Addis Ababa University
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Graduate Seminar Report

On

HYPERBOLIC EQUATION AND FLOW IN PIPELINES

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Preface

This seminar report is devoted to "Hyperbolic equations and flow in pipelines."

The aim of this seminar report is to show that equations describing fluid flow in pipelines subjected to certain conditions is hyperbolic.

The material is arranged in the following way:

Chapter I : presents the elements of quantities that describe the flow of fluids and the Transport theorem.

Chapter II : discuss about conservational laws and some thermodynamics results.

Chapter III: deals about the hyperbolicity of the equations governing the fluid flow in pipelines.

I owe a special dept of gratitude to my Advisor and instructor **Dr. Tsegaye Gedif** for his unreserved material support, invaluable and continuous advice as well as I would like to express my sincere appreciation to **Agegnehu Atena(Proc.Ph.D)** for his many helpful suggestions, which substantially improved this seminar.

My special thanks must go to my fiancé **Hirut Assaye** for bearing with me through all the hours spent on this. With out her encouragement and counsel this seminar would never have come to reality

It is my best pleasure to thank my family who has helped me in both finance and advice.

To: ***Hiry*** and my mother ***Minalu Tiruneh***.

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1 Preliminaries

1.1 Some Mathematical Concepts and Notations:

1.1.1 Basic Notation:

$\Re^n (n \geq 1)$ = Euclidean space with axes $x_1, x_2, \ldots, x_n$.

e_i $(i = 1,2,\ldots,n)$ = the unit vectors in the direction s of the coordinate axes.

X, y $\in \Re^n$ will be denoted as x= $(x_1, x_2, \ldots, x_n)$ and y=$(y_1, y_2 \ldots, y_n)$.

If P and Q are two sets and f is the mapping (function) defined on P with its values lying in Q, we write f: $P \to Q$. For such a mapping and a subset $M \subseteq P$ the symbol

$f/_M$,then g: $M \to Q$ and g(x)=f(x) $\forall x \in M$.

For $M \subseteq \Re^n$, C(M) = The space of all continuous on M.

If $\Omega \subseteq \Re^n$ is an open set and $k \geq 0$ is an integer, then

$C^k(\Omega)$= The space of all functions which have continuous partial derivatives up to the order k in Ω.

$\partial\Omega$ = The boundary of the set Ω.

$\overline{\Omega}$ = The closure of the set Ω.

$C^k(\overline{\Omega})$= The space of all functions from $C^k(\Omega)$ whose all derivatives up to the order k can be continuously extended onto $\overline{\Omega}$.

f $\in [C^k(\Omega)]^n$, if **f**: $\Omega \to \Re^n$. that is, **f**=$(f_1, f_2, \ldots, f_n)$, $f_i : \Omega \to \Re^1$ for i= 1,2,…n. and $f_i \in C^k(\Omega) \, \forall i = \overline{1,n}$. Similarly we define the space $[C^k(\overline{\Omega})]^n$.

If P is a set, then P^n = PxPxPx…xP (n times), That is $P^n = \{ (a_1, a_2, \ldots, a_n) / a_i \in P, i = \overline{1,n}. \}$

Quantities describing fluid flow are functions of space and time. It means we write such a quantity as a function f= f(x,t) , where t is the time and x=(x_1, x_2, x_3) denotes points of a set Ω_t occupied by the fluid at time t.

Let (T_1, T_2) with $T_1 < T_2$ be a time interval during which we follow the fluid motion. Then the domain of definition of the function f is the set

$$\mu = \{ (\mathbf{x},\mathbf{t}) / \mathbf{x} \in \Omega_t, \mathbf{t} \in (\mathbf{T_1}, \mathbf{T_2}) \} \subseteq \Re^4$$

Let us assume that the set μ is open. If $t \in (T_1, T_2)$ is fixed, then f(. ,t) denotes the function $x \mapsto f(x,t)$ whose value at $x \in \Omega_t$ equals f(x,t).

1.1.2 **Differential Operators**: Let us consider a function f= f(x,t) defined on the μ and having continuous partial derivatives with respect to x_1, x_2, x_3 in μ, then

$$\text{(1.1.3)} \qquad \text{grad f} = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$$

If $\mathbf{f} = (f_1, f_2, f_3) : \mu \to \Re^3$ is a vector function with components which have continuous first order partial derivatives with respect to

x_1, x_2, x_3 in μ, then we set

$$\text{(1.1.4)} \qquad \text{div } \mathbf{f} = \sum_{i=1}^{3} \frac{\partial f_i}{\partial x_i}, \qquad \text{rot f} = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)$$

we define the scalar product in $\Re^n$ as

$$\mathbf{a.b} = \sum_{i=1}^{n} a_i b_i \quad \text{where } a = (a_1, a_2, \ldots, a_n) \text{ and } b = (b_1, b_2, \ldots, b_n) \in \Re^n.$$

$|a| = (a.a)^{1/2}$ (Euclidean norm)

For $\mathbf{a}, \mathbf{b} \in \Re^3$

$$\mathbf{axb} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

$$= \det\begin{pmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

Now, using the differential operator nabla , $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ we can express the operators grad, div and rot in the form

$$\text{(1.1.7)} \qquad \begin{aligned} \text{Gradf} &= \nabla f \\ \text{Divf} &= \nabla . f \\ \text{Rotf} &= \nabla x f \end{aligned}$$

Let f: $\mu \rightarrow \Re^3$ be a mapping whose components have continuous first order partial derivatives with respect to x_1,x_2,x_3 . We define the Jacobi matrix of the mapping f(., t) ($t \in (T_1, T_2)$ is fixed) as the matrix

(1.1.8)
$$\frac{Df(x,t)}{Dx} = \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix} (x,t)$$

Let f, f_i: $\mu \rightarrow \Re^1$, $\mathbf{f}=(f_1,f_2,f_3)$ and let f, f_i have continuous second order partial derivatives with respect to x_1,x_2,x_3 . then we set

(1.1.9) $\Delta f = \sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i^2}$, $\Delta \mathbf{f} = (\Delta f_1 , \Delta f_2 , \Delta f_3)$ where Δ is called Laplace operator.

Let the mapping **u:** $(T_1, T_2) \rightarrow \Re^n$: if $\mathbf{U} = (u_1,u_2,\ldots,u_n)$, then

1.1.10) $U' = (u'_1,u'_2,\ldots,u'_n)$.

1.1.11 **Transformations of Cartesian coordinates**: Let us consider two Cartesian coordinate systems $x= (x_1 ,x_2, \ldots,x_n)$ and $x^*=(x^*_1 ,x^*_2, \ldots,x^*_n) \in \Re^n$. Then the transition from x_i to x^*_i is realized by the relation

(1.1.12) $x^*_i = \sum_{j=1}^{n} a_{ij} x_j + c_i$, $i=\overline{1,n}$ where $A= (a_{ij})^n_{i,j=1}$ is an orthonormal matrix.

That is, (1.1.13) $AA^T = I$, where I denotes the unit matrix and $A^T =(a_{ji})^n_{i,j=1}$ is the transpose of A. Relation (1.1.12) can be written in the vector form

(1.1.14) $x^*=Ax + C$, $C =(c_1, c_2, \ldots, c_n)$

The transformation inverse to (1.1.12) is

(1.1.15) $$X_k=\sum_{j=1}^{n} a_{ik}\, x_i{}^* + c^*_k \quad k=\overline{1,n} \text{ where } c^*_k = -\sum_{i=1}^{n} a_{ik} c_i \,.$$

The vector form of (1.1.15) is

(1.1.16) $$X=A^T x^* + C^* , \qquad C^*=(c^*_1, c^*_2, \ldots, c^*_n)$$

1.1.17 Tensors: The quantity represented by an element $b=(b_1,b_2,\ldots,b_n) \in \Re^n$ is called a tensor of order one or a vector, if the transformation (1.2.12) changes its components b_i in to

(1.1.18) $$b^*_i = \sum_{j=1}^{n} a_{ij} b_j \,, \quad i=\overline{1,n}$$ which can be written as

(1.1.19) $$b^*=Ab.$$ The quantity represented in the coordinate system $x_1, x_2, \ldots, x_n$ by a matrix $T =(T_{ij})^n_{i,j=1}$ is called a tensor of order 2.

If the transformation (1.1.12) changes the entries of T in to

(1.1.20) $$T_{ij}^* = \sum_{k,m=1}^{n} a_{ik}\, a_{jm} T_{km}$$ we see that we get

the matrix

(1.1.21) $$T^* = ATA^{-1} = ATA^T.$$

1.1.22 **A curve in $\Re^n$**: is defined as a continuous mapping φ of compact interval $\subseteq \Re^1$ in to $\Re^n$. It means that $\varphi:[a,b] \rightarrow \Re^n$, $a,b \in \Re^1$, $a<b$, $\varphi=(\varphi_1, \varphi_2, \ldots, \varphi_n)$ and the functions $\varphi_i: [a,b] \rightarrow \Re^1$ are continuous for $i=\overline{1,n}$. The set

(1.1.23) $\langle\varphi\rangle = \{ \varphi(\tau) \in \Re^n / \tau \in [a,b]\}$ is called the geometric image of the curve φ.

We say that a curve $\varphi:[a,b] \rightarrow \Re^n$ is smooth , if $\varphi \in [C^1([a,b])]^n$ and $\varphi(\tau) \neq 0$ for each $\tau \in[a,b]$.

Then at every point $\varphi(\tau) \in \langle\varphi\rangle$ it is possible to define a unit tangent vector $\mathbf{t}(\tau) = \dfrac{\varphi(\tau)}{|\varphi'(\tau)|}$, oriented in the direction of the course along the curve φ.

- A curve $\varphi:[a,b] \to \Re^n$ is of the class C^k ($k \geq 1$) if it is smooth and $\varphi \in [C^k([a,b])]^n$.
- The symbol $\int_\varphi f ds$ denotes the curvilinear integral of the function

f: $\langle\varphi\rangle \to \Re^1$ along the curve φ. If $\varphi:[a,b] \to \Re^n$ is piecewise curve and $f \in C(\langle\varphi\rangle)$,

then the integral $\int_\varphi f ds$ exist , is finite and $\int_\varphi f ds = \int_a^b f(\varphi(\tau))|\varphi'(\tau)|d\tau$

-An open set $\Omega \subseteq \Re^n$ is connected, if its arbitrary two points can be connected with a line which is a geometric image of a piecewise linear curve φ with $\langle\varphi\rangle \subseteq \Omega$. An open and connected set is called a domain.

Definition: a function **f:** $M \to \Re^n$ is Lipschitz- continuous, if there exists a constant L such that $|\mathbf{f}(x) - \mathbf{f}(y)| \leq L\,|x-y|, \forall\, x,y \in M$.

1.1.24 **Domains with Lipschitz- continuous boundaries**: Let $\Omega \subseteq \Re^n$ be a bounded domain . Its boundary $\partial\Omega$ is called Lipschitz- continuous, if there exist numbers $\alpha \succ 0, \beta \succ 0$, and a finite number of local Cartesian coordinate systems $x_1^r, x_2^r, \ldots, x_n^r$ and Lipschitz- continuous functions a_r: $\mu_r \to \Re^1$ where $\mu_r = \{\hat{x}^r = ((x_2^r, \ldots, x_n^r) \in \Re^{n-1}; |\hat{x}^r| \leq \alpha\}$, $r = \overline{1, R}$ such that

$$\partial\Omega = \bigcup_{r=1}^{R} \Lambda_r \quad \text{where } \Lambda_r = \{ (x_1^r, \hat{x}^r) / x_1^r = a_r(\hat{x}^r), |\hat{x}^r| < \alpha \}$$

$$\text{and } \Gamma_r = \{ (x_1^r, \hat{x}^r) / a_r(\hat{x}^r) < x_1^r < a_r(\hat{x}^r) + \beta, |\hat{x}^r| < \alpha \} \subset \Omega, r = \overline{1, R}.$$

$$\Delta_r = \{(x_1^r, \hat{x}^r) / a_r(\hat{x}^r) - \beta < x_1^r < a_r(\hat{x}^r), |\hat{x}^r| < \alpha \} \subset \Re^n \setminus \Omega, r = \overline{1, R}.$$

If $a_r \in C^k(\mu_r)$ for all $r=\overline{1,R}$ then we write $\partial\Omega \in C^k$. The boundary $\partial\Omega$ is called smooth, if $\partial\Omega \in C^1$.

1.1.25 **Green's Theorem:** Let $\Omega \subseteq \Re^n$ be a bounded domain with a Lipschitz-continuous boundary and let $u,v \in C^1(\overline{\Omega})$. Then

$$\int_{\Omega} \frac{\partial u}{\partial x_i} v dx = \int_{\partial\Omega} uvn_i ds - \int_{\Omega} u\frac{\partial v}{\partial x_i} dx \tag{1.1.26}$$

Proof: it is suffice to show that $\int_{\Omega} \frac{\partial u}{\partial x_i} v dx + \int_{\Omega} u\frac{\partial v}{\partial x_i} dx = \int_{\partial\Omega} uvn_i ds$

Now, $\int_{\Omega}((v\nabla u)+(u\nabla v))dx = \int_{\Omega}((v\frac{\partial u}{\partial x_1}, v\frac{\partial u}{\partial x_2}, v\frac{\partial u}{\partial x_3})+(u\frac{\partial v}{\partial x_1}, u\frac{\partial v}{\partial x_2}, u\frac{\partial v}{\partial x_3}))dx$

$$= \int_{\Omega}((v\frac{\partial u}{\partial x_1}+u\frac{\partial v}{\partial x_1}),(v\frac{\partial u}{\partial x_2}+u\frac{\partial v}{\partial x_2}),(v\frac{\partial u}{\partial x_3}+u\frac{\partial v}{\partial x_3}))dx$$

$$= \int_{\Omega}\nabla(uv)dx$$

$$= \int_{\partial\Omega}(uv)n_i ds \quad \text{(using Gradient theorem)}$$

Hence, the proof of the theorem.

