NUMERICAL SOLUTION OF BOUNDARY INTEGRAL EQUATIONS



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Abstract

This paper considers the numerical solution of boundary integral equations of the second kind, for Laplace's equation $\Delta u = 0$ *on connected regions* Ω *in* R^3 *with boundary* $\partial\Omega$*. The boundary* $\partial\Omega$ *is allowed to be smooth or piecewise smooth; and we let* $\{\triangle_k | 1 \leq k \leq n\}$ *be a triangulation of* $\partial\Omega$*. The paper will focus on the derivation of the boundary integral equation (BIE) for the Laplace equation, and numerical results are presented for some examples using constant elements.*

Key words: Boundary integral equations, Laplace equation, Numerical solutions using Matlab.

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Notations and Abrivations

Ω	Domain
$\partial\Omega$	Boundary of the domain
$\bar{\Omega}$	Closurity of the domain
DE	Differential equation
PDE	Partial Different Equation
BVP	Boundary Value Problem
BIE	Boundary Integral Equation
BEM	Boundary Element Method
Δ	Laplacian oprator
∇	Gradient operator
$\sum$	Summation
$\Gamma(P)$	interior solid angle at $P \in \partial\Omega$
$\mathcal{K}$	Double layer operator
$\mathcal{S}$	Single layer operator
κ	Kernel function
R	Real number
$\mathcal{F}$	Fourier transform
n_σ	The unit outward normal at the boundary point σ

Introduction

Boundary integral equations (BIEs) are reformulations of boundary value problems (BVPs) for partial differential equations (PDEs). They have been presented in the mathematics literature for almost two centuries, and engineers have made much use of them for solving a wide variety of real-world problems.

In spite of the widespread use of BIE, much of their numerical analysis has not been well-understood until much more recently, say from 1975 onwards. One of the major contributions to the subject was the recognition that most BIE can be regarded as strongly elliptic pseudo-differential operator equations on suitably chosen Sobolev spaces, and further, that boundary element methods can then be applied to their solution.

The boundary element method (BEM) has become an efficient and popular alternative, because of its ability, at least for some problems with constant coefficients, of reducing a boundary-value problem (BVP) for a linear partial differential equation (PDE) defined in a domain to an integral equation defined on the boundary, leading to a simplified discretisation process with boundary elements only. The main requirement for the reduction of the PDE to a boundary integral equation (BIE) is that a fundamental solution to the PDE must be available.

In this work, we concentrate on the study of boundary integral equations of the second kind for solving Laplace's equation ($\Delta u = 0$) on connected regions Ω in R^3. The boundary element method (BEM) is the numerical method being analyzed. BEM is an important computational tool used by researchers in many fields in the physical and engineering sciences [9].

The BEM is one of the methods for the numerical solution of boundary integral equations. It presents many advantages, first of all, only the boundary of the domain needs to be discretised. Especially in two dimensions, where the boundary is just a curve, this allows very simple data input and storage methods. Also, the smaller computer time due to a lesser number of nodes and elements. Moreover, high accuracy and the stability of numerical computations because of the utilization of fundamental solutions.

A major consideration in the error analysis of numerical methods for these boundary integral equations is whether the boundary of Ω, call it $\partial\Omega$ is smooth or piecewise smooth.If $\partial\Omega$ is smooth, then the associated integral operator is compact and there is a wealth of results available for the error analysis. But if $\partial\Omega$ is only piecewise smooth, then the integral operator is not compact; and moreover, the operator can be viewed as involving a Dirac delta function in its definition, see[8]. In addition, most numerical methods require approximations of a piecewise polynomial nature over triangulations of the surface. These numerical methods lead to the need to solve very large linear systems, and until recently most computers had great difficulty in handling such problems.

The practical aspects of setting up and solving such linear systems are more onerous for BIEs in three dimensions, and this means that the numerical analysis problems of concern are often of a different nature than for planar BIEs. In addition, there are some quite difficult and deep problems in understanding the behavior of BIEs on piecewise smooth surfaces, and it is important to have this understanding both to prove stability of numerical schemes and to know how to properly grade the mesh in the vicinity of edges and corners.

In the limited space of this project we can only begin to lay a foundation for the study of numerical solution of BIEs on R^3, by boundary element method.

Chapter 1

Basic Definitions and Mathematical tools

1.1 Definitions

Definition 1.1.1. *An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a* ***differential equation*** *(DE), (Zill - A First Course in Differential Equations).*

Definition 1.1.2. *A differential equation involving partial derivatives with respect to more than one independent variables is a* ***partial differential equation*** *(PDE).*

Definition 1.1.3. *Let we define $u : R^3 \to R$. Then the Laplace equation of u which denoted by (Δu) is given by*

$$\Delta u(x) = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} u(x) = 0 \quad for \quad x \in \Omega \subset R^3 \tag{1.1}$$

Definition 1.1.4. *A function $f : A \to B$ is one-to-one if for each $b \in B$ there is at most one $a \in A$ with $f(a) = b$. It is onto if for each $b \in B$ there is at least one $a \in A$ with $f(a) = b$. It is a one-to-one correspondence or bijection if it is both one-to-one and onto.*

Definition 1.1.5. *Assume that $u \in L^1(R^n)$. The Fourier transform of u is defined by*

$$\mathcal{F}u(\xi) = (2\pi)^{\frac{-n}{2}} \int_{R^n} u(x) e^{-ix\xi} dx$$

And the inverse Fourier transform of u is

$$\mathcal{F}^{-1}u(x) = (2\pi)^{\frac{-n}{2}} \int_{R^n} u(\xi) e^{ix\xi} d\xi$$

where, $x\xi = \sum_{i=1}^{n} x_i \xi_i$.

Definition 1.1.6. *Let $p(x, D_x) \in R^n$ is an operator, whose value on the function $u(x)$ is the function of x, such that;*

$$p(x, D_x)u(x) = \frac{1}{(2\pi)^n} \int_{R^n} e^{ix\cdot\xi} p(x, \xi)\hat{u}(\xi)d\xi$$

is called a ***pseudo-differential operator****, where $\hat{u}(\xi) = \mathcal{F}u(x)$ is the Fourier transform of u and the symbol $p(x, \xi)$ in the integrand belongs to a certain symbol class.*

Definition 1.1.7. ***(Norms)****:- A norm on a vector space X is a mapping $\|\cdot\| : X \to [0, \infty)$ with the following properties:*

(i) $\|x\| \geq 0, \quad \forall x \in X$ *and* $\|x\| = 0 \Leftrightarrow x = 0$

(ii) $\|\lambda x\| = |\lambda|\|x\|, \quad \forall x \in X$ *and* $\lambda \in \mathbb{K}$ *for* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$

(iii) $\|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in X$

The pair $(X, \|\cdot\|)$ is called a ***normed space****.*

Definition 1.1.8. *A function (operator) $A : U \subset X \to Y$ is called continuous at a point $x \in U$, if $\lim_{n\to\infty} Ax_n = Ax \quad \forall\{x_n\} \in U$, with $\lim_{n\to\infty} x_n = x$.*
An equivalent definition is: a function $A : U \subset X \to Y$ is continuous at a point $x \in U$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\|Ax - Ay\| < \epsilon$, $\forall y \in U$, with $\|x - y\| < \delta$.
Function A is called uniformly continuous, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|Ax - Ay\| < \epsilon, \quad \forall x, y \in U$, with $\|x - y\| < \delta$.

Remark 1.1.1. *Every norm $\|\cdot\| : X \to [0, \infty)$ is continuous since we have from (iii) the reverse triangle inequality*
$|\|x\| - \|y\|| \leq \|x - y\| \quad \forall x, y \in X,$ *is hold.*

1.2 Open and closed sets

For each element x of a normed space X and each positive number r, the set $B(x; r) := \{y \in X : \|y - x\| < r\}$ is called the open ball of radius r with center x, and the set $B[x; r] = \{y \in X : \|y - x\| \leq r\}$ is called a closed ball.

Definition 1.2.1. *A subset U of a normed space X is called open if for each element $x \in U$, there exists $r > 0$ such that $B(x; r) \subset U$.*

Definition 1.2.2. *A subset U of a normed space X is called closed if it contains all limits of convergent sequences of U.*

Definition 1.2.3. *A linear operator $U : X \to Y$ (from a normed space X into a normed space Y) is called bounded if there exists a positive number C such that;*

$$\|U\varphi\| \leq C\|\varphi\|, \quad \forall\varphi \in X.$$

Definition 1.2.4. *A sequence x_n in a normed space X is convergent to the limit x_0 if and only if $\lim_{n\to\infty} x_n = x_0$ in X.*

Definition 1.2.5. *A sequence x_n in a normed space X is called a* ***Cauchy sequence*** *if and only if for every $\epsilon > 0$ there exists an integer N such that $\|x_n - x_m\| < \epsilon$, $\forall m, n > N$, that is $\lim_{n,m\to\infty} \|x_n - x_m\| = 0$.*

Every convergent sequence is a Cauchy sequence whereas the converse in general is not true.

Example 1.2.1. *Let $x_n = \frac{1}{n}\sqrt{2}$ for each $n \in N$. Note that each x_n is an irrational number (Q^c) and that $\{x_n\}$ converges to 0 as $n \to \infty$. Thus, $\{x_n\}$ converges in R. But 0 is a rational number. $(Thus\ 0 \notin Q^c)$, so although the sequence $\{x_n\}$ is entirely in Q^c, it does not converge in Q^c.*

Definition 1.2.6. *A subset U of a normed space X is called complete if every Cauchy sequence of elements in U converges to an element in U.*

Definition 1.2.7. *A complete normed space is called* ***Banach space****.*

Definition 1.2.8. *Let X be a vector space. A mapping $(\cdot,\cdot) : X \times X \to K$ is called an inner product on X if*

(i) $(x, x) > 0, \quad \forall x \in X\backslash\{0\}$

(ii) $(\lambda x + y, z) = \lambda(x, z) + (y, z), \quad \forall \lambda \in K, x, y, z \in X$

(iii) $(x, y) = \overline{(y, x)}, \quad \forall x, y \in X$

A Banach space $(X, \|\cdot\|_X)$ is called a ***Hilbert space*** *if there exists an inner product on X such that $\|x\|_X = (x, x)^{\frac{1}{2}}$ for all $x \in X$.*

Definition 1.2.9. *(Holder's Inequality for Integrals)*
If $1 < p < \infty,\ x \in L^p,\ y \in L^q$ and $\frac{1}{p} + \frac{1}{q} = 1$,then

$$\int_\Omega |f(x)g(x)|dx \le \|f(x)\|_p \|g(x)\|_q$$

is called **Holder's inequality**.

Definition 1.2.10. *Let $\mathcal{K}$ be a subset of a normed space V. We say $\mathcal{K}$ has an open covering by a collection of open sets $\{U_\alpha|\alpha \in \Lambda,\ \ \Lambda$ an index set$\}$, if*

$$\mathcal{K} \subset \bigcup_{\alpha\in\Lambda} U_\alpha.$$

We say $\mathcal{K}$ is ***compact*** *if for every open covering $\{U_\alpha\}$ of $\mathcal{K}$, there is a finite subcover $\{U_{\alpha j}|j = 1, ..., m\} \subset \{U_\alpha|\alpha \in \Lambda\}$ which also cover $\mathcal{K}$.*

Definition 1.2.11. *Let $k \in N$ and $1 \le p < \infty$.Then*

$$W_p^k(R^n) = \{u \in L_p(R^n) : D^\alpha u \in L_p(R^n), \forall \alpha \in N_0^n, |\alpha| \le k\} \tag{1.2}$$

are the classical sobolev spaces.

Definition 1.2.12. *Let $u \in L^1(\Omega)$ and $\alpha \in N_0^n$. The function u is said to have a weak derivative $D_w^\alpha u$, if there exists a function $\nu \in L^1(\Omega)$ such that*

$$\int_\Omega u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega \nu \varphi dx, \varphi \in C_0^\infty(\Omega). \tag{1.3}$$

We then set $D_w^\alpha u := \nu$.

Definition 1.2.13. *Let $k \in N$ and $1 \leq p \leq \infty$. Then*

$$W_p^k(\Omega) := \{u \in L_p(\Omega) | D_w^\alpha u \in L_p(\Omega), |\alpha| \leq k\} \tag{1.4}$$

*is called **sobolev space**.*

Definition 1.2.14. *Let $\Omega \in R^n$ be an open set, $k \in N, 1 \leq p \leq \infty$. The function u belongs to Sobolev space $W_p^k(\Omega)$ and if $u \in L_p(\Omega)$, and if it has weak derivatives $D_w^\alpha u$ on Ω, $\forall \alpha \in N_0^n$ satisfying $|\alpha| = k$ and*

$$\|u\|_{W_p^k(\Omega)} = \|u\|_{L_p(\Omega)} + \sum_{|\alpha|=k} \|D_w^\alpha u\|_{L_p(\Omega)} < \infty; \tag{1.5}$$

$$\|u\|_{W_\infty^k(\Omega)} = \max_{|\alpha| \leq k} \|D^\alpha u\|_\infty. \tag{1.6}$$

Theorem 1.2.1. *$W_p^k(\Omega)$ is a Banach space [4].*

Proof. let $\{u_n\}$ be a cauchy sequence in $W_p^k(\Omega)$. Then $\{D^\alpha u_n\}$ is a cauchy sequence in $L^p(\Omega)$ for $0 \leq |\alpha| \leq k$. Since $L^p(\Omega)$ is complete there exist functions u and u_α, $0 \leq |\alpha| \leq k$ such that $u_n \to u$ and $D^\alpha u_n \to u_\alpha$ in $L^p(\Omega)$ as $n \to \infty$. Now $L^p(\Omega) \subset L_{loc}^1(\Omega)$ and so u_n determine a distribution $T_{u_n} \in D'(\Omega)$. For any $\phi \in D(\Omega)$, we have $|T_{u_n}(\phi) - T_u(\phi)| \leq \int_\Omega |u_n(x) - u(x)||\phi(x)| dx \leq \|\phi\|_{p'} \|u_n - u\|_p$ by Holder's inequality, where p' is the exponent conjugate to p.
Therefore $T_{u_n} \to T_u$ for every $\phi \in D(\Omega)$ as $n \to \infty$. Similarly, $T_{D^\alpha u_n}(\phi) \to T_{u_\alpha}(\phi)$ for every $\phi \in D(\Omega)$. It follows that

$$\begin{aligned} T_{u_\alpha}(\phi) &= \lim_{n\to\infty} T_{D^\alpha u_n}(\phi) = \lim_{n\to\infty} (-1)^{|\alpha|} T_{u_n}(D^\alpha \phi) \\ &= (-1)^{|\alpha|} T_u(D^\alpha \phi) \quad forevery \quad \phi \in D(\Omega) \end{aligned}$$

Thus $u_\alpha = D^\alpha u$ in the distributional sense on Ω for $0 \leq |\alpha| \leq k$, hence $u \in W_p^k(\Omega)$.
Since $\lim_{n\to\infty} \|u_n - u\|_{k,p} = 0$,
the space $W_p^k(\Omega)$ is complete. □

Definition 1.2.15. *Let $W_0^{k,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W_1^{k,p}(\Omega)$. When $p = 2$, we denote $H_0^k(\Omega) \equiv W_0^{k,2}(\Omega)$. We interpret $W_0^{k,p}(\Omega)$ to be the space of all the functions v in $W_0^{k,p}(\Omega)$ with the property that*

$$D^\alpha v(x) = 0 \quad on \quad \partial\Omega, \forall \alpha \quad with \quad |\alpha| \leq k - 1.$$

Definition 1.2.16. *A function f from $S \subset R^n$ into R^m is **Lipschitz continuous** at $x \in S$ if there is a constant C such that*

$$\|f(y) - f(x)\| \leq C\|y - x\| \tag{1.7}$$

for all $y \in S$ sufficiently near to x.

