalent to one donkey load) as a unit of measurement. Depending on the size of the house, they estimated that a small house would require roughly 10 donkeys, a medium house would require 15 donkeys, and a large house would require 20 to 25 donkeys. This indicates that estimating the amount of straw to cover the roof of a straw house characterizes counting.

While creating various types of pottery, pottery makers mix different soil types in varying ratios. For instance, when making mitad, they might mix an equal amount of sandy soil and clay soil, or combine 25 kg of sandy soil with 100 kg of clay soil (a 1:4 ratio). Others mixed seven kuna of red soil, four kuna of black soil, and three kuna of sandy soil. This illustrates one of the processes of mathematizing called counting.

The local community extensively utilizes the measurement procedure in the creation of various cultural products. They utilized a non-standard units like bundle of straw (one donkey of straw) as a unit of measurement to determine how much straw was needed to cover the roof. They utilized kilograms and kuna for measuring the amount of different types of soil; feet and ropes for measuring the circular house's radius and circumference; and a tape measure and rope for measuring the rectangular house's side lengths and diagonals. They also utilized non-standards units like their yards and figure to determine the size of a given hat and the diameter of ginglicha, kuna, and other materials made from migigira and utilized their either cubit or stick to measure the size of the gabi or buluko they made. The local community also frequently used estimations when creating various local materials. For instance, they estimated the size of the material they created and its capacity; in particular, they estimated how many cups of coffee a particular jebena would contain. These findings show that measuring activity is a widespread activity that is entrenched in the local activity of making cultural products.

Explaining is investigated differently in various cultural activities. Local community explains, for example, why they chose to make a circular mitad instead of a rectangular or triangular one, and why it gradually slopes down toward the center after rising at the edges. Furthermore, the local community logically justifies why they attempted to make the diagonals equal when building a four-sided house, and how they made all four inner walls equal when building a circular house. Additionally, they attempted to reason out why Jebena's neck is narrow and its bottom is spherical. Additionally, the local community explains why vertical strings must be strung in pairs when producing gingilcha. These imply that the local community uses the mathematizing process of explaining on a daily basis when creating different cultural products.

Local communities use various techniques and generate distinctive patterns for crafting gingilcha nets and spiral designs for hats. Similarly, while building straw buildings, they use radial symmetry to maintain equal partitioning and visual balance, as well as cylindrical walls with conical or frustum-shaped roofs. All of these instances demonstrate Bishop's design concepts, which are used in the production of cultural products.

The process of mathematization is firmly rooted in local communities' cultural activities, particularly the production of traditional goods. These activities demonstrate a variety of mathematical principles used in practical and meaningful contexts. For example, geometric progression can be seen in hat-making. The hat maker starts by tying two threads at their midpoint, then doubles the number of threads: 4 to 8, then 16, 32, 64, 128, and finally 256 threads.

Similarly, in the making of *gingilcha*—particularly its net—arithmetic sequences are embedded. Women start the net with four threads: two horizontal and two vertical. They then continue adding two vertical threads at a time, eventually reaching around 96 to 100 threads, depending on the desired size of the *gingilcha*, which is typically determined by customer preferences. During the weaving process, weavers create numerous congruent tiny triangles, forming the net structure. This process illustrates a mathematical understanding similar to the concept of integral calculus, as the total area of the *gingilcha* net is essentially the sum of the areas of these small triangles.

Basketry incorporates other mathematical ideas such as radial symmetry, solid geometry, and volume. In Basket formation, there are different Euclidean and non-Euclidean geometries, such as triangles and quadrilaterals that were formed while making different designs/decorations on them. These shapes, as well as the decorative patterns that have been added to them, demonstrate the usage of geometric design concepts.

This study's findings are compatible with those of Tesfamicael et al. (2021), Sari et al. (2023), and Shimwandi et al. (2024), who also observed the presence of mathematical concepts in cultural product development.

Straw house construction also combines a variety of mathematical principles. These include arc length, arc and triangle congruence, radial symmetry, volume, conical shapes, slope, and the use of non-standard units like feet and rope, triangle congruency and similarly in a roof construction, quadrilateral and its properties, and many others. These findings are consistent with those of Dominikus et al. (2022) and Susanti et al. (2023), who both highlighted the complex mathematical procedures involved in traditional house construction.

Gabi, netela, bulluko, and degemo are widely popular cultural products throughout Ethiopia, especially in Oromia, Selale, Arsi, and Bale. The Selale community is well-known for weaving, pottery, and basketry. Weavers develop various designs and ornamentation while producing these cultural artifacts. These designs incorporate geometric rigid transformations such as translation, rotation, and reflection that result in the congruency and similarity of plane figures such as triangles, rhombuses, and others, as well as non-rigid transformations such as dilation that result in the formation of similar plane figures such as triangles, squares, and rhombuses. This indicated that in there are several mathematical concepts particularly plane figures embedded in weaving activity of the community. The finding of this study is consistent with the finding of Aunga et al. (2025), Dominikus et al. (2017), Daniel et al. 2016 and Salma et al. (2022).

Furthermore, there are also different mathematical concepts like the concepts of different solid figures: half sphere and its volume, elliptical shapes and its volume like the *dist* and *jebena* formation, embedded in pottery making. The results of this study agree with the results of Budiarto et al. (2019), Hidayah and Pard (2022).

In general, the cultural activities of basketry, weaving, pottery, and house construction collectively provide rich and familiar contexts for realistic mathematics education (RME)–based mathematization, where mathematical ideas emerge naturally from everyday practices. Throughout these processes, the artisans carry out universal mathematical activities as Bishop describes. Counting, measuring, planning, explaining, and reasoning are all part of these activities. Whether weaving basketry or pottery or building a house, artists intuitively employ mathematical reasoning. These processes reflect the idea that mathematics is a part of the culture and that it emerges as a result of solving real and meaningful problems. This concept is in line with the principles of RME.

Many aspects of geometric thought, especially from plane geometry, are incorporated in these cultural activities. Geometric patterns and designs in weaving and basket-making, for example, cover aspects of symmetry, congruence, similarity, and various geometric transformations such as reflection, rotation, and translation. The circular designs in pottery and houses cover aspects of radii, diameters, arcs, and their areas. The various surfaces and spaces cover aspects of angles, proportions, and part-whole relationships. Although aspects of volume and surface area of solids are incorporated in these activities, plane geometry is dominant in these activities, and it is an integral part of teacher education. Abstraction and generalization are naturally incorporated in these activities as artisans justify their designs and balance functionality and beauty.

With these integrated cultural activities as contexts for mathematization, the pre-service teachers are given opportunities to relate formal mathematical ideas to real-life experiences. In keeping with the RME approach, learning begins with realistic problem situations taken from cultural activities, so that learners can reinvent mathematical ideas through guided mathematization. Pre-service teachers will gain a deeper understanding of geometric concepts, develop spatial reasoning skills, and enhance their problem-solving abilities by learning in socially and culturally meaningful contexts. Through their observations and analysis of these activities, pre-service teachers will come to understand that geometry is a living body of knowledge that is immersed in human action and is not simply a set of abstract principles disconnected from human experience.

Engaging in mathematization through these cultural practices is also significant in facilitating a transformation in pre-service teachers' epistemological beliefs about mathematics. This is because pre-service teachers come to appreciate indigenous knowledge systems as well as the significance of mathematics in daily life. This positively influences their epistemological beliefs about mathematics as a sense-making process rather than a process of rote memorization. This builds pre-service teachers' confidence in solving geometrical problems. Thus, incorporating cultural activities as a collective context for mathematization in RME not only builds pre-service teachers' performance in geometry but also equips them for providing meaningful mathematics education. This study's findings are compatible with Tesfamicael et al. (2021), Ali et al. (2018), and Weldeana (2016). The results of the study indicated that mathematics is a cultural product, which is in line with Weldeana's (2016) notion that a learner's knowledge of mathematics is created or given meaning through their interactions with their surroundings, where their prior knowledge and experiences serve as pillars for future learning.

The initial goal of this study was to investigate the mathematization process and the embedded mathematical concepts inside selected local community cultural activities. While the proposal phase used the term "mathematics," the first phase of the study—a qualitative analysis of cultural practices—showed that the majority of these mathematical conceptions were based on geometry geometry both planar and the solid one. Symmetry, angles, proportional reasoning, circular and spiral patterns, and two-dimensional shapes and their similarities and congruencies are among the topics covered, which are especially obvious in gingilcha weaving, weaving patterning, and house construction.

These findings provided a strong empirical basis for narrowing the intervention phase's focus to plane geometry within the context of Realistic Mathematics Education (RME). Thus, the intervention integrated culturally grounded problems involving geometric reasoning, which not only reflected the participants' lived experiences and local knowledge, but also aligned with the geometric concepts discovered in the first phase. As evidenced by Ali et al. (2018) cultural activities based realistic mathematics education can be a transforming effort to bridge mathematics with the realities and perceptions of pupils in learning mathematics.

To implement the intervention, preservice mathematics teachers from two teacher education colleges were assigned to two groups: a comparison group that taught conventionally and an intervention group that taught using an RME-based mathematization approach supported with a REACT strategy. Before implementing the intervention, the researcher administered a pre-geometry achievement test, pre-problem-solving questions, and a pre-epistemological beliefs questionnaire for the intervention group and comparison group. The researcher computed both descriptive statistics as well as inferential statistics to determine the level of both groups in terms of problem-solving skills, geometry achievement, and epistemological beliefs. In relation to problem-solving and epistemological beliefs, the independent t-test indicated that there was no significant mean difference between the intervention group and comparison group (See Table 5.10). This indicated that groups treated with the RME-based mathematical approach supported with REACT stratergy and groups treated with the conventional approach were approximately on the same level in terms of problem-solving skills and epistemological beliefs. Thus, the observed final difference is ascribed to the intervention's effectiveness rather than the pre-existing differences among pre-service mathematics teachers. However, for geometry achievement, the pre-test result indicated that there was significant mean difference between intervention group and comparison group (See Table 5.10). This suggested that pre-geometry achievement scores of pre-service mathematics teachers were utilized as a covariate in the analysis of post-geometry achievement, as it showed that the intervention and comparison groups did not perform at the same level in the pre-geometry achievement test.

We conducted a one-semester intervention with a four-credit hour per week. Prior to the intervention, the researcher trained mathematics instructors on how to execute it. Furthermore, the researcher maintained contact with the intervention implementer throughout the entire intervention period. During the intervention period, the researcher conducted interviews and observations with pre-service mathematics teachers and the classroom instructor to gather qualitative data on the mathematization process, epistemological beliefs, and assessment for learning. Following the intervention, both the intervention and comparison groups received a post-test consisting of a geometry achievement test, problem-solving questions, and epistemological beliefs. Following data collection, the researcher analyzed the data in light of the research questions. Following the intervention, the researcher conducted interviews with some pre-service mathematics teachers. For quantitative data, the researcher utilized mean gain and inferential statistics such as multiple regression, analysis of covariance, independent t-test, and multivariate analysis after thoroughly reviewing their statistical assumptions. For the analysis of qualitative data, the researcher utilized thematic analysis.

6.1.2. Treatments' effect on pre-service mathematics teachers' problem-solving skills

On the pre-test of problem-solving skills, the p-value was 0.053, which is almost non-significant; therefore, an analysis of covariance was performed. After controlling for the covariate (pre-test score), there was a significant mean difference between the group taught using the RME-based mathematization approach and the group taught via the conventional approach. Once the covariate (pre-test score) was controlled, the mean and standard error of problem-solving skills for the intervention group were M = 21.5 and SE = 1.66, respectively, while the comparison group had a mean score of M = 9.39 and SE = 1.59, indicating that the comparison group had less problem-solving ability than the intervention group (Table 5.11). Furthermore, the ANCOVA results showed a significant difference in means between the intervention and comparison groups. The eta value also indicated that 14.2% of the variation in problem-solving skills is attributable to the intervention. Pre-service mathematics teachers were able to solve various geometrical problems with close guidance from their instructor and through different methods, such as determining the relationship between a polygon's sides and its total number of diagonals. This aligns with the guided reinvention concept of realistic mathematics education principles, where pre-service teachers practice in a manner similar to mathematics invention through progressive mathematization. It enabled them to deduce that raising the area of a regular polygon while maintaining a fixed perimeter, with an increasing number of sides, results in a circle with the same perimeter having the largest area. Learners are more likely to engage in discussions during the problem-solving phase of RME-based mathematization when activities are personally relevant. This helps pre-service teachers connect learners' everyday experiences with geometric concepts, making abstract geometrical ideas more concrete and memorable (Uskun et al., 2021). Additionally, it is plausible that learners develop geometric concepts by working through real-world problems, which provides essential knowledge. Since RME-based mathematization emphasizes an interconnected approach, it helps pre-service teachers relate geometric concepts to other mathematical ideas, and these principles collectively enhance problem-solving skills.

The results of this study are in line with Nursiddik et al. (2017), who claim that the RME approach is a teaching-learning strategy that begins with real-world situations for learners and emphasizes the skills of doing mathematics, discussing and working cooperatively, and arguing with peers to help them come up with their own unique solutions. This study's findings support the findings of Anggraini and Fauzan (2020), who stated that the RME-based mathematization technique contributes to enhancing learners' mathematical problem-solving skills. Furthermore, the findings of this study are consistent with those of Taufina et al. (2019), who revealed that learners taught utilizing an RME-based mathematization strategy fared much better in problem solving than learners taught conventional approaches.

6.1.3. Treatments' effect on pre-service mathematics teachers' geometry achievement

On the pre-test geometry achievement score, there was a significant mean difference between the group taught using the RME-based mathematization approach and the group taught via the conventional approach. Thus, instead of independent t-test, the researcher computed an analysis of covariance to examine the significant effect of the treatment. After controlling the covariate (pre-test score), the post-test geometry achievement mean and standard error of the intervention group were $M = 14.75$ and $SE =.64$, whereas the mean score and standard error of the comparison group were $M = 11.6$ and $SE =.6$ indicating that virtually the comparison group have less geometry achievement than intervention group (Table 5.13). The ANCOVA result indicated that after controlling the effect of the pre-geometry achievement, there was significant mean difference between intervention group and the comparison group in geometry achievement, F (1, 56), $p <.001$, $\eta 2 =.23$. This showed that RME-based mathematization brought 23% variation the geometry achievement (Table 4.14).

This result obtained because of RME-based mathematics indicated that RME-based mathematics has a significant effect on the geometric performance of pre-service mathematics teachers. The RME-based mathematization approach plays a fundamental role in enhancing pre-service teachers' level of geometric understanding and their mastery of the overall geometrical concepts, as realistic mathematics education provides an opportunity for pre-service teachers to reinvent or rebuild geometrical concepts. Furthermore, in this approach, instead of starting with abstractions or definitions that pre-service teachers will utilize later, studying geometry starts with problem scenarios that are meaningful to learners, which helps them connect meaning to the geometrical constructs they generate while solving problems. The findings of this study is consistent with the finding of Laurens et al. (2017) as in their finding revealed that RME-based mathematization approaches boosts motivation, self-confidence, problem-solving skills, and reasoning, all of which contribute to improved cognitive achievement. In addition, the finding of this study supports the findings of Afthina and Pramudya (2017), Tarım, and Kütküt (2021). Furthermore, this study coincided with the findings of Aksu and Colak (2021), who concluded that the RME-based mathematization strategy greatly improves learners' academic performance in geometry. Other research findings also indicated that the effect of applying RME-based mathematization could improve learners' geometry performance (Jupri, 2017; Loc & Tien, 2020; Sholikhah & Rasmita, 2020). Since RME-based mathematization provides an opportunity for pre-service teachers to interact in the teaching and learning process, it guarantees a productive learning environment and creates the foundation for improved academic achievement in learners (Kaylak, 2014).

6.1.4.Treatments' effect on pre-service mathematics teachers' epistemological beliefs

In order to examine the effect of the intervention on pre-service mathematics teachers, the researcher utilized an independent t-test after carefully examining all its assumptions. Table 5.15 indicated that participants in the comparison group scored mean values of 2.41, 2.35,and 2.39 on source, stability, and structure of mathematical knowledge, respectively, indicating that they resided at the absolutist end of the continuum. Furthermore, in relation to general mathematical epistemological beliefs, the mean value of the comparison group was 2.38 show that positioned at the absolutist end of the continuum. Furthermore, mean gain also computed and the result indicated that large improvements in all aspects of mathematical epistemological beliefs of the intervention group over the comparison group, establishing the effectiveness of the RME-based mathematization strategy.

The result indicated that group treated with mathematization-based instruction scored mean values of 2.99, 2.65, 2.56 and 2.74 on source, stability, structure of mathematical knowledge and the general mathematical epistemological beliefs respectively. This showed that these mean values posited participants treated with mathematization-based instruction at the lowest end of the fallibilist spectrum.

The results in Table 5.15 reveal that individuals treated with the mathematization-based method showed a significant shift in their mathematical epistemological views when compared to those treated with the conventional approach.

Furthermore, to determine the difference between the means of two groups, the researcher computed Cohen's d. Results showed a significant effect size of 2.32 for SoMK and a substantial effect size of 2.45 for MEB, indicating that there is a 2.32 and 2.45 standard deviations between the mean of the comparison group and the mean of the intervention group. Similarly, for StaMK, a d value of 1.15 showed that there was a 1.15 standard deviation between the mean of the intervention group and the mean of the comparison group. Furthermore, for the StrMk, the d value of 1.31 indicated that there was a 1.31 standard deviation between the mean of the intervention group and that of the comparison group.

