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Fuzzy Programming Approach To Multilevel Linear Programming Problems

Graduate Project

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ABSTRACT

Multilevel programming is characterized as mathematical programming to solve decentralized planning problems with multiple decision makers in a hierarchical organization. They become more important for contemporary decentralized organization where each unit or department seeks its own interests. In this report we have considered a multilevel programming problem and applied fuzzy mathematical programming (FMP) approach to obtain the solution of the system. We have suggested FMP method for the minimization of the objectives using linear membership functions. FMP is a supervised search procedure (supervised by the higher level decision maker (DM)). The higher level DM provides the preferred values of decision variables under his control (to enable the lower level DM to search for his optimum in a wider feasible space) and the bounds of his objective function (to direct the lower level DM to search for his solutions in the right direction).

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INTRODUCTION

The use of the fuzzy set theory for decision problems with several conflicting objectives was first introduced by Zimmermann. Thereafter, various versions of fuzzy programming (FP) have been investigated and widely circulated in literature. The use of the concept of *tolerance membership function* of fuzzy set theory to MLP problems for satisfactory decisions was first introduced by Lai in 1996 [3]. Shih and Lee further extended Lai's concept by introducing the compensatory fuzzy operator for solving MLPPs [5]. Sinha studied alternative MLP techniques based on *fuzzy mathematical programming (FMP).* The basic concept of these fuzzy mathematical programming (FMP) approaches is the same as fuzzy goal programming(FGP) approach which implies that the lower level DMs optimizes his/her objective function, taking a goal or preference of the higher level DMs into consideration. In the decision process, considering the membership functions of the fuzzy goals for the decision variables of the higher level DM, the lower level DMs solves a FMP problem with a constraint on an overall satisfactory degree of the higher level DMs. If the proposed solution is not satisfactory to the higher level DMs, the solution search is continued by redefining the elicited membership functions until a satisfactory solution is reached [5]. The main difficulty that arises with the FMP approach of Sinha is that there is possibility of rejecting the solution again and again by the higher level DMs and reevaluation of the problem is repeatedly needed to reach the satisfactory decision, where the objectives of the DMs are over conflicting [5]. In this FMP techniques, last(lower) level is most important and decision of lowest level remains either unchanged or closest to individual best decision, which leads to paradox, that the decision power of the lowest level DM dominates the higher level DM. To overcome, these difficulties Tapan Kumer Roy propose "FGP approach to MLPPs" is presented for proper distribution of decision powers to the DMs to arrive at satisfaction decision for overall benefit of the organization[7]. The resulting multi-level programming techniques approach is a powerful one and can be used to solve practical problems encountered in large hierarchical organizations with decentralized operations [4].

Multilevel programming is characterized as mathematical programming to solve decentralized planning problems. We have considered a multilevel linear programming problem and applied fuzzy mathematical programming (FMP) approach to obtain the solution of the system. Over the last three decades, many methodologies have been proposed to solve Multilevel programming problems (MLPPs) potentially for agriculture, bio-fuel production, economic systems, governmental policy, network flow design, transportation design and etc. Although many approaches have been proposed during the last three decades, none of the approaches are computationally efficient. Some important existing solution approaches such as, the *vertex enumeration approach and transformation approaches* [5]. The former is to seek a comprise vertex by simplex algorithm based on adjusting higher level control variables. The computational effort increases exponentially and thus is very inefficient for large problems. The latter involves transforming the lower-level programming problem to be the constraints of the higher level by its KKT conditions or penalty function. Because of the non-linearity or the appearance of the Lagrangian multipliers, the resulting problem becomes complex and sometimes unmanageable.

In this project, we discuss a procedure for solving MLPPs in large hierarchal decentralized organization through linear fuzzy mathematical programming (FMP) approach. In order to reach the optimal solution of MLPPs using fuzzy programming approach, the report contains three chapters. In chapter I, we discuss the basic concept of fuzzy set, membership function and binary operation on fuzzy numbers, in chapter II, the basic concept of Multi-level programming, characteristics and mathematical formulation of MLLP's will be presented, in chapter III the procedure for solving MLPPs by FMP approaches (i.e. the formulate of FMP models of the problems and FMP solution approach) are discussed; moreover, the selection of compromise solution to FMP models and comparison of optimal solution with other FP approach, is also included.

Chapter 1: Fuzzy Set

1. Fuzzy Set Theory

Fuzzy set theory has been developed to solve problems where the descriptions of activities and observations are imprecise, vague, or uncertain. The term "fuzzy" refers to a situation in where there are no well-defined boundaries of the set of activities or observations to which the descriptions apply. For example, one can easily assign a person 180cm tall to the "class of tall men". But it would be difficult to justify the inclusion or exclusion of a 173cm tall person to that class, because the term "tall" does not constitute a well-defined boundary. This notion of fuzziness exists almost everywhere in our daily life, such as a "class of red flowers," a "class of good shooters," a "class of comfortable speeds for traveling, a "numbers close to 10," etc. These classes of objects cannot be well represented by classical set theory. In classical set theory, an object is either in a set or not in a set. An object cannot partially belong to a set.

1.1. Definition of a fuzzy set

Definition 1.1: Let X be a space of pointes (objects) called universal or referential set. An ordinary (crisp) subset $\tilde{A}$ is characterized by a characteristics function $X_{\tilde{A}}$ as mapping from the element of X to the element of the set {0, 1}, defined by

$$X_A(x) = \begin{cases} 1 \ if \ x \in \tilde{A} \\ 0 \ if \ x \notin \tilde{A} \end{cases} \qquad (1)$$

where {0, 1} is called a valuation set..

If the valuation set is allowed to be the real interval [0, 1], $\tilde{A}$ is called a fuzzy set.

Definition 1.2: Let X be any referential set and $\tilde{A}$ is a subset of X. A fuzzy set of $\tilde{A}$ in X is a pairs

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \epsilon X\} \qquad (2)$$

where $\mu_{\tilde{A}}: X \to [0,1]$ is called the membership function at x representing "*the grade of membership*" of x in $\tilde{A}$.

Remark:

1. The value zero is used to represent a complete non-membership, the value one is used to represent complete membership, and values in between are used to represent intermediate degrees of membership.
2. A fuzzy set is an extension or the generalization of a crisp set. Crisp sets allow only full Membership or no membership at all, whereas fuzzy sets allow also partial degree of membership.

1.2. Basic Concepts of Fuzzy Sets

The basic concepts presented here include complement, intersection, union, algebraic product, algebraic sum, difference, support, α-cut, convexity, normality, cardinality, and the m^{th} power of a fuzzy set $\tilde{A}$.

Complement

The complement of a fuzzy set $\tilde{A}$, denoted by $\tilde{A}^c$, is defined as:

$$\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x)\ ,\forall x \in X \tag{3}$$

Intersection

The intersection of $\tilde{A}$ and $\tilde{B}$ denoted by $\tilde{A} \cap \tilde{B}$ which is the largest fuzzy subset contained in both fuzzy subsets $\tilde{A}$ and $\tilde{B}$. When the min operator is used to express the logic "and", its corresponding membership is then characterized by:

$$\mu_{\tilde{A}\cap\tilde{B}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\},\ \forall x \in X \tag{4}$$

Union

The union of $\tilde{A}$ and $\tilde{B}$ which denoted by $\tilde{A} \cup \tilde{B}$ is dual to the notation of intersection. Thus, the union of $\tilde{A}$ and $\tilde{B}$ is defined as the smallest fuzzy set containing both $\tilde{A}$ and $\tilde{B}$. The membership function of $\tilde{A} \cup \tilde{B}$ is given by:

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\},\ \forall x \in X \tag{5}$$

Algebraic product

The algebraic product $\tilde{A}\tilde{B}$ of $\tilde{A}$ and $\tilde{B}$ is characterized by the following membership function:

$$\mu_{\tilde{A}\tilde{B}}(x) = \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x),\ \forall x \in X \tag{6}$$

This algebraic product is considered as a soft "and".

Algebraic sum

The algebraic sum $\tilde{A} \oplus \tilde{B}$ of $\tilde{A}$ and $\tilde{B}$ is defined by the following membership function:

$$\mu_{\tilde{A} \oplus \tilde{B}}(x) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) . \mu_{\tilde{B}}(x), \ \forall x \in X \quad (7)$$

This algebraic sum is considered as a soft "or"

Difference

The difference $\tilde{A} - \tilde{B}$of $\tilde{A}$ and $\tilde{B}$ is defined by:

$$\mu_{\tilde{A} \cap \tilde{B}^c}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}^c}(x)\}, \ \forall x \in X \quad (8)$$

where $\tilde{B}^c$ is the complement of $\tilde{B}$.

Example 1. Concider X= {Abutu, Addis, Dave, Bake} Suppose A is the Fuzzy subset of "***good looking students***" and B is the fuzzy subset of "***intelligent students***". Then

X	Abutu	Addis	Dave	Bake
$\mu_{\tilde{A}}$	0.2	0.3	0.6	0.8
$\mu_{\tilde{B}}$	0.7	0.4	0.1	0.5
$\mu_{\tilde{A}^c}$	0.8	0.7	0.4	0.2
$\mu_{\tilde{A} \cap \tilde{B}}$	0.2	0.3	0.1	0.8
$\mu_{\tilde{A} \cup \tilde{B}}$	0.7	0.4	0.6	0.8
$\mu_{\tilde{A}\tilde{B}}$	0.14	0.12	0.06	0.4
$\mu_{\tilde{A} \oplus \tilde{B}}$	0.76	0.58	0.64	0.9
$\mu_{\tilde{B}^c}$	0.3	0.6	0.9	0.5
$\tilde{A} - \tilde{B} = \mu_{\tilde{A} \cap \tilde{B}^c}$	0.2	0.3	0.6	0.5

Support and α-cut

Sometimes, we might only need objects of a fuzzy set instead of its characteristic function, that is, to transfer a fuzzy set into a crisp set. In order to do so, we need two concepts, support, and α-cut.

It is often necessary to consider those elements in a fuzzy set which have non-zero membership grades. These elements are called the support of that fuzzy set.

Definition1.2.1 Given a fuzzy set A, its support S(A) is an ordinary crisp subset on X defined as

$$S(A) = \{x: \mu_A(x) > 0 \; and \; x \in X\} \tag{10}$$

Definition1.2.2 Given a fuzzy set A, its α-cut A_α defined as

$$A_\alpha = \{x: \mu_A(x) > \alpha \; and \; x \in X\} \tag{11}$$

Where α is the confidence level.