Note that Lipschitz continuity at a point depends only on the behavior of the function near that point. For f to be Lipschitz continuous at x, an inequality (1.7) must hold for all y sufficiently near x, but it is not necessary that (1.7) hold if y is not near x. Also, f may be Lipschitz continuous at other points, but different values of C may be required for (1.7) to hold near those points.

For example, $f(x) = \frac{1}{x}$ for $x > 0$ is Lipschitz continuous at each $x > 0$, but there is no single C for which (1.7) holds for all $x >0$. If f is Lipschitz continuous at x, then it is continuous at x, and if f is a real-valued function defined on $S \subset R$ that is differentiable at $x \in S$, then f is Lipschitz continuous at x.
i.e. differentiable at $x \Rightarrow$ Lipchitz continuous at $x \Rightarrow$ continuous at x.
But the converse implications don't hold.

Example 1.2.2. *$f(x) = \sqrt{|x|}$ is continuous at $x = 0$ but not Lipschitz continuous there because its derivative is unbounded as $x \to 0$. And $f(x) = |x|$ is Lipschitz continous at $x = 0$ but not differentiable there.*

Definition 1.2.17. *A domain $\Omega \subset R^3$ is a Lipschitz polyhedron if $\Omega \in C^{0,1}(\Omega)$ and there exist finitely many disjoint, relatively open surface patches $\partial\Omega_j \subset \partial\Omega$, $1 \le j \le q$ that are smooth, $\partial\Omega_j \in C^\infty(\Omega)$, and satisfy $\partial\Omega = \bigcup_{j=1}^{q} \partial\bar{\Omega}_j$.*

Theorem 1.2.2. *Assume Ω is a Lipschitz domain in $R^n, 1 \le p < \infty$. Then there exists a continuous linear operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ with the following properties.*

(a) *$\gamma v = v|_{\partial\Omega}$ if $v \in W^{1,p}(\Omega) \bigcap C(\bar{\Omega})$.*

(b) *For some constant $c > 0, \|\gamma v\|_{L^p(\partial\Omega)} \le c\|v\|_{W^{1,p}}(\Omega), \forall v \in W^{1,p}(\Omega)$.*

(c) *The mapping $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is compact; i.e.,for any bounded sequence $\{v_n\}$ in $W^{1,p}(\Omega)$, there is a subsequence $\{v_{n_1}\} \subset \{v_n\}$ such that $\{\gamma v_{n_1}\}$ is convergent in $L^p(\partial\Omega)$.*

The operator γ is called the trace operator, and γv can be called the generalized boundary value of v. The trace operator is neither an injection nor a surjection from $W^{1,p(\Omega)}$ to $L^p(\partial\Omega)$. The range $\gamma(W^{1,p(\Omega)})$ is a space smaller than $L^p(\partial\Omega)$.
We need to use traces of the $H^1(\Omega)$ functions, that form the space $H^{1/2}(\partial\Omega)$; in other words,

$$H^{1/2}(\partial\Omega) = \gamma(H^1(\Omega))$$

Correspondingly, we can use the following as the norm for $H^{1/2}(\partial\Omega)$:

$$\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} = \inf_{\substack{v \in H^1(\Omega) \\ \gamma v = g}} \|v\|_{H^1(\Omega)}$$

Definition 1.2.18. *Let $f, g \in D(R^n)$. The convolution of f and g, denoted by $f * g$, is the function*

$$(f * g)(y) = \int_{R^n} f(y - x)g(x)dx. \tag{1.8}$$

Definition 1.2.19. *A distribution is a linear map $T : D(\Omega) \to R$ which is continuous in the sense that ;*

$$\lim_{n\to\infty} T(\varphi_n) = T(\varphi) \tag{1.9}$$

for any sequence $\varphi_n \to \varphi$ on $D(\Omega)$ and denoted by $D'(\Omega)$.

Definition 1.2.20. *The distribution define by*

$$\langle \delta, \varphi \rangle = \varphi(0) \tag{1.10}$$

$\forall \varphi \in D(\Omega)$ *is called dirac's delta function.*

Theorem 1.2.3. ***(Banach fixed-point theorem)*** *Assume that K is a nonempty closed set in a Banach space V, and further, that $T : K \to K$ is a contractive mapping with contractivity constant α, for $\alpha \in [0, 1)$. Then the following results hold.*

1. *Existence and uniqueness: There exists a unique $u \in K$, such that*

$$u = T(u).$$

2. *Convergence and error estimates of the iteration: For any $u_0 \in K$,the sequence $\{u_n\} \in K$ defined by $u_{n+1} = T(u_n), n = 0, 1, ...,$ converges to u:*

$$\|u_n - u\|_V \to 0 \quad as \quad n \to \infty.$$

For the error, the following bounds are valid.

$$\|u_n - u\|_V \leq \frac{\alpha^n}{1-\alpha} \|u_0 - u_1\|_V \tag{1.11}$$

$$\|u_n - u\|_V \leq \frac{\alpha}{1-\alpha} \|u_{n-1} - u_n\|_V \tag{1.12}$$

$$\|u_n - u\|_V \leq \alpha \|u_{n-1} - u\|_V. \tag{1.13}$$

Proof. see [2 pg 204] □

1.3 The Lax-Milgram Lemma

The Lax-Milgram Lemma is employed frequently in the study of linear elliptic boundary value problems of the form $a(u,v) = \ell(v), \quad \forall u, v \in V$. For a real Banach space V, let us first explore the relation between a linear operator $A : V \to V'$ and a bilinear form $a : V \times V \to R$ related by

$$\langle Au, v\rangle = a(u,v), \qquad \forall u, v \in V. \tag{1.14}$$

The bilinear form $a(\cdot,\cdot)$ is continuous if and only if there exists $M > 0$ such that

$$|a(u,v)| \le M\|u\|\|v\| \quad \forall u, v \in V.$$

Theorem 1.3.1. *There exists a one-to-one correspondence between linear continuous operators $A : V \to V'$ and continuous bilinear forms $a : V \times V \to R$, given by the formula* (1.14)

Proof. see [2 pg 330] □

Theorem 1.3.2. *Assume K is a nonempty, closed, convex subset of the Hilbert space $V, \ell \in V'$. Let*

$$E(v) = \frac{1}{2}\|v\|^2 - \ell(v), \quad \forall v \in V.$$

Then there exists a unique $u \in K$ such that

$$E(u) = \inf_{v\in K} E(v).$$

The minimizer u is uniquely characterized by the inequality

$$u \in K, \qquad (u, v-u) \ge \ell(v-u), \quad \forall v \in K.$$

If additionally, K is a subspace of V, then u is equivalently defined by

$$u \in K, \qquad (u,v) = \ell(v), \quad \forall, v \in K.$$

Theorem 1.3.3. *Assume K is a nonempty, closed, convex subset of the Hilbert space V, $a(\cdot,\cdot) : V \times V \to R$ bilinear, symmetric, bounded and V-elliptic, $\ell \in V'$. Let*

$$E(v) = \frac{1}{2}a(v,v) - \ell(v), \quad v \in V.$$

Then there exists a unique $u \in K$, such that

$$E(u) = \inf_{v\in K} E(v), \tag{1.15}$$

which is also the unique solution of the variational inequality

$$u \in K, \qquad a(u, v-u) \ge \ell(v), \quad \forall v \in K. \tag{1.16}$$

or

$$u \in K, \qquad a(u, v-u) = \ell(v), \quad \forall v \in K. \tag{1.17}$$

in the special case where K is a subspace.

Proof. see [2 pg 342] □

Theorem 1.3.4. ***(Lax-Milgram Lemma)*** *Assume V is a Hilbert space, $a(\cdot,\cdot)$ is a bounded, V-elliptic bilinear form on $V, \ell \in V'$. Then there is a unique solution of the problem*

$$u \in V, \quad a(u,v) = \ell(v) \qquad \forall v \in V. \tag{1.18}$$

Before proving the result, let us consider the simple real linear equation

$$x \in R, \quad ax = \ell.$$

Its weak formulation is

$$x \in R, \quad axy = \ell y, \qquad \forall y \in R.$$

We observe that the real linear equation has a solution if and only if $0 < a < \infty$ (we multiply the equation by (-1) to make a positive, if necessary) and $|\ell| < \infty$, i.e., if and only if the bilinear form $a(x,y) \equiv axy$ is continuous and R-elliptic, and the linear form $\ell(y) \equiv \ell y$ is bounded. Thus the assumptions made in this Theorem are quite natural.

Several different proofs are possible for this important result. Here we present one of them.

Proof. For any $\theta > 0$, the problem (1.18) is equivalent to

$$(u,v) = (u,v) - \theta[a(u,v) - \ell(v)] \quad \forall v \in V,$$

i.e., the fixed-point problem

$$u = P_\theta(u),$$

where $P_\theta(u) \in V$ is defined through the relation

$$(P_\theta(u), v) = (u,v) - \theta[a(u,v) - \ell(v)], \quad v \in V.$$

We will apply the Banach fixed-point theorem with a proper choice of θ.
Let $A : V \to V'$ be the linear operator associated with the bilinear form $a(\cdot,\cdot)$; see (1.14). Then A is bounded and strongly positive: $\forall v \in V$,

$$\|Av\| \le M\|v\|,$$
$$\langle Av, v\rangle \ge \alpha\|v\|^2.$$

Denote $J : V \to V$ the isometric dual mapping from the Riesz representation theorem. Then

$$a(u,v) = \langle Au, v\rangle = (JAu, v) \quad \forall u, v \in V,$$

and

$$\|JAu\| = \|Au\| \quad \forall u \in V.$$

For any $u \in V$, by Theorem 1.2.3, the problem

$$(w,v) = (u,v) - \theta[a(u,v) - \ell(v)] \qquad \forall v \in V$$

has a unique solution $w = P_\theta(u)$. Let us show that for $\theta \in (0, 2\alpha/M^2)$, the operator P_θ is a contraction. Indeed let $u_1, u_2 \in V$, and denote $w_1 = P_\theta(u_1),\ w_2 = P_\theta(u_2)$. Then

$$\begin{aligned}
(w_1 - w_2, v) &= (u_1 - u_2, v) - \theta a(u_1, u_2, v) = ((I - \theta JA)(u_1 - u_2), v), \\
i.e., w_1 - w_2 &= (I - \theta JA)(u_1 - u_2).\ \ We\ \ then\ \ have \\
\|w_1 - w_2\|^2 &= \|u_1 - u_2\|^2 - 2\theta(JA(u_1 - u_2), u_1 - u_2) \\
&\quad + \theta^2 \|JA(u_1 - u_2)\|^2 \\
&= \|u_1 - u_2\|^2 - 2\theta a(u_1 - u_2, u_1 - u_2) \\
&\quad + \theta^2 \|A(u_1 - u_2)\|^2 \\
&\leq (1 - 2\theta\alpha + \theta^2 M^2)\|u_1 - u_2\|^2.
\end{aligned}$$

since $\theta \in (0, 2\alpha/M^2)$, we have

$$1 - 2\theta\alpha + \theta^2 M^2 < 1$$

and the mapping P_θ is a contraction. By the Banach fixed-point theorem, P_θ has a unique fixed-point $u \in V$, which is the solution of the problem (1.18). □

1.4 The Fundamental solution

A fundamental solution is a function that satisfies a PDE everywhere except for a point.

Definition 1.4.1. *Let L_x be a second-order differential operator acting on the x variable only. Then $\Phi(x, y)$ is a **fundamental solution** of L_x if $L_x\Phi(x, y) = \delta_y(x)$, where δ_y is the Dirac delta distribution at point y. We refer to y as the source of the fundamental solution, and x as the target.*

Proposition 1.4.1. *Let L_x be a second-order general elliptic operator. Then any fundamental solution for L_x is of the form*

$$\Phi(x, y) = A(x, y) log \frac{1}{|r|} + B(x, y)$$

Where $r = x - y$, and coefficients A,B are continuous functions. Furthermore

$$\lim_{r \to 0} A(x, y) = \frac{1}{2\pi}$$

That is, the singularity at $x = y$ is logarithmic.

Definition 1.4.2. *The fundamental solution Φ corresponding to the Laplacian operator (Δ) is*

$$\Phi(x, y) := \begin{cases} \frac{1}{2\pi} ln|x - y|, & n = 2, \\ \frac{1}{\omega_n |x-y|^{n-2}}, & n \geq 3, \end{cases} \tag{1.19}$$

where as $y \in R^n$ and ω_n is the surface area of a unit ball in R^n. In particular the fundamental solution of Laplacian's equation in R^3 is given by

$$\Phi(x, y) := \frac{1}{4\pi|x - y|}, \qquad y \in R^3.$$

Definition 1.4.3. *Let $\Omega \subset R^n$ be a domain. A Green function in Ω is defined to be a function of $(x, y) \in \Omega \times \Omega$ verifying the following conditions for each fixed $x \in \Omega$:*

$$\Delta_y G(x, y) = \delta(x), \qquad y \in \Omega \tag{1.20}$$

$$G(x, \sigma) = 0 \qquad \sigma \in \partial\Omega \tag{1.21}$$

Definition 1.4.4. *A twice continuously differentiable real valued function u, defined on a domain $\Omega \subset R^3$, is called* ***harmonic*** *if it satisfies Laplace's equation*

$$\Delta u = 0 \quad in\ \Omega$$

where

$$\Delta u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2}.$$

1.5 Interpolation and numerical integration on surfaces

An important category of integral equations is that defining integral equations on surfaces in R^3.
We assume $\partial\Omega$ is a connected piecewise smooth surface in R^3. By this, we mean $\partial\Omega$ can be written as

$$\partial\Omega = \partial\Omega_1 \cup ... \cup \partial\Omega_J \tag{1.22}$$

with each $\partial\Omega_j$ the continuous image of a polygonal region in the plane:

$$F_j : R_j \xrightarrow[onto]{1-1} \partial\Omega_j, \qquad j = 1, ..., J. \tag{1.23}$$

where R_j is a polygonal domain in the plane and $F_j \in C^6(R_j)$. In this case we say $\partial\Omega$ is piecewise smooth. By a smooth surface, we mean that for each point $p \in \partial\Omega$, there is a neighborhood on $\partial\Omega$ of p with the neighborhood having a local six-times continuously differentiable parametrization in R^2.
We create triangulations for $\partial\Omega$ by first triangulating each R_j and then mapping this triangulation onto $\partial\Omega_j$. Let $\{\hat{\triangle}_{j,k} | k = 1, ..., n_j\}$ be a triangulation of R_j, and then define

$$\triangle_{j,k} = F_j(\hat{\triangle}_j, k), \quad k = 1, ..., n_j, \quad j = 1, ..., J. \tag{1.24}$$

This yields a triangulation of $\partial\Omega$, which we refer to collectively as $T_n = \{\triangle_1, ..., \triangle_n\}$. We make the following assumptions concerning this triangulation.

T1. The set of all vertices of the surface $\partial\Omega$ is a subset of the set of all vertices of the triangulation T_n.

T2. The union of all edges of $\partial\Omega$ is contained in the union of all edges of all triangles in T_n.

T3. If two triangles in T_n have a nonempty intersection, then that intersection consists either of (i) a single common vertex, or (ii) all of a common edge.

We refer to triangulations satisfying **T1-T3** as conforming triangulations, while noting this definition may vary from that found elsewhere in the literature. We include **T3** because it simplifies some arguments by allowing them to be done in the framework of $C(\partial\Omega)$.