For $u \in [C^1(\overline{\Omega})]^n$, we get from Green's theorem the identity

$$\int_{\Omega}\nabla.u dx = \int_{\partial\Omega} u.n ds \tag{1.1.27}$$

1.1.28 **Lemma**: Let $\Omega \subseteq \Re^n$ be an open set, $f \in C(\Omega)$. Then

$f=0$ in Ω if and only if $\int_{\sigma} f dx = 0$ for any open bounded set $\sigma \subset \overline{\sigma} \subset \Omega$.

Proof: $(\Rightarrow)$ suppose $f=0$ in Ω.

We shall show that $\int_{\sigma} f dx = 0$ for every open set $\sigma \subset \overline{\sigma} \subset \Omega$.

Since f=0 in Ω then so does on any open subset σ of $\sigma \subset \overline{\sigma} \subset \Omega$.

$\Rightarrow$ $\int_{\sigma} f dx$ =0 (trivially)

($\Leftarrow$) Suppose $\int_{\sigma} f dx$ =0 for all open subset σ of $\sigma \subset \overline{\sigma} \subset \Omega$.

We shall show f=0 in Ω.

Suppose not! That is, f$\neq$ 0 in some open subset σ * in Ω such that

$\sigma * \subset \overline{\sigma} \subset \Omega$.

$\Rightarrow$ |f|>0 in σ *

Now since f$\in$ C(Ω), so does |f| $\in$ C(Ω) and $\int_{\sigma} | f | dx$ >0 in σ *, which

is a contradiction .

$\Rightarrow$ f=0 in Ω.

CHAPTER- 1

Quantities and Concepts Describing the Flow

1.1 INTRODUCTION

If one casually glances around, most things seem to be solid, but when one thinks of the oceans, the atmosphere and on out into space it becomes rather obvious that a good portion of the earth's surface and of the entire universe is in the fluid state. Indeed, it is difficult to think of any machine, device or tool which does not have some fluid hidden in it somewhere and some fluid mechanics behind its design.

Fluid Dynamics forms one of the foundations of aeronautics and astronautics, mechanical engineering, marine engineering, civil engineering ,bio- engineering and in fact, just about every scientific and engineering field.

1.2 Fundamental Conceptions about Fluids

1.2.1 **Fluid as a continuous Medium**: It is clear that the mass consists of discrete particles – molecules and atoms. In the study of fluid motion it is usually not necessary to consider the real discrete structure of the mass, but on the contrary, it is suitable to start from the idea that the fluid particles have no size and are continuously distributed. This means that we assume that each point of the domain occupied by the fluid represents exactly one fluid particle. Hence, we say that the fluid is continuously medium or simply a continuum.

Therefore, we assume that functions describing the behavior and motion of fluid are continuous or sufficiently smooth.

1.2.2 **Viscosity:** Fluids differ from solids by their microstructure. A significant role is played by intermolecular forces and molecular mean free path, characterizing heat phenomena. In the fluids the molecular mean free path is essentially larger and the intermolecular forces are smaller than in solids. These properties causes that fluid volume does not have its own definite shape and is easily divisible. The fluid puts up a relatively small resistance against forces causing the change of their form. The property of putting up a resistance against the change of shape is called Viscosity.

- All fluids have viscosity, which causes friction. The importance of this friction in physical situations depends on the type of the fluid and the physical configuration or the flow pattern.

If we consider a small element of the fluid as shown below in Fig 1.1 the shear stress τ on the top (which is numerically the same as on the bottom in this case) may be written:

$\tau = \mu \frac{\partial u}{\partial y}$ Where μ = viscosity

$\frac{\partial u}{\partial y}$ = Velocity gradient.

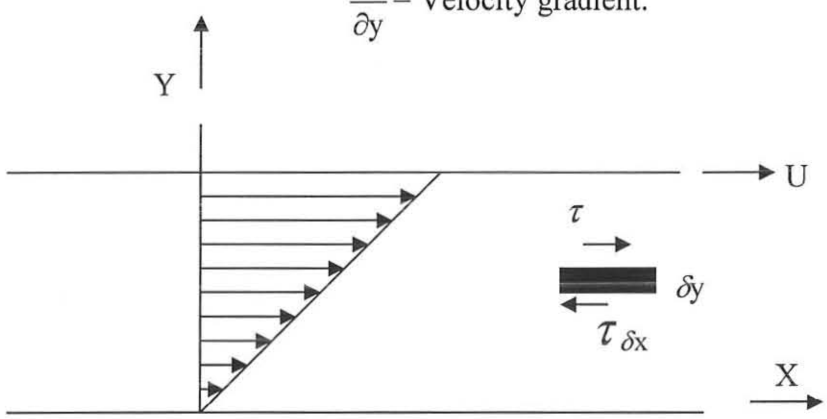


Fig 1.1 Flow b/n parallel plates to illustrate viscosity.

1.2.3 Laminar and Turbulent flow: In laminar flow the trajectories of fluid particles are well organized (ordered) and the particles moves in layers, which slides on each other. That is, no random vortex structure in the flow pattern. (Occurs when in wide pipes and slow velocities). On, the contrary, in turbulent flow the motion of the fluid particles is unorganized, chaotic. The trajectories of particular particles intersect each other and their velocity has a character of random fluctuations around their mean value. That is, there is random vortex structure of the flow pattern.(occurs when high velocities and in narrow pipes.)

Fig 1.2 Laminar and turbulent flow

The factors that decide whether the flow will be laminar or turbulent, are

- The fluid velocity
- The shape and size of the regions occupied by the fluid
- The viscosity coefficient.

1.2.4 Compressibility: It is customary to divide fluids into two groups as gases and liquids. Gases are more compressible and their density changes readily with temperature and pressure. Liquids, on the other hand, are rather difficult to compress and for most problems are might consider then incompressible.

In a gas the density is related to temperature and pressure by the law $p=\rho R\theta$ (for perfect gas) .For liquids, the density is related to temperature by a coefficient of expansion just as for a solid, and the pressure dependence as $dp=\beta\frac{d\rho}{\rho}$ where β is bulk compression modulus.

In compressible flow there is a great distinction b/n flow involving velocities less than that of sound (subsonic) and flow involving velocities greater than that of sound (supersonic flow).

- It is important to remember that shock waves can only occur in flows, which are supersonic.

1.2.5 Character and types of a flow: According to the region occupied by a fluid, we can also classify the flow as inner and outer flow. By inner flow we mean pipe or channel flow and the like, where fluid flows in a confining structure.

On the other hand, outer flow is the flow of a fluid over an object, such as in aerodynamics. Now on wards, we shall focus on the internal (inner) flow.

1.3 **Kinematics of Fluids**: Let $(T_1,T_2) \subseteq \Re^1$ be the time interval, during which we follow the fluid motion, and let $\Omega_t \subseteq \Re^3$ denote the domain occupied by the fluid at a time $t\in(T_1,T_2)$.

The set Ω_t need not represent the whole region occupied by the fluid at a time t. It can be, a section of a pipeline, a part of a turbine or a small part of the ocean surrounding a ship. In many cases the domain Ω_t does not change its form with time and then $\Omega_t=\Omega$ $\forall\ t\in(T_1,T_2)$. Sometimes, however, we must take in to account the time dependant geometry of the domain filled by the fluid. It is suitable to suppose that its shape changes continuously with time. This requirement represents the assumption that there exist time $t_o\in(T_1,T_2)$ and a continuous mapping $\phi=\phi(x,t)$, $\phi: \Omega_{to}x\ (T_1,T_2) \rightarrow \Re^3$ such that $\phi(.,t)$ is a one to one transformation of Ω_{to} on to Ω_t for each $t\in(T_1,T_2)$.

- Now on we will always use the fundamental hypothesis that

(1.3.1) Exactly one fluid particle passes through each point $x\in\Omega_t$ at any time t.

There are two possibilities of describing the fluid motion:

1.3.2 **Lagrangian description of the flow**: Considers the motion of each particular fluid particle. The trajectories of the particles can be described by the equation

(1.3.3) $x= \varphi(X, t)$ (i.e $x_i=\varphi_i(X,t)$, i= 1,2,3) where X represents the reference determining the particle under consideration.

Equation (1.3.3) determines the position of the particle given by the reference X at a time t. Sometimes we use a more detailed description of the motion of fluid particles in the form

(1.3.4) $x= \varphi(X, t_o; t)$ which determines, at time t, the position x of the particle passing through the point (given by the reference) X at time t_o.

Then of course, $X=\varphi(X, t_o; t_o)$ provided the references are identical with the coordinates of particles at time t_o.

The velocity and the acceleration of the fluid particle given by the reference X are defined as

$$\text{a)}\quad \hat{v}(X,t) = \frac{\partial \varphi(X,t)}{\partial t} \qquad \left(= \frac{\partial \varphi(X,t_0,t)}{\partial t}\right)$$

(1.3.5)

$$\text{b)}\quad \hat{a}(X,t) = \frac{\partial^2 \varphi(X,t)}{\partial t^2} \qquad \left(= \frac{\partial^2 \varphi(X\,t_0,t)}{\partial t^2}\right) \text{ respectively}$$

provided the above derivatives exist.

1.3.6 **Eulearian description**: In the investigation of fluid flow we are rarely interested in the motion of each particular fluid particle , but we want to know the state of the flow and its change in dependence on time. Therefore, we usually use the so called Eulearian description based on the determination of the velocity **V**(x, t) of the fluid particle passing through the point x at time t.

Now, using (1.3.3) and (1.3.5) we can write

$$(1.3.7) \qquad \mathbf{V}(x,t) = \hat{v}(X,t) = \frac{\partial \varphi(X,t)}{\partial t} \text{ where } x = \varphi(X,t)$$

Since it is not important that the fluid particle considered is given by the reference X, we can use the notation $\tilde{\varphi}(t) = \varphi(X,t)$ and (1.3.7) can be rewritten as

$$(1.3.7^*) \qquad \mathbf{V}(x,t) = \frac{d\tilde{\varphi}(t)}{dt}, \qquad x = \tilde{\varphi}(t) = (\tilde{\varphi}_1(t), \tilde{\varphi}_2(t), \tilde{\varphi}_3(t)).$$

By V_i, i= 1,2,3 we will denote the components of the velocity vector **V** in the Cartesian coordinate system (x_1, x_2, x_3). Time t together with coordinates x_1, x_2, x_3 are the so- called Eulearian coordinates.

The domain of the mapping **V** =**V**(x, t) is the set μ , i.e.

$$(1.3.8) \quad \mathbf{V} : \mu \to \Re^3, \qquad \mu = \{ (x,t) \,/\, x \in \Omega_t, t \in (T_1, T_2) \}.$$

Since the set Ω_t depends continuously on time , the set μ is open. Then, it follows that we can assume **V** has continuous first order derivatives on μ : that is

$$(1.3.9) \qquad \mathbf{V} \in [C^1(\mu)]^3 .$$

For $(x,t) \in \mu$ we denote by **a**(x,t) the acceleration of the particle passing through the point x at time t. Then using relations (1.3.5b) and (1.3.7*) and the chain rule , we find that

$$\mathbf{a}(x,t) = \frac{d^2\widetilde{\varphi}(t)}{dt^2} = \frac{d}{dt}\left(v(\widetilde{\varphi}(t),t)\right)$$

$$= \frac{\partial v(\widetilde{\varphi}(t),t)}{\partial t} + \sum_{i=1}^{3} \frac{\partial v(\widetilde{\varphi}(t),t)}{\partial x_i} \frac{d\widetilde{\varphi}_i}{dt}$$

Hence, (1.3.10) $\mathbf{a}(x,t) = \dfrac{\partial v(x,t)}{\partial t} + \sum_{i=1}^{3} \dfrac{\partial v(x,t)}{\partial x_i} v_i(x,t)$, which can be written in the form :

(1.3.11) $$\mathbf{a} = \frac{\partial v}{\partial t} + (\mathbf{V}.\text{grad})\,\mathbf{V} = \frac{\partial v}{\partial t} + (\mathbf{V}.\ \nabla)\,\mathbf{V}$$

Let us introduce the symbol

(1.3.12) $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + (\mathbf{V}.\text{grad}) = \dfrac{\partial}{\partial t} + (\mathbf{V}.\ \nabla)$ is called the material (total)derivative with respect to the time. The partial derivative $\dfrac{\partial}{\partial t}$ is called the local derivative and the term $(\mathbf{V}.\ \nabla)$ is referred to the convective derivative. We see that the acceleration of a fluid particle is expressed in the Eulearian coordinates as the material derivative of the velocity:

(1.3.13) $$\mathbf{a} = \frac{dv}{dt} : = \frac{\partial v}{\partial t} + (\mathbf{V}.\ \nabla)\,\mathbf{V}$$

1.3.14 **Relations between the Lagrangian and Eulearian description of flow**: if we want to pass from the Lagrangian description of the flow to the Eulearian one, we must express the velocity **V**(x, t) in the Eulearian coordinates on the basis of (1.3.7) . Since in virtue of (1.3.1) , the mapping φ(., t) (for fixed t) is one to one , its inverse exists- let us denote it by ϕ(.,t). Then

(1.3.15) x= φ(X, t) $\Leftrightarrow$ X= ϕ (x,t) and we have

(1.3.16) $\mathbf{V}(x,t) = \hat{v}(\phi(x,t),t)$.

The transition from the Eulearian to the Lagragian one is equivalent to the determination of the paths of the fluid particles on the basis of a given velocity field **V**(x,t). The trajectory of the fluid particles passing through a point $X \in \Omega_{t0}$ at time $t_o \in (T_1,T_2)$ is given as the solution of the initial value problem

$$\frac{dx}{dt} = \mathbf{V}(x,t) \quad , x(t_o) = X \tag{1.3.17}$$

1.3.18 **Theorem:** Under assmption (1.3.9) the following statements hold:

1) For each $(X, t_o) \in \mu$ problem (1.3.17) has exactly one maximal solution $\varphi(X, t_{o;}\, t)$.
2) The mapping φ has continuous first oreder partial derivatives with respect to X_1, X_2, X_3, t_o, t and continuous derivatives $\frac{\partial^2 \varphi}{\partial t \partial X_i}, \frac{\partial^2 \varphi}{\partial t_0 \partial X_i}$, i= 1,2,3 in its domain of the definition $\{(X, t_o, t) / (X, t_o) \in \mu\}$

1.3.19 **Stationary Flow**: The flow is called stationary, if the quantities characterizing in the E ulearian description depend on space variables only and are independant of time. i.e. $\frac{\partial}{\partial t}=0$ and $\mathbf{V} = \mathbf{V}(x)$, a=a(x), etc where $x \in \Omega_t = \Omega$ for any time instant t.

