ADDIS ABABA UNIVERSITY

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On Counting Spanning Trees of the Graph

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Abstract

In this paper, we will observe how to find the spanning trees of a graph and the methods that we use to calculate. The methods that we use for calculating the spanning tree of the graph are deletion and contraction, matrix tree theorem and combinatorial interpretation of matrix tree theorem. In the matrix tree theorem we will explore an interesting relationship between linear algebra and graphs. In combinatorial interpretation we develop a new approach by constructing the polynomial that enumerates the spanning trees of the graph according to degrees of all vertices. The matrix tree theorem does not give the final answer to all problems concerning enumeration of trees because finding the determinant of a matrix is a complicated task for complicated graphs. So, this paper is useful to find the number of spanning trees of the complicated graph simply.

Table of Contents

Part I

Introduction

Introduction

In a connected graph G, it is easy to find a tree that contains all the vertices and some edges of G; such a subgraph is called a spanning tree. How many spanning trees does that graph contain? That is what Gustav Robert Kirchhoff was wondering. Kirchhoff was a German physicist, who contributed to the fundamental understanding of electrical circuits, spectroscopy and radiation. Kirchhoff found an answer to this question, which is formulated in the Matrix Tree Theorem. By means of this theorem we find the number of spanning trees of a graph.

A spanning tree T in a graph G is a connected subgraph in G that contains all vertices of G and has no cycle.

In the real world knowing the spanning trees of a graph is very essential. For example, a telecommunications company has n ground stations, to be linked with its two orbiting satellites. In how many ways can we link the ground stations to the satellites so that all parts of the resulting system can communicate with one another efficiently? These types of problems are solved by using spanning trees of a graph

Preliminaries

Some Definitions

- A graph G is an ordered triple (V(G), E(G), Ψ_G) consisting of a nonempty set V(G) of vertices, a set E(G), disjoint from V(G), of edges, and an incidence function Ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G.
- Each edge e has either one or two vertices as endpoints; an edge with only one endpoint is called a loop.
- Two vertices are adjacent if they are connected by an edge and the edge is incident to the vertices.
- Two edges e, e' are said to be parallel edges if they have the same pair of vertices.
- A graph with no edges is called an empty graph.
- A graph is called a simple graph if it has no parallel edges and loops.
- The number of edges incidence to the vertex v in the graph G is called the degree of the vertex v, denoted by $d_G(v)$.
- A vertex of degree 0 in G is called isolated vertex and a vertex of degree 1 in G is called a pendant vertex.
- A graph is connected if there is a path connecting every pair of vertices.
- A graph H is said to be a subgraph of a graph G if V(H)⊆ V(G) and E(H)⊆ E(G).
- A cycle is a closed path
- Tree is a connected graph with no cycle.
- A spanning tree of a graph G is a connected subgraph of G that contains all the vertices of G and has no cycle.

Part II

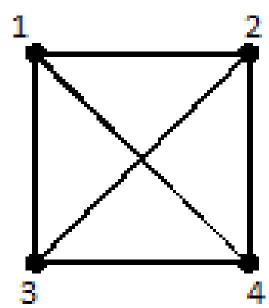
Methods of Counting of Spanning Trees of a Graph

Chapter One

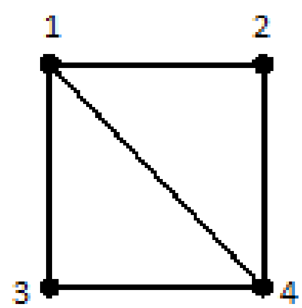
1. Deletion and Contraction

Let $G = (V, E)$ be a connected graph and $e = vw \in E$. The deletion of a graph G, denoted by $G - e$ is just the graph $(V, E - \{e\})$. The contraction of the graph G, denoted by $G \cdot e$ is obtained from $G - e$ by identifying v and w (or "fusing" the two vertices together). For contraction to make sense, we usually require that e not be a loop.

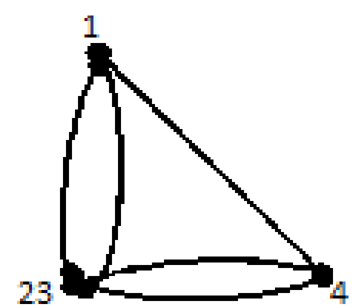
For example, if $G = K_4$ and $e = 23$, then the deletion and contraction are as follows:



$Graph\ G$



$G - e$



$G \cdot e$

Definition: A graph is connected if there is a connecting every pair of vertices. A graph that is not connected can be divided into connected components. The component of a graph G is denoted by $\omega(G)$ is a collection of disjoint connected subgraphs.

Remark: If e is an edge of the connected graph G, then

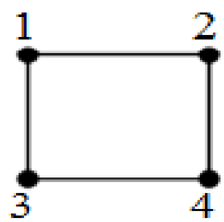
1. $|V(G - e)| = |V(G)|, |E(G - e)| = |E(G)| - 1$ and $\omega(G - e) \geq \omega(G)$
2. $|V(G \cdot e)| = |V(G)| - 1, |E(G \cdot e)| = |E(G)| - 1$ and $\omega(G \cdot e) = \omega(G)$

Theorem: If e is an edge of the connected graph G, then $t(G) = t(G - e) + t(G \cdot e)$ where $t(G)$ denotes the number of spanning trees of a connected graph G.

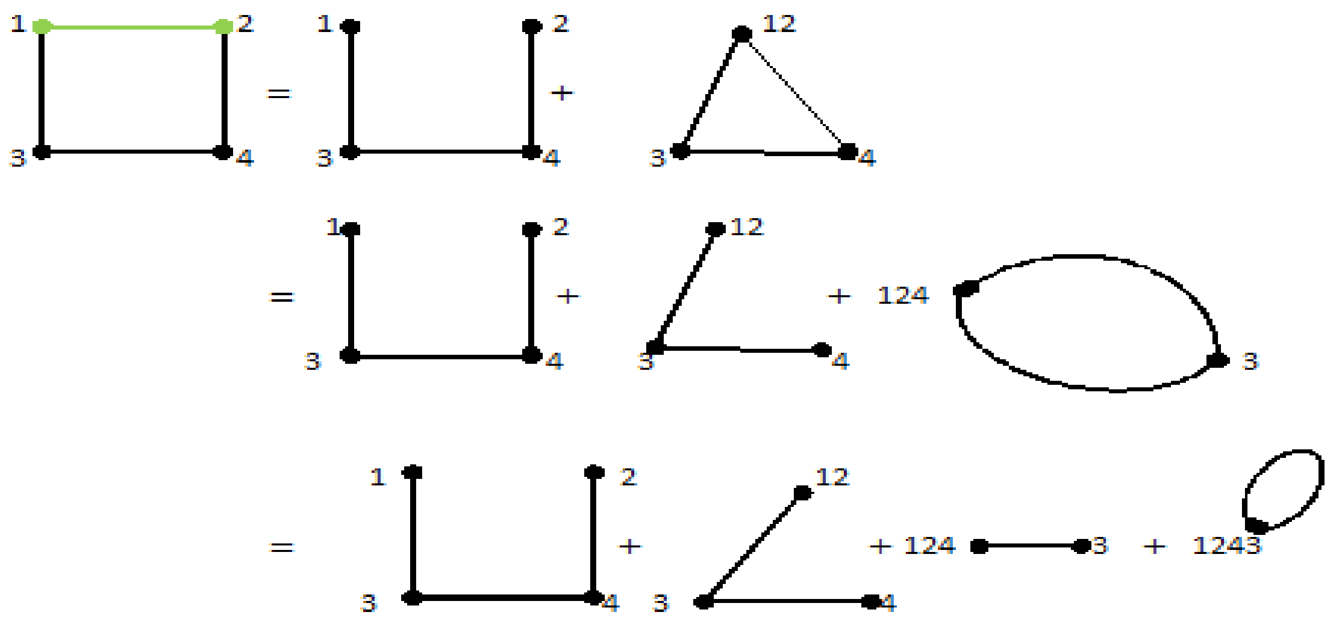
Proof: Every spanning trees of G that does not contain an edge e is also a spanning trees of $G - e$, and conversely, $t(G - e)$ is the number of spanning trees of G that do not contain e.

Now to each spanning T of G that contain e, there corresponds a spanning trees $T.e$ of $G \cdot e$. This correspondence is clearly a bijection. Therefore, $t(G \cdot e)$ is exactly the number of spanning trees of G that contain e. It follows that $t(G) = t(G - e) + t(G \cdot e)$

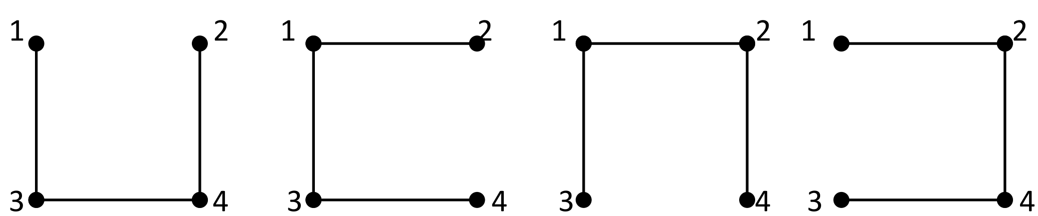
Example: Find the number of spanning trees of the following graph using deletion and contraction method.



$$t(G) = \quad t(G - e) \quad + \quad t(G \cdot e)$$



Therefore, the number of spanning trees of the above graph is 4. The spanning tree of the above graph is listed below:



Remark: The limitation of deletion and contraction method is not suitable for large graphs.

Chapter Two

2. Matrix Tree Theorem

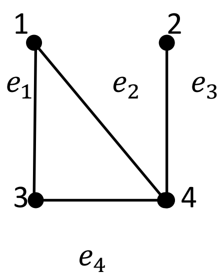
Before we can consider how to use linear algebra to find the number of spanning trees of a graph, we need to discuss some of the matrices that can be associated to a graph. And also we introduce the relationship between graphs and matrices and some properties of the graphs. Behind this we will prove matrix tree theorem algebraically.

2.1 Graphs and Matrices

Definition: - *let* $G = (V, E)$ *be a graph with vertex set* $V = [n]$*. Then the adjacency matrix of G, denoted by* $A(G) = (a_{ij})_{n\times n}$, *where*

$$a_{ij} = \begin{cases} 1, & if\ i\ is\ adjacent\ to\ j\ in\ G \\ 0, & otherwise \end{cases}$$

Example: Let G is a graph shown below



$Figure\ 1$

The adjacency matrix $A(G)$ of this graph is

$$A(G) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

The adjacency matrix of a graph gives everything that we need to know about the graph, since it completely specifies which pairs of vertices are adjacent and which are not.

Definition: - For any graph G with vertex set $V = [n]$, the diagonal matrix of the graph G, denoted by $D(G) = (d_{ij})_{n\times n}$, where

$$d_{ij} = \begin{cases} \deg(i), & i = j \\ 0, & otherwise \end{cases}$$

Definition: - Laplacian matrix is given by

$$L(G) = D(G) - A(G)$$

Example: For the graph in figure 1

$$L(G) = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

We use the Laplacian matrix for computing the number of spanning trees of a graph G.