The surface $\partial\Omega$ of (1.22) is divided into a triangular mesh

$$\{\triangle_{k,n} | 1 \leq k \leq n\} \tag{1.25}$$

for a sequence $n = n_1, n_2, ...$ Each $\partial\Omega_j$ is to be broken apart into a set of non over lapping triangular shaped elements $\triangle_{k,n_j's}$. Define the mesh size of (1.25) by

$$\delta_n = \max_{1\leq k\leq n} diam(\triangle_k)$$

$$diam(\triangle_k) = \max_{p,q\in\triangle_k} |p-q| \tag{1.26}$$

Let σ denote the unit simplex in the st-plane

$$\sigma = \{(s,t) | 0 \leq s, t, s+t \leq 1\}.$$

Let $\rho_1, ..., \rho_6$ denote the three vertices and three midpoints of the sides of σ, numbered according to Figure 1.1

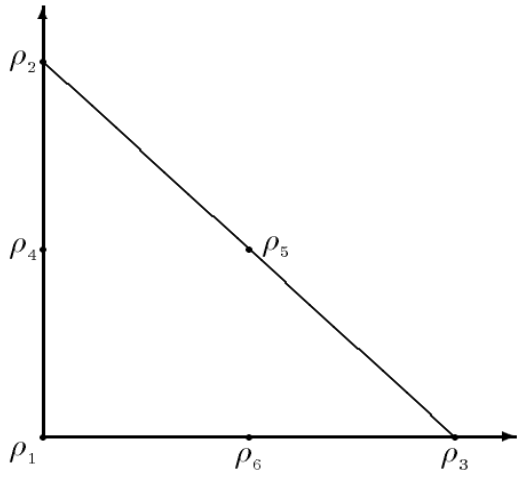


Figure 1.1: The unit simplex

Let $\hat{\triangle}_k$ be an element in the triangulation of R_j, and let $\{\hat{v}_1, \hat{v}_2, \hat{v}_3\}$ denote the vertices of $\hat{\triangle}_k$. Define $m_k : \sigma \xrightarrow[onto]{1-1} \triangle_k$ by

$$m_k(s,t) = F_j(u\hat{v}_1 + t\hat{v}_2 + s\hat{v}_3),\ (s,t) \in \sigma, \quad u = 1-s-t. \tag{1.27}$$

and let $\triangle_k$ be the image of $\hat{\triangle}_k$ under this mapping. Also, if any two elements in this triangulation have a side in common, then their intersection will be an entire side of both triangles. Most surfaces $\partial\Omega$ of interest can be decomposed as in (1.22) with each representable as in (1.23). Also the surface $\partial\Omega$ could be smooth and we would often still want to decompose it as in (1.22).

The mapping (1.27) used in defining interpolation and numerical integration $\triangle_k$. Introduce the node points for $\triangle_k$ by

$$v_{j,k} = m_k(\rho_j), \qquad j = 1, ..., 6$$

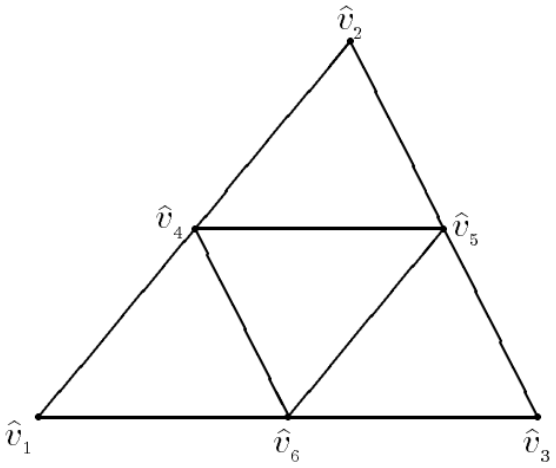


Figure 1.2: Refinement

Collectively, the node points of the triangulation $\{\triangle_k\}$ will be denoted by

$$\{v_i | 1 \leq i \leq n_v\},$$

with n_v the number of distinct node points.

The sequence of triangulations (1.25) will usually be obtained by successive refinements. The refinement process is based on connecting the midpoints of the sides of a given element $\hat{\triangle}_k$. Given $\{\hat{v}_1, ..., \hat{v}_6\}$, connect $\hat{v}_4, \hat{v}_5, \hat{v}_6$ by straight lines, as in Figure 1.2, producing four new triangular elements. The new elements all are congruent, and they are similar to $\hat{\triangle}_k$. More importantly any symmetric pair of triangles as shown in Figure 1.3, have the following property:

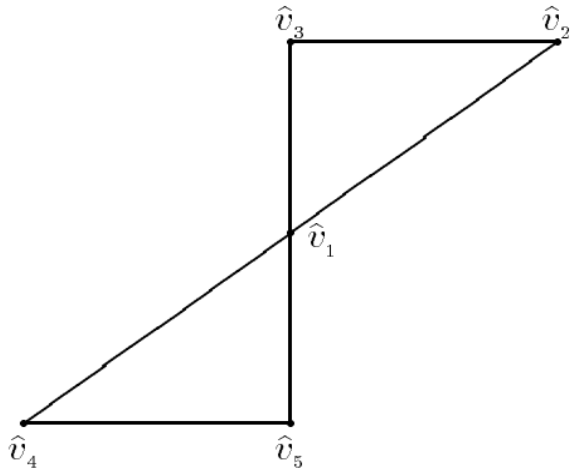


Figure 1.3: A Symmetric pair of triangles

$$\hat{v}_1 - \hat{v}_2 = -(\hat{v}_1 - \hat{v}_4) \quad and \quad \hat{v}_1 - \hat{v}_3 = -(\hat{v}_1 - \hat{v}_5) \tag{1.28}$$

The assumptions on $\partial\Omega$ and the node points that we made in this section are for the use of quadratic interpolation. There are other degrees of interpolation that can be used and the assumptions on the smoothness of $\partial\Omega$ and the definition of the nodes will change appropriately. But the general process of refinement will still remain the same, and we still subdivide $\triangle_{k's}$ in the same way as we do for the quadratic interpolation.

To define interpolation, introduce the six basis functions for quadratic interpolation on σ. Letting $u = 1 - (s + t)$, define

$$\begin{aligned} \ell_1(s,t) &= u(2u-1), & \ell_2(s,t) &= t(2t-1), & \ell_3(s,t) &= s(2s-1), \\ \ell_4(s,t) &= 4tu, & \ell_5(s,t) &= 4st, & \ell_6(s,t) &= 4su. \end{aligned}$$

Define a corresponding set of basis functions $\{\ell_{j,k(q)}\}$ on $\triangle_k$:

$$\ell_{j,k(m_k(s,t))} = \ell_j(s,t), \quad 1 \le j \le 6, \quad 1 \le k \le n.$$

Given a function $f \in C(\partial\Omega)$, define

$$P_n f(q) = \sum_{j=1}^{6} f(v_{j,k})\ell_{j,k(q)}, \quad q \in \triangle_k, \tag{1.29}$$

for $k = 1, ..., n$. This is called the piecewise quadratic isoparametric function interpolating f on the nodes of the mesh $\{\triangle_k\}$ for $\partial\Omega$.

It is straightforward that P_n is a bounded projection operator and $\|P_n\| = 5/3$. Also, for any $f \in C(\partial\Omega)$,

$$\|f - P_n f\|_\infty = O(\hat{\delta}_n^3) \tag{1.30}$$

where $\hat{\delta}_n$ is the mesh size of the triangulation $\{\hat{\triangle}_{k,n}\}$ of $R_{j's}$.

Other kinds of interpolation can be used such as piecewise cubic isoparametric interpolation. In this case, we need ten node points, $\rho_1, ..., \rho_{10}$, and ten basis functions for the interpolation on σ. The error analysis is the same, although some what more complicated.

We also use the same quadratic interpolation scheme to construct an approximate surface $\partial\tilde{\Omega}$ for $\partial\Omega$. The approximate surface $\partial\tilde{\Omega}$ is composed of elements $\tilde{\triangle}_1, ..., \tilde{\triangle}_k$, with $\tilde{\triangle}_k$ an interpolant of $\triangle_k$. Write

$$m_k(s,t) = \begin{bmatrix} x_k^1(s,t) \\ x_k^2(s,t) \\ x_k^3(s,t) \end{bmatrix}, \quad (s,t) \in \sigma \tag{1.31}$$

The reference to k will be omitted but understood implicitly. Define

$$\tilde{m}_k(s,t) = \sum_{j=1}^{6} m_k(\rho_j)\ell_j(s,t) = \begin{bmatrix} \sum_{j=1}^{6} x_k^1(\rho_j)\ell_j(s,t) \\ \sum_{j=1}^{6} x_k^2(\rho_j)\ell_j(s,t) \\ \sum_{j=1}^{6} x_k^3(\rho_j)\ell_j(s,t) \end{bmatrix}, \quad (s,t) \in \sigma \tag{1.32}$$

Thus, $\tilde{m}_k(s,t)$ interpolates $m_k(s,t)$ at $\{\rho_1, ..., \rho_6\}$, and each component is quadratic in (s,t).

We introduce two major numerical integration schemes that we have used. The first numerical integration method is the 3-point rule

$$\int_\sigma h(s,t)d\sigma \approx \frac{1}{6}\sum_{j=4}^{6} h(\rho_j). \tag{1.33}$$

This method has degree of precision two, integrating exactly all quadratic polynomials.

The 3-point rule is mainly for computing integrals if the integrands are continuous. In order to get the above results, the integrands are required to be four times continuously differentiable. If the integrands are continuous or smooth on the $\triangle_k$, but there is a nearby singularity, we need to use a better numerical integration method. The second method is

$$\int_\sigma h(s,t)d\sigma \approx \sum_{j=1}^{7} w_j h(r_j). \tag{1.34}$$

the weights w_j and nodes r_j given in the above reference. This formula has degree of precision five.

Chapter 2

Boundary Integral Equations

Let Ω denote a bounded open simply connected region in R^3, and let $\partial\Omega$ denote its boundary. Assume $\bar{\Omega} = \Omega \cup \partial\Omega$ is a region to which the divergence theorem can be applied. Let $\Omega' = R^3 \backslash \bar{\Omega}$ denote the region complementary to Ω. Sometimes, we will use the notations Ω_i and Ω_e in place of Ω and Ω', respectively, to refer to the regions interior and exterior to $\partial\Omega$. At a point $\sigma \in \partial\Omega$, let n_σ denote the unit normal directed into Ω, provided such a normal exists. In addition, we will assume that $\partial\Omega$ is a piecewise smooth surface. This means that $\partial\Omega$ can be written as

$$\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup ... \cup \partial\Omega_j \tag{2.1}$$

with each $\partial\Omega_j$ being the image of a smooth parametrization over a planar polygon. More precisely, for each j, there is a function

$$F_j : R_j \xrightarrow[onto]{1-1} \partial\Omega_j \tag{2.2}$$

with R_j a closed planar polygon and $F_j \in C^m(R_j)$ for some $m \geq 2$. For simplicity, we often take $m = \infty$. Moreover, we assume that the only possible intersection of $\partial\Omega_i$ and $\partial\Omega_j$, $i \neq j$, is along some portion of their boundaries (which correspond to the boundaries of R_i and Rj, respectively). When $\partial\Omega$ is a smooth surface, we may still choose to write it in the form (2.1), to simplify constructing triangulations of $\partial\Omega$ and to introduce a surface parametrization into the equation being studied. The assumption that Ω is simply connected is more restrictive than necessary, but it simplifies the presentation of the solvability theory of many of the boundary integral equations we derive.

The Dirichlet Problem:Find $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ that satisfies

$$\begin{aligned} \Delta u(P) &= 0, \qquad P \in \Omega \\ u(P) &= f(P), \qquad P \in \partial\Omega \end{aligned} \tag{2.3}$$

with $f \in C(\partial\Omega)$ a given boundary function.

The Neumann Problem:Find $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ that satisfies

$$\begin{aligned} \Delta u(P) &= 0, \qquad P \in \Omega \\ \frac{\partial u(P)}{\partial n_p} &= f(P), \qquad P \in \partial\Omega \end{aligned} \tag{2.4}$$

with $f \in C(\partial\Omega)$ a given boundary function.

Theorem 2.0.1. *Let $\partial\Omega$ be a piecewise smooth surface, with $\bar{\Omega}$ a region to which the divergence theorem can be applied. Assume the function $f \in C(\partial\Omega)$. Then*
(a) The Dirichlet problem (2.3) has a unique solution, and
(b) The Neumann problem (2.4) has a unique solution, up to the addition of an arbitrary constant, provided

$$\int_{\partial\Omega} f(Q)d\partial\Omega = 0 \tag{2.5}$$

Proof. see [7 p.64,66] □

2.1 Green's identities and representation formula

A very important tool for studying elliptic partial differential equations is the divergence theorem or Gauss's theorem.

Theorem 2.1.1. ***(The Divergence Theorem)***. *Assume $F : \bar{\Omega} \to R^3$ with each component of F contained in $C^1(\bar{\Omega})$ .Then the following is true.*

$$\int_{\Omega} \nabla \cdot F(X)d\Omega = \int_{\partial\Omega} F(X) \cdot n(X)d\sigma \tag{2.6}$$

Proof. Suppose that,

$$F(X) = F(x, y, z) = \begin{pmatrix} f(x, y, z) \\ g(x, y, z) \\ h(x, y, z) \end{pmatrix}$$

We have

$$\nabla \cdot F(X) = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right)(x, y, z)$$

Thus,

$$\int_{\Omega} \nabla \cdot F(X)dX = \int_{\Omega} \frac{\partial f}{\partial x}dX + \int_{\Omega} \frac{\partial g}{\partial y}X + \int_{\Omega} \frac{\partial h}{\partial z}dX$$

We will also initially suppose that the bounding surface $\partial\Omega$ for Ω consists of an upper surface and a lower surface:

$$\partial\Omega_\tau : \phi(x, y, z) = z - \tau(x, y) = 0; \quad \partial\Omega_\rho : \psi(x, y, z) = z - \rho(x, y) = 0.$$

together with some possible surface components parallel to the z-axis which we will denote collectively by $\partial\Omega_z$. If we take a point on the upper surface, namely $(x, y, \tau(x, y))$, the unit normal vector there is

$$n_\tau(X) = \frac{-i\frac{\partial\tau}{\partial x} - j\frac{\partial\tau}{\partial y} + k}{\sqrt{(\frac{\partial\tau}{\partial x})^2 + (\frac{\partial\tau}{\partial y})^2 + 1}}$$

On the lower surface the unit outward normal vector is

$$n_\rho(X) = \frac{i\frac{\partial\rho}{\partial x} + j\frac{\partial\rho}{\partial y} - k}{\sqrt{(\frac{\partial\rho}{\partial x})^2 + (\frac{\partial\rho}{\partial y})^2 + 1}}$$

On the upper surface, $z = \tau(x, y)$, the element of surface area is

$$d\sigma = \sqrt{(\frac{\partial \tau}{\partial x})^2 + (\frac{\partial \tau}{\partial y})^2 + 1}\ \ dxdy$$

, while on the lower surface the corresponding surface area element is

$$d\sigma = \sqrt{(\frac{\partial \rho}{\partial x})^2 + (\frac{\partial \rho}{\partial y})^2 + 1}\ \ dxdy$$

Therefore, with Ω_{xy} denoting the projection of Ω onto the x, y plane,

$$\begin{aligned}\int_\Omega \frac{\partial h}{\partial z}(x, y, z)dxdydz &= \int_{\Omega_{xy}} \left(\int_{z=\rho(x,y)}^{z=\tau(x,y)} \frac{\partial h}{\partial z}(x, y, z)dz \right) dxdy \\ &= \int_{\Omega_{xy}} (h(x, y, \tau(x, y)) - h(x, y, \rho(x, y)))dxdy \\ &= \int_{\partial\Omega} h(x, y, \tau(x, y) \frac{d\sigma}{\sqrt{(\frac{\partial \tau}{\partial x})^2 + (\frac{\partial \tau}{\partial y})^2 + 1}} \\ &- \int_{\partial\Omega} h(x, y, \rho(x, y) \frac{d\sigma}{\sqrt{(\frac{\partial \rho}{\partial x})^2 + (\frac{\partial \rho}{\partial y})^2 + 1}}\end{aligned}$$

But we clearly have

$$\frac{1}{\sqrt{(\frac{\partial \tau}{\partial x})^2 + (\frac{\partial \tau}{\partial y})^2 + 1}} = k \cdot n_\tau(X), \quad \frac{-1}{\sqrt{(\frac{\partial \rho}{\partial x})^2 + (\frac{\partial \rho}{\partial y})^2 + 1}} = k \cdot n_\rho(X)$$