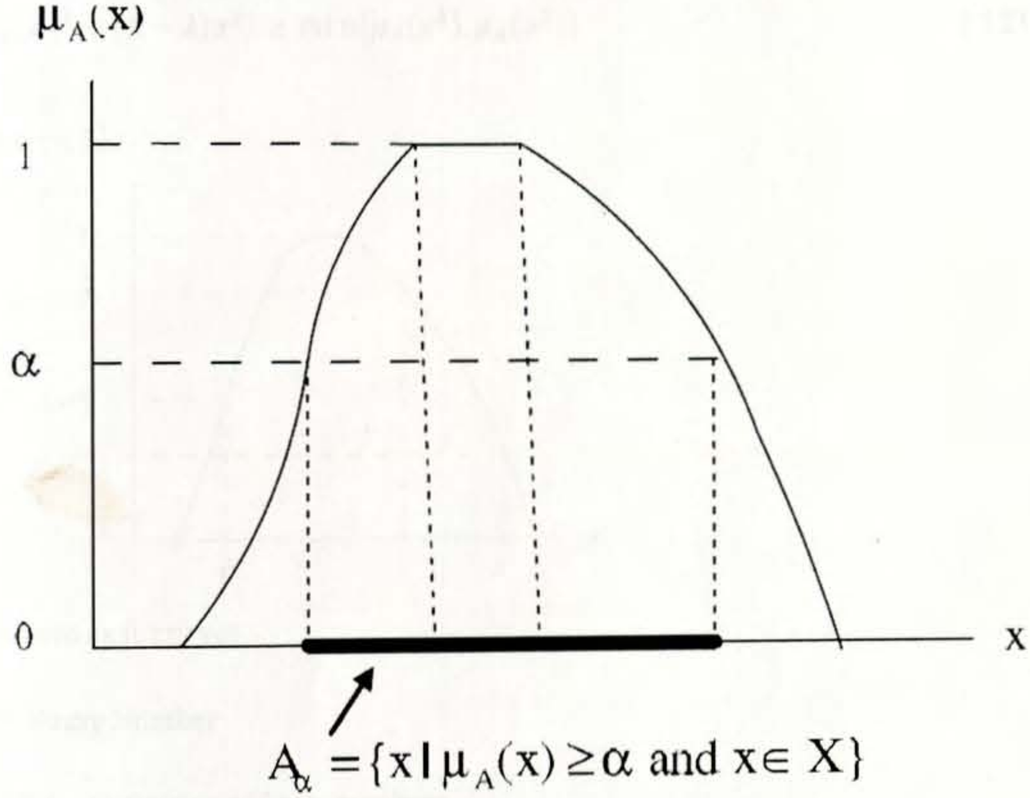


Fig 1.1 An α-set

Example1.2.2:Let X = {10,20,30,40,50,60,70,80,90,100},be the set of possible speeds(mph) at which people feel comfortable in traveling a long distance. Then the fuzzy set "comfortable speed for long distance travel" may be defined by an individual as:

A = {(0.7, 30), (0.75, 40), (0.8, 50), (0.8, 60), (1, 70), (0.8, 80), (0.3, 90)}.

Then S (A) = {30, 40, 50, 60, 70, 80, 90} and

$A_{0.5}$ = {30, 40, 50, 60, 70, 80}

<u>Normality</u>

A fuzzy set A is normal if and only if $\mathbf{sup}\boldsymbol{\mu_A}(\boldsymbol{x})$**=1** , $\boldsymbol{x} \in \boldsymbol{X}$, that is, the supremum is unity.

Note: A fuzzy set is subnormal if it is not normal. A non-empty subnormal fuzzy set can be normalized by dividing each $\mu_A(x)$ by the factor sup$\mu_A(x)$. (***A fuzzy set is empty if and only if*** $\mu_A(x) = 0, \forall x \in X$)

Convexity

A fuzzy set A in X is convex if and only if for every pair of points $x^1, x^2 \in X$ the membership function of A satisfies the inequality:

$$\mu_A(\lambda x^1 + (1-\lambda)x^2) \geq \min\{\mu_A(x^1), \mu_A(x^2)\} \qquad (12)$$

where $\lambda \in [0, 1]$.

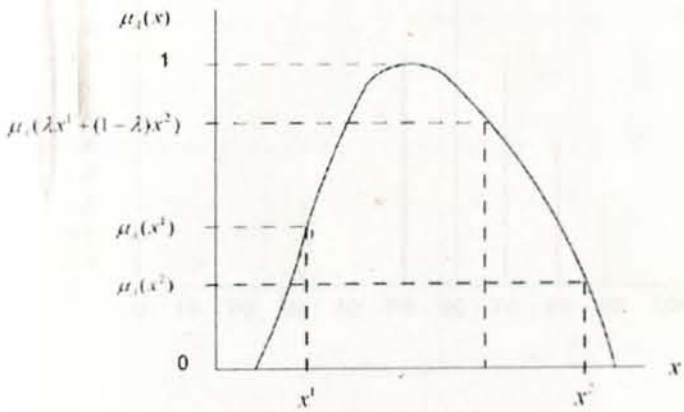


fig 1.2. A convex fuzzy set

1.3. Fuzzy Number

1.3.1. Definition of fuzzy numbers

The term fuzzy number is used to handle imprecise numerical quantities, such as "Close to 10," "about 60," "several," etc. A general definition of a fuzzy number is given by Dubois and Prade[1]: any fuzzy subset M = {(x, μ(x))} where x takes its number from the real line R and, $\mu_M(x) \in [0,1]$.

Definition1.3.1 Let A be a fuzzy set, its membership function is $\mu_A: R \rightarrow [0,1]$, if

i. A is upper semi-continuous, i.e. α-cut A_α is close set, for $0 < \alpha \leq 1$.

ii. A is normal, i.e., $A_1 \neq \emptyset$.

iii. A_α is a convex subset of R, for $0 < \alpha \leq 1$.

then A is a fuzzy number.

A fuzzy number may be represented in discrete or continuous form. For example, Let M be the fuzzy number "about 60" which may be given as either one of the Following

(1) **Discrete membership function**: Given the universal

$$X=\{10, 20, 30, 40, 50, 60, 70, 80, 90\}.$$

the fuzzy number M may be represented as shown in Figure1.3

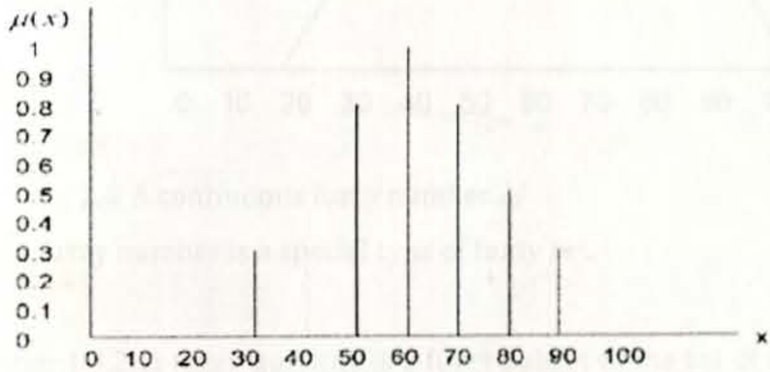


Fig 1.3 A discrete fuzzy number *M*

(2) **Continuous membership function**: Given the universe X= {the set of allreal numbers}, the continuous membership function for M may be represented as (see Figure):

$$\mu_M(x) = (1 + \frac{(x-60)^2}{10^2})^{-1} \tag{13}$$

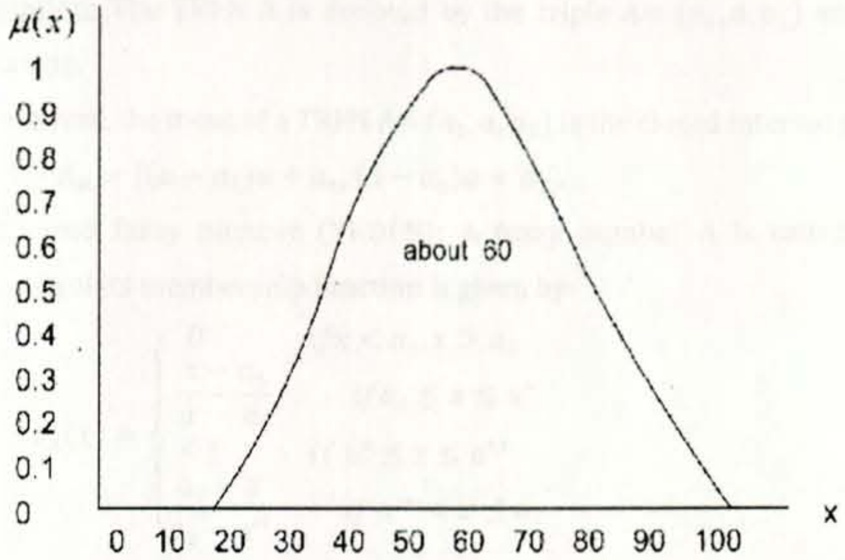


Fig 1.4. A continuous fuzzy number M

Note: A fuzzy number is a special type of fuzzy set.

Definition1.3.2. A fuzzy quantity is a fuzzy subset of the set of real numbers. The family of all fuzzy quantities usually denoted by F(R).

Remark:

1. $S(A) = \{x \in R: \mu_A(x) > 0\}$ is the support of A
2. $A_\alpha = \{x: \mu_A(x) > \alpha \text{ and } x \in R\}$ is the α-cut of A
3. $\sup\mu_A(x)=1, x \in R$ if and only if A fuzzy set A is normal
4. If A is not a fuzzy number then there exists α ∈ [0, 1] such that A_α is not a convex subset of R.

1.3.2. Types of Fuzzy Numbers

1. Triangular fuzzy number(TRFN): A fuzzy number A is called triangular fuzzy number if its membership function μ_A is given by

$$\mu_A(x) = \begin{cases} 0 & if\, x < a_1, x > a_2 \\ \dfrac{x - a_1}{a - a_1} & if a_1 \leq x \leq a \\ \dfrac{a_2 - x}{a_2 - a} & if\, a < x \leq a_2 \end{cases} \tag{14}$$

Notation: The TRFN A is denoted by the triple A= (a_1, a, a_2) and has a shape of triangle.

Moreover, the α-cut of a TRFN A= (a_1, a, a_2) is the closed interval given by

$$A_\alpha = [(a - a_1)\alpha + a_1, (a - a_2)\alpha + a_2].$$

2. Trapezoid fuzzy number (TRDFN): A fuzzy number A is called trapezoid fuzzy number if its membership function is given by

$$\mu_A(x) = \begin{cases} 0 & if x < a_1, x > a_2 \\ \dfrac{x - a_1}{a' - a_1} & if a_1 \le x \le a^{*} \\ 1 & if\ a^{*} \le x \le a^{**} \\ \dfrac{a_2 - x}{a_2 - a''} & if\ a^{**} < x \le a_2 \end{cases} \tag{15}$$

Notation: The TRDFN is denoted by the quadruplet

$$A = (a_1, a^{*}, a^{**}, a_2)$$

and has the shape of trapezoid. The α-cut of a TRDFN

$$A = (a_1, a^{*}, a^{**}, a_2)$$

is a closed interval given by

$A_\alpha = [a_1(\alpha), a_2(\alpha)][(a^{*} - a_1)\alpha + a_1, (a^{**} - a_2)\alpha + a_2].$

3. L-R fuzzy number: A fuzzy number A is called L-R fuzzy number if its membership function is given by

$$\mu_A(x) = \begin{cases} L\left(\frac{x-a}{\alpha}\right); & if (a - \alpha) \le x < a, \alpha > 0 \\ 1 \quad ; & if\ a \le x \le b \\ R\left(\frac{x-b}{\beta}\right); & if b < x \le (b + \beta), \beta > 0 \\ 0 \quad ; & otherwise \end{cases} \tag{16}$$

where $L(.)$ and $R(.)$ are picewise continues function such that $L(.) = R(.)=1$; $L(.)$ is increasing and $R(.)$ decreasing. L is called the left reference function and R is called the right referential function. α *and* β are the left and the right spreads respectively

1.3.3. **Binary operation.**

Definition1.3.3.1. Let o be a binary operation on R, then o induces a binary operation on F(R). i.e. $o: F(R) \times F(R) \rightarrow F(R)$ is given by

$$\mu_{AoB}(X) = {}_{X=AoB}^{max}\{\min(\mu_A(a), \mu_B(b))\}$$ or simply we write as

$$\mu_{AoB}(X) = {}_{X=aob}^{\vee}\{\wedge(\mu_A(a), \mu_B(b))\}.$$

Arithmetic Operation on Fuzzy Numbers

In this section we will see how ordinary arithmetic operation, addition, subtraction, multiplication, and division on R can be extended to F(R) and performed. Let A and B be fuzzy numbers $\alpha \in [0,1], and, A_\alpha, B_\alpha$ be α-cut of A and B respectively.