Definition: - *A directed graph is a graph in which each edge* (i,j) *has an orientation, that is the edge* (i,j) *goes from* $vertex\ i$ *to* $vertex\ j$.

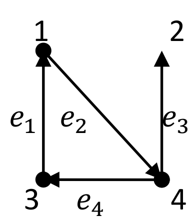
Definition: An orientation of graph is simply a directed graph in which only one of the edges (i,j) *or* (j,i) *can occur.*

Definition: - *let* G *be a connected graph with vertex set* $[n]$ *and edge set* $\{e_1, e_2, \dots, e_m\}$*. We orient each edge randomly. The incidence matrix of* G *is the* $n \times m$ *matrix, with rows indexed by the vertices and columns indexed by the edges, denoted by* $Q(G) = (q_{ij})$*, where*

$$q_{ij} = \begin{cases} 1 \ \ if\ e_j\ goes\ to\ vertex\ i; \quad e_j = (j,i) \\ -1\ \ if\ e_j\ goes\ from\ vertex\ i; e_j = (i,j) \\ 0 \quad otherwise. \end{cases}$$

Definition: - *The reduced incidence matrix* $\widetilde{Q(G)}$ *of the graph G is a matrix constructed by deleting any of the rows from the incidence matrix* $Q(G)$.

Example: In figure 2 we show an arbitrary orientation of the graph of figure 1.



$Figure\ 2$

The incidence matrix $Q(G)$ corresponding to this orientation is

$$Q(G) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix} \qquad Q(G)^T = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$Q(G)Q(G)^T = L(G) = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

We obtain the reduced incidence matrix $\widetilde{Q(G)}$ of G by deleting the 3^{rd} row of $Q(G)$

$$\widetilde{Q(G)} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix}$$

Lemma 2.1: For any orientation of a graph $G = (V, E)$

$$L(G) = Q(G)Q(G)^T$$

where $Q(G)^T$ is the transpose of $Q(G)$.

Proof: we consider the entries q_{ij} of $Q(G)Q(G)^T$ in to two cases.

<u>Case1</u>: For the entries q_{ii}; we note that this is the dot product of row i of $Q(G)$ and column i of $Q(G)^T$, by definition of transpose; row i of $Q(G)$ is equal to column i of $Q(G)^T$. Thus, when computing the dot product, all the terms are either of the form $1 \cdot 1, (-1) \cdot (-1), or\ 0 \cdot 0$. Thus, for every edge incident with vertex i that is the degree of the vertex i, we get a contribution of 1 to the sum in each dot product. Therefore, $q_{ii} = \deg(i)\ for\ all\ i$

<u>Case 2</u>: The entries $q_{ij}\ for\ i \neq j$. Here we take the dot product of row i of $Q(G)$ with column j of $Q(G)^T$, which is the same as row j of $Q(G)$. These rows are different, and the product of the k^{th} entry of these two rows is zero if edge e_k does not have $i\ and\ j$, as its ends.

On the other hand, if e_k does have i and j, as its ends, then the product of the k^{th} entry is -1. Thus

$$q_{ij} = \begin{cases} -1 \; if \; and \; only \; if \; ij \in E(G) \; and \; i \neq j \\ 0 \; otherwise \end{cases}$$

From here

$q_{ii} = \deg(i) \; for \; all \; i$ form a diagonal matrix in which each diagonal element has a value of the degree of vertex i.

i.e. $D(G) = (q_{ii}) \; for \; all \; i \in V(G)$

$A(G) = (q_{ij})$ where

$q_{ij} = \begin{cases} -1 \; if \; and \; only \; if \; ij \in E(G) \; and \; i \neq j \\ 0 \; otherwise \end{cases}$

Therefore

$$Q(G)Q(G)^T = L(G) = D(G) - A(G)$$

2.2 Some properties of graphs

Before we prove matrix tree theorem, we prove some properties of $Q(G)$ and $\widetilde{Q(G)}$.

2.2.1 Rank

Definition: - This number of linearly independent rows or columns is called the **rank** of A.

The procedure for computing the rank of the matrix A is as follows

Step 1 use elementary row operations, transform matrix A in to matrix B in reduced row echelon form.

Step 2 Rank of A is equal to the number of nonzero rows of B.

Example: Find rank of $Q(G)$ from example 3

$$Q(G) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix} \xrightarrow{R_2 = R_3} \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix} \xrightarrow{R_2 = R_4} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_4 = R_1 +} $$

$$R_4 \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \xrightarrow{R_4 = R_2 + R_4} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \xrightarrow{R_4 = R_3 + R_4} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, Rank of $Q(G)$= 3.

If the j^{th} edge goes from vertex i_1 to i_2, then $q_{i_1,j} = -1$ and $q_{i_2,j} = 1$ and the other column elements are zero. So every column of $Q(G)$ contains exactly one $+1$ and -1. Hence the sum of the rows is the zero-row, so a row of $Q(G)$is -1 times the sum of all the other rows. So this row is dependent of all the rows of $\widetilde{Q(G)}$.

Lemma 2.2: Let G be a graph with n vertices. Then G is connected if and only if the rank of its incidence matrix $Q(G)$ is $n-1$.

Proof:

(⇒)Suppose G is connected.

The incidence matrix $Q(G)$ of the graph G has n rows.

Let $r < n$. The sum of any r rows must contain at least one non-zero entry. This is because otherwise there would be no edge connecting any of these r $vertices$ to any vertices outside the set of r vertices, which would contradict the connectedness of G.

Because of this, r rows are linearly independent if $r < n$. The sum of n rows is a zero-row, because each column contains one $+1$, one -1 and the other is zero.
Therefore, $rank\big(Q(G)\big) = n - 1$.

(⇐)Suppose $rank\big(Q(G)\big) = n - 1$ and $r < n$,

when you add all r rows we get a nonzero row. So there are r vertices which are connected to the remaining $n - r$ vertices. Hence G is connected.

2.2.2 Determinants

Definition: A unimodular matrix M is a matrix with integer entries having a determinant $+1$ or -1.

Definition:- *An $n \times m$ matrix $Q(G)$ is said to be totally unimodular (TU) if every $k \times k$ submatrix has determinant equal to* $-1, 1$ or 0 $(1 \leq k \leq min\{n, m\})$.

Lemma 2.3: Let $Q(G)$ be a matrix with entries $-1, 1$ or 0 , such that each column contains at most one -1 and at most one $+1$. Then $Q(G)$ is totally unimodular (TU).

Proof: We prove by induction on k that every $k \times k$ submatrix of $Q(G)$ has the determinant. When $k = 1$, we have a 1×1 submatrix of $Q(G)$ with entries$-1, 1$ or 0. So the determinant is either $-1, 1$ or 0.

When $k \geq 2$, Assume that $det\big(Q_{k-1}(G)\big) = -1, 1$ or 0, where $Q_{k-1}(G)$ is a $(k - 1) \times (k - 1)$ submatrix of $Q(G)$. Let $Q_k(G)$ be a $k \times k$ submatrix of $Q(G)$.

- If $Q_k(G)$ has a column with exactly one non-zero entry, we can develop $\det(Q_k(G))$ with respect to that column. This gives that

  $$\det\left(Q_k(G)\right) = \pm\det\left(Q_{k-1}(G)\right) = -1, 1 \ or \ 0$$

- If $Q_k(G)$ has no column with exactly one non-zero entry, then the rows of $Q_k(G)$ add up to all-zero row. Then $det\left(Q_k(G)\right) = 0$

Therefore, $\det\left(Q_k(G)\right) = -1, 1 \ or \ 0$

Lemma 2.4: If B is a non-singular square submatrix of $Q(G)$, then the determinant of B is ± 1.

Proof: Suppose B is a non-singular square submatrix of $Q(G)$. In the matrix $Q(G)$, every column contains exactly one -1, exactly one $+1$ and the other entries are all 0. By $lemma$ 2.3 $Q(G)$ is totally unimodular. By definition of TU, $det(B) = -1, 1 \ or \ 0$. Since B is $non-singular$ $\det(B) \neq 0$. Therefore,$\det(B) = \pm 1$

2.2.3 Matrices and Trees

Lemma 3.3.1: If B is a square submatrix of order $n-1$ of $Q(G)$, then:
B is non-singular if and only if the edges corresponding to the columns of B determine a spanning tree of G.

Proof: Suppose H is a subgraph of G determined by B and B is of order $n-1$. We need to show that H is a spanning tree of G.

Here H contains $n-1$ edges because our matrix B is $(n-1) \times (n-1)$ square matrix. Hence H is a tree if and only if H is connected. But by $lemma$ 3.1.1, we have that H is connected if and only if $rank(B) = n-1$. But $rank(B) = n-1$ if and only if B is non-singular, because B is $(n-1) \times (n-1)$ matrix. Therefore B is non-singular if and only if H is a spanning tree of G.

2.3 Cauchy-Binet

Let R be a $p \times q$ matrix and S be a $q \times p$ matrix, where $p \leq q$. Then

$$\det(RS) = \sum \det(B) \cdot \det(C)$$

where the sum is over all $p \times p$ submatrices B of R and C of S. The numbers of columns deleted from R to get B are the same as the numbers of the rows deleted from S to get C.

Example: $R = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{pmatrix}$, $S = \begin{pmatrix} -1 & 3 \\ 2 & 0 \\ 4 & 5 \end{pmatrix}$

$$RS = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 2 & 0 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 15 & 18 \\ 0 & 6 \end{pmatrix};$$

$$\det(RS) = 15 \times 6 = 90$$

- Deleting the first column from R and the first row from S. we get

  $B = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 0 \\ 4 & 5 \end{pmatrix}$, $\det(B) = -3$ *and* $\det(C) = 10$ *then*

  $\det(B) det(C) = -30$

- Deleting the second column from R and the second row from S. we get

  $B = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 0 \\ 4 & 5 \end{pmatrix}$, $\det(B) = -6$ *and* $\det(C) = -17$ *then*

  $\det(B)\det(C) = 102$

- Deleting the third column from R and the third row from S. we get

  $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$, $\det(B) = -3$ *and* $\det(C) = -6$ *then*

  $\det(B) det(C) = 18$

  Therefore, $\det(RS) = \sum \det(B) . \det(C) = (-30) + 102 + 18 = 90$

Theorem :(Leibniz formula)

$$\mathrm{Det}(B) = \sum_{\sigma} sgn(\sigma) \prod_{i=1}^{n} b_{i\sigma(i)}$$

Proof:

$\mathrm{Det}(B) = \det(B_1, B_2, \dots, B_n)$

$$= \det\left(\begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{pmatrix}, \begin{pmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{pmatrix}, \dots, \begin{pmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{nn} \end{pmatrix}\right)$$

$$= det\left(\begin{pmatrix} b_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b_{21} \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b_{n1} \end{pmatrix}, \begin{pmatrix} b_{12} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b_{22} \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b_{n2} \end{pmatrix}, \dots, \begin{pmatrix} b_{1n} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b_{2n} \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b_{nn} \end{pmatrix}\right)$$

