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ON

NON-OSCILLATION CRITERIA ON HALF- LINEAR DIFFERENTIAL EQUATION USING RICCATI TECHNIQUE

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Abstract

In this paper we study some of the non-oscillatory criteria on half- linear differential equation of the form

$$(r(t)\Phi(x^{'}))^{'} + c(t)\Phi(x) = 0 \text{ where } \Phi(x) = |x|^{p-2}.x, p > 1$$

Using modified Riccati technique. In addition to that we are going to see some of the characterization of determining principal solution of half-linear differential equation.

Chapter 1

Introduction

Ordinary differential equation has an important role in solving many problems. This makes them an essential topic in mathematics, science and/or social science. Even though most physical problems involve nonlinear differential equation, it can be possible to reduce to linear and find the solution of different problems by changing nonlinear to linear ones.
Consider the differential equation of the form

$$L[x] := (r(t)\Phi_p(x^{'}))^{'} + c(t)\Phi_p(x) = 0 \tag{1.1}$$

$$\Phi_p(x) = |x|^{p-2}.x, \quad p > 1$$

Where r and c are continuous functions for $t \geq 0$ and $r(t) \geq 0$ is known as *second order half-linear differential equation.* The domain of the operator on the left hand side of (1.1) is defined to be the set of all continuous real-valued functions x defined on I such that x and $r(t)\Phi(x^{'})$ are continuously differentiable on I.

Some special results on qualitative theory of differential equation, with a homogeneous (but not generally additive) solution space first appeared in Bihar papers in 1957-58. Equation of the form (1.1) was perhaps for the first time considered by beesack in 1961 in connection with an extension of hardy inequality. He understood half-linear equations as Euler-lagrange equations . In this paper, Riccati type transformation was introduced. The term Half-linear differential equation, which attracted considerable attention in recent years , introduced by the Hungarian mathematician I Bihari in 1964 and systematically used by A' Elbert who has also substantially contributed to the development of the qualitative theory of Half-linear differential equation[2] . One of the reason for the research in the field of half-linear equation is that variety of physical, biological, and chemical phenomena (like non-Newtonian fluid theory or some model in glaciology) are described by the partial differential equations with the so called p-Laplacian, and this partial differential equation can be reduced under some assumption to the differential equations of the form half-linear.

The reason for this terminology is that the solution space of the differential equation (1.1) has just one half of the properties which characterize linearity, namely homogeneity i.e if x is a solution of (1.1) then λx,$\lambda \in R$, is a solution as well but generally not additive. We suppose that the function r, c are continuous and $r(t) > 0$ in the interval under consideration. However, since we are interested in solution (1.1) in the classical sense (i.e, a solution x of (1.1) is a C^1 function such that $r\Phi(x^{'}) \in C^1$ and satisfies (1.1) in an interval under consideration). If $p = 2$,then (1.1) reduces to linear equation is known as *Sturm- Liouville second order linear differential equation*, which is written of the form

$$(r(t)x^{'})^{'} + c(t)x = 0 \tag{1.2}$$

But the linearity property is lost in the general case $p \neq 2$. And for a given $t_0, x_0, x_1 \in R$, there exists the unique solution of (1.1) satisfying the initial conditions $x(t_0) = x_0, x^{'}(t_0) = x_1$ which is extensible over the whole interval where the function r, c are continuous and $r(t) > 0$. This follows example from the fact that (1.2) can be written as the 2-dimensional first order linear system

$$x^{'} = \frac{1}{r(t)}u, u^{'} = -c(t)x$$

and the linearity (hence the Lipschitz property) of this system implies the unique solvability of (1.2).

Example 1 $u^{''} + \lambda u = 0,\ \ u(0) = u(\pi) = 0$

It is easy to see that in the cases $\lambda = 0$ (general solution $u = c_1 + c_2x$) and $\lambda = -\mu^2 < 0$ (general solution $u = c_1e^{\mu x} + c_2e^{-\mu x}$) ,the boundary value problem does not have a nontrivial solution. In the case $\lambda = \mu^2 > 0$ (general solution $u = c_1cos(\mu x) + c_2sin(\mu x)$ the boundary conditions are satisfied if $c_1 = 0$ and $sin(\mu\pi) = 0$ .Thuse we obtain the

$$Eigenvalues \quad \lambda_n = n^2(n = 1, 2, 3, .)$$

and the corresponding

$$Eigenfunctions \qquad u_n(x) = sin(nx)$$

In the first two cases i.e for $\lambda = 0$ and $\lambda = -\mu^2 < 0$ the equation is non-oscillatory. But in the third case i.e. for $\lambda = \mu^2 > 0$, It is clear that the eigenvalues are uniquely determined by the problem, but that the eigenfunctions are not unique, because for any nonzero constant multiples of eigenfunction, say $a_1 sinx, a_2 sin2x.$, will also are eigenfunctions.

The eigenvaues form an increasing sequence of positive numbers that approaches ∞ ; and the n^{th} function, $sin(nx)$, vanishes at the end points of the interval $[0, \pi]$ and has exactly $n-1$ zeros inside this interval. This implies that, if n tends to infinity then there are infinitely many zeros in the given interval. This means that, this equation would be oscillatory for $\lambda = \mu^2 > 0$.

According to classical Hill-Nehari ,oscillation criteria, special half-linear differential equation (1.2) can be determined by the continuous function r and c . i.e. if $\int_t^\infty r^{-1}(t)dt = \infty$ and $\int_t^\infty c(t)dt = lim_{b\to\infty} \int_0^b c(t)dt < \infty$, then the linear Sturm-liouville differential equation (1.2) is non-oscillatory provided that

$$\lim_{t\to\infty} sup(\int_0^t r^{-1}(s)ds) \int_t^\infty c(t)dt < \frac{1}{4}$$

and

$$\lim_{t\to\infty} inf(\int_0^t r^{-1}(s)ds) \int_t^\infty c(t)dt > \frac{-3}{4}$$

On the other hand, if we rewrite (1.1) into a first order system (substituting $u = r\Phi_p(x^{'})$, since for q which is the conjugate number of p , i.e, $\frac{1}{p} + \frac{1}{q} = 1$ and $\Phi^{-1}(i.e, \Phi_p^{-1}(x) = |x|^{q-1}.sgn\, x = \Phi_q(x)\, for\, \Phi_p(x) = |x|^{p-1}.sgnx)$ is the inverse of Φ we get the system

$$u = r\Phi_p(x^{'})$$

$$\Rightarrow \frac{1}{r(t)}u = \Phi_p(x^{'})$$

$$\Rightarrow \Phi_p^{-1}(\frac{1}{r(t)}u) = x^{'}$$

$$\Rightarrow x^{'} = |r^{-1}(t)u|^{q-1} = r^{1-q}(t)|u|^{q-1}$$

$$\Rightarrow x^{'} = r^{1-q}(t)\Phi_p^{-1}(u), \quad u^{'} = -c(t)\Phi_p(x) \tag{1.3}$$

The right hand side of (1.3) is no longer Lipschitzian in x, u hence the standard existence and uniqueness theorems do not apply directly to this system.

The investigation of solutions of half-linear has attracted considerable attention in the last two decades, and it was shown that solutions of this equation behave in many aspects like those of the Sturm-Liouville differential equation. In particular, the methods of the half-linear oscillation theory are similar to those of the oscillation theory of Sturm-Liouville linear equations and that the Sturmian theory extends similar to half-linear equation (1.1) and can be classified as oscillatory or non-oscillatory according to whether any non trivial solution of (1.2) does or doesn't have infinitely many zeros on any interval of the form $[T, \infty]$.

There are some elementary half-linear differential equation. These are equation with constant coefficients and Euler-type half-linear differential equation. The constant coefficient type of equation (1.1) is when $r(t) \equiv r$ and $c(t) \equiv c$. This can be written in the form

$$(\Phi_p(x^{'}))^{'} + \frac{c}{r}\Phi_p(x) = 0$$

And the other half-linear differential equation type is Euler-type half-linear differential equation which is written of the form

$$(\Phi_p(x^{'}))^{'} + \frac{\gamma}{t^p}\Phi_p(x) = 0$$

where γ is a real constant. Its oscillatory property can be determined by the constant γ (i.e) if $\gamma > \gamma_p := (\frac{p-1}{p})^p$, then the equation is oscillatory and non-oscillatory in the opposite case. This constant is known as oscillation constant. The value t^{-p} is a border line between oscillation and non-oscillation in the half-linear oscillation theory. More precisely, if $r(t) \equiv 1$ in (1.1), by Kneser-type (non)oscillation criteria: This equation is oscillatory provided

$$\lim_{t\to\infty} inf\, t^p c(t) > \gamma_p$$

and non-oscillatory if ,

$$\lim_{t\to\infty} sup\, t^p c(t) < \gamma_p$$

These two formulas show that "borderline" means.

There are some criteria of non-oscillation of equation (1.1) with respect to the continuous functions r and c where $r(t) \geq 0$ in the interval under consideration, which is an extension of Hill-Nehari criterion in the linear case.[2] i.e. if $\int_t^\infty r^{1-q}(t)dt = \infty$ and $\int_t^\infty c(t)dt = lim_{b\to\infty} \int_0^b c(t)dt$ converges, then equation (1.1) is non-oscillatory provided that

$$\lim_{t\to\infty} sup(\int_0^t r^{1-q}(t)dt)^{p-1}(\int_t^\infty c(t)dt) < \frac{1}{p}(\frac{p-1}{p})^{p-1}$$

and

$$\lim_{t\to\infty} inf(\int_0^t r^{1-q}(t)dt)^{p-1}(\int_t^\infty c(t)dt) > -\frac{2p-1}{p}(\frac{p-1}{p})^{p-1}$$

If the integral $\int_t^\infty r^{1-q}(t)dt < \infty$, then the above statement can be modified as follows. i.e. equation (1.1) is non-oscillatory provided that

$$\lim_{t\to\infty} sup(\int_t^\infty r^{1-q}(t)dt)^{p-1}(\int_0^t c(t)dt) < \frac{1}{p}(\frac{p-1}{p})^{p-1}$$

and

$$\lim_{t\to\infty} inf(\int_t^\infty r^{1-q}(t)dt)^{p-1}(\int_0^t c(t)dt) > -\frac{2p-1}{p}(\frac{p-1}{p})^{p-1}$$

The aim of this paper is to study some of the non-oscillatory criteria for half-linear equations by using the modified Riccati technique and show some properties of the so called principal solution of (1.1).
Through out the paper we suppose that $h(t) \in C^1([t_0,\infty),(0,\infty))$ is a positive function such that $h^{'}(t)$ has no zero in some neighborhood of infinity. Given equation (1.1) and the function $h(t)$, and define

$$G(t) = r(t)h(t)\Phi_p(h^{'}(t))$$

The paper is organized as follows In the next section we introduce basic facts related to the modified Riccati technique and derive estimates to the non linear term in modified Riccati equation. Chapter 3 a short introduction to the principal solution and criterion which allows us to detect a solution of non-oscillatory equation as a principal. And the last, chapter show some non-oscillation criteria of half linear differential equation by using modified Riccati technique.

Chapter 2

Preliminaries

In this chapter we concentrate on different definitions, theorems and lemmas which are important in order to apply modified Riccati technique to determine Half-linear differential equation is non-oscillatory.