Let $A_\alpha = [a_1^\alpha, a_2^\alpha]$ and $B_\alpha = [b_1^\alpha, b_2^\alpha]$

Definition 1.3.3.1(Addition (+) and subtraction (−)). If $x \in [a_1^\alpha, a_2^\alpha]$, $y \in [b_1^\alpha, b_2^\alpha]$ then $x + y \in [a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha]$ and $x - y \in [a_1^\alpha - b_1^\alpha, a_2^\alpha - b_2^\alpha]$.

Therefore the addition of A and B, denoted by A (+) B, is defined as

$$A\,(+)\,B = [a_1^\alpha, a_2^\alpha]\,(+)[b_1^\alpha, b_2^\alpha] = [a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha]$$

Similarly, the subtraction of A and B, denoted by A (−) B is defined as

$$A\,(-)\,B = [a_1^\alpha, a_2^\alpha]\,(-)[b_1^\alpha, b_2^\alpha] = [a_1^\alpha - b_1^\alpha, a_2^\alpha - b_2^\alpha].$$

Definition1.3.3.2. (Multiplication (·)). The multiplication of two closed intervals $A = [a_1^\alpha, a_2^\alpha]$ and $B = [b_1^\alpha, b_2^\alpha]$ of R, denoted by A (·) B, is defined as

$$A\,(\cdot)\,B = [a_1^\alpha, a_2^\alpha]\,(.)[b_1^\alpha, b_2^\alpha]$$
$$= [\min(a_1^\alpha b_1^\alpha, a_1^\alpha b_2^\alpha, a_2^\alpha b_1^\alpha, a_2^\alpha b_2^\alpha), \max(a_1^\alpha b_1^\alpha, a_1^\alpha b_2^\alpha, a_2^\alpha b_1^\alpha, a_2^\alpha b_2^\alpha)]$$

In case these intervals are in R^+, the non-negative real line, the multiplication formula gets simplified to

$$A\,(\cdot)\,B = [a_1^\alpha b_1^\alpha, a_2^\alpha b_2^\alpha].$$

Definition1.3.3.3. (Scalar multiplication and inverse). Let $A = [a_1^\alpha, a_2^\alpha]$ be a closed interval in R^+ and $k \in IR^+$ identifying the scalar k as the closed interval $[k_1, k_2]$, the scalar multiplication $k \cdot A$ is defined as

$$k.A = [k_1, k_2](.)[a_1^\alpha, a_2^\alpha] = [k_1 a_1^\alpha, k_2 a_2^\alpha]$$

For $A = [a_1^\alpha, a_2^\alpha]$ inR^+ if $x \in [a1, a2]$ and $0 \notin [a_1^\alpha, a_2^\alpha]$ then

$\frac{1}{x} \in [\frac{1}{a_2^\alpha}, \frac{1}{a_1^\alpha}]$ Therefore, the inverse of A, denoted by A^{-1}, is defined to be

$$A^{-1} = [a_1^\alpha, a_2^\alpha]^{-1} = [\frac{1}{a_2^\alpha}, \frac{1}{a_1^\alpha}],$$

provide $0 \notin [a_1^\alpha, a_2^\alpha]$.

Definition1.3.3.4. (Division (:)). The division of two closed intervals $A = [a_1^\alpha, a_2^\alpha]$ and $B = [b_1^\alpha, b_2^\alpha]$ of R, denoted by A(:)B, is defined as the multiplication of $[a_1^\alpha, a_2^\alpha]$ and $[\frac{1}{b_2^\alpha}, \frac{1}{b_1^\alpha}]$

$$A(:)B = [a_1^\alpha, a_2^\alpha]\,(:)[b_1^\alpha, b_2^\alpha]$$
$$= [a_1^\alpha, a_2^\alpha](.)[\frac{1}{b_2^\alpha}, \frac{1}{b_1^\alpha}]$$
$$= [\min(\frac{a_1^\alpha}{b_2^\alpha}, \frac{a_1^\alpha}{b_1^\alpha}, \frac{a_2^\alpha}{b_2^\alpha}, \frac{a_2^\alpha}{b_1^\alpha}), \max(\frac{a_1^\alpha}{b_2^\alpha}, \frac{a_1^\alpha}{b_1^\alpha}, \frac{a_2^\alpha}{b_2^\alpha}, \frac{a_2^\alpha}{b_1^\alpha})]$$

Definition1.3.3.5. (Max (V) and min (Λ) operations). Let $A = [a_1^\alpha, a_2^\alpha]$ and $B = [b_1^\alpha, b_2^\alpha]$ be two closed intervals in R. Then the max (V) *and* min (Λ) operations on A and B are defined as

$$A(\vee)B = [a_1^\alpha, a_2^\alpha]\ (\vee)[b_1^\alpha, b_2^\alpha] = [a_1^\alpha \vee b_1^\alpha, a_2^\alpha \vee b_2^\alpha]$$
$$A(\wedge)B = [a_1^\alpha, a_2^\alpha]\ (\wedge)[b_1^\alpha, b_2^\alpha] = [a_1^\alpha \wedge b_1^\alpha, a_2^\alpha \wedge b_2^\alpha].$$

Chapter 2: Multileveled programming

2. Definition of Multileveled programming

Multilevel programming has been studied extensively since the 1980s, especially for the bi-level case. The p-level,($p \geq 3$), programming problems were investigated by Bard, Benson, Ruan and others [8]. Multilevel programming (MLP) techniques are developed to solve decentralized planning problems with multiple decision makers (DMs) arranged in a hierarchical structure. Multilevel programming has been applied widely to many decision situations, for example, water quality problems, traffic planning, tax credits determination, pollution control policy and etc [2]. The model proposes to solve problems where the decision is executed in a top-to-down level sequential manner, and where the lower decision makers do have freedom to make the decision within the broad range set by the top managers or decision makers. The basic concept of the approach is that an upper (higher) level DM sets the goal and then asks each lower level of the organization for their own optimum one which are calculated in isolation. The lower-level managers' decisions are then submitted and modified by the higher level with consideration of the overall benefit for the organization. This process is continued until a satisfaction is reached. Multilevel programming is particularly appropriate for problems with the following characteristics:

- Interaction: Interactive decision-making units within a predominantly hierarchical
- Hierarchy: Execution of decision is sequential, from upper to lower level.
- Full information: Each DM is fully informed about all prior choices when it is his turn to move.
- Nonzero sum: The loss for the cost of one level is unequal to the gain for the cost of the other level. External effect on a DM's problem can be reflected in both the objective function and the set of feasible decision space.
- Each DM controls only a subset of the decision variables in an organization.

2.1. Mathematical formulation of MLP Models

We consider a P-level programming problem of maximization type objective function at each level. Mathematically, the problem can be represented as follows.

$$\max_{x_1} f_1(x_1, x_2, \dots, x_p) = \sum_{j=1}^{p} c_{1j} x_j$$

$$where\ x_2, x_3, \dots, x_4\ solve$$

$$\max_{x_2} f_2(x_1, x_2, \dots, x_p) = \sum_{j=1}^{p} c_{2j} x_j$$

$$where\ x_3, x_4, \dots, x_4\ solve$$

....

$$\max_{x_p} f_p(x_1, x_2, \dots, x_p) = \sum_{j=1}^{p} c_{pj} x_p$$

$$s.t\ A_{i1}x_1 + A_{i2}x_2 + \dots + A_{ip}x_p \begin{pmatrix} \le \\ = \\ \ge \end{pmatrix} b_i, \quad i = 1{:}m$$

$$x_1, x_2, \dots, x_p \ge 0$$

where $x_i, i = 1,2,\dots,p$, is an n_i-dimensional decision variable column vector; c_{ij} where $i = j = 1,2,\dots,p$ is an n_j-dimensional coefficient row vector; $A_j\ j = 1,2,\dots,p$ is an $m \times n_j$ coefficient matrix; $x_i\ and\ f_i(x_1, x_2, \dots, x_p)\ i = 1,2,\dots,p$ are the decision variable vector and the objective function of the decision maker at the i^{th} level, respectively.

2.2. MLP Problem Description.

Let the vector of decision variable $x = (x_1, x_2, \dots, x_n) \in R^n$ be partitioned among p decision makers and let $x^k = (x_1^k, x_2^k, \dots, x_{n_k}^k) \in R^{n_k}$ for $k = 1,2,\dots,p$ where $\sum_1^p n_k = n$. Let

$$Max\ \{f(x): (x^k \mid x^1, x^2, \dots, x^{k-1})\}$$

denote the maximization of a function $f(x)$ over a compact region $S \subseteq R$ by varying only $x^k \in R^{n_k}$ given fixed $(x^1, x^2, \dots, x^{k-1}) \in R^{n_1} \times R^{n_2} \times \dots \times R^{n_{k-1}}$. Note that x^{k+i} is a function of $(x^1, x^2, \dots, x^{k+i-1})$ for $i = 1,2,\dots,p-k$.

Definition 2.2.1. The set of $W_f(S)$ given by

$$W_f(S) = \{\hat{x} \in S: f(\hat{x}) = \max\{f(x): (x^k \mid x^1, x^2, \dots, x^{k-1})\}\}$$

is known as the **set of rational reactions** of f over S.

The Problem Description for MLP problems generally follows the following procedure. First define level one problem and solve it the with given feasible set (S)

$$(P^1)\begin{cases} \max\limits_{x_1} f_1(X) \\ \text{St.} \quad X \in S = S^1 \end{cases}$$

The feasible region, $S=S^1$, is defined as the level-one feasible region. The solution to (P^1) in R_1^n and for fixed parameter $x_2, x_3, \dots, x_p$, from a set,

$$S^2 = \{X \in S^1; f_1(X) = {}^{Min}_{x_1} f_1(X): x_2, x_3, \dots, x_p\},$$

called the level-two feasible region over which $f_2(X)$ is then maximized by varying x_2 for fixed $x_3, x_4, \dots, x_p$.