In addition, a one-way MNOVA was also computed to see the effect of treatment on the linear combination of the dependent variables (source of mathematical knowledge, stability of mathematical knowledge, and structure of mathematical knowledge). From Table 5.16, the Wilk's Lambda value =.361 with $F (2, 54) = 31.266$, $p <.001$, and =.639 indicating that there was a statistically significant difference in source, stability, and structure of mathematical knowledge across the two groups (treatment group and comparison group). This result showed that mathematization-based strategies play a significant role in changing the mathematical epistemological beliefs of pre-service mathematics teachers. In this regard, the finding of this study supports the study conducted by Yuanita, et al. (2018) and their findings showed that the use of mathematization-based approach has a positive effect in changing students' mathematical epistemological beliefs as compared to the traditional approach.

The results obtained from classroom observation and interviews support those of the post-epistemological beliefs questionnaire. According to their interview results, before intervention, pre-service teachers believed in one-size-fits-all pedagogy and transmission mathematical knowledge from teachers to students. The result obtained from classroom observation also indicated that during intervention, pre-service mathematics teachers independently constructed their mathematical knowledge through guided reinvention and progressive mathematization. In doing so, pre-service mathematics teachers established congruency criteria for triangles, solved different realistic geometry problems regarding triangles, circles, and quadrilaterals, designed their own way of solving or approaching realistic problems, and formulated an algorithm regarding the relationship between sides of a polygon and its total number of diagonals.

Furthermore, their interview description stated that the mathematization strategy assisted them in modeling realistic problems informally as well as formally, allowing them to solve problems in a variety of ways. This indicated that the mathematization-based instructional approach helped them get many opportunities to reinvent mathematical ideas and concepts by themselves with the close assistance of their instructor. It even let them take responsibility for their own learning.

Their interview result indicated that the mathematization strategy helped them to consider mathematical knowledge as the product of social construction; and it enabled pre-service mathematics teachers to conceptualize mathematics as the daily activity of society; letting them sense the daily cultural activity of society, their interactions, their prior experience, and even mistakes as sources of mathematical knowledge. We believe that such achievement is because of interactivity and the use of student work-alone principles in the mathematization -based strategy. In addition to what was mentioned earlier, some pre-service mathematics teachers considered teachers and books as the source of mathematical knowledge.

With regard to discovery and the invention of mathematical knowledge, the interview result indicated that pre-service mathematics teachers believed both the discovery and invention nature of mathematical knowledge. Including history of mathematics like history of Pythagoras theorem, trigonometry, calculus, coordinate geometry, Pi (π) and some others in mathematization-based instruction not only help pre-service mathematics teachers to inspire (Ashari , 2021; Uyen et al., 2021; Sumirattana, 2017) but also let them held the co-existence nature of absolutist and fallibilist perspectives. Thus, this finding is consistent with the finding of (Butuner & Baki, 2020; Kapofu & Kapofu, 2020; Perk, 2018).

Even though the mathematization strategy enabled pre-service mathematics teachers to construct different mathematical concepts, there were areas they failed to mathematize. For example, pre-service mathematics teachers failed to prove the relationship between sides of a polygon and its total number of diagonals using mathematical induction, and the instructor tried to show them how to prove the given problems using mathematical induction principles by taking into consideration the guiding principles of mathematization strategy. This indicated that learners acquired mathematical knowledge from their instructor, indicating that the teacher is the source of mathematical knowledge. The majority of pre-service math teachers were thus led to accept the coexistence of absolutist and fallibilist viewpoints because of mathematization instruction's substantial influence in altering their epistemological views about the source of mathematical knowledge. In this regard, the finding of this study is consistent with the finding of (Şahin et al., 2019).

Since reinvention through progressive mathematization is the core principle of mathematization-based strategy and such strategy facilitates informal solution strategies and provide pre-service mathematics teachers an opportunity to invent formal mathematical practices. Additionally, in the framework of mathematization-based strategy, mathematics is human activity, and the activity principle gave pre-service mathematics teachers the chance to create mathematical objects. This suggests that students build mathematical knowledge through activity, as it is consistent with Perk's 2018 study.

As described in interview of some pre-service mathematics teachers, mathematization-based strategy enabled them in conceptualizing the co-existence nature of stability of mathematical knowledge (i.e., the existence of both permanent; and work-in-progress and incomplete nature of mathematical knowledge). Even though they were unable to support their argument with evidence, some pre-service mathematics teachers stated that mathematical knowledge is subjected to revision. This idea is consistent with the study of (De Toffoli, 2021). In his study, De Toffoli justify his argument by taking some examples on the modification of mathematical proof as follows: In 1993, Wiles tried to prove the Diophantus' theorem for no three non-zero integers a, b and c that satisfy equation $a^n + b^n = c^n$ for any n >2, yet his proof was fallacious and there was a significant gap in proof. He decided to fix his proof and after intensive work in collaboration with his former student Richard Taylor, they arrived at the correct proof in 1995. Similarly, this author also stated that the proof provided by D'Alembert in 1746 for Fundamental Theorem of Algebra had many gaps but the proof done by Gauss in 1997 after many years had no such gaps indicating that mathematical proofs are subjected to revisions.

In their interview, pre-service mathematics teachers stated that the truth of mathematical knowledge is not absolute rather time and space dependent. They asserted their argument by considering 9+7=16 and 9+7= 4 in addition modulo 12 indicating that 9+7 is 16 in the real arithmetic and it becomes four (4) in modular arithmetic. The modular arithmetic example demonstrates how it is quite conceivable for various mathematics to exist across cultures and time, suggesting that the veracity of mathematical knowledge is condition-dependent, as claimed by Ernest (1996) and Peck (2018). This finding corroborates their findings.

The result obtained from interview observation indicated that mathematization-based instruction helped pre-service mathematics teachers to construct their mathematical concepts by interconnecting different mathematical concepts. This is because 'intertwining of various learning strands or units' is the main feature of mathematization-based strategy. While constructing mathematical concepts they used different mathematical concepts like coordinate geometry, algebra, calculus, combinatory, algebra indicating that the structure of mathematical knowledge is interconnected (intra-mathematical connections). Mathematization-based strategy also enabled them to conceptualize mathematics as the daily activity of the society which indicated the connection between mathematics and daily activity of the society (extra-mathematical connections). This finding is consistent with the finding of (Hasbi et al., 2019; Febriyanti et al., 2019; Menanti et al., 2018).

The finding obtained from both quantitative and qualitative data indicated that mathematization-based instruction play a significant role changing the epistemological beliefs of pre-service mathematics teachers and this finding is consistent with the finding of (Yuanita, & Zakaria, 2016; Yuanita et al., 2018).

6.1.5. Pre-service mathematics teachers' geometry achievement as a function of epistemological beliefs and problem solving skills

The researcher utilized multiple linear regression to examine the relationship between the dependent variable (geometry achievement) and the predictor variables, which are mathematical epistemological beliefs together with problem solving skills. In group taught through RME-based mathematization approach, only problem-solving skills significantly predict pre-service mathematics teachers' geometry performance. However, the mathematical epistemological beliefs did not significantly predict geometry achievement of pre-service mathematics teachers. In comparison group, both problem-solving skills and mathematical epistemological beliefs were non-significant predictor of geometry achievement of pre-service mathematics teachers.

For Intervention group, the study's findings indicated that pre-service teachers' components of mathematical epistemological beliefs together with problem-solving skills accounted for 35.4% of the difference in their geometry achievement. However, with in intervention group, only problem solving siklls significantly predict geometry achievement, but none of the components of the epistemological beliefs is significantly predict geometry achievement.

In comparison group, components of mathematical epistemological beliefs together with problem solving skills only accounted 5.4% for the variation in pre-service teachers' geometry performance.

The study's findings showed a favorable correlation between pre-service mathematics teacher achievement in geometry and their problem-solving skills. The findings of this study are consistent with those of many other studies (Hsiao et al., 2018; Md. Hassan & Rahman, 2017; Kamalimoghaddam et al., 2016). This result makes sense when one considers how the problem-solving skills of pre-service mathematics teachers enhance their geometry achievement. This is because solving mathematical problems enhances higher-order thinking skills such as analysis, reasoning, interpretation, evaluation, reflection, and prediction (Rahman & Md. Hassan, 2017). This suggests that learners who develop these higher-order thinking skills may view mathematical concepts, particularly geometrical ideas, holistically and grow into capable thinkers, which enhances their geometry achievement in particular and mathematics achievement in general (Tanujaya et al., 2017). In relation to mathematical epistemological beliefs, the findings of this study is in agreement with Shoaib et al. (2021) and Sürmeli & Ünver (2017) but against the findings of Belecina and Ocampo (2016), Schommer Aikins, and Duell (2013).

The study found that mathematical knowledge was not a significant predictor of pre-service mathematics teachers' geometry achievement. Several studies report that mathematical epistemological beliefs either show weak or non-significant predictive relationships with mathematics achievement outcomes. For example, research with secondary and tertiary students found no significant correlation between epistemological beliefs and mathematics performance, while other variables like academic self-concept were stronger predictors (Sürmeli & Ünver, 2017). Additionally, Shoaib (2021) reported that beliefs about mathematical knowledge had only a minor association with academic achievement, further indicating that epistemological beliefs alone may not strongly predict performance. These results align with the present study's findings and support the interpretation that the short intervention period and small sample size may limit the detection of significant effects of epistemological beliefs on geometry achievement.

Previous studies however revealed that there is a strong relationship between mathematical beliefs learners' academic performance (Belecina & Ocampo, 2016; Suthar & Tarmizi, 2010) which against the findings of this study. Furthermore, other studies also acknowledged the connection between epistemological beliefs and learners' mathematics achievement (Schommer-Aikins et al., 2005) and that epistemological beliefs have a fundamental role in student mathematics achievement (Briley et al., 2009; Schommer-Aikins & Duell, 2013; Shoaib et al., 2021), yet the findings of this study are contrary.

6.1.6. Mathematization Process of Pre-Service Mathematics Teachers

In relation to triangle inequality, in their investigation pre-service teachers identified two unique scenarios: a) Open curve formation: This occurs when the sum of the lengths of any two sides is less than or equal to the length of the third side. b) Triangle formation criterion: A triangle is created when the sum of any two sides' lengths exceeds the length of the third.

Fascinatingly, some pre-service teachers claimed that the length of a triangle may be achieved with three stated side lengths if and only if the length difference between two sides is smaller than the length of the third. In contrast, several pre-service teachers proposed a different criterion for triangle construction. When the difference is greater than or equal to the third side's length, an open curve forms.

During the problem-solving process, pre-service teachers demonstrated horizontal mathematization by identifying and using mathematical concepts including sum, difference, and open and closed curves. Their investigation went beyond just identifying concepts; they additionally developed a formula or algorithm that encompassed the constraints for triangle side lengths, formalizing the notion.

Their success in establishing the required and sufficient conditions for three lengths to form a triangle correlates with the findings of studies by Genc and Ergan (2022) and Yenil et al. (2023). The discovery of the triangle inequality theorem and the associated conditions to generate a triangle from three side lengths align with Anwar and Rofikis' (2018) discoveries.

Throughout the exploratory process, the pre-service teachers delved deeply into the mathematical concepts. They went on to explain that possessing three criteria does not ensure congruence. For example, AAA may not always indicate congruence. They also recognized that these congruence criteria apply to all types of triangles, including equilateral, isosceles, scalene, and right angle triangles. This understanding demonstrates horizontal mathematization. In contrast, the creation of congruency criteria suggests vertical mathematization.

Patkin and Plaksin (2011) note that presenting a minimum set of identical triangle components to determine congruency allows learners to discern between their preconceived concepts and the real concept definition, thus improving their comprehension. Shabari and Daher (2020) also emphasize the benefits of incorporating ethnomathematics into geometry teaching. Such approaches allow students to define triangle congruence, find congruency criteria, infer pertinent theorems, and engage in theorem-based proofs.

During their exploratory techniques, pre-service teachers attempted to uncover regularities, discern linkages, and locate patterns relevant to problems. For example, when tasked with creating diagonals from multiple decagon vertices such as v1 and v2, they discovered an emergent pattern of 7, 7, 6, 5, 4, 3, 2, 1, 0, 0. Similarly, for an eight-sided polygon, the sequence was 5, 5, 4, 3, 2, 1, 0, 0, and for a seven-sided polygon, it was 4, 4, 3, 2, 1, 0. They applied this to an n-sided polygon, obtaining (n - 3) diagonals for V1 and V2, (n - 4) for V3, and so on, with no diagonals for Vn-1 and Vn. V1–Vn denote the vertices of an n-sided polygon, where n is an element of W.

The pre-service teachers also displayed an ability for vertical mathematization. This was evident in their usual routine activities. One group used the counting principle to determine that a hexagon's total number of diagonals is 1/2 (6 × 3) = 9. Another group used the concept of combinations to determine the total diagonals in an octagon: Td = 8C2 - 8 = 20. Another set dealt with the decagon using the formula '7 plus the sum of the first seven natural numbers.' This resulted in a thorough equation that, when simplified, revealed a relationship between the total number of diagonals and the polygon’s sides.

These aspiring teachers began their mathematization by drawing various polygons and methodically following down every possible diagonal from each vertex. They also employed a variety of models and tables to provide visual aids. As Freudenthal (2002) pointed out, modeling acts as a gateway to more generic equations or models, emphasizing its importance in the mathematization process. The preservice teachers' drawings from the activity are exhibited below.

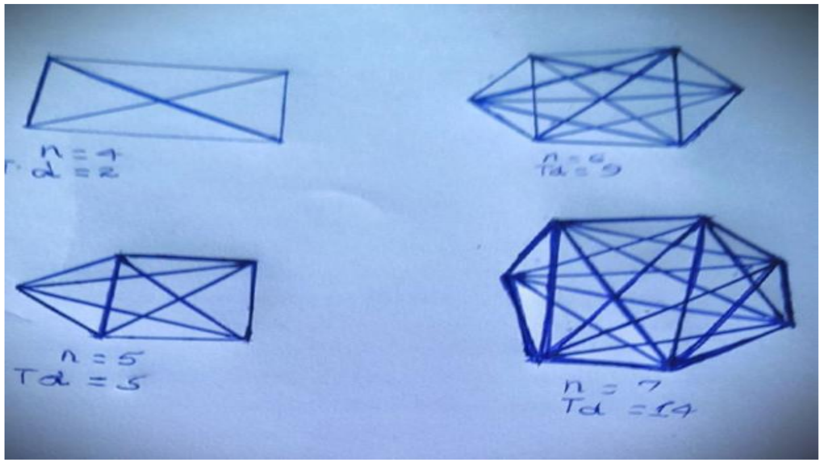

Figure 6.1 Total number of diagonals formed from four, five, six, and seven-sided polygons

Pre-service mathematics teachers engaged in problem-solving and re-invention. They attempted to develop or construct a formula for determining the total number of diagonals in an n-sided polygon. Remarkably, they selected diverse ways rather than simply applying the formula deductively. One group, for example, utilized a strategy known as "counting the diagonals once." A different group used the "combination principle," while the last group used the formula "n + the sum of the first n natural numbers," where n is the maximum number of diagonals that may be drawn from a starting vertex. As a result, they came up with the equation Td = (n (n-3))/2, where n is the number of a polygon’s n sides. The various tactics employed by pre-service mathematics teachers to associate the total number of diagonals with the number of polygon sides demonstrate varied cognitive processes. This conclusion is consistent with the findings of Jupri and Drijvers (2016), who discovered differences in students' methods for generating equations or diagrams. These differences in approach are critical for understanding student comprehension.

In their exploration of a carpenter's task to build a four-sided house, pre-service teachers used informal methods to analyze a quadrilateral. They measured interior angles with protractors and used paper folding to check angle congruency, with some using A4 paper and right-angle triangles for verification. These techniques, emphasizing horizontal mathematization, turned real-world scenarios into initial mathematical models. Following their hands-on approach, they formalized their findings through vertical mathematization, applying mathematical reasoning and the SAS congruency criterion to show that the quadrilateral is a parallelogram. They determined that the diagonals form isosceles triangles, leading to the conclusion that each angle measures 90°, which proves the figure is a rectangle and is equiangular. The congruency of the diagonals emerged as a vital criterion for this determination. This indicated that RME-based mathematization approach that relies on the cultural activity of the society helped them to develop problem-solving skills through horizontal and vertical mathematization. The findings of this study is consistent with the finding of Zhang and Zhang (2010), who argued that integrating cultural activity of the society provide learners a rich background knowledge. Furthermore, the finding of this study corporates the findings of Afriansyah and Arwadi (2021) and Inayatusufi and Aziz (2021) who found that learners develop geometric understanding because of RME-based instruction.

Pre-service mathematics teachers used several mathematical ideas to ensure the congruency of the four entries (the gate, the shelving area, the sleeping area, and the cooking area) of a cylindrical wall, as well as the four portions in which the inner wall falls. These include the congruency of a triangle, vertically opposing angles, subtended arcs, the isosceles triangle, the perpendicular bisector, the idea of central angle and its relationship with the subtended arc, the relationship between arcs subtended by two parallel chords, and many more. They sought to demonstrate the congruency of the straw house's four entries and the four portions in which the inner wall falls using an informal mathematizing approach, such as measuring using ropes, a protractor, and a sector as a unit of measurement.