2.1 Modified Riccati Equation

Let $x(t)$ be a solution of equation (1.1) such that $x(t) \neq 0$ in an interval I then

$$w(t) = r(t)\Phi_p(\tfrac{x^{'}(t)}{x(t)})$$

Is a solution of the Riccati type differential equation (or the generalized Riccati differential equation)

$$R[w] := w^{'} + c(t) + (p-1)r^{1-q}(t)|w|^q = 0 \tag{2.1}$$

where q is the conjugate number of p , i.e $q = \frac{p}{p-1}$,indeed , in view of (1.1) we have

$$\begin{aligned}
w^{'} &= \frac{(r\Phi_p(x^{'}))^{'}\Phi_p(x) - (p-1)r\Phi_p(x^{'})|x|^{p-2}x^{'}}{\Phi_p^2(x)} \\
&= \frac{(r\Phi_p(x^{'}))^{'}\Phi_p(x)}{\Phi_p^2(x)} - \frac{(p-1)r\Phi_p(x^{'})|x|^{p-1}x^{-1}x^{'}}{\Phi_p^2(x)} \\
&= \frac{(r\Phi_p(x^{'}))^{'}}{\Phi_p(x)} - \frac{(p-1)|x|^{p-1}x^{-1}x^{'}}{\Phi_p(x)} . \frac{r\Phi_p(x^{'})}{\Phi(x)} \\
&= \frac{(-c(t)\Phi_p(x))}{\Phi_p(x)} - (p-1)\frac{x^{'}}{x}.w
\end{aligned}$$

$$Since\, w(t) = r(t)\Phi_p(\frac{x^{'}(t)}{x(t)}) \Rightarrow \frac{x^{'}}{x} = \Phi_p^{-1}(\frac{w}{r}) = r^{1-q}|w|^{q-1}$$
$$= -c - (p-1)r^{1-q}|w|^q$$
$$R[w] := w^{'} + c(t) + (p-1)r^{1-q})|w|^q$$

The technique based on the connection between nonexistence of zeros of a solution to equation (1.1) and the solvability of generalized Reccati equation is very useful tool of oscillation theory.

Lemma 1 *let h and w be differentiable functions and $V = h^p w - G$, then we have the identity*

$$h^p R[w] = V^{'} + hL[h] + (p-1)r^{1-q}h^{-q}H(t,V) \tag{2.2}$$

Where $H(t,V) = |V+G|^q - q\Phi_q(G)V - |G|^q \geq 0$

Proof since $G(t) = h^p(t)w_h(t) = h^p(t)r(t)\Phi_p(\frac{h^{'}(t)}{h(t)}) = h(t)r(t)\Phi_p(h^{'}(t))$

$$G^{'}(t) = (r\Phi_p(h^{'}(t))^{'}h + r|h^{'}|^p$$

and since $\Phi_q(G) = r^{q-1}(t)h^{q-1}(t)h^{'}$, $V = h^p w - G = h^p(w - w_h)$
and $hL[h] = h(t)[(r(t)\Phi_p(h^{'}(t)))^{'} + c(t)\Phi_p(h(t))]$

To show $h^p R[w] = V^{'} + hL[h] + (p-1)r^{1-q}h^{-q}H(t,V)$
Differentiating $V = h^p w - G = h^p w - h(t)r(t)\Phi_p(h^{'}(t))$

$$V^{'} = ph^{p-1}h^{'}w + h^p w^{'} - (r\Phi_p(h^{'}))^{'}h - r|h^{'}|^p \tag{2.3}$$

$$Similarly\, H(t,V) = |V+G|^q - q\Phi_q(G)V - |G|^q$$
$$= |V+G|^q - q\Phi_q(G)(h^p w - G) - |G|^q$$
$$= |h^p w|^q - qr^{q-1}h^{q-1}h^{'}(h^p w - G) - |G|^q$$
$$= |h^p w|^q - qr^{q-1}h^{q-1}h^{'}h^p w + qr^{q-1}h^{q-1}h^{'}G - |G|^q$$
$$= h^{pq}|w|^q - qr^{q-1}h^{q-1}h^{'}h^p w + qr^{q-1}h^{q-1}h^{'}hr|h^{'}|^{p-1} - |hr(h^{'})^{p-1}|^q$$
$$= h^{pq}|w|^q - qr^{q-1}h^{q-1}h^{'}h^p + qr^q h^q|h^{'}|^p - h^q r^q|h^{'}|^p$$

$$H(t,V) = h^{pq}|w|^q - qr^{q-1}h^{q-1}h^{'}h^p w + (q-1)r^q h^q|h^{'}|^p \tag{2.4}$$

$$(p-1)r^{1-q}h^{-q}H(t,V) = (p-1)r^{1-q}h^{-q}(h^{pq}|w|^q - qr^{q-1}h^{q-1}h^{'}h^p w + (q-1)r^q h^q|h^{'}|^p)$$
$$= (p-1)r^{1-q}h^{(p-1)q}|w|^q - (p-1)qh^{-q-1+q}h^{'}h^p w + (p-1)(q-1)r|h^{'}|^p$$
$$= (p-1)r^{1-q}h^{p-1}|w|^q - ph^{'}h^p w + r|h^{'}|^p$$

$$(p-1)r^{1-q}h^{-q}H(t,V) = (p-1)r^{1-q}h^p|w|^q - ph^{'}h^p w + r|h^{'}|^p \tag{2.5}$$

Adding (2.3) and (2.5) and adding the term $hL[h]$ to both sides of the equation

$$V^{'} + (p-1)r^{1-q}h^{-q}H(t,V) + hL[h]$$

$$= ph^{p-1}h^{'}w + h^p w^{'} - (r\Phi_p(h^{'}))^{'}h - r|h^{'}|^p + (p-1)r^{1-q}h^p|w|^q - ph^{'}h^p w + r|h^{'}|^p$$

$$= h^p w^{'} + (p-1)r^{1-q}h^p|w|^q - (r\Phi_p(h^{'}))^{'}h + h[(r\Phi_p(h^{'}))^{'} + c\Phi_p(h)]$$

$$= h^p w^{'} + (p-1)r^{1-q}h^p|w|^q + h(-(r\Phi_p(h^{'}))^{'} + (r\Phi_p(h^{'}))^{'} + c\Phi_p(h))$$

$$= h^p w^{'} + (p-1)r^{1-q}h^p|w|^q + hc\Phi_p(h)$$

$$= h^p w^{'} + (p-1)r^{1-q}h^p|w|^q + hch^{p-1}$$

$$= h^p[w^{'} + (p-1)r^{1-q}|w|^q + c]$$

$$= h^p R[w] \qquad \square$$

$$\therefore h^p R[w] = v^{'} + (p-1)r^{1-q}h^{-q}H(t,V) + hL[h]$$

Is known as *modified* version of the Riccati equation (2.1).[1]

Lemma 2 *: let $V(t)$ and $G(t)$ be real functions defined on $[t_0, \infty)$ such that*

$$\lim_{t\to\infty} \frac{V(t)}{G(t)} = 0 \tag{2.6}$$

Let $\gamma \in (1,2)$ and $k > 0$ be real numbers. There exist $t_1 \geq t_0$ such that

$$H(t, v(t)) \leq k|G(t)|^q \left|\frac{V(t)}{G(t)}\right|^\gamma \; for\, t > t_1 \tag{2.7}$$

proof from the definition of the function H we have

$$\begin{aligned} H(t,v) &= |V+G|^q - q\Phi_q(G)V - |G|^q \\ &= |V+G|^q - q|G|^{q-1}V - |G|^q \\ &= |V+G|^q - q|G|^q \frac{V(t)}{G(t)} - |G|^q \end{aligned}$$

[1]**N.B:** the function $H(t,v)$ satisfies $H(t,v) \geq 0$ for every $V, G \in R$ and $H(t,v) = 0 = H_v(t,v)$ if and only if $V = 0$. Observe also the Riccati equation is a special case of modified Riccati equation with $h(t) \equiv 1$, that is , $G(t) \equiv 0$.

$$
\begin{aligned}
&= |\frac{V(t)}{G(t)}+1|^q|G|^q - q|G|^q\frac{V(t)}{G(t)} - |G|^q \\
&= q|G|^q(\frac{1}{q}|\frac{V(t)}{G(t)}+1|^q - \frac{V(t)}{G(t)} - \frac{1}{q}) \\
H(t,V) &= q|G|^q g(\frac{V(t)}{G(t)})
\end{aligned}
$$

where $g(x) = \frac{|x+1|^q}{q} - x - \frac{1}{q}$.
The function $g(x)$ satisfies

$$g(x) = (q-1)\tfrac{1}{2!}x^2 + O(x^3)$$

in the neighborhood of $x = 0$ and

$$
\begin{aligned}
g(x) &= \frac{|x+1|^q}{q} - x - \frac{1}{q} = \frac{|x+1|^q}{q} - x - 1 + \frac{1}{p}. \\
&= \frac{|x+1|^q}{q} - (x+1) + \frac{1}{p}. \\
&= P(1, x+1) \geq 0
\end{aligned}
$$

let us define a function

$$\varphi(x) = \begin{cases} \frac{g(x)}{|x|^\gamma}, & x \neq 0 \\ \lim_{x\to 1} \frac{g(x)}{|x|^\gamma}, & x = 0 \end{cases}$$

let $x_1 = x_2$, $\forall x_1, x_2$,then
$g(x_1) = \frac{1}{p} - (x_1+1) + \frac{|x_1+1|^q}{q} = \frac{1}{p} - (x_2+1) + \frac{|x_2|^q}{q} = g(x_2)$. Therefore, $\varphi(x_1) = \varphi(x_2)$
since $g(x) \geq 0$, then $\varphi(x) \geq 0$
and according to the definition, $\forall x \in (-x_0, x_0)$, $\varphi(x)$ is continuous and for $\gamma > 2$

$$\lim_{x\to 0} \varphi(x) = 0.$$

Therefore, it is well defined ,nonnegative and continuous on $[-x_0, x_0]$ hence for every $\gamma \in (1,2)$ and every $k > 0$ there exists x_0 such that

$$g(x) \leq \frac{k}{q}|x|^\gamma \ \forall x \quad which\ satisfies\ |x| \leq x_0.$$

This inequality together with (2.6) implies that there exists t_1 such that

$$H(t,v) \leq k|G(t)|^q|\frac{V(t)}{G(t)}|^\gamma \; holds \; \forall t > t_1. \qquad \square$$

Remark 1 *The inequality*

$$H(t,V(t)) \leq q\beta_{\gamma,p}|G(t)|^q|\frac{V(t)}{G(t)}|^\gamma \tag{2.8}$$

which hold for $p \geq 2$,$\gamma \in [q,2]$, a convenient number $\beta_{\gamma,p}$ and every $t \in R$. And the inequality

$$H(t,V(t)) \geq q\beta_{\gamma,p}|G(t)|^q|\frac{V(t)}{G(t)}|^\gamma \tag{2.9}$$

which hold for $p \in (1,2]$,$\gamma \in [2,q]$, a convenient number $\beta_{\gamma,p}$ and every $t \in R$. In contrast to (2.8) and (2.9), inequalities (2.7) holds only for the restricted values of the quotient $\frac{V(t)}{G(t)}$ and from this point of view it can be consider as a local version of (2.8). We restricted ourselves to the case $\frac{V(t)}{G(t)} \to 0$ as $t \to \infty$.

Since there is a close correspondence between the function H and the function P given by

$$H(t,v) = qP(\Phi_q(G), V+G) \tag{2.10}$$

where

$$P(a,b) := \frac{|a|^p}{p} - ab + \frac{|b|^q}{q} \tag{2.11}$$

$$
\begin{aligned}
That\ is\ P(\Phi_q(G), V+G) &= \frac{|\Phi_q(G)|^p}{p} - \Phi_q(G)(V+G) + \frac{|V+G|^q}{q} \\
&= \frac{|G|^{p(q-1)}}{p} - \Phi_q(G)V - G\Phi_q(G) + \frac{|V+G|^q}{q} \\
&= \frac{|G|^q}{p} - \Phi_q(G)V - G|G|^{q-1} + \frac{|V+G|^q}{q} \\
&= \frac{|G|^q}{p} - \Phi_q(G)V - |G|^q + \frac{|V+G|^q}{q} \\
&= |G|^q(\frac{1}{p} - 1) - \Phi_q(G)V + \frac{|V+G|^q}{q} \\
&= |G|^q(\frac{1-p}{p}) - \Phi_q(G)V + \frac{|V+G|^q}{q} \\
&= -\frac{|G|^q}{q} - \Phi_q(G)V + \frac{|V+G|^q}{q} \\
qP(\Phi_q(G), V+G) &= |V+G|^q - q\Phi_q(G)V - |G|^q = H(t, V(t)) \\
\therefore H(t, V(t)) &= qP(\Phi_q(G), V+G)
\end{aligned}
$$

The non-negativity of $H(t, v)$ follows from the non-negativity of P.