Thus the problem at level two is given by

$$(P^2)\begin{cases} \max\limits_{x_2}\{f_2(X) : (x^2 \mid x^1)\} \\ \text{St.} \quad X \in S^2 \end{cases}$$

In general, the level k^{th} feasible region is defined as

$$S^k = \left\{X \in S^{k-1}: f_{k-1}(X) = \max_{x_{k-1}}\{f_{k-1}(X): x_k, x_{k+1}, \dots, x_p\}\right\}$$

The problem at each level is

$$(P^k)\begin{cases} \max\limits_{x_k}\{f_k(X) : (x^k \mid x^1, x^2, \dots, x^{k-1})\} \\ \text{St.} \quad X \in S^k \end{cases}$$

Which is a function of $x_{k+1}, \dots, x_p$, and

$$(P^p)\begin{cases} \max\limits_{x_p}\{f_k(X) : (x^p \mid x^1, x^2, \dots, x^{p-1})\} \\ \text{St.} \quad X \in S^p \end{cases}$$

This establishes a collection of nested mathematical programming problems $\{P^1, P^2, \dots, P^p\}$.

> **Note that:** The optimal solution of problem (P^k) depends on the optimal solution of t^{th} level (t=k+1, k+2,..., p) and for given values of $x_1, \dots, x_{k-1}$ after the decisions of level 1 to level k-1 are made. That means: $x_k^*(x_{k+1}^*, x_{k+2}^*, \dots, x_p^*, x_1, \dots, x_{k-1})$.
>
> The objective at level *k*, $f_k(X)$ is defined over the decision space of all levels. Thus, the level *k* planner may have his objective function determined in part by variables controlled at other levels. However; by controlling x_k after decisions from levels *k+*1 to p have been made, level *k* may influence the policies at *k* -1 and hence all lower levels improve his own objective functions.

In order to make the above approaches is clear consider the three level programming problems.

Three level linear programming problems

It is part of MLPPs having three level DMs such as: top level, middle level and lower level.

Let $X = (x_1, x_2, x_3)$

$$\max_{x_1} f_1(X) = C_{11}x_1 + C_{12}x_2 + C_{13}x_3$$

where x_2, x_3 solve,

$$\max_{x_2} f_2(X) = C_{21}x_1 + C_{22}x_2 + C_{23}x_3$$

where x_3 solves,

$$\max_{x_3} f_3(X) = C_{31}x_1 + C_{32}x_2 + C_{33}x_3$$

Subject to:

$$S = \{X \in R^3 \colon A_1x_1 + A_2x_2 + A_3x_3 \le b, X \ge 0\}$$

The nested optimization problem of three levels can be written as: [3]

$$(P^1)\left\{\begin{array}{l} \max\ f_1(X) = C_{11}x_1 + C_{12}x_2 + C_{13}x_3 \\ \text{where } x_2, x_3 \text{ solve} \\ (P^2)\left\{\begin{array}{l} \max\ f_2(X) = C_{21}x_1 + C_{22}x_2 + C_{23}x_3 \\ \text{where } x_3 \text{ solves} \\ (P^3)\left\{\begin{array}{l} \max\ f_3(X) = C_{31}x_1 + C_{32}x_2 + C_{33}x_3 \\ S = \{X \in R^3 \colon A_1x_1 + A_2x_2 + A_3x_3 \le b, X \ge 0\} \end{array}\right. \end{array}\right. \end{array}\right.$$

We show the basic concepts of the three-level linear programming problem as follows:

A three-level linear programming problem for obtaining the Stackelberg solution is formulated as:

$$\max_{x_1} f_1(X) = C_{11}x_1 + C_{12}x_2 + C_{13}x_3$$

where x_2, x_3 solve,

$$\max_{x_2} f_2(X) = C_{21}x_1 + C_{22}x_2 + C_{23}x_3$$

where x_3 solves,

$$\max_{x_3} f_3(X) = C_{31}x_1 + C_{32}x_2 + C_{33}x_3$$

$$\text{Subject to } S = \{X \in R^3 : A_1x_1 + A_2x_2 + A_3x_3 \leq b, X \geq 0\}$$

i. The feasible set of the middle level for a fixed $x_1 \epsilon X_1$

$$S(x_1) = \{(x_2, x_3): A_2x_2 + A_3x_3 \leq b - A_1x_1; x_1, x_2 \geq 0\}$$

ii. The feasible set for lower level for fixed $(x_1, x_2)\epsilon X_1 \times X_2$

$$S(x_1, x_2) = \{x_3: A_3x_3 \leq b - A_1x_1 - A_2x_2; x_1, x_2 \geq 0\}$$

iii. The rational reaction set for middle level for given $x_1 \in X_1$

$$S^2 = \{(x_2, x_3)\epsilon X_2 \times X_3: (x_2, x_3)\epsilon argmax\{f_2(x_1, \hat{x}_2, \hat{x}_3): (\hat{x}_2, \hat{x}_3)\epsilon S(x_1)$$
$$and\ \hat{x}_3 \epsilon argmax\{f_2(x_1, \hat{x}_2, \hat{x}_3): \hat{x}_3 \epsilon S(x_1, \hat{x}_2)\}\}\}$$

iv. The rational reaction set for lower level for given $(x_1, x_2) \in X_1 \times X_2$

$$S^3 = \{x_3 \epsilon X_3: x_3 \epsilon argmax\{f_3(x_1, x_2, \hat{x}_3): \hat{x}_3 \epsilon S(x_1, x_2)\}\}$$

v. Inducible region (IR)

$$IR = \{(x_1, x_2, x_3): (x_1, x_2, x_3)\epsilon S\ and\ (x_2, x_3)\epsilon S^2\}$$

vi. Stackelberg solution

$$\{(x_1, x_2, x_3): (x_1, x_2, x_3)\epsilon argmax\{f_1(x_1, x_2, x_3): (x_1, x_2, x_3)\epsilon IR\}$$

<u>Note:</u>

Determining optimal solution to above problem is equivalent to solving

$$\max_{x_1} f_1(X) = C_{11}x_1 + C_{12}x_2 + C_{13}x_3$$

$$(x_1, x_2, x_3)\epsilon IR$$

3. Fuzzy linear programming

A crisp linear programming problem is a problem of maximizing or minimizing a crisp objective function subject to crisp constraints (crisp-linear inequalities and/or equations). However, in many practical situations, it may not be possible for the decision maker to specify the objective and/or the constraint in crisp manner rather he/she may have put them in "fuzzy sense". In this case, the type of the problem he/she put the fuzziness. That means there is no general or unique definition of fuzzy linear programming problems. The fuzziness may appear in a liner programming problem in several way such as the inequality may be fuzzy(P1-FLP), the objective function may be fuzzy(P2-FLP), or the parameters C, A, b may be fuzzy (P3-FLP)and so on.

We consider the general model of a linear programming

$$\begin{gathered} Max\ C^T x \\ subject\ to\ A_i x \leq b_i, (i = 1,2, \dots, m), \\ x \geq 0 \end{gathered} \tag{3.1}$$

where A_i is an n-vector, C is an n-column vector and $x \in R^n$.

To a standard linear programming problem (3.1), taking into account the imprecision, or fuzziness of a decision maker's judgment, Zimmermann consider the following linear programming problem with a fuzzy goal (objective function) and fuzzy constraints.

$$C^T x \lesssim z_0 \tag{3.1a}$$

$$A_i x \lesssim b_i, (i = 1,2, \dots, m) \tag{3.1b}$$

$$x \geqq 0,$$

where the symbol $\lesssim$ denotes a relaxed or fuzzy version of the ordinary inequality $\leqq$. From the decision maker's preference, the fuzzy goal (3.1a) and the fuzzy constraints (3.1b) mean that the objective function $C^T x$ should be essentially smaller than or equal to a certain level z_0, and that the values of the constraints *Ax* should be essentially smaller than or equal to *b*, respectively. Assuming that the fuzzy goal and the fuzzy constraints are equally important, he employed the following unified formulation.

$$Bx \lesssim b'$$

$$x \gtrsim 0$$

where $B = [\begin{smallmatrix} C \\ A_i \end{smallmatrix}]$ and $b' = [\begin{smallmatrix} z_0 \\ b_i \end{smallmatrix}]$

3.1. Solution techniques of solving fuzzy linear programming problems

The solution techniques for LP problems follow the following procedure. First fuzzify the objective functions by calculating the upper and the lower bounds of the optimal value. This is done by solving the following crisp LPP.

(LP (b))

$$z_1 = Max\ C^T x$$
$$subject\ to\ A_i x \leq b_i, (i = 1,2, \dots, m) \tag{3.1.1}$$
$$x \geq 0$$

And

(LP (b+p))

$$z_2 = Max\ C^T x$$
$$subject\ to\ A_i x \leq b_i + p_i, (i = 1,2, \dots, m) \tag{3.1.2}$$
$$x \geq 0$$

where p_i is the maximum tolerance of the i^{th} constraint i.e. $A_i x \in (b_i, b_i + p_i)$.

Assume that z_1 and z_2 are finite. We can now construct a continuous non decreasing linear membership function μ_G for the objective function by using z_1 and z_2 as follows.

$$\mu_G(x) = \begin{cases} 1 \ ; & if\ C^T x > z_2 \\ 1 - \frac{z_2 - C^T x}{z_2 - z_1} \ ; & if z_1 \leq C^T x \leq z_2, \\ 0 \ ; & if\ C^T x < z_1 \end{cases} \tag{3.2.3}$$

The membership functions of the constraints are

$$\mu_{C_i}(x) = \begin{cases} 1 \; ; & if\ if\ A_i x < b_i \\ 1 - \dfrac{A_i x - b_i}{z_2 - z_1} \; ; & if\ b_i \le A_i x \le b_i + p_i, \\ 0 \; ; & if\ A_i x > b_i + p_i \end{cases} \qquad (3.2.4)$$

Now using the above membership function, $\mu_{C_i}(x)$ and $\mu_G(x)$ and following **Bellman and Zadech** approach we have

$$\mu_D(x) = {}^{min}_{i}(\mu_G(x), \mu_{C_i}(x)) \qquad (3.2.5)$$

Where$\mu_D(.)$ is the membership function of the fuzzy decision set.

Then the optimal decision x^* is the solution of

$$\mu_D(x^*) = {}^{max}_{x \ge 0} \mu_D(x)$$

Consequently, the problem (p1-FLP) becomes the following optimization problem.

$$\max \ \alpha$$

$$subject\ to \qquad \mu_G(x) \le \alpha$$

$$\mu_{C_i}(x) \le \alpha, (i = 1,2, \dots, m) \qquad (3.2.6)$$

$$x \ge 0$$

$$\alpha \in [0,1]$$

Substituting $\mu_{C_i}(x)$ and $\mu_G(x)$ in the above problem, the above problem is equivalent t

$$\max \ \alpha$$

$$s.t \ \ C^T x \geq z_2 - (1-\alpha)(z_2 - z_1) \tag{3.2.7}$$

$$A_i x \leq b_i + (1-\alpha)p_i, (i = 1{:}\,m)$$

$$x \geq 0, \alpha \in [0,1]$$

Here solving the above linear programming problem gives us an optimum$\alpha^* \in [0,1]$. Then the solution of the problem (P1-FLP) is any $x \geq 0$ satisfying the problem constraint with

$\alpha = \alpha^*$.