1.3.20 **Tensor charactor of the velocity:** Let us show that under the transformation

$$x^*_i = \sum_{j=1}^{n} a_{ij} x_j \tag{1.3.21}$$

, i= 1,2,3. of the cartesian coordinates the components V_i of the velocity are transformed as a tensor of order one. Actually, if V_i and V^*_i are the velocity components in the coordinate systems x_i and x^*_i , respectively, then

$V_i(x, t) = \frac{d\tilde{\varphi}_i(t)}{dt}$, $x_i = \tilde{\varphi}_i(t)$ and $V^*_i(x,t) = \frac{d\tilde{\varphi}_i^*(t)}{dt}$, $x^*_i = \tilde{\varphi}^*_i(t)$ where $\tilde{\varphi}_i$ and $\tilde{\varphi}^*_i$ (i= 1,2,3)represents the components of particle trajectories in the coordinates x_i and x^*_i respectively.

Then using (1.3.21) we get

$$\tilde{\varphi}^*_i(t) = \sum_{j=1}^{3} a_{ij}\tilde{\varphi}_j(t) + c_i \quad , i = 1,2,3.$$

(1.3.22) $$V^*_i(t) = \sum_{j=1}^{3} a_{ij} v_j(t)$$

1.4 **Transport Theorem**:

Let a function F= F (x,t) : $\mu \rightarrow \Re^1$ be the Eulearian representation of some physical quantity transported by the fluid particles. And let us consider a system of fluid particles filling a bounded domain $\sigma(t) \subset \Omega_t$ at time t. The total amount of the quantity given by the function F that is contained in the volume $\sigma(t)$ at time t equals the integral

(1.4.1) $$f(t) = \int_{\sigma(t)} F(x,t)dx$$

Now, we shall be interested on the rate of change of the quantity F bound on the system of particles considered.

i.e. $$\frac{df(t)}{dt} = \frac{d}{dt}\int_{\sigma(t)} F(x,t)dx$$

It would be very nice, if the differentiation of the integral is with respect to a parameter under the assumption that the integration domain is fixed. But this is not the case here, as both the integrand F and the integration domain $\sigma(t)$ depends on time t. Now, to tackle this problem let us proceed as follows:

In our further consideration let us suppose that the velocity satisfies assumption (1.3.9) . i.e. $V \in [C^1(\mu)]^3$. Then Theorem (1.3.18) can be applied showing that the mapping $\varphi = \varphi(X, t_0; t)$ has continuous first order derivatives with respect to all its variables and continuous second order derivatives $\frac{\partial^2 \varphi}{\partial t \partial X_i}$, i= 1,2,3. The mapping φ defines the change of the domain $\sigma(t)$ with time: let $t_o \in (T_1, T_2)$ be an arbitrary fixed time instant and $\sigma(t_0) \subseteq \Omega_{t0}$. Then

(1.4.3) $\sigma(t) = \{ \varphi(X, t_{o;} t) / X \in \sigma(t) \}$ (provided $\varphi(X, t_{o;} t)$ is defined for all $X \in \sigma(t_0)$.

By J(X,t) we will denote the jacobian of the mapping $X \in \sigma(t) \to \varphi(X, t_{o;} t) \in \sigma(t)$ such that

$$(1.4.4) \quad J(X,t) = \det \frac{D\varphi(X, t_0; t)}{DX} = \det \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \frac{\partial \varphi_1}{\partial x_3} \\ \frac{\partial \varphi_2}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_2} & \frac{\partial \varphi_2}{\partial x_3} \\ \frac{\partial \varphi_3}{\partial x_1} & \frac{\partial \varphi_3}{\partial x_2} & \frac{\partial \varphi_3}{\partial x_3} \end{pmatrix} (X, t_0, t)$$

1.4.5 Lemma: Let $t_o \in (T_1, T_2)$, $\sigma(t_0)$ be a bounded domain and let $\overline{\sigma(t_0)} \subseteq \Omega_{t0}$.

Then there exists an interval $t_o \in (t_1, t_2)$ such that the following conditions are satisfied:

a) The mapping „$t \in (t_1, t_2)$, $X \in \sigma(t_0) \to x = \varphi(X, t_{o;} t) \in \sigma(t)$" has continuous first order derivatives with respect to t, X_1, X_2, X_3 and continuos second order derivatives $\frac{\partial^2 \varphi}{\partial t \partial X_i}$, i= 1,2,3.

b)The mapping „$X \in \sigma(t_0) \to x = \varphi(X, t_{o;} t) \in \sigma(t)$" is continuously differentioable one-to one mapping of $\sigma(t_0)$ on to $\sigma(t)$ with the jacobian (1.4.4) which is continuous and bounded and satisfies the condition :

$$\mathcal{J}(X,t) > 0 \ \forall X \in \sigma(t_0), \quad \forall t \in (t_1, t_2).$$

C) The inclusion $\{(x,t)/t \in [t_1, t_2], x \in \overline{\sigma(t)}\} \subset \mu$ holds and thus the mapping **V** has continuous and bounded first order derivatives on $\{(x,t)/t \in [t_1, t_2], x \in \sigma(t)\}$.

d) $\mathbf{V}(\varphi(X, t_{o;} t), t) = \frac{\partial \varphi(X, t_0, t)}{\partial t} \quad \forall X \in \sigma(t_0), \forall t \in (t_1, t_2)$.

Proof: Since $x \in \overline{\sigma(t0)}X\{t_o\}$ is compacct subset of the open set μ, then there exists an open bounded set M such that $\overline{\sigma(t0)}X\{t_o\} \subset M \subset \overline{M} \subset \mu$.

In virtue of (1.3.9) , the vector function **V** is bounded and Lipshcitz-continuous with respect to x on M. By the proof of the local solvability of the problem (1.3.17) , there exists $\varepsilon > 0$ such that foe each $X \in \sigma(t_0)$ the solution $\varphi(X, t_{o;} t)$of (1.3.17) is defined for $\forall t \in [t_1 - \varepsilon, t_2 - \varepsilon]$.

Now, due to Theorem (1.3.18) , the mapping $\varphi(., t_{o;} .)$ has continuous bounded derivatives $\frac{\partial\varphi}{\partial t}$, $\frac{\partial\varphi}{\partial x_i}$ and $\frac{\partial^2\varphi}{\partial t\partial X_i}$ (i= 1,2,3.) on $\overline{\sigma(t0)}X$ I.

Hence, the proof of a).

The uniqueness of the solution of problem (1.3.17) , implies that the mapping „$X \in \sigma(t_0) \rightarrow x = \varphi(X, t_{o;} t) \in \sigma(t)$" is one-to-one . Its continuous differentiability follows from assertion 2) of the theorem (1.3.18).

And moreover, since $\varphi(X, t_{o;} t_0) = X$, we have J(X,t) =det(1)= 1. By the continuity of J , there exists $(t_1, t_2) \subset I$ such that J is positive and bounded on the set $\sigma(t_0)X(t_1, t_2)$.

Hence, the proof of assertion b).

Assertion c) is a consequence of assumption (1.3.9) and the fact that $(\varphi(X, t_{o;} t), t) \in \mu$ for $X \in \overline{\sigma(t0)}$ and $t \in [t_1, t_2]$.

Finally, assertion d) means that $\varphi(X, t_{o;} .)$ is the solution of problem (1.3.17), and is defined on the interval (t_1, t_2) provided $X \in \sigma(t_0)$.

1.4.6 **Theorem:** Let conditions a)-d) from 1.4.5 be satisfied. Then the function J=J (X,t) has a continuous and a bounded partial derivative $\frac{\partial J}{\partial t}$ for $X \in \sigma(t_0), t \in (t_1, t_2)$, and

$$\frac{\partial J}{\partial t}(X,t) = J(X,t)\,\mathrm{div}\mathbf{V}(x,t)\text{ , where } x = \varphi(X, t_{o;} t).$$

Proof: Now, J (X,t) $= \det \frac{D\varphi(X,t_0;t)}{DX} = \det \begin{pmatrix} \frac{\partial\varphi_1}{\partial x_1} & \frac{\partial\varphi_1}{\partial x_2} & \frac{\partial\varphi_1}{\partial x_3} \\ \frac{\partial\varphi_2}{\partial x_1} & \frac{\partial\varphi_2}{\partial x_2} & \frac{\partial\varphi_2}{\partial x_3} \\ \frac{\partial\varphi_3}{\partial x_1} & \frac{\partial\varphi_3}{\partial x_2} & \frac{\partial\varphi_3}{\partial x_3} \end{pmatrix} (X,t_0,t)$.

Now the Jacobian J (X,t) can be expanded with respect to the ith row as

$J(X,t)=\sum_{\alpha=1}^{3}\left(\frac{\partial\varphi_i}{\partial x_\alpha}(X,t_0;t)D_{i,\alpha}(X,t)\right)$ where $D_{i,\alpha}$ denotes the cofactor of the element $\frac{\partial\varphi_i}{\partial x_\alpha}$. For $\alpha,\beta=1,2,3$, cofactors $D_{i,\beta}$ are independent of $\frac{\partial\varphi_i}{\partial x_\alpha}$. Hence $\frac{\partial J}{\partial\left(\frac{\partial\varphi_i}{\partial x_\alpha}\right)}=D_{i,\alpha}$

In order to calculate the derivative $\frac{\partial J}{\partial t}$, we consider the determinant J(X,t) as a function dependant on the elements $\frac{\partial\varphi_i}{\partial x_\alpha}$ which depend on t(i.e. $J \rightarrow \frac{\partial\varphi_i}{\partial x_\alpha} \rightarrow t$ and

$$J=\sum_{\alpha=1}^{3}\left(\frac{\partial\varphi_i}{\partial x_\alpha}D_{i,\alpha}\right)$$

$$\Rightarrow \frac{\partial J}{\partial t}=\sum_{i,\alpha=1}^{3}\left(\frac{\partial\left(\frac{\partial\varphi_i}{\partial x_\alpha}\right)}{\partial\left(\frac{\partial\varphi_i}{\partial x_\alpha}\right)}\cdot\frac{\partial\left(\frac{\partial\varphi_i}{\partial x_\alpha}\right)}{\partial t}.D_{i,\alpha}+\frac{\partial(D_{i,\alpha})}{\partial\left(\frac{\partial\varphi_i}{\partial x_\alpha}\right)}\cdot\frac{\partial\left(\frac{\partial\varphi_i}{\partial x_\alpha}\right)}{\partial t}\cdot\frac{\partial\varphi_i}{\partial x_\alpha}\right)$$ (Using the

derivative of product functions and chain rule. But, $D_{i,\alpha}$ is independent of $\frac{\partial\varphi_i}{\partial x_\alpha}$, $\alpha=1,2,3$. Then its derivative becomes zero.

$\therefore$ (1.4.8) $$\frac{\partial J}{\partial t}=\sum_{i,\alpha=1}^{3}\left(\frac{\partial(J)}{\partial\left(\frac{\partial\varphi_i}{\partial x_\alpha}\right)}\cdot\frac{\partial\left(\frac{\partial\varphi_i}{\partial x_\alpha}\right)}{\partial t}\right)=\sum_{i,\alpha=1}^{3}\left(D_{i,\alpha}\frac{\partial^2\varphi_i}{\partial x_\alpha\partial t}\right)$$

Now, let us deal with the derivative $\frac{\partial^2 \varphi_i}{\partial x_\alpha \partial t}$(X, t_o; t). Under conditions 1.4.5 a)-b) we

get the relation

$$\frac{\partial^2 \varphi_i}{\partial x_\alpha \partial t}(X, t_o; t) = \frac{\partial^2 \varphi_i}{\partial t \partial x_\alpha}(X, t_o; t) = \frac{\partial}{\partial x_\alpha} V_i(\varphi(X, t_0; t), t) =$$

$$\sum_{j=1}^{3}\left(\frac{\partial(v_i)}{\partial(x_j)}(x,t)\frac{\partial(\varphi_j)}{\partial x_\alpha}(X,t_0;t)\right)$$

Substituting into (1.4.8), we have

$$\frac{\partial J}{\partial t} = \sum_{i,\alpha=1}^{3}\left(D_{i,\alpha}\sum_{j=1}^{3}\frac{\partial \varphi_j}{\partial x_\alpha}\frac{\partial v_i}{\partial x_j}\right) = \sum_{i,j=1}^{3}\left(\sum_{\alpha=1}^{3}\frac{\partial \varphi_j}{\partial x_\alpha}D_{i,\alpha}\right)\frac{\partial v_i}{\partial x_j}$$ But, from the theory

of determinant we know that $\sum_{\alpha=1}^{3}\frac{\partial \varphi_j}{\partial x_\alpha}D_{i,\alpha} = J\,\delta_{ij}$ where $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

and thus $\frac{\partial J}{\partial t} = J\sum_{i,j=1}^{3}\delta_{ij}\frac{\partial v_i}{\partial x_j} = J\sum_{i=1}^{3}\frac{\partial v_i}{\partial x_i} = J \operatorname{div} \mathbf{V}$

Now we can proof the so-called transport theorem on the derivative of integral (1.4.1)

(1.4.9) **Theorem**: Let conditions 1.4.5 a) – d) be satisfied and let the function F = F(x, t) have continuous and bounded first order derivatives on the set $\{(x, t) | t \in (t_1, t_2), x \in \sigma(t)\}$. Then for each $t \in (t_1, t_2)$ there exists a finite derivative

$$(1.4.10)\ \frac{df}{dt}(t) = \frac{d}{dt}\int_{\sigma(t)} F(x,t)dx =$$

$$\int_{\sigma(t)}\left[\frac{\partial F}{\partial t}(x,t) + V(x,t).\operatorname{grad} F(x,t) + F(x,t)\operatorname{div} V(x,t)\right]dx$$