Consequently

$$\int_\Omega \frac{\partial h}{\partial z}(x, y, z)dxdydz = \int_{\partial\Omega} h(x, y, z)k \cdot n_\tau(X)d\sigma + \int_{\partial\Omega} h(x, y, z)k \cdot n_\rho(X)d\sigma$$

On $\partial\Omega_z$ we have $k \cdot n(X0)$, so the above can be extended to

$$\int_\Omega \frac{\partial h}{\partial z}(x, y, z)dxdydz = \int_{\partial\Omega} h(x, y, z)k \cdot n(X)d\sigma$$

In the same way we can see that

$$\int_\Omega \frac{\partial f}{\partial x}(x, y, z)dxdydz = \int_{\partial\Omega} f(x, y, z)k \cdot n(X)d\sigma$$

$$\int_\Omega \frac{\partial g}{\partial y}(x, y, z)dxdydz = \int_{\partial\Omega} g(x, y, z)k \cdot n(X)d\sigma$$

Adding the three results separately obtained for f, g and h we have

$$\begin{aligned}\int_\Omega \nabla F(X)dX &= \int_\Omega \left(\frac{\partial f}{\partial x}(x, y, z) + \frac{\partial g}{\partial y}(x, y, z) + \frac{\partial h}{\partial z}(x, y, z) \right) dxdydz \\ &= \int_{\partial\Omega} (f(x, y, z)i + g(x, y, z)j + h(x, y, z)k) \cdot n(X)d\sigma \\ &= \int_{\partial\Omega} F(X) \cdot n(X)d\sigma\end{aligned}$$

□

Using this divergence theorem we can obtain Green's identities and Green's representation formula.
Assume $u \in C^1(\bar{\Omega})$, and $w \in C^2(\bar{\Omega})$
Let $F = u\nabla w$ in (2.6), then $\int_\Omega \nabla(u\nabla w)d\Omega = \int_{\partial\Omega} u\nabla w \cdot n(\sigma)d\partial\Omega$

$$\int_\Omega \nabla u \cdot \nabla w d\Omega + \int_\Omega u\Delta w d\Omega = \int_{\partial\Omega} u\frac{\partial w}{\partial n_\sigma} d\partial\Omega \tag{2.7}$$

is **first Green's identity**.
Next,assume $u, w \in C^2(\bar{\Omega})$. Interchanging the roles of u and w in (2.7) and we obtain;

$$\int_\Omega \nabla u \cdot \nabla w d\Omega + \int_\Omega w\Delta u d\Omega = \int_{\partial\Omega} w\frac{\partial u}{\partial n_\sigma} d\partial\Omega \tag{2.8}$$

Then subtract equation (2.7) from (2.8), we get the following.

$$\int_\Omega (w\Delta u - u\Delta w)d\Omega = \int_{\partial\Omega} [u\frac{\partial w}{\partial n_\sigma} - w\frac{\partial u}{\partial n_\sigma}]d\partial\Omega \tag{2.9}$$

, which is known as the **second Green's identity**.

Proposition 2.1.1. *Let Φ be the fundamental solution for $\Delta \in R^n$, and let $\Omega \in R^n$ be a domain. Then the Green function $G(x, y)$ for Ω can be decomposed as*

$$G(x, y) = \Phi(x - y) - \phi(x, y) \tag{2.10}$$

where for each $x \in \Omega$, $\phi(x, y)$ solve the Dirichlet problem

$$\Delta_y \phi(x, y) = 0, \qquad y \in \Omega \tag{2.11}$$

$$\phi(x, \sigma) = \Phi(x - \sigma), \qquad \sigma \in \partial\Omega \tag{2.12}$$

Proof. As we seen in the definition, $\Delta\Phi = \delta$. Also using (2.11), we compute that

$$\Delta_y(\Phi(x - \sigma) - \phi(x, y)) = \Delta_y\Phi(x - \sigma) - \Delta_y\phi(x, y) = \delta(x - y) \tag{2.13}$$

Therefore, $\Phi(x - \sigma) - \phi(x, y)$ verifies equation (1.20).
Furthermore, using (2.12), we have that $\Phi(x - \sigma) - \phi(x, y) = 0$ whenever $\sigma \in \partial\Omega$. Thus, $\Phi(x - \sigma) - \phi(x, y)$ also verifies the boundary condition (1.21). □

2.1.1 Green's representation formula

Assume that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and $\Delta u(P) = 0, \quad \forall P \in \Omega$.
Then:

$$\int_{\partial\Omega} u(\sigma)\frac{\partial}{n_\sigma}\left[\frac{1}{|P - \sigma|}\right] d\sigma - \int_{\partial\Omega} \frac{\partial u(\sigma)}{\partial n_\sigma}\frac{d\sigma}{|P - \sigma|} = \begin{cases} 4\pi u(P), & P \in \Omega \\ \Gamma(P)u(P), & P \in \partial\Omega \\ 0 & P \in \Omega' \end{cases} \tag{2.14}$$

here, n_σ denotes the unit normal to $\partial\Omega$ at point σ, directed outward from Ω; $\Gamma(P)$ denotes the interior solid angle at $P \in \partial\Omega$. To measure it, begin by defining

$$\hat{\Omega} = \{\sigma \in \Omega | L[P, \sigma] \subset \Omega\}$$

where $L[P,\sigma]$ denotes the open line segment joining P and σ:

$$L[P,\sigma] = \{\lambda P + (1-\lambda)\sigma | 0 < \lambda < 1\}$$

The set $\hat{\Omega}(P)$ consists of all points of Ω that can be seen from P on a direct line that does not intersect the boundary $\partial\Omega$ of Ω. Define the interior tangent cone at $P \in \partial\Omega$ by

$$\Upsilon_i(P) = \{P + \lambda(\sigma - P) | \lambda > 0,\ \sigma \in \hat{\Omega}(P)\} \tag{2.15}$$

and define the exterior tangent cone at $P \in \partial\Omega$ as the complement of $\Upsilon_i(P)$:

$$\Upsilon_e(P) = R^3 \backslash \overline{\Upsilon_i(P)} \tag{2.16}$$

The boundary of $\Upsilon_i(P)$ is denoted by $\Upsilon(P)$:

$$\Upsilon(P) = \textit{boundary}\ [\Upsilon_i(P)]$$

Now intersect $\Upsilon_i(P)$ with the unit sphere centered at P. The surface area of this intersection is called the interior solid angle at $P \in \partial\Omega$, and it is denoted by $\Gamma(P)$. If $\partial\Omega$ is smooth at P, then $\Gamma(P) = 2\pi$. For a cube the corners have interior solid angles of $\frac{1}{2}\pi$, and the edges have interior solid angles of π. We always assume

$$0 < \Gamma(P) < 4\pi \tag{2.17}$$

which eliminates the possibility of a cuspoidal type of behavior on $\partial\Omega$.

The first and third relationships in (2.14) follow from the Green's identities, which in turn follow from the divergence theorem. The second relationship follows from the following limit:

$$\lim_{\substack{A \to P \\ A \in \Omega}} \int_{\partial\Omega} u(\sigma) \frac{\partial}{\partial n_\sigma} \left[\frac{1}{|A-\sigma|}\right] d\sigma$$
$$= [4\pi - \Gamma(p)]u(P) + \int_{\partial\Omega} u(\sigma) \frac{\partial}{\partial n_\sigma} \left[\frac{1}{|P-\sigma|}\right] d\sigma \tag{2.18}$$

Referring to (2.14), integrals of the form

$$v(A) = \int_{\partial\Omega} \psi(\sigma) \frac{1}{|A-\sigma|} d\sigma, \qquad A \in R^3 \tag{2.19}$$

$$w(A) = \int_{\partial\Omega} \rho(\sigma) \frac{\partial}{\partial n_\sigma} \left[\frac{1}{|A-\sigma|}\right] d\sigma, \qquad A \in R^3 \tag{2.20}$$

are called single layer and double layer potentials, respectively; and the functions ψ and ρ are called single layer density and double layer density functions, respectively. The functions $v(A)$ and $w(A)$ are harmonic at all $A \notin \partial\Omega$.

2.2 Kelvin Transformation

Define a transformation $T : R^3\backslash\{0\} \to R^3\backslash\{0\}$ by

$$T(x,y,z) = (\xi,\eta,\zeta) \equiv \frac{1}{r^2}(x,y,z), r = \sqrt{x^2+y^2+z^2} \tag{2.21}$$

In spherical coordinates,

$$T(rcos\phi sin\theta, rsin\phi sin\theta, rcos\theta) = \frac{1}{r}(cos\phi sin\theta, sin\phi sin\theta, cos\theta)$$

for $0 < r < \infty, 0 \le \phi \le 2\pi, 0 \le \theta \le \pi$ .The inverse transformation is given by

$$T^{-1}(\xi,\eta,\zeta) = (x,y,z) \equiv \frac{1}{\rho^2}(\xi,\eta,\zeta), \qquad \rho = \sqrt{\xi^2+\eta^2+\zeta^2}$$

and $T^{-1} = T$. The point (ξ,η,ζ) is called the reflection of (x,y,z) through the unit sphere, and $r\rho = 1$.

Assume the bounded open region $\Omega_i = \Omega$ contains the origin 0. For a function $u \in C(\bar{\Omega}_e),\ \Omega_e \equiv \Omega'$, define

$$\hat{u}(\xi,\eta,\zeta) = \frac{1}{\rho}u(x,y,z) \quad with \quad (x,y,z) = T(\xi,\eta,\zeta) \tag{2.22}$$

and $\rho = \sqrt{\xi^2+\eta^2+\zeta^2}$. The function $\hat{u}$ is called the **Kelvin transformation of** u. Introduce the region

$$\hat{\Omega} = \{T(x,y,z)|(x,y,z) \in \Omega_e\}$$

and let $\partial\hat{\Omega}$ denotes its boundary. By our assumption on Ω, the region $\hat{\Omega}$ is an open, bounded region containing the origin 0. Then it can be shown that

$$\Delta\hat{u} = r^5\Delta u, \qquad r = \sqrt{x^2+y^2+z^2}$$

In this,

$$\Delta u(x,y,z) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \quad \Delta\hat{u}(\xi,\eta,\zeta) = \frac{\partial^2 \hat{u}}{\partial \xi^2} + \frac{\partial^2 \hat{u}}{\partial \eta^2} + \frac{\partial^2 \hat{u}}{\partial \zeta^2}$$

Thus

$$\Delta u \equiv 0 \quad on \quad \Omega_e \quad \Leftrightarrow \quad \Delta\hat{u} \equiv 0 \quad on \quad \hat{\Omega} \tag{2.23}$$

This can be used to better understand the Dirichlet problem for Laplace's equation on the exterior region Ω_e.

The exterior Dirichlet Problem: Find $u \in C^1(\bar{\Omega}_e) \cap C^2(\Omega_e)$ that satisfies

$$\begin{aligned} \Delta u(P) &= 0, && P \in \Omega_e \\ u(P) &= f(P), && P \in \partial\Omega \\ |u(P)| &= O(|P|^{-1}), && |P| \to \infty \end{aligned} \tag{2.24}$$

with $f \in C(\partial\Omega)$ a given boundary function.

Using the above discussion on the Kelvin transform, the exterior Dirichlet problem is equivalent to the following interior Dirichlet problem:

$$\begin{aligned} \Delta\hat{u}(\xi,\eta,\zeta) =& 0 \qquad\qquad (\xi,\eta,\zeta) \in \hat{\Omega} \\ \hat{u}(\xi,\eta,\zeta) =& \frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}} f(T(\xi,\eta,\zeta)), \quad (\xi,\eta,\zeta) \in \partial\hat{\Omega} \end{aligned} \tag{2.25}$$

The condition on $u(P)$ as $|P| \to \infty$ can be used to show that $\hat{u}(\xi,\eta,\zeta)$ has a removable singularity at the origin, and Theorem 2.0.1 guarantees the unique solvability of this interior Dirichlet problem.Therefore, the exterior Dirichlet problem (2.24) also has a unique solution; moreover,

$$u(P) = \frac{\hat{u}(0)}{|P|} + (|P|^{-2}) \quad as \quad |P| \to \infty. \tag{2.26}$$

will be the value of $u(P)$ as $|P| \to \infty$. It would seem advantageous to use the Kelvin transformation to reformulate all exterior Dirichlet problems as interior problems on an equivalent domain, but this seems to be a seldom used approach to the numerical solution of such problems.

The exterior Neumann problem does not convert to an equivalent interior Neumann problem.

The exterior Neumann Problem: Find $u \in C^1(\bar{\Omega}_e) \cap C^2(\Omega_e)$ that satisfies

$$\begin{aligned} \Delta u(P) =& 0, \qquad P \in \Omega_e \\ \frac{\partial u(P)}{\partial n_P} =& f(P), \qquad P \in \partial\Omega \end{aligned} \tag{2.27}$$

$$\begin{aligned} |u(P)| =& O(|P|^{-1}), \qquad \frac{\partial u(P)}{\partial r} = O(|P|^{-2}) \\ as \quad r =& |P| \to \infty, \quad uniformly \quad in \quad P/|P|. \end{aligned} \tag{2.28}$$

with $f \in C(\partial\Omega)$ a given boundary function.

2.2.1 Green's representation formula for exterior regions

Assume $u \in C^2(\Omega_e) \cap C^1(\bar{\Omega}_e)$, and assume $\Delta u \equiv 0$ on Ω_e. Further assume u satisfies the growth conditions of (2.28). Then

$$\int_{\partial\Omega} u(\sigma) \frac{\partial}{n_\sigma} \left[\frac{1}{|P-\sigma|}\right] d\sigma - \int_{\partial\Omega} \frac{\partial u(\sigma)}{\partial n_\sigma} \frac{d\sigma}{|P-\sigma|} = \begin{cases} 0 & p \in \Omega \\ -[4\pi - \Gamma(P)]u(P), & P \in \partial\Omega \\ -4\pi u(P), & P \in \Omega_e \end{cases} \tag{2.29}$$

Let $R \to \infty$ and use the growth conditions (2.28) to obtain (2.29). We can combine the two representation formulas (2.14) and (2.29) to obtain another quite useful representation formula. Let $u_i \in C^2(\Omega_i) \cap C^1(\bar{\Omega}_i)$ and $u_e \in C^2(\Omega_e) \cap C^1(\bar{\Omega}_e)$, and further assume u_e satisfies the growth conditions of (2.28).

We define,

$$u_i(\sigma) = \lim_{\substack{A\to\sigma \\ A\in\Omega_i}} u_i(A), \qquad u_e(\sigma) = \lim_{\substack{A\to\sigma \\ A\in\Omega_e}} u_e(A), \qquad \sigma \in \partial\Omega$$

$$[u(\sigma)] = u_i(\sigma) - u_e(\sigma), \qquad \sigma \in \partial\Omega \tag{2.30}$$

Similarly, define

$$\frac{\partial u_i(\sigma)}{\partial n_\sigma} = \lim_{\substack{A\to\sigma \\ A\in\Omega_i}} n_\sigma \cdot \nabla u_i(A), \qquad \frac{\partial u_e(\sigma)}{\partial n_\sigma} = \lim_{\substack{A\to\sigma \\ A\in\Omega_e}} n_\sigma \cdot \nabla u_e(A), \qquad \sigma \in \partial\Omega$$

$$\left[\frac{\partial u(\sigma)}{\partial n_\sigma}\right] = \frac{\partial u_i(\sigma)}{\partial n_\sigma} - \frac{\partial u_e(\sigma)}{\partial n_\sigma}, \qquad \sigma \in \partial\Omega \tag{2.31}$$

Subtract (2.29) from (2.14), we obtaining

$$\int_{\partial\Omega} [u(\sigma)] \frac{\partial}{\partial n_\sigma}\left[\frac{1}{|P-\sigma|}\right] d\sigma - \int_{\partial\Omega}\left[\frac{\partial u(\sigma)}{\partial n_\sigma}\right]\frac{d\sigma}{|P-\sigma|} = \begin{cases} 4\pi u(P), & \text{if} \quad P \in \Omega_i \cup \Omega_e \\ \Gamma(P)u_i(P) + [4\pi - \Gamma(P)]u_e(P), & \text{if } P \in \partial\Omega \end{cases} \tag{2.32}$$

This is used in obtaining indirect boundary integral equation methods.