3.2. Fuzzy Bi-level Linear Programming.

As shown in chapter 2, a two-level linear programming problem is formulated as:

$$\max_{x_1} f_1(x_1, x_2) = C_{11}x_1 + C_{12}x_2$$

$$where\ x_2\ solves$$

$$\max_{x_2} f_2(X) = C_{21}x_1 + C_{22}x_2$$

$$s.t \quad A_1x_1 + A_2x_2 \leq b,$$

$$x_1, x_2 \geq 0$$

Where $x_{i\,,}\, i = 1,2;$ is an n_i dimensional decision variable column vector, $C_{i1\,,}\, i = 1,2;$ is an n_1 dimensional constant column vector, $C_{i2\,,}\, i = 1,2.$ is an n_2 dimensional constant column vector, b is an m-dimensional constant column vector and $A_i, i = 1,2;$ is an $m \times n_i$ coefficient matrix.

We obtain optimal solution of each DM1 and DM2 calculated in isolation. If the individual optimal solution $x_i^0, i = 1,2;$ are the same then a satisfactory solution of the system has been attained. But this rarely happens due to conflicting objective functions of two DMs. The decision-making process then begins at the first level. Thus, the first -level

DM provides his preferred ranges for f_1 and decision vector x_1 to the second level DM. This information can be modeled by fuzzy set theory using membership functions [3].

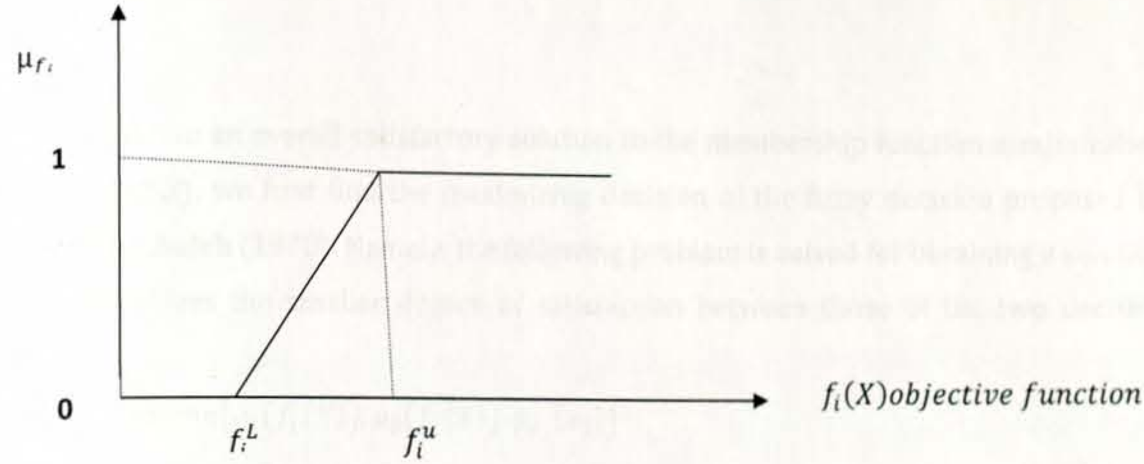
To build membership functions, goals and tolerance should be determined first. However, they could hardly be determined without meaningful supporting data. Using the individual best solutions, we find the values of all the objective functions at each best solution and construct a payoff matrix as[4]:

$$\begin{array}{c} \\ x_1^0 \\ x_2^0 \end{array} \begin{pmatrix} f_1(x) & f_2(x) \\ f_1(x_1^0) & f_2(x_2^0) \\ f_1(x_1^0) & f_2(x_2^0) \end{pmatrix}$$

The maximum value of each column $(f_i(x_i^0))$ gives **upper tolerance limit or aspired level of achievement** for the i^{th} objective function where $f_i^u = f_i(x_i^0) = \max_{x \in S} f_i(x_i^0)$ $, i = 1,2$.

The minimum value of each column gives **lower tolerance limit or lowest acceptable level of achievement** for the i^{th} objective function where $f_i^L = min f_i(x_i^0), t = 1,2$.

For the maximization- type objective function, the upper tolerance limit f_i^u, $t = 1,2$. are kept constant at their respective optimal values calculated in isolation but the lower tolerance limit f_i^L are changed. The idea being that $f_i(X) \to f_i^u$, then the fuzzy objective goals take the form $f_i(X) \gtrapprox f_i(X_i^u), t = 1,2$. and the fuzzy goal for the control vector x_i is obtained as $X_i \cong X_i^u$. Now, in the decision situation, it is assumed that the all DMs that are up to i^{th} a motivation to cooperate each other to make a balance of decision powers, and they agree to give a possible relaxation of their individual optimal decision. The i^{th} level DM must adjust his/her goal by assuming the lowest acceptable level of achievement f_i^L based on indefiniteness of the decentralized organization. Thus, all values of $f_i(X)$ with $f_i(X) \geq Z_i^u$ are absolutely acceptable to objective function $Z_i(X)$ satisfactory to the i^{th} level DM. All values of $f_i(X)$ with $f_i(X) \leq f_i^L$ are absolutely unacceptable to the objective function $f_i(X)$ for $i = 1,2$. Based on this interval of tolerance, we can establish the following linear membership functions for the defined fuzzy goals as fig 3.2.1.



$$\mu_{f_i}(f_i(X)) = \begin{cases} 1 \ ; & if\ f_i(X) \geq f_i^u, \\ \frac{f_i(X)-f_i^L}{f_i^u-f_i^L} \ ; & if\ f_i^L \leq f_i(X) \leq f_i^u, i = 1,2 \\ 0 \ ; & if\ f_i(X) \leq f_i^L \end{cases} \tag{3.2.2}$$

By identifying the membership functions $\mu_1(f_1(X))$ and $\mu_2(f_2(X))$ for the objective functions$f_1(X))$ and$f_2(X))$, the original two-level linear programming problem (3.2) can be interpreted as the membership function maximization problem defined by:

$$\begin{aligned} &\min_{x_1}\ \mu_1(f_1(X)) \\ &\min_{x_2}\ \mu_2(f_2(X)) \\ &s.t\quad A_1x_1 + A_2x_2 \leq b, \\ &\qquad x_1, x_2 \geq 0 \end{aligned} \tag{3.2.3}$$

Then the linear membership functions for decision vector x_1 can be formulated as:

$$\mu_{x_1}(x_1) = \begin{cases} \dfrac{x_1 - (x_1^0 - e_1^-)}{e_t^-} \ ; if\ x_1^0 - e_1^- \leq x_1 \leq x_1^0 \\ \dfrac{(x_1^o + e_1^+) - x_1}{e_1^+} \ ; if\ x_1^o \leq x_1 \leq (x_1^0 + e_1^+) \\ 0 \quad ;\ otherwise \end{cases} \tag{3.2.4}$$

where x_1^0 is the optimal solution of first level DM

e_1^- the negative tolerance value on x_1

e_1^+ the negative tolerance value on x_1

To derive an overall satisfactory solution to the membership function maximization problem (3.2.3), we first find the maximizing decision of the fuzzy decision proposed by **Bellman** and **Zadeh (1970)**. Namely, the following problem is solved for obtaining a solution which maximizes the smaller degree of satisfaction between those of the two decision makers:

$$
\begin{aligned}
&Max \min\{\mu_1(f_1(X)), \mu_2(f_2(X)), \mu_{x_1}(x_1)\} \\
s.t \quad & A_1x_1 + A_2x_2 \leq b, \qquad\qquad (3.2.5)\\
& x_1, x_2 \geq 0
\end{aligned}
$$

By introducing an auxiliary variable λ, this problem can be transformed into the following equivalent problem:

$$
\begin{aligned}
\max \quad & \lambda \\
s.t \quad & \mu_1(f_1(X)) \geq \lambda \\
& \mu_2(f_2(X)) \geq \lambda \\
& \mu_{x_1}(x_1) \geq \lambda \qquad\qquad (3.2.6)\\
& A_1x_1 + A_2x_2 \leq b, \\
& x_1, x_2 \geq 0
\end{aligned}
$$

If DM1 is satisfied with the optimal solution x^*, it follows that the optimal solution x^*, becomes a satisfactory solution; however, DM1 is not always satisfied with the solution x^*, it is quite natural to assume that DM1 specifies the minimal satisfactory level $\delta \in [0,1]$ for the membership function $\mu_1(f_1(X))$ subjectively. Consequently, if DM1 is not satisfied with the solution x^* to problem (3.2.5), the following problem is formulated:

$$
\begin{aligned}
max \quad & \mu_2(f_2(X)) \\
s.t \quad & \mu_1(f_1(X)) \geq \delta \\
& \mu_{x_1}(x_1) \geq \delta \qquad\qquad (3.2.7)\\
& A_1x_1 + A_2x_2 \leq b, \\
& x_1, x_2 \geq 0
\end{aligned}
$$

If an optimal solution to problem (3.2.7) exists, it follows that DM1 obtains a satisfactory solution having a satisfactory degree larger than or equal to the minimal satisfactory level specified by DM1's own self. However, the larger the minimal satisfactory level is assessed, the smaller DM2's satisfactory degree becomes. Consequently, a relative difference between the satisfactory degrees of DM1 and DM2 becomes larger and it is feared that overall satisfactory balance between both levels cannot be maintained.

To take into account the overall satisfactory balance between both levels, DM1 needs to compromise with DM2 on DM1's own minimal satisfactory level. To do so, a ratio of satisfactory degree between both DMs is defined as:

$$\Delta = \frac{\mu_2(f_2(x^*))}{\mu_1(f_1(x^*))} \tag{3.2.8}$$

which is defined by Lai [12], is useful

Let Δ^L and Δ^U denote the lower bound and the upper bound of Δ specified by DM1. If $\Delta > \Delta^U$, i.e. $\mu_2(f_2(x^*)) > \Delta^U \mu_1(f_1(x^*))$ then DM1 updates the minimal satisfactory level δ by increasing δ. Then DM1 obtains a larger satisfactory degree and DM2 accepts a smaller satisfactory degree. Conversely, if $\Delta < \Delta^L$, i.e. $\mu_2(f_2(x^*)) < \Delta^L \mu_1(f_1(x^*))$, then DM1 updates the minimal satisfactory level δ by decreasing δ, and DM1 accepts a smaller satisfactory degree and DM2 obtains a larger satisfactory degree.

At an iteration l , let $\mu_1\left(f_1(x^l)\right), \mu_2\left(f_2(x^l)\right)$, λ^l and $\Delta = \frac{\mu_2(f_2(x^l))}{\mu_1(f_1(x^l))}$ denote DM1's and DM2's satisfactory degrees, a satisfactory degree of both levels and the ratio of satisfactory degrees between both DMs, respectively, and let a corresponding solution be x^l. The iterated interactive process terminates if the following two conditions are satisfied and DM1 concludes the solution as a satisfactory solution.

Termination conditions of the interactive processes for two-level linear programming problems.

i. DM1's satisfactory degree is larger than or equal to the minimal satisfactory level δ specified by DM1, i.e.,$\mu_1\left(f_1(x^l)\right) \leq \delta$.
ii. The ratio Δ^l of satisfactory degrees is in the closed interval, the lower and the upper bounds of which are specified by DM1.