Building on their informal mathematizing, pre-service mathematics teachers explicitly demonstrated the congruence of the four arcs produced by the gate, cooking area, shelf, and sleeping space. They also calculated the congruency of the four portions of the interior home where the wall falls. They used reasoning, problem-solving, and argumentation abilities, which are typical of vertical mathematization, to prove the arcs produced by the four gates and the wall. These are the outcomes of realistic mathematics education, which begins with the reality and activity principles and allows students to mathematize practical situations horizontally and vertically while working closely with their teachers.

The findings of this study are consistent with those of Juniarti et al. (2022), Iskandar et al. (2022), Kabuye Batiibwe (2024), and Suh et al. (2017), which found that learning in environments related to learners' lives improves problem-solving abilities and grasp of circular ideas. Furthermore, the findings of this study reflect the work of Shahbari and Daher (2020) and Massarwe et al. (2012), who underline the need of incorporating learners' real-life experiences into geometry teaching. The findings of this study support the findings of Nursyahidah et al. (2020), which show that learners' context leads to students' more concrete grasp of ideas and that design helps to instructional theory on circular concepts.

When pre-service mathematics teachers are given tasks that have personal significance for them, they are more likely to engage in discussions during the problem-solving stage of RME-based mathematization instruction. This resulted to the concretization of abstract geometrical concepts by allowing pre-service math teachers to establish a connection between students' real-world experiences and geometrical concepts (Uskun et al., 2021).

6.1.7. Assessing the Mathematization Processes of Pre-service Mathematics Teachers

Through argumentation and mathematical discourse, pre-service mathematics teachers constructed knowledge about how single congruent criteria do not ascertain the congruency of two or more triangles. From the perspective of Realistic Mathematics Education, such knowledge building would represent aspects of what is termed horizontal and vertical mathematization. An indication of horizontal mathematization would be, for instance, pre-service teachers' efforts to understand the situation embedded in problems and make comparisons between diagrams while experimenting on specific conditions for congruency between triangles (AA, AAA, SSA, etc.). On the other hand, pre-service teachers transcending their specific experiences with those conditions to justify formally why they are not sufficient for determining congruence between triangles would manifest vertical mathematization. They also justified how a random but two congruent criteria may not ensure triangle congruency like the AA congruency criteria. Furthermore, they also identified how three congruency criteria may not guarantee the congruency of triangles like AAA, SSA.

Thus, while developing necessary and sufficient conditions of triangle congruency, pre-service mathematics teachers developed mathematical reasoning, generalization, and communication skills by means of sustained argumentation and dialogue. These dialogical acts gave rise to opportunities for progressive mathematisation, in which informal mathematics reasoning and visual arguments were refined to yield formal mathematics conclusions. In this movement, there is a movement from horizontal mathematisation, typified by diagrammatic reasoning and explorations based on a mathematics task, to vertical mathematisation, characterized by symbolic reasoning supported logically and generalized congruence principles.

The embedded use of assessment for learning strategies within argumentation and informal dialogue allowed the instructor to examine not only procedural knowledge but also how the pre-service teachers reasoned, justified, and generalized mathematical ideas. Thus, the argumentation served simultaneously as both a learning process and an assessment mechanism for mapping the depth and quality of mathematization.

The study further revealed that argumentation and mathematical discourse enabled pre-service mathematics teachers to gain a more profound understanding of geometric concepts, particularly the criteria for triangle congruency. Through discussion, they were able to construct and justify why certain criteria, such as AA or AAA, do not guarantee congruency, while others, such as SSA, require further examination. Engaging in these dialogues improved their reasoning, generalization, and communication skills, all of which are central to mathematization.

These findings suggest that mathematization is intensified where the learners are required to make their reasoning explicit and account for their conclusions. Argumentation challenged pre-service teachers to interrogate the general validity of congruency criteria beyond merely empirical success in isolated cases, and it thus fostered vertical mathematization through abstraction and formal reasoning.

These dialogical interactions thus gave observable data on the mathematization processes of the pre-service teachers and reduced dependence on instructor self-report by grounding findings in students' articulated reasoning and justifications. These findings support the use of argumentation and dialogue as being not only significant teaching-learning strategies but also powerful tools for assessment. Encouraging discussions that involve justification and reasoning raises the tendency of an educator to assess mathematical thinking in generalization by learners. This corroborates the views of Uygun and Akyüz (2019), who showed that collective argumentation, enhances the reasoning and generalization skills of pre-service teachers.

Similarly, Jonassen and Kim (2010) emphasized that argumentation functions as both a pedagogical tool and an alternative assessment strategy, particularly for evaluating problem-solving and reasoning processes. The use of interviews and informal dialogue to assess mathematization, rather than relying solely on traditional assessments, is consistent with prior studies (Ardiyani et al., 2018; Araújo & de Lima, 2020; Deniz & Kabael, 2017; Upu et al., 2017; Widada et al., 2019), which highlight the effectiveness of dialogic assessment in revealing learners’ mathematization processes.

Considering these insights, reexamining the role of argumentation, dialogue, and feedback in assessment practices can significantly enhance the mathematization processes of pre-service mathematics teachers. Integrating formative, dialogic assessment strategies—even in theoretical domains such as geometry and algebra—creates opportunities for deeper engagement, meaningful feedback, and the development of both horizontal and vertical mathematization. Such practices support learners in moving from contextual reasoning toward formal mathematical understanding

6.1.8. Challenges Hindering Pre-service Teachers' Successful Mathematization

The results of the classroom observation demonstrated that some pre-service teachers fail to relate school geometry to what they notice in their surroundings. One pre-service teacher stated that the inability to teach mathematics in the context of what learners experienced in their daily activities was the primary reason for the failure to localize mathematics in general and geometry in particular. This in turn implied that pre-service teachers were prevented from mathematizing due to their limited exposure to contextual learning. As a result, they were forced to rote memorization, and consequently their learning was not meaningful, but rather preparing themselves for tests, and this in turn caused them to fail in constructing geometrical concepts. According to Pavloviová et al. (2022), the problem of spatial ability influences the development of pre-service teachers' geometric thinking, which in turn influences the construction of geometrical concepts. Other studies revealed that learns having problem of spatial ability unable to solve geometry problems and fail to conceptualize or understand geometry in meaningfully (Delice et al., 2009; Makamure & Jojo, 2021).

There was noticeable challenges pre-service teacher encountered while establishing the necessary and sufficient conditions for triangle congruency. They challenged in diagrammatically modelling the triangles. Similar finding was also obtained by Dedeoğlu (2022) and Jupri and Drijvers (2016). As they stated such failure is the problem of mathematizing horizontally since pre-service teachers encountered challenges in translating real world problem into mathematical formulation (diagrammatically). One possible contributing factor to the creation of the notion connecting side length and angle measure is the prevalent practice of sketching triangles with arbitrary side lengths and measures.

Pre-service teachers faltered when tasked with proving this relationship through mathematical induction. The observations revealed that the root reason for this failure is a complete failure to internalize mathematical induction processes and notions. The finding of this study is consistent with the findings of Danny and Jacob (2018) and Lestyanto et al. (2022). The findings of this study are congruent with those of Danny and Jacob (2018) and Lestyanto et al. (2022). Failure to apply proof by mathematical induction indicates inability in both horizontal and vertical mathematization.

Without adequate time for each topic, learners cannot develop a deep understanding of mathematical knowledge, the end goal of mathematization since the mathematization process needs critical thinking, which involves identifying mathematical concepts, modeling both informally and formally, and forming concepts.

Pre-service teacher further highlighted the importance of practical sessions, much like lab sessions in biology, chemistry, and physics. These sessions would provide students with ample time for mathematization, thereby solving real-world problems. He associated this need with the broad content and the misallocation of time. Studies by OECD (2010), Papadopoulos (2020), and Pöhler et al. (2017) support this, noting that insufficient time is a significant challenge in the mathematization process.

Another issue emerges when the presentation of mathematics content does not correspond to students' experiences and everyday activities, since most presentations are mostly symbolic and abstract. This disparity can be ascribed to the country's overall educational system. Laurens et al. (2018) and Tefera (2022) support this viewpoint. According to Tefera (2022), ignoring cultural values and beliefs in the school system reduces educational quality. Freudenthal (1971) agreed that such a framework does not allow pupils to rediscover mathematics by analyzing real-world circumstances.

The knowledge and perceptions of teachers also pose challenges to the mathematization process. How a teacher views and approaches the subject influences the students' journey in understanding mathematical knowledge. Saleh et al. (2018) critiqued the traditional didactic approach, which views teachers as mere knowledge dispensers. A shift towards a more practical approach that relates mathematics to students' daily lives is necessary. Research shows that teachers' self-perception affects their willingness to adopt pedagogical practices that foster independent student learning (Althauser, 2018). Şahin et al. (2019) emphasized the importance of providing a conducive environment for students to study and rediscover mathematical concepts.

Regarding students' perceptions, if they view themselves as passive recipients of information and rely solely on rote memorization, they stifle their understanding and application of mathematical concepts in real life. Freudenthal (2002) warned against such a passive mindset, stating that it restricts mathematics to bookish knowledge.

Furthermore, interviews with pre-service mathematics teachers highlighted challenges that could affect he entire mathematization process. If they cannot capture the mathematization and appreciate teaching with RME-based mathematization, it is less likely for them to use it when they go out to teach at schools as teachers.

6.1.9. Meta- Inferences

The findings of this study are aligned with the findings of Uygun and Akyüz (2019). These authors utilized collective argumentation as a lens to assess pre-service teachers' mathematization process in establishing the triangle inequality, and they found that this collective argumentation promotes pre-service teachers' reasoning skills, their justification, and the way they come to generalization.

Furthermore, this study is consistent with the study of Jonassen and Kim (2010), and the authors stated that argumentation does not only serve as a teaching and learning process but also as an alternative assessment strategy for assessing learners problem-solving skills.

In using interview/informal dialogue as a means of assessing the mathematization process of learners, the findings agree with the findings of (Araújo et al., 2020; Deniz & Kabael, 2016; Upu et al., 2017; Widada et al.,2019). The above-stated research findings by different researchers indicated that they did not confine themselves to a single test or a single open-ended problem type; rather, they utilized interviews or informal dialogue to assess the mathematization process of their learners while carrying out their research. Therefore, if we reconsider the use of those approaches as assessment for learning strategies preservice teachers can improve their mathematization and can explicate a better thought process during mathematics teaching and learning.

In summary, the quantitative phase results showed that pre-service teachers who participated in the RME-based geometry intervention improved much more in geometry achievement and problem-solving skills than those in the control group. Furthermore, participants in the intervention group transitioned from absolutist ideas (viewing mathematics as rigid and rule-based) to fallibilist perspectives (seeing mathematics as dynamic and contextual).

This alignment between Phase I's cultural mathematical practices and Phase II's geometry-focused intervention not only validates the decision to narrow the focus, but also demonstrates the efficacy of an exploratory sequential design in responding to data as the research progresses.

The study's findings lead to several overall conclusions about the interaction between cultural knowledge, mathematical understanding, and educational design. First, the rise of plane geometry as the main mathematical concept within the cultural activities examined in Phase I supported and justified the decision to focus the intervention on geometry. This data-driven modification increased the intervention's cultural relevance and conceptual depth by ensuring it was based on actual, regionally meaningful practices. By drawing on such lived experiences, the intervention evolved into a culturally infused teaching experience rather than simply delivering knowledge.

Furthermore, incorporating culturally contextualized geometric problems resulted in significant alterations in participants' epistemological beliefs about the nature of mathematics, as well as advances in geometry content knowledge. Mathematizing popular cultural behaviors encouraged students to perceive mathematics as a dynamic and human activity built via investigation, reasoning, and real-world application, rather than as a fixed set of rules. This combined impact on both subject mastery and epistemic growth emphasizes the pedagogical power of culturally responsive mathematics instruction.

The study also emphasizes the value of the exploratory sequential mixed-methods approach. The ability to revise and alter the intervention in response to the findings of the initial qualitative phase indicates the approach's flexibility and responsiveness. It allowed the research process to remain open and context-sensitive, allowing for the development of an intervention that was not predetermined but rather emerged from and aligned with the data.

Finally, geometry proved to be an especially successful subject for bridging the divide between informal culture knowledge and formal mathematics instruction. Geometry, due to its intrinsic visual, spatial, and pattern-based nature, was immensely approachable and meaningful when delivered through culturally relevant activities. This made it an ideal vehicle for connecting community-based knowledge to school-based learning, resulting in a more integrated and holistic understanding of mathematics.

6.2. Conclusion

Mathematics learning is meaningful when it starts with learners' real-life situations. According to Freudenthal, mathematics is a human activity; therefore, the objects of mathematics in learning should begin with real-world problems or those that real to students' experiences. Traditional methods of teaching mathematics, which emphasize memorization and mechanical procedures without linking to real-life contexts, not only diminish the significance of learning but also cause learners to see mathematics as irrelevant to their daily lives, leading to low motivation and poor performance.

This gap focuses on the need for research into the RME-based mathematization approach, which emphasizes the use of real-world scenarios and cultural variables in mathematics learning in general and geometry learning in particular. Furthermore, the Ethiopian Ministry of Education also emphasizes the integration of ethno-mathematics in mathematics learning. This RME-based mathematization approach is intended to assist learners in connecting the concept of geometry to their daily lives, changing their epistemological beliefs, and improving their geometry performance and problem-solving skills, thereby closing existing gaps and improving the quality of geometry instruction in teacher education.

This study intends to explore the mathematization and mathematical concepts incorporated in society's cultural activities, as well as to conduct an intervention based on RME-oriented mathematization. The intention is to investigate its effect on pre-service teachers' mathematical epistemological beliefs, problem-solving skills, and geometry achievement. This is because a realistic mathematics-learning model based on ethno-mathematics greatly aids students' processes of abstraction, idealization, and generalization through the employment of both the model off and the model for. It emphasizes student participation, real-world problems, and interactive and collaborative activities—all of which are hallmarks of a high-quality mathematics education.

As a research paradigm, this study employed dialectical pluralism by valuing perspectives from both Phases I and II while also acknowledging and integrating the opposing views of absolutists and fallibilists. An exploratory sequential study design led to the use of a mixed-methods methodology. Phase I data were collected through semi-structured interviews and field observations. In Phase II, data were collected using an epistemological beliefs questionnaire, semi-structured interviews, and observations. Quantitative data of phase II were analyzed using one-way MANOVA and an independent t-test, and the qualitative data of both phase I and phase II were analyzed thematically

From the qualitative data analysis and discussion outlined earlier in phase I, it can be concluded that various mathematical knowledge exists in the weaving, basketry making activity, straw house construction, and pottery making of Selale, Arsi, and Bale of cultural activities. Geometric elements and concepts like points, lines, angles, parallel lines, quadrilateral shapes, circles, arc length, radial symmetry, rotational symmetry, and geometric and arithmetic sequences are found in societal cultural activities. Moreover, the concepts of integral calculus, the concepts of like cones, frustums, cylinders, and their volume were also found in societal cultural activities. Similarly, ideas of rigid and non-rigid transformational motion, such as rotation, translation, reflection, and dilation, are also present in society's cultural practices. Additionally, the similarity and congruency of plane figures like triangles and quadrilaterals are often observed in various cultural activities such as weaving, house construction, and basketry. These mathematical concepts are consistent with those used in mathematics in a teacher education program. Furthermore, the mathematization process of counting, measuring, locating, designing, and explaining were utilized in the process of making cultural products.

Based on the findings of phase I, an intervention relying on only a plane figure was implemented by taking cultural activities of the society as a source of a realistic situation in the RME-based mathematization approach. Thus, from the data analysis and discussion of phase II, particularly based on pre- and post-test results, the study found a substantial difference between the groups that were taught using the RME-based mathematization strategy and the group that was taught using a traditional approach. MANOVA result indicated that there were significant mean differences between the treatment group and comparison group on a linear combination of dependent variables (source, stability, and structure of mathematical knowledge) with $\eta^2 = .639$ indicating that the group factor explained about 63.9% of the variation in the dependent variables.

Based on the qualitative findings, it was found that mathematization-based instruction helped most pre-service math teachers hold both absolutist and fallibilist views about the origins and stability of mathematical knowledge. However, fallibilist perspectives were more prevalent when it came to the structure of mathematical knowledge.

Pre-service teachers' mathematical epistemological views were impacted by this distinction, which also enhanced their geometry and problem-solving skills. The study's most noteworthy finding is that pre-service teachers learn more when using an RME-based mathematization approach that is rooted in society's culture. Additionally, the application of this strategy by teacher educators was shown to favorably influence the beliefs of pre-service teachers, helping to establish a productive learning environment that develops a deeper comprehension of geometry and fortifies problem-solving skills. Furthermore, personalized learning is supported by the RME-based mathematization strategy, which makes the teaching and learning process more efficient and adaptable. It can be inferred that problem-solving skills had a significant predictive power of geometry achievement within the intervention group; however, mathematical epistemological beliefs were not a significant predictor. Within the comparison group, neither problem-solving skills nor mathematical epistemological beliefs were significant predictors of geometry achievement.