$= \det(b_{11}e_1 + b_{21}e_2 + \cdots + b_{n1}e_n, b_{12}e_1 + b_{22}e_2 + \cdots + b_{n2}e_n, \dots, b_{1n}e_1 + b_{2n}e_2 + \cdots + b_{nn}e_n)$

$$= det\left(\sum_{k_1=1}^{n} b_{k_1 1}e_{k_1}, \sum_{k_2=1}^{n} b_{k_2 2}e_{k_2}, \dots, \sum_{k_n=1}^{n} b_{k_n n}e_{k_n}\right)$$

$$= \sum_{k_1=1}^{n} b_{k_1 1}\, det\left(e_{k_1}, \sum_{k_2=1}^{n} b_{k_2 2}e_{k_2}, \dots, \sum_{k_n=1}^{n} b_{k_n n}e_{k_n}\right)$$

$$= \sum_{k_1=1}^{n} b_{k_1 1} \sum_{k_2=1}^{n} b_{k_2 2}\, det\left(e_{k_1}, e_{k_2}, \dots, \sum_{k_n=1}^{n} b_{k_n n}e_{k_n}\right)$$

$$= \sum_{k_1=1}^{n} b_{k_1 1} \sum_{k_2=1}^{n} b_{k_2 2} \cdots \sum_{k_n=1}^{n} b_{k_n n}\, det\big(e_{k_1}, e_{k_2}, \dots, e_{k_n}\big)$$

$$= \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \cdots \sum_{k_n=1}^{n} b_{k_1 1}\, b_{k_2 2} \cdots b_{k_n n} det\big(e_{k_1}, e_{k_2}, \dots, e_{k_n}\big)$$

$$= \sum_{\sigma} b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)} \det\big(e_{\sigma(1)}, e_{\sigma(2)}, \cdots, e_{\sigma(n)}\big)$$

$$= \sum_{\sigma} sgn(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)}$$

$$= \sum_{\sigma} sgn(\sigma) \prod_{i=1}^{n} b_{i\sigma(i)}$$

Note: We write the k^{th} column of matrix R as R_k in the following theorem

Theorem: (Cauchy-Binet) Let R be a $p \times q$ matrix and S be a $q \times p$ matrix, were $p \leq q$. Then

$$\det(RS) = \sum_{1 \leq k_1 < k_2 < \cdots < k_p \leq q} \det(R_{k_1}, R_{k_2}, \dots, R_{k_p}) \cdot \det(S_{k_1}^T, S_{k_2}^T, \dots, S_{k_p}^T)$$

Proof: The element in row i and column j of RS is $\sum_{k=1}^{q} r_{ik} \cdot s_{kj}$, so the j^{th} column is

$(RS)_j = \sum_{k=1}^{q} R_k s_{kj}$.

So:

$$\det(RS) = \det\big(RS_1, RS_2, \dots, RS_p\big)$$

$$= \det\left(\sum_{k_1=1}^{q} R_{k_1} s_{k_1 1}, \sum_{k_2=1}^{q} R_{k_2} s_{k_2 2}, \dots, \sum_{k_p=1}^{q} R_{k_p} s_{k_p p} \right)$$

$$= \sum_{k_1=1}^{q} s_{k_1 1} \det\left(\mathrm{R}_{\mathrm{k}_1}, \sum_{k_2=1}^{q} R_{k_2} s_{k_2 2}, \dots, \sum_{k_p=1}^{q} R_{k_p} s_{k_p p} \right)$$

$$= \sum_{k_1=1}^{q} \sum_{k_2=2}^{q} \dots \sum_{k_p=1}^{q} s_{k_1 1} s_{k_2 2} \dots s_{k_p p} \det(R_{k_1}, R_{k_2}, \dots, R_{k_p})$$

It is clear that $\det\left(R_{k_1}, R_{k_2}, \dots, R_{k_p}\right) = 0$ when $k_i = k_j for\ some\ i \neq j$ that means if two columns are the same. Hence, we only have to sum over all different $k_i's$. We get all choices by choosing p numbers $k_1, \dots, k_p\ from\ 1, \dots, q\ with\ k_1 < k_2 < \dots < k_p$ and then permute these numbers by a permutation $\sigma \epsilon S_p$. The sign of the permutation is $sgn(\sigma)$. In general, we have

$$\det\left(B_{\sigma(1)}, B_{\sigma(2)}, \dots, B_{\sigma(n)}\right) = sgn(\sigma) \det(B_1, B_2, \dots, B_n)$$

Exchange the columns of the matrix changes the sign of determinant and

In our case, we get:

$$\det(RS) = \sum_{1 \le k_1 < k_2 < \dots < k_p \le q} \sum_{\sigma \epsilon S_p} s_{k_{\sigma(1)} 1} s_{k_{\sigma(2)} 2} \dots s_{k_{\sigma(p)} p} \det\left(R_{k_{\sigma(1)}}, R_{k_{\sigma(2)}}, \dots, R_{k_{\sigma(p)}}\right)$$

$$= \sum_{1 \le k_1 < k_2 < \dots < k_p \le q} \sum_{\sigma \epsilon S_p} s_{k_{\sigma(1)} 1} s_{k_{\sigma(2)} 2} \dots s_{k_{\sigma(p)} p} \operatorname{sgn}(\sigma) \det(R_{k_1}, R_{k_2}, \dots, R_{k_p})$$

$$= \sum_{1 \le k_1 < k_2 < \dots < k_p \le q} \det(R_{k_1}, R_{k_2}, \dots, R_{k_p}) \sum_{\sigma \epsilon S_p} \operatorname{sgn}(\sigma) s_{k_{\sigma(1)} 1} s_{k_{\sigma(2)} 2} \dots s_{k_{\sigma(p)} p}$$

$$= \sum_{1 \le k_1 < k_2 < \dots < k_p \le q} \det(R_{k_1}, R_{k_2}, \dots, R_{k_p}) \cdot \det\left(S_{k_1}^T, S_{k_2}^T, \dots, S_{k_p}^T\right)$$

Theorem: (Matrix Tree Theorem) $If\ \widetilde{Q(G)}\ is\ a\ reduced\ incidnce\ matrix\ of$

$the\ connected\ graph\ G, then\ the\ number\ of\ spanning\ tree\ of\ G\ equals\ the$

$determinant\ of\ \widetilde{Q(G)} \cdot \widetilde{Q(G)}^T.$

Proof: (Algebraic Proof) To make the proof simple $\widetilde{Q(G)} = \tilde{Q}\ and\ \widetilde{Q(G)}^T = \tilde{Q}^T$.

$$\det\left(\tilde{Q}\cdot\tilde{Q}^T\right) = \sum \det(B)\cdot\det(B^T)$$

$$= \sum (\det(B))^2$$

$$= \sum_{B\ non-singular} (\det(B))^2 + \sum_{B\ singular} (\det(B))^2$$

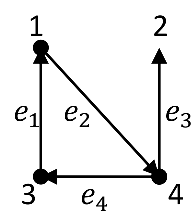
$$= \sum_{B\ non-singular} \det(B)^2$$

$$= \sum_{B\ non-singular} 1 \qquad since\ \det(B) = \pm 1$$

$= \#$ non-singular $(n-1)\times(n-1)$ $submatrices\ B\ of\ \tilde{Q}$

$= \#\ spanning\ trees\ of\ G$

Example:



$Figure\ 2$

The incidence matrix $Q(G)$corresponds to this orientation is

$$Q(G) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix}$$

We obtain the reduced incidence matrix $\widetilde{Q(G)}$ of G is obtaind by deleting the 3^{rd} row of $Q(G)$

$$\widetilde{Q(G)} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix}$$

$$\det(\tilde{Q} \cdot \tilde{Q}^T) = \sum (\det((B))^2$$

Where the sum is overall 3×3 submatrices B of $\widetilde{Q(G)}$ obtained by deleting the i^{th} column of $\widetilde{Q(G)}$.

$$\det(\tilde{Q} \cdot \tilde{Q}^T) = \sum (\det(B))^2$$

$$= \begin{vmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}^2 + \begin{vmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix}^2 + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{vmatrix}^2 + \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{vmatrix}^2$$

$$= (-1)^2 + 0^2 + (-1)^2 + (1)^2$$

$$= 1 + 1 + 1 = 3$$

Therefore the number of spanning tree of the graph G is 3.

Some Examples

Example1. **(Cayley's Formula)**. The number of spanning trees of the complete graph K_n is n^{n-2}.

Proof: $\deg(v) = n - 1 \ \ for\ all\ v \in V(K_n)$

Laplacian matrix $L(K_n)$ of the complete graph K_n .It is the $n \times n$ matrix

$$L(K_n) = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}$$

We can find the number of spanning trees by computing $\det(L_n)$ where L_n is a matrix obtained from $L(K_n)$ by deleting the n^{th} row and n^{th} column. Since adding rows together does not change the determinant, we first add all the rows together and make the new first row, which gives that

$$\det(L_n) = \begin{vmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{vmatrix} \xrightarrow{R_1 = R_1 + \cdots + R_{n-1}} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{vmatrix}$$

$$\xrightarrow{R_i = R_1 + R_i (\forall i \in \{2, \ldots, n-1\})} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{vmatrix}$$

Now we get upper triangular matrix, so the determinant is the product of the diagonal entries. There are $n-2$ $n's$ and a single 1 on the diagonal.

$$\det(L_n) = det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{pmatrix} = 1 \cdot n \cdot n \cdot \cdots \cdot n = n^{n-2}$$

Example2. The number of spanning trees of the complete bipartite graph $K_{m,n}$ with $m+n$ vertices and $m \cdot n$ edges is $m^{n-1} \cdot n^{m-1}$.

Proof: (Algebraic proof) Let $K_{m,n}$ be a complete bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y.

$$\deg(v) = \begin{cases} n \ if \ v \in X \\ m \ if \ v \in Y \end{cases}$$

From this we get a Laplacian matrix $L(K_{m,n})$ of the complete

$$L(K_{m,n}) = \begin{pmatrix} n & 0 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\ 0 & n & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\ 0 & 0 & n & \cdots & 0 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & n & -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & -1 & \mathrm{m} & 0 & \cdots & 0 \\ -1 & -1 & \cdots & -1 & -1 & 0 & \mathrm{m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & -1 & 0 & 0 & \cdots & m \end{pmatrix}$$

$$= \begin{pmatrix} nI_m & -1 \\ -1 & mI_n \end{pmatrix}$$

By deleting any one of the row and column of the Laplacian matrix $L(K_{m,n})$ we get a matrix $L(K_{m,n-1})$

$$\mathrm{L}(K_{m,n-1}) = \begin{pmatrix} nI_m & -1 \\ -1 & mI_{n-1} \end{pmatrix}$$

Since adding rows together does not change the determinant, we first add all the rows together and make the new first row similarly we add the i^{th} row with the first row and make the new i^{th} row which gives that

$$L(K_{m,n-1}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & n & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\ 0 & 0 & n & \cdots & 0 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & n & -1 & -1 & \cdots & -1 \\ 0 & 0 & \cdots & 0 & 0 & \mathrm{m} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \mathrm{m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & m \end{pmatrix}$$

$$Let\ A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & n & 0 & \cdots & 0 \\ 0 & 0 & n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & n \end{pmatrix} and\ B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ -1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & -1 \end{pmatrix}$$

$$det\left(L\left(K_{m,n-1}\right)\right) = det\begin{pmatrix} A & \mathrm{B} \\ 0 & mI_{n-1} \end{pmatrix} = \det(\mathrm{A}) \cdot \det(mI_{n-1}) = n^{m-1} \cdot m^{n-1}$$

Chapter Three

3. Combinatorial Interpretation of Matrix Tree Theorem

3.1 Polynomials Enumerating Spanning Trees

We define and study polynomials that enumerate spanning trees in terms of the degrees of all vertices.