Condition (i) is DM1's required condition for solutions, and condition (ii) is provided in order to keep overall satisfactory balance between both levels. Unless the conditions are satisfied simultaneously, DM1 needs to update the minimal satisfactory level δ.

Procedure for updating the minimal satisfactory level δ

1. If condition (i) is not satisfied, then DM1 decreases the minimal satisfactory level by δ.
2. If the ratio Δ^l exceeds its upper bound, then DM1 increases the minimal satisfactory level δ. Conversely, if the ratio Δ^l is below its lower bound, then DM1 decreases the minimal satisfactory level δ.

3.2.1. Algorithm of interactive fuzzy programming

Step1: find the solution of the first level and second level independently with the same feasible set given.

Step2: Do these solutions coincide?

. If yes, an optimal solution is reached.

. If No, go to step 3

Step3: construct a payoff matrix, and then find upper tolerance limit f_t^U and lower tolerance limit f_t^L.

Step4: Build membership functions for maximization objective functions $\mu_{f_i}(f_i(X))$ and decision vector x_1 using equation (3.2.2) and (3.2.4) respectively.

Step5: $\boldsymbol{l} = \mathbf{1}$ solve the auxiliary problems (3.2.6). If DM1 is satisfied with the optimal solution, the solution becomes a satisfactory solution x^*. Otherwise, ask DM1 to specify the minimal satisfactory level δ together with the lower and the upper bounds $[\Delta_{min}, \Delta_{max}]$ of the ratio of satisfactory degrees Δ^l with the satisfactory degree λ^* of both decision makers and the related information about the solution in mind.

Step6: Solve problem (3.2.7), in which the satisfactory degree of DM2 is maximized under the condition that the satisfactory degree of DM1 is larger than or equal to the minimal satisfactory level δ, and then an optimal solution x^l to problem (3.2.7) is proposed to DM1 together with λ^l, $\mu_1\left(f_1(x^l)\right), \mu_2\left(f_2(x^l)\right)$ and Δ^l.

Step7: If the solution x^l satisfies the termination conditions and DM1 accepts it, then the procedure stops, and the solution x^l is determined to be a satisfactory solution.
Step8: Ask DM1 to revise the minimal satisfactory level δ in accordance with the procedure of updating minimal satisfactory level. Return to Step 7.

3.3. Fuzzy programming approach for multilevel linear problem

In this subsection, we extend fuzzy programming approach stated for two-level linear programming problems to that of multilevel linear programming problems.
A multi-level linear programming problem is formally represented as:

$$\max_{x_1} f_1(X) = \sum_{j=1}^{p} c_{1j} x_j$$

$$\text{where } x_2, x_3, \ldots, x_4 \text{ solve} \qquad (3.3.1)$$

$$\max_{x_2} f_2(X) = \sum_{j=1}^{p} c_{2j} x_j$$

$$\text{where } x_3, x_4, \ldots, x_4 \text{ solve}$$

....

$$\max_{x_p} f_p(X) = \sum_{j=1}^{p} c_{pj} x_p$$

$$s.t\ A_{i1}x_1 + A_{i2}x_2 + \cdots + A_{ip}x_p \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b_i, \qquad i = 1{:}m$$

$$x_1, x_2, \ldots, x_p \geq 0$$

where $x_i, i = 1,2,\ldots,p.$ is an n_i dimensional decision variable column vector, $C_{ij}, i = j = 1,2,\ldots,p.$ is an n_j dimensional constant column vector, b is an m-dimensional constant column vector and $A_i, i = 1,2,\ldots,p$, is an $m \times n_i$ coefficient matrix.

We obtain optimal solution of each DM calculated in isolation. If the individual optimal solution $x_P^o; P = 1,2,\ldots,p$ are the same then a satisfactory solution of the system has been attained. But this rarely happens due to conflicting objective functions of two or more DMs. The decision-making process then begins at the first level. Thus, the first-level DM provides his preferred ranges for f_1 and decision vector x_1 to the second level DM. This information can be modeled by fuzzy set theory using membership functions [3]. The membership functions that can be used to represent any objective function $f_P; P = 1,2,\dots,p$ may be linear, piecewise linear, exponential, logarithmic, hyperbolic, inverse hyperbolic, quadratic, etc. The tolerance interval that may be assigned to any decision vector $x_P^0; P = 1,2,\dots,p-1$ can be represented by linear membership function only.

Considering the top two levels ($p = 2$), the first-level DM must specify his/her objective function within the stipulated bounds to the second-level DM in order to direct supervise him to search for his/her solution in the correct direction. The upper bound (f_1^U) and lower bound (f_1^L) on objective function f_1 can be obtained from the (2×2) payoff- matrix formed at x_1^0 and x_2^0 where $f_1^L \le f_1 \le f_1^U$. We use linear membership function to model this information. Diagrammatically we illustrate the membership function of f_p; p=1, as (see Fig. 3.3.1)

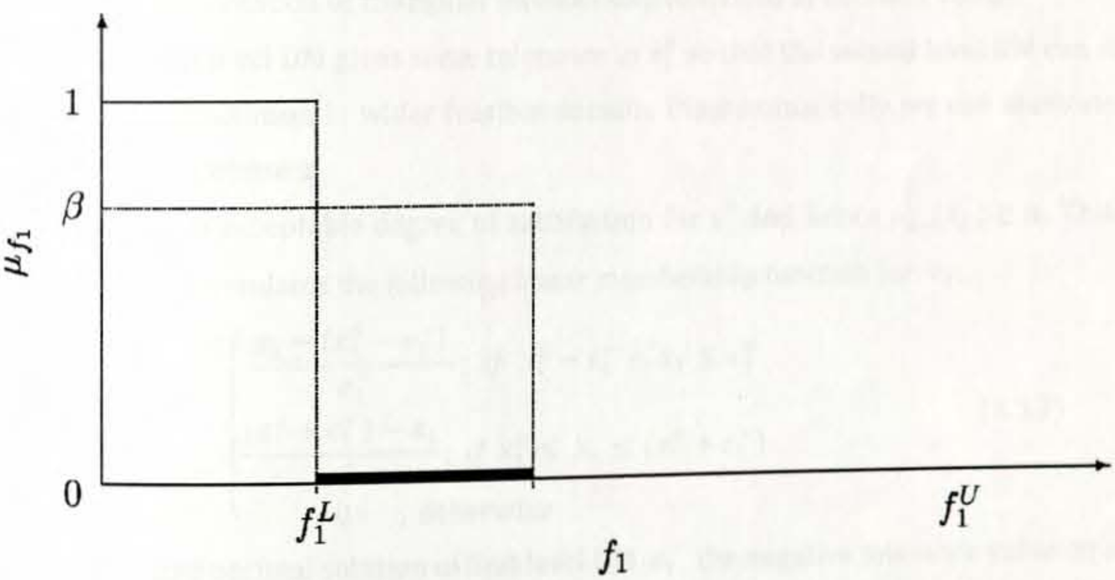


Fig 3.3.1 Representation of membership function of minimization objective function.

Where β is the minimum acceptable degree of satisfaction for objective function f_1 and hence $\mu_{f_1}(f_1(X)) \ge \beta$. Therefore, the linear membership function is

$$\mu_{f_1}(f_1(X)) = \begin{cases} 1; & \text{if } f_1(X) \geq f_1^U, \\ \frac{f_1(X)-f_1^L}{f_1^U-f_1^L}; & \text{if } f_1^L \leq f_1(X) \leq f_1^U \\ 0; & \text{if } f_1(X) \leq f_1^L \end{cases} \quad (3.3.2)$$

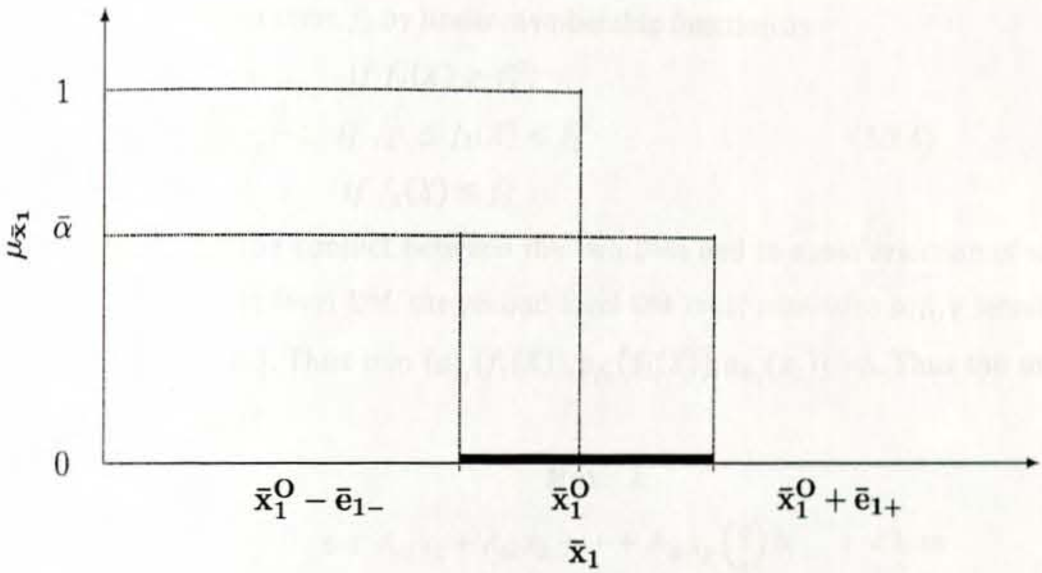


Fig 3.3.2.Representation of triangular membership functions of decision vector.

The first level DM gives some tolerance to x_1^0 so that the second level DM can search for his optimal solution in wider feasible domain. Diagrammatically we can illustrate it as (see Fig. 3.3.2), where α
is the minimum acceptable degree of satisfaction for x_1^0 and hence $\mu_{x_1}(x_1) \geq \alpha$. Thus, the first-level DM formulates the following linear membership function for x_1:

$$\mu_{x_1}(x_1) = \begin{cases} \dfrac{x_1 - (x_1^0 - e_1^-)}{e_1^-}; & \text{if } x_1^0 - e_1^- \leq x_1 \leq x_1^0 \\ \dfrac{(x_1^o + e_1^+) - x_1}{e_1^+}; & \text{if } x_1^o \leq x_1 \leq (x_1^o + e_1^+) \\ 0; & \text{otherwise} \end{cases} \quad (3.3.3)$$

Where x_1^0 is the optimal solution of first level DM, e_1^- the negative tolerance value on $x_1^0$and e_1^+ the positive tolerance value on x_1^0. This is a triangular fuzzy number [5]. $e_1^-$and e_1^+ are not necessarily the same values. x_1^0 generally lies between $(x_1^0 - e_1^-)$ and $(x_1^0 + e_1^+)$. But the DM may desire to shift the range of x_1^0 altogether which need not be aroundx_1^0. The desired range can be shifted to the right or left of x_1^0 depending on the discretion of the DM.