The efficacy of the RME-based Mathematization approach is evident in its application. Its grounding in real-world scenarios renders the content highly pertinent to students, ensuring that their mathematical experiences are authentic and meaningful. For instance, it facilitates the formulation of the triangle inequality theorem related to triangle sides; delineation of the essential and supplementary conditions for the congruency of triangles; and derivation of the relationship between the number of sides of a polygon and its total diagonals. This process is enhanced by rigorous exercises in both horizontal and vertical mathematization.

The findings of the study also revealed that some mathematical concepts and skills, like mathematization, need further assessment for learning strategies in addition to the usual tests. Thus, mathematical discourse like interview/informal dialogue, presentation, argumentation, and observation should be used as assessment for learning strategies in assessing the mathematization process of pre-service mathematics teachers, and at promoting their thought processes. Thus, teacher educators should not rely only on the usual test or exam to assess the learners' higher order thinking skills like problem-solving skills, critical thinking skills, argumentation, reasoning skills, and communication skills but rather use alternative assessment mechanisms like informal discussion, argumentation, and presentation.

6.3. Recommendations

The Ethiopian Ministry of Education highlighted the integration of ethno-mathematics into the mathematics curriculum and mathematics classroom (MoE, 2022). However, there was limited ethno-mathematics that was explored and documented. The strategies that help explore and document ethno-mathematics to strengthen the effective practices of teacher education programs and methodologies need to be designed and implemented.

Pre-service mathematics teachers should be well-prepared and equipped with mathematical and geometric knowledge, as poor preparation of pre-service mathematics graduates contributes to the next generation of mediocre teachers, perpetuating the cycle. This study's findings lead to the following recommendations for teachers, curriculum designers, and teacher educators:

Curriculum Designers

When developing curricula or modules for mathematics and geometry, curriculum designers may want to incorporate ethnomathematics through RME-based mathematization, which allows students to mathematize mathematics and reality into the teaching methodology and mathematical content. Curriculum designers should provide examples of using the RME-based mathematization approach in real-world contexts to effectively engage learners in mathematics classes.

The Ministry of Education

To improve the learning outcomes of pre-service mathematics teachers, the Ministry of Education (e.g., Division of Curriculum Supervisory, Teacher Professional Provision Division) should look into the College of Teacher Education to introduce professional development opportunities for mathematics teacher educators, as well as how to incorporate ethnomathematics into math classes using RME-based mathematization. Mathematics teachers must receive continual professional development and training in the use of RME-based mathematization, which blends students' real-life experiences into mathematics instruction.

Teacher Education Organizers and CTE Proprietors

Training is necessary for teachers and educators to employ RME-based mathematization, which incorporates learners' practical experiences with the instructional strategy and mathematical content. Furthermore, college teacher education should give periodical seminars, symposiums, workshops, and training sessions for teacher educators on how to employ RME-based mathematization in their mathematics teaching and learning.

Teacher education colleges need to inculcate RME-based mathematization strategy rigorously and adequately with the reformed mathematics curriculum. Preservice teacher training programs should also involve a course to inform prospective teachers about the benefits of RME-based mathematization approach that is rooted in ethno-mathematics, and pre-service mathematics education courses need to incorporate cultural responsive and pedagogies.

Teachers Educator

To help their pre-service teachers understand the nature of mathematics and geometric concepts in particular, teacher educators must be prepared with RME-based mathematization. This will help them change their mathematical beliefs and learn mathematics in general and plane geometry in particular. To have a synthesis of mathematical knowledge, teacher educators must also understand how to mathematize reality and mathematics itself.

This, in turn, helped them introduce RME-based mathematization, which incorporates ethno-mathematics into the mathematics classroom to increase teaching and learning efficacy. Similarly, to adopt RME-based mathematization in their future classrooms, pre-service mathematics teachers must get instruction on its concepts and characteristics. This study suggests implications for the assessment methodologies employed in teacher education programs to improve mathematization. Effective mathematization also transcends to better teaching and learning in classrooms.

6.4. Suggestions for future research

More research is needed to assess the impact of RME-based mathematization education on learners' learning outcomes using mathematical topics other than plane geometry, as used in this study. More research is also needed to determine how RME-based mathematization might be used in educational settings other than Teachers Education College. There is a need for research on learners' ontological beliefs. Furthermore, future research should examine the relationship between learners' mathematical epistemological beliefs and geometry achievement within RME-based mathematization grounded in cultural mathematics contexts. In this study, the intervention lasted four months and involved a relatively small sample (N = 57), which may have limited the ability to detect a significant effect. Since RME emphasizes learning through guided reinvention and meaningful cultural contexts, longer intervention periods and larger sample sizes may better capture the gradual influence of epistemological beliefs on learners' geometry achievement.

6.5. Limitation of the study

Two teacher educators from two different teacher education colleges carried out the interventions. Despite the fact that the researcher endeavored to choose participants with almost identical teaching experiences from multiple viewpoints, offered training on instructional methods and approaches for implementation, and performed classroom observations with the teacher educator throughout implementation. This is to guarantee that the research protocols were followed; nonetheless, teacher educators may affect the intervention and the college, as it is difficult to control all factors. Another limitation of this study is the clear bias in qualitative data interpretation. The last limitation was related to the study's design, which was hampered by the small sample size. As a result, the study's findings may not apply to the whole population of pre-service mathematics teachers in Ethiopia. Furthermore, the discussion was constrained due to a lack of resources on mathematical epistemological beliefs

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APPENDIX

Some transcriptions of Phase I

Making Gingilcha

Researcher: to start making a sieve, how many strings do you need?

Respondent: It needs four strings; one pair of strings are downward and one pair is horizontal.

Researcher: can't you start with one or two or three strings? What will happen if it is one rope or two strings, or three?

Respondent: I can not make the sieve if it is one or three strings.

Researcher: while adding vertical paired strings how many strings do you add at a time? And why?

Respondent: two strings which tied together will be added at a time. To make the strings intertwining.

Researcher: how many horizontal strings do you need while making a net sieve at each row at a time? And why?

Respondent: A pair of strings. This is to tie or intertwine the vertical strings and make a gap among these string so as to separate flour from hubs.

Researcher: How many vertical strings (columns) do you need to make a sieve?

Respondent: it depends on the size of the sieve. For example, for this case we can count them.

Researcher: how do you count them? Is it one by one or any else?

Respondents: of course, it is possible to count one by one, yet since I insert two strings (a pair of strings) at a time I can count them as 2, 4,6, …96. Or I can consider a pair of string as one and I can count them as 1, 2, 3, …, 48. Meaning that there are 48 pairs of string which is totally 96 strings.

Researcher: What will happen if you add a single string at a time?

Respondent: Since one is one there is no intertwining of strings and there is a wide gap between the vertical (column) adjacent strings from starting till the end and hence it makes difficult to separate different finer particles of mixture of floor or others like teff from coarser (i.e., the flour passes through with the hub as it is).

Researcher: While doing the sieve, what is the purpose of the pair horizontal strings?

Respondent: while making a sieve, the main purpose of horizontal strings (pair of strings) is to tie those vertical strings added one after the other. At a time, you insert a pair of strings in between the two horizontal strings. When we make a sieve, from the two horizontal strings one is at the top and the other string is at the bottom. In the next step, the one at the top will be at the bottom and the bottom one will be at the top. This top, bottom, top, bottom, … pattern will continue until the required distance.

Researcher: why do you make a gap between two adjacent rows?

Respondents: if there is no gap between two adjacent rows flour can not pass through the sieve.

Researcher: how do you preserve the gaps between the two adjacent rows?

Respondent: This is by estimation which developed through experiences.

Researcher: Why do you intertwine the horizontal strings?

Respondent: to make the gap between the strings smaller.

Researcher: How much distance would you cover? How do you determine the size of the sieve?

Respondent: there are different sizes of the sieve. Small, medium, and large. I make a small or large size depending on the need of the customer.

Researcher: while making the net of the sieve since you take and insert a pair of vertical strings in the horizontal strings at the same time, how do you intertwine these vertical strings till the end?

Respondent: during the formation of net of the sieve, starting from the second row to the end, I start with a single string and then the remaining of the string from the first pair is paired with the one from the second pair, and the remaining of the second pair is intertwine with the one from the third pair and the remaining of the third one is intertwined with the one from fourth pair. This pattern continues up to the last of the 48^{th} pair yet ends with a single string of the 48^{th} pair. However, the next row will be started with the first pair and then the second pair, third fourth … until the 48^{th} pair.

F1, F1F2,F2F3,F3F4,F4F5,…,F47F48, F48

F1F1, F2F2,F3F3, F4F4, …, F48F48.

F1, F1F2,F2F3,F3F4,F4F5,…,F47F48, F48

F1F1, F2F2,F3F3, F4F4, …, F48F48.

F1, F1F2,F2F3,F3F4,F4F5,…,F47F48, F48

…

….

….

Researcher: why you start and end with one vertical string and then a pair strings in the middle?

Respondent: this is to separate a pair of strings so as to make a gap among the strings.

Researcher: why you separate the strings? What will happen if you do not separate them?

Respondent: to make a gap among them. if we do not separate a pair of strings the gap between two adjacent strings will be large and everything will pass through easily and the sieve unable to separate hubs from flour.

Researcher: In making the net, after intertwining the strings of a given row why you tie the horizontal strings on the last vertical string or on the last pair of strings together? And then you start the new row at the end of the previous row? What will happen if all the starting point is on the same side and all the end points on the other side?

Respondent: So that it does not disperse and has strength. The reason a new line starts where the previous one ends is to make it easier for the net to be stretched over a circular piece to start sewing on the net.

Researcher: why do you tie many strings together? Why you need a circular dish?

Respondent: unless otherwise tie the strings together it is impossible to start sewing. The need of circular dish is for the purpose of stretching the net to start sewing on it.

Researcher: why you select only circular dish since there are different shapes?

Respondent: if I select shape different from circular one, the so called 'suta' will not bend rather easily broken. Since for sewing purpose we need dry'suta' but not the wet one for the purpose strength.

Researcher: why do you need strings at this stage? And if you need them how many of them at a time?

Respondent: to prevent the net not detaching from sewing part. At a time, I use four strings one pair is forward and one pair backward to increase the beauty of the sieve. And they will make a design on the sieve. These four strings will be used only at the beginning cycle.

Researcher: on the upper side how many cycles is necessary? And what about the bottom side?

What are their purposes?

Respondent: It dependent. On the upper side we can go up to the seven or eight cycle so as to hold much amount of flour. At the bottom side we need two or, three or four cycles to prevent the net from dusts and other unnecessary things. Otherwise, the net will be exposed to many things which shorten its lifespan. The other purpose of the bottom side for strength.

Researcher: while sewing, are you sewing clockwise direction or counterclockwise direction? And why?

Respondent: I am sewing counterclockwise direction and this is its rule. Even I have seen left-handed women and their sewing direction is counterclockwise.

Hat making

Researcher: the ultimate purpose of this is to understand and identify the mathematization process and mathematical concepts embedded in hat making. Let us start with your age. Would you tell us your age?

Participant: I was born in 1977 in EC.

Researcher: would tell us your educational background?

Participant: My schooling was up to grade 8.

Researcher: from where did you get the experience of making hat? From whom you get this skill? How do you start making hat?

Participant: I started this when we were herding cattle. I discussed with my colleagues on how to make hat while we were herding cattle.

Researcher: Was there anyone doing it, and did you learn from him/her? How did you learn?

Participant: there was individuals who made hat from different materials like wool rope, long straw (migira ykn coqorsa). We tried to make hats using trial and error.

Researcher: You mean that you obtained the skills from those who made hat from wool rope or straw.

Participant: Yes, there was individuals who made hat from the materials I mentioned earlier. We developed the skill through time. The beginning was not attractive. However, through time we made attractive hat. At that time since there was no shopping, we made it for ourselves. However, through times, even elders bought from us and put it on their head just to protect themselves from heat of the sun and cold.

Researcher: what objective did you set every time before starting making hat?

Participant: I did not do it the whole day; rather I performed it at morning and at the nighttime. However, while we were herding cattle, we performed it the whole day.

Researcher: how did you start making hat? What materials did you need to start making hat?

Participant: First, I have to prepare many ropes made from a pair of chokorsa or migira (ilkole or fo'aa) up to 400 hundred in numbers. After collecting the so called chokorsa or migira and chewing them to make soften and make a rope. I have to identify two very large ropes and then, I have to tie them together or pass one rope through the other rope at their mid-point (at their center) and they make four ropes. Chokorsa and migira are naturally diverse in size; therefore, I'll use larger ropes at first, medium-sized ropes in the middle, and smaller ropes in the end.

Researcher: What other alternative do you use in the absence of chokorsa and migira?

Participant: we may make ropes from the so called 'madebera'. We may use wool rope yet it is not as strong as chokorsa or migira.

Researcher: what is the purpose of the hat?

Participant: just to protect the heat of the sun. It is a cultural beauty of a given society and it is a traditional jewelry.

Researcher: Are going to overlap the two ropes? Alternatively, are you going to tie them together? On the other hand, any mechanism?

Participant: we intertwine the two ropes and these two ropes made four wings. After intertwining the two ropes together and we will start making hat by a pair of straws together with the intertwining the two ropes. During the process, one straw will be above a given rope, and the other straw will be below it. Furthermore, in the following stage, the straw below the prior rope gets above the succeeding rope, the straw above the preceding rope becomes below the subsequent rope, and this process is repeated, resulting in a spiral form.

Researcher: You started making hat using two ropes. Did you proceed the process with these two ropes until the end? How many ropes we need to complete the hat?

Participants: We start with two ropes. These two ropes are tied at their halfway point, making four ropes and four distinct spaces. As the procedure progresses, the distance between two successive ropes grows greater and larger. Then we place four ropes into these four spots, bringing the total number of ropes to eight and creating eight spaces. In the same way, after a few steps, the number of ropes gets dispersed, leaving a big gap between two successive ropes. Then we insert 8 ropes in the preceding 8 spaces, bringing the total number of ropes to 16, with 16 unique spaces. Then 'tata'u' is continuing and the size of the hat become larger and larger and making the gap between two consecutive ropes larger indicating to insert additional ropes. Then I will insert 16 ropes making the total number of ropes 32 with 32 unique spaces. The same process continues and I will insert 32 ropes making the total number of ropes 64 with 64 unique spaces. This process continues until the hat is completed.

Researcher: How many ropes will you need to finish the hat in the chosen size? How many ropes have you used in total so far?

Participant: we can approximate the number of ropes. Until now, I've used 128 ropes. I don't think I'll insert ropes more than once on this hat because I'm almost done. I hope I can finish the hat with 256 ropes.

Researcher: Since you informed me that you first require two ropes and then insert four, eight, sixteen, thirty-two, and so on, does this procedure reveal any patterns?

Participant: Yes, of course. I began with two ropes and proceeded to add four, eight, sixteen, thirty-two, sixty-four, and so on until the desired size of the hat is maintained.

Researcher: Is it feasible to return to the start point during the process? Why?

Participant: We'd rather produce a spiral (maramaa) form, similar to how we make "sefede." When building a spiral shape, it progressively expands until it reaches the appropriate size.

Researcher: how many days it will take to complete the hat?

Participant: If you do it all day, it will take one week. However, since I only do it in the morning and in the evening, it will take around 3 to 4 weeks.

Researcher: why you utilize the so called ‘kula’ wool rope?

Participant: Just to make the hat beautiful and attractive. Furthermore, ‘kula’ or wool rope is more durable than chokorsa but interms of strength chokorsa is preferable.

Researcher: As you mentioned earlier, you need different materials like chokorsa, ropes and wool ropes. Is it possible to now the amount you need to complete the hat?

Participant: we can approximate the number of ropes. For example up to now we have utilized 128 ropes.

Researcher: How many times do you insert the ropes to complete the hat?

Participant: I added four ropes at a time after starting with two. I suppose I've inserted the ropes five times so far. After all, I doubt I'll be inserting the ropes more than once.

Researcher: what does it look like?

Participant: it look like the circular roof of the straw house.

Basketary

Ink, Migira, muta, suta. Fire, water, needle, madabera, threads and dish are some of the material necessary for basket weaving. To make mesobwork, I started weaving by the inner circular part called gindo. After completing five cycles of weaving, I have tried to bend it down ward to weave the leg first. In Mesobwork making, there are three parts: the inner circular part, the leg- a frustum of cone-shaped and the upper part, which starts from the inner circular part after weaving five cycles. Weaving the upper part is done after completing the leg of the mesobwork, or after weaving many cycles of the leg of the mesobwork, or the leg should be large enough to start the upper part.

I start the leg of the mesobwork by counting the number of cycles I have woven the inner circular part. Most of the time I have started the leg after weaving the inner circular part around four to five cycles. Sometimes I may start weaving the leg of the mesobwork after weaving the inner part around 6 to 7 cycles. If we start weaving the leg after weaving the inner part seven cycles, it becomes the large mesobwork, and such a type of mesobwork is very much in demand in the market and easily sold. In this mesobwork I bent and start weaving the leg on the fifth cycle of the inner part.

In mesob weaving, the inner circular part is woven from inside to outside, the leg and the upper part are woven from outside to inside. It is difficult to weave the upper and the leg from inside to outside. However, it is possible to weave the upper part from inside to outside, but its beauty is not that much. Weaving the leg from inside to outside is unthinkable. For the sake of beauty and ease in weaving, the inner one is woven from inside to outside, the leg and the upper are woven from outside to inside. If it is woven in such a way, it becomes very beautiful and attractive. Furthermore, the inner part of the circular part and the outer part of the leg and the upper part which are visible and customers can visualize it.