Proof: By using the substitution theorem the integral F(t) can be written in the form

$$f(t) = \int_{\sigma(t_0)} F(\varphi(X, t_0; t) J(X, t) dX$$

(Because from $f(t) = \int_{\sigma(t)} F(x,t)dx$ and $x=\varphi(X, t_0; t)$

But $\varphi(X, t_0; t_0) = X$

$$\Rightarrow \frac{dx}{dX} = \frac{d(\varphi(X,t_0;t_0)}{dX} = \frac{dX}{dX} = 1 = J(X,t_0)$$

$$\Rightarrow dx = J(X,t_0)dX$$

Now fixing our domain at t_0, we get

$$f(t) = \int_{\sigma(t_0)} F(\varphi(X,t_0;t))J(X,t)dX \quad \frac{\partial F}{\partial t}(\varphi(X,t_0;t$$

Since t_0 is fixed and the integration domain $\sigma(t_0)$ does not depend on time t, we can apply the well known theorem on differentiation of an integral with respect to a parameter:

$$\frac{df}{dt}(t) =$$

$$\int_{\sigma(t_0)} \left[\left(\frac{\partial F}{\partial t}(\varphi(X,t_0;t) + \sum_{i=1}^{3}\frac{\partial F}{\partial x_i}(\varphi(X,t_0;t),t)\frac{\partial \varphi_i}{\partial t}(X,t_0,t)\right)J(x,t)) + F(\varphi(X,t_0;t),t)\frac{\partial J}{\partial t}(X,t)\right]dX$$

Now using 1.4.6 and condition 1.4.5 d) we get

$$\frac{df}{dt}(t) = \int_{\sigma(t_0)} \left[\frac{\partial F}{\partial t}(\varphi(X,t_0;t) + \sum \frac{\partial F}{\partial x_i}(\varphi(X,t_0;t),t)v_i(\varphi(X,t_0,t),t)\right.$$

$$\left. + F(\varphi(X,t_0.;t),t)\operatorname{div} v(\varphi(X,t_0,t)t)\right]J(X,t)dx$$

because, $\frac{\partial J}{\partial t} = J\operatorname{div} v$ & $v(\varphi(X, t_0; t) = \frac{\partial\varphi(X,t_0;t)}{\partial t}$

Now using the inverse substitution and transforming the integral over $\sigma(t_0)$ on to the integral over $\sigma(t)$, we have

$$\frac{df(t)}{dt} = \int_{\sigma(t)} \left[\frac{\partial F}{\partial t}(x,t) + v(x,t).grad\ F(x,t) + F(x,t)div\,v(x,t)\right]dx$$

Hence the proof of the theorem.

1.4.11 **Flux formulation of Transport theorem:** provided the assumptions of theorem (1.4.9) are satisfied, identity (1.4.10) can be written in the form

$$(1.4.12) \qquad \frac{df(t)}{dt} = \int_{\sigma(t)} \left[\frac{\partial F}{\partial t}(x,t) + \operatorname{div}(FV)(x,t)\right]dx$$

Proof: $\frac{df(t)}{dt} = \int_{\sigma(t)} \frac{\partial F}{\partial t}(x,t)dx + \int_{\sigma(t)} div(FV)(x,t)dx$

But using the divergence theorem

$$\int_{\sigma(t)} div(FV)(x,t)dx = \int_{\partial\sigma(t)} (FV)(x,t).n(x)ds = \int_{\partial\sigma(t)} F(x,t)v(x,t).n(x)ds$$

Where $\mathbf{n}(x)$ denotes the unit outer normal to $\partial\sigma(t)$ at the point x.

$$(1.4.13) \qquad \therefore \frac{df(t)}{dt} = \int_{\sigma(t)} \frac{\partial F}{\partial t}(x,t)dx + \int_{\partial\sigma(t)} F(x,t)V(x,t).n(x)ds.$$

The first integral on the right hand side of (1.4.13) determines the rate of change of the quantity f is virtue of the dependence of the Function F on time t. The second integral that appears here as a consequence of the dependence of the domain $\sigma(t)$ on t, represents the flux of the quantity F through the boundary $\partial\sigma(t)$.

Note: If we fix the time instant t and write $\sigma = \sigma(t)$, the transport theorem can be formulated in the following way: The rate of change of the quantity F contained in the domain σ is equal to the sum of the total amount of the derivative of the quantity F contained in σ : $\int_{\sigma} \frac{\partial F}{\partial t} dx$ and of the flux of the quantity F through the boundary $\partial\sigma$:

$$\int_{\partial\sigma} F(x,t)V(x,t).n(x)ds$$

1.4.14 Remark: Since $\overline{\sigma(t)} \subset \Omega_t$ for all $t \in (t_1, t_2)$,the conditions for F from theorem(1.4.9) are guaranteed. Example by the condition that $F \in C^1(\mu)$

Proof: <u>WTS</u> F is continuous and bounded.

Since $F \in c^1(\mu)$ it is clear that F is continuous and its first order partial derivative exist on $\{(x, t) \mid t \in (t_1, t_2), x \in \Omega_t\}$.

and since $\overline{\sigma(t)}$ is closed and bounded subset of $\Omega_t \subseteq \Re^4$, then it is compact.

Hence, F is bounded on $\overline{\sigma(t)}$.

CHAPTER 2

Conservation Laws

We Shall discuss first the law of conservation of mass, momentum and energy and from which we will derive the fundamental differential equations of fluid dynamics: The continuity equation, the equation of motion and the energy equation.

2.1 The continuity Equation

2.1.1 The Density of Fluid: is a function ρ: $\mu = \{x, t)/t \in (T_1, T_2), x \in \Omega_t\} \rightarrow (0, +\infty)$ which allows to the determine the mass (σ; t) of the fluid contained in any sub domain $\sigma \subset \Omega_t$:

$$(2.1.2) \qquad m(\sigma; t) = \int_{\sigma} \rho(x,t)dx .$$

The density $\rho(x, t)$ can not measured directly. The measurement allows to determine the mass $m(\sigma; t)$ and the volume $\mu(\sigma)$ = meas. (σ) of the domain $\sigma \subset \Omega_t$. If the density of fluid is constant, then $\rho = \frac{m(\sigma;t)}{\mu(\sigma)}$. In a general case the density can be defined in the precise mathematical way as the Radon-Nikodym derivative $\frac{dm}{d\mu}$ of the mass measure $m(\sigma; t)$ with respect to the lebesgue measure $\mu(\sigma)$.

2.1.3 **Assumptions:** Let $\rho \in c^1(\mu)$ and $v \in [c^1(\mu)]^3$. Let us considered an arbitrary time instant to $\in (T_1, T_2)$ and a moving piece of the fluid formed by the same particles at each instant and filling at time t_o a bounded domain $\sigma \subset \overline{\sigma} \subset \Omega_{t_0}$ with lipschitz continuous boundary $\partial\sigma$. Such a domain will be called the control volume in the domain Ω_{t_0}. By $\sigma(t)$ we denote the domain occupied by this piece of fluid at time t $\in (t_1, t_2)$, where (t_1, t_2) is sufficiently small time interval containing t_0 with properties from Lemma 1.4.5. Hence, $\sigma(t_0) = \sigma$ and conditions 1.4.5, a) – d) are satisfied.

2.1.4 The law of conservation of Mass

Since the domain σ(t) is formed by the same particles at each time instant, the conservation of mass can be formulated as follows:

"The mass of the piece of fluid represented by the domain σ(t) does not depend on time t". The means that

$$(2.1.5) \qquad \frac{dm(\sigma(t))}{dt} = 0 \; t \in (t_1, t_2).$$

Using (2.1.2)

$$(2.1.6) \qquad m(\sigma(t)) = \int_{\sigma(t)} \rho(x,t)dx$$

Now, using the transport them (1.4.9), where assumptions are satisfied for the function F = ρ(using Remark 1.4.14), from (2.1.5) and (2.1.6) we get

$$\frac{dm(\sigma(t))}{dt} = \frac{d}{dt} \int_{\sigma(t)} \rho(x,t)dx$$

$$= \int_{\sigma} \left[\frac{\partial \rho}{\partial t}(x,t) + v(x,t).\mathrm{grad}\rho(x,t) + \rho(x,t)\mathrm{div}v(x,t) \right] dx = 0, t \in (t_1, t_2)$$

Now, if substitute t: = t_0 and taking into account that $\sigma(t_0) = \sigma$ and

di(ρv) = ρdiv v + ∇ρ.v we conclude that

$$(2.1.7) \qquad \int_{\sigma} \left[\frac{\partial \rho}{\partial t}(x,t_0) + \mathrm{div}(\rho v)(x,t_0) \right] dx = 0$$ for an arbitrary $t_0 \in (T_1, T_2)$ and an arbitrary control volume σ in Ω_{t_0}.

Now, using the continuity of the integrad in (2.1.7) and assertion 1) of Lemma 1.1.28 and writing t instead of t_0, we conclude that

$$(2.1.8) \qquad \frac{\partial \rho}{\partial t}(x, t) + \mathrm{div}(\rho(x, t)\, v\,(x, t) = 0, \; t \in (T_1, T_2), \; x \in \Omega_t.$$

This equation is the differential form of the law of conservation of mass and is called the continuity equation.

On the basis of the flux form (1.4.11) of the transport theorem, the law of conservation of mass can be written in the form

(2.1.9) $$\int_{\sigma} \frac{\partial \rho}{\partial t}(x,t)dx + \int_{\partial\sigma} \rho(x,t)v(x,t).n(x)ds = 0$$ for an arbitrary

$t \in (T_1, T_2)$ and an arbitrary control volume σ is Ω_t.

2.1.10 **Remark.** If the flow is stationary in the domain Ω, then ρ, v_i: $\Omega \to \Re^1$.

$\rho = \rho(x)$, $v_i = v_i(x)$ and the continuity equation has the form

(2.1.8*) $\operatorname{div}(\rho v) = 0$ in Ω

or (2.1.9*) $\int_{\partial\sigma} \rho v.nds = 0$ for an arbitrary control volume σ in Ω.

If the fluid is incompressible, (i.e. its density is constant: $\rho = \rho_0$ = constant, $\rho_0 > 0$) then the continuity equation becomes

(2.1.11) $\operatorname{div} v = 0$

2.2. The Equations of Motion

Basic dynamical equation describing fluid motion will be derived from the law of conservation of momentum which can be formulated as follows:

The rate of change of the total momentum of a piece of fluid formed by the same particles at each time and occupying the domain $\sigma(t)$ at instant t is equal to the force acting on $\sigma(t)$.

Let assumption 2.1.3 be satisfied. The total momentum of particles contained in σ is given by

(2.2.1) $$\underline{H}(\sigma(t)) = \int_{\sigma(t)} \rho(x,t)v(x,t)dx$$

Moreover, denoting by $F(\sigma(t)$ the force acting on the volume $\sigma(t)$, the law of conservation of momentum reads

(2.2.2) $$\frac{d\underline{H}(\sigma(t))}{dt} = \underline{F}(\sigma(t)),\ t \in (t_1, t_2)$$

It is clear that the assumption of transport theorem (1.4.9) are satisfied for functions F equal to the components of the vector ρv. Using 1.4.14, we get the equation

$$\frac{d\underline{H}(\sigma(t))}{dt} = \int_{\sigma(t)} [\frac{\partial}{\partial t}(\rho(x,t)v_i(x,t)) + \operatorname{div}(\rho(x,t)\, v_i(x,t)\, v(x,t))]dx = f_i(\sigma(t)$$

$i = 1, 2, 3$, $t \in (t_1, t_2)$

Now, taking into account that $t_0 \in (T_1, T_2)$ is an arbitrary time instant and $\sigma(t_0) = \sigma \subset \overline{\sigma} \subset \Omega_{t_0}$, where σ is an arbitrary control volume, we get the law of conservation of momentum in the form where we write t instead of t_0.

$$(2.2.3) \qquad \int_{\sigma(t)} \left[\frac{\partial}{\partial t}(\rho(x,t)v_i(x,t)) + \operatorname{div}(\rho(x,t)v_i(x,t)v(x,t) \right] dx = f_i(\sigma; t)$$

$i = 1, 2, 3$, for an arbitrary $t \in (T_1, T_2)$ and an arbitrary control volume σ in Ω_t.

The vector $\underline{F}(\sigma; t)$ with components $F_i(\sigma; t)$ denotes the force acting on the volume σ at time t.

Now, first we shall specify the character of the vector $\underline{F}(\sigma; t)$

2.2.4 **Forces Acting in Fluids**: There are two types of forces acting in Fluids, the so called volume (or body) forces and surfaces forces.

a) **Volume forces**: are forces acting on volume elements of fluid, we express them usually with the use of the density of volume force **f**: $\mu \rightarrow \Re^3$, related to using mass.

Now, for $\mathbf{f} \in [c(\mu)]^3$, the volume force $\underline{F}_v(\sigma; t)$ acting at time on the particles contained in a control volume $\sigma \subset \overline{\sigma} \subset \Omega_t$ is given by the relation

$$(2.2.5) \qquad \underline{F}_v(\sigma; t) = \int_{\sigma} \rho(x,t) f(x,t) dx.$$

b) **Surface forces**: represent the action of the fluid contained outside the considered volume σ onto the fluid occupying the domain σ through its boundary $\partial\sigma$. As surface forces result from the inner interaction b/n fluid volumes, it is commonly called inner forces. Now let us consider the simplest example of surface force, which is the pressure force. Assume the boundary $\partial\sigma$ is lipschitz continuous, S C $\partial\sigma$ is a measurable subset (wrt the measure defined on $\partial\sigma$) and denoted by n(x) the unit outer normal to $\partial\sigma$ at a point

$x \in \partial\sigma$. Then the pressure force acting at time on the set S from the exterior of the domain σ is expressed in the form

$$(2.2.6) \qquad F_{p,s} = - \int_{s} p(x,t) n(x) ds.$$

The quantity p(which is assumed p $\in c(\mu)$) is called pressure. The vector $-p(x, t)\, n(x)$, representing the density of the pressure force, is orthogonal to the surface $\partial\sigma$ at each point x $\in \partial\sigma$ and its tangential component to $\partial\sigma$ is zero.