2.3 Direct boundary integral equations

Methods based on (2.14) and (2.29) are generally referred to as direct methods. The unknown function being sought on $\partial\Omega$ is either u or $\frac{\partial u}{\partial n}$, both usually of direct interest to the person solving Laplace's equation.

2.3.1 The interior Dirichlet problem

Using (2.14), the interior Dirichlet problem (2.3) can be solved by first solving the integral equation

$$\mathcal{S}\psi(P) \equiv \int_{\partial\Omega} \frac{\psi(\sigma)}{|P-\sigma|} d\sigma = g(P), \quad P \in \partial\Omega \tag{2.33}$$

In which

$$\psi(\sigma) = \frac{\partial u(\sigma)}{\partial n_\sigma}$$
$$g(P) = \int_{\partial\Omega} f(\sigma) \frac{\partial}{\partial n_\sigma}\left[\frac{1}{|P-\sigma|}\right] d\sigma - \Gamma(P)f(P)$$

We need to know $\frac{\partial u}{\partial n_\sigma}$ on $\partial\Omega$ to be able to use this formula. The integral equation (2.33) is an integral equation of the first kind. When $\partial\Omega$ is a smooth surface, the integral equation (2.33), has a unique solution $\psi(\sigma)$ for any continuous function $f(P)$. After obtaining $\psi(\sigma)$, the solution $u(P)$ can be generated from (2.14).

$$u(P) = \frac{1}{4\pi}\int_{\partial\Omega} f(\sigma)\frac{\partial}{\partial n_\sigma}\left[\frac{1}{|P-\sigma|}\right] - \frac{1}{4\pi}\int_{\partial\Omega}\psi(\sigma)\frac{1}{|P-\sigma|}d\sigma, \quad P \in \Omega \tag{2.34}$$

where $f(\sigma) = u(\sigma)$, *and* $\psi(\sigma) = \frac{\partial u(\sigma)}{\partial n_\sigma}$.

2.3.2 The interior Neumann problem

The interior Neumann problem (2.4) has the solution (2.34) , but here we need to know $u(\sigma)$ on $\partial\Omega$. The boundary condition is $\frac{\partial u(\sigma)}{\partial n_\sigma} = f$ on $\partial\Omega$. Using (2.14), we obtain the integral equation of the second kind

$$-\Gamma(P)u(P) + \int_{\partial\Omega} u(\sigma)\frac{\partial}{\partial n_\sigma}\left[\frac{1}{|P-\sigma|}\right] d\sigma \\ = \int_{\partial\Omega} \frac{g(\sigma)}{|P-\sigma|} d\sigma \equiv \hat{g}(P), \quad P \in \partial\Omega. \tag{2.35}$$

For functions g satisfying the standard compatibility condition

$$\int_{\partial\Omega} g(\sigma)d\sigma = 0 \tag{2.36}$$

the integral equation (2.35) is not solvable for those right-hand sides $\hat{g}$, where (2.36) is not satisfied; and the difficulty with using this equation is that it is not uniquely solvable. In particular, $u \equiv 1$ is a solution of the homogeneous form of the equation, and the null space for the homogeneous equation consists entirely of the constant functions. The simplest way to deal with the lack of uniqueness is to introduce an additional condition such as;

$$u(P^*) = 0$$

for some fixed point $P^* \in \partial\Omega$. Combine this with the discretization of the integral equation to obtain a suitable numerical approximation for u.

For the exterior Dirichlet and Neumann problems the equation (2.32) also leads to integral equations of the first and second kind, respectively. For the Dirichlet problem this leads again to the first kind equation (2.33). For the Neumann problem, we obtain the integral equation (2.35).

$$[4\pi - \Gamma(P)]u(P) + \int_{\partial\Omega} u(\sigma)\frac{\partial}{\partial n_\sigma}\left[\frac{1}{|P-\sigma|}\right] d\sigma \\ = -\int_{\partial\Omega} \frac{\partial u(\sigma)}{\partial n_\sigma}\frac{d\sigma}{|P-\sigma|}, \qquad p \in \partial\Omega \tag{2.37}$$

We usually write it in the form

$$(2\pi + \mathcal{K})u = g \tag{2.38}$$

$$g(P) = -\int_{\partial\Omega} \frac{\partial u(\sigma)}{\partial n_\sigma}\frac{d\sigma}{|P-\sigma|}$$

$$\mathcal{K}u(P) = \int_{\partial\Omega} u(\sigma)\frac{\partial}{\partial n_\sigma}\left[\frac{1}{|P-\sigma|}\right] d\sigma + [2\pi - \Gamma(P)]u(P), \qquad P \in \partial\Omega \tag{2.39}$$

The equation (2.38) is uniquely solvable in most function spaces of interest, and it is a practical approach to solving the exterior Neumann problem for Laplace's equation. A more serious concern with both (2.33) and (2.35) is that the right-hand functions must be evaluated by numerical integration of a singular surface integral.

An important practical aspect of direct methods is that the quantity being found on $\partial\Omega$, either u or $\partial u/\partial n$, may be all that is needed by the user. In such a case, it will not be necessary to consider the numerical integration of the quantities in (2.34), for $P \in \Omega$.

2.4 Indirect boundary integral equations

Combining (2.32) with the type of arguments used in (2.31), we can justify representing harmonic functions as single or double layer potentials. For the interior Dirichlet problem, assume the solution u is a double layer potential,

$$u(p) = \int_{\partial\Omega} \rho(\sigma) \frac{\partial}{\partial n_\sigma} \left[\frac{1}{|P-\sigma|} \right] d\sigma, \quad P \in \Omega \tag{2.40}$$

Then the unknown function ρ must satisfy

$$2\pi\rho + \mathcal{K}\rho = f \tag{2.41}$$

with f the given Dirichlet boundary data and $\mathcal{K}$ the integral operator of (2.39). This is the same uniquely solvable integral equation as in (2.38), with a different right-hand side.

For the interior Dirichlet problem we can also assume u can be represented as a single layer potential,

$$u(P) = \int_{\partial\Omega} \frac{\psi(\sigma)}{|P-\sigma|} d\sigma, \qquad P \in \Omega \tag{2.42}$$

In this case, the density function ψ must be chosen to satisfy the first kind integral equation

$$\mathcal{S}\psi = f \tag{2.43}$$

which is basically the same equation as in (2.33). The equation (2.41) is the classical BIE approach to solving the interior Dirichlet problem, but recently there has been a great deal of interest in (2.43).

For the interior Neumann problem, consider representing its solution as a single layer potential, as in (2.42). Then imposing the boundary condition from (2.4), we have

$$\lim_{\substack{A \to P \\ A \in \Omega}} n_P \cdot \nabla \left[\int_{\partial\Omega} \frac{\psi(\sigma)}{|A-\sigma|} \right] d\sigma = f(P), \quad \forall P \in \partial\Omega$$

at which the normal n_P exists (which implies $\Gamma(P) = 2\pi$). Using this limiting argument, we obtain the second kind integral equation

$$-2\pi\psi(P) + \int_{\partial\Omega} \psi(\sigma) \frac{\partial}{\partial n_\sigma} \left[\frac{1}{|P-\sigma|} \right] d\sigma = f(P), \ P \in \partial\Omega^* \tag{2.44}$$

The set $\partial\Omega^*$ is to contain all points $P \in \partial\Omega$ for which $\Gamma(P) = 2\pi$. For $\partial\Omega$ a smooth surface, $\partial\Omega^* = \partial\Omega$; otherwise, $\partial\Omega \backslash \partial\Omega^*$ is a set of measure zero.

As with the earlier equation (2.35) for the Neumann problem, this equation has the difficulty that it is not uniquely solvable, and moreover, it is solvable if and only if the Neumann data f satisfy the integral condition (2.5). The solutions can be shown to differ by a constant. For this particular equation, there is a simple fix to the lack of unique solvability. Consider the integral equation

$$2\pi C(P) + \int_{\partial\Omega} C(\sigma) \frac{\partial}{\partial n_\sigma} \left[\frac{1}{|P-\sigma|} \right] d\sigma = f(P), \ P \in \partial\Omega^* \tag{2.45}$$

with P^* some fixed point from $\partial\Omega^*$.

We can also develop similar integral equations for the exterior Dirichlet and Neumann problems, but we omit the details as they are quite similar to what has already been done.

Chapter 3

Numerical Solution of Boundary Integral Equations

There are a wide variety of numerical methods that have been used for solving three-dimensional BIEs. This is partly a reflection of the wide variety of regions and problems, where one wants to take into account the special nature of the problem at hand. It also reflects the limited general knowledge concerning numerical methods for BIEs for the three-dimensional Laplace equation.

Until the early 1970s, the most commonly studied BIE in numerical analysis was the classic second kind equation with a smooth planar boundary $\partial\Omega$ in R^2. The study of BIE on smooth surfaces in R^3 is a well-developed area of classical mathematics. The general theory of numerical methods for solving BIE of the second kind with a compact operator on smooth surfaces is basically well-understood. But significant practical problems remain, especially concerning numerical integration. It is straightforward to speak of both Galerkin and collocation methods for solving (2.40). The convergence theory follows easily from the compactness of $\mathcal{K}$ and $\mathcal{S}$. For BIE of the first kind, only Galerkin method is well-understood.

Let the boundary integral equation being solved be denoted by [10]

$$\ell\rho = f. \tag{3.1}$$

Both Galerkin and collocation methods (and all other projection methods) approximate ρ with functions from a finite-dimensional family, denoted here by R_n. The methods attempt to minimize

$$r_n = \ell\rho_n - f,$$

as ρ_n ranges over R_n, in the hope that the minimizer ρ_n will also be close to ρ. The methods differ in the criterion for making r_n small. Galerkin methods minimize r_n by forcing the Fourier coefficients of r_n to be zero for all elements of some basis of R_n. If $\{\varphi_1, ..., \varphi_n\}$ is such a basis, then we choose $\rho_n \in R_n$ such that

$$(r_n, \varphi_i) = 0, \quad i = 1, ..., n. \tag{3.2}$$

where $(\cdot, \cdot)$ denotes some type of inner product for the functions under consideration. Writing

$$\rho_n = \sum_{j=1}^{n} \alpha_j \varphi_j, \tag{3.3}$$

the coefficients $\{\alpha_j\}$ are determined from

$$\sum_{j=1}^{n} \alpha_j(\ell\varphi_1, \varphi_i) = (f, \varphi_i), \quad i = 1, ..., n. \tag{3.4}$$

The inner products are integrals which must be evaluated numerically in most cases.

Collocation methods make r_n small by forcing $r_n(P)$ to be zero at a preselected set of points P in the integration domain $\partial\Omega$. Letting $\{P, ..., P_N\}$ be points in $\partial\Omega$, we require

$$r_n(P_i) = 0, \quad i = 1, ..., n. \tag{3.5}$$

If $\{\varphi_1, ..., \varphi_n\}$ is a basis of R_n, then we again seek the solution ρ_n in the form (3.3). The coefficients are now determined from

$$\sum_{j=1}^{n} \alpha_j \ell\varphi_j(P_i) = f(P_i), \quad i = 1, ..., n. \tag{3.6}$$

When the functions of R_n are defined globally, in some sense, as smooth functions on the integration domain $\partial\Omega$, we say the numerical method is a global method. In contrast, if the region is divided into small elements Δ_i,

$$\partial\Omega = \cup_{j=1}^{N} \Delta_i \tag{3.7}$$

and if the elements of R_n are piecewise polynomial functions of a fixed degree on the elements of this subdivision, we say the method is a local method. Most of the numerical methods used to solve BIEs in the engineering literature have been local methods. They are generally referred to as boundary element methods (or BEMs) in the engineering literature, and therefore, we will also use that name.

The boundary element methods are very flexible in handling a variety of surfaces $\partial\Omega$, boundary conditions, and changes in the smoothness of the unknown density function ρ. For surfaces that are only piecewise smooth, BEMs are almost mandatory, since the elements can be graded more finely near edges and vertices to compensate for the lack of smoothness in the density function ρ. In contrast, for surfaces that are smooth, the global methods can be much more efficient as regards computing time and storage, when compared to the BEMs. But we can't see this method in this paper.

3.1 Boundary element methods

In this section, we shall show how a simple numerical implementation of the BEM may be obtained to find approximations to the solution of the BVP for the Laplace equation [9]. The first step of the BEM is to approximate the boundary $\partial\Omega$ by a polygon with N straight-line segments. let the surface $\partial\Omega$ be subdivided into a fixed number of subregions,

$$\partial\Omega = \partial\Omega_1 \cup ... \cup \partial\Omega_N,$$

with each $\partial\Omega_j$ closed and the interiors of the $\partial\Omega_{j's}$ pairwise disjoint.
We construct these segments by choosing N points, say $x^k = (x_1^k, x_2^k)$, $k = 1, 2, ..., N$, on the boundary, and we define $x^{k+1} = x^1$. Then, $\partial\Omega_k$ is the straight line joining x^k *to* x^{k+1}. These straight line segments $\partial\Omega_k$ for $k = 1, 2, ..., N$, are known as
boundary elements. There are different types of boundary elements. But, here we are going to see only one of them.

3.1.1 Constant elements

We approximate the values of u and $\frac{\partial u}{\partial n}$ by constants over each boundary element. More specifically, we take the values at the midpoint of each element to be the values over the whole element. That is, for $k = 1, 2, ..., N$, $u \approx u_k$ and $\frac{\partial u}{\partial n} \approx q_k$, where u_k and q_k are the values of u and $\frac{\partial u}{\partial n}$ at the midpoint of boundary element $\partial\Omega_k$.
To proceed let us consider the following equation

$$c(P)u(P) = \int_{\partial\Omega} u(\sigma)\frac{\partial\Phi(\sigma, P)}{\partial n_\sigma}d\sigma - \int_{\partial\Omega} \frac{\partial u(\sigma)}{\partial n_\sigma}\Phi(\sigma, P)d\sigma \tag{3.8}$$

where

$$c(P) = \begin{cases} 1, & \text{if } P \in \Omega \\ \frac{1}{2}, & \text{if } P \in \partial\Omega \textit{ and } \partial\Omega \textit{ smooth at } P \\ \frac{\alpha}{2\pi}, & \text{if } P \in \partial\Omega \textit{ and } \partial\Omega \textit{ not smooth at } P \textit{ (corner with internal angle } \alpha) \\ 0, & \text{if} P \notin \Omega \cup \partial\Omega. \end{cases}$$

Then using the midpoint values, we can write (3.8) as an approximation,

$$c(P)u(P) = \sum_{k=1}^{N}(u_k H_k(P) - q_k G_k(P)) \tag{3.9}$$

where

$$G_k(P) = \int_{\partial\Omega} \Phi(\sigma, P)d\sigma \tag{3.10}$$

$$H_k(P) = \int_{\partial\Omega} \frac{\partial}{\partial n_\sigma}[\Phi(\sigma, P)]d\sigma \tag{3.11}$$

If we know u_k and q_k, then we can use equation (3.9) to find the values of $u(P)$ for $P \in \Omega$. However, in a properly posed boundary value problem, either u_k or q_k (but not both) is known on any given portion of the boundary.