Lemma 3 *For every $\gamma \geq 2$ and every $k_0 \in (0, \infty)$ there exists a constant $k > 0$ such that if $G(t) = 0$ or $|\frac{V(t)+G(t)}{G(t)}| \leq k_0$, then*

$$H(t, v(t)) \geq k|G(t)|^{q-\gamma}|V(t)|^\gamma \tag{2.12}$$

proof let $\gamma \geq 2$ and $k_0 \in (0, \infty)$ be arbitrary. If $G(t) = 0$ then (2.12) holds. Suppose that $G(t) \neq 0$ and $|\frac{V(t)+G(t)}{G(t)}| \leq k_0$ using (2.10) ,(2.11) and the obvious fact

$$p(a, b) = |a|^p P(1, \frac{b}{\Phi_p(a)})$$

$$
\begin{aligned}
i.e. since\, P(a,b) :&= \frac{|a|^p}{p} - ab + \frac{|b|^q}{q} \\
&= |a|^p(\frac{1}{p} - \frac{ab}{|a|^p} + \frac{|b|^q}{q|a|^p}) \\
&= |a|^p(\frac{1}{p} - |a|^{1-p}b + \frac{|b|^q|a|^{-p}}{q}) \qquad since - p = q(1-p) \\
&= |a|^p(\frac{1}{p} - |a|^{1-p}b + \frac{|b|^q|a|^{q(1-p)}}{q}) \\
Let\, x = |b|^q|a|^{(1-p)q} &= \frac{|b|^q}{|\Phi_p(a)|^q} \\
&= |a|^p(\frac{1}{p} - x + \frac{|x|^q}{q}) = |a|^p P(1,x) = |a|^p f(x)
\end{aligned}
$$

$$
Where\ f(x) = \frac{1}{p} - x + \frac{|x|^q}{q} = p(1,x) \geq 0
$$

Therefore, we write the function H in the form

$$
H(t, V(t)) = q|\Phi_q(G)|^p P(1, \frac{V}{G} + 1) = q|G|^q f(\frac{V}{G} + 1)
$$

let us define the function

$$
\varphi(x) = \begin{cases} \frac{|x-1|^\gamma}{f(x)}, & x \neq 1 \\ \lim_{x\to 1} \frac{|x-1|^\gamma}{f(x)}, & x = 1 \end{cases}
$$

let $x_1 = x_2$, $\forall x_1, x_2$,then
$f(x_1) = \frac{1}{p} - x_1 + \frac{|x_1|^q}{q} = \frac{1}{p} - x_2 + \frac{|x_2|^q}{q} = f(x_2)$. Therefore, $\varphi(x_1) = \varphi(x_2)$
since $f(x) \geq 0$, then $\varphi(x) \geq 0$
and according to the definition, $\forall x \in (-k_0, k_0)$, $\varphi(x)$ is continuous and for $\gamma > 2$

$$
\lim_{x\to 1} \varphi(x) = 0.
$$

Therefore, it is well defined ,nonnegative and continuous on $[-k_0, k_0]$ and there exists C such that

$$
\begin{aligned}
&\varphi(x) \leq C\, on\, [-k_0, k_0]. \\
&\Rightarrow \frac{|x-1|^{\gamma}}{f(x)} \leq C \\
&\Rightarrow |x-1|^{\gamma} \leq Cf(x) \\
&\Rightarrow \frac{1}{c}|x-1|^{\gamma} \leq f(x) \quad let \quad \frac{k}{q} = \frac{1}{c} \quad and \quad |x| \leq k_0 \\
&\Rightarrow f(x) \geq \frac{k}{q}|x-1|^{\gamma} \\
&Let \;\; x = \frac{V}{G} + 1 \\
&\Rightarrow f(\frac{V}{G}+1) \geq \frac{k}{q}|\frac{V}{G}+1-1|^{\gamma} \\
&\Rightarrow f(\frac{V}{G}+1) \geq \frac{k}{q}|\frac{V}{G}|^{\gamma} \\
&\Rightarrow q|G|^{q} f(\frac{V}{G}+1) \geq k|G|^{q}|\frac{V}{G}|^{\gamma} \\
&\Rightarrow H(t,V) \geq k|G|^{q}|\frac{V}{G}|^{\gamma} \qquad \square
\end{aligned}
$$

2.2 Necessary and sufficient condition for non-oscillation

Definition 1 ***(non-oscillation equation)*** *Equation (1.1) is said to be non-oscillatory if there exists a number $T \geq t_0$ and solution $x(t)$ of (1.1) which satisfies $x(t) > 0$ for every $t \geq T$.* [2]

Theorem 1 ***(Sturm comparison theorem)*** *if $\beta, \alpha \in I$ are the consecutive zeros of a nontrivial solution $y(t)$ of equation of the form*

$$y'' + q(t)y = 0, \quad where \quad q(t) > 0, \quad \forall t > 0 \tag{2.13}$$

and if $q_1(t)$ is continuous and $q_1(t) \geq q(t)$, $q_1(t) \not\equiv q(t)$ in $[\alpha, \beta]$, then every nontrivial solution $z(t)$ of the differential equation

$$z'' + q_1(t)z = 0 \tag{2.14}$$

[2] **(N.B)** equation (1.1) is said to be *non-oscillatory* if all solutions are *non-oscillatory*, and *oscillatory* if at least one solution is *oscillatory*.

has a zero in (α, β).

proof Multiplying (2.13) by $z(t)$ and (2.14) by $y(t)$ and subtracting, we obtain

$$z(t)y^{''}(t) - y(t)z^{''}(t) + (q(t) - q_1(t))y(t)z(t) = 0$$

which is the same as

$$(z(t)y^{'}(t) - y(t)z^{'}(t))^{'} + (q(t) - q_1(t))y(t)z(t) = 0$$

Since $y(\alpha) = y(\beta) = 0$, an integration yields

$$z(\beta)y^{'}(\beta) - z(\alpha)y^{'}(\alpha) + \int_{\alpha}^{\beta} (q(t) - q_1(t))y(t)z(t)dt = 0 \tag{2.15}$$

From the linearity of (2.13) we assume that $y(t) > 0$ in (α, β), then $y^{'}(\alpha) > 0$ and $y^{'}(\beta) < 0$. thus, from (2.15) it follows that $z(t)$ cannot be fixed sign in (α, β) ,i.e it has zero in (α, β). □

Example 2 . *Obviously the differential equation*

$$y^{''} + y = 0$$

is oscillatory. Thus, by the above Sturm-comparison theorem, it follows that the differential equation

$$y^{''} + (1 + t)y = 0$$

is also oscillatory in $I = [0, \infty)$.

Definition 2 *Equation(1.1) is said to disconjugate on the closed interval* I *if every its nontrivial solution has at most one zero in* I.

Theorem 2 *(Roundabout theorem) the following statement are equivalent*
(i) Equation (1.1) is disconjugat on the closed interval $I = [a, b]$.
(ii) There exist a solution of (1.1) having no zero in $[a, b]$.
(iii) There exist a solution of w of the generalized Riccati equation which is defined on the whole interval [a,b].
(iv) The energy functional F(X;a,b) is positive for energy $0 \not\equiv X \in w_0^{1,p}(a, b)$.

proof:see the proof of the theorem from ([1],theorem 1.2.2)

Definition 3 *Equation (1.1) is said to be* non-oscillatory *(more precisely, non-oscillatory at* ∞ *), if there exists* $T_0 \in R$ *such that (1.1) is* disconjugate *on* $[T_0, T_1]$ *for every* $T_1 > T_0$*.In the opposite case ,(1.1) is said to be* oscillatory.

The following theorem shows that the Riccati operator from the equation $R[w]$ is closely related to the non-oscillatory equation (1.1).

Theorem 3 *The following statements are equivalents:*
(i) Equation (1.1) is non-oscillatory
(ii) There is $a \in R$ *and a continuously differentiable function* $w : [a, \infty) \to R$ *such that*

$$R[w](t) = 0 \; for \; t \in [a, \infty) \tag{2.16}$$

(iii) There is $a \in R$*, a constant* $A \in R$ *and a (continuous) function* $w : [a, \infty) \to R$ *such that*

$$w(t) = A - \int_a^t (c + S[w, r])(s) ds \; for \; t \in (a, \infty)$$

(iv) There is $a \in R$ *and a continuously differentiable function* $w : [a, \infty) \to R$ *such that*

$$R[w](t) \leq 0 \; for \; t \in [a, \infty)$$

(v) There is $a \in R$ *and a positive function* $X : [a, \infty) \to R$ *(with* $r\Phi(x'))$ *continuously differentiable such that*

$$L[X](t) \leq 0 \; for \; t \in [a, \infty) \tag{2.17}$$

proof $(i) \Rightarrow (ii)$
This implication follows from the roundabout theorem since non-oscillation of (1.1) implies the existence of $a \in R$ such that (1.1) is disconjugate on $[a, \infty)$.
$(ii) \Rightarrow (iii)$
Suppose that $R[w](t) = 0$ for $t \in [a, \infty)$ from the Riccati type equation

$i.e.\ R[w](t) := w^{'}(t) + c(t) + (p-1)r^{1-q})|w|^q = 0\ for\ t \in [a, \infty)$

$$\Rightarrow w^{'}(t) = -c(t) - (p-1)r^{1-q}|w|^q$$
$$\Rightarrow w^{'}(t) = -((c(t) + (p-1)r^{1-q})|w|^q)$$
$$\Rightarrow dw = -((c(t) + (p-1)r^{1-q})|w|^q)dt$$
$$\Rightarrow dw = -((c(t) + S[w, r]))dt$$
$$\Rightarrow w(t) = A - \int_a^t (c + S[w, r](s))ds\ for\ some\ constant\ A \in R$$

$(iii) \Rightarrow (iv)$

$$Suppose \quad that \quad w(t) = A - \int_a^t (c + S[w, r](s))ds$$
$$\Rightarrow \frac{dw}{dt} \leq -((c(t) + S[w, r]))$$
$$\Rightarrow w^{'}(t) \leq -((c(t) + (p-1)r^{1-q})|w|^q)$$
$$\Rightarrow R[w](t) \leq 0$$

$(iv) \Rightarrow (v)$
Let w satisfy $R[w] \leq 0\ on\ [a, \infty)$. The function

$$x(t) = exp\int_a^t \Phi_p^{-1}(\frac{w(s)}{r(s)})ds$$

is a positive solution of the initial value problem

$$x^{'}(t) = \Phi_p^{-1}(\frac{w(t)}{r(t)})x\ , x(a) = 1.$$

by applying ordinary differential equation, then we have

$$xL[x] = x(r\Phi_p(x^{'}))^{'} + xc\Phi_p(x) - (p-1)r|x^{'}|^p + (p-1)r|x^{'}|^p$$

$$= |x|^p \frac{(r\Phi_p(x^{'}))^{'}\Phi_p(x) - (p-1)r\Phi_p(x^{'})|x|^{p-2}x^{'}}{(\Phi_p(x)\Phi_p(x))} + |x|^p c + |x|^p(p-1)r^{1-q}(\frac{(r|x^{'}|^{p-1})}{|x|^{p-1}})^q$$
$$= |x|^p R[w] \leq 0$$

Since x is positive, it follows that

$$L[x] \leq 0$$

$(v) \Rightarrow (i)$
suppose that a function x satisfies equation(2.17) on $[a, \infty)$. Then

$$\varphi(x) := -xL[x](t)$$

is a nonnegative function on this interval. Set $\bar{c}(t) = c(t) + \frac{\varphi(t)}{|x|^p}$,then $\bar{c} \geq c$ and

$$(r(t)\Phi_p(x^{'}))^{'} + \bar{c}(t)\Phi_p(x) = (r(t)\Phi_p(x^{'}))^{'} + (c(t) + \frac{\varphi(t)}{|x|^p})\Phi_p(x)$$

$$= (r(t)\Phi_p(x^{'}))^{'} + c(t)\Phi_p(x) + \frac{\varphi(t)}{|x|^p}\Phi_p(x)$$

$$= (r(t)\Phi_p(x^{'}))^{'} + c(t)\Phi_p(x) + \frac{\varphi(t)}{|x|^p}|x|^{p-1}$$

$$= L[x] + \frac{\varphi(t)}{|x|} = 0$$

Thus the equation

$$(r(t)\Phi_p(x^{'}))^{'} + \bar{c}(t)\Phi_p(x) = 0$$

is disconjugate on $[a, \infty)$.Therefore, equation (1.1) is disconjugate on $[a, \infty)$ as well by Sturmian comparison theorem and hence it is nonoscillation.Therefore, from this theorem the following lemma is true. □

Lemma 4 *Equation (1.1) is non-oscillatory if and only if there exists a differentiable function w which satisfies the Riccati type inequality*

$$R[w] \leq 0$$

for large t.