Definition: In the context of polynomials, the word monomial is the product of variables or any value obtained by finitely many multiplications of variables.

Definition: - A homogeneous polynomial is a multiplicative polynomial (i.e. a polynomial in more than one variable) in which each term or monomial has the same degree.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. In this section we construct a polynomial f_G for a graph G. We associate a variable x_v to the vertex v an element of the vertex set V. For each spanning tree T of $G = (V, E), |V| \geq 2$, labeled on [n] we define a monomial of variables x_v:

$$m(T) = \prod_{v \in V(T)} x_v^{d_T(v)-1} \qquad 3.1$$

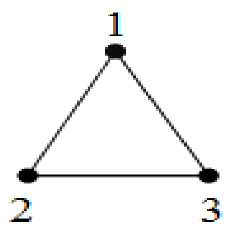
where $d_T(v)$ is the degree of vertex v in T.

For the graph G, we construct a polynomial t_G of variables x_v:

$$t_G := \sum_T m(T) \qquad 3.2$$

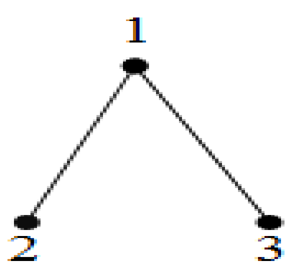
where the sum is overall spanning trees T of the graph G.

Example: - *Graph G*

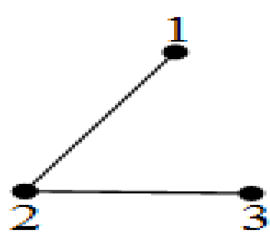


The above graph contains three vertices and three edges. Then find $m(T)$ *and* t_G

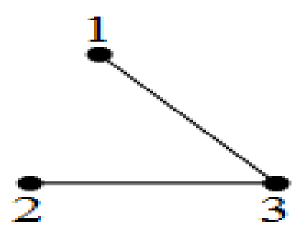
Solution: - *As showen below* T_1, T_2 *and* T_3 *are spanning trees of the graph* G.



T_1



T_2



T_3

We associate the variables $x_1,\ x_2$ *and* x_3 to the vertices $1, 2$ *and* 3 respectively. Now let us find the monomial of each tree T_1, T_2 *and* T_3.

$$m(T_1) = {x_1}^{d_{T_1}(1)-1}{x_2}^{d_{T_1}(2)-1}{x_3}^{d_{T_1}(3)-1} = {x_1}^{2-1}{x_2}^{1-1}{x_3}^{1-1} = {x_1}^{1}{x_2}^{0}{x_3}^{0} = x_1$$

$$m(T_2) = {x_1}^{d_{T_2}(1)-1}{x_2}^{d_{T_2}(2)-1}{x_3}^{d_{T_2}(3)-1} = {x_1}^{1-1}{x_2}^{2-1}{x_3}^{1-1} = x_2$$

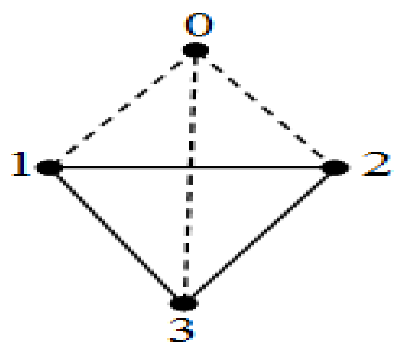
$$m(T_3) = x_1{}^{d_{T_3}(1)-1} x_2{}^{d_{T_3}(2)-1} x_3{}^{d_{T_3}(3)-1} = x_1{}^{1-1} x_2{}^{1-1} x_3{}^{2-1} = x_3$$

$$t_G(x_1, x_2, x_3) = m(T_1) + m(T_2) + m(T_3) = x_1 + x_2 + x_3$$

Let $0 \notin V$ and $\tilde{V} = V \cup \{0\}$. For a graph G on the set V the extended graph $\tilde{G}$ on the set $\tilde{V}$ is obtained from G by adding edges $\{0, v\}$ for all vertices $v \in V$. Let the variable x be associated with the added vertex zero. For a non-null graph, we construct a polynomial f_G of variables x and x_v, for all $v \in V$.

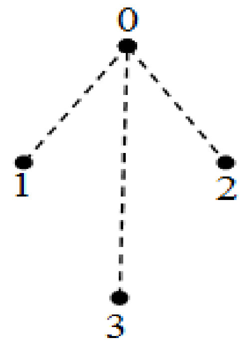
$$f_G(x;\ x_1, x_2, \dots, x_n) := t_{\tilde{G}}(x, x_1, x_2, \dots, x_n) \qquad 3.3$$

From the above example, the extended graph is shown below:



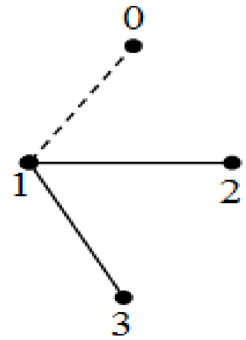
The extended graph $\tilde{G}$ from the above example

We associate the variables $x, x_1,\ x_2$ *and* x_3 to the vertices $0, 1, 2$ *and* 3 respectively. The spanning trees of the extended graph $\tilde{G}$ are listed below:



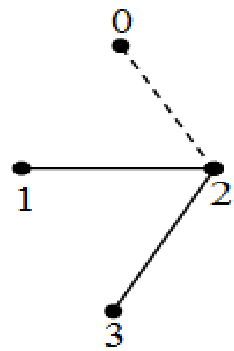
T_1

$m(T_1) = x^2$



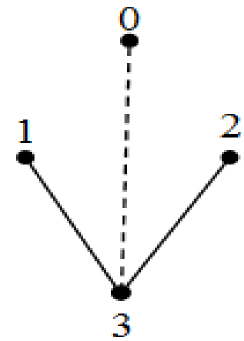
T_2

$m(T_2) = {x_1}^2$



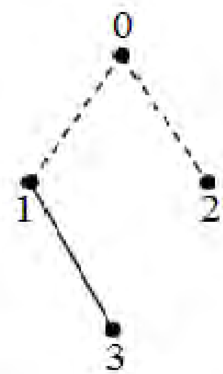
T_3

$m(T_3) = {x_2}^2$



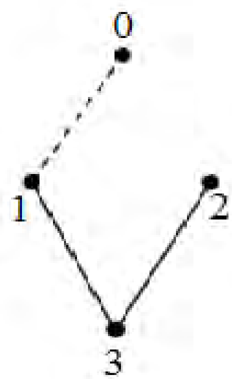
T_4

$m(T_4) = {x_3}^2$



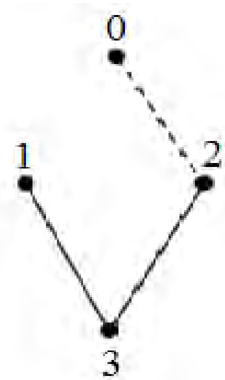
T_5

$m(T_5) = xx_1$



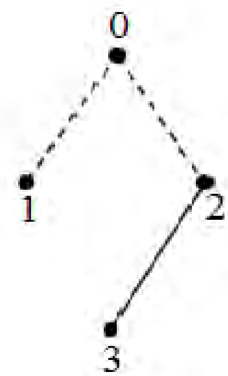
T_6

$m(T_6) = x_1x_3$



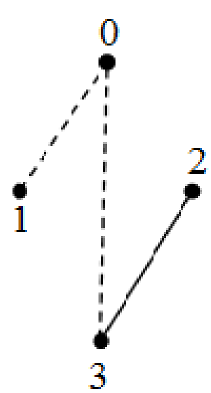
T_7

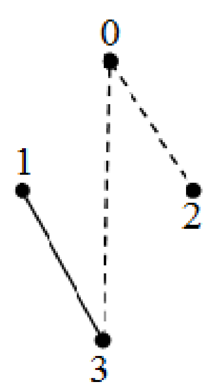
$m(T_7) = x_2x_3$

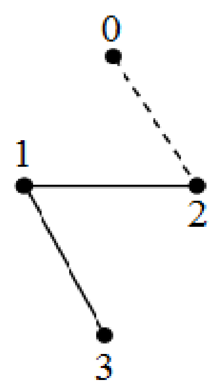


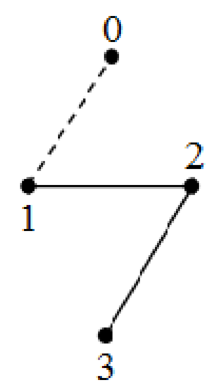
T_8

$m(T_8) = xx_2$



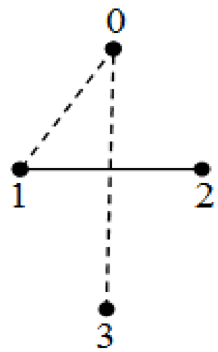






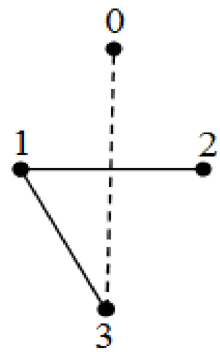
T_9 $\quad$ T_{10} $\quad$ T_{11} $\quad$ T_{12}

$m(T_9) = xx_3$ $\quad$ $m(T_{10}) = xx_3$ $\quad$ $m(T_{11}) = x_1x_2$ $\quad$ $m(T_{12}) = x_1x_2$



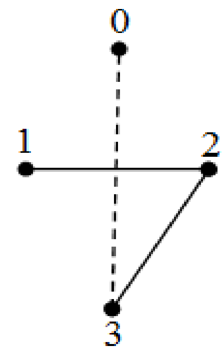
T_{13}

$m(T_{13}) = xx_1$



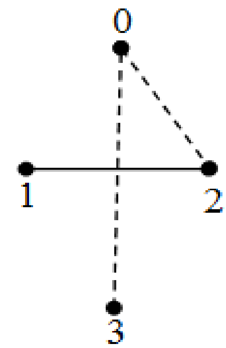
T_{14}

$m(T_{14}) = x_1x_3$



T_{15}

$m(T_{15}) = x_2x_3$



T_{16}

$m(T_{16}) = xx_2$

$$t_{\tilde{G}}(x, x_1, \dots, x_n) = \sum_{i=1}^{16} m(T_i)$$

$$= x^2 + {x_1}^2 + {x_2}^2 + {x_3}^2 + xx_1 + x_1x_3 + x_2x_3 + xx_2 + xx_3 + xx_3 + x_1x_2 + x_1x_2 + xx_1 + x_1x_3 + x_2x_3 + xx_2$$

$$= (x^2 + xx_1 + xx_2 + xx_3) + (xx_1 + {x_1}^2 + x_1x_2 + x_1x_3) + (xx_2 + x_1x_2 + {x_2}^2 + x_2x_3) + (xx_3 + x_1x_3 + x_2x_3 + {x_3}^2)$$

$$= x(x + x_1 + x_2 + x_3) + x_1(x + x_1 + x_2 + x_3) + x_2(x + x_1 + x_2 + x_3) + x_3(x + x_1 + x_2 + x_3)$$