2.2.7 The Euler Equations of Motion of Ideal fluid

In ideal (i.e. inviscid fluid) we don't consider inner friction. It means that the mutual interaction between fluid volumes is given by the pressure force only. If the volume force with the density $f = (f_1, f_2, f_3)$ is also taken into account, then the total force acting on a control volume σ at time t achieves the form

$$(2.2.8)\quad \underline{F}(\sigma; t) = \int_{\sigma} \rho(x,t) f(x,t)dx - \int_{\partial\sigma} p(x,t)n(x)ds .$$

Let assumptions 2.1.3 be satisfied and let P $\in c^1(\mu)$. Substituting (2.2.8) into (2.2.3) and transforming the surface integral in (2.2.8) into a volume integral by Green's identity, we get

$$\int_{\sigma}\left[\frac{\partial}{\partial t}(\rho(x,t)v_i(x,t) + div(\rho(x,t)v_i(x,t)v(x,t))\right]dx = \int_{\sigma} \rho(x,t) f_i(x,t)dx - \int_{\sigma}\frac{\partial p(x,t)}{\partial x_i}dx.$$

(2.2.9) i= 1, 2, 3, for an arbitrary t $\in (T_1, T_2)$ and an arbitrary control volume σ.

$$\left(\text{because} \int_{\partial\sigma} p(x,t)n(x)ds = \int_{\sigma} \nabla(p(x,t))dx = \int_{\sigma}\frac{\partial p}{\partial x_i}(x,t)dx\right)$$

Now , using Lemma 1.1.28, this implies that

$$(2.2.10)\quad \frac{\partial}{\partial t}(\rho v_i)(x,t) + div(\rho v_i v)(x,t) = \left(\rho f_i(x,t) - \frac{\partial p}{\partial x_i}(x,t)\right)$$

i = 1, 2, 3, x $\in \Omega_t$ t $\in (T_1, T_2)$

The above equations are called the Euler equations of motion of ideal fluid. Using the continuity equation (2.1.8), we easily derive the convective form of the Euler equations:

$$(2.2.11)\quad \frac{\partial v_i}{\partial t} + \sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j} = f_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i},\ i = 1, 2, 3$$

The expression $\sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j}$, which is the convective derivative of the function v_i, called the convective term of the equation of motion. The left hand side of (2.2.11) represents the total derivative $\frac{dv_i}{dt}$.

We can also rewrite (2.2.11) in vector from as

$$(2.2.12)\ \frac{\partial v}{\partial t} + \sum_{j=1}^{3} v_j \frac{\partial v}{\partial x_j} = f - \frac{1}{\rho}\nabla p \Rightarrow \frac{dv}{dt} = f - \frac{1}{\rho}\nabla p$$

$$\Leftrightarrow \frac{dv}{dt} = f - \frac{1}{\rho}\nabla p$$

Using the operator "nabla"

$$(2.2.13)\quad \frac{\partial v}{\partial t} + (v.\nabla)V = f - \frac{1}{\rho}\nabla p$$

Now, Applying the Green's theorem to the integral of the expression div(ρv_i v) is (2.2.3) and using (2.2.8), we get the flux formulation of the equations of motion:

$$(2.2.14)\qquad \int_{\sigma} \frac{\partial(\rho v_i)}{\partial t}(x,t)dx + \int_{\partial\sigma} (\rho v_i)(x,t)v(x,t).n(x)ds = \int_{\sigma} (\rho f_i)(x,t)dx - \int_{\partial\sigma} p(x,t)n_i ds,$$

i =1,2, 3 for an arbitrary control volume σ contained in Ω_t and an arbitrary $t \in (T_1, T_2)$.

2.3 **Stress Tensor**

In this section we will investigate the character of general internal forces acting on the boundary $\partial\sigma$ of a control volume σ c Ω_t at time $t \in (T_1, T_2)$.

Let S C $\partial\sigma$ be a subset measurable with respect to the measure defined on $\partial\sigma$. The force F_s, by which the fluid contained outside the domain σ acts on the set S, will be expressed with the use of the stress vector T(x, t, n) characterizing the density of the surface for σ:

$$(2.3.1)\quad F_s = \int_{s} T(x,t,n(x)ds$$

The vector T is supposed to depend on the point x, time t and the unit outer normal n(x) to $\partial\sigma$ at the point x. (i.e. T depends on the orientation of the surface $\partial\sigma$.)

In the following we will assume that T $\in$[c(μ)x s] where s is the surface of the unit sphere with center at the origin.

Now, using Newton's third Law, the fluid acts on the surface S from the opposite side (i.e. from the interior of σ) by the force $\underline{-F_s}$ (S; t). This implies that

(2.3.2) $\quad T(x, t, n) = -T(x, t, -n)$

by (2.3.1), the total force acting at time t on the control volume σ from outside has the form

(2.3.3) $$\underline{F}_S(\sigma; t) = \int_{\partial\sigma} T(x,t,n(x)ds$$

2.3.4 **Components of the stress Tensor**: we show that the stress vector T(x, t, n) is determined by some of its values for certain normals.

Let us choose the noramls parallel to the coordinate axes and set

(2.3.5) $\tau_{ji} = T_i(x, t, e_j)$. i, j = 1, 2, 3

$e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$

The quantities $\tau_{ji} = \tau_{ji}(x, t)$, i, j = 1, 2, 3, are called the components of the stress tensor

(2.3.6) $$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}$$

The component τ_{ii}, i = 1, 2, 3, and the components τ_{ij}, $i \neq j$, i, j= 1, 2, 3, are called normal stresses and shear (tangential) stresses, respectively.

Let us study the connection b/n the components $\tau_{ji}(x,t)$ and the vector T(x, t, n). For this purpose we consider an arbitrary point $O \in \Omega_t$ and choose, in particular, $\sigma = v_h$ where $v_h \subset \Omega_t$ is a tetrahedron OABC as shown below, three edges of which are parallel to the coordinate axes.

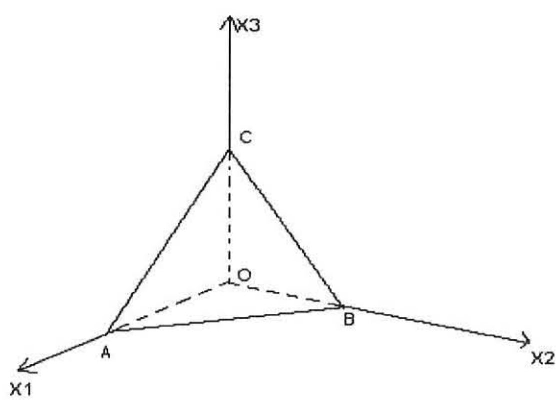


Fig. 1.3

Let Ω be the normal to the surface ABC and denote by $(\partial\sigma)_0$, $(\partial\sigma)_1$, $(\partial\sigma)_2$ and $(\partial\sigma)_3$ the surfaces ABC, OBC, OAC and OAB respectively.

Further denote by $|(\partial\sigma)_i|$ the area of $(\partial\sigma)_i$ and by h the distance of the origin from the plane ABC. Then $|(\partial\sigma)_i| = |(\partial\sigma)_0|n_i$ and the volume $\phi|v_h| = \frac{1}{3} h\, |(\partial\sigma)_0|$.

Now we consider equation (2.2.3) where we substitute the sum of the volume force (2.2.5) and the surface force (2.3.3) for $F_i(\sigma; t)$:

$$\text{(2.3.7)} \qquad F_i(\sigma, t) = \int_{\sigma} \rho(x,t) f_i(x,t)dx + \int_{\partial\sigma} T_i(x,t,n(x))dx$$

Let us put $F_i = \frac{\partial}{\partial t}(\rho v_i) + \mathrm{div}(\rho v_i v) - \rho f_i$ and assume that ρ, $v_i \in c^1(\mu)$ and $f_i \in C(\mu)$. Then $F_i \in c(\mu)$ and

$$\text{(2.3.8)} \qquad \int_{\partial v_h} T_i(x,t,n(x))ds + \int_{v_h} F_i(x,t)dx = 0,\ i = 1, 2, 3$$

But $\int_{\partial v_h} T_i(x,t,n(x))ds = \int_{(\partial\sigma)_0} T_i(x,t,n)ds - \int_{(\partial\sigma)_1} T_{1i}(x,t)dx_2 x_3 - \int_{(\partial\sigma)_2} T_{2i}(x,t)dx_1 x_3$

$- \int_{(\partial\sigma)_3} T_{3i}(x,t)dx_1 x_i$

By (2.3.2)and (2.3.5), the negative signs appear, since the outer normal to $\partial\sigma$ i equal to $- e_i$ on the surface $(\partial\sigma)_i = 1, 2, 3$. Substituting into 2.3.8 and dividing by $|(\partial\sigma)_0|$, we get

$$\text{(2.3.9)} \quad \frac{1}{|(\partial\sigma)_0|} \int_{(\partial\sigma)_0} T_i(x,t,n)ds = \frac{n_1}{|(\partial\sigma)_1|} \int_{(\partial\sigma)_1} \tau_{1i}(x,t)dx_2 dx_3 + \frac{n_2}{|(\partial\sigma)_1|} \int_{(\partial\sigma)_2} \tau_{2i}(x,t)dx_3 dx_3$$

$$+ \frac{n_3}{|(\partial\sigma)_3|} \int_{(\partial\sigma)_3} \tau_{3i}(x,t)dx_1 dx_2 + \frac{1}{|(\partial\sigma)_0|} \int_{v_h} f_i(x,t)dx .$$

Let us Prove that for h $\rightarrow$ 0, we get

$$\lim_{h\to 0} \frac{n_1}{|(\partial\sigma)_2|} \int_{(\partial\sigma)_0} T_i(x,t,n)ds = T_i(0, t, n)$$

(2.3.10) $$\lim_{h\to 0} \frac{n_1}{|(\partial\sigma)_1|} \int_{(\partial\sigma)_1} \tau_{1i}(x,t)dx_2dx_3 = n_1\tau_{1i}(0, t)$$

$$\lim_{h\to 0} \frac{n_2}{|(\partial\sigma)_2|} \int_{(\partial\sigma)_2} \tau_{2i}(x,t)dx_1dx_3 = n_2\tau_{2i}(0, t)$$

$$\lim_{h\to 0} \frac{n_3}{|(\partial\sigma)_3|} \int_{(\partial\sigma)_3} \tau_{3i}(x,t)dx_1dx_2 = n_3\tau_{3i}(0, t).$$

The function $T_i(., t, n)$ is continuous (t is a fixed time instead and n is a constant vector). Hence, for each $\varepsilon > 0$ there exist $\delta > 0$ such that

$T_i(0, t, n) - \varepsilon < T_i(x, t, n) < T_i(0, t, n) + \varepsilon$, provided $x \in \overline{v}_h$ and $0 < h < \delta$.

Integrating these quantities over $(\partial\sigma)_0$ and dividing the result by $(\partial\sigma)_0$, we find that

$$\left| \frac{1}{|(\partial\sigma)_0|} \int_{(\partial\sigma)_0} \tau_i(x,t,n)ds - T_i(0,t,n) \right| < \varepsilon \text{ for } 0 < h < \delta.$$

Hence, the first equality is (2.3.10) holds. The reaming relations can be proved in the same way.

The continuity of the function $F_i(.,t)$ on Ω_t and the compactness of the set $\overline{\sigma} = \overline{v}_h$ c Ω_t imply the existence of a number $M > 0$ (independent of h) such that $|F_i(x,t)| \leq M$ for an $x \in \overline{v}_h$ and hence

(2.3.11) $$\left| \frac{1}{|(\partial\sigma)_0|} \int_{v_h} F_i(x,t)dx \right| \leq \frac{1}{|(\partial\sigma)_0|} M \frac{1}{3} |(\partial\sigma)_0| h \rightarrow 0 \text{ for } h \rightarrow 0$$

then from (2.3.9) – (2.3.11), the identity

(2.3.12) $T_i(o, t, n) = \sum_{j=1}^{3} n_j \tau_{ji}(o,t)$ immediately follows.

Now, writing $x \in \Omega_t$ instead of O, we set the following expression for the components of the stress vector:

$$(2.3.13)\quad T_i(x, t, n) = \sum_{j=1}^{3} n_j \tau_{ji}(x,t).$$

2.3.14 **Tensor character of stress:** we will show that under an orthonormal transformation of coordinates

$x_i^* = \sum_{j=1}^{3} a_{ij} x_j + c_i$, i = 1, 2, 3, the components of the stress tensor τ

are transformed as the components of a tensor of order two:

$$(2.3.15)\qquad \tau_{ij}^* = \sum_{t,m=1}^{3} a_{ik} a_{jm} \tau_{km}.$$

(2.3.16) The equations of motion of General fluids: Let us assume that ρ, v_i, $\tau_{ij} \in C^1(\mu)$ and $f_i \in C(\mu)$ (i = 1, 2, 3,). Expressing the total force acting on the fluid contained in a control volume σ according to (2.3.7) and (2.3.12,) we get

$$(2.3.17)\qquad f_i(\sigma;t) = \int_{\sigma} \rho(x,t) f_i(x,t)dx + \int_{\partial\sigma} \sum_{j=1}^{3} \tau_{ji}(x,t) n_j(x) ds$$

substituting this in (2.2.3) gives us

$$(2.3.18)\quad \int_{\sigma} \left[\frac{\partial}{\partial t}(\rho(x,t) v_i(x,t) + \operatorname{div}(\rho(x,t) v_i(x,t) v(x,t)) \right] dx = \int_{\sigma} \rho(x,t) f_i(x,t)dx$$

$$+ \int_{\partial\sigma} \sum_{j=1}^{3} \tau_{ji}(x,t) n_j(x) ds$$

where i = 1, 2, 3, for each $t \in (T_1, T_2)$ and an arbitrary control volume σ in Ω_t.
Moreover, applying Green's theorem and Lemma 1.1.28, we get the equations of motion of general fluid in differential conservative form:

(2.3.19) $$\frac{\partial}{\partial t}(\rho v_i) + \mathrm{div}(\rho v_i\, v) = \rho f_i + \sum_{j=1}^{3} \frac{\partial \tau_{ji}}{\partial x_j}, \; i = 1, 2, 3,.$$

Using the continuity equation, the convective from of the equations of motion:

(2.3.20) $$\rho \frac{\partial v_i}{\partial t} + \rho \sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j} = \rho f_i + \sum_{j=1}^{3} \frac{\partial \tau_{ji}}{\partial x_j}, \; i = 1, 2, 3.$$

which can be written in the vector form as:

(2.3.21) $$\rho \frac{\partial v}{\partial t} + \rho \sum_{j=1}^{3} v_j \frac{\partial v}{\partial x_j} = \rho f + \mathrm{div}\, \tau$$ where divergence of the tensor $\tau = \left(\tau_{ji}\right)_{ij=1}^{3}$ we understand the vector $\mathrm{div}\ \tau = \left(\sum_{j=1}^{3} \frac{\partial \tau_{j1}}{\partial x_j}, \sum_{j=1}^{3} \frac{\partial \tau_{j2}}{\partial x_j}, \sum_{j=1}^{3} \frac{\partial \tau_{j3}}{\partial x_j} \right)$

Moreover, the flux formulation of (2.3.18) is

(2.3.22) $$\int_{\sigma} \frac{\partial}{\partial t}(\rho(x,t) v_i(x,t)dx + \int_{\partial\sigma} \rho(x,t) v_i(x,t) v(x,t).\, n(x)dx$$

$$= \int_{\sigma} \rho(x,t) f_i(x,t)dx + \int_{\partial\sigma} \sum_{j=1}^{3} \tau_{ji}(x,t) n_j(x)ds$$

where i = 1, 2, 3, for each $t \in (T_1, T_2)$ and an arbitrary control volume σ in Ω_t.