Chapter 3

Principal Solution of non-oscillatory equation

If a half-linear differential equation is non-oscillatory, then there is a solution of the associated Riccati equation which can be extended to some neighborhood of infinity. The qualitative theory of (1.1) was shown that many properties of solutions of (1.1) are very similar to those of the linear sturm-liouville differential equation (1.2).

3.1 Principal Solution of Sturm-Liouville equation

The concept of the principal solution of the linear equation (1.2) was introduced by W. Leighton and M.Morse in 1936 [1]. Later on, in the sixties of the last century, P.Hartman and Wintner established basic properties of this solution. Since principal solution is one of the properties of solution in this equation, so, to distinguish among all solutions of Sturm-liouville differential equation (1.2) (i.e. for p=2 of (1.1)) a solution $\tilde{x}$, is called the *principal solution* (determine uniquely up to a multiplicative factor), which is near ∞ less than any other solution of this equation. We have different approach to determine principal solution from all other solution, for instance, limit and integral approach. In case of limit approach, that

$$\lim_{t\to\infty}\frac{\tilde{x}(t)}{x(t)}=0$$

For any solution x which is linearly independent solution of $\tilde{x}$.
Let x, y be eventually positive linearly independent solution of (1.2), then

$$r(t)[x^{'}(t)y(t)-x(t)y^{'}(t)] =: k,\ where\, k\neq 0\, is\, a\, real\, constant.$$

This means that the function $\frac{x}{y}$ is monotonic ($since\ (\frac{x}{y})' = \frac{k}{ry^2}$) and hence there exists (finite or infinite) limit

$$\lim_{t\to\infty} \frac{x(t)}{y(t)} = L$$

if $L = 0$, x is the principal solution of (1.2).
if $L = \infty, y$ is the principal solution of (1.2)
if $0 < L < \infty$, we set $\tilde{x} = x - Ly$. Then obviously

$$\lim_{t\to\infty} \frac{\tilde{x}(t)}{y(t)} = 0 \quad and \quad \tilde{x}\; of\; (1.2)\; is\, the\, principal.$$

In case of integral approach, the principal solution $\tilde{x}$ of (1.2) can be equivalently characterized as a solution satisfying

$$\int_0^\infty \frac{dt}{r(t)\tilde{x}^2(t)} = \infty$$

Indeed, let y be a solution linearly independent of $\tilde{x}$. Then by the previous argument $\frac{y}{\tilde{x}}$ tends monotonically to ∞ as $t \to \infty$, hence ,

$$\int_0^\infty \frac{dt}{r(t)\tilde{x}^2(t)} = \lim_{t\to\infty} \int_0^t \frac{ds}{r(s)\tilde{x}^2(s)} = \lim_{t\to\infty} \frac{y(t)}{\tilde{x}(t)} = \infty.[1]$$

3.2 Principal solution of non-oscillatory half-linear equation

If a half-linear differential equation is non-oscillatory, then there is a solution of the associated Riccati equation which can be extended to some neighborhood of infinity. Among all solutions of the Riccati equation there exists the so called *principal solution* $\tilde{w}$ with the property that $\tilde{w}(t) < w(t)$ for all solution $w(t)$ which is defined in the whole interval $[T, \infty)$. And a Similar situation holds for the principal solution of half-linear differential equation (1.1) by way of the principal solution of the associated Riccati equation. Non-oscillation of (1.1) implies that there exist $T \in R$ and a solution $\tilde{w}(t)$ of (2.1) which is defined in the whole interval $[T, \infty)$.

The principal solution $\tilde{x}$ of (1.1) is defined as the solution which determines the principal solution $\tilde{w}(t)$ of (2.1) through the substitution

$$\tilde{w}(t) = r(t)\Phi_p(\frac{\tilde{x}'(t)}{\tilde{x}(t)})$$
$$\frac{\tilde{w}(t)}{r(t)} = \Phi_p(\frac{\tilde{x}'(t)}{\tilde{x}(t)})$$
$$\Phi_p^{-1}(\frac{\tilde{w}(t)}{r(t)}) = \frac{\tilde{x}'(t)}{\tilde{x}(t)}$$
$$\tilde{x}'(t) = \Phi_p^{-1}(\frac{\tilde{w}(t)}{r(t)})\tilde{x}(t)$$
$$= r^{1-q}(t)\Phi_p^{-1}(\tilde{w}(t))\tilde{x}(t)$$

Solve the ordinary differential equation

$$\frac{d\tilde{x}}{\tilde{x}} = r^{1-q}(t)\Phi_p^{-1}(\tilde{w}(t))dt$$
$$ln\tilde{x} = \int_0^\infty r^{1-q}(t)\Phi_p^{-1}(\tilde{w}(t))dt$$
$$\tilde{x}(t) = ce^{\int_0^\infty r^{1-q}(t)\Phi_p^{-1}(\tilde{w}(t))dt}$$
$$where \quad \Phi_p^{-1}(x) = |x|^{q-2}.x \; is\, the\, inverse\; function\; of\, \Phi_p$$

This principal solution is unique up to a nonzero constant multiple.The integral way of determining the principal solution of Sturm-Loiuvlle can be apply in the case to distinguish the principal solution of half-linear differential equation with some condition.The following theorem is used to distinguish the principal solution of Half-linear equation .

Theorem 4 *suppose that (1.1) is non-oscillatory and $h(t)$ is its positive solution which satisfies $h'(t) \neq 0$ for large t.*
(i) Let $p \geq 2$. If h is a principal solution, then $\forall\gamma \in [q, 2]$

$$\int_0^\infty \frac{dt}{r^{\gamma-1}(t)h^\gamma(t)|h'(t)|^{(p-1)(\gamma-q)}} = \infty \tag{3.1}$$

(ii) Let $p \in (1, 2]$. if (3.1) holds for some $\gamma \in [2, q]$, then h is a principal solution.

Proof i) suppose not,that is for $\gamma \in [q, 2]$ such that (3.1) does not hold. By (2.9) $\beta_{\gamma,p} > 0$ such that $H(t, V) \leq q\beta_{\gamma,p}|G(t)|^q|\frac{V(t)}{G(t)}|^\gamma$, for every $t, V \in R$ holds. Suppose that T is so large that

$$\int_T^\infty \frac{dt}{r^{\gamma-1}(t)h^\gamma(t)|h'(t)|^{(p-1)(\gamma-q)}} < \frac{1}{2\beta_{\gamma,p}(\gamma-1)} \tag{3.2}$$

Denote $w_h := r(t)\Phi_p(\frac{h'(t)}{h(t)})$ and consider the solution w of Riccati equation satisfying the initial condition $w(T) = w_h(T) - h^{-p}(T)$. By this definition we have $w(T) < w_h(T)$ and the unique solvability of Riccati equation imply that $w(t) < w_h(t)$ $\forall t \in [T, T^*]$ is the maximal interval of existence of the solution w. since $L[h] = 0$ and the function $V = h^p(w - w_h)$ is a solution of

$$v' + (p-1)r^{1-q}(t)h^{-q}(t)H(t, v(t)) = 0.$$

This solution satisfies $V(T) = -1$ and $V(t) < 0$ $\forall t$ for which w (and hence also V) is defined for $t \in [T, T^*]$. Since $p \geq 2$ and $\gamma \in [q, 2]$, by (2.8) the inequality $H(t, V) \leq q\beta_{\gamma,p}|G(t)|^q|\frac{V(t)}{G(t)}|^\gamma$ implies

$$\begin{aligned}(p-1)r^{1-q}(t)h^{-q}(t)H(t, v(t)) &\leq q(p-1)r^{1-q}(t)h^{-q}(t)\beta_{\gamma,p}|G(t)|^q|\frac{V(t)}{G(t)}|^\gamma \\ &= pr^{1-q}(t)h^{-q}(t)\beta_{\gamma,p}|G(t)|^{q-\gamma}|V(t)|^\gamma \\ &= p\beta_{\gamma,p}h^{-\gamma}(t)r^{1-\gamma}(t)|h'(t)|^{(p-1)(q-\gamma)}|V(t)|^\gamma \,, t \in [T, T^*] \\ v' + (p-1)r^{1-q}(t)h^{-q}(t)H(t, v(t)) &\leq v'(t) + p\beta_{\gamma,p}h^{-\gamma}(t)r^{1-\gamma}(t)|h'(t)|^{(p-1)(q-\gamma)}|V(t)|^\gamma, \\ t \in [T, T^*] \\ 0 &\leq v'(t) + p\beta_{\gamma,p}h^{-\gamma}(t)r^{1-\gamma}(t)|h'(t)|^{(p-1)(q-\gamma)}|V(t)|^\gamma\end{aligned}$$

v is a solution of the inequality
$v'(t) + p\beta_{\gamma,p}h^{-\gamma}(t)r^{1-\gamma}(t)|h'(t)|^{(p-1)(q-\gamma)}|V(t)|^\gamma \geq 0$ on $[T, T^*]$ and consequently
i.e

$$-v'(t) \leq p\beta_{\gamma,p}h^{-\gamma}(t)r^{1-\gamma}(t)|h'(t)|^{(p-1)(q-\gamma)}|V(t)|^\gamma$$

$$\frac{-dv}{|V(t)|^\gamma} \leq p\beta_{\gamma,p}h^{-\gamma}(t)r^{1-\gamma}(t)|h'(t)|^{(p-1)(q-\gamma)}dt$$

Integrate this inequality over $[T, t]$ we obtain

$$\frac{1}{(\gamma-1)|V(T)|^{\gamma-1}} - \frac{1}{(\gamma-1)|V(t)|^{\gamma-1}} \leq p\beta_{\gamma,p}\int_T^t h^{-\gamma}(s)r^{1-\gamma}(s)|h'(s)|^{(p-1)(q-\gamma)}ds$$

Letting $t \to \infty$, we have

$$\frac{1}{(\gamma-1)|V(T)|^{\gamma-1}} \leq p\beta_{\gamma,p}\int_T^\infty h^{-\gamma}(s)r^{1-\gamma}(s)|h'(s)|^{(p-1)(q-\gamma)}ds$$

From (3.2) we have

$$p\beta_{\gamma,p}\int_T^\infty \frac{ds}{h^\gamma(s)r^{\gamma-1}(s)|h'(s)|^{(p-1)(\gamma-q)}} \leq \frac{1}{2(\gamma-1)}$$