Let γ be the minimum acceptable degree of satisfaction of second level DM. So $\mu_{f_1}(f_1(X)) \geq \gamma$. The bounds f_2^U and f_2^L on objective function f_2 are also found from the same payoff matrix. To rate the satisfaction of his/her solution, the second-level DM also represents his/her objective function f_2 by linear membership function as

$$\mu_{f_2}(f_2(X)) = \begin{cases} 1 \ ; & if\ f_2(X) \geq f_2^U, \\ \frac{f_2(X)-f_2^L}{f_2^U-f_2^L} \ ; & if\ f_2^L \leq f_2(X) \leq f_2^U \\ 0 \ ; & if\ f_2(X) \leq f_2^L \end{cases} \tag{3.3.4}$$

To resolve the conflict between the two DMs and to avoid rejection of satisfactory solution by the first-level DM, the second-level DM must maximize α, β, γ simultaneously. Let λ=min $\{ \alpha, \beta, \gamma \}$. Thus min $\{\mu_{f_1}(f_1(X)), \mu_{f_2}(f_2(X)), \mu_{x_1}(x_1)\}$ =λ. Thus the second-level auxiliary problem is

$$\begin{gathered} Max \ \ \lambda \\ s.t\ A_{i1}x_1 + A_{i2}x_2 + \cdots + A_{ip}x_p \left(\begin{smallmatrix}\leq\\=\\\geq\end{smallmatrix}\right) b_i\,, \quad i = 1{:}m \\ \mu_{f_1}(f_1(x)) \leq \lambda, \mu_{f_2}(f_2(x)) \leq \lambda, \mu_{x_1}(x_1) \leq \lambda \qquad (3.3.5) \\ x_1, x_2, \dots, x_p \geq 0 \\ \lambda \in [0,1] \end{gathered}$$

If the center is satisfied with this solution then the third level is also included. If not, then the center modifies the tolerance values or may even change the membership functions and the second-level DM solves a new auxiliary problem again. The process continues until the satisfactory solution is attained for top two levels after which the third level is included. Again both the higher level DMs pass their preferred values of their decision variables and objective functions separately to the third-level DM. For p=3 and above, we follow the procedure outlined in the following paragraphs.

For the minimization-type objective function, the lower bounds f_p^L , $p = 1,2, \dots, P$ are kept constant at their respective optimal values calculated in isolation but the upper bounds f_p^U are changed. The idea being that $f_p(x) \to f_p^L$ for minimizing the objective functions. The values of f_p^U are found at the satisfactory solution obtained at the $(p-1)$th level. Let λ denote the minimum of all the minimum acceptable levels of satisfaction. Then for P objective functions, min $\{\mu_{f_1}(f_1(X)), \mu_{f_2}(f_2(X)), \dots, \mu_{f_p}(f_p(x))\}$ =λ. Thus each membership function should individually be $\geq \lambda$. The auxiliary problem is solved by the P th-level DM. Thus the auxiliary problem for the Pth-level DM is

$$Max \quad \lambda$$

$$s.t \ A_{i1}x_1 + A_{i2}x_2 + \cdots + A_{ip}x_p \left(\begin{smallmatrix}\leq \\ = \\ \geq\end{smallmatrix}\right) b_i, \quad i = 1{:}m$$

$$\mu_{f_1}(f_1(x)) \leq \lambda I, \mu_{f_2}(f_2(x)) \leq \lambda I, \ldots, \mu_{f_1}\left(f_p(x)\right) \leq \lambda I \tag{3.3.6}$$

$$\mu_{x_1}(x_1) \leq \lambda I, \mu_{x_2}(x_2) \leq \lambda I, \ldots, \mu_{f_{p-1}}(x_{p-1}) \leq \lambda I$$

$$x_1, x_2, \ldots, x_p \geq 0$$

$$\lambda \in [0,1]$$

Solving problem (3.3.6), a satisfactory solution for P-levels is found. This satisfactory solution is finalized on approval by all the higher level DMs, after which the succeeding lower level is included into the system (i.e. P is incremented by 1) and the process repeated. The process continues until the last level is included into the system.

Example1 solve (Linear BLPP)

$$\underset{x_1,x_2}{Max} f_1(X) = 5x_1 + 6x_2 + 4x_3 + 2x_4$$

$$where \ x_{3,}x_4 \ solves$$

$$\underset{x_3,x_4}{Max} f_2(X) = 8x_1 + 9x_2 + 2x_3 + 4x_4$$

$$3x_1 + 2x_2 + x_3 + 3x_4 \leq 40$$

$$3x_1 + 2x_2 + x_3 + 2x_4 \leq 30$$

$$2x_1 + 4x_2 + x_3 + 2x_4 \leq 35$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Solution

Step1: Fined the solution of the top-level and lower level independently with the same feasible set. i.e.

Max $f_1(X) = 5x_1 + 6x_2 + 4x_3 + 2x_4$

s.t $3x_1 + 2x_2 + x_3 + 3x_4 \leq 40$

$3x_1 + 2x_2 + x_3 + 2x_4 \leq 30$

$2x_1 + 4x_2 + x_3 + 2x_4 \leq 35$

$$x_1, x_2, x_3, x_4 \geq 0$$

and

Max $f_2(X) = 8x_1 + 9x_2 + 2x_3 + 4x_4$

s.t $3x_1 + 2x_2 + x_3 + 3x_4 \leq 40$

$$3x_1 + 2x_2 + x_3 + 2x_4 \leq 30$$

$$2x_1 + 4x_2 + x_3 + 2x_4 \leq 35$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Then we find the optimal solution

$f_1 = 125$ at $x_1^0 = (5, 0, 25, 0)$

$f_2 = 118.125$ at $x_2^0 = (11.25, 3.125, 0, 0)$

But this is not a satisfactory solution (since $x_1^0 \neq x_2^0$)

Step2: construct the pay-off matrix and we need to find the upper and lower tolerance limit

$$\begin{pmatrix} & f_1(x_1^0) & f_2(x_2^0) \\ x_1^0 & 125 & 90 \\ x_2^0 & 75 & 118.125 \end{pmatrix}$$

Then $f_1^U = 125$,$f_2^U = 118.125$, $f_1^L = 75$ and $f_2^L = 90$

Step3. Build membership functions for maximization objective functions

$$\mu_{f_1}(f_1(X)) = \begin{cases} 1 & if\ f_1(X) \geq 125, \\ \dfrac{f_1(X) - 75}{125 - 75} & if 75 \leq f_1(X) \leq 125, \\ 0 & if\ f_1(X) \leq 75 \end{cases}$$

$$\mu_{f_2}(f_2(X)) = \begin{cases} 1 & if\ f_2(X) \geq 118.125, \\ \dfrac{f_1(X) - 90}{118.125 - 90} & if\ 90 \leq f_2(X) \leq 119.125, \\ 0 & if\ f_2(X) \leq 90 \end{cases}$$

Let the upper level DM specify x_1 to be around 5 with 2.5 (negative) and 2.5 (positive) tolerance and x_2 to be around 0 with 0 (negative) and 3 (positive) tolerance.

$$\mu_{x_1}(x_1) = \begin{cases} \frac{x_1 - (5-2.5)}{2.5} & if\ 2.5 \le x_1 \le 5 \\ \frac{(5+2.5) - x_1}{2.5} & if\ 5 \le x_1 \le 7.5 \\ 0 & otherwise \end{cases}$$

$$\mu_{x_2}(x_2) = \begin{cases} x_2 & if\ x_2 \le 3 \\ \frac{3 - x_2}{3} & if\ 0 \le x_2 \le 3 \\ 0 & otherwise \end{cases}$$

Step4: solve the auxiliary problems

$$\begin{aligned} &Max\ \ \lambda \\ &s.t\ \mu_{f_1}(f_1(x)) \le \lambda \\ &\mu_{f_2}(f_1(x)) \le \lambda \\ &\mu_{x_1}(x_1) \le \lambda \\ &\mu_{x_2}(x_2) \le \lambda \\ &3x_1 + 2x_2 + x_3 + 3x_4 \le 40 \\ &3x_1 + 2x_2 + x_3 + 2x_4 \le 30 \\ &2x_1 + 4x_2 + x_3 + 2x_4 \le 35 \\ &x_1, x_2, x_3, x_4 \ge 0, \lambda \in [0,1]. \end{aligned}$$

The result of the first iteration including an optimal solution to problem is

x_1=6.41, x_2=1.95, x_3=10.52, x_4=1.42 and λ=0.316

Suppose that DM1 is not satisfied with the solution obtained in Iteration 1, and then he specifies the minimal satisfactory level at $\delta = 0.4$ and the bounds of the ratio at the interval $[\Delta_{min}, \Delta_{max}] = [0.3, 0.4]$, taking account of the result of the first iteration. Then, the problem with the minimal satisfactory level is written as

$$\begin{aligned} &Max\ \ \mu_{f_2}(f_2(x)) \\ &s.t\ \mu_{f_1}(f_1(x)) \le 0.4 \\ &x \in S \end{aligned}$$

Appling simplex algorithm, solution is x_1=6.71, x_2=2.05, x_3=10.52, x_4=1.42 and λ=0.316.Therefore, this solution satisfies the termination conditions, and it becomes a satisfactory solution for both decision makers if DM1 accepts the solution.

Example2. Solve

$$\underset{x_1}{Max} f_1(X) = x_1 + x_2 + 2x_3 + x_4$$

$$where\ x_2, x_3, x_4\ solves$$

$$\underset{x_2}{Max} f_2(X) = -x_1 + 3x_2 - 2x_3 - 4x_4$$

$$where\ x_3, x_4\ solves$$

$$\underset{x_3}{Max} f_3(X) = -x_1 - x_2 + 3x_3 - x_4 \qquad (1)$$

$$where\ x_4\ solves$$

$$\underset{x_4}{Max} f_4(X) = -x_1 - x_2 - x_3 + 3x_4$$

$$3x_1 + 3x_2 \leq 30$$

$$2x_1 + x_2 \leq 20$$

$$x_2 \leq 10$$

$$x_1 + 2x_2 + 2x_3 + x_4 \leq 40$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Solution

STEP1. Find the solution from top to bottom-levels independently with same feasible set given.

$f_1 = 35$ at $x_1^0 = (10,0,10,5)$

$f_2 = 30$ at $x_2^0 = (0,10,0,0)$

$f_3 = 30$ at $x_3^0 = (0,0,10,0)$,

$f_4 = 30$ at $x_4^0 = (0,0,0,10)$

STEP2. Define the fuzzy goal, construct payoff matrix and find upper and lower tolerance limit.

- Objective function as $f_1 \gtrsim 35, f_2 \gtrsim 30, f_3 \gtrsim 30, f_4 \gtrsim 30$
- decision variable as $x_1 \cong 5$,

$$\text{payoff matrix} = \begin{bmatrix} 35 & -5 & 15 & -5 \\ 10 & 30 & -10 & -10 \\ 20 & -10 & 30 & -10 \\ 10 & -10 & -10 & 30 \end{bmatrix}$$

Upper tolerance limit are: $f_1^U = 35,\ f_2^U = 30, f_3^U = 30\ f_4^U = 30$

Lower tolerance limit are: $f_1^L = 10,\ f_2^L = -10, f_3^L = -10\ f_4^L = -10$

It may be noted here that $f_2^L = -10\ \ f_3^L = -10,\ \ f_4^L = -10$ which are quite unacceptable to the second level DMs since negative objective values are not preferred in the decision making situation. Hence $f_2^L = 0\ \ f_3^L = 0,\ \ f_4^L = 0$ are taken into account.