To increase its beauty I utilize alela by inking migira into the ink. I need around one litter of water to mix different inks so as to get different colors. Thus, with different types of alela I can decorate the mesob by making different designs on it. Mesobwork is for beauty and it also for shinote ceremony during wedding. It can also sever as a food storage while the bride departs from her family to her husband's home. I can compete this within one day if I exert all my effort on it. The amount of each material for weaving is through estimation. For example, to complete one mesobwork I may need 3 to 4 bundle of migira.

I know its size by counting the number of cycle in each part of the mesobwork. For example, I can complete its leg by weaving 18 cycles. It can be achieved with 13 to 15 cycles, but it is a small improvement. We can use threads (kula) and medabera instead of alela and migira. If one utilizes these two materials instead of migira, she must use as many needles as possible. In the case of migira and alela one can only utilize muta. But I am not familiar with them. In terms of suta, we can use senbeleta.

Building a straw house

Researcher: How do you start the straw house?

Participant: Since we build a house with a cylindrical shape, we need a rope, a small stick with a sharp edge, and a 'sickle' to make a circular shape. First, we have to decide the place to put the utuba in the house. Then we insert the small stick into the ground at the place we want to place the utuba. Then we will start measuring from where we want to put the utuba. Starting from utuba, we have to measure 9 to 14 feet for the build 'mekides' and to build 'gaadaa we then measure 6 to 8 feet from the mekides to gaadaa. Then we tie the rope on the stick (chikalii) we stretch a rope up to the desired height. Then we make a circular shape for ‘mekides’ and ‘gaadaa’ using ‘hamtu’ and the rope. Since the straw house has four partitions (a place for sleeping, a cooking place, a place for a shelf, and a gate).

Researcher: Why do you want to partition them into four?

Each of them serves a specific purpose. For example, the sleeping area is situated in front of the cooking area, and the gate is in front of the shelf area. The sleeping area is where we rest, the cooking area is where we prepare food, and the shelf is where we place our jebena, rekebot, and, occasionally, it serves as a place they pour a little coffee before drinking (iddoo buna itti dhibasanii).

Researcher: How do you make these partitions?

Participant. To divide the inner circular wall into four sections, we measure four feet up from the center and we measure four feet down to the opposite side, put the pegs, four feet to the left and four feet to the right of the center, and put the pegs. Finally, through pegs we stretch the ropes from one end of the inner circle to the other end to get four free spaces in the four directions. Place for sleeping, for cooking, for shelving the gate.

Researcher: What is the ultimate purpose of measuring three feet up and down and three feet to the right and three feet to the left?

Participant: This is to make all ‘mekides’ equal and all four partitions equal, and this in turn makes the house beautiful and smart.

Researcher: What is the next step after making the partition?

Participant: We dig the outer and the inner circular part by jumping all four spaces (Bed, fire, shelf, and gate).

Researcher: How far did you dig into the ground? And why do you dig the ground?

Participant: It is around an arm's length, and this is just to insert wood and make the house strong.

Researcher: Does constructing a straw house require special skills?

Participant: No. It is teamwork, and we build it in teamwork called ‘jigi’ within one day whenever the householder prepares all the necessary materials for building. However, there are different stages. The first stage is building the wall of the house using wood, both the inner and the outer part, by jigi within one day. The second stage is building the roof of the house, and this is made from wood and straw. To construct the roof of the house, we need to prepare the so-called’ walta’, which frustum of a cone shape made from wood, and it has a hole at the center to pass the ‘utuba’ through it. This cone-shaped material together with the height of both the inner and the outer and the pole determine the steepness of the roof of the house. The third stage is covering the roof of the house using straw. At this stage, a well-skilled person is needed to cover the roof of the house using straw unless otherwise the rain will sink. The amount of the straw depends on the size of the roof of the house. For the small house, we need around ten bundles of straw (10 donkeys), for medium, it needs around 15 donkeys, and for the big house, we need around 20 to 25 bundles (20 to 25 donkeys). The fourth stage involves covering the house's wall with mud, a task that requires no special skills and can be completed in a team within one day.

The straw house is a two-walled structure with a cylindrical foundation of radius 16 feet for the outer wall and radius 10 feet for the inner wall, as well as a conical roof created from local materials such as wood, wattle, mud, stone, tusha (shiboo), and grass or straw thatching (Sanbaleta). Since it is a double-walled round, the inner wall is partitioned into four equal parts: for sleeping, for cooking, for a gate, and shelving, locally named as 'Boro’o, which serves for placing a shelf (locally named as gorbo). Each inner wall has its sitting area (known as wedebi in the local language), which is used for sitting during the coffee ceremony and eating time, as well as a space for the children to sleep. They utilize the space between the outer and inner walls to store various goods such as cooking materials, grain, and water, as well as to prepare Tella for different celebrations. The pole in the middle of this house balances the roof's steepness, keeping rain from getting through.

After preparing the building materials, the house owner decided on the location where the house pole would be placed and put a small stick. Then, after measuring the radius of the inner and outer walls using their feet, they started making concentric circles.

Building a steel house or straw house of different shapes

A house with a quadrilateral base with a roof of a pyramid with four triangular faces

Researcher: How do you start building?

Respondent: Before starting, the owner of the house must first select the type of home. They may be interested in a house with a quadrilateral base and a roof shaped like a pyramid with four triangular faces; a house with a hexagonal base and a roof shaped like a pyramid with six triangular faces; or a house with a quadrilateral base and a roof shaped like a pyramid with two trapeziums and two triangular faces. After fixing the type of the house, the design will follow. For example, if my customer is interested in a house with a quadrilateral base and a roof shaped like a pyramid with four triangular faces, I will start designing using ropes and pegs.

Researcher: How do you design it? How many people do you need for designing? What about the necessary materials needed for designing?

Respondents: regarding the number of assistants, I need three assistants. I need rope, a meter, pegs, and a hammer. Then, after determining the area of the house, the design will follow. Since the area of the house varies based on the interest of the customer, we need a very long rope, which is around 50m or 100 arms. First, we extend the rope in the form of a four-sided shape. Then, we plant two pegs at the two corners. Then, to secure the 'squadra', we have to make the length of the two diagonals equal.

Researcher: Why do you make the length of the two diagonals equal? What will happen if they are not equal?

Respondent: If the two diagonals are not equal, there is no 'squadra,' and it may not affect a straw house, but it does impact a steel house. Extra steel will be needed, affecting how to make the cornice and how to cover the floor with ceramics. While building the house, we may cut the steel into two equal parts. If the squadra is secured, then we can effectively use the steel, which divided into two equal parts. Half of the steel covers one part of the roof, and the remaining half will be used to cover the opposite side.

Researcher: If the lengths of the two diagonals are not equal, what measures do you take to make them equal?

Respondent: We will divide the differences of the two diagonals into two equal parts. By keeping the two adjacent corners fixed, we can find the remaining two corners, each of them half of the difference between the two diagonals. For example, if the difference is 10cm, we shift one corner 5cm towards the inner and the other corner 5cm towards the outer.

Some Transcriptions of in phase II

C-022

Interview on the perception of RME-based mathematization

Researcher name: Girma

Name of the interviewee: Ismael (**C-022**)

Occupation: Pre-service mathematics teacher

Level of education: Degree program

Department: Mathematics

Name of the institution: CTE1

Location of the interview: Education department, Room10

Date of the interview and time: 18 June 2023,4:18-4:56 pm

Researcher: Thank you for your invaluable information. The ultimate goal of this interview is to gather information regarding pre-service teachers' perceptions of the RME-based Mathematization Approach and the challenges they face during the intervention. During this interview, no need of mentioning your name, and the information you will give form me is only for research purposes and will be encoded to protect your identity and all results will be kept confidential. Saying this, I directly proceed to the first question.

Researcher: For the last three months you have learned plane geometry via an RME-based mathematization approach. In your opinion, do you think that there is a difference between this approach and the approach you were taught in the previous semester and even in the lower school?

C-022: Thank you for giving me this chance. Next to this, I will directly go to your question. Starting from primary school to college level, I have learned a lot. In my previous career, the majority of my teaching-learning process was theoretical yet there were some laboratory activities. However, over the last three months, I have constructed knowledge by doing practical activities and observing and it was a learner-centered approach and the teacher's role was the initiation. The approach initiated us for creation. Since it brought a radical change in our minds, in our future careers after graduation, it will help us on how to teach our students in a practical manner so as to develop conceptual understanding. As a student of Robe CTE, starting from the first day of plane geometry class up to now, without missing our class, we have been learning each and every concept practically and we were involved in field experimentation by measuring using local measurements and the modern one. In general, there was a big difference.

Researcher: Does the RME-based mathematization approach you have been using in the teaching and learning process of geometry improve the mathematical epistemological belief you had before? How?

C-022: I have seen a lot. The mathematics I was learning so far didn't seem real. But now when I actually do it practically in the classroom when I went out on the field and demonstrated itin action, I believed the real mathematics. For example, from the intervention we have learned and understood Mathematics as the practical activity of society or in their/our/ daily life. Thus, we had got real mathematics. Herregni qabatamummaan dhugaa hawaasa keessa jiru isa sirriifi kan hawwasa keenya utubee deemuudha (Mathematics is the right reality in society and the pillar of our society).

Researcher: As a pre-service of mathematics teacher what mathematical epistemological belief did you have before? And what about now? Is there any change with respect to such beliefs?

C-022: Before I learned in this way, I thought I would teach my students in a conventional way by talking what written in the book or business as usual in the future. After this intervention, my thought has been completely changed and now I have developed the belief that mathematics is the practical activity of our society and students should be taught practically. I have also developed the beliefs that Mathematics exists in a proven or practical form in our society and we can take them from our environment and use them for teaching our students. For example, in our teaching and learning process particularly for the concepts of similarity and similarity of triangles, we used different materials that our fathers and mothers prepared for their daily activity. Thus, this intervention has enabled to develop the beliefs on how to teach mathematics using different daily practical activities practiced in of the society.

Researcher: Therefore, do you think that this approach has changed your mathematical epistemological beliefs?

C-022: Definitely. Even if my students ask me different questions, I will use the real and practical activity of the society.

Researcher: Taking this in to consideration, how do you understand mathematics? What is mathematics for you?

C-022: the way I understand mathematics and for what mathematics mean for me is that we cannot part with mathematics. It is the activity I will perform daily or it is what we perform in our daily life. It is the pillar of our future life.

Researcher: Do you think mathematics is related to daily life? How?

C-022: Yes, there mathematics is related to daily life. It is what we discuss now. From our teaching learning process, we have seen that mathematics is related to our environment. Our teaching learning process we have been learned so far has been linked with real life our society. For example, in our teaching learning process we have seen the way our family prepare and 'oogdii' and the reason why they prefer circular form. When our mothers make different ornaments, even if they do not know, they are doing mathematics. Furthermore, in the construction of different building there are many different mathematics.

Researcher: In your opinion where do you think the source mathematical knowledge is?

C-022: The source of mathematical knowledge is the society itself. Even if it is written in the text book, the source of mathematical knowledge is our society and we have seen this practically. The activity in the society is organized in modern form and written in the book then taught as mathematics in the classroom. Thus, the source of mathematical knowledge is the society.

Researcher: can't we consider teacher as a source of mathematical knowledge? Can't you acquire mathematical knowledge from different books, videos and teachers? Or can we consider them as a source of mathematical knowledge?

C-022: Teacher can explore and bring the mathematical knowledge from the society to classroom and hence we can consider them as the source mathematical knowledge. We can also acquire mathematical knowledge from different books and videos.

Researcher: When you say that the teacher brings what is in the community to the classroom, do you mean the teacher himself digs and brings mathematical knowledge that exists in the community to the classroom, or do you consider the teacher himself or herself the source of mathematical knowledge?

C-022: He/she brings the real mathematics practiced in to classroom. He /she is the one who show where the source of mathematical knowledge is. Thus, the source of mathematical knowledge is the society and the teacher as well.

Researcher: Is previous experience regarded as a source of mathematical knowledge?

C-022: Of course, yes. Prior experience can be the source of mathematical knowledge, new knowledge can be constructed based on the previous one.

Researcher: How do you see the role of teacher and the role of the student?

C-022: From what we have been learning for the last three months, I understood what the role of a teacher should be. However, in the past, I viewed teachers to be the only knowledge transmitters. But these days, I think the majority of teachers roles are as facilitators and guides, where the source of mathematical knowledge is. Students arrive to the goal as a result of their own effort and prior experience.

Researcher: Is mathematical knowledge evolve from time to time? Or do you think that mathematical knowledge is immutable and that yesterday's mathematical knowledge is the same as today and tomorrow? Why?

C-022: Suppose if we consider the mathematical knowledge of yesterday and today, it is the one which evolves over time. Thus, mathematical knowledge is evolving from time to time but not fixed and achieved through the efforts of the student and the teacher. It is related with creation and it means that as generations become civilized, mathematical knowledge will be invented. As we struggle to derive different formulas during the last three months, mathematics can be constructed. Therefore, it changes from time to time. It needs in-depth research, and the teaching-learning process should take into account the realities of society. Students as well as teachers should struggle with mathematical knowledge construction, and a learner-centered teaching and learning approach that relies on the real-life context should be promoted.

Researcher: how does the structure of mathematical knowledge be organized? Is it in simple form, in complex, in hierarchical order? Is it fragmented or interconnected?

C-022: I have seen that the structure of mathematical knowledge is organized from simple to complex. For example, in the lower class, there is simple mathematical knowledge that I can manipulate in my mind, but at this stage, the mathematical knowledge that roots society as a source of knowledge is becoming complex. Mathematical knowledge is also existed in a fragmented manner in different places.

Researcher: In your future career as a teacher, are you going to make students remember different facts, formulas and principles and solve mathematical problems in only one unique way, or are you going to help them solve the problems in different ways and with different strategies?

C-022: During our teaching and learning process for the last three months, we were encouraged to solve mathematical problems by ourselves, and our teacher guided us. We did everything by ourselves. The teaching and learning process made us derive different formulas by using our prior experience and our own strategies. For my future career, I will encourage and facilitate different conditions for my students to derive different formulae in their text book but rather not to memorize them.

Researcher: What is your opinion about mathematics after you have been learned geometry using RME-based mathematization approach?

C-022: I know myself that the opinion I and my classmate had on mathematics was not that great. I have been studying mathematics for twelve years, but this year, in the 2nd semester, I saw that the way I studied mathematics is real and that it can be achieved in practice.

Researcher: What benefits did you get from this approach? Particularly, does it contribute to the development of your problem-solving skills, the change of your mathematical epistemological beliefs, and the improvement of your mathematical achievement?

C-022: since mathematics is related with our daily activity, and from this point of view I have develop how to solve real life problems I will face in my future career. As I mentioned before, it also helped me to mathematical epistemological beliefs. Regarding achievement, it does not only help me to improve my mathematical achievement, but it also makes me competent enough in everything.

Researcher: In this learning approach, what impresses you most? Why?

C-022: It is not a teacher centered approach but it is learner centered approach. With the guidance of our teacher, we have solved problems with our efforts and done different experiments. What I've learned from this is that the learning-teaching process has transformed me.

Researcher: Do you want other future instructors to learn in this manner? What is it for?

C-022: Of course, yes. If teachers teach their students theoretically or business as usual, they will forget. Thus, they have to learn practically, and what they learn should be related to what society practices in their daily-life since the one they do practically is unforgettable.

Researcher: Do you think such an approach to teaching will be effective in the future?

C-022: Teaching in this way changes not only the teacher but also the student. It also solves the problems in daily activity. It also promotes creativity.

Researcher: In your future career, are you going to use such an instructional learning approach?

C-022: After my graduation, in my future career, I will plan to explore real mathematics practiced in society and use it in my teaching and learning process.

Researcher: So far, you have learned plane geometry using RME-based mathematization; would you please mention some challenges you encounter in the teaching and learning process? Would you please mention some challenges in implementing this approach in the teaching and learning process in your future career?

C-022: For example, in this approach, we have been using different materials prepared in society, but in my case, what I fear is that I may face the scarcity of teaching aids, which will affect my success.

Researcher: So far, you mentioned that mathematics is the daily activity practiced in society, and there are different mathematical concepts in various resources prepared by society for their daily use; therefore, can't you get and use them as teaching and learning materials, as we have done for the last three months?

C-022: Of course, we agree that we can get mathematics and different materials from society, but if there is no support and willingness from the school as well as from different education offices, it affects the proper implementation and success of such a strategy. In our society, the teacher may not have access to how a given piece of material is prepared.

Researcher: Regarding time, materials, broadness of the content, the curriculum itself, how do you see RME-based mathematization approach?

C-022: Yes, it challenged us. Since this approach needs to do mathematics practically and relate school mathematics to the every-day activities of society, it definitely takes time. The way the text book was prepared may not allow you to do mathematics practically. It had an impact on us when we learned. If time is not taken into consideration in curriculum preparation and if the curriculum does not invite teachers to teach and students to learn mathematics practically, change may not be brought on not only to students mathematical problem-solving skills but also to their mathematical beliefs.

Researcher: As a prospective teacher, during your class in your future career, what will be expected of you to implement RME-based mathematization?