2.4 The Energy Equation

First let us introduce some important concepts.

2.4.1 **Work and power**: It is clear that a constant force acting on a mass particle moving along a straight line performs work equal to the product "force x length of the part".

In general, If a particle moves along a path $\varphi(x_1, x_2)$ connecting points x_1 and x_2 due to the action of a (generally non constant) force F, work is defined as the integral

(2.4.2) $\int_{\varphi(x_1,x_2)} F.t\, ds$, where t is the unit tangent vector to the course $\varphi(x_1, x_2)$.

It means a force field F=. F(x, t) acting on a moving fluid particle with the trajectory given by the equation $x = \tilde{\varphi}(t)$ performs, during a time interval [to, t], the work

$$(2.4.3) \qquad A(t) = \int_{t_0}^{t} F(\tilde{\varphi}(t), \tau.). \frac{d\tilde{\varphi}(t)}{dt} d\tau = \int_{t_0}^{t} F\left(\tilde{\varphi}_{(\tau)}, \tau\right) v\left(\left(\tilde{\varphi}_{(\tau)}, (\tau\right) d\tau.$$

Power: is defined as the time derivative of work, i.e. $\tilde{w}(t) = \frac{dA(t)}{dt} = F(\tilde{\varphi}(t), t) \,.\, v(\tilde{\varphi}(t), t)$.

i.e. The power of the force F acting on the particle passing through the point x at time t equals

$$(2.4.4) \qquad W(x, t) = \mathbf{F}(x, t) \,.\, \mathbf{V}(x, t)$$

2.4.5 The Law of conservation of Energy

Let us consider a piece of fluid represented by a control value $\sigma(t)$ satisfying assumptions from (2.1.3). The Law of conservation of energy stated as follows:

"The rate of change of the total energy of the fluid particles, occupying the domain $\sigma(t)$ at time t, is equal to the sum of powers of the volume force acting on the volume $\sigma(t)$ and the surface force acting on the surface $\partial\sigma(t)$, and of the amount of heat transmitted to $\sigma(t)$".

Notation:

$\varepsilon(\sigma(t))$ = The total energy of the fluid particles contained in the domain $\sigma(t)$

$Q(\sigma(t))$ = amount of heat transmitted to $\sigma(t)$ at time t.

T = stress tensor

and F = density of the volume force.

Now the law of conservation of energy:

$$(2.4.6)\ \frac{d}{dt}(\varepsilon(\sigma(t))) = \int_{\sigma_{(t)}} \rho(x,t)f(x,t).v(x,t)\,dx + \int_{\partial\sigma_{(t)}} T(x,t,n(x)).v(x,t)ds + Q(\sigma_{(t)})$$

Further, we can write

$$\text{a) } \varepsilon(\sigma(t)) = \int_{\sigma_{(t)}} \rho(x,t)E(x,t)dx$$

$$(2.4.7) \qquad \text{b) } E = U + \frac{|v|^2}{2}$$

$$\text{c) } Q(\sigma(t)) = \int_{\sigma_{(t)}} \rho(x,t)q(x,t)dx - \int_{\partial\sigma_{(t)}} q(x,t).n(x)ds$$

Where E is the density of energy (related to unit of mass) which consists of internal

energy U associated with molecular and atomic behavior and the density of kinetic energy $\frac{|v|^2}{2}$, q presents the density of heat sources (related to unit mass) and of **q** is the heat flux.

Now, using Fournier's Law,

$$(2.4.8) \qquad \mathbf{q} = -k \operatorname{grad}\theta, \text{ we have}$$

$$(2.4.9) \qquad \int_{\partial\sigma_{(t)}} q(x,t).n(x)ds = -\int_{\partial\sigma_{(t)}} k(x,t)\frac{\partial\theta(x,t)}{\partial n}ds,$$ where $k \geq 0$ is the

coefficient of heat conductivity and θ is the absolute temperature.

Substituting 2.3.12 and 2.4.7, a – c) into (2.4.6,) we get

$$(2.4.10)\ \frac{d}{dt}\int_{\sigma_{(t)}} \rho(x,t)E(x,t)dx = \int_{\sigma_{(t)}} \rho(x,t)f(x,t).v(x,t)dx + \int_{\partial\sigma(t)} \sum_{i,j}^{3} \tau_{ji}(x,t)n_j(x)v_i(x,t)ds$$

$$+ \int_{\sigma_{(t)}} \rho(x,t)q(x,t)dx - \int_{\partial\sigma(t)} q(x,t).n(x)ds$$

Like in the preceding considerations we will assume some smoothness of functions describing the flow. i.e. let ρ, u, v_i, τ_{ij}, $q_i \in c^1(\mu)$, and f_i, $q \in c(\mu)(i, j = 1, 2, 3,)$.

Now, using the transport theorem (1.4.9)

$$(2.4.11)\quad \frac{d}{dt}\int_{\sigma(t)} \rho(x,t)E(x,t)dx = \int_{\sigma(t)} \left[\frac{\partial}{\partial t}(\rho E)(x,t) + \mathrm{div}(\rho E v)(x,t)\right]dx .$$

Substituting this into(2.4.10) we have

$$(2.4.12)\quad \int_{\sigma(t)} \left[\frac{\partial}{\partial t}(\rho E)(x,t) + \mathrm{div}(\rho E v)(x,t)\right]dx = \int_{\sigma(t)} \rho(x,t)f(x,t).v(x,t)dx$$

$$+ \int_{\partial\sigma(t)} \sum_{i,j=1}^{3} \tau_{ji}(x,t)n_j(x)v_i(x,t)ds$$

$$+ \int_{\sigma(t)} \rho(x,t)q(x,t)dx - \int_{\partial\sigma(t)} q(x,t).n(x)ds$$

Now let $t \in (T_1, T_2)$ be a fixed time instant and $\sigma(t) = \sigma$, where $\sigma \subset \overline{\sigma} \subset \Omega_t$ is arbitrary fixed control volume.

Then, using Green's theorem and Lemma (1.1.28), then equation (2.4.12)can be written as:

$$(2.4.13)\quad \frac{\partial}{\partial t}(\rho E) + \mathrm{div}(\rho E v) = \rho f.v + \sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}(\tau_{ji} v_i) + \rho q - \mathrm{div} q.$$

This is called the law of conservation of energy.

Now, using the continuity equation (2.1.8), we find that

$$\frac{\partial(\rho E)}{\partial t} + \mathrm{div}(\rho\, E v) = \rho\left(\frac{\partial E}{\partial t} + v.gradE\right) = \rho\,\frac{dE}{dt}$$

then the energy equation can be rewritten as (in convective form)

$$(2.4.14)\quad \rho\,\frac{dE}{dt} = \rho\,\frac{\partial E}{\partial t} + \rho\sum_{j=1}^{3} v_j \frac{\partial E}{\partial x_j} = \rho\ \mathbf{f.v} + \mathrm{div}(\boldsymbol{\tau v}) + \rho\, q - \mathrm{div}\ \mathbf{q}$$

$$\text{where } \tau v = \left(\sum_{i=1}^{3}\tau_{1i}v_i, \sum_{i=1}^{3}\tau_{2i}v_i, \sum_{i=1}^{3}\tau_{3i}v_i\right)$$

The flux formulation of the law of conservation of energy has the form

$$\int_{\sigma}\frac{\partial(\rho E)}{\partial t}dx + \int_{\partial\sigma}\rho E v.n ds = \int_{\sigma}\rho f.v dx + \int_{\sigma}\rho q dx + \int_{\partial\sigma}(\tau v).n ds - \int_{\partial\sigma}q.n ds$$

for each $t \in (T_1, T_2)$ and an arbitrary control volume σ in Ω_t.

Remark! The number of conservation Law equations including the continuity, the equations of motion and the energy equation $(1 + 3 + 1 = 5)$ does not amount the number of unknowns: v_1, v_2, v_3, ρ, u, θ, p. Now to make the system is closed let us see some results of thermodynamics.

2.5 Basic Thermodynamics Quantities

The absolute temperature θ, the density ρ and the pressure P are called the state Variables. All the quantities are positive functions. We now consider the so-called perfect gas whose state variables satisfy the state equation

(2.5.1) $P = R\theta\rho$, $R > 0$ is the gas constant, which can be expressed in the form

(2.5.2) $R = c_p - c_v$, where c_p and c_v denote the specific heat at constant pressure and the specific heat at constant volume, respectively.

It means that c_p or c_v can be expressed as the ratio of the increment of the amount of heat ,related to unit mass, to the increment of the temperature, at constant pressure or at constant volume, respectively.

The internal energy, is given by

(2.5.3) $u = c_v\theta$,hence

(2.5.4) $u = c_p\theta - R\theta = h - \frac{p}{\rho}$, where $h = c_p\theta$ is called enthalpy.

CHAPTER -3
HYPERBOLICITY OF THE SYSTEM OF EQUATIONS

3.1 Using eigenvalues of the matrix for the system of the equation:

It is necessary to consider the complete system of differential equation of conservation laws consisting of the continuity equation, Euler equation of motion and energy equation. This system is nonlinear, first order and it is hyperbolic in case of unsteady flow.

3.2 Properties of non linear hyperbolic systems.

Let us consider unsteady flow of ideal perfect gas in domain $\Omega \subseteq \Re^N$ $(1 \leq N \leq 3)$ and time interval (0, T) $(0 < T \leq +\infty)$. As we showed in Chapter 2, it is governed by the continuity equation (2.1.8), the Euler equation of motion (2.2.10) and the energy equation (2.4.13) to which we add expression (2.5.3) for internal energy and the state equation (2.5.1). Since the gas is inviscid, we put $\tau_{ij} = -p\delta_{ij}$ and introduce the assumptions of adiabatic flow, so that $\mathbf{q} = 0$ and $q = 0$ in (2.4.13). Moreover, because the gas is light, we neglect the volume force. The system of governing equations has the form:

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^{N} \frac{\partial(\rho v_j)}{\partial x_j} = 0, \tag{3.2.1}$$

$$\frac{\partial(\rho v_i)}{\partial t} + \sum_{j=1}^{N} \frac{\partial(\rho v_i v_j + P\delta_{ij})}{\partial x_j} = 0, \quad i=1,2,\ldots,N \tag{3.2.2}$$

$$\frac{\partial e}{\partial t} + \sum_{j=1}^{N} \frac{\partial((e+P)v_j)}{\partial x_j} = 0, \tag{3.2.3}$$

(3.2.4) $P = (k-1)\left(e - \frac{\rho |v|^2}{2}\right)$, where V_j are the components of the velocity vector $\mathbf{V}$ in the directions x_j (j=1,2,....,N), ρ is density, P is pressure.

(3.2.5) $e = \rho E = \rho\left(C_v \theta + \frac{|V|^2}{2}\right)$ is the density of energy and θ denotes the absolute temperature. Equation (3.2.4) for the pressure easily follows from (3.2.5) and the state equation (2.5.1).

System (3.2.1) -(3.2.3) can be written in the vector form

$$\frac{\partial W}{\partial t} + \sum_{j=1}^{N} \frac{\partial F_j(W)}{\partial x_j} = 0, \text{ where} \tag{3.2.6}$$

$$W = (\rho, \rho V_1, \rho V_2, \ldots, \rho V_N, e)^T \in \Re^m, \ m = N+2 \tag{3.2.7}$$

and

$$F_j(W) = (F_{j1}(W), F_{j2}(W), \ldots, F_{jm}(w))^T = (\rho V_j,\ \rho V_1 V_j + \delta_{1j}P, \ldots, \rho V_N V_j + \delta_{Nj}P,\ (e+P)V_j)^T, \ i = 1,2,\ldots,N, \tag{3.2.8}$$

$F_{jk} \in C^1(D)$

The domain of the definition $D \subset \Re^m$ of the functions F_{jk} is an open set. In particular , for system (3.2.1) -(3.2.4) we have

$$D=\{W \in \Re^m / \ w_1 = \rho > 0, w_j = \rho V_{j-1} \in \Re^1 \text{ forj} = 2,\dots, N+1, w_m - \sum_{j=2}^{N+1} \frac{w_j^2}{2w_1} = \frac{P}{k-1} > 0\}$$

In general, an equation of the type (3.2.6) is referred to as a conservation law written in a differential form. The vector function F_j is called the flux of the quantity W in the direction x_j (j=1,2,....,N) .

Differentiating (3.2.6) with respect to the primitive variables and applying the chain rule to the composite functions $F_j(W)$ (x,t) =(F_joW) (x,t), we obtain a first order quasilinear system of partial differential equations:

$$\frac{\partial W}{\partial t} + \sum_{j=1}^{N} \left(\frac{\partial F_j(W)}{\partial W} \cdot \frac{\partial W}{\partial x_j} \right) = 0$$

(3.2.10) $\Rightarrow \frac{\partial W}{\partial t} + \sum_{j=1}^{N} \left(A_j(W) \frac{\partial W}{\partial x_j} \right) = 0$, where $A_j(W)$ are m x m matrices defined for each $W \in D$:

(3.2.11) $$A_j(W) = \frac{DF_j(W)}{DW} = \left(\frac{\partial F_{ji}(W)}{\partial w_k} \right)_{i,k=1}^{m}$$ =The Jacobian matrix of the mapping F_j, j=1,2,....,N.