$\frac{1}{(\gamma-1)|V(T)|^{\gamma-1}} - \frac{1}{(\gamma-1)|V(t)|^{\gamma-1}} \leq \frac{1}{2(\gamma-1)}$ since $V(T) = -1$ we get

$\frac{1}{(\gamma-1)|V(t)|^{\gamma-1}} \leq \frac{1}{(\gamma-1)} - \frac{1}{2(\gamma-1)} = \frac{1}{2(\gamma-1)}$ and hence $|V(t)| < 2^{\frac{1}{(\gamma-1)}}$ this means that $V(t)$ is continuable up to infinity ($T^* = \infty$). Hence $V(t) < 0$. $\forall t \geq T$,i.e. $w(t) < w_h(t)$ $\forall t \geq T$. This shows that w_h is not the minimal solution of Riccati equation and hence h is not the principal solution of (1.1).

ii) Suppose not, i.e, the assumption of the theorem hold and h is not principal solution. Denote $w_h := r(t)\Phi_p(\frac{h'(t)}{h(t)})$ the corresponding solution of (2.1). Since h is not principal, $\exists T > 0$ and a solution w of (2.1) such that $w(t) < w_h(t)$ for $t \geq T$. We see that $V = h^p(w - w_h)$ is a solution of equation $v' + (p-1)r^{1-q}(t)h^{-q}(t)H(t, v(t)) = 0$ since $p \in (1, 2]$ and $\gamma \in [2, q]$ by (2.9) $\exists \beta_{\gamma,p} > 0$ such that

$$\begin{aligned}(p-1)r^{1-q}(t)h^{-q}(t)H(t, v(t)) &\geq q(p-1)r^{1-q}(t)h^{-q}(t)\beta_{\gamma,p}|G(t)|^q|\frac{V(t)}{G(t)}|^\gamma \\ &= pr^{1-q}(t)h^{-q}(t)\beta_{\gamma,p}|G(t)|^{q-\gamma}|V(t)|^\gamma \\ &= p\beta_{\gamma,p}h^{-\gamma}(t)r^{1-\gamma}(t)|h'(t)|^{(p-1)(q-\gamma)}|V(t)|^\gamma, t \in [T, T^*]\end{aligned}$$

$v' + (p-1)r^{1-q}(t)h^{-q}(t)H(t, v(t)) \geq v'(t) + p\beta_{\gamma,p}h^{-\gamma}(t)r^{1-\gamma}(t)|h'(t)|^{(p-1)(q-\gamma)}|V(t)|^\gamma$, $t \in [T, T^*]$

$$0 \geq v'(t) + p\beta_{\gamma,p}h^{-\gamma}(t)r^{1-\gamma}(t)|h'(t)|^{(p-1)(q-\gamma)}|V(t)|^\gamma$$

v is a solution of the inequality

$$v'(t) + p\beta_{\gamma,p}h^{-\gamma}(t)r^{1-\gamma}(t)|h'(t)|^{(p-1)(q-\gamma)}|V(t)|^\gamma \leq 0$$

on $[T, T^*]$ and consequently

$$-v'(t) \geq p\beta_{\gamma,p}h^{-\gamma}(t)r^{1-\gamma}(t)|h'(t)|^{(p-1)(q-\gamma)}|V(t)|^\gamma$$

$$\frac{-dv}{|V(t)|^\gamma} \geq p\beta_{\gamma,p}h^{-\gamma}(t)r^{1-\gamma}(t)|h'(t)|^{(p-1)(q-\gamma)}dt$$

The inequality $w < w_h$ for $t \geq T$ implies that $V(t) < 0$ for $t \geq T$ Integrate this inequality over $[T, t]$ we obtain

$$\frac{1}{(\gamma-1)|V(T)|^{\gamma-1}} - \frac{1}{(\gamma-1)|V(t)|^{\gamma-1}} \geq p\beta_{\gamma,p}\int_T^t h^{-\gamma}(s)r^{1-\gamma}(s)|h'(s)|^{(p-1)(q-\gamma)}ds$$

Letting $t \to \infty$, we have

$$\frac{1}{(\gamma-1)|V(T)|^{\gamma-1}} \geq p\beta_{\gamma,p} \int_T^\infty h^{-\gamma}(s) r^{1-\gamma}(s) |h^{'}(s)|^{(p-1)(q-\gamma)} ds$$

$$p\beta_{\gamma,p} \int_T^\infty h^{-\gamma}(s) r^{1-\gamma}(s) |h^{'}(s)|^{(p-1)(q-\gamma)} ds \leq \frac{1}{(\gamma-1)|V(T)|^{\gamma-1}}$$

This contradict (3.2) i.e

$$\int_0^\infty \frac{dt}{r^{\gamma-1}(t) h^\gamma(t) |h'(t)|^{(p-1)(\gamma-q)}} = \infty$$

$\therefore h(t)$ is a principal solution

Example 3 *consider equation*

$$(\Phi_{\frac{3}{2}}(x^{'}))^{'} + \frac{15t^{\frac{-3}{2}}}{(t^9-1)^{\frac{1}{2}}} \Phi_{\frac{3}{2}}(x) = 0, t > 1.$$

In this setting we have $p = \frac{3}{2}$ *,*$q = \frac{p}{p-1} = 3, r(t) = 1$ *and* $c(t) = \frac{15t^{\frac{-3}{2}}}{(t^9-1)^{\frac{1}{2}}}$ *. This equation has a solution* $h(t) = 1 - \frac{1}{t^9}$ *. Direct computation shows*

$$\int_0^\infty \frac{dt}{r^{\gamma-1}(t) h^\gamma(t) |h'(t)|^{(p-1)(\gamma-q)}} = \int_0^\infty \frac{dt}{3^{\gamma-1}(1-\frac{1}{t^9})^\gamma t^{15-5\gamma}} = \int_0^\infty \frac{dt}{3^{\gamma-1} t^{15-5\gamma}}$$

and the integral diverges for $\gamma \in [\frac{14}{5}, 3]$*. Hence* h *is a principal solution (by ii) since for* $\gamma \in [\frac{14}{5}, 3] \subseteq [2, q] = [2, 3]$ *that (3.1) diverge.*

The following theorem shows that under some additional assumptions we can drop the restriction $p \leq 2$ and $\gamma \leq q$ from the implication of (ii) and we get the following statement which is in the same sense close to the opposite implication of the statement in (i) of the above theorem.

Theorem 5 *suppose that (1.1) is non-oscillatory and* $h(t)$ *is its positive solution which satisfies* $h^{'}(t) \neq 0$ *for large* t*. Further suppose that*

$$\int_t^\infty c(s)ds \geq 0, \int_t^\infty c(s)ds \not\equiv 0$$

for large t *and*

$$\int_0^\infty r^{1-q}(t)dt = \infty$$

If there exist a real number $\gamma \geq 2$ *such that (3.1) holds, then* h *is a principal.*

Proof suppose not, i.e. the assumption of the theorem hold and h is not principal solution. Denote $w_h := r(t)\Phi_p(\frac{h^{'}(t)}{h(t)})$ the corresponding solution of (2.1). Since h is not principal, there exists $T > 0$ and a solution $\tilde{w}$ of (2.1) such that $\tilde{w}(t) < w_h(t)$ for $t \geq T$. Condition $\int_0^\infty r^{1-q}(t)dt = \infty$ and the convergence of $\int_t^\infty c(s)ds$ imply that$\int_0^\infty r^{1-q}(t)|\tilde{w}(t)|^q dt < \infty$ and

$$\tilde{w}(t) = \int_t^\infty c(s)ds + (p-1)\int_0^\infty r^{1-q}(s)|\tilde{w}(s)|^q ds \; for \, t \geq T$$

Since $\int_t^\infty c(s)ds \geq 0$, we have $\tilde{w}(t) \geq 0$ and hence $0 \leq \frac{\tilde{w}(t)}{w_h(t)} < 1$. consequently, consider the function $v = h^p\tilde{w} - G = h^p(\tilde{w} - w_h)$. It holds $v(t) < 0$ for $t \geq T$ and since $L[h] = 0$, we see from identity (2.2) that v is a solution of the modified Riccati equation

$$v^{'} + (p-1)r^{1-q}(t)h^{-q}(t)H(t, v(t)) = 0 \; for \; t \geq T$$

We have $\frac{V}{G} = \frac{h^p\tilde{w}}{G} - 1 = \frac{\widetilde{w}}{w_h(t)} - 1 \, i.e. \; -1 \leq \frac{V}{G} < 0$ and $|\frac{V+G}{G}| \leq 1$ for $t \geq T$. now, using (2.12) $\exists K > 0$ such that

$$H(t, v(t)) \geq K|G(t)|^q|\frac{V(t)}{G(t)}|^\gamma \;, t \geq T \; hence$$

$$(p-1)r^{1-q}(t)h^{-q}(t)H(t, v(t)) \geq (p-1)r^{1-q}(t)h^{-q}(t)K|G(t)|^q|\tfrac{V(t)}{G(t)}|^\gamma$$

$$= (p-1)r^{1-q}(t)h^{-q}(t)K|G(t)|^{q-\gamma}|V(t)|^\gamma$$

$$= (p-1)Kh^{-\gamma}(t)r^{1-\gamma}(t)|h^{'}(t)|^{(p-1)(q-\gamma)}|V(t)|^\gamma, t \geq T$$

$v^{'}+(p-1)r^{1-q}(t)h^{-q}(t)H(t, v(t)) \geq v^{'}(t)+(p-1)Kh^{-\gamma}(t)r^{1-\gamma}(t)|h^{'}(t)|^{(p-1)(q-\gamma)}|V(t)|^\gamma$, $t \geq T$

$$0 \geq v^{'}(t) + (p-1)Kh^{-\gamma}(t)r^{1-\gamma}(t)|h^{'}(t)|^{(p-1)(q-\gamma)}|V(t)|^\gamma, t \geq T$$

v is a solution of the inequality

$$v^{'}(t) + (p-1)Kh^{-\gamma}(t)r^{1-\gamma}(t)|h^{'}(t)|^{(p-1)(q-\gamma)}|V(t)|^\gamma \leq 0$$

$$-v^{'}(t) \geq (p-1)Kh^{-\gamma}(t)r^{1-\gamma}(t)|h^{'}(t)|^{(p-1)(q-\gamma)}|V(t)|^\gamma, \qquad t \geq T$$

$$\frac{-dv}{|V(t)|^\gamma} \geq (p-1)Kh^{-\gamma}(t)r^{1-\gamma}(t)|h^{'}(t)|^{(p-1)(q-\gamma)}dt, \qquad t \geq T$$

Integrate this inequality over $[T, t]$ we obtain

$\frac{1}{(\gamma-1)|V(T)|^{(\gamma-1)}} - \frac{1}{(\gamma-1)|V(t)|^{(\gamma-1)}} \geq (p-1)K \int_T^t h^{-\gamma}(s) r^{1-\gamma}(s) |h^{'}(s)|^{(p-1)(q-\gamma)} ds$
Letting $t \to \infty$, we have
$\frac{1}{(\gamma-1)|V(T)|^{(\gamma-1)}} \geq (p-1)K \int_T^t h^{-\gamma}(s) r^{1-\gamma}(s) |h^{'}(s)|^{(p-1)(q-\gamma)} ds$

$$(p-1)K \int_T^{\infty} \frac{ds}{h^{\gamma}(s) r^{\gamma-1}(s) |h^{'}(s)|^{(p-1)(q-\gamma)} ds} \leq \frac{1}{(\gamma-1)|V(T)|^{(\gamma-1)}}$$

This contradict (3.1) i.e $\int_T^{\infty} \frac{ds}{h^{\gamma}(s) r^{\gamma-1}(s) |h^{'}(s)|^{(p-1)(q-\gamma)}} = \infty$
$\therefore h(t)$ is a principal solution. $\square$

Note that the case when $G(t) \to 0$ as $t \to \infty$ is delicate in some sense, since this case the integral in (3.2) may fail to be divergent if γ is not sufficiently large. Hence the "usual" integral criteria to detect principality which deals with $\gamma = 2$ may fail. Using the definition of $G(t)$ we can write (3.1) in the form

$$\int_0^{\infty} |G(t)|^{1-\gamma} \frac{|h^{'}(t)|}{h(t)} = \infty \tag{3.3}$$

The following simple corollary shows that if the function $G(t)$ approaches 0 sufficiently fast and the fraction $\frac{|h^{'}(t)|}{h(t)}$ does not tend to zero faster than a power function, then h is a principal solution.