C-022: My first role is to let the students participate in learning mathematics practically and try to facilitate different conditions.

Researcher: when you say 'facilitate different condition' does it include preparing realistic problems from real context?

C-022: So far in our teaching and learning process, which lasted for three months, there have been real-life problems prepared from the daily activities of our society and provided to us. From this, I have learned that I have to prepare for real-life problems before starting my teaching.

Researcher: In our teaching and learning process, which lasted for three months, what were some of the challenges you encountered?

C-023

Interview on the perception of RME-based mathematization

Researcher name: Girma

Name of the interviewee: Birru(**C-023**)

Occupation: Pre-service mathematics teacher

Level of education: Degree program

Department: Mathematics

Name of the institution: Robe CTE

Location of the interview: Education department, Room10

Date of the interview and time: 18 June 2023,5:13-5:46 pm

Researcher: Thank you for your invaluable information. The ultimate goal of this interview is to gather information regarding pre-service teachers' perceptions of the RME-based Mathematization Approach and the challenges they face during the intervention. During this interview, no need of mentioning your name, and the information you will give form me is only for research purposes and will be encoded to protect your identity and all results will be kept confidential. Saying this, I directly proceed to the first question.

Researcher: For the last three months you have learned plane geometry via an RME-based mathematization approach. In your opinion, do you think that there is a difference between this approach and the approach you were taught in the previous semester and even in the lower school?

C-023: I want to say thank you. To tell you the truth, there is a big difference. I haven't seen such an approach before. I learned mathematics theoretically, like history, or in the form of lectures. In mathematics, we memorize different formulas and use them to solve some application problems without knowing or understanding where the formula itself is derived from. For example, if we consider a triangle, the sum of its interior angles is 180 degrees. For a long time, I considered it as an 'axiom'. I've never confirmed that with a protractor. I have got a lot of things from this approach. In the past, the meaning of mathematics in my mind was different from that of today. But now we have seen different things practically.

Researcher: Did RME-based mathematization approach changes your mathematical epistemological beliefs? what was your previous mathematical epistemological beliefs? Is there any change regarding these beliefs?

C-023: this is right. Regarding the beliefs I have on Mathematics before and now are different. Prior to this, I believed that foreigners or developing countries were imposing something on underdeveloped countries like us in order to numb our minds, and therefore I perceived that mathematics had no significant value for us. But now I have seen the benefit of mathematics in practice and even where we get it. The belief I had in the past was that in order to get mathematical knowledge, we had to go abroad. I assumed there was a hidden secret to gaining this knowledge by a simple method in foreign country.

Researcher: Does this imply that you believe there is no mathematics in our society?

C-023: There is mathematics in the society. But I have no idea about the existence of practical mathematics in the society. I perceive the activities in the society is related to physics, survey and I perceive as if measuring the land, the building and the height of something does not reflect mathematics. But now as we have seen practically for the last three months, everything is inside mathematics and mathematics is in the society and everything is in our hand to prove it. The reason why I thought in such a way, we did not learned mathematics practically rather we learned like history. For example, we did not measure the height of the building using shadow, mirror rather reading problems related to the height of the building. Now I have developed the knowledge to measure the height of the building, the height of the pole, the height of tree without measuring its height using measurement. Even I can use stick, yard, foot and other for measuring different the length of things. In the past all these were out of my sight. But now since I have seen and practiced mathematics within different plane geometry concepts for the last three months my epistemological beliefs are completely changed.

Researcher: From this, what is mathematics for you?

C-023: it is the one which is highly related to the daily activity of the society. It is the one which is not apart from the life every society. It is the one found in the daily activity of human being. It is found in our activity we perform in the 24 hours. It is the culture of the society or it is what is found in the society's daily activity.

Researcher: How does mathematics related with daily activity of the society?

C-023: mathematics is highly interconnected with daily life. For example, abbootin keenymidhaan haammatanii oogditti caalleesu. Dirqama oogditu isaan barbaachisa. Oogdin kun akkamitti qophaa'a gaafa jennu, oogdin kun akkamitti qophaa'a kan jedhu herregni adaa wajjin akka inni itti walqabatu, fakkenyaaf daangaa namaa bira jira yoo ta'e e essatan oogdii godhachuu qabaa midhaan kiyya osoo alatti hin bahin naa callaa'uu qabaa. Abbootin keenya maal godhu muko hordaniitti chikaala hordanii faanan ykn tarkaafitiin bakka waliqixatti safaru bifa geengoo ykn bifa rog-arfee uumaniitu midhaan calleeffatu. kun ammoo beekumsa herregaa isa guddaadha. Furthermore, in house building, in rural aera when the society build their house, they use mathematics. For example, in building the rural house made from straw, first the put the center and then using rope they make a circle and then they dig and build the house. Thus, there is mathematical concepts here, in making door and roof there are mathematics there.

Researcher: for you, where is the source of mathematics?

C-023: mathematics is originated from the daily activity of human being. Mathematics exist naturally.

Researcher: Do we take it as the one exists before or created by God? Or is it the one which is used/practiced by human being in his/her daily activity or the invention of human being?

C-023: It is the one which is created by human being but not exist before. Our parents make 'harqoota fi moofara/gindii' for ploughing. they have their own measures. For example, where do I build my house, what should be its area to accommodate all my family and even how wide be the house of goats/sheep as well as cattle so as to accommodate them, all these are the one human being think and create it.

Researcher: In your opinion does mathematical knowledge evolving? Or it is the one which is fixed permanent and ever changing? Or do you believe that there is mathematical knowledge which evolves over time and there is mathematical knowledge which is not changed over time? Why?

C-023: in my opinion there is mathematical knowledge which evolves over time and there is also mathematical knowledge which is permanent. In my opinion majority of mathematical knowledge is corrigible. Earlier our parents build circular houses but now a days, they build the houses in the form of rectangle, square, and other different polygons. This indicated that they developed the knowledge of how their house is built in different shapes. And this indicates change in mathematical knowledge. The new house has different shapes, has different areas. Thus, in my opinion it is evolving overtime and there is also permanent mathematical knowledge.

Researcher: when we say permanent, there are different mathematical concepts like Pythagoras relation, Euclidian geometry, for example in Euclidean geometry, the degree measure of the interior angles of a triangle is 180 degrees. I think you have learned that in non-Euclidian geometry, the degree measure of the interior angles of a triangle is either less than 180 degrees or greater than 180 degrees. Thus, from this point of view what opinion do you have regarding mathematical knowledge?

C-023: In my opinion, this is happened through time. On prior experience if there is strong or new experience added, then the existing knowledge ca be corrigible.

Researcher: how does the structure of mathematical knowledge be organized? Is it in simple form, in complex, in hierarchical order? Is it fragmented or interconnected?

C-023: in my opinion the structure of mathematical knowledge is in hierarchical order.

Researcher: Do you mean that there is one and only one way to solve a given mathematical problems?

C-023: I mean that the simple knowledge we possess in the previous becomes complex through time. It means that it goes from simple to complex one through process. It does not mean that there is only one way to solve a given problems.

Researcher: In your future career as a teacher, are you going to make students remember different facts, formulas and principles and solve mathematical problems in only one unique way, or are you going to help them solve the problems in different ways and with different strategies?

C-023: In my opinion, students should be encouraged to derive a formula or construct mathematical concepts but not tell them and let them memorize the formula. They should have to construct new mathematical knowledge by their own efforts with close assistance of their me and I will let them derive /construct mathematical formula/concepts with their different strategies. The solution set of given problem can be computed in many different ways.

Researcher: Does a given problem have different solutions?

C-023: One problem may have one solution or many solutions. It depends on the problem. If it is open, it has many solutions, for example $x + y = 9$ has many solutions and the students can approach it in different ways. if it is closed like $x + 3 = 7$, has one solution however the way students approach it may vary.

Researcher: What is your opinion about mathematics after you have been learned geometry using RME-based mathematization approach?

C-023: After we have learned geometry practically, the opinion I have on mathematics has been changed. My attitude towards mathematics is also changed. Furthermore, my understanding as well as my thinking on how to solve a given problem are also changed. For example, while we were dealing with the diagonals of polygons, how to find the diagonal of constructed from a given vertex or the total number of diagonals constructed from all vertices, from the first and the second vertices we can connect to all non-adjacent vertices, however, in the third vertex there is one overlap. In so doing there is two overlaps in the fourth vertex, three overlaps in the fifth vertex and sixth vertices for six-sided polygon and hence from the last two vertices (v5 and v6) there is diagonals formed since all are overlapped. For any n-sided polygon, what I observed that from the first two vertices we can construct ‘n-3’ vertices and as the result of overlaps there are no vertices on the last two vertices. In the past I except memorizing the formula $n(n-3)/2$ of the total number of diagonals constructed from an n-vertices I no idea where this formula come from. Hence this approach helps me on how to derive this formula. Furthermore, it helps me to think deeply and critically.

Researcher: From this onwards, in your opinion where is the source of mathematical knowledge? Is teacher the source of mathematical knowledge or is it a student who him/herself construct mathematical knowledge?

C-023: As to me I did not consider textbook as a source of mathematical knowledge since text book is written by an individual. Thus, the source of mathematical knowledge is an individual person.

Researcher: is a teacher?

C-023: It may not be a teacher. Since teacher develop such profession after a series of learning. And hence mathematical knowledge is the common commodity of the society at large. Mathematical knowledge is the one individual create.

Researcher: In the teaching learning process, is it a teacher who transmit mathematical knowledge or a student who construct mathematical concepts by him/herself?

C-023: Teacher is a facilitator. We all know that without the guidance of teacher, it is difficult for the student to construct mathematical knowledge. He can assist us to construct mathematical knowledge. Thus, it the student who construct mathematical knowledge with close assistance of the teacher.

Researcher: Have you got some benefits from the way you have learned for the last three months? Particularly, regarding on the development of your problem-solving skills, on change of your mathematical epistemological beliefs, and even your improvement in your mathematics achievement.

C-023: To tell you the truth, I have improved my mathematical epistemological beliefs. even if it might be easy for other yet I had a challenge with the concept of triangle. How it becomes equilateral triangle, isosceles triangle, I had only knowhow on the sides. For example, In RME-based mathematization, the first problem is about the sides of the triangle. A carpenter has a rope with 24m long. With this rope he wants to construct a triangle with sides 12m, 8m and with the remaining one. Can he construct the triangle with such length? Of all students I was the first students who said ‘the carpenter can construct a triangle with such sides. What I knew is that any triangle has three sides. I was the first who argued that he can construct the triangle with such sides. But in during field work I disprove my argumentation that it is impossible to construct triangle with such lengths. I will never ever forget such experimentation since I have done it practically. Thus, it helps me in changing my mathematical epistemological beliefs.

Researcher: Does this approach helps you to improve your problem-solving skills and your creativity?

C-023: Yes. this approach helped me to enhance my problem-solving skills. For example, to measure the height of a building or a tree it does need to measure the building or the tree directly but rather I can use shadow and mirror easily. Thus, I can solve difficult problems easily and this the as a result of this approach. Regarding achievement, to solve different

problems we have done so many things practically like cutting angles, measuring angles using protractor, by putting one triangle on the other so as to see the congruency of sides and angles; by folding paper to know whether the side or the diagonals overlap and so many. Thus, all these things helped to easily understand different mathematical concepts

and therefore, I hope I will improve my achievement.

Researcher: From this approach which one impresses you more? Why?

C-023: What impresses me more in this approach is that doing mathematics practically since we have learned in a practical manner.

Researcher: Do you want other pre-service teachers or students to learn is this approach? Why?

C-023: If possible, I recommend others to learn not only mathematics but also other subjects in this approach since this approach take in to account the real-life of the society and use contextual problems as a starting point for learning.

Researcher: In the future, teaching learning process based mathematization approach is promising? After graduation, are going to use this approach in your teaching learning process?

C-023: Yes of course. It is very promising approach since in such approach teaching learning process is practical. For example, there are mathematical knowledges found in the society like the mathematical concepts in the local material called ‘gulilat’ which put on the straw roof. This is unquestionable since it is a practical science and therefore, I will use it to teach my students. Teaching-learning mathematics should be practical in order for the students to construct mathematical concepts by themselves with close assistance of their teacher.

Researcher: In the teaching learning process carried out for the last three months, have you faced some challenges? If so, would you mention some of them? In your future career, would you mention some of the obstacles that hider in implementing this approach in you teaching learning process?

C-023: There may be a shortage of protractors and rulers, meters. Furthermore, interest of the students may be one challenge since this approach is new for them but not practiced. There are always challenge when new thing is implemented.

Researcher: Can we consider time as well as curriculum itself as one challenge?

C-023: As to me time is not considered to be an obstacle to implement such approach. Just like that of biology and chemistry, there should be a lab session for mathematics. If independent time is given for lab session, it does not take time. Otherwise, time may be considered as one obstacle. The way curriculum is prepared may also considered to be one obstacle.

Researcher: In your future career, what are those activities expected from you to fully implement this approach?

C-023: I have to facilitate all things for the students to do mathematics practically.

C-029

Interview on the perception of RME-based mathematization

Researcher name: Girma

Name of the interviewee:Gudde (**C-029**)

Occupation: Pre-service mathematics teacher

Level of education: Degree program

Department: Mathematics

Name of the institution: Robe CTE

Location of the interview: Education department, Room10

Date of the interview and time: 21, June 2023,6:09-6:44 pm

Researcher: Thank you for your invaluable information. The ultimate goal of this interview is to gather information regarding pre-service teachers' perceptions of the RME-based

Mathematization Approach and the challenges they face during the intervention. During this interview, no need of mentioning your name, and the information you will give form me is only for research purposes and will be encoded to protect your identity and all results will be kept confidential. Saying this, I directly proceed to the first question.

Researcher: For the last three months you have learned plane geometry via an RME-based mathematization approach. In your opinion, do you think that there is a difference between this approach and the approach you were taught in the previous semester and even in the lower school?

C-029: I want to say thank you. There isa big difference. The way we learned before is not learner-centered. It was the teacher who transmit knowledge. The teacher shared the knowledge in his mind yet we were not given an opportunity to research, think, practice by ourselves. However, the way we have been learning for the last three months are learner-centered since the role of the teacher is to give only direction and facilitate conditions. We ourselves have been practicing and constructing mathematical concepts.

Researcher: Does this change your mathematical epistemological beliefs? If so, how?

C-029: Yes, definitely. In the previous time I was studying books just for memorizing different formulas. However, it was not stay with me for a long time, rather I forgot them. The reason why I forgot was that I did not give due attention and dig out the secret behind the formula and did not think from where and how it was derived rather than memorizing. However, with these few months, I have got a lot of things. With in these few months, we have got an opportunity to construct mathematical concepts by ourselves and tried to analyze from where and we have derived different formulas. We have derived formulas and construct mathematical concepts with our effort and with close assistance of our teacher. Since it was me who derive formulas and construct mathematical concepts, they will stay a long time with me.

Researcher: What was your previous Mathematical epistemological beliefs? And bout now?

C-029: In the previous time I was passive listener. We did not get an opportunity to create/ construct different mathematical knowledge. As a result, what we memorized did not stay with us for a long period of time rather we memorize for a short period of time. Yet, for the last three months the way we have learned give us an opportunity to practice, research, argue, criticize and evaluate and it initiate us even to create different things and formulas.

Researcher: Taking this into consideration, what is mathematics for you?

C-029: mathematics is the basis for everything. Mathematics is in our daily life. If we give due attention, everything is mathematics. Therefore, for me, mathematics is my daily life. There no thing which is out of mathematics. If we go to rural areas, we our family practice in their daily activity are full of mathematics.

Researcher: Is mathematics related to daily life? How?

C-029: As I mentioned earlier, mathematics is found in all things. Particularly, in different skills practiced by our societies like making jewelries, house construction and many others are full of mathematics whether they have/have not an awareness on the existence of mathematics in their activity. if we deeply see it what our society have done in their daily activity is full of mathematics.

Researcher: Taking this into consideration, for you where is the source of mathematical knowledge?

C-029: For me if everyone of us gives due attention, the source of mathematics is everywhere. For example, our curriculum is teacher centered. Many people even in our college, in our class, we are from different areas and different background, different culture and hence have different knowledge. If our curriculum is learner-centered, everyone has an opportunity to think critically and relate what he/she has learned with his/her environment. And every one will come with different mathematical knowledge and share among with his/her classroom. And we can learn many things from each other. If the role of the teacher is simple to guide and assist, student not only learn by him/herself but also create many things.

Researcher: So where is the source of mathematical knowledge?

C-029: The source of mathematical knowledge is the society.

Researcher: can't we consider teacher, books, videos and internet as a source of knowledge?

C-029: Of course, yes, we can get mathematical knowledge from our teacher since he/she guide and give direction. Furthermore, he/she can facilitate different conditions. Therefore, teacher has a significant role. The same is true in case books internet and videos.

Researcher: Is mathematical knowledge develop from time to time or everlasting? Why?

C-029: Mathematical knowledge evolves from time to time. For example, my friend in Asella created a formula and therefore, it evolves from time to time.

Researcher: Is there mathematical knowledge that is everlasting?

C-029: may be. Yet I think it is corrigible since technology become advancing. For example, 20 century and the twenty first century are not the same. As the world changes, everything will be in dynamism. Today's technology is not the same as the yesterday one and tomorrow's technology become more and more advance.