Definition: system (3.2.10) is called hyperbolic, if for arbitrary vectors $W \in D$ and $\underline{\upsilon} = (\upsilon_1, \upsilon_2, \upsilon_3, \dots, \upsilon_N) \in \Re^N$ the matrix

(3.2.12) $\mathbf{P}(W, \underline{\upsilon}) = \sum_{j=1}^{N} \upsilon_j A_j(W)$ has real eigenvalues $\lambda_j = \lambda_j(W, \underline{\upsilon})$,

j=1,2,...,m and is diagonalizable.

This means that there exists a non singular matrix $T = T(W, \underline{\upsilon})$ such that

$$T^{-1}\mathbf{P}T = D = D(W, \underline{\upsilon}) = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_m) = \begin{pmatrix} \lambda_1 & 0 & 0 & . & . & . & 0 \\ 0 & \lambda_2 & 0 & . & . & . & 0 \\ 0 & 0 & \lambda_3 & . & . & . & 0 \\ . & & & & . & & . \\ . & & & & & . & . \\ . & & & & & . & . \\ 0 & 0 & . & . & . & 0 & \lambda_m \end{pmatrix}$$

Now, let us show the system in (3.2.1) -(3.2.4) is hyperbolic.

3.3 **Hyperbolicity of the invisid flow system :**

Consider the system (3.2.1) -(3.2.4) . For simplicity consider two dimensional flow. (i.e N=2, m=4)

Now, using the notation (3.2.7)and (3.2.8) and making use of the notation V_1=U, V_2=V we have

$$W=(w_1,w_2,w_3,w_4)^T=(\rho,\ \rho V_1,\rho V_2,e)^T=(\rho,\ \rho u,\rho v,e)^T \tag{3.3.1}$$

and

$$F_1(W)=F(W)=(F_{11}(W),F_{12}(W),F_{13}(W),F_{14}(W))^T$$
$$=(\rho u,\rho u^2+\delta_{11}P,\rho vu+\delta_{21}P,(e+P)u)^T$$

But, $w_1=\rho$

$w_2=\rho u$ and $w_4 - \sum_{j=2}^{3}\frac{w_j^2}{2w_1}=\frac{P}{k-1}$ moreover, $\delta_{ij}=\begin{cases}1, \text{for } i=j\\0, \text{for } i\neq j\end{cases}$

$w_3=\rho v$

$w_4=e$

Then using these relations , we have the following:

* $\rho u = w_2$

* $\rho u^2+\delta_{11}P=(\rho u)u+P=(w_2)(\frac{w_2}{w_1})+P=\frac{w_2^2}{w_1}+(k-1)(w_4-\frac{(w_2^2+w_3^2)}{2w_1})$

* $\rho vu+\delta_{21}P=\rho uv=(w_3)(\frac{w_2}{w_1})=\frac{w_2w_3}{w_1}$

* $(e+P)u=\left(w_4+(k-1)(w_4-\frac{(w_2^2+w_3^2)}{2w_1})\right)\frac{w_2}{w_1}=\frac{w_2\left(kw_4-\frac{(k-1)(w_2^2+w_3^2)}{2w_1}\right)}{w_1}$

Hence, F(W)= (w_2,

$$\frac{w_2^2}{w_1}+(k-1)(w_4-\frac{(w_2^2+w_3^2)}{2w_1}),\frac{w_2w_3}{w_1},\frac{w_2\left(kw_4-\frac{(k-1)(w_2^2+w_3^2)}{2w_1}\right)}{w_1})^T$$

And similarly , we can get

$$F_2(W)\quad=G(W)=(w_3,\frac{w_2w_3}{w_1},\frac{w_3^2}{w_1}+(k-1)(w_4-\frac{(w_2^2+w_3^2)}{2w_1}),$$
$$,\frac{w_3\left(kw_4-\frac{(k-1)(w_2^2+w_3^2)}{2w_1}\right)}{w_1})^T$$

Now, setting $A=\frac{DF}{DW}$ and $B=\frac{DG}{DW}$, we obtain

$$A=\frac{DF}{DW}=\begin{pmatrix} \frac{\partial F_{11}(W)}{\partial w_1} & \frac{\partial F_{11}(W)}{\partial w_2} & \frac{\partial F_{11}(W)}{\partial w_3} & \frac{\partial F_{11}(W)}{\partial w_4} \\ \frac{\partial F_{12}(W)}{\partial w_1} & \frac{\partial F_{12}(W)}{\partial w_2} & \frac{\partial F_{12}(W)}{\partial w_3} & \frac{\partial F_{12}(W)}{\partial w_4} \\ \frac{\partial F_{13}(W)}{\partial w_1} & \frac{\partial F_{13}(W)}{\partial w_2} & \frac{\partial F_{13}(W)}{\partial w_3} & \frac{\partial F_{13}(W)}{\partial w_4} \\ \frac{\partial F_{14}(W)}{\partial w_1} & \frac{\partial F_{14}(W)}{\partial w_2} & \frac{\partial F_{14}(W)}{\partial w_3} & \frac{\partial F_{14}(W)}{\partial w_4} \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{(k-3)u^2}{2}+\frac{(k-1)v^2}{2} & (3-k)u & (1-k)v & k-1 \\ -uv & v & u & 0 \\ \frac{-keu}{\rho}+(k-1)u(u^2+v^2) & \frac{ke}{\rho}-\frac{(k-1)(3u^2+v^2)}{2} & (1-k)uv & ku \end{pmatrix}$$

and similarly ,

$$B=\frac{DG}{DW}=\begin{pmatrix} \frac{\partial F_{21}(W)}{\partial w_1} & \frac{\partial F_{21}(W)}{\partial w_2} & \frac{\partial F_{21}(W)}{\partial w_3} & \frac{\partial F_{21}(W)}{\partial w_4} \\ \frac{\partial F_{22}(W)}{\partial w_1} & \frac{\partial F_{22}(W)}{\partial w_2} & \frac{\partial F_{22}(W)}{\partial w_3} & \frac{\partial F_{22}(W)}{\partial w_4} \\ \frac{\partial F_{23}(W)}{\partial w_1} & \frac{\partial F_{23}(W)}{\partial w_2} & \frac{\partial F_{23}(W)}{\partial w_3} & \frac{\partial F_{23}(W)}{\partial w_4} \\ \frac{\partial F_{24}(W)}{\partial w_1} & \frac{\partial F_{24}(W)}{\partial w_2} & \frac{\partial F_{24}(W)}{\partial w_3} & \frac{\partial F_{24}(W)}{\partial w_4} \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 0 & 1 & 0 \\ -uv & v & u & 0 \\ \frac{(k-3)v^2}{2}+\frac{(k-1)u^2}{2} & (1-k)u & (3-k)v & k-1 \\ \frac{-kev}{\rho}+(k-1)v(u^2+v^2) & (1-k)uv & \frac{ke}{\rho}-\frac{(k-1)(3v^2+u^2)}{2} & kv \end{pmatrix}$$

Now, let $\alpha,\beta\in\Re^1$ and P= $\alpha A+\beta B$. We now show that the matrices

$$T = \begin{pmatrix} 1 & 0 & \frac{1}{2a^2} & \frac{1}{2a^2} \\ u & \tilde{\beta} & \frac{u+a\tilde{\alpha}}{2a^2} & \frac{u-a\tilde{\alpha}}{2a^2} \\ v & -\tilde{\alpha} & \frac{v+a\tilde{\beta}}{2a^2} & \frac{v-a\tilde{\beta}}{2a^2} \\ \frac{u^2+v^2}{2} & \tilde{\beta}u-\tilde{\alpha}v & \frac{H+a(\tilde{\alpha}u+\tilde{\beta}v)}{2a^2} & \frac{H-a(\tilde{\alpha}u+\tilde{\beta}v)}{2a^2} \end{pmatrix}$$

(3.3.2) And

$$T^{-1} = \begin{pmatrix} 1-\frac{(k-1)(u^2+v^2)}{2a^2} & \frac{(k-1)u}{a^2} & \frac{(k-1)v}{a^2} & \frac{-(k-1)}{a^2} \\ \tilde{\alpha}v-\tilde{\beta}u & \tilde{\beta} & -\tilde{\alpha} & 0 \\ -a(\tilde{\alpha}u+\tilde{\beta}v)+\frac{(k-1)(u^2+v^2)}{2} & a\tilde{\alpha}-(k-1)u & a\tilde{\beta}-(k-1)v & k-1 \\ a(\tilde{\alpha}u+\tilde{\beta}v)+\frac{(k-1)(u^2+v^2)}{2} & -a\tilde{\alpha}-(k-1)u & -a\tilde{\beta}-(k-1)v & k-1 \end{pmatrix}$$

Where $\tilde{\alpha}=\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$,$\tilde{\beta}=\frac{\beta}{\sqrt{\alpha^2+\beta^2}}$, $a=\sqrt{\frac{kp}{\rho}}$ and $H=\frac{e+P}{2}=\frac{a^2}{k-1}+\frac{u^2+v^2}{2}$ (Here, a is the speed of sound)

These matrices realizes the diagonalization of **P.**

i.e. $T^{-1}\mathbf{P}T$ =D= diag $(\lambda_1 \ ,\lambda_2 \ ,\lambda_3, \ \lambda_4)$.

The columns of the matrix T are eigenvectors of the matrix associated with its eigenvalues

(3.3.3) $\lambda_1 = \lambda_2 = \alpha u + \beta v$,

$\lambda_3 = \lambda_1 + a(\alpha^2+\beta^2)^{1/2}$,

$\lambda_4 = \lambda_1 - a(\alpha^2+\beta^2)^{1/2}$. (which are real numbers.)

To prove these results , let us proceed in the following way:

First let us show that the matrix $A=\frac{DF}{DW}$ has the eigenvalues

(3.3.4) $\tilde{\lambda}_1=\tilde{\lambda}_2=u$, $\tilde{\lambda}_3=u+a$, $\tilde{\lambda}_4=u-a$ associated with the vectors

$$\mathbf{r}_1 = (1, u, v, \frac{u^2+v^2}{2})^T,$$

$$\mathbf{r}_2 = (0, 0, -1, -v)^T,$$

(3.3.5) $$\mathbf{r}_3 = \frac{1}{2a^2}(1, u+a, v, H+au)^T,$$

$$\mathbf{r}_4 = \frac{1}{2a^2}(1, u-a, v, H-au)^T$$

The matrix $\widetilde{T}$, whose columns are vectors (3.2.5), and $\widetilde{T}^{-1}$ diagonalize $A : \widetilde{T}^{-1} A \widetilde{T} = \text{diag}$ $(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{\lambda}_3, \widetilde{\lambda}_4)$.

Furthermore, we use the rotational invariance of the equations of inviscid flow:

(3.3.6) $Q(\alpha F(W) + \beta G(W)) = (\alpha^2 + \beta^2)^{1/2} F(QW)$, where

(3.3.7) $$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \widetilde{\alpha} & \widetilde{\beta} & 0 \\ 0 & -\widetilde{\beta} & \widetilde{\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Differentiating relation (3.3.6) , we obtain

(3.3.8) $\mathbf{P}(W) = (\alpha^2 + \beta^2)^{1/2} Q^{-1} A(QW) Q$

And hence,

(3.3.9) $\mathbf{P}(W) = Q^{-1} \widetilde{T}(QW) (\alpha^2 + \beta^2)^{1/2} \widetilde{D}(QW) \widetilde{T}^{-1}(QW) Q$

$= T(W) D(W) T^{-1}(W)$,

where, $T(W) = Q^{-1} \widetilde{T}(QW)$,

(3.3.10) $T^{-1}(W) = \widetilde{T}^{-1}(QW) Q$,

$D(W) = (\alpha^2 + \beta^2)^{1/2} \widetilde{D}(QW)$ which are the matrices (3.3.2).

The eigenvalues of the matrix $\mathbf{P}(W)$ are the diagonal entries of the matrix $D(W)$ from (3.3.10) [Theory of diagonalization]

Therefore, the eigenvalues are real numbers as in (3.3.3), which implies the system (3.2.1) -(3.2.4) for the considered case is hyperbolic.

3.4 NON-DIAMENSIONALIZATION

In the viscous , compressible flow the Navier-Stokes equations read as

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho V) = 0 \quad \text{(continuity equation)}$$

$$\frac{\partial V}{\partial t} + (V.\text{grad})V + \frac{1}{\rho}\text{grad}\, P = k\,\Delta\, V, \quad \text{(equation of motion)}$$

where ρ =density of fluid ,t= time
V= velocity of the fluid , div= divergence operator
P= pressure , grad= gradient operator
K= kinematic viscosity , Δ= Laplacian operator

Additionally, we have an equation of the state P= P (ρ) relating pressure and density (since the case is isothermal)

In case of ideal gas , state equation is

$$P = \rho\, R_s T, \text{ where } \quad R_s = \frac{R}{M} = \text{specific gas constant such that}$$

R=8.314J/kmol is the universal gas constant , M is molar mass.
T= temperature

And in the case of a liquid, the equation of the state is $P = -k\frac{\Delta V}{V}$

Where K=compression modulus.
V= volume and
ΔV =change of volume.

Overall, There are 5 equations (1 mass conservation +3 momentum conservation +1 equation of state) for the 5 unknowns quantities P, ρ, 3 components of V; and therefore the system is closed.