Corollary 1 *Suppose that* $\int_t^{\infty} c(s)ds \geq 0, \int_t^{\infty} c(s)ds \not\equiv 0$ *for large* t *and* $\int_0^{\infty} r^{1-q}(t)dt = \infty$ *suppose that there exist a positive solution* h *of equation (1.1), real number* β *and positive real number* ϵ *and* K *such that*

$$h^{'}(t) \neq 0 \; and \; \frac{|h^{'}(t)|}{h(t)} \geq K t^{\beta} \; for \; large \; t$$

$$G(t) = O(t^{-\epsilon}) \; as \quad t \to \infty$$

Then h *is the principal solution of (1.1).*

proof let $\gamma \geq 2$. From the assumption it follows that there exist K_1, such that $|G(t)| \leq K_1 t^{-\epsilon}$ and hence

$$|G(t)|^{1-\gamma} \leq K_2 t^{\epsilon(\gamma-1)} \quad for \quad t \geq T_0. This\, shows\, that$$

$$|G(t)|^{1-\gamma} \frac{|h^{'}(t)|}{h(t)} \geq K K_2 t^{\beta+\epsilon(\gamma-1)}$$

And if $\gamma \geq \gamma_0 := max\{2, 1 - \frac{\beta+1}{\epsilon}\}$, then (3.3) holds and the solution h is principal by theorem 5. $\square$

Example 4 *conceder the above example*

$$(\Phi_{\frac{3}{2}}(x^{'}))^{'} + \frac{15t^{\frac{-3}{2}}}{(t^9-1)^{\frac{1}{2}}}\Phi_{\frac{3}{2}}(x) = 0, t > 1$$

the function $h(t) = 1 - \frac{1}{t^9}$. *is a solution of this equation. This solution is a principal solution, as follows from the corollary and from the fact that* $h(t) \sim 1$,$h^{'}(t) \sim 9t^{-10}$ *and* $G(t) \sim 3t^{-5}$ *near infinity.*

Chapter 4

Non-oscillation criteria on half-linear equation using Riccati Technique

In this chapter we concentrate our attention to non-oscillatory half-linear differential equation. We present non-oscillation criteria for (1.1) which are based on the Riccati technique.

Recall that (1.1) is said to be non-oscillatory if there exists $T \in R$ such that (1.1) is disconjugate on $[T, \infty)$; i.e every non trivial solution of this equation has at most one zero in this interval and this means that every nontrivial solution is eventually positive or negative.
N.B: Due to homogeneity of the set of all solutions we can restrict ourselves to a solutions which are positive in a neighborhood of infinity.
The roundabout theorem offers two basic methods of the investigation of oscillation properties of (1.1). The first one, usually referred to as the variational principle, is based on the equivalence of disconjugacy of (1.1) the positivity of the associated energy function F. The second main method that we are concentrated, is the Riccati technique uses the equivalence of disconjugacy (1.1) and solvability of the generalized Riccati equation (2.1). Therefore, due to roundabout theorem it follows that non-oscillation of (1.1) is equivalent to solvability of the associated Riccati equation . i.e. solvability of Riccati type first order differential equation.

$$w^{'} + c(t) + (p-1)r^{1-q}(t)|w|^q = 0$$

Let x be a solution of (1.1) , then the function $w = r\Phi(\frac{x^{'}(t)}{x(t)})$ solves the Riccati equation and it is well known that equation (1.1) is non-oscillatory

if and only if there exist a solution of Riccati type on some interval of the form $[T, \infty)$.
Due to the Sturm comparison theorem, non-oscillation of (1.1) is actually equivalent to solvability of the Riccati inequality. i.e. equation (1.1) is non-oscillatory if and only if there exists continuously differentiable function w defined on an interval $[T, \infty)$ and satisfying the inequality

$$w^{'} + c(t) + (p-1)r^{1-q}(t)|w|^{q} \leq 0$$

Therefore, non-oscillation criteria using Riccati inequality means establishing sufficient condition which guarantee that

$$R[w](t) \leq 0 \; for \; large \; t.$$

has a solution in a neighborhood of infinity.This implies that for $h(t) > 0$ the modified version of Riccati inequality

$$h^{p}(t)R[w](t) \leq 0 \; for \; large \; t$$

also satisfied.

Theorem 6 *let $h(t)$ be a function such that $h(t) > 0$ and $h^{'}(t) \neq 0$, both for large t. Suppose that the following condition hold*

$$\int_{0}^{\infty} \frac{dt}{r(t)h^{2}(t)|h^{'}(t)|^{p-2}} < \infty$$

$$\lim_{t\to\infty} |G(t)| \int_{t}^{\infty} \frac{ds}{r(s)h^{2}(s)|h^{'}(s)|^{p-2}} = \infty$$

If

$$\lim_{t\to\infty} sup \int_{t}^{\infty} \frac{ds}{r(s)h^{2}(s)|h^{'}(s)|^{p-2}} \int_{0}^{t} h(s)L[h](s)ds < \frac{1}{q}(-\alpha + \sqrt{2\alpha})$$

$$\lim_{t\to\infty} inf \int_{t}^{\infty} \frac{ds}{r(s)h^{2}(s)|h^{'}(s)|^{p-2}} \int_{0}^{t} h(s)L[h](s)ds > \frac{1}{q}(-\alpha - \sqrt{2\alpha})$$

for some $\alpha > 0$, then equation (1.1) is non-oscillatory.

Proof Denote $G(t) = r(t)h(t)\Phi_{p}(h^{'}(t)), R(t) = r(t)h^{2}(t)|h^{'}(t)|^{p-2}$ and

$$V(t) = \frac{-\alpha}{q}(\int_{t}^{\infty} R^{-1}(s)ds)^{-1} - \int_{0}^{t} h(s)L[h](s)ds, we \; have$$

$$= \frac{\frac{-\alpha}{q} - \int_t^\infty R^{-1}(s)ds \int_0^t h(s)L[h](s)ds}{\int_t^\infty R^{-1}(s)ds}$$

$$\frac{V(t)}{G(t)} = \frac{\frac{-\alpha}{q} - \int_t^\infty R^{-1}(s)ds \int_0^t h(s)L[h](s)ds}{G(t)\int_t^\infty R^{-1}(s)ds}$$

and

$$V^{'}(t) = \frac{-\alpha}{q}(\int_t^\infty R^{-1}(s)ds)^{-2}R^{-1}(t) - h(t)L[h](t)$$

$$since \qquad H(t,V(t)) = |V(t)+G(t)|^q - q\Phi_q(G(t))V(t) - |G(t)|^q$$

$$= |G(t)|^q(|1+\frac{V(t)}{G(t)}|^q - q\frac{V(t)}{G(t)} - 1)$$

$$= r^q(t)h^q|h^{'}|^{(p-1)q}(|1+\frac{V(t)}{G(t)}|^q - q\frac{V(t)}{G(t)} - 1)$$

$(p-1)r^{1-q}(t)h^{-q}H(t,V(t)) = (p-1)r^{1-q}(t)h^{-q}(t)[|V(t)+G(t)|^q - q\Phi_q(G(t))V(t) - |G(t)|^q]$

$$= (p-1)r(t)|h^{'}|^p(|1+\frac{V(t)}{G(t)}|^q - q\frac{V(t)}{G(t)} - 1)$$

$$= (p-1)r(t)|h^{'}(t)|^p[|1+\frac{V(t)}{G(t)}|^q - q\frac{V(t)}{G(t)} - 1]$$

Consider the function

$$F(x) = |1+x|^q - qx - 1.$$

This function satisfies $F(0) = 0 = F^{'}(0)$ and $F^{''}(0) = q(q-1)$. Hence ,by the Taylar's formula, the function $F(x)$ can be approximated by $(\frac{q(q-1)}{2})x^2$ in the neighborhood of $x = 0$. Condition of the theorem imply that

$$\frac{V(t)}{G(t)} = \frac{\frac{-\alpha}{q} - \int_t^\infty R^{-1}(s)ds \int_0^t h(s)L[h](s)ds}{G(t)\int_t^\infty R^{-1}(s)ds} \to 0\, as\, t \to \infty$$

$$F(\frac{V(t)}{G(t)}) = \frac{q(q-1)}{2}\frac{V^2(t)}{G^2(t)}$$

Hence ,$\forall \epsilon > 0$, $\exists T \in R$ such that

$$F(\frac{V(t)}{G(t)}) \leq \frac{q(q-1)}{2}(1+\epsilon\frac{q}{\sqrt{\alpha}})\frac{V^2(t)}{G^2(t)} \quad holds\; for\;\; t > T$$

At the same time ϵ can be take so small and T so large that for $t > T$ we have

$$\epsilon - \frac{\alpha}{q} - \frac{\sqrt{2\alpha}}{q} < \int_t^\infty R^{-1}(s)ds \int_0^t h(s)L[h](s)ds < -\epsilon - \frac{\alpha}{q} + \frac{\sqrt{2\alpha}}{q}$$

Consequently,

$$(p-1)r^{1-q}(t)h^{-q}(t)H(t,V(t)) = (p-1)r(t)|h^{'}|^p F(\frac{V(t)}{G(t)})$$

$$\leq (p-1)r(t)|h^{'}|^p \frac{q(q-1)}{2}(1+\epsilon\frac{q}{\sqrt{\alpha}})\frac{V^2(t)}{G^2(t)}$$

$$= \frac{q}{2}r(t)|h^{'}|^p(1+\epsilon\frac{q}{\sqrt{\alpha}})\frac{V^2(t)}{G^2(t)}$$

$$= \frac{q}{2}(1+\epsilon\frac{q}{\sqrt{\alpha}})r(t)|h^{'}|^p\frac{V^2(t)}{G^2(t)}$$

$$= \frac{q}{2}(1+\epsilon\frac{q}{\sqrt{\alpha}})\frac{(\frac{\alpha}{q} + \int_t^\infty R^{-1}(s)ds \int_0^t h(s)L[h](s)ds)^2}{R(t)(\int_t^\infty R^{-1}(s)ds)^2}$$

$$< \frac{\frac{q}{2}(1+\epsilon\frac{q}{\sqrt{\alpha}})(\frac{\sqrt{2\alpha}}{q} - \epsilon)^2}{R(t)(\int_t^\infty R^{-1}(s)ds)^2}$$

Let $w(t) = h^{-p}(t)(V(t) + G(t))$, we have by identity(2.2)

$$h^p(t)R[w](t) = V^{'} + h(t)L[h](t) + (p-1)r^{1-q}(t)h^{-q}H(t,V)$$

$$= \frac{-\alpha}{q}(\int_t^\infty R^{-1}(s)ds)^{-2}R^{-1}(t) - h(t)L[h](t)ds + h(t)L[h](t) + (p-1)r^{1-q}(t)h^{-q}H(t,V)$$

$$< \frac{-\alpha}{q}(\int_t^\infty R^{-1}(s)ds)^{-2}R^{-1}(t) + \frac{\frac{q}{2}(1+\epsilon\frac{q}{\sqrt{\alpha}})(\frac{\sqrt{2\alpha}}{q} - \epsilon)^2}{R(t)(\int_t^\infty R^{-1}(s)ds)^2}$$

$$= \frac{\frac{q}{2}(1+\epsilon\frac{q}{\sqrt{\alpha}})(\frac{\sqrt{2\alpha}}{q} - \epsilon)^2 - \frac{\alpha}{q}}{R(t)(\int_t^\infty R^{-1}(s)ds)^2} \tag{4.1}$$