Researcher: Let us consider, Euclidian geometry, there are around five axioms. These axioms were true yesterday and they are true today, they may be true tomorrow. Do you think that these axioms will be corrigible?

C-029: If someone study it deeply, there may be changes. All these axioms were formulated by human being and therefore, other can improve it or disprove these axioms. Even if I myself carry out deep investigation/study/, I may improve them. They are not naturally existing rather they were formulated by human being.

Researcher: In your opinion, how the structure of mathematical knowledge is organized? Are they organized in simple form, complex, hierarchical, or interconnected?

C-029: The structure of mathematical knowledge both simple, complex, hierarchical and interconnected (beekumsi caasaa herregaa waluma keessa). Some are simple, some are complex and some other are hierarchical, and others are interconnected. But most of the time what I have seen is hierarchical.

Researcher: Suppose you are given a realistic problem; do you think that there is one approach to solve the problem or there are different strategies since different people have different strategies?

C-029: different people may have different strategies to solve the problem. For example, in our calculus class we have got an opportunity to solve problems with different approach so as to get solution.

Researcher: After you have learned plane geometry using RME-based mathematization, is there any improvement in your perception regarding mathematics?

C-029: The approach needs extra time. If the course we have been learning is only plane geometry, I assure you that I will bring a very significant change. For example, I was surprised with some problems, particularly the problem regarding the width of the river, I argued that it is impossible to do in such a way. Even after I depart from my teacher, I was thinking over it through out night, how we can find out the width by measuring only one side of the river, without crossing to the other side and measuring it. Yet after I have seen it practically, I talk to myself that it is possible and nothing is impossible and I have got confidence on mathematics. There are many people who blame mathematics, this is due to the fear they have towards mathematics. But I get confidence.

Researcher: for the last three months you have been taught using RME-based mathematization? Have you got benefit from this approach regarding problem solving skills, mathematical epistemological beliefs and mathematics achievement?

C-029: Yes of course. Even it helps me not only to improve my problem-solving skills, but also to solve my real-life problem I will encounter in my life. It also helps me to improve my critical thinking skills and obsa akkan horadhu na gargaaree jira. Jireenya keenya kessattillee osoo rakkoon tokko nu mudatee, waan sana dafanii murtessuu osoo hin taane, for example if we consider the width of the river, we were arguing that it was impossible. But we practically did it and confirmed that it is possible.

Researcher: How about your mathematical communication skills?

C-029: Yes, I have got many things regarding this area. In our previous time the place we gave for different mathematical symbols, diagrams, graphs and other was not that much and we did not use them in the appropriate place and time. However, from this approach I have learned a lot and improved my mathematical communication skills. for example, we used equal sign and the bar line on the letters to indicate, the length of the line segment which was not common. I have also improved even how to draw different diagrams like circle quadrilateral polygons and others.

Researcher: from RME-based mathematization, what impresses you more? Why? Do you want other students to learn different course in such manner? In your future career are you going to use such approach?

C-029: The way we were taught did not bring changes since it was not learner-centered. The learner is passive listener and there were no practical activity and teacher is considered to be the source of knowledge. What this make different is that this approach gives an opportunity to do mathematics practical, to give them a chance to create things. Even the problems themselves are constructed from the daily activity of our society and these help us as to know more. If others even our country use this approach, I think we will proceed to changes. Itti yaada, ni kalaqqa, waan naannoo isaa jiru waliin walqabsiisa. What are those things in my environment, and one can ask himself and create new thing. In this situation the role of the students is that he/she is the one who finds everything by himself as a consequence he can develop self-confidence. However, if it is teacher- centered, the role the student is passive listener.

Researcher: in your opinion what are the challenges that affect successful implementation of RME-based mathematization?

C-029: It need different mathematical materials. and these may be linked with budget constraints.

Researcher: can't we get them from our society?

C-029: of course, we can get from our environment, yet, we can not get materials like meters.

Researcher: can we consider time as a challenge? What about the way curriculum is prepared?

C-029: regarding time, it is obvious that it takes long time since it need critical thinking and it is practical and needs discussion. It asks patience. The way the curriculum is prepared is one factor. If we consider, our module, the objective stated in it were demarked and you cannot go beyond. If it was open and enough time is given there will be great change.

Researcher: In your future, career, what is expected from you to implement this approach in you teaching learning process?

C-029: to make teaching learning practical by using different materials and I will do what we have done for the last three months.

INFORMED CONSENT FORM

Dear ________

This study entitled " *Realistic Mathematics Education-based Mathematization, Epistemological Beliefs, Problem-Solving Skills and Geometry Achievement of Pre-Service Mathematics Teachers* " is being conducted to understand how daily activities are mathematized. You are thus invited to participate in the study. I will ask questions that are helpful to understanding the daily mathematical practices that take place in your activities. I have the plan to make repeated visits to your site, and conduct one-on-one audio-taped interviews with you; make observations while you are working in your everyday routine activities and audio/video record the practices; have copies and pictures of some of your products and documents, if any, and following these visits I will record field notes. Your information will be encoded to protect your identity and all results will be kept confidential. Data will be stored securely and will be made available only to persons conducting the study, unless participants specifically give permission in writing to do otherwise. No reference will be made in oral or written reports which could link the participants to the study, but aggregated results will be used to fulfill an educational requirement such dissertation, publication in a professional journal, or presentation at a professional meeting without any identifier of the participants. Participation is completely voluntary. You can withdraw at any time and if you withdraw from the study before data collection is completed, your data will be destroyed. Your decision about whether or not to participate will not jeopardize your future relations in any form. If you have any questions at any time about the study or the procedures, or if you experience any negative effects as a result of participating in this study, you may contact the researcher, Girma Tessema via (251) 913513199, or the supervisor at (251) 11220767 and (251) 926765528. If you have questions regarding your rights as a participant, contact the Office of the Department of Science and Mathematics Education at (251) 111220767. I would like you to confirm your participation by putting your signature.

Confirmation

I have read or listened the above information and agree to participate in this study.

Name: ________________________________

Signature: _____________________________

Date: _________________________________

Addis Ababa University

College of Education and Behavioral Studies

Department of Science and Mathematics Education

Semi-structured Interview Questions

Instruction:

I. Introduction Questions

1. Saala: _____________ Umrii: ______ Afaan dhalootaa __________

2. Mee waa'ee barnoota kee yaada nuuf qood.

3. Mee waa'ee hojii kee, muuxxaanoo, eenyuu irraa akka barte, akkamitti akka eegaltuufi akkamitti akka hojjattu nuuf ibsi.

II. Meeshaalee aadaa qopheessuu qoodaa fudhachuu

a) Hojii kee guyyaa guyyaan akkamitti hojjata?

b) Hojii kana akkamiin eegaalta? Akkamiin hojjatta? Adeemsa mataa isaa qabaa?

c) Hojii kana hojjachuuf meeshaalee beekumsaafi ogummaa akkamiittu si barbaachisa?

d) Meeshaa akkamii hojjachuuf?

e) Meeshaa aadaa kana hojjachuuf wantoota akkamii fi hanga sibarbaachisu akkamiin beekta? Yeroo hangamii sitti fudhata? Maaliif?

f) Hanga(guddina) meeshichaa akkamitti akka beektu tooftaa akkamii akka gargaaramtu nutti himuu dandeessaa?

g) Maaliif akkaataa Kanaan hojjatamuun barbaachise? Akkaataa birootin osoo hojjatamee maaltu ta'a laata?

NB. There will be follow-up questions for each round of visit based on the information gathered previously.

Addis Ababa University

College of Social Science and Behavioral Studies

Department of Science and Mathematics Education

BargaafileeAmantaa Falaasama Herregaa irratti jiru (Mathematical Epistemological Beliefs Questionnaire)

Mata duree qoranichaa "**Realistic Mathematics Education-based Mathematization, Epistemological Beliefs, Problem-Solving Skills and Geometry Achievement of Pre-Service Mathematics Teachers**" yoo ta'u kaayyoon isaas haawwasni hojiilee idilee isaanii yeroo gaggeessan akkaataa itti herreganiifi herreega gochaalee sana keessa jiru baasuu; Akkasumaas isaan bahan kana kolleejjii barnootaatti fiduun dhiibbaa inni gama amanta barsiisonni hojiin duraa herrega irratti qaban, ogummaa rakkoo hiikuu isaan qabaniifi qabxii herregaa isaanii irratti dhiibbaa inni fiduu malu ilaaluudha).

Qajeelfama: Himamootin armaan gadii kun amantaa barsisootni hojiin duraa qaban kan sakkata'uudha. Kanaaf tokko tokkoo himaaf filannoowwan afur siif kennaman keessaa yaada kiyya naaf ibsa kan jettu (√) barreessuun agarsiisi.

Hubadhu! Deebiin sirrii ykn dogoggora jedhamu hinjiru).

Maqaa: ______________ LWE. _____________Koorniyaa. ______Umrii_____

1- Gonkuma itti walii hingalu, 2- Itti walii hingalu, 3 –Ittin walii gala, 4-Sirriittan itti walii gala

La k.	Himoota	1	2	3	4
1	Beekumsi herregaa miira, nama ykn hawaasa gidduu hingaleeffatu				
2	Daree herregaa keessatti kalaqniifi yaadun iddoo hinqabu.				
3	Yaad-rimeewwan herregaa uumamuu ilma namaatin duraayyuu nijiru.				
4	Yaad-rimeewwan herregaa kan namni uumuu osoo hintaane kan argamaniidha				
5	Daree herregaa keessatti beekumsi herregaa barsiisaa irraa gara barattootaatti kan daddarbuudha.				
6	Beekumsi herregaa muuxannoo barattoonni duraan qaban irraa kan ijaaramudha.				

7	Herrega hojjachuun kalaqaafi itti yaaduu gaafata				
8	Beekumsi herregaa kan ijaaramu adeemsa waliitti dhufeenya hawaasaatiin.				
9	Beekumsi herregaa faaya aadaati.				
10	Beekumsi herregaa kan argamu pirobileemota qabatamaa haala garagaratin hojjachuudhaani.				
11	Beekumsi herregaa baay'ee tasgabbaa'aa, kanhinjijjiramneefi dhaabbataa akka ta'eetti fudhatama.				
12	Seerotni/ amalootni/ herregaa yeroo hunda dhugaadha.				
13	Mirkanni herregaafi yaad-rimeewwan isaa akka dogoggora irraa bilisa akka ta'eefi kan dhumaa akka ta'etti ilaalama.				
14	Dhugaawwan bu'uuraa herregaa yomiyyuu kan hinjijjiiramneedha.				
15	Bu'aaleen beekumsa herregaa yeroofi iddoo irratti hundaa'a.				
16	Beekumsi herregaa karaa mirkanaafi yaad-rimee isaatin fooyya'iinsaaf banaadha.				
17	Beekumsi herregaa kan ijaaramu adeemsa keessa.				
18	Beekumsi herregaa turti yeroo keessa kan jijjiiramuudha.				
19	Herregni caaseffama tartiiba'aa addaafi dhaabbataa ta'e qaba.				
20	Herregni sirriitti kan baratamu bifa tartiiba'aa ta'een yoo qindaa'eedha.				
21	Caasefamni beekumsa herregaa salphaafi yaadolee bittinnaa'oo ta'eedha.				
22	Herregni walitti qabama haqawwan, seerotafi ogummaalee garagaraati.				
23	Beekumsa herregaa argachuuf dur-duubeen hinbarbaachisu				
24	Herregni kan baratamu qabu walitti hidhamiinsa qabiyyeewwan walitti dhufoo ta'een.				
25	Beekumsi herregaa walxaxaafi kan walutubeedha.				
26	Herregni netwoorkii walitti hidhamiinsa yaad-rimeewwaaniifi ogumoota garagaraati.				

Addis Ababa University

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Department of Science and Mathematics Education

Battallee Gahumsa Herregaa (Geometry Achievement Test)

Gaafilee kun Qorannoo digrii 3ffaaf mata dureen isaa "**Realistic Mathematics Education-based Mathematization, Epistemological Beliefs, Problem-Solving Skills and Geometry Achievement of Pre-Service Mathematics Teachers**" jedhu irratti barsiisota hojiin dura kan sadarkaa digirii jalqabatu irratti hirmaataniif qophaa'e

Odeeffannoo Dhuunfaa

LWE__________ Saala_______ Umrii_________

Ajaja: Baattallee kun filannoo afur qabu. Filanoowwan afran siif kennaman keessaa isa sirrii jettu filachuun iddoo deebii siif kenname irratti barreesi.

1. ΔDEF∼ΔABC yoo ta'eefi DE= 10cm, AB=12cm, BC=18cm yoo ta'an, EF=_______ta'a.

A. 5cm
C. 36cm
B. 15cm
D. 30cm

2. Rog-sadeen ΔTUV keessatti yoo s∠(UTV) = 10x°− 9°, s(∠TUV) = 7x°− 1° fi $s(\angle TVU)= s\ (TUV)-49\ ^{o}$ ta'an, ΔTUV rog-sadee akkamiiti?

A. Rog-sadee kofa sirrii
C. Rog-sadee Akiyuutii
B. Rog-sadee Obtiyusii
D. Deeebihinqabu

3. Tuqaaleen D (-7,1), A (-1,7), R (-1, -5) varteeksota ΔDAR yoo ta'an, ΔDAR ___ta'a.

A. Rog-sadee Iskeelenii
C. Rog-sadee Ikulataralii
B. Rog-sadee Ayisosilasii
D. B fi C

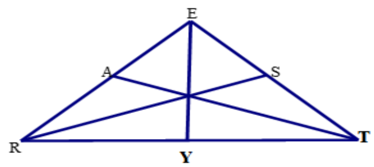
4. Yoo sarbiiwwan rombasii 18cm fi 24cm ta'an, naannaww fi balinni rombasichaa wal- duraa duuban_____fi ___________ta'u.

A. 216cm fi 60$cm2$
C. 84cm fi 432$cm2$
B. 60cm fi 216$cm2$
D. 60cm fi 432$cm2$

5. Rog-sadee ΔRTE armaan gadii irratti, RY=(2b+13)cm, YT=(b+26)cm, $s(\angle RT$A) =3yo−21o, $s(\angle ATE)$ =2yo−8o fi $s(\angle RST)$=4xo−10o, yoo $R\overline{S}$, $T\overline{A}$ fi $E\overline{Y}$ duraa duuban iiroo (altitude), qixa qoodaa kofaa (angle bisector) miidiyaanii (median) ΔRTE ta'an,kan armaan gadii kessaa kamtu soba?

A. x=20
C. b= 13
B. y= 13

D. YT= 39cm

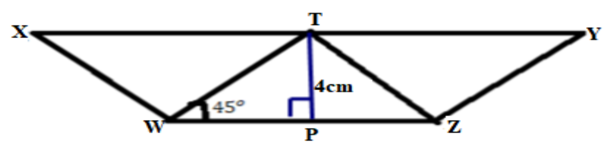


6. Rog-arfee XYZW keessatti, M fi N tuqaalee $\overline{XW}$ $irra\ jiran\ yoo\ ta'aniifi$ $\overline{XY}\cong\overline{MZ}$, $\overline{YN}\cong\overline{ZW}$ fi $\overline{XM}\cong\overline{NM}$ ta’an kan armaan gadii keessaa kamtu soba ta’a?

   A. $\angle YXN\cong\angle ZMW$ C. $\angle XNY\cong\angle MZW$

   B. ΔXYN ≅ ΔMZW D. $\overline{XN}\cong\overline{MW}$

7. Tiraapiziyemii XYZW armaan gadii irratti, ΔTWZ rog−sadee ayisosilasii hundeen isaa $Z\overline{W}ta'eedha$. Yoo $\angle WXT\cong\angle ZYT$, $\angle XTW\cong\angle YTZ$, XT = (0.75$a$−5)$m$fi YT=(0.5a+3)cm fi $T\overline{P}\perp W\overline{Z}$ ta’an, bal’inni tiraapiziyeemiicha meeqa ta’a?



A. 32cm2 D. 92cm2

B. 46cm2

C. 38cm2

8. Tuqaaleen A (−6, 4), B (4, 12), C (−6,12) $varteeksota$ ΔABC yoo ta’an, bal’inni rog-sadichachaa yuunitii iskuweeriidhaan meeqa ta’a?

   A. 40 C. 82

   B. 80 D. 164

9. Kanneen armaan gadii keessaa kamtu yeroo hunda dhugaa ta’a.

   A. Rog-sadeen ikulaataraalii lama kamuu wal-fakkaatodha

   B. Rog-sadeen lama kamuu wal-fakkaatodha.

   C. Rog-sadeen ayisoosilasii lama kamuu wal-fakkaatodha

   D. Rog-sadeen kofa sirri lama kamuu wal-fakkaatodha.

10. A (−6, −5), B (4, −5), C (−6, −8), fi D (4, −8) varteeksota rog-arfee yoo ta’an, gosti rog-arfichaa fi bal’inni isaa duraa duuban______.