However, the above equations are absolutely impractical to model and simulate fluid flow in pipelines. This is because they are

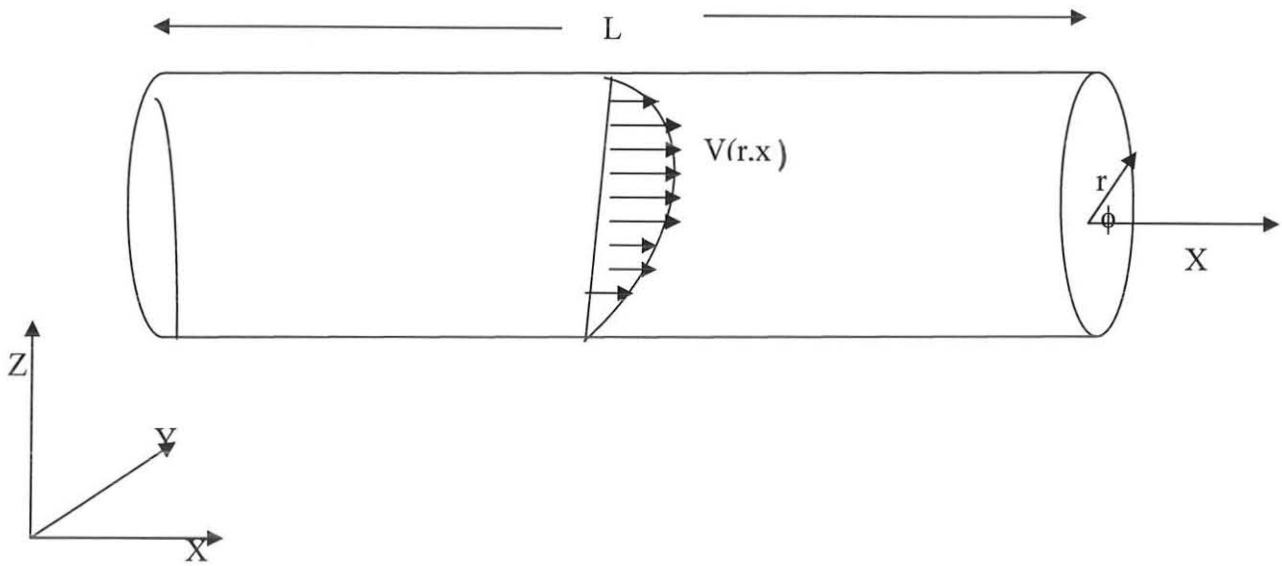
-very complex, due to non-linearity $(V.\nabla)V$

- extremely difficult and expensive to solve(4 time dependant PDE's in $\Re^3$.

-There is no cut and dried rule (Mathematical theory) concerning their solvability.

And, therefore , there is a need to simplify the equations and to adapt them to our flow problem in pipelines.

First of all, we choose a coordinate system that resembles the geometry of our pipeline. Since we will consider straight pipes with circular cross-section, it seems to be advisable to use cylindrical coordinates(r, ϕ ,x) instead of the Cartesian coordinates (x,y,z). Here x is the longitudal coordinate , r is the radial and ϕ the angular coordinate.



Secondly, we assume that the velocity has only a longitudal component. i.e. $V = ve_x + ue_r + we_\phi$ and u=w=0.
Furthermore, we assume rotational symmetry of the fluid flow (i.e the velocity has no angular dependence)

$V(t, r, \phi, x) = V(t,r,x)\, e_x.$

The pressure is assume to vary only along the longitudinal coordinate,(i.e. P=P(t,x) and using the equation of the state $\rho = \rho(t,x)$)
Using the above assumptions, the Navier-Stokes equations reduces to

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho V) = 0$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = k\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial x^2}\right)$$

where $\Delta = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial x^2}$ in cylindrical coordinate.

This system is still very complex because of
- 2D-space (r,x)
- time dependant

- non linearity $v\frac{\partial v}{\partial x}$

In order to simplify the problem further, let us consider the simplest possible flow in an infinitely long circular pipe, the laminar flow, incompressible and viscous flow. Then the above equations reduced to

$$\frac{\partial v}{\partial x}=0$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x}=k(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r})$$ with the boundary condition V(r=R=0) to

$V(r)=\frac{-1}{4\rho k}(R^2-r^2)\frac{\partial p}{\partial x}$, which describes the well- known parabolic velocity profile in a circular pipe.

This is because $\Delta=\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial x^2}$ in cylindrical coordinate

and $\frac{\partial \rho}{\partial t}+\mathrm{div}(\rho V)=0$

$\Rightarrow \frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho V)=0$, as the velocity has only a longitudal component

and $(V.\nabla)V=v\frac{\partial v}{\partial x}$ and $\frac{1}{\rho}(\nabla P)=\frac{1}{\rho}(\frac{\partial P}{\partial x})$

and for incompressible , ρ is constant

$\Rightarrow \frac{\partial \rho}{\partial t}=0$ and $\frac{\partial}{\partial x}(\rho V)=\rho\frac{\partial v}{\partial x}=0 \Rightarrow \frac{\partial v}{\partial x}=0$

$$\therefore \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=k(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial x^2})$$

$$\Rightarrow \frac{1}{\rho}\frac{\partial p}{\partial x}=k(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r})$$

$$\Rightarrow \frac{1}{\rho}\frac{\partial p}{\partial x}=k(V_{rr}+\frac{1}{r}V_r)=k(\frac{1}{r}(rV_r)_r)$$

$$\Rightarrow \frac{1}{r}(rV_r)_r=\frac{1}{\rho k}\frac{\partial P}{\partial x}$$

$$\Rightarrow (rV_r)_r=\frac{r}{\rho k}\frac{\partial P}{\partial x}$$

$$\Rightarrow \int (rV_r)_r\,dr=\int\frac{r}{\rho k}\frac{\partial P}{\partial x}dr$$

$$\Rightarrow rV_r=\frac{r^2}{2\rho k}\frac{\partial P}{\partial x}+A$$

$$\Rightarrow V_r=\frac{r}{2\rho k}\frac{\partial P}{\partial x}+\frac{A}{r}$$

$$\Rightarrow V(r) = \frac{r^2}{4\rho k}\frac{\partial P}{\partial x} + A \ln r + B$$

But A must be zero, for not to become the velocity infinite along the longitudal axis, and B can be evaluated from V(r=R)=0.

$$\text{i.e. } V(R) = \frac{R^2}{4\rho k}\frac{\partial P}{\partial x} + B$$

$$\Rightarrow B = -\frac{R^2}{4\rho k}\frac{\partial P}{\partial x}$$

$$\therefore \quad V(r) = \frac{r^2}{4\rho k}\frac{\partial P}{\partial x} - \frac{R^2}{4\rho k}\frac{\partial P}{\partial x} = \frac{-1}{4\rho k}(R^2 - r^2)\frac{\partial P}{\partial x}. \quad (*)$$

Now, introducing the average velocity $\overline{V}$,

$$\overline{V} = \frac{\int_0^R \rho(2\pi r)V(r)dr}{(\pi R^2)\rho}$$

$$= \frac{2\pi}{\pi R^2}\int_0^R V(r)rdr = \frac{2}{R^2}\int_0^R r\left(\frac{r^2 - R^2}{4\rho k}\right)\frac{\partial P}{\partial x}dr$$

$$= \frac{2}{4\rho k R^2}\frac{\partial P}{\partial x}\int_0^R (r^3 - rR^2)dr$$

$$\Rightarrow \overline{V} = \frac{-1}{8\rho k}R^2\frac{\partial P}{\partial x}. \quad (**)$$

We can write the cross-sectional velocity distribution as the average velocity multiplied by the shape of the function, using (*) and (**), $V(r) = \overline{V} \,.\, 2(1 - \frac{r^2}{R^2})$

$$\text{i.e. } V(r) = \frac{-1}{4\rho k}(R^2 - r^2)\frac{\partial P}{\partial x}$$

$$= \frac{-2}{2(4)\rho k}R^2\frac{\partial P}{\partial x} + \frac{1}{4\rho k}r^2\frac{\partial P}{\partial x}$$

$$= 2\left(\frac{-R^2}{8\rho k}\frac{\partial P}{\partial x}\right) + \frac{1}{4\rho k}\frac{\partial P}{\partial x}r^2 \quad ,$$

But $\quad \overline{V} = \frac{-1}{8\rho k}R^2\frac{\partial P}{\partial x}$

$$\Rightarrow \frac{-\overline{V}}{R^2} = \frac{1}{8\rho k}\frac{\partial P}{\partial x} = \frac{1}{2(4)\rho k}\frac{\partial P}{\partial x}$$

$$\Rightarrow \frac{-2\overline{V}}{R^2} = \frac{1}{4\rho k}\frac{\partial P}{\partial x}$$

$\therefore\ V(r)=2(\overline{V}) - \frac{2\overline{V}}{R^2}r^2 = 2\overline{V}(1-\frac{r^2}{R^2})$ i.e $V(r)=\overline{V}.2(1-\frac{r^2}{R^2})$., as we want to show.

In our case , we expect a similar decomposition of the velocity field $V(t,r,x)=\overline{V}(t,x).2(1-\frac{r^2}{R^2})$.

Now ,let us try to use this decomposition of the velocity to solve the Navier-stokes equations.

Plugging this decomposition for V into the Navier-Stokes equations we obtain the system

$$\frac{\partial \rho}{\partial t} + 2(1-\frac{r^2}{R^2})\frac{\partial}{\partial x}(\rho\overline{V}) = 0$$

$$2(1-\frac{r^2}{R^2})(\frac{\partial \overline{V}}{\partial t} + \overline{V}\frac{\partial \overline{V}}{\partial x}) = \frac{-8k\overline{V}}{R^2} + 2k(1-\frac{r^2}{R^2})(\frac{\partial^2 \overline{V}}{\partial x^2}).$$

Averaging over the cross-sectional area reduces the equations to

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho\overline{V}) = 0$$

$$\frac{\partial \overline{V}}{\partial t} + \overline{V}\frac{\partial \overline{V}}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{-8k\overline{V}}{R^2} + k\frac{\partial^2 \overline{V}}{\partial x^2}.$$

as a last step we would like perform a dimensional analysis and scaling of the above system. We use the length L of the pipeline as a reference scale for the spatial coordinate and the time $\frac{L}{c}$ a sound wave needs to travel through the pipe as our time scale. Since we expect the fluid velocity V to be small compared to our reference velocity c , we can write

$$\overline{V} = \varepsilon c \tilde{V} \text{ where } \varepsilon \ll 1.$$

Accordingly, we scale the pressure by $P = P_0 + \varepsilon \rho c^2 \tilde{P}$, where P_0 is some outside pressure. Summarizing these scaling, we have

$x = L\tilde{x}$, $t = \frac{L}{c}\tilde{t}$, $\overline{V} = \varepsilon c \tilde{V}$, $P = P_0 + \varepsilon \rho c^2 \tilde{P}$, where _ denotes the dimensionless quantities. From this time on we will skip the _.

Using the equation of state $P=P(\rho)$ and the equation for the speed of sound $c^2 = \frac{dP}{d\rho}$, we obtain

$$\frac{\partial \rho}{\partial t} + V\frac{\partial \rho}{\partial x} + \rho\frac{\partial V}{\partial x} = 0 \longrightarrow \frac{\partial P}{\partial t} + V\frac{\partial P}{\partial x} + c^2\rho\frac{\partial V}{\partial x} = 0$$

$$\Rightarrow \frac{\varepsilon\rho c^3}{L}(\frac{\partial P}{\partial t} + \varepsilon V\frac{\partial P}{\partial x} + \frac{\partial V}{\partial x}) = 0 \text{ and } \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{-8kV}{R^2} + k\frac{\partial^2 V}{\partial x^2}$$

$$\Rightarrow \frac{\varepsilon c^2}{L}\left(\frac{\partial v}{\partial t}+\varepsilon v\frac{\partial v}{\partial x}+\frac{\partial p}{\partial x}\right)=\frac{\varepsilon c^2}{L}\left(\frac{-8k}{R^2}\frac{L}{c}V+\frac{k}{L^2}\frac{L}{c}\frac{\partial^2 V}{\partial x^2}\right)$$

Introducing the Reynolds number

$R_e = \frac{Rc}{k}$ which is for the flows under consideration typically of the order of 10^7 ,

and the aspect ratio of the pipe $\delta = \frac{L}{R}$ which is typically 10^5. We observe that the first term on the right hand side scales like $\frac{\delta}{\mathbf{Re}}$ and the second one scales like $\frac{1}{\mathbf{Re}.\delta}$.

(Dropping $O(\varepsilon)$ terms) of our pipe flow equations

$$\frac{\partial P}{\partial t}+\frac{\partial V}{\partial x}=0$$

$$\frac{\partial v}{\partial t}+\frac{\partial p}{\partial x}+\frac{8\delta}{\mathbf{Re}}V=0$$

Or rewritten in dimensional variables,

$$\frac{\partial P}{\partial t}+c^2\rho\frac{\partial V}{\partial x}=0$$

$$\frac{\partial v}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial x}=-\eta V, \text{ where } \eta=\frac{8k}{R^2} \text{ is the friction factor.}$$

The above equations are pipeline flow equations with linear friction and commonly called Waterharmmer equations.

This system of firs t order PDE's can be rewritten as a second order PDE for V (or P respectively)

i.e. $\frac{\partial^2 V}{\partial t^2}+\eta\frac{\partial V}{\partial t}-c^2\frac{\partial^2 V}{\partial x^2}=0$, the so-called Telegraph equation.

In the invisid case k=0 and thus $\eta=0$,the telegraph equation reduces to the well known wave equation

$$\frac{\partial^2 V}{\partial t^2}-c^2\frac{\partial^2 V}{\partial x^2}=0$$

$-c^2V_{xx}+V_{tt}=0$, here , A= $-c^2$, B= 0, C= 1

Then $B^2-4AC=4c^2>0$

Hence this equation (which is obtained from system of equations describing fluid flow in pipelines) represents hyperbolic equations for all x and for all t and it is solvable (By either Fourier method or D'Alembert's Method)

REFERENCES

1. D.J.Tritton (1988). Physical Fluid Dynamics. St Edmunds bury Press Ltd, Great Britain . 2nd edition.

2.Fritz John (1979). Partial Differential Equations. Narosa Publishing House, New Delhi. 3rd edition.

3.M.Freistaure (1993). Mathematical Methods in Fluid Dynamics . Long man Group UK limited , London.

4.P.D.M_c Cormark and Lawrence Crane (1973). Physical Fluid Dynamics Academic Press, Inc.,New York and London

5.W.F.Hughes and J.A. Brighton (1967). Theory and Problems of Fluid Dynamics. Schaum publishing Co.,New York.

6. Thomas Gotz and Christian Schick (2002). Hyperbolic Equations and Flow in Pipelines. Lecture notes ITB Bandung.