By using the collocation method, we collocate P in (3.9) at the midpoints of $\partial\Omega_1, \partial\Omega_2, ..., \partial\Omega_N$. Therefore, one can use equation (3.9), firstly to find approximations to these unknown boundary values, before using it again to approximate the values at the interior points, so we get,

$$\frac{1}{2}u_m = \sum_{k=1}^{N}[G_k(\bar{\sigma}_m, \bar{P}_m)q_k - u_k H_k(\bar{\sigma}_m, \bar{P}_m)], \ for\ m = 1, 2, ..., N, \tag{3.12}$$

where $(\bar{\sigma}_m, \bar{P}_m)$ is the midpoint of $\partial\Omega_m$, and when P is on $\partial\Omega_k$, $c(P) = 1/2$.
It is convenient to write (3.12) as a system of equations in the form **Az=b**.
The unknowns in $\mathbf{z}$ are either u_k or q_k. The right hand side $\mathbf{b}$ is made up of all the known values. In other words, for each $m = 1, 2, ..., N$, we obtain:

$$\sum_{k=1}^{N} A_{mk}z_k = b_m, \ where\ b_m = \sum_{k=1}^{N} \tilde{b}_{mk}. \tag{3.13}$$

We construct $\mathbf{A}$ and $\tilde{\mathbf{b}}_{mk}$ by considering the two cases, namely (a) when u is given (Dirichlet condition) and (b) when $q = \frac{\partial u}{\partial n}$ is given (Neumann condition) over the

boundary $\partial\Omega_k$, for each equation $m = 1, 2, ..., N$ in (3.12).
So, we have the following cases for the construction of matrix $\mathbf{A}$ and vector $\tilde{\mathbf{b}}_{mk}$,

$$A_{mk} = \begin{cases} -G_k(\bar{\sigma}_m, \bar{P}_m), & \text{if } u \textit{ is given over } \partial\Omega_k; \\ H_k(\bar{\sigma}_m, \bar{P}_m), & \text{if } q \textit{ is given over } \partial\Omega_k \textit{ and } k \neq m \\ H_k(\bar{\sigma}_m, \bar{P}_m), & \text{if } q \textit{ is given`over`} \partial\Omega_k \textit{ and } k = m; \end{cases} \tag{3.14}$$

$$\tilde{b}_{mk} = \begin{cases} q_k G_k(\bar{\sigma}_m, \bar{P}_m), & \text{if } q \textit{ is given over } \partial\Omega_k; \\ -u_k H_k(\bar{\sigma}_m, \bar{P}_m), & \text{if } u \textit{ is given over } \partial\Omega_k \textit{ and } k \neq m; \\ u_k(-H_k(\bar{\sigma}_m, \bar{P}_m) + 1/2) & \text{if } u \textit{ is given over } \partial\Omega_k \textit{ and } k = m; \end{cases} \tag{3.15}$$

Therefore, after $\mathbf{A}$ and $\mathbf{b}$ are formed, we can solve for $\mathbf{z}$. When all the values on the boundary are obtained, we can then use equation (3.9) to obtain values at any interior point in the domain Ω.

3.1.2 Collocation methods on smooth surfaces

In this section we consider the numerical solution of the second kind double layer equation $2\pi\rho + \mathcal{K}\rho = f$, from (2.38) or (2.41). The methods of discretization can be used in approximating a wide variety of BIE, but the convergence theory presented here will generalize to only second kind BIE. We also restrict the methods of discretization to piecewise polynomial forms of approximating ρ. Methods of discretization for BIE that break the surface into small elements (as in this section) are often called boundary element methods(BEMs).

The smoothness of $\partial\Omega$ implies that $\mathcal{K}$ is a compact operator both from $C(\partial\Omega)$ to $C(\partial\Omega)$ and from $L^2(\partial\Omega)$ to $L^2(\partial\Omega)$. The compactness of $\mathcal{K}$ implies that the full body of theory developed in §1.5 for projection methods can be brought to bear on the numerical solution of $2\pi\rho + \mathcal{K}\rho = f$.

The collocation methods for solving $2\pi\rho + \mathcal{K}\rho = f$ are written abstractly as

$$(2\pi + P_n\mathcal{K})\rho_n = P_n f \tag{3.16}$$

with P_n an interpolatory projection or orthogonal projection, respectively.

An important auxiliary solution for the collocation method is the iterated collocation solution:

$$\hat{\rho}_n = \frac{1}{2\pi}(f - \mathcal{K}\rho_n)$$

It satisfies the equations

$$(2\pi + \mathcal{K}P_n)\hat{\rho}_n = f \tag{3.17}$$

$$P_n\hat{\rho}_n = \rho_n \tag{3.18}$$

The questions of stability for (3.16) and (3.17) are linked by the identities

$$\begin{aligned} (2\pi + \mathcal{K}P_n)^{-1} &= \frac{1}{2\pi}[I - \mathcal{K}(2\pi + \mathcal{K}P_n)^{-1}P_n] \\ (2\pi + \mathcal{K}P_n)^{-1} &= \frac{1}{2\pi}[I - P_n(2\pi + P_n\mathcal{K})^{-1}\mathcal{K}] \end{aligned} \tag{3.19}$$

The solvability of (3.16) is determined from the standard theory for projection methods. With the assumption of (a) compactness for $\mathcal{K} : C(\partial\Omega) \to C(\partial\Omega)$, and (b) pointwise convergence on $C(\partial\Omega)$ of the projections P_n to I we have that

$$\|(I - P_n)\mathcal{K}\| \to 0 \quad as \quad n \to \infty$$

From this, we have the standard result that if $(2\pi + \mathcal{K})^{-1}$ exists on $C(\partial\Omega)$, then $(2\pi + P_n\mathcal{K})^{-1}$ exists and is uniformly bounded for all sufficiently large n, say $n \geq N$. The existence of uniform boundedness of $(2\pi + P_n\mathcal{K})^{-1}$ then follows from (3.19). For the error in ρ_n and $\hat{\rho}_n$, use

$$\begin{aligned}\rho - \rho_n =& 2\pi(2\pi + P_n\mathcal{K})^{-1}(\rho - P_n\rho)\\ \rho - \hat{\rho}_n =& -(2\pi + \mathcal{K}P_n)^{-1}\mathcal{K}(\rho - P_n\rho)\end{aligned}$$

The quantity $\mathcal{K}(\rho - P_n\rho)$ often converges to zero more rapidly than does $\rho - P_n\rho$. Using (3.18) this will show that ρ_n is superconvergent to ρ at the collocation node points.

Theorem 3.1.1. *Consider the integral equation (2.37) with soluton ρ. Let $\partial\Omega$ be a smooth surface in R^3, and assume the unknown function $\rho \in C^4(\partial\Omega)$. Then*

$$\max_{1\leq i\leq n_v} |\rho(v_i) - \rho_n(v_i)| = O(\hat{\delta}_n^4 log\hat{\delta}_n). \tag{3.20}$$

Proof. **(a)** Since the solid angle $\Gamma(P) = 2\pi$ for every P on a smooth surface, then recall (2.44)

$$2\pi\rho(P) + \int_{\partial\Omega} \rho(\sigma)\frac{\partial}{\partial n_\sigma}\left[\frac{1}{|P-\sigma|}\right] d\sigma = f(P),\ P \in \partial\Omega^* \tag{3.21}$$

Using the triangulation scheme in §1.5 the compact operator $\mathcal{K}$ can be written as

$$\mathcal{K}\rho(P) = \sum_k \int_Q \rho(m_k(s,t))\frac{\partial}{\partial n_\sigma}\left[\frac{1}{|P - m_k(s,t)|}\right] |D_s m_k \times D_t m_k| dQ \tag{3.22}$$

For $\sigma = m_k(s,t)$

$$n(s,t) = n_\sigma = \pm\frac{D_s m_k \times D_t m_k}{|D_s m_k \times D_t m_k|}$$

with the sign chosen so that n_σ points into the bounded domain Ω.

Without loss of generality, we assume the sign of n_σ is always positive, and (3.22) becomes

$$\mathcal{K}\rho(P) = \sum_k \int_Q \rho(m_k(s,t))\frac{(D_s m_k \times D_t m_k)\cdot(P - m_k(s,t))}{|P - m_k(s,t)|^3} dsdt \tag{3.23}$$

In order to measure the error $\mathcal{K}(I - P_n)\rho(P)$ for P a node point we need to examine the local error which is contributed by each $\triangle_k$.

For each $\triangle_k$, the integrand of the equation (3.23) has one singularity at P when $P \in \triangle_k$ and it is smooth over $\triangle_k$ with $P \notin \triangle_k$, although it is increasingly peaked as P and $\triangle_k$ become closer together. We first compute the error for those

$\triangle_{k's}$ which contain P.

For simplifying notation, we assume $p = (0,0,0)$ and $m_k(0,0) = (0,0,0)$. The error integrating over $\triangle_k$ equals

$$\int_Q (\rho(m_k(s,t)) - P_n\rho(m_k(s,t)))\frac{(D_s m_k \times D_t m_k)\cdot(m_k(s,t))}{|m_k(s,t)|^3}dsdt \tag{3.24}$$

This integral exists even though $|m_k(0,0)| = 0$. To see this, use the Taylor error formula for the x_k^i about $(s,t) = (0,0)$. Then (3.24) equals

$$\int_Q \frac{h(s,t,\hat{\delta}^7)}{g(s,t,\hat{\delta}^3)}dQ \tag{3.25}$$

where h and g are polynomials in s and t, and their coefficients are of size $O(\hat{\delta}^7)$ and $O(\hat{\delta}^3)$, respectively. Also, h and g are polynomials of degrees two and three, respectively, which shows the existence of integral (3.25), and it is $O(\hat{\delta}^4)$ [When speaking of an order of convergence, say one based on $\hat{\delta}$, the order of convergence is uniform with respect to any absent variable or index].

When $P \notin \triangle_k$, $P - m_k$ never equals zero for $(s,t) \in Q$. The kernel function, $\kappa(s,t)$,

$$\kappa(s,t) = \frac{(D_s m_k \times D_t m_k)\cdot(P - m_k(s,t))}{|P - m_k(s,t)|^3}$$

is smooth. Compute the partial derivative κ_s before expanding $\kappa(s,t)$ about $(0,0)$.

$$\begin{aligned}
\kappa_s(s,t) =& \frac{[D_s(D_s m_k \times D_t m_k)]\cdot(P - m_k) - (D_s m_k \times D_t m_k)\cdot D_s m_k}{|P - m_k|^3} \\
&-3\frac{[(D_s m_k \times D_t m_k)\cdot(P - m_k)][D_s m_k \cdot(|P - m_k)]}{|P - m_k|^5} \\
=& \frac{[D_s(D_s m_k \times D_t m_k)]\cdot(P - m_k)}{|P - m_k|^3} \\
&-3\frac{[(D_s m_k \times D_t m_k)\cdot(P - m_k)][D_s m_k \cdot(|P - m_k)]}{|P - m_k|^5}
\end{aligned}$$

the term $(D_s m_k \times D_t m_k)\cdot D_s m_k$ was dropped because $(D_s m_k \times D_t m_k) \perp D_s m_k$. Also

$$|(D_s m_k \times D_t m_k)\cdot(P - m_k)| = |(D_s m_k \times D_t m_k)|\cdot|(P - m_k)|\cdot|cos\theta|$$

θ is the angle between the vectors $D_s m_k \times D_t m_k$ and $P - m_k$, and θ is a function of s and t;

$$|cos\theta| \le |P - m_k(s,t)|\cdot constant \quad \forall(s,t) \tag{3.26}$$

Therefore, κ_s is $O(\hat{\delta}^3/d_k^2)$ where $d_k = |P - m_k(0,0)|$. Use a similar calculation, κ_t is also $O(\hat{\delta}^3/d_k^2)$. We now expand $\kappa(s,t)$ about $(0,0)$ and have the following formula:

$$\kappa(s,t) = \kappa(0,0) + O(\hat{\delta}^3/d_k^2) \tag{3.27}$$

The error of $\rho(m_k) - P_n\rho(m_k)$ is

$$\rho(m_k) - P_n\rho(m_k) = H_k(s,t) + O(\hat{\delta}^4) \tag{3.28}$$

Where

$$H_k(s,t) = \frac{1}{3!}\left[(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t})^3 x^i(0,0) - \sum_{j=1}^{6}(s_j\frac{\partial}{\partial s} + t_j\frac{\partial}{\partial t})^3 x^i(0,0)\ell_j(s,t)\right].$$

Note that $\kappa(0,0)$ and $H_k(s,t)$ are $O(\hat{\delta}^2/d_k^2)$ and $O(\hat{\delta}^3)$, respectively.

Combining (3.27) and (3.28) we have

$$\begin{aligned}\int_Q \kappa(s,t)(\rho(m_k) - P_n\rho(m_k))dQ &= \int_Q \left(\kappa(0,0) + O(\frac{\hat{\delta}^3}{d_k^2})\right) H_k(s,t)dQ \\ &= O(\frac{\hat{\delta}^5}{d_k^2}) + O(\frac{\hat{\delta}^6}{d_k^2})\end{aligned}$$

for every $\triangle_k$ which does not contain P.

We now add all errors contributed by each $\triangle_k$. Let T' be the set of $\triangle_{k's}$ which contain P, and let T be the set of the remaining $\triangle_{k's}$, which do not contain p. Then,

$$\begin{aligned}\mathcal{K}(s,t)(\rho - P_n\rho)(P) &= \int_{\partial\Omega} \kappa(P,\sigma)(\rho(\sigma) - P_n\rho(\sigma))d\partial\Omega_\sigma \\ &= \sum_{\triangle_k \in T'} \int_Q \kappa(s,t)(\rho(m_k) - P_n\rho(m_k))dQ \\ &+ \sum_{\triangle_k \in T} \int_Q \kappa(s,t)(\rho(m_k) - P_n\rho(m_k))dQ \\ &= O(\hat{\delta}^4) + \sum_{\triangle_k \in T} \int_Q \kappa(s,t)(\rho(m_k) - P_n\rho(m_k))dQ\end{aligned}$$

$O(\hat{\delta}^4)$ is contributed by $\triangle_{k's}$ which are in T', and T' has at most six elements.

The error contributed by each $\triangle_k$ in T is $O(\hat{\delta}^5/d_k^2)$. Examining the error carefully, we find that cancellation happens on each symmetric pair of triangles. Thus for the dominant terms in the error

$$\kappa(0,0)H_i(s,t) + \kappa(0,0)H_j(s,t) = 0$$

if $\triangle_i$ and $\triangle_j$ are a symmetric pair of triangles. This improves the error from $O(\hat{\delta}^5/d_k^2)$ to $O(\hat{\delta}^6/d_k^2)$ for each $\triangle_k$ that is part of a symmetric pair of triangles. Let T_1 be the set of these kinds of triangles. Let T_2 be the set of triangles that are not in T_1.

The error being contributed by triangles in T_2 arises from the term

$$\begin{aligned}
&|\kappa(0,0)\cdot H_k(s,t)| \\
&\qquad = \left|\frac{(D_s m_k(0,0)\times D_t m_k(0,0))\cdot(P-m_k(0,0))}{|P-m_k(0,0)|^3}\right| |H_k(s,t)| \\
&\qquad = \left|\frac{|(D_s m_k(0,0)\times D_t m_k(0,0))|\cdot|(P-m_k(0,0))|\cdot cos\theta}{|p-m_k(0,0)|^3}\right| |H_k(s,t)| \\
&\qquad \leq \frac{|(D_s m_k(0,0)\times D_t m_k(0,0))|}{|P-m_k(0,0)|}|H_k(s,t)| = O(\frac{\hat{\delta}^5}{d_k}).
\end{aligned}$$

Thus, the error analysis has been improved from $O(\hat{\delta}^5/d_k^2)$ to $O(\hat{\delta}^5/d_k)$ which is contributed by each triangle in T_2.
Let

$$d(P)\equiv d=\min\{d_k|P\notin\triangle_k,\ k=1,...,n\}.$$

For simplicity we take $d_k \doteq d, 2d, ...$, depending on how far the $\triangle_k$ is from the point p. [A somewhat more complicated argument can be based on a lower bound of a similar type for d_k]. Let $r=\hat{\delta}/d$, which is finite for our uniform mesh subdivision scheme; and $d=O(\hat{\delta}_n)$. Note the indexing $\triangle_1,...,\triangle_n$ does not indicate distance from P. But there is an arrangement of $\{\triangle_k\}$ where the number of triangles at a distance R is proportional to R, with $R=d,2d,3d,....$

The number c_i of triangles in T_1 at a distance $i\cdot d$ is proportional to i for $i=1,...,t_j$. Note that for some integer t_j, $t_j\cdot d$ is the longest possible distance from P to triangles in R_j. Adding the error contributed by each triangle in T_1, we have

$$\sum_k O(\frac{\hat{\delta}^6}{d_k^2})=\sum_{i=1}^{t_j} c_i\cdot O(\frac{\hat{\delta}^6}{(i\cdot d)^2}=O(\hat{\delta}^4)\sum_{i=1}^{t_j} r^2\frac{i}{i^2}=O(\hat{\delta}^4 log\hat{\delta}).$$

For the triangles in T_2, the error contributed by each of them is $O(\hat{\delta}^6/d_k)$. The number of triangles of this type at a distance $i\cdot d$ is a finite number, and it usually is two or three; but the proof is omitted. Therefore,

$$\sum_k O(\frac{\hat{\delta}^5}{d_k})=\sum_{i=1}^{t_j} c_j'\cdot O(\frac{\hat{\delta}^4}{(i\cdot d)}=O(\hat{\delta}^4)\sum_{i=1}^{t_j} r\frac{1}{i}=O(\hat{\delta}^4 log\hat{\delta}).$$

where c_j' is either two or three. This completes the proof that

$$\mathcal{K}(I-P_n)\rho(P)=O(\hat{\delta}^4 log\hat{\delta}) \tag{3.29}$$

uniformly for P a node point in the triangulation $\{\triangle_{n,k}\}$ of $\partial\Omega$.