Consider the function in the numerator of the last fraction

$$f(\epsilon) = \frac{q}{2}(1+\epsilon\frac{q}{\sqrt{\alpha}})(\frac{\sqrt{2\alpha}}{q} - \epsilon)^2 - \frac{\alpha}{q}$$

we have $f(0) = 0$ and by direct computation

$$f^{'}(\epsilon) = \frac{q}{2}\frac{q}{\sqrt{\alpha}}(\frac{\sqrt{2\alpha}}{q} - \epsilon)^2 - q(1+\epsilon\frac{q}{\sqrt{\alpha}})(\frac{\sqrt{2\alpha}}{q} - \epsilon),$$

and hence $f^{'}(0) = (1-\sqrt{2})\sqrt{\alpha} < 0$. This means that ϵ can be taken so small that

$$f(\epsilon) < \frac{1-\sqrt{2}}{2}\epsilon\sqrt{\alpha} \quad (4.2)$$

Combining (4.1)and(4.2) we have

$$h^p(t)R[w](t) < \frac{\frac{1-\sqrt{2}}{2}\epsilon\sqrt{\alpha}}{R(t)(\int_t^\infty R^{-1}(s)ds)^2} < 0\, fort > T$$

and (1.1) is non-oscillatory by lemma 4. □

In view of the above Theorem, the non-oscillation property related with

$$\int \frac{dt}{r(t)h^2(t)|h^{'}(t)|^{p-2}}\ .$$

And in the following theorems we use the integral

$$\int_0^\infty \frac{dt}{r^{\gamma-1}(t)h^\gamma(t)|h^{'}(t)|^{(p-1)(\gamma-q)}} = \infty.$$

Note that we do not allow $\gamma = 2$ in Theorems 8 and 9. and we can see that finite non-oscillation constants which are replaced by ∞ and $-\infty$. And also note that we use opposite estimates than in the previous chapter and thus the condition $\gamma \geq 2$ is replaced by the condition $\gamma \in (1,2)$ in the following theorems.

Theorem 7 *let $\gamma \in (1,2)$ be a real number and $\overline{\gamma} = \frac{\gamma}{\gamma-1}$ be the conjugate number of γ. Let h be a positive continuously differentiable function such that $h^{'}(t) \neq 0$ in some neighborhood of infinity. Denote*

$$R(t) = r^{\gamma-1}(t)h^\gamma(t)|h^{'}(t)|^{(\gamma-q)(p-1)}\ and\ suppose\ that$$

$$\int_t^\infty R^{-1}(s)ds < \infty \qquad and \qquad \lim_{t\to\infty}|G(t)|[\int_t^\infty R^{-1}(s)ds]^{\bar{\gamma}-1} = \infty$$

$$If\ \lim_{t\to\infty} sup[\int_t^\infty R^{-1}(s)ds]^{\bar{\gamma}-1}\int_0^t h(s)L[h](s)ds < \infty \quad (4.3)$$

and

$$\lim_{t\to\infty} inf[\int_t^\infty R^{-1}(s)ds]^{\bar{\gamma}-1}\int_0^t h(s)L[h](s)ds > -\infty \quad (4.4)$$

then (1.1) is non-oscillatory.

Proof let

$$Y(t) = [\int_t^\infty R^{-1}(s)ds]^{\bar{\gamma}-1} \int_0^t h(s)L[h](s)ds$$

then condition (4.3) and (4.4) imply that there exist $t_0 \in R$ and a positive constants α, c_0 such that

$$|Y(t) + \alpha|^\gamma < \frac{\alpha}{c_0} \ , for\ t \geq t_0$$

Define the function

$$V(t) = -\alpha[\int_t^\infty R^{-1}(s)ds]^{1-\overline{\gamma}} - \int_0^t h(s)L[h](s)ds$$

$$= \frac{-\alpha}{[\int_t^\infty R^{-1}(s)ds]^{\overline{\gamma}-1}} - \int_0^t h(s)L[h](s)ds$$

$$= \frac{-\alpha - [\int_t^\infty R^{-1}(s)ds]^{\overline{\gamma}-1} \int_0^t h(s)L[h](s)ds}{[\int_t^\infty R^{-1}(s)ds]^{\overline{\gamma}-1}}$$

$$= \frac{-\alpha - Y(t)}{[\int_t^\infty R^{-1}(s)ds]^{\overline{\gamma}-1}}$$

$$\frac{V(t)}{G(t)} = \frac{-\alpha - Y(t)}{G(t)[\int_t^\infty R^{-1}(s)ds]^{\overline{\gamma}-1}} \quad and$$

$$V'(t) = \frac{\alpha(1-\overline{\gamma})}{R(t)[\int_t^\infty R^{-1}(s)ds]^{\overline{\gamma}}} - h(t)L[h](t)$$

Condition of the theorem imply that $\frac{V(t)}{G(t)} \to 0$ as $t \to \infty$, hence, using inequality (2.7)

$$H(t, V(t)) \leq K|G(t)|^q|\frac{V(t)}{G(t)}|^\gamma \qquad with\ K = c_0(\overline{\gamma}-1)(q-1)$$

$\exists t_1 \geq t_0$ such that

$$H(t, V(t)) \leq c_0(\overline{\gamma}-1)(q-1)|G(t)|^q|\frac{V(t)}{G(t)}|^\gamma$$

$$(p-1)r^{1-q}(t)h^{-q}(t)H(t, V(t)) \leq c_0(\overline{\gamma}-1)r^{1-q}(t)h^{-q}(t)|G(t)|^q|\frac{V(t)}{G(t)}|^\gamma$$

$$since\ (p-1)(q-1) = 1$$

$$and \quad G(t) = r(t)h(t)|h'(t)|^{p-1}$$

$$|G(t)|^q = r^q(t)h^q(t)|h'(t)|^{p-1}q = r^q(t)h^q(t)|h'(t)|^p \qquad , since \quad (p-1)q = p.$$

$\Rightarrow (p-1)r^{1-q}(t)h^{-q}(t)H(t,V(t)) \leq c_0(\overline{\gamma}-1)r^{1-q}(t)h^{-q}(t)|G(t)|^q|\frac{V(t)}{G(t)}|^\gamma$
holds for $t \geq t_1$.
Consequently, if $w = h^{-p}(V+G)$, we have by identity (2.2)

$$i.e.\ h^pR[w] = v' + hL[h] + (p-1)r^{1-q}h^{-q}H(t,V)$$

$$= \frac{\alpha(1-\overline{\gamma})}{R(t)[\int_t^\infty R^{-1}(s)ds]^{\overline{\gamma}}} - h(t)L[h](t) + hL[h] + (p-1)r^{1-q}h^{-q}H(t,V)$$

$$= \frac{\alpha(1-\overline{\gamma})}{R(t)[\int_t^\infty R^{-1}(s)ds]^{\overline{\gamma}}} + (p-1)r^{1-q}h^{-q}H(t,V)$$

$$\leq \frac{\alpha(1-\overline{\gamma})}{R(t)[\int_t^\infty R^{-1}(s)ds]^{\overline{\gamma}}} + c_0(\overline{\gamma}-1)r(t)|h'(t)|^p|\frac{V(t)}{G(t)}|^\gamma$$

$$= \frac{\alpha(1-\overline{\gamma})}{R(t)[\int_t^\infty R^{-1}(s)ds]^{\overline{\gamma}}} + c_0(\overline{\gamma}-1)r(t)|h'(t)|^p|\frac{-\alpha-Y(t)}{G(t)[\int_t^\infty R^{-1}(s)ds]^{\overline{\gamma}-1}}|^\gamma$$

$$= \frac{\alpha(1-\overline{\gamma})}{R(t)[\int_t^\infty R^{-1}(s)ds]^{\overline{\gamma}}} + c_0(\overline{\gamma}-1)r(t)|h'(t)|^p\frac{|-\alpha-Y(t)|^\gamma}{|G(t)|^\gamma}[\int_t^\infty R^{-1}(s)ds]^{-\overline{\gamma}}$$

$$= \frac{1}{R(t)[\int_t^\infty R^{-1}(s)ds]^{\overline{\gamma}}}[\alpha(1-\overline{\gamma}) + c_0(\overline{\gamma}-1)\frac{r(t)|h'(t)|^pR(t)}{|G(t)|^\gamma}|\alpha+Y(t)|^\gamma]$$

$$Since\ \frac{r(t)|h'(t)|^pR(t)}{|G(t)|^\gamma} = \frac{r(t)|h'(t)|^pr^{\gamma-1}(t)h^\gamma(t)|h'(t)|^{(\gamma-q)(p-1)}}{|G(t)|^\gamma}$$

$$= \frac{r(t)|h'(t)|^pr^{\gamma-1}(t)h^\gamma(t)|h'(t)|^{\gamma(p-1)-p}}{|G(t)|^\gamma}$$

since $(\gamma-q)(p-1) = \gamma(p-1)-p$

$$= \frac{r^\gamma(t)h^\gamma(t)|h'(t)|^{\gamma(p-1)}}{|G(t)|^\gamma} = \frac{(r(t)h(t)|h'(t)|^{p-1})^\gamma}{|G(t)|^\gamma} = \frac{|G(t)|^\gamma}{|G(t)|^\gamma} = 1 \qquad (4.5)$$

$= \frac{1}{R(t)[\int_t^\infty R^{-1}(s)ds]^{\overline{\gamma}}}[\alpha(1-\overline{\gamma}) + \gamma(\overline{\gamma}-1)|\alpha+Y(t)|^\gamma] \qquad since\ |\alpha+Y(t)|^\gamma < \frac{\alpha}{c_0}$

$$< \frac{1}{R(t)[\int_t^\infty R^{-1}(s)ds]^{\overline{\gamma}}}[\alpha(1-\overline{\gamma}) + \alpha(\overline{\gamma}-1)] = 0$$

$$\Rightarrow h^pR[w] < 0$$

This means that (1.1) is non-oscillation by lemma 4. □

Theorem 8 *let* $\gamma \in (1,2)$ *be a real number and* $\overline{\gamma} = \frac{\gamma}{\gamma-1}$ *be the conjugate number of* γ. *Let* h *be a positive continuously differentiable function such that* $h^{'}(t) \neq 0$ *in some neighborhood of infinity. Define*

$$R(t) = r^{\gamma-1}(t)h^{\gamma}(t)|h^{'}(t)|^{(\gamma-q)(p-1)} \text{ and suppose that}$$

$$\int_0^{\infty} h(t)L[h](t)dt < \infty \text{ and } \lim_{t\to\infty} |G(t)|[\int_0^t R^{-1}(s)ds]^{\bar{\gamma}-1} = \infty$$

$$If \lim_{t\to\infty} sup[\int_0^t R^{-1}(s)ds]^{\bar{\gamma}-1} \int_t^{\infty} h(s)L[h](s)ds < \infty \tag{4.6}$$

and

$$\lim_{t\to\infty} inf[\int_0^t R^{-1}(s)ds]^{\bar{\gamma}-1} \int_t^{\infty} h(s)L[h](s)ds > -\infty \tag{4.7}$$

then (1.1) is non-oscillatory.