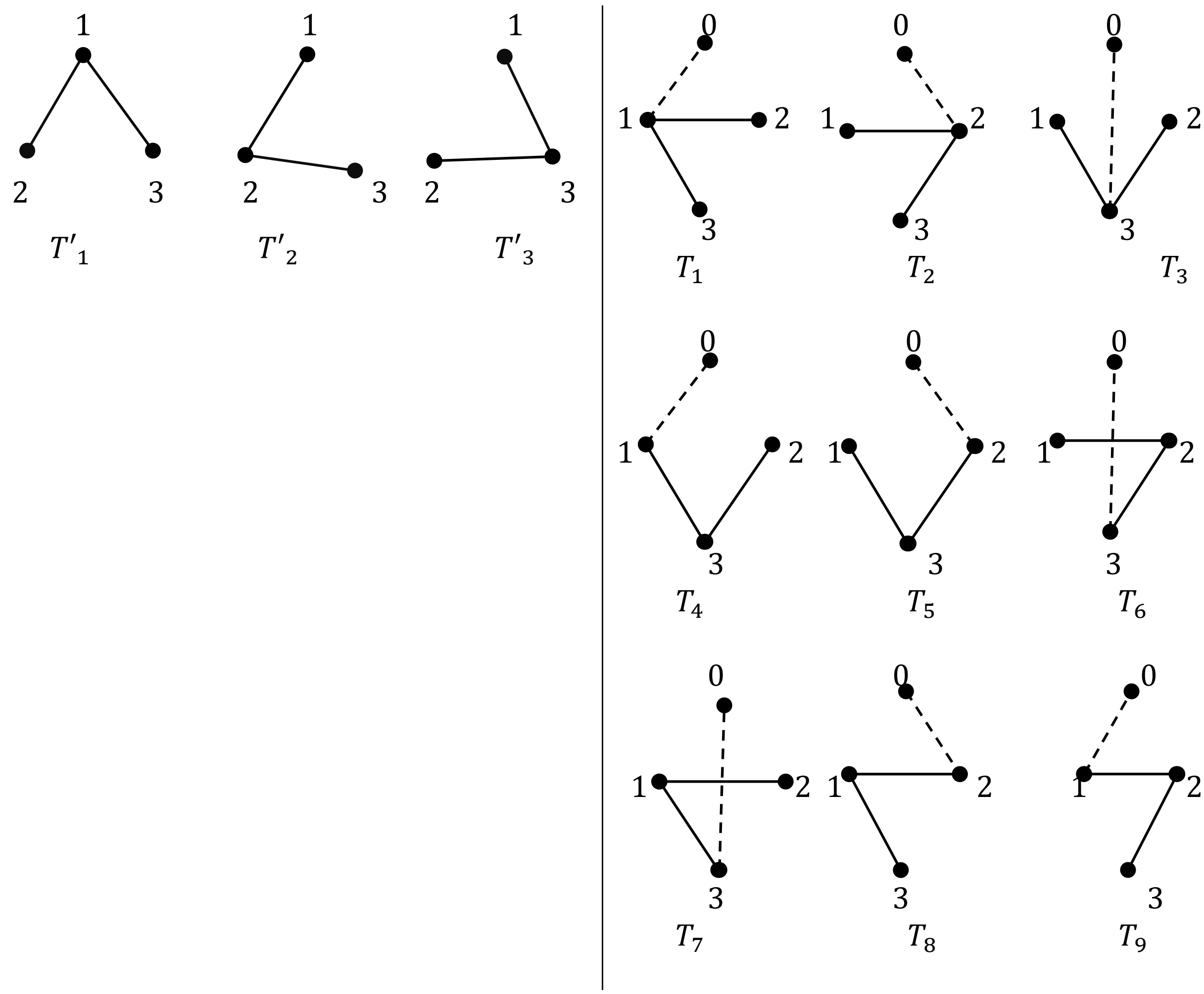
$$= (x + x_1 + x_2 + x_3)^2$$

$Therefore, \quad f_G(x;\ x_1, x_2, x_3) = (x + x_1 + x_2 + x_3)^2$

The number of spanning tree of the extended graph is 16. The monomials of the polynomial f_G which do not include x, correspond to the spanning trees of the graph G such that the degree of the vertex 0 is equal to one that means the vertex zero is pendent. Hence the vertex 0 is connected by an edge with certain vertex i of the graph $G, i = 1, 2, \dots, n$. Therefore,

$$f_G(0;\ x_1, x_2, \dots, x_n) = t_G\,(x_1, x_2, \dots, x_n)(x_1 + x_2 + \cdots + x_n) \qquad 3.4$$

The spanning trees of the graph G	The monomials of the polynomial f_G which do not include x



$$t_G(x_1, x_2, x_3) = m(T'_1) + m(T'_2) + m(T'_3) = x_1 + x_2 + x_3$$

$$f_G(0;\ x_1, x_2, x_3)$$

$$= m(T_1) + m(T_2) + m(T_3) + m(T_4) + m(T_5) + m(T_6) + m(T_7) + m(T_8) + m(T_9)$$

$$= {x_1}^2 + {x_2}^2 + {x_3}^2 + x_1x_3 + x_2x_3 + x_1x_2 + x_1x_2 + x_1x_3 + x_2x_3$$

$$= ({x_1}^2 + x_1x_2 + x_1x_3) + (x_1x_2 + {x_2}^2 + x_2x_3) + (x_1x_3 + x_2x_3 + {x_3}^2)$$

$$= x_1(x_1 + x_2 + x_3) + x_2(x_1 + x_2 + x_3) + x_3(x_1 + x_2 + x_3)$$

$$= (x_1 + x_2 + x_3)^2 = t_G(x_1, x_2, x_3)(x_1 + x_2 + x_3)$$

$$Therefore, \quad f_G(0;\ x_1, x_2, x_3\ ) = t_G(x_1, x_2, x_3)(x_1 + x_2 + x_3)$$

Formulas (3.3) and (3.4) show that the polynomials f_G and t_G define each other. But formulas for f_G are usually simpler than the corresponding formulas for t_G. We will use the polynomial f_G further on. We rewrite the above equation as follows:

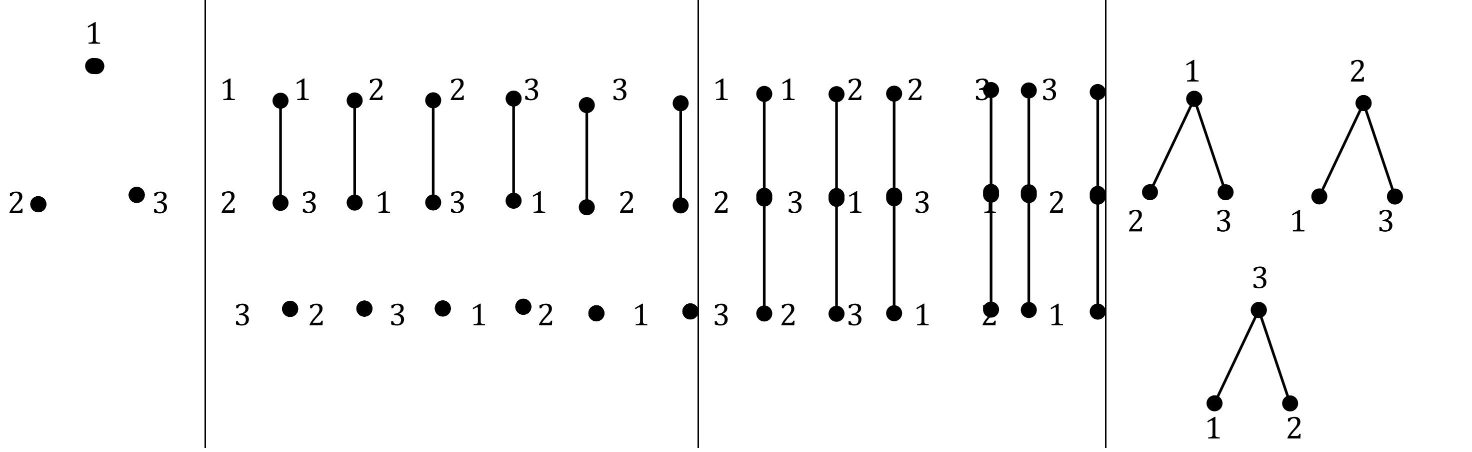
$$\frac{f_G(x; x_1, x_2, \dots, x_n)}{x_1 + x_2 + \cdots + x_n} = t_G(x_1, x_2, \dots, x_n) \qquad 3.5$$

Remark: - It is easy to see that the spanning trees in $\tilde{G}$ correspond to the spanning rooted forests in G (G is a complete graph)i.e. subgraphs in G without cycles containing all vertices of G, with a root chosen in each component. Hence f_G is the sum over all spanning rooted forests in G.

Example: - *Find the spanning rooted forests in G from the above example.*

Solution:-We list below the spanning rooted forests in G as follows:

Labeled trees of G with no edge.	Labeled trees of G with 1 edge and each rooted vertices are degree one.	Labeled trees of G with 2 edge and each rooted vertices are degree one.	Labeled trees of G with 2 edge and each rooted vertices are degree 2.



In the above example the number of spanning trees of the extended graph $\tilde{G}$ is 16 and spanning rooted forests G are 16. This shows the above remark is correct.