(b) To show (3.20), we first note that the error equation for the iterated collocation solution $\hat{\rho}_n$ is given by

$$(2\pi+\mathcal{K}P_n)(\rho-\hat{\rho}_n)=-\mathcal{K}(I-P_n)\rho \tag{3.30}$$

The linear system associated with this is

$$(2\pi+K_n)e_n=-\epsilon_n \tag{3.31}$$

With

$$\begin{aligned} e_{n,i} =& \rho(v_i) - \hat{\rho}_n(v_i) = \rho(v_i) - \rho_n(v_i) \\ \epsilon_{n,i} =& \mathcal{K}(I - P_n)\rho(v_i), \quad i = 1, ..., n_v. \end{aligned}$$

The matrix of coefficients $2\pi + K_n$ is also the same as that for the linear system associated with the collocation equation (3.16).

As noted earlier following (3.19), $(2\pi + \mathcal{K}P_n)^{-1}$ is uniformly bounded for all sufficiently large n. Also since the iterated collocation equation can be considered as being a Nystrom method it is a standard derivation that

$$\|(2\pi + K_n)^{-1}\| \leq \|(2\pi + \mathcal{K}P_n)^{-1}\|$$

where the matrix norm is the standard row norm. Combining these results,

$$\|(2\pi + K_n)^{-1}\| \leq c < \infty, \quad n \geq n_0$$

for some sufficiently large n_0 and some $c > 0$.

Using this result with (3.31), and using (3.29) to bound $\|\epsilon_n\|_\infty$, we have the desired result (3.20).

□

3.1.3 The single layer integral

For the exterior problem we need to evaluate the corresponding single layer integrals on the right hand side of (2.37). Write

$$\int_{\partial\Omega} \frac{f(\sigma)}{|P - \sigma|} d\sigma \approx \sum_{k=1}^{n} \int_Q \frac{f(\tilde{m}_k(s,t))}{|P - \tilde{m}_k(s,t)|} |D_s\tilde{m}_k(s,t) \times D_t\tilde{m}_k(s,t)| dQ \tag{3.32}$$

where P is one of node points. Note we are including the use of the approximating surface.

We can see the integrand in (3.32) varies from singular to quite smooth. To handle this varied behavior, we use two ways to study errors. The first case is for those $\triangle_{k's}$ that contain the point P, and the second case is for the remaining $\triangle_{k's}$.

Lemma 3.1.1. *Let P be a node point in $\triangle_k$ for some k. Then*

$$\begin{aligned} &\int_Q \frac{f(m_k(s,t))}{|P - m_k(s,t)|} |D_s m_k(s,t) \times D_t m_k(s,t)| dQ \\ -&\int_Q \frac{f_n(\tilde{m}_k(s,t))}{|P - \tilde{m}_k(s,t)|} |D_s\tilde{m}_k(s,t) \times D_t\tilde{m}_k(s,t)| dQ = O(\hat{\delta}_k^3) \end{aligned}$$

where $\hat{\delta}_k$ is the diameter of $\hat{\triangle}_k$.

Proof. There are two cases. The first case is that P is a vertex in some $\hat{\triangle}_k$, and the second case is that P is a midpoint of a side of $\hat{\triangle}_k$.

Begin with the first case and without loss of generality, assume that

$$P = m_k(0,0) = \tilde{m}_k(0,0) = (p_1, p_2, p_3).$$

Before proving the theorem, we show that

$$\int_Q \frac{1}{|P - m_k(s,t)|} dQ = O(\hat{\delta}_k^{-1}).$$

Compute

$$\begin{aligned}
&\int_Q \frac{1}{|P - m_k(s,t)|} dQ \\
&\qquad = \int_Q \frac{1}{[(P_1 - x^1(s,t))^2 + (P_2 - x^2(s,t))^2 + (P_3 - x^3(s,t))^2]^{1/2}} dQ \\
&\qquad = \int_Q [(sx_s^1(0,0) + tx_t^1(0,0))^2 + (sx_s^2(0,0) + tx_t^2(0,0))^2 \\
&\qquad + (sx_s^3(0,0) + tx_t^3(0,0))^2 + O(\hat{\delta}_k^3)]^{-1/2} dQ
\end{aligned}$$

After integrating the dominant part of the above equation by using polar coordinates in the st-plane about $(0,0)$, we obtain

$$\int_Q \frac{1}{|P - m_k(s,t)|} dQ = O(\hat{\delta}_k^{-1}).$$

Now, we break the error analysis into three parts.

$$\begin{aligned}
&\int_Q \frac{f(m_k(s,t))}{|P - m_k(s,t)|} |D_s m_k(s,t) \times D_t m_k(s,t)| dQ \\
&- \int_Q \frac{f_n(\tilde{m}_k(s,t))}{|P - \tilde{m}_k(s,t)|} |D_s \tilde{m}_k(s,t) \times D_t \tilde{m}_k(s,t)| dQ \\
&= E_1 + E_2 + E_3
\end{aligned}$$

with

$$E_1 = \int_Q \frac{f(m_k(s,t)) - f_n(\tilde{m}_k(s,t))}{|P - m_k(s,t)|} |D_s m_k(s,t) \times D_t m_k(s,t)| dQ \tag{3.33}$$

$$\begin{aligned}
E_2 = \int_Q \frac{f_n(\tilde{m}_k(s,t))}{|P - m_k(s,t)|} (&|D_s m_k(s,t) \times D_t m_k(s,t)| \qquad (3.34) \\
&- |D_s \tilde{m}_k(s,t) \times D_t \tilde{m}_k(s,t)|) dQ \qquad (3.35)
\end{aligned}$$

$$\begin{aligned}
E_3 = \int_Q \left[\frac{1}{|P - m_k(s,t)|} - \frac{1}{|P - \tilde{m}_k(s,t)|} \right] \cdot \qquad (3.36) \\
|D_s \tilde{m}_k(s,t) \times D_t \tilde{m}_k(s,t)| f_n(\tilde{m}_k(s,t)) dQ \qquad (3.37)
\end{aligned}$$

In equation (3.33),

$$|E_1| \leq O(\hat{\delta}_k^3) \cdot O(\hat{\delta}_k^2) \int_Q \frac{1}{|P - m_k(s,t)|} dQ \leq O(\hat{\delta}_k^5) \cdot O(\hat{\delta}_k^{-1}) = O(\hat{\delta}_k^4)$$

For the equation (3.34), we can easily see it has order three:

$$|E_1| \leq \max_{s,t\in Q} |(|D_s m_k(s,t) \times D_t m_k(s,t)| - |D_s \tilde{m}_k(s,t) \times D_t \tilde{m}_k(s,t)|)| \cdot \int_Q \frac{f_n(\tilde{m}_k(s,t))}{|P - m_k(s,t)|} dQ = O(\hat{\delta}_k^4) \cdot O(\hat{\delta}_k^{-1}) = O(\hat{\delta}_k^3)$$

For E_3, expand each x^i about $(0,0)$,and then integrate it over Q. With a very lengthy calculation, we can show that E_3 is of order three.

If P is a midpoint of a side of $\triangle_k$, we split Q into two triangles, Q_1 and Q_2, and we put the singular point at a vertex in each of the new triangles; see figure 3.1 for the case with $P = m_k(\rho_4)$. We apply an affine change of variables, to move again to an integral over Q. Applying the first case to these two subtriangles, we again can show the error is of order three. Thus, the error contributed by the integral over $\triangle_k$, which contains P, is always of order three, no matter whether P is a vertex or a midpoint of a side of $\triangle_k$.

□

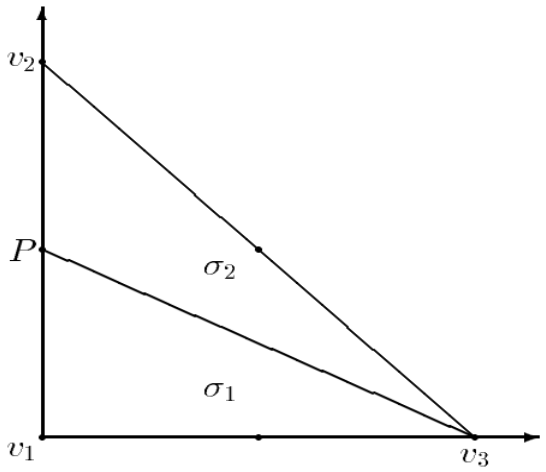


Figure 3.1: Splitting triangles

In the next lemma, we examine the errors from integrating over those triangles $\triangle_k$ which do not contain P.

Lemma 3.1.2. *Let P be a node point, and consider all $\triangle_k$ for which $P \notin \triangle_k$. Then*

$$\sum_k \int_Q \frac{f(m_k(s,t))}{|P - m_k(s,t)|} |D_s m_k(s,t) \times D_t m_k(s,t)| dQ - \sum_k \int_Q \frac{f_n(\tilde{m}_k(s,t))}{|P - \tilde{m}_k(s,t)|} |D_s \tilde{m}_k(s,t) \times D_t \tilde{m}_k(s,t)| dQ = O(\hat{\delta}_k^3) \tag{3.38}$$

where $\hat{\delta}_k$ is the diameter of $\triangle_k$.

Proof. Since $P \notin \triangle_k$, we can treat the function $1/|P - m_k(s,t)|$ as a smooth function. All results from Lemma 3.1.1, can be applied with slight changes. Let (3.38) be decomposed as $E_1 + ... + E_5$ where

$$\begin{aligned}
E_1 =& \sum_k \int_Q \frac{f(m_k(s,t))[|D_s m_k(s,t) \times D_t m_k(s,t)| - |D_s \tilde{m}_k(s,t) \times D_t \tilde{m}_k(s,t)|]}{|P - m_k(s,t)|} dQ \\
E_2 =& \sum_k \int_Q \frac{[f(m_k(s,t)) - f_n(\tilde{m}_k(s,t))]|D_s \tilde{m}_k(s,t) \times D_t \tilde{m}_k(s,t)|}{|P - m_k(s,t)|} dQ \\
& - \sum_k \int_Q \frac{[f(m_k(s,t)) - f_n(\tilde{m}_k(s,t))]|D_s m_k(s,t) \times D_t m_k(s,t)|}{|P - m_k(s,t)|} dQ \\
E_3 =& \sum_k \int_Q \frac{[f(m_k(s,t)) - f_n(\tilde{m}_k(s,t))]|D_s m_k(s,t) \times D_t m_k(s,t)|}{|P - m_k(s,t)|} dQ \\
E_4 =& \sum_k \int_Q \left[\frac{f(m_k(s,t))}{|P - m_k(s,t)|} - \frac{f_n(\tilde{m}_k(s,t))}{|P - \tilde{m}_k(s,t)|} \right] |D_s \tilde{m}_k(s,t) \times D_t \tilde{m}_k(s,t)| ds dt \\
& - \sum_k \int_Q \left[\frac{f(m_k(s,t))}{|P - m_k(s,t)|} - \frac{f_n(\tilde{m}_k(s,t))}{|P - \tilde{m}_k(s,t)|} \right] |D_s m_k(s,t) \times D_t m_k(s,t)| ds dt \\
E_5 =& \sum_k \int_Q \left[\frac{f(m_k(s,t))}{|P - m_k(s,t)|} - \frac{f_n(\tilde{m}_k(s,t))}{|P - \tilde{m}_k(s,t)|} \right] |D_s m_k(s,t) \times D_t m_k(s,t)| dQ
\end{aligned}$$

The integrand of E_1 is $O(\hat{\delta}^6/d_k^3) + O(\hat{\delta}^5/d_k^2)$. Using the calculation, we had in Theorem 3.1.1, we get that E_1 is of order three.

For E_2,

$$\begin{aligned}
\int_Q \frac{(f(m_k(s,t)) - f_n(\tilde{m}_k(s,t)))}{|P - m_k(s,t)|} \{|D_s m_k(s,t) \times D_t m_k(s,t)| - |D_s \tilde{m}_k(s,t) \times D_t \tilde{m}_k(s,t)|\} dQ \\
= \sum_k O(\hat{\delta}^3) \cdot O(\hat{\delta}^4) \cdot \int_Q \frac{1}{|P - m_k(s,t)|} dQ = O(\hat{\delta}^7) \cdot \frac{1}{d_k}
\end{aligned}$$

for every k where $P \notin \triangle_k$. Adding errors from each triangle, we have that E_2 is $O(\hat{\delta}^5)$.

For E_3, we have the error from

$$\int_Q \frac{(f(m_k(s,t)) - f_n(\tilde{m}_k(s,t)))}{|P - m_k(s,t)|} |D_s m_k(s,t) \times D_t m_k(s,t)|$$

is $O(\hat{\delta}^6/d_k)$ for every triangle. Again, following the argument in Theorem 3.1.1, E_3 is $O(\hat{\delta}^4)$.

Analyzing E_4, we have

$$\frac{1}{|P - m_k(s,t)|} - \frac{1}{|P - \tilde{m}_k(s,t)|} = O(\frac{\hat{\delta}^3}{d_k^2}). \tag{3.39}$$

and

$$\int_Q \left[\frac{1}{|P-m_k(s,t)|}-\frac{1}{|P-\tilde{m}_k(s,t)|}\right] f_n(m_k(s,t))|D_s m_k(s,t)\times D_t m_k(s,t)|-$$
$$\int_Q \left[\frac{1}{|P-m_k(s,t)|}-\frac{1}{|P-\tilde{m}_k(s,t)|}\right] f_n(m_k(s,t))|D_s \tilde{m}_k(s,t)\times D_t \tilde{m}_k(s,t)|$$
$$= O(\frac{\hat{\delta}^7}{d_k^2})$$

for each $\triangle_k$. After adding up errors $E_4 = O(\hat{\delta}^5 ln\hat{\delta})$.

For E_5, each triangle give us an error of $O(\hat{\delta}^5/d_k^2)$. When adding errors together, cancellation happens at every symmetric pair of triangles and errors become $O(\hat{\delta}^6/d_k^3)$. Thus, as we discussed in computing E_1, E_5 is $O(\hat{\delta}^3)$. After going through $E_1 - E_5$, the global error for the single layer integral, in which $P - m_k(s,t)$ is nonzero for every k, is $O(\hat{\delta}^3)$. This result is uniform as P ranges over the node points of the triangulation. □

3.1.4 Using the approximate surface

When using the approximate surface $\partial\tilde{\Omega}_n$, the linear system for (2.44) for the Dirichlet problem becomes

$$2\pi\tilde{\rho}_n(v_i) + [2\pi - \Gamma(v_i)]\tilde{\rho}_n(v_i) +$$
$$\sum_{k=1}^{n}\sum_{j=1}^{6}\tilde{\rho}_n(v_{j,k})\int_Q \ell_{j,k}(s,t)\frac{(D_s\tilde{m}_k(s,t)\times D_t\tilde{m}_k(s,t))\cdot(v_i-\tilde{m}_k(s,t))}{|v_i-\tilde{m}_k(s,t)|^3}dQ$$
$$= f(v_i), \quad i = 1,...,n. \tag{3.40}$$

For a smooth surface $\partial\Omega$, we would expect to use $\Gamma_n(P) = \Gamma(P) = 2\pi$, thus simplifying the above system.