Proof let

$$Y(t) = [\int_0^t R^{-1}(s)ds]^{\bar{\gamma}-1} \int_t^{\infty} h(s)L[h](s)ds$$

then condition (4.6) and (4.7) imply that there exist $t_0 \in R$ and a positive constants α, c_0 such that

$$|Y(t) + \alpha|^{\gamma} < \frac{\alpha}{c_0} \qquad , for\ t \geq t_0$$

Define the function

$$\begin{aligned} V(t) &= \alpha[\int_0^t R^{-1}(s)ds]^{1-\bar{\gamma}} + \int_t^{\infty} h(s)L[h](s)ds \\ &= \frac{\alpha}{[\int_0^t R^{-1}(s)ds]^{\bar{\gamma}-1}} + \int_t^{\infty} h(s)L[h](s)ds \\ &= \frac{\alpha + [\int_0^t R^{-1}(s)ds]^{\bar{\gamma}-1} \int_t^{\infty} h(s)L[h](s)ds}{[\int_0^t R^{-1}(s)ds]^{\bar{\gamma}-1}} \\ &= \frac{\alpha + Y(t)}{[\int_0^t R^{-1}(s)ds]^{\bar{\gamma}-1}} \\ \frac{V(t)}{G(t)} &= \frac{\alpha + Y(t)}{G(t)[\int_0^t R^{-1}(s)ds]^{\bar{\gamma}-1}} \quad and \\ V^{'}(t) &= \frac{\alpha(1-\overline{\gamma})}{R(t)[\int_0^t R^{-1}(s)ds]^{\bar{\gamma}}} - h(t)L[h](t) \end{aligned}$$

Condition of the theorem imply that $\frac{V(t)}{G(t)} \to 0$ as $t \to \infty$, hence, using inequality

$$H(t, V(t)) \leq K|G(t)|^q|\tfrac{V(t)}{G(t)}|^\gamma \text{ with } K = c_0(\overline{\gamma} - 1)(q - 1)$$

there exist $t_1 \geq t_0$ such that

$$H(t, V(t)) \leq c_0(\overline{\gamma} - 1)(q - 1)|G(t)|^q|\frac{V(t)}{G(t)}|^\gamma$$

$$\Rightarrow (p - 1)r^{1-q}(t)h^{-q}(t)H(t, V(t)) \leq c_0(\overline{\gamma} - 1)r^{1-q}(t)h^{-q}(t)|G(t)|^q|\tfrac{V(t)}{G(t)}|^\gamma$$

$$since\ (p - 1)(q - 1) = 1$$

$$Since\ G(t) = r(t)h(t)|h^{'}(t)|^{p-1}$$

$$|G(t)|^q = r^q(t)h^q(t)|h^{'}(t)|^{(p-1)}q = r^q(t)h^q(t)|h^{'}(t)|^p, since\ (p - 1)q = p.$$

$\Rightarrow (p-1)r^{1-q}(t)h^{-q}(t)H(t, V(t)) \leq c_0(\overline{\gamma}-1)r(t)|h^{'}(t)|^p|\tfrac{V(t)}{G(t)}|^\gamma$ holds for $t \geq t_1$.
Consequently, if $w = h^{-p}(V + G)$, we have by identity (2.2)

$$\begin{aligned}
h^p R[w] &= v^{'} + h(t)L[h](t) + (p - 1)r^{1-q}h^{-q}H(t, v) \\
&= \frac{\alpha(1 - \overline{\gamma})}{R(t)[\int_0^t R^{-1}(s)ds]^{\bar{\gamma}}} - h(t)L[h](t) + h(t)L[h](t) + (p - 1)r^{1-q}h^{1-q}H(t, V(t)) \\
&= \frac{\alpha(1 - \overline{\gamma})}{R(t)[\int_0^t R^{-1}(s)ds]^{\bar{\gamma}}} + (p - 1)r^{1-q}h^{1-q}H(t, v(t)) \\
&\leq \frac{\alpha(1 - \overline{\gamma})}{R(t)[\int_0^t R^{-1}(s)ds]^{\bar{\gamma}}} + c_0(\overline{\gamma} - 1)r(t)|h^{'}(t)|^p|\frac{V(t)}{G(t)}|^\gamma \\
&= \frac{\alpha(1 - \overline{\gamma})}{R(t)[\int_0^t R^{-1}(s)ds]^{\bar{\gamma}}} + c_0(\overline{\gamma} - 1)r(t)|h^{'}(t)|^p|\frac{\alpha + Y(t)}{G(t)[\int_0^t R^{-1}(s)ds]^{\bar{\gamma}-1}}|^\gamma \\
&= \frac{\alpha(1 - \overline{\gamma})}{R(t)[\int_0^t R^{-1}(s)ds]^{\bar{\gamma}}} + c_0(\overline{\gamma} - 1)r(t)|h^{'}(t)|^p\frac{|\alpha + Y(t)|^\gamma}{|G(t)|^\gamma[\int_0^t R^{-1}(s)ds]^{\bar{\gamma}}} \\
&= \frac{1}{R(t)[\int_0^t R^{-1}(s)ds]^{\bar{\gamma}}}[\alpha(1 - \overline{\gamma}) + c_0(\overline{\gamma} - 1)\frac{r(t)|h^{'}(t)|^p R(t)}{|G(t)|^\gamma}|\alpha + Y(t)|^\gamma]
\end{aligned}$$

$$Since \quad \frac{r(t)|h'(t)|^p R(t)}{|G(t)|^\gamma} = 1 \quad by \quad (4.5)$$

$$= \frac{1}{R(t)[\int_t^\infty R^{-1}(s)ds]^{\overline{\gamma}}}[-\alpha(1-\overline{\gamma}) + c_0(\overline{\gamma}-1)|\alpha + Y(t)|^\gamma]$$

$$since\ |\alpha + Y(t)|^\gamma < \frac{\alpha}{c_0}$$

$$< \frac{1}{R(t)[\int_t^\infty R^{-1}(s)ds]^{\overline{\gamma}}}[-\alpha(1-\overline{\gamma}) + \alpha(\overline{\gamma}-1)] = 0$$

$$\Rightarrow h^p R[w] < 0. \qquad \square$$

This means that (1.1) is non-oscillation by lemma 4.

Conclusion

Eventhough there are different non-oscillation criterias of half-linear differential equation, out of these, we consider only the modified Riccati technique. Non-oscillation criteria using Riccati technique means establishing sufficient condition which guarantee that

$$R[w](t) \leq 0 \; for \; large \; t.$$

There are conditions satisfying the above statement i.e. for $\gamma \in (1,2)$ be a real number and $\overline{\gamma} = \frac{\gamma}{\gamma-1}$ be the conjugate number of γ. Let h be a positive continuously differentiable function such that $h^{'}(t) \neq 0$ both for large t. if

$$\int_t^{\infty} R^{-1}(s)ds < \infty \qquad and \qquad \lim_{t\to\infty} |G(t)|[\int_t^{\infty} R^{-1}(s)ds]^{\bar{\gamma}-1} = \infty$$

then (1.1)is non-oscillatory prvided that

$$\lim_{t\to\infty} sup[\int_t^{\infty} R^{-1}(s)ds]^{\bar{\gamma}-1} \int_0^t h(s)L[h](s)ds < \infty$$

and

$$\lim_{t\to\infty} inf[\int_t^{\infty} R^{-1}(s)ds]^{\bar{\gamma}-1} \int_0^t h(s)L[h](s)ds > -\infty$$

If the integral $\int_0^{\infty} h(t)L[h](t)dt < \infty$,then the above statement can be modified as follows i.e. equation (1.1) is non-oscillatory provided that

$$\lim_{t\to\infty} sup[\int_0^t R^{-1}(s)ds]^{\bar{\gamma}-1} \int_t^{\infty} h(s)L[h](s)ds < \infty$$

and

$$\lim_{t\to\infty} inf[\int_0^t R^{-1}(s)ds]^{\bar{\gamma}-1} \int_t^{\infty} h(s)L[h](s)ds > -\infty.$$

Appendix

Asymptotic Relation

Definition let $f, g : D \subseteq R \to R, x_0 \in \overline{D}$, the asymptotic behavior of f for $x \to x_0$ is given by the value of $f(x)$ for $x \to x_0$.
We shall use the following notations to describe the behavior of a function $f(x)$in the limit as $x \to x_0$.

1. $f \sim g$ as $x \to x_0$ read as f is asymptotic to g or g is an asymptotic approximation to f as $x \to x_0$. This means that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$$

2. $f = o(g)$ as $x \to x_0$, is read as f is order less than g as $x \to x_0$. This means
$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$. Also we write $f \ll g$ as $x \to x_0$.

3. $f = O(g)$ as $x \to x_0$, is read as as f is order not exceeding g as $x \to x_0$. This means $\frac{f}{g}$ is bounded in the neighborhood of x_0.

Note:

- $f = o(1)$ as $x \to x_0$, meaning f vanish as $x \to x_0$.
- $f = O(1)$ as $x \to x_0$, meaning that $|f|$ is bounded as $x \to x_0$.
- **Taylor's formula**
  When f is a function and $k \geq 0$ is an integer the notation $f^{(k)}$ denotes k^{th} derivative off f. Thus

$$f^{(0)}(x) = f(x), \quad f^{(1)}(x) = f^{'}(x),\ f^{(2)}(x) = f^{''}(x),\ and\, so\, on.$$

  Given a number a in the domain of f and an integer $n \geq 0$, the polynomial

$$p_n(x) = \sum_{k=0}^{n} \frac{(f^{(k)}(a)(x-a)^k)}{k!}$$

is called the degree n Taylor polynomial of f centered at a. The Taylor polynomial $p_n(x)$ is the unique polynomial of degree n which has the same derivatives as f at a up to order n :

$$p_n^{(k)}(a) = f^{(k)}(a)\, fork = 0, 1, 2, n$$

- **Young inequality**: If $p, q \in [1, \infty]$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $a, b \geq 0$ then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ where q is the conjugate of p. Therefore the expression $P(a, b) := \frac{a^p}{p} - ab + \frac{b^q}{q}$ is positive for $p, q \in [1, \infty]$.
For $a \neq 0$, $b \neq \Phi_p(a)$ and consider the function

$$Q(a, b) = \frac{p(a,b)}{|a|^{(P-1)(q-\gamma)}|b-\Phi_p(a)|^{\gamma}}$$

it is bounded below by a positive constant if $1 < p \leq 2 \leq \gamma \leq q$ and bounded above if $1 < q \leq \gamma \leq 2 \leq p$.direct computation shows that $Q(a, b) = f(\frac{b}{\Phi_p(a)})$ where

$$f(x) = \frac{\frac{1}{p} - x + \frac{|x|^q}{q}}{|x-1|^{\gamma}}$$

the function f is positive and continuous on $R \setminus \{1\}$ and has the following properties

$$\lim_{x \to \overset{+}{-}\infty} f(x) = \begin{cases} \infty, & for \quad \gamma < q \\ \frac{1}{q}, & for \quad \gamma = q \\ 0 & for \quad \gamma > q \end{cases}$$

and

$$\lim_{x \to 1} f(x) = \begin{cases} \infty, & for \quad \gamma > 2 \\ \frac{q-1}{2}, & for \quad \gamma = 2 \\ 0 & for \quad \gamma < 2 \end{cases}$$

- **Energy functional**:The p-degree functional

$$F(y; a, b) = \int_a^b [r(t)|y^{'}|^p - c(t)|y|^p]dt$$

Consider over the Soblove space $w_0^{1,p}(a, b)$ is usually called the *Energy functional* of (1).

- **Soblove space** $W_0^{1,p}(a,b)$:is the space of absolutely continuous functions x whose derivative is in $L^p(a,b)$ and $x(a) = 0 = x(b)$, with the norm
$$\|x\| = (\int_a^b [|x^{'}|^p + |x|^p]dt)^{\frac{1}{p}}$$

- $L^p(a,b)$ **Space**: the vector space of all continuous real-valued functions on (a,b) forms a normed space X with norm defined by
$$\|x\| = (\int_a^b |x|^p)^{\frac{1}{p}}$$

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