3.2 Reciprocity Theorem for Polynomials f_G

Definition: - Let G_1 *and* G_2 be two graphs on the disjoint sets of vertices. Let $G_1 + G_2$ denote the disjoint union of the graphs. We associate variables $y_1,\ y_2, \dots, y_r$ to the vertices of G_1 and variables $z_1, z_2, \dots, z_s$ to the variables of G_2. Then the following formula holds:

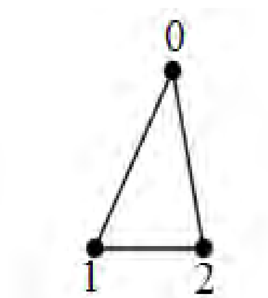
$$f_{G_1+G_2}(x;\ y_1, y_2, \dots, y_r, z_1, z_2, \dots, z_s)$$
$$= x f_{G_1}(x;\ y_1, y_2, \dots, y_r) \cdot f_{G_2}(x; z_1, z_2, \dots, z_s) \qquad 3.6$$

Example:

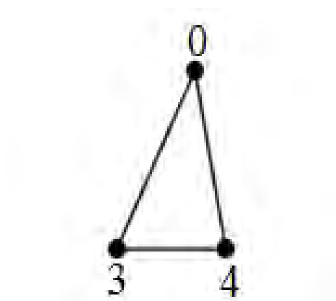
Graph G_1

Graph G_2

From here the two graphs have disjoint sets of vertices and their extended graphs are given below

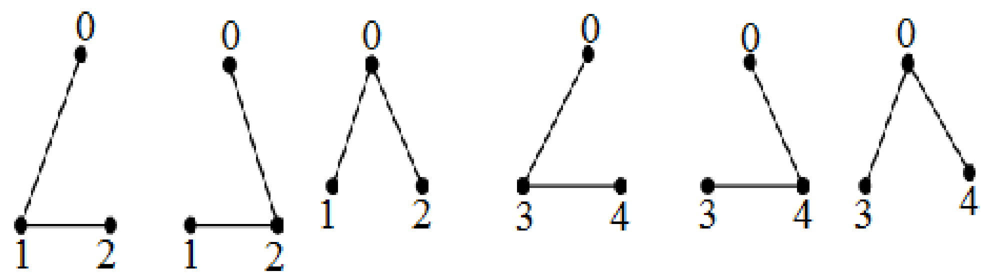


Extended graph $\tilde{G}_1$



Extended graph $\tilde{G}_2$

The spanning trees of the extended graph $\tilde{G}_1$ and$\tilde{G}_2$ are three in number and listed below:



T_1 T_2 T_3 T_4 T_5 T_6

$$m(T_1) = y_1, \ m(T_2) = y_2, m(T_3) = x, m(T_4) = z_3, m(T_5) = z_4, m(T_6) = x$$

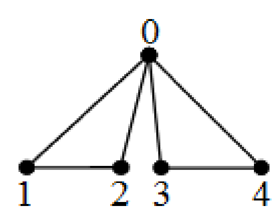
$t_{\tilde{G}_1}(x, y_1, y_2) = m(T_1) + m(T_2) + m(T_3) = y_1 + y_2 + x$ *and*

$t_{\tilde{G}_2}(x, z_3, z_4) = m(T_4) + m(T_5) + m(T_6) = z_3 + z_4 + x$

$f_{G_1}(x; y_1, y_2) = x + y_1 + y_2$ *and* $f_{G_2}(x; z_3, z_4) = z_3 + z_4 + x$

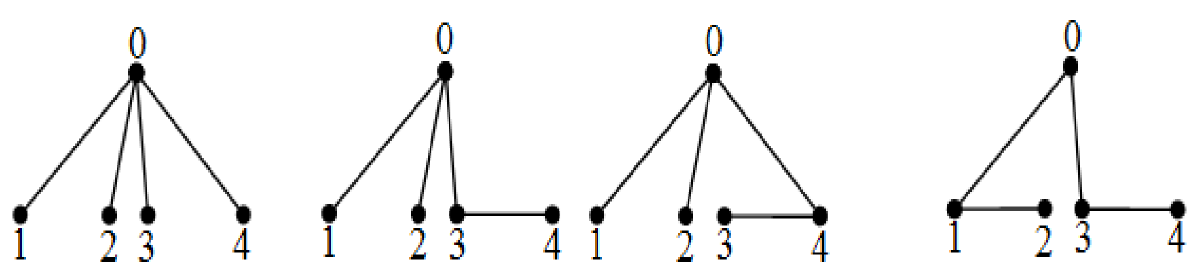
$$f_{G_1}(x; y_1, y_2) f_{G_2}(x; z_3, z_4) = (x + y_1 + y_2)(x + z_3 + z_4) \qquad (*)$$

Next to this we find the polynomial $f_{G_1+G_2}(x; y_1, y_2, z_3, z_4)$.

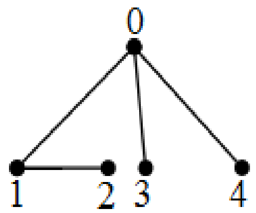


Extended graph $\widetilde{G_1 + G_2}$

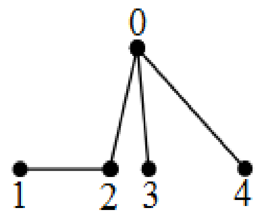
The spanning trees of the extended graph $\widetilde{G_1 + G_2}$ are listed below:



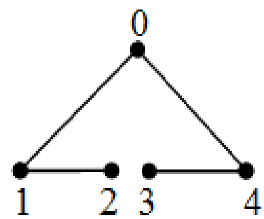
T'_1 T'_2 T'_3 T'_4



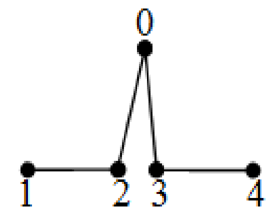
T'_5



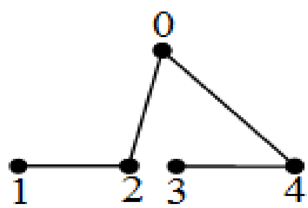
T'_6



T'_7



T'_8



T'_9

$m(T'_1) = x^3,\ \ m(\ T'_2) = x^2 \cdot z_3,\ \ m(\ T'_3) = x^2 z_4,\ \ \ m(\ T'_4) = xy_1z_3,\ \ m(\ T'_5) = x^2y_1$

$m(T'_6) = x^2y_2,\ \ m(\ T'_7) = xy_1z_4,\ \ \ \ \ \ m(\ T'_8) = xy_2z_3,\ \ \ m(\ T'_9) = xy_2z_4,$

$$t_{\widetilde{G_1+G_2}}(x, y_1, y_2, z_3, z_4) = m(T'_1) + m(\ T'_2) + m(\ T'_3) + m(\ T'_4) + m(\ T'_5) + m(T'_6) + m(\ T'_7) + m(\ T'_8) + m(\ T'_9)$$

$$= x^3 + x^2 \cdot z_3 + x^2z_4 + xy_1z_3 + x^2y_1 + x^2y_2 + xy_1z_4 + xy_2z_3 + xy_2z_4$$

$$= x(x^2 + xz_3 + xz_4 + y_1z_3 + xy_1 + xy_2 + y_1z_4 + y_2z_3 + y_2z_4)$$

$$= x(x(x + z_3 + z_4) + y_1(x + z_3 + z_4) + y_2(x + z_3 + z_4))$$

$$= x(x + y_1 + y_2)(x + z_3 + z_4)$$

Therefore, $f_{\widetilde{G_1+G_2}}(x, y_1, y_2, z_3, z_3) = x(x + y_1 + y_2)(x + z_3 + z_4)$

$$= x \cdot f_{G_1}(x; y_1, y_2) \cdot f_{G_2}(x; z_3, z_3) \quad from\ (*)$$

Definition: - *A graph* $\bar{G} = (V, \bar{E})$ *is said to be a complement of a graph* $G = (V, E)$ *if* e *is an edge of* $\bar{G}$ *if and only if* e *is not an edge of* G*.*

Theorem 1: Let G be a graph on the set of vertices $V = \{1, 2, \dots, n\}$. Then Definition: - *A graph* $\bar{G} = (V, \bar{E})$ *is said to be a complement of a graph* $G = (V, E)$ *if* e *is an edge of* $\bar{G}$ *if and only if* e *is not an edge of* G*.*

Theorem 1: Let G be an oriented net (graph) on the set of vertices $V = \{1, 2, \dots, n\}$. Then

$$f_{\bar{G}}(x;\ x_1, x_2, \dots, x_n\ ) = (-1)^{n-1} \cdot f_G(-x - x_1 - x_2 - \cdots - x_n;\ x_1, x_2, \dots, x_n\ )$$

3.3 Generalization to oriented nets

Definition: - An oriented net (or simply net) on the set of vertices V is defined by the set of conductivities $g_{vw} \epsilon \mathbb{R}$ assigned to every ordered pair of vertices $v, w \epsilon V$. We associate a net with each graph as follows:

$$g_{vw} = g_{wv} = \begin{cases} 1 & if\ \{v, w\}\ is\ an\ edge\ of\ the\ graph; \\ 0 & otherwise. \end{cases}$$

Do not confuse, we denote the graph and the corresponding net by the same latter G. When displaying a net graphically, we draw an oriented graph with assigned edge conductivities. Net $G = (g_{vw})\ for\ v, w\ \epsilon\ V\ \ and\ g_{vv} = 0, \forall v \in V$. Let T be a tree on the set of vertices $\tilde{V} = V \cup \{0\}$. Let us orient the tree T from the root in vertex 0. The multiplicity of T in the net G is the number:

$$k_G(T) = \prod_{(v,w)} g_{vw} \qquad 3.7$$

where the product is over all ordered pairs $(v, w) \in V \times V$ such that (v, w) is an edge of T in the orientation. If T consists only of edges $(0, v)$ and doesn't contain any edge $(v, w) \in V \times V$, we assume that $k_G(T) = 1$.

Note that: If G is the net associated with a graph then

$$k_G(T) = \begin{cases} 1 & \text{if } T \text{ is the spanning tree of the graph } \tilde{G} \\ 0 & \text{otherwise} \end{cases}$$

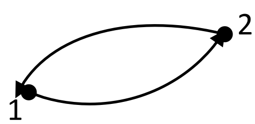
Now we define a polynomial f_G for the net G

$$f_G(T) := \sum_T k_G(T) \cdot m(T)$$

where the sum is over all spanning trees on the set of vertices $\tilde{V}$; and the monomial $m(T)$.

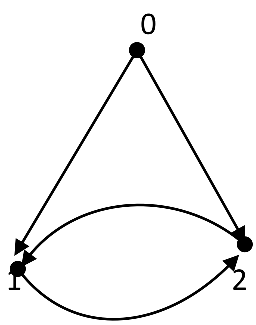
Definition: - A net $\bar{G} = (\bar{g}_{vw})$ on the set V is called a complementary to the net $G = (g_{vw})$ on the same set V if $\bar{g}_{vw} = \begin{cases} 1 - g_{vw} & \text{for all } v \neq w \\ 0 & \text{for all } v = w \end{cases}$

Example: - Let $V = \{1, 2\}$

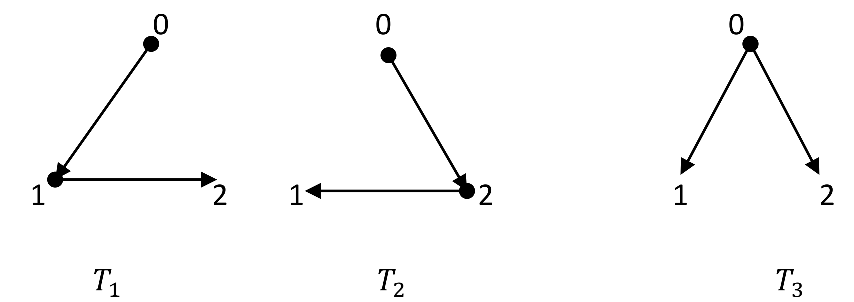


oriented net G

The extended net $\tilde{G}$ of the net G is given below:



The spanning tree of the above graph is listed as follows:



$Associate\ the\ variables\ x, x_1$ and x_2 with respect to the vertices $0, 1\ and\ 2$.