Using the identity

$$\Gamma(P) = \int_{\partial\Omega}\frac{\partial}{\partial n_\sigma}\left[\frac{1}{|P-\sigma|}\right]d\partial\Omega_\sigma \tag{3.41}$$

We define

$$\Gamma_n(P) = \sum_{k=1}^{n}\sum_{j=1}^{6}\int_Q \ell_{j,k}(s,t)\frac{(D_s\tilde{m}_k(s,t)\times D_t\tilde{m}_k(s,t))}{|v_i-\tilde{m}_k(s,t)|^3}dQ, \quad P\in\partial\Omega \tag{3.42}$$

The linear system (3.40) is denoted here by

$$(2\pi + \tilde{K}_n)\tilde{\rho}_n = g_n \tag{3.43}$$

with

$$\tilde{\rho}_{n,i} \equiv \tilde{\rho}_n(v_i), \quad g_{n,i} \equiv f_n(v_i), \quad i = 1,...,n_v.$$

When solving the integral equation (2.37) for the exterior Neumann problem, we also approximate the right-hand side, now a single layer integral, using (3.32). In the above frame work, and consistent with earlier notation, we write

$$\tilde{g}_{n,i} = \sum_{k=1}^{n}\sum_{j=1}^{6}\int_Q \ell_{j,k}(s,t)\frac{f(\tilde{m}_k(s,t))|D_s\tilde{m}_k(s,t)\times D_t\tilde{m}_k(s,t)|}{|v_i-\tilde{m}_k(s,t)|^3}dQ \tag{3.44}$$

for $i = 1,...,n_v$.

3.1.5 Numerical examples

The collocation method, §3.1.2, with the use of the quadratic isoparametric interpolation of the surface $\partial\Omega$ was implemented with a package of programs which work for a wide variety of smooth and piecewise smooth surfaces.

There are two crucial aspects of the practical implementation that were not discussed in §3.1.2: the calculation of the collocation integrals and the solution of the large linear systems that often arise from the discretization. Here, we restrict our attention to the numerical integration of the collocation integrals.

The numerical integration of the collocation integrals is by far the most time-consuming part in solving the boundary integral equation. One must have integrals that are sufficiently accurate, to match the accuracy of the "pure" collocation solution ρ_n. But it is very wasteful of computing time to calculate these integrals with more accuracy than is needed.

The collocation integrals in the matrix of coefficients of (3.40) are given by

$$\int_Q \kappa(v_i, \tilde{m}_k(s,t))\ell_j(s,t)|D_s\tilde{m}_k(s,t) \times D_t\tilde{m}_k(s,t)|dQ \tag{3.45}$$

In this, $i = 1, ..., n_v, \quad j = 1, ..., 6 \quad and \quad k = 1, ..., n$; and $\kappa(P, q)$ denotes the kernel function for the double layer integral operator. For the exterior Neumann problem, we also need to evaluate the corresponding single layer integrals

$$\int_Q \frac{f(\tilde{m}_k(s,t))}{|v_i - \tilde{m}_k(s,t)|}|D_s\tilde{m}_k(s,t) \times D_t\tilde{m}_k(s,t)|dQ \tag{3.46}$$

Recall from §1.5 that $\tilde{m} : \sigma \to \tilde{\triangle}_k$ is a one-to-one and onto parametrization of the triangle approximating $\triangle_k$. We consider two cases in evaluating (3.45), depending on whether v_i is inside or outside of $\triangle_k$.

If $v_i \in \triangle_k$, then $\kappa(P, q)$ is singular. We use a change of variable, where we noted that it removed all singular behavior in both the double layer integrals (3.45), and the corresponding single layer integrals. This paper carries out a detailed analysis of the method and an extension of the transformation to other singular integration problems arising in solving boundary integral equation.

Assuming the collocation node $v_i = m_k(0, 0)$, introduce the change of variables

$$s = (1 - y)x \qquad t = xy \qquad 0 \le x, y \le 1$$

With this, the new integrands in (3.45) and (3.46) will be well-behaved. For $\triangle_k$ a surface with C^m differentiability, $m \ge 3$, the transformed integrand for (3.45) will be C^{m-2} times differentiable; and if the density $f(\sigma)$ is m-times differentiable on $\triangle_k$, the transformed integrand for (3.46) will be C^{m-1} times differentiable. We then evaluate the transformed integral using a product Gaussian quadrature formula with n_g nodes in both the x and y coordinates (thus using n_g integration nodes).

The above change of variables is used to remove the singularity in the integration over each triangle. For cases of $n = 512$ faces, we have found $n_g = 10$ to be very sufficient to preserve the accuracy of the collocation solution; and smaller values of n_g are sufficient for smaller values of n. Note that the number of integrals (3.45) with $v_i \in \triangle_k$, for some i and k, is of order n_v whereas the total number of integrals to

be computed is of order n_v^2. Thus when considering operation counts, the singular integrals are the less important of the integrals (3.45) to be considered.

For $v_i \notin \triangle_k$, the integrand in (3.45) is analytic; but it is increasingly peaked as the distance between v_i and $\triangle_k$ decreases. A method to evaluate integrals such as (3.45) and (3.46) over σ is based on (1.34). Let an integer parameter $n_d \geq 0$ be given. If $v_i \notin \triangle_k$ and

$$dist(v_i, \triangle_k) \leq \delta_n,$$

where δ_n is the mesh size of $\triangle_k$ as defined in (1.26), then integrate (3.45) using (1.34) with n_d levels of subdivision of σ [thus dividing σ into 4^{n_d} subtriangles, with (1.34) applied to the integral over each of the corresponding subintegrals]. If $v_i \in \triangle_k$ and

$$\delta_n < dist(v_i, \triangle_k) \leq 2\delta_n,$$

then integrate (3.45) using (1.34) with $max\{n_d - 1, 0\}$ levels of subdivision of σ. If $v_i \notin \triangle_k$ and

$$2\delta_n < dist(v_i, \triangle_k) \leq 3\delta_n,$$

then integrate (3.45) using (1.34) with $max\{n_d - 2, 0\}$ levels of subdivision of σ. Continue with this in the obvious way.

We have found that as n is increased to $4n$, then raising n_d to $n_d + 1$ is sufficient to integrate (3.45) and (3.46) with the needed accuracy. For all of our examples, for both smooth and piecewise smooth surfaces, the largest value of n_d that we have needed to use has been $n_d = 2$. We have used larger values of n_d in our experiments to check the accuracy when using the lower values of n_d. Nonetheless, the integrations of (3.45) and (3.46) are still the most expensive parts of our computation, far exceeding the cost of solving the linear system (3.22) for the discretized boundary integral equation.

3.2 Numerical solutions using Matlab

It is clear from the previous discussion that a method such as the BEM would best be implemented with a tool that can perform matrix computations and numerical integration efficiently and produce graphical output. Matlab is one such tool that can perform all these tasks with a simple code.
For simplicity and convenience, we shall solve the boundary integral equations using constant elements.
It is very useful to divide the Matlab program into three stages:

3.2.1 First stage:

During this stage, the program creates a uniform discretisation on the boundary of the domain. It also computes the coordinates of all the midpoints and the lengths of each boundary element, as well as the unit normal vector to each element.
This information will be needed for the next stage.

3.2.2 Second stage:

During this stage, the matrix **A** and column vector **b** are constructed, so that the system for **z** can be solved to find the approximate solutions to the unknown boundary values. There are two integrals to be computed. These are,

$$\int_{\partial\Omega_k} u(x,y)d\sigma \qquad and \qquad \int_{\partial\Omega_k} \frac{\partial u(x,y)}{\partial n_\sigma} d\sigma.$$

where $k = m$, the integrals can be evaluated analytically to give,

$$\int_{\partial\Omega_k} u(x,y)d\sigma = \frac{L_k}{2\pi}\left[ln(\frac{L_k}{2}) - 1\right] \tag{3.47}$$

where L_k, represents the length of the element $\partial\Omega_k$.

$$\int_{\partial\Omega_k} \frac{\partial u(x,y)}{\partial n_\sigma} d\sigma = 0. \tag{3.48}$$

When $k \neq m$, we can evaluate the integrals using numerical methods such as Gauss quadrature because the analytic calculation of the integrals is more complicated. Then, we use the Gauss elimination method to solve the final algebraic system of equations.

3.2.3 Third stage:

In this stage, we essentially use (3.9) to calculate the solutions at the interior points. We also calculate the exact solution and the difference between the exact and approximate solutions. We can draw and generate a surface plot for the exact and approximate solutions and the difference between them, also the relative error is calculated as

$$r(J) = \frac{\max_{1\leq j\leq J} |u_{approx}(x^j) - u_{exact}(x^j)|}{\max_{1\leq j\leq J} |u_{exact}(x^j)|} \tag{3.49}$$

where u_{approx}, u_{exact} are the numerical and exact solutions, respectively.

3.3 Example

In this section, we shall examine test example to assess the performance of the BEM Matlab program. In order to verify the convergence of the method, we applied the BEM to some test problems with different domains as discussed below. The surface plots of the numerical solutions were obtained with the most refined mesh in each example. The graph of relative error has the number of nodes on the horizontal axis and the relative error on the vertical axis.
Let us solve the following problem, see figure 3.2

$$\Delta u = 0, \quad for \quad 0 < x_1 < 1, \quad 0 < x_2 < 1,$$

with the boundary conditions:

$$\begin{aligned} u = 0 \ when \ \ x_1 = 0, \ for \ \ 0 < x_2 < 1 \\ u = cos(\pi x_2) \ when \ \ x_1 = 1, \ for \ \ 0 < x_2 < 1 \\ q = \frac{\partial u}{\partial n} = 0, \ when \ \ x_2 = 0 \ \ and \ \ \ x_2 = 1, \ \ for \ \ 0 < x_1 < 1. \end{aligned}$$

The exact solution for this problem is

$$u_{exact}(x_1, x_2) = \frac{sinh(\pi x_1)cos(\pi x_2)}{sinh(\pi)}$$

A surface plot of the exact and approximate solutions and the difference between them is shown in Figure 3.3.

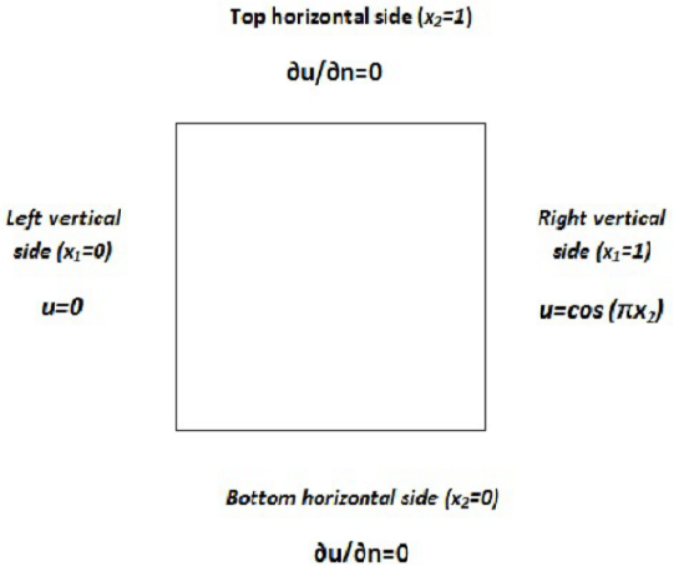


Figure 3.2: Domain plot

x_1	32 elements	64 elements	128 elements	256 elements	Exact
0.10	-0.022383	-0.022375	-0.022373	-0.022372	-0.022371
0.20	-0.047005	-0.046981	-0.046973	-0.046970	-0.046969
0.30	-0.076310	-0.076263	-0.076248	-0.076243	-0.076241
0.40	-0.113207	-0.113134	-0.113110	-0.113103	-0.113099
0.50	-0.161361	-0.161261	-0.161228	-0.161217	-0.161212
0.60	-0.225558	-0.225431	-0.225388	-0.225373	-0.225367
0.70	-0.312171	-0.312024	-0.311973	-0.311956	-0.311948
0.80	-0.429788	-0.429648	-0.429597	-0.429579	-0.429571
0.90	-0.590047	-0.589986	-0.589958	-0.589947	-0.589941

Table 3.1. Computed temperatures along line $x_2 = 0.8$

Table 3.1 lists the computed values of $u(x)$ along the line $x_2 = 0.8$. The values at interior points are obtained by using 32 to 256 boundary elements, compared with the exact solution. The improvement in the accuracy of the numerical results can be seen clearly when the number of boundary elements is increased from 32 to 256.

Number of elements	Relative error
32	0.0437
64	0.0145
128	0.0030
256	0.0007

Table 3.2. Convergence of the approximate solution

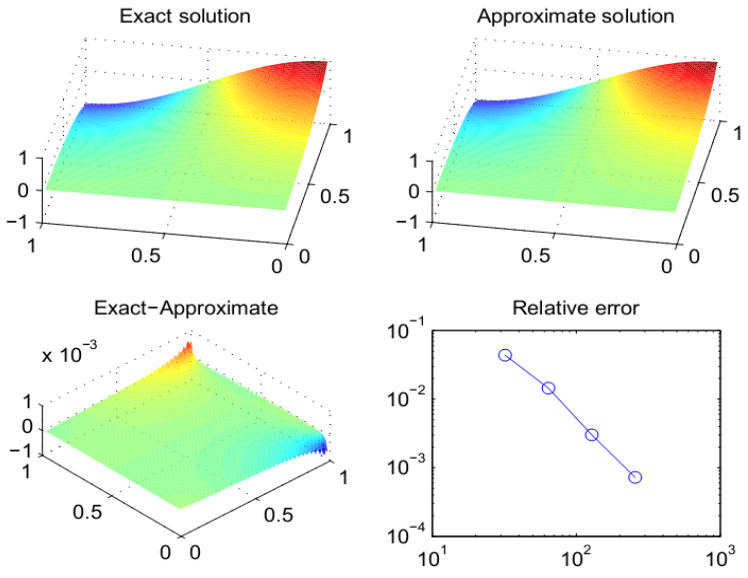


Figure 3.3: Surface plot of solution

Table 3.2 provides the relative error as the number of boundary elements is increased. It can be clearly seen that the relative error is reduced by increasing the number of boundary elements, demonstrating the convergence of the solution.

Conclusion

The BIE formulation was presented for solving the three-dimensional Boundary value problem with Dirichlet and Neumann boundary conditions. Numerical solutions for BIE on surfaces $\partial\Omega$ in R^3 lead to computationally expensive procedures, and a great deal of the research for such BIE has looked at the efficient numerical evaluation of integrals, which use iterative methods for solving the associated linear systems.

The numerical solution of the integral equation would be presented by using constant boundary elements, because at this stage understanding the numerical implementation of the BEM overrides the need to incorporate more advanced numerical techniques, which improve the accuracy and efficiency of the BEM. The functions of BEM was to discretize the boundary into a finite number of segments, which were called boundary elements.

In constant elements the boundary segment was approximated by a straight line, which connects its extreme points. The node was placed at the mid-point of the straight line and the boundary quantity was assumed to be constant along the element and equal to its value at the nodal point. Constant elements depict the boundary quantities discontinuously from element to element.

The boundary $\partial\Omega$ was discretized into N constant elements, which were numbered in the counter-clockwise sense. The values of the boundary quantity u and its normal derivative $\partial u/\partial n$ were assumed constant over each element and equal to their value at the mid-point of the element.

Appendix

Main program for example 1

This program is taken from [9, p.203-207]

```
%%%
```

```
%%%
```

```
%%%
```

```
%%%
```

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