   A. Rektaangilii, bal’inni iskuweerii yuunitii 30

   B. Parallelogiramii,bal’inni iskuweerii yuunitii 30

C. Iskuwerii, bal'inni iskuweerii yuunitii 30

D. Rombasii, bal'inni iskuweerii yuunitii 30

11. Kanneen armann gadii keessaa rog-baayyee sirnaawaa kan **hin ta'iin** isa kami?

A. Reektaangilii C. Iskuweerii

B. Rog-sadee ikulaateraalii D. B fi C

12. Waa'ee paaraalelogiramii kan soba ta'e isa kam?

A. Rogni walii fullee walitti galoodha.

B. Sarbiiwwan isaa walitti galoodha.

C. Kofti walitti aanaan lamaa hirkoodha.

D. Kofti walii fuullee walitti galoodha.

13. Kamtu dhugaa ta'a?

A. Reektaangilii hundi iskuweeridha.

B. Paaraalelogiramii hundi reektangiliidha

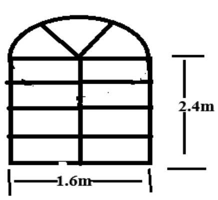
C. Rombasii hundi paaraalelogiraamidha.

D. Paaraalelogiraamii hundi iskuweeridha.

14. Fooddaan mana obbo Obsaa akkaataa fakkii armaan gadii kannatti yoo ijaarameefi gubbaan isaa walakkaa geengoo yoo ta'e, ball'inni fooddichaa ________ta'a.

A. $0.64\pi m2+3.84m2$ C. $0.32\pi m2+3.84m2$

B. $0.64m2+3.84\pi m2$ D. $0.32m2+3.84\pi m2$

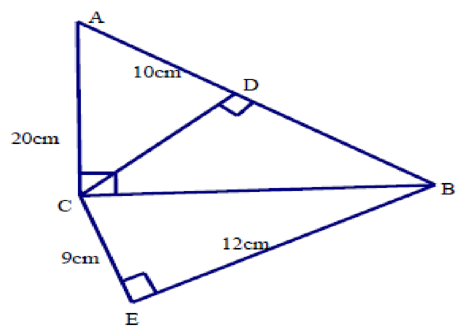


15. Rog-arfeewwan armaan gadii keessaa amaloota parallelogiramii **hunda kan hin qabne** kam?

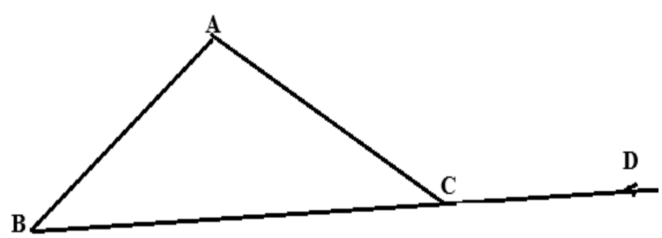
A. Iskuwerii B. Rombasii C. Tirapizemi D. Reektaangilii

16. Danaa armaan gadii irratti, yoo AC= 20cm, BE= 12cm, CE= 9cm fi AD= 10cm ta'an, dheerinni AB=__________

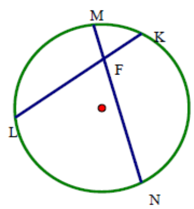
A. 15cm B. 50cm C. 25cm D. 30cm



17. Paaraaleelogramii ABCD irratti, yoo AB=3x−4, BC=2x+7 fi CD=x+18 ta'an, Paaraaleelogramii ABCD
    A. Reektaangilii
    B. Roombosii
    C. Tiraapiziyeemii
    D. Rog-arfeen gosa kanaa hinjiru
18. Dheerinni rogoota rog-sadee 21, 27cmfi xcm yoo yoo ta'an, filannoowwan armaan gadii keessaa isa kamtu gatii $'x'$ hunda hammachuu danda'a?
    A. 21 < x < 27 B. x < 6 C. 6 < x < 48 D. x > 48
19. Danaa armaan gadii keessatti yoo $(\angle A)$ = 3x°, $m(\angle B)$ = 2x°, fi s∠ (ACD) digirii soddomaan (30 $_o$) kofa qajeelaa (straight angle) gadi yoo ta'e, rog-sadeen kun rog-sadee akkamii ta'a?
    A. Rog-sadee Akiyuutii
    B. Rog-sadee kofa sirri
    C. Rog-sadee obtiyusii
    D. .Rog-sadee ikulataralii



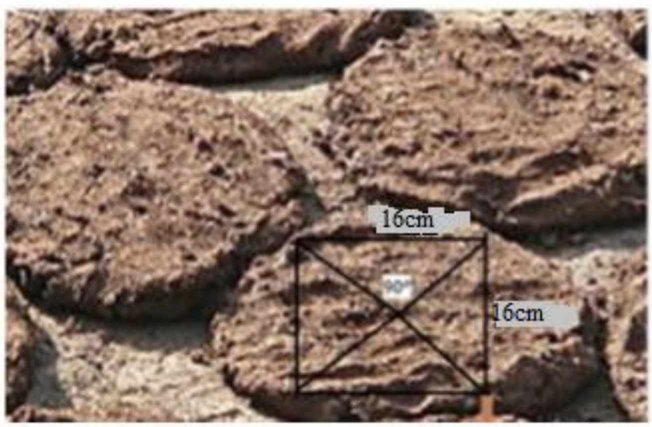
20. Danaa armaan gadii irratti, yoo $s(\widehat{LN}) = 90_o, s\ (\widehat{KM}\ ) = 60_o\ ta'an,$
    $(\angle LFN)$ = ___ta'a.
    A. 30^o B.15^oC. 75^o D. 150 ^{o}

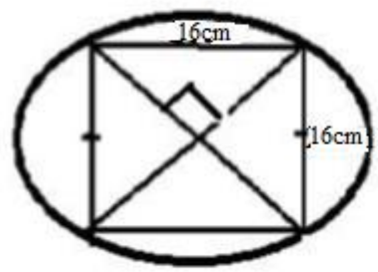


21. Fakki armaan gadii irratti inni mul'atu kun koboota dhoqqee horii irraa hojjetamuudha Diyaameenshiniin isaa haala fakkii irratti mul'atuun yoo siif kennameefi isaan guutuu ta'an walqixa jenne yoo fudhanne, bal'inni koboota sadan waliigalaan meeqa ta'a?

    A. $384(\pi-1)cm^2$ C. $384((\pi-\sqrt{2})cm^2$

    B. $384((\pi+1)cm^2$ D. $384((\pi+\sqrt{2})cm^2$





22. Baayinni sarbii rog-bayyee rogoota 12 qabuu______ ta'a

    A. 48 B. 36 C.54 D.72

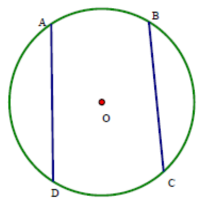
23. Dana armaan gadii kessatti yoo $\overline{AD}$ *fi* $\overline{BC}$ handhuura geengoorra waliqixa fagaatu ta'e, kanneen armaan gadii keessaa kamtu soba ta'a?

    A. $AD \cong BC$

    B. $s(\widehat{AB}) \cong s(\widehat{DC})$

    C. $\Delta AOD \cong \Delta BOC$

    D. $s(\widehat{AD}) \cong s(\widehat{DC})$



24. Yoo kofti alee rog-baay'ee sirnaawaa tokko 30° ta'e, baay'ini rogoota isaa ____.

A. 9 B. 8 C. 12 D. 10

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Department of Teacher Education

Pirobileemota Ji'oomeetirii qabatamaa (Realistic geometry problems)

Qorannoo digrii 3ffaaf mata dureen isaa "**Realistic Mathematics Education-based Mathematization, Epistemological Beliefs, Problem-Solving Skills and Geometry Achievement of Pre-Service Mathematics Teachers**" jedhu irratti barsiisota hojiin duraa kan sadarkaa digirii jalqabaatu irratti hirmaata.

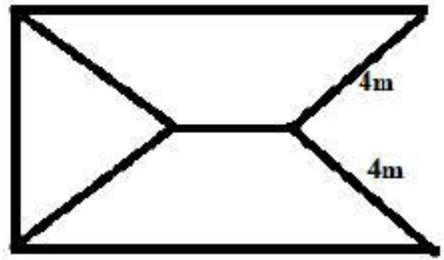
Odeeffannoo Dhuunfaa

LWE__________ Saala_______ Umrii_________

Ajaja: waliigala gaafilee 10tu jira. Tokkoo tokkoo gaafileef deebii ta'uudhaa mulu jette yaaddu adeemsa barbaachisu hunda agarsiisuun dalagi.

1. Qotee bulaan tokko saaxaraa dheerina 36m bite. Ooyiruu isaa kan bifa rog-sadee ta'e saaxaraa kanaa dallaa itti ijaaruu barbaade. Qotee bulaan kun saaxaraa kana 12m,8m fi 4m itti yoo kukkutte, saxaarawwan iddoo saditti qoqqoode kanaan os walirra hin dabarsin osoo walhinhaqisin ooyiruu isaa ijaaruu danda'aa? Maaliif? Osoo si ta'e saaxaraa 36m qabu kana irraa rog-sadee safarrii dheerina rogoota isaa lakkoofsa lakkaawwii ta'e ijaari osoo jedhamtee, rog-sadee meeqa ijaaruu dandeessa?

2. Ogeessi mana ijaaru tokko baaxii manaa boca tiraapiziyeemii ayisoosilasii lamaafi rog-sadee ayisoosilasii lama ijaaruu barbaade. Tirapiziyeemii lamaaniifi rog-sadee tokko erga ijaaree xumuree booda rog-sadee isa lammataatiif muka dheerinni isaa 10m ta'e qofatu lafatti hafeef. Ogeessi kun muka hafeef kana osoo irraa hin muriin itti fayyadamee rog-sadicha hafe xumuruu danda'aa? Yoo deebin kee lakkii jetta ta'e maaliif? Ogeessi kun baaxii mana kanaa ijaaree xumuruuf maal godhuu qaba jetta? Filanoo ogeessichi qabaachuu qabu hamma tokko kennuu

3. A (2,2) fi B (2,8) varteeksota rog-sadee yoo ta’an varteeksii sadaffaa isaa barbaadi yoo rog-sadichi rog-sadee kofa sirrii hayipootinasiin isaa yuunitii 10 ta’e?

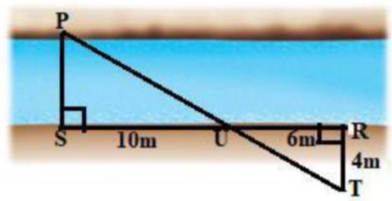
4. Fuullee daree barnootaa Kolleejjii Barnoota Barsiisotaa Roobee iddoo magariisa boca rog-sadee qabutu jira. Iddoo magarisaa kana keessatti kolleejjichi biqiltuu maangoo dhaabuu barbaade. Rog-sadeen kun daandii sadiin kan daangeffameedha. Daandiin tokko kan gara daree barnootaatti geessuudha. Daandin inni biroo immoo kan gara kutaa baay’isuutti geessu yoo ta’u daandiin inni hafee ammoo gara kutaa istriimii afaanii kan geessuudha. Biqiltuun kun daandii sadan irraa walqixa fagaatee argamuu qaba yoo ta’e, biqiltuun kun eessatti dhaabachuu qaba? Maaliif? Hojii barbaachisu hunda agarsiisun mirkaneessi.

5. Ogeessi mana ijaaru tokko osoo gara ijaarsaa hin galin dura boca barbaadu sanaan lafa safarachuu qaba. Bocni inni barbaade rog-arfee cimdiin rogoota walii fuullee isaa walqixaafi fageenya walqixarratti argamaniidha. Sarbiiwwan rog-arfee kanaa walqixa yoo ta’an,

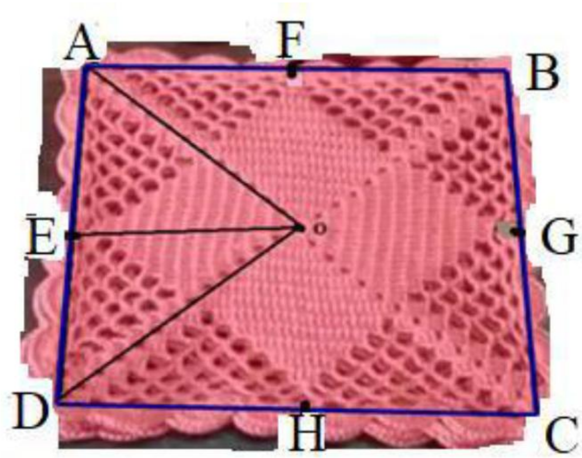
a) Ogeessi kun manni inni ijaaruu barbaade boca rog-arfee isa akkamiiti? Maaliif?

b) Yoo rogootni arfanuu walitti galoo ta’an, rog-arfichi isa boca akkamii qabuudha? Maaliif?

6. Ogeessi tokko laga irratti riqicha ijaaruu barbaade. Dalgee riqichaa argachuuf safartuuwwan fakkii armaan gadii irratti argamu kana argate. Kennama fakkii irra jiru kana bu'uura godhachuun yaad-rimee herregaa akkamii fayyadameeti dalgee lagichaa argachuu danda'a? Dalgeen lagichaa hoo dheerina hangamii qaba? Akkamiin arganna? Hojii barbaachisu agarsiisun dalagi.



7. Faayaan asii gaditti jiru kun boca rog-arfee qaba. Rog-arfeen 'ABCD'irratti yoo AC fi BD qixa qoodaa parpeendikulaarii walii (perpendicular bisector of each other) ta'aniifi $\overline{AC} \cong \overline{BD}$ yoo ta'e,



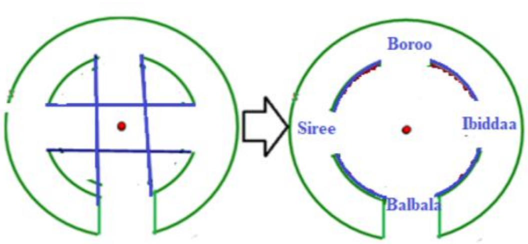
a. ΔDOC ≅ ΔDOA ta'aa? Akkamitti? hojii barbaachisu hunda agarsiisun dalagi.

b. F, G, H fi E walduraa duuban tuqaalee walakkeessaa $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ fi $\overline{DA}$ yoo ta'an, EFGH rog-arfeen akkamiiti? Maaliif? Hojiin agarsiis?

c. ΔDEO rog-sadee akkamiiti? Maaliif? Agarsiisi.

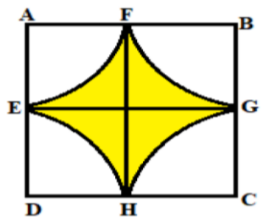
d. Hariiron $\overline{AD}$, $\overline{AE}$ fi $\overline{AO}$ gidduu jiru maal? Akkamitti?

8. Baadiyaa keessatti mana citaa boca siliindarii lama kan handuura tokko waliin qaban ijaaru. Adeemsa ijaarsaa mana citaa kana keessatti keessoo isaa iddoo afuritti qoodu: Iddoo sirree, iddoo ibiddaa, boroo, fi iddoo balbalaa jedhanii qoodu. Kana gochuudhafis iddoo ibiddaafi siree walii fullee, boroofi balballi walii fuullee ta'uu qaba. Kanaaf iddoo sireefi fi kan ibiddaa baasuuf handhura irraa ka'uun faana lama (2feet) handhuraa gadiifi handhuraa ol safaruun muka xixxiqqoo lama dhaabu. Sana booda wadaroon mukkeen dhaaban kana irraa dabarsuun qarqara geengoo keessoo gara qarqara isa biroo ni hidhu akka fakkii jalqabaa armaan gadii kanatti. Isa Balbalaafi kan boroos akkasummatti godhu. Kan kana godhaniif iddoowwan arfan(Iddoo sirree, iddoo ibiddaa, fi iddoon balbalaa) kun hin qotaman muknis keessa hin dhaabatu. Ijaarsa mana citaa kana keessatti iddoowwan afur fakkii lammaffaa irratti argamu kan bifa golbootin jiran kana qotuun muka keessa dhaabu. Mee ati safarri dheerina golboowwan mukkeen keessa dhaabbatan walqixa ta'u jttee yaaddaa? Akkamitti? Mee dura golboowwan waliif fuulee jiran irraa ka'uun itti adeemi.



9. Yaadataan koornisii mana isaa jiipsamii irraa hojachuu barbaada. Walakkaa koornisii irratti boca armaan gadii kana gochuu barbaade. Dheerinni roga

iskuweerii guddaa 1m yoo ta’eefi E, F, G fi H tuqaalee walakkeessoo rogoota iskuweerii isaa guddaa yoo ta’an bal’ina isa dibamee barbaadi.



10. Ogeessi mana ijaaru tokko mana roga afur qabu irratti baaxii manaa boca tiraapiziyeemii ayisoosilasii qabuufi rog-sadee ayisoosilasii qabu ijaare. Safarri dheerina hundeewwan tirapiziyeemii 2m fi 8m. Akkasumas miilli tiraapiziyeemiifi hundeen rog-sadoota walqixaafi 5m yoo ta’an, bal’inni bal’iinsa waliigalaa dirra baaxichaa hangam ta’a?

11. Tolaan mana naannawwaan isaa faana 48tii ijaaruu barbaade. Filannoowwan sadii qaba. Mana boca reektaangilii ta’e, mana boca heeksaagonaalii sirnaawaa ta’eefi mana boca geengoo yoo ta’an, mana bal’aa ijaarrachuuf filannoowwan sadan keessaa isa kam filachuu qaba jettee yaadda? Maaliif? Hojiin agarsiis.