$m(T_1) = x_1,\ m(T_2) = x_2\ and\ m(T_3) = x$

$k_G(T_1) = k_G(T_2) = k_G(T_3) = 1$

$Therefore, f_G(x; x_1, x_2) = k_G(T_1)m(T_1) + k_G(T_2)m(T_2) + k_G(T_3)m(T_3) = x_1 + x_2 + x$

Before proving this theorem first we must see some lemmas.

Definition: - Let G be a net on the set V , $v\epsilon V$ and let $G\backslash v$ be a restriction of G onto the set $V\backslash\{v\}$ is defined by the net on the set $V\backslash\{v\}$ with the same conductivities as the net G.

Lemma1:

$$f_G|_{x_v=0} = \left(x + \sum_{w\epsilon V} x_w g_{wv}\right) f_{G\backslash v}$$

Proof: The polynomials $f_G|_{x_v=0}$ consist of monomials corresponding to trees T such that the vertex V is an end point of T, that is v is pendent. This vertex connected by the edge in T with some vertex $w\ \epsilon\tilde{V}\backslash\{v\}$. Let $T'\ b$e a tree on the set $\tilde{V}\backslash\{v\}$ obtained by deleting the vertex v $(T' = T\backslash\{v\})$. Then

$$k_G(T)m(T) = \begin{cases} xk_{G\backslash v}(T')m(T') & if w = 0 \\ x_w g_{wv} k_{G\backslash v}(T')m(T')\ \ if\ w\epsilon V \end{cases}$$

- $k_G(T)m(T)$ is useful for only one spanning tree T of the graph G.

We know that

$$f_G(T) = \sum_{T} k_G(T)m(T)$$

$$f_G|_{x_v=0}(T) = \left(\sum_{T} k_G(T)m(T)\right)\Bigg|_{x_v=0}$$

$$= \sum_{T\prime}(xk_{G\backslash v}(T')m(T') + x_w g_{wv} k_{G\backslash v}(T')m(T'))$$

$$= \sum_{T'}(x + x_w g_{wv})k_{G\backslash v}(T')m(T')$$

$$= \left(x + \sum_{w\epsilon V} x_w g_{wv}\right)\sum_{T'} k_{G\backslash v}(T')m(T')$$

$$= \left(x + \sum_{w\epsilon V} x_w g_{wv}\right)f_{G\backslash v}(T') \quad since\ f_{G\backslash v}(T') = \sum_{T'} k_{G\backslash v}(T')m(T')$$

$Therefore,\ f_G|_{x_v=0} = (x + \sum_{w\epsilon V} x_w g_{wv})f_{G\backslash v}$

Lemma 2:

Let G be a graph (or a net) with $n - vertices$. Then f_G is a homogeneous polynomial of degree $n - 1$. (We assume that $degx = degx_v = 1, for\ all\ v\epsilon V$.)

Proof: Let T be a tree on the set of vertices $\tilde{V} = V \cup \{0\}$, $|\tilde{V}| = n + 1$. We have a monomial of variables x_v:

$$m(T) = \prod_{v \in V(T)} x_v^{d_T(v)-1}$$

$$\deg\big(m(T)\big) = \sum_{v \epsilon \tilde{V}} (d_T(v) - 1))$$

$$= \sum_{v \epsilon \tilde{V}} d_T(v) - \sum_{v \epsilon \tilde{V}} 1$$

$$= 2n - (n+1) \ \ since \ \ the \ tree \ T \ has \ n - edges \ and \ n+1 \ vertices$$

$$= n - 1$$

Therefore, f_G is a homognous polynomial of degree $n-1$ because f_G is a sum of monomials.

Lemma 3:

Let h be a polynomial of variables $x_1, x_2, \dots, x_n$ with degree strictly less than n, and $h|_{x_i=0} = 0 \ for \ all \ i = 1, 2, \dots, n$.Then $h \equiv 0$.

Proof: suppose that $h \neq 0$ and $h|_{x_i=0} = 0 \ for \ all \ i = 1, 2, \dots, n$. Then we can find a monomial in h with nonzero coefficient. Since $\deg(h) < n$, we can find an integer $i \in \{1, 2, \dots, n\}$ such that the variable x_i does not appear in the monomial. Hence $h|_{x_i=0} \neq 0$, that contradicts to the assumption.

Proof of theorem1:

We prove by induction on $n = |V|$

$h_G(x;\ x_1, x_2, \dots, x_n) := f_{\bar{G}}(x;\ x_1, x_2, \dots, x_n) - (-1)^{n-1} \cdot f_G(-x - x_1 - x_2 - \cdots - x_n;\ x_1, x_2, \dots, x_n)$ is equal to zero.

Let $n = 1$. There is a unique graph (net) with one vertex $G = K_1$for which $f_{K_1}(x;\ x_1) = 1 = f_{\bar{K}_1}(x;\ x_1)$. A complete graph with one vertex has no edge so its complementary graph is also the same.

$Hence,\ h_G(x;\ x_1) = f_{\bar{G}}(x;\ x_1) - (-1)^{1-1} \cdot f_G(-x - x_1;\ x_1) = 1 - 1 = 0$

This shows that $f_{\bar{G}}(x;\ x_1) = (-1)^{1-1} \cdot f_G(-x - x_1;\ x_1)$

Assume that $n \geq 2$ and $h_{G\backslash i}(x; x_1, x_2, \dots, \hat{x}_i, \dots, x_n) = 0$. We want to show $h_G(x;\ x_1, x_2, \dots, x_n\ ) = 0$. We obtain by lemma 1:

$h_G(x;\ x_1, \dots, x_n\ )|_{x_i=0} = f_{\bar{G}}(x;\ x_1, \dots, x_n\ )|_{x_i=0} -$
$(-1)^{n-1}\cdot f_G(-x - x_1 - \cdots - x_n;\ x_1, x_2, \dots, x_n\ )\ |_{x_i=0}$

$$= (x + \sum_{j:j\neq i}(1 - g_{ji})x_j)f_{\overline{G\backslash i}}\,(x; x_1, x_2, \dots, \hat{x}_i, \dots, x_n)$$
$$- (-1)^{n-1}((-x - x_1 - \cdots - \hat{x}_i - \cdots - x_n)$$
$$+ \sum_{j:j\neq i} g_{ji}x_j)f_{G\backslash i}(-x - x_1 - \cdots - \hat{x}_i - \cdots - x_n;\ x_1, x_2, \dots, \hat{x}_i, \dots, x_n\ )$$

$$= \left(x + \sum_{j:j\neq i}(1 - g_{jii})x_j\right)f_{\overline{G\backslash i}} - (-1)^{n-1}\left(-x - \sum_{j:j\neq i} x_j + \sum_{j:j\neq i} g_{ji}x_j\right)f_{G\backslash i}$$

$$= \left(x + \sum_{j:j\neq i}(1 - g_{jii})x_j)\right)f_{\overline{G\backslash i}} - (-1)^{n-2}\left(x + \sum_{j:j\neq i}\big(1 - g_{ji}\big)x_j\right)f_{G\backslash i}$$

$$= \left(x + \sum_{j:j\neq i}(1 - g_{jii})x_j\right)\big(f_{\overline{G\backslash i}} - (-1)^{n-2}f_{G\backslash i}\big)$$

$$= \left(x + \sum_{j:j\neq i} (1 - g_{jii}) x_j \right) h_{G\setminus i}(x; x_1, x_2, \dots, \hat{x}_i, \dots, x_n)$$

$$= \left(x + \sum_{j:j\neq i} (1 - g_{jii}) x_j \right) \cdot 0$$

$$= 0$$

Here the symbol ^ denotes that the corresponding element is omitted. $Therefore,$ $h_G(x;\ x_1, \dots, x_n)|_{x_i=0} = 0\ for\ all\ i = 1, 2, \dots, n$. By lemma 2: We know G be a graph (or a net) with $n - vertices$. Then h_G is a homogeneous polynomial of degree $n - 1$. $degh_G = n - 1 < n$. And by lemma 3: h_G be a polynomial of variables $x_1,\ x_2, \dots, x_n$ with degree strictly less than n, and $h|_{x_i=0} = 0\ for\ all\ i = 1, 2, \dots, n$.

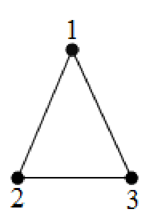
$Therefore,\ \ h_G(x;\ x_1, x_2, \dots, x_n) = 0.$

$$h_G(x;\ x_1, x_2, \dots, x_n) = f_{\bar{G}}(x;\ x_1, x_2, \dots, x_n) - (-1)^{n-1} \cdot f_G(-x - x_1 - x_2 - \cdots - x_n;\ x_1, x_2, \dots, x_n)$$

$$0 = f_{\bar{G}}(x;\ x_1, x_2, \dots, x_n) - (-1)^{n-1} \cdot f_G(-x - x_1 - x_2 - \cdots - x_n;\ x_1, x_2, \dots, x_n)$$

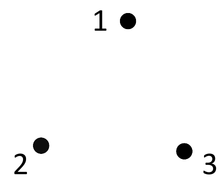
$$Therefore,\ \ f_{\bar{G}}(x;\ x_1, x_2, \dots, x_n) = (-1)^{n-1} \cdot f_G(-x - x_1 - x_2 - \cdots - x_n;\ x_1, x_2, \dots, x_n)$$

Example: Graph G

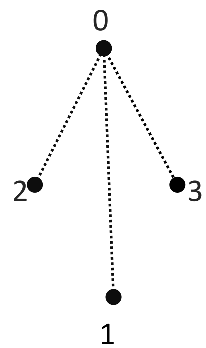


From this graph we know $f_G(x;\ x_1, x_2, x_3) = (x + x_1 + x_2 + x_3)^2$

The complement graph of G is an empty graph showed below:



The extended graph of the complement graph $\bar{G}$ is a tree indicated below:



$$m(T) = x^2$$

$$t_{\tilde{G}}(x, x_1, x_2, x_3) = x^2$$

$$f_{\bar{G}}(x;\ x_1, x_2, \dots, x_n) = x^2$$

But we know this formula

$$f_{\bar{G}}(x;\ x_1, x_2, \dots, x_n) = (-1)^{n-1} \cdot f_G(-x - x_1 - x_2 - \cdots - x_n;\ x_1, x_2, \dots, x_n)$$

$$f_{\bar{G}}(x;\ x_1, x_2, x_3) = (-1)^{3-1} \cdot f_G(-x - x_1 - x_2 - x_3;\ x_1, x_2, x_3)$$

$$f_{\bar{G}}(x;\ x_1, x_2, x_3) = (-x - x_1 - x_2 - x_3 + x_1 + x_2 + x_3)^2 = x^2$$

3.4 Expressing a Polynomial as a Determinant of Matrix

We can express the polynomial f_G as a determinant of a matrix. With any tree T on the set of vertices $\tilde{V} = V \cup \{0\}$ we can associate a monomial $M(T)$ of variables z_{vw}. z_{vw} is a collection of commutative variables, we assume that $z_{vv} = 0\ for\ v \in \tilde{V}$. Let us orient the tree T from the root in the vertex 0 and assume that

$$M(T) = \prod_{(v,w)\in E(T)} z_{vw}$$

where the product is overall pairs $(v, w) \in \tilde{V} \times \tilde{V}$ which are oriented edges of T. We denote

$$F_V = \sum_T M(T)$$

where the sum is over all trees on the set of vertices $\tilde{V}$. Without loss of generality we can assume that $V = \{1, 2, \dots, n\}$. In this case $F_n := F_{\{1,2,\dots,n\}}$ is a polynomial of $z_{ij}, 1 \le i, j \le n$.