STEP3. Built membership function for:

- **Objective function as**

$$\mu_{f_1}(f_1(x)) = \begin{cases} 1 & if\ f_1(x) \geq 35, \\ \dfrac{f_1(x) - 10}{25} & if\ 10 \leq f_1(x) \leq 35, \\ 0 & if\ f_1(x) \leq 10 \end{cases}$$

$$\mu_{f_t}(f_t(x)) = \begin{cases} 1 & if\ f_t(\bar{X}) \geq 30, \\ \dfrac{f_t(x)}{30} & if\ 0 \leq f_t(x) \leq 30, for\ t = 2,3,4. \\ 0 & if\ f_t(x) \leq 0 \end{cases}$$

- **Decision variable function as:**

  The top-level DM formulates the linear membership function for decision variable x_1 based on the tolerance limit of x_1 which decide by top-level. For instance, the first level DM decides $x_1 = 5$ with -0.5 (negative) and 0.5 (positive) tolerance values [8].

$$\mu_{x_1}(x_1) = \begin{cases} \frac{x_1 - 4.5}{-0.5} \ if\ 4.5 \le x_1 \le 5 \\ \frac{5.5 - x_1}{0.5} \ if\ 5 \le x_1 \le 5.5 \\ 0 \quad \text{otherwise} \end{cases}$$

Step4: First solve the auxiliary problems for p=2

$$\begin{aligned} & Max \quad \lambda \\ & s.t\ \mu_{f_1}(f_1(x)) \le \lambda \\ & \mu_{f_2}(f_2(x)) \le \lambda . \\ & \mu_{x_1}(x_1) \le \lambda \\ & 3x_1 + 3x_2 \le 30 \\ & 2x_1 + x_2 \le 20 \\ & x_2 \le 10 \\ & x_1 + 2x_2 + 2x_3 + x_4 \le 40 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned} \tag{2a}$$

that is

$$\begin{aligned} & Max \quad \lambda \\ & s.t\ \frac{f_1(x) - 10}{25} \ge \lambda \\ & \frac{f_2(x)}{30} \ge \lambda \\ & \frac{5.5 - x_1}{0.5} \ge \lambda \\ & 3x_1 + 3x_2 \le 30 \\ & 2x_1 + x_2 \le 20 \\ & x_2 \le 10 \\ & x_1 + 2x_2 + 2x_3 + x_4 \le 40 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned} \tag{2b}$$

The result of the first iteration including an optimal solution to problem (2)is

$\lambda = 0.71;\ x_1^1 = 0.45, x_2^1 = 7.95, x_3^1 = 7.445, x_4^1 = 7.5\ ;\ f_1^1 = 30.76, f_2^1 = 7.665;$
$\mu_{f_1}(f_1(x)) = 0.8,\ \mu_{f_2}(f_2(x)) = 0.25$ and Δ^1=0.0321.

Suppose that DM1 is not satisfied with the solution obtained in Iteration 1, and then he specifies the minimal satisfactory level at $\delta = 0.8$ and the bounds of the ratio at the interval $[\Delta_{min}, \Delta_{max}] = [0.8,\ 0.9]$, taking account of the result of the first iteration. Then, the problem with the minimal satisfactory level is written as

$$\begin{aligned} &Max \quad \mu_{f_2}(f_2(x)) \\ &s.t\, \mu_{f_1}(f_1(x)) \leq 0.8 \\ &\frac{5.5 - x_1}{0.5} \geq \lambda \\ &x \in S \end{aligned} \tag{3}$$

The result of the second iteration including an optimal solution to problem (3) is
$\lambda = 0.079\ ; x_1^2 = 0.0, x_2^2 = 7.857, x_3^2 = 7.445,\ x_4^2 = 9.286\ ;$
$f_1^2 = 32.143, f_2^2 = 8.571;$
$\mu_{f_1}(f_1(x)) = 0.886,\ \mu_{f_2}(f_2(x)) = 0.486$ and Δ^2=1.861344.

At the second iteration, the satisfactory degree $\mu_{f_1}(f_1(x)) = 0.886$ of DM1 becomes less than or equal to the minimal satisfactory level δ = 0.800, but the ratio Δ = 1.861344 of satisfactory degrees is not in the valid interval [0.8,0.9] of the ratio. Therefore, this solution does not satisfy the second condition of termination of the interactive process. Suppose that DM1 updates the minimal satisfactory level at δ = 0.9.

$$\begin{aligned} &Max \quad \mu_{f_2}(f_2(x)) \\ &s.t\, \mu_{f_1}(f_1(x)) \leq 0.9 \\ &x \in S \end{aligned} \tag{4}$$

Then, the problem with the revised minimal satisfactory level (4) is solved, and the result of the third iteration is

$\lambda = 0.079\ ; x_1^3 = 0.0, x_2^3 = 7.857, x_3^3 = 7.445, x_4^3 = 9.286\ ;\ f_1^3 = 32.143, f_2^3 = 8.514;$
$\mu_{f_1}(f_1(x)) = 0.8,\ \mu_{f_2}(f_2(x)) = 0.286$ and Δ^2=1.861344.

At the third iteration, the satisfactory degree $\mu_{f_1}(f_1(x)) = 0.800$ of DM1 becomes equal to the minimal satisfactory level δ= 0.85, and the ratio $\Delta = 0.8492$ of satisfactory degrees is in the valid interval [0.8, 0.9] of the ratio. Therefore, this solution satisfies the termination conditions of the interactive process, and it becomes a satisfactory solution for both decision makers if DM1 accepts the solution.

Next solve the auxiliary problems for p=3

Let the first-level DM decide x_1=0 with 0 (negative) and 2 (positive) tolerances, x_2=7.857 with 6 (negative) and 8 (positive) tolerances.

$$\begin{aligned} & Max \ \ \lambda \\ & s.t\, \mu_{f_1}(f_1(x)) \leq \lambda \\ & \quad \mu_{f_t}(f_t(x)) \leq \lambda\,, t = 2,3 \\ & \quad \mu_{x_1}(x_1) \leq \lambda \\ & \quad \mu_{x_2}(x_2) \leq \lambda \\ & \quad x \in S \end{aligned} \tag{5}$$

The result of the last iteration including an optimal solution to problem (5) is

$$\lambda = 1.083, x_1 = 0.857, x_2 = 1.857, x_3 = 0, x_4 = 0.714$$

$$f_1 = 13, f_2 = 4.71, f_3 = 4.28$$

$$\mu_{f_1} = \frac{13-5}{11.25} = 0.711, \mu_{f_2} = \frac{4.71-1}{4} = 0.928, \mu_{f_3} = \frac{4.28-2}{3} = 0.762$$

Now, solve the auxiliary problems for p=4

Let the first-level DM decide x_1=0.857 with 0 (negative) and 1 (positive) tolerances, x_2=1.857 with 1.00 (negative) and 2.02 (positive) tolerances and x_3=0 with 0 (negative) and 1.02 (positive) tolerances.

$$
\begin{aligned}
& Max \;\; \lambda \\
& s.t\, \mu_{f_1}(f_1(x)) \le \lambda \\
& \mu_{f_t}(f_t(x)) \le \lambda, t = 2, 3, 4 \\
& \mu_{x_1}(x_1) \le \lambda \\
& \mu_{x_2}(x_2) \le \lambda \\
& \mu_{x_3}(x_3) \le \lambda \\
& x \in S
\end{aligned} \tag{6}
$$

The result of the first iteration including an optimal solution to problem (6) is

$\lambda = 0.93;\ x_1^1 = 0., x_2^1 = 7.875, x_3^1 = 9.286, x_4^1 = 5.714$;

$f_1^1 = 32.143, f_2^1 = 8.571, f_3^1 = 14.284, f_4^1 = 0$;

$\mu_{f_1}(f_1(x)) = 0.71,\ \mu_{f_2}(f_2(x)) = 0.93$, $\mu_{f_3}(f_2(x)) = 0.76,\ \mu_{f_4}(f_2(x)) = 0.74$ and $\Delta_1^1 = 1.3098, \Delta_2^1 = 1.0271, \Delta_3^1 = 0.9737$.

Suppose that DM3 is not satisfied with the solution obtained in Iteration 1, and then he specifies the minimal satisfactory level at $\delta = 0.7$ and the bounds of the ratio at the interval $[\Delta_{min}, \Delta_{max}] = [0.7,\ 0.8]$, taking account of the result of the first iteration. Then, the problem with the minimal satisfactory level is written as

$$
\begin{aligned}
& max \;\; \mu_{f_2}(f_2(x)) \\
& s.t\, \mu_{f_1}(f_1(x)) \le \delta \\
& \mu_{f_t}(f_t(x)) \le \delta, t = 3,4 \\
& \mu_{x_1}(x_1) \le \lambda \\
& \mu_{x_2}(x_2) \le \lambda \\
& \mu_{x_3}(x_3) \le \lambda \\
& x \in S
\end{aligned} \tag{7}
$$

The result of the second iteration including an optimal solution to problem (7) is

$\lambda = 0.7;\ x_1^1 = 1.07, x_2^1 = 7.5, x_3^1 = 7.5, x_4^1 = 7.5$;

$f_1^1 = 31.07, f_2^1 = 6.43, f_3^1 = 6.43, f_4^1 = 6.430$;

$\mu_{f_1}(f_1(x)) = 0.61,\ \mu_{f_2}(f_2(x)) = 0.08$, $\mu_{f_3}(f_2(x)) = 0.01,\ \mu_{f_4}(f_2(x)) = 0.7$

and $\Delta_1^1 = 0.135, \Delta_2^1 = 0.125, \Delta_3^1 = 0.7$.

At the third iteration, the satisfactory degree $\mu_{f_4}(f_4(x)) = 0.700$ of DM4 becomes equal to the minimal satisfactory level $\delta = 0.7$, and the ratio $\Delta = 0.7$ of satisfactory degrees is in the valid interval [0.7, 0.8] of the ratio. Therefore, this solution satisfies the termination conditions of the interactive process, and it becomes a satisfactory solution for both decision makers if DM4 accepts the solution.

Conclusion

The FMP approach is simple to implement, interactive and applicable to MLPP. The satisfactory solution obtained is realistic. A satisfactory solution obtained at P=2 does not guarantee the same for P=3. We can take any membership function other than linear. The results will hold good, however, the problem will become a nonlinear programming problem. We observe that even though the decision-making process is from higher to lower level, the last level becomes most important. This is because the decision vector under the control of the lowest level DM is not given any tolerance limits. Hence this decision vector either remains unchanged or closest to its value obtained in isolation. But at higher level, the decision vectors are given some tolerance and hence they are free to move within the tolerance limits. The tolerance levels can also be considered as variables and if the DMs cooperate then the entire system as a whole can be optimized. We can easily apply the same approach to nonlinear MLPPs.

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