Let Kirchhoff's matrix be $n \times n$ matrix $A = (a_{ij}), i, j \in \{1, 2, \dots, n\}$, where

$$a_{ij} = \begin{cases} \sum_{l=1}^{n} z_{li} & if\ i = j \\ -z_{ij} & if\ i \ne j \end{cases} \qquad 4.2.1$$

Theorem: (Matrix Tree Theorem)

$$F_n = detA$$

Let now G be a net on the set of vertices V with conductivities g_{vw}. Assume that

$$z_{vw} = x_v g_{vw}, v, w \in V\ and\ z_{ov} = x, for\ v \in V \qquad 4.2.2$$

Therefore, $xk_G(T)m(T) = M(T)$

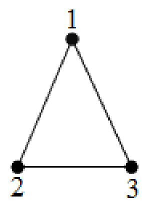
Corollary:

$$xf_G(x_1, x_2, \dots, x_n) = \det B$$

where $B = (b_{ij}), 1 \le i, j \le n, is\ n \times n\ matrix$

$$b_{ij} = \begin{cases} x + \sum_{l=1}^{n} x_l g_{li}, & i = j \\ -x_i g_{ij}, & i \neq j \end{cases}$$

Example:



Solution: We need to express the Laplacian matrix as a determinant.

$$L(K_3) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$b_{11} = x + x_1 g_{11} + x_2 g_{21} + x_3 g_{31}$$

$$= x + x_2 + x_3$$

$$b_{12} = -x_1 g_{12} = -x_1;$$

$$b_{13} = -x_1 g_{13} = -x_1;$$

$$b_{21} = -x_2 g_{21} = -x_2$$

$$b_{22} = x + x_1 g_{12} + x_2 g_{22} + x_3 g_{32}$$
$$= x + x_1 + x_3;$$

$$b_{23} = -x_2 g_{23} = -x_2$$

$$b_{31} = -x_3 g_{31} = -x_3;$$

$$b_{32} = -x_3 g_{32} = -x_3;$$

$$b_{33} = x + x_1 g_{13} + x_2 g_{23} + x_3 g_{33}$$
$$= x + x_1 + x_2$$

We express the Laplacian matrix in terms of variables if you assume $x = 0$, $x_1 = x_2 = x_3 = 1$ we get the matrix $L(K_3)$

$$L(K_3) = \begin{pmatrix} x + x_2 + x_3 & -x_1 & -x_1 \\ -x_2 & x + x_1 + x_3 & -x_2 \\ -x_3 & -x_3 & x + x_1 + x_2 \end{pmatrix}$$

$$\det(L(K_3)) = x f_G(x;\ x_1,\ x_2, x_3\ )$$

Delete 1st row and column and take the determinant

$$det\begin{pmatrix} x + x_1 + x_3 & -x_2 \\ -x_3 & x + x_1 + x_2 \end{pmatrix} = (x^2 + xx_1 + xx_2 + xx_1 + {x_1}^2 + x_1x_2 + xx_3 + x_1x_3)$$

$$= x\big(x + x_1 + x_2 + x_1 + xx_3\big) + ({x_1}^2 + x_1x_2 + x_1x_3\big)$$

Some examples

Example: **(Cayley's Formula)**. The number of spanning trees of the complete graph K_n is n^{n-2}.

Solution: - we know that for any an oriented net (graph) G on the set of vertices $V = \{1, 2, \ldots, n\}$. Then

$$f_{\bar{G}}(x;\ x_1, x_2, \ldots, x_n\ ) = (-1)^{n-1} \cdot f_G(-x - x_1 - x_2 - \cdots - x_n;\ x_1, x_2, \ldots, x_n\ )$$

Definition: An empty graph O_n on the set of $n\, vertices$ is a graph which has no vertices is adjacent (there is no edge between any of two vertices).

$$f_{O_n}(x; x_1, x_2, \cdots, x_n) = x^{n-1}$$

$$f_{K_n}(x; x_1, x_2, \cdots, x_n) = f_{\bar{O}_n}(x; x_1, x_2, \cdots, x_n)$$

$$= (-1)^{n-1} f_{O_n}(-x - x_1 - \ x_2 - \cdots - x_n; x_1, x_2, \cdots, x_n)$$

$$= (-1)^{n-1}(-x - x_1 - \ x_2 - \cdots - x_n)^{n-1}$$

$$= (-1)^{n-1}(-1)^{n-1}(x + x_1 + \ x_2 + \cdots + x_n)^{n-1}$$

$$= (x + x_1 + x_2 + \cdots + x_n)^{n-1}$$

$$Therefore, \quad f_{K_n}(x; x_1, x_2, \cdots, x_n) = (x + x_1 + x_2 + \cdots + x_n)^{n-1}$$

$$f_{K_n}(0; x_1, x_2, \cdots, x_n) = (x_1 + x_2 + \cdots + x_n)^{n-1}$$

$t_G(x_1, x_2, \cdots, x_n) = \sum_T m(T)$ where the sum is overall spanning tree T in the graph G. And we know that

$$t_G(x_1, x_2, \dots, x_n) = \frac{f_G(0; x_1, x_2, \dots, x_n)}{x_1 + x_2 + \cdots + x_n}$$

$$t_{K_n}(x_1, x_2, \dots, x_n) = \frac{f_{K_n}(0; x_1, x_2, \dots, x_n)}{x_1 + x_2 + \cdots + x_n}$$

$$= \frac{(x_1 + x_2 + \cdots + x_n)^{n-1}}{x_1 + x_2 + \cdots + x_n}$$

$$= (x_1 + x_2 + \cdots + x_n)^{n-2}$$

Assume $x_1 = x_2 = \cdots = x_n = 1$. We get

$t_{K_n}(1, 1, \dots, 1) = (1 + 1 + \cdots + 1)^{n-2} = n^{n-2}$.

$Therefore, the\ number\ of\ spanning\ trees\ of\ \ the\ complete\ graph K_n\ is\ \ t_{K_n} = n^{n-2}$.

Example:-The number of spanning trees of the complete bipartite graph $K_{m,n}$ with $m + n$ vertices and $m \cdot n$ edges is $m^{n-1} \cdot n^{m-1}$.

<u>Solution</u>:- Let $K_{m,n}$ be a complete bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y. We associate variables $y_1, y_2, \dots, y_m$ to the vertices of X and variables $z_1, z_2, \dots, z_n$ to the variables of Y. Then the following formula holds:

$$f_{K_m+K_n}(x;\ y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_n)$$
$$= xf_{K_m}(x;\ y_1, y_2, \dots, y_m) \cdot f_{K_n}(x; z_1, z_2, \dots, z_n)$$
$$= x(x+\ y_1+\ y_2+\cdots+y_m)^{m-1} \cdot (x+z_1+z_2+\cdots+z_n)^{n-1}$$

$$f_{K_{m,n}}(x;\ y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_n) = f_{\overline{K_m+K_n}}(x;\ y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_n)$$

$$= (-1)^{m+n-1} f_{K_m+K_n}(-x-y_1-y_2-\cdots-y_m-z_1-z_2-\cdots-z_n;\ y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_n)$$

$$= (-1)^{m+n-1}(-x-y_1-y_2-\cdots-y_m-z_1-z_2-\cdots-z_n)$$
$$\cdot ((-x-y_1-y_2-\cdots-y_m-z_1-z_2-\cdots-z_n)+\ y_1+\ y_2+\cdots+y_m)^{m-1}$$
$$\cdot ((-x-y_1-y_2-\cdots-y_m-z_1-z_2-\cdots-z_n)+z_1+z_2+\cdots+z_n)^{n-1}$$

$$= (-1)^{m+n-1}(-x-y_1-y_2-\cdots-y_m-z_1-z_2-\cdots-z_n) \cdot$$

$$(-x-z_1-z_2-\cdots-z_n)^{m-1} \cdot (-x-z_1-z_2-\cdots-z_n)^{n-1}$$

$$= (-1)^{m+n-1}(-1)^{m+n-1}(x+y_1+y_2+\cdots+y_m+z_1+z_2+\cdots+z_n)$$
$$\cdot (x+z_1+z_2+\cdots+z_n)^{m-1}$$
$$\cdot (x+y_1+y_2+\cdots+y_m)^{n-1}(x+y_1+y_2+\cdots+y_m)^{n-1}$$

$$= (x+y_1+y_2+\cdots+y_m+z_1+z_2+\cdots+z_n) \cdot$$

$$(x+z_1+z_2+\cdots+z_n)^{m-1} \cdot (x+y_1+y_2+\cdots+y_m)^{n-1}$$

$$\therefore\ f_{K_{m,n}}(x;\ y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_n)$$
$$= (x+y_1+y_2+\cdots+y_m+z_1+z_2+\cdots+z_n) \cdot (x+z_1+z_2+\cdots+z_n)^{m-1}$$
$$\cdot (x+y_1+y_2+\cdots+y_m)^{n-1}$$

$$f_{K_{m,n}}(0;\ y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_n)$$
$$= (y_1+y_2+\cdots+y_m+z_1+z_2+\cdots+z_n) \cdot (z_1+z_2+\cdots+z_n)^{m-1}$$
$$\cdot (y_1+y_2+\cdots+y_m)^{n-1}$$

$We\ know\ \ t_{K_{m,n}}(y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_n)$

$$= \frac{f_{K_{m,n}}(0;\ y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_n)}{(y_1 + y_2 + \cdots + y_m + z_1 + z_2 + \cdots + z_n)}$$

$t_{K_{m,n}}(y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_n)$

$$= \frac{(y_1 + y_2 + \cdots + y_m + z_1 + z_2 + \cdots + z_n) \cdot (z_1 + z_2 + \cdots + z_n)^{m-1} \cdot (y_1 + y_2 + \cdots + y_m)^{n-1}}{(y_1 + y_2 + \cdots + y_m + z_1 + z_2 + \cdots + z_n)}$$

$$t_{K_{m,n}}(y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_n) = (z_1 + z_2 + \cdots + z_n)^{m-1} \cdot (y_1 + y_2 + \cdots + y_m)^{n-1}$$

Assume that $z_1 = z_2 = \cdots = z_n = y_1 = y_2 = \cdots = y_m = 1$

$$t_{K_{m,n}}(1, 1, \ldots, 1, 1, \ldots, 1) = (1 + 1 + \cdots + 1)^{m-1} \cdot (1 + 1 + \cdots + 1)^{n-1}$$

$$Therefore, \qquad t_{K_{m,n}} = n^{m-1} m^{n-1}$$

4. References

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