COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES

DEPARTMENT OF MATHEMATICS

Multiplicity of Positive Solutions and The Method of Upper and Lower Solutions for Nonlinear Elliptic PDEs

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BY

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The undesigned here by certify that they have read and recommended to the College of Natural and Computational Sciences for acceptance of a project entitled **Multiplicity of positive Solutions and the Method of Upper and Lower solutions for Nonlinear Elliptic PDEs** by **Dagmawit Mershaye** in partial fulfillment of the requirement for degree of master of science in mathematics.

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Abstract

In this paper, we are interested in the positive solutions of nonlinear elliptic boundary value problems on a bounded domain. We establish the existence, uniqueness and multiplicity of positive solutions for the nonlinear elliptic boundary value problem of autonomous type,

$$\begin{cases} -\mathcal{L}u = f(\xi, u) & in \quad \Omega \\ \mathcal{B}u = c(\xi) & on \quad \partial\Omega \end{cases}$$

and then proceed to establish conditions for existence and uniqueness of positive global solutions for the non-autonomous type elliptic BVP,

$$\begin{cases} u_t - \triangle u = f(\xi, t, u) & in \quad \mathcal{D} \\ \mathcal{B}u = c(\xi, t) & on \quad \partial\mathcal{D} \\ u(\xi, 0) = u_0(\xi) & in \quad \Omega \end{cases}$$

We employ the method of lower and upper solutions along with the monotone iterative techniques to this end.

Contents

Notation

$\mathbb{R}^n$	n-tuple of real number
$\mathbb{C}$	Complex numbers
Ω	Open set in $\mathbb{R}^n$
$\overline{\Omega}$	Closure of Ω
$\partial\Omega$	Boundary of Ω
$C(\Omega)$	$\{f : f \text{ continuous } \text{ in } \Omega\}$
$C(\overline{\Omega})$	Continuouse extension of $C(\Omega)$ to $\partial\Omega$
supp f	$\overline{\{x \in \Omega : f(x) \neq 0\}}$
$C_0(\Omega)$	$\{f \in C(\Omega)$: supp f is a compact subset of $\Omega\}$
$C^k(\Omega)$	$\{f \in C(\Omega)$: f is k times continuously differentiable in$\Omega\}$
$C_0^k(\Omega)$	$C^k(\Omega) \cap C_0(\Omega)$
$C^\infty(\Omega)$	$\cap_{k=1}^{\infty} C^k(\Omega)$
$C_0^k(\Omega)$	$C^\infty(\Omega) \cap C_0(\Omega)$; test functions
$H_0^1(\Omega)$	$\overline{C_0^\infty(\Omega)}$ w.r.t the sobolev norm $H^1(\Omega)$
$L^p(\Omega)$	the space of p-integrable functions in Ω $(1 \leq p < \infty)$
$L^p_{loc}(\Omega)$	the space of locally p-integrable functions in Ω $(1 \leq p < \infty)$
$W^{k,p}(\Omega)$	Sobolev spaces on Ω
$\|.\|$	norm
$\langle , \rangle$	Inner product
∇	$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)$
Δ	$\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$
u^+	$\max\{u, 0\}$
u^-	$\max\{-u, 0\}$

Chapter 1

Introduction

The method of lower and upper solutions is known to be efficient for analyzing existence results for nonlinear problems. This method helps one to ensure the existence of a solution that lies between the lower and upper solutions by appealing to a monotone iterative scheme [24]. In other words, we have a knowledge of whereabouts of the solutions. As a result, the problem of finding a solution to the considered problem is replaced by the problem of finding two well-ordered functions that satisfy a given appropriate inequalities.

Monotone iterative schemes along with the method of lower and upper solutions render an efficient technique to prove existence of solution of differential equations of first and higher order. This method has been employed to prove various types of differential equations, namely ODEs, PDEs and FDEs [24]. It has found wider application in the study of boundary value problems of nonlinear elliptic and parabolic types long back [4] while its treatment for hyperbolic partial differential equations is relatively recent until it is considered by V. Lakshmikanthan [24].
Nonlinear elliptic boundary value problems are a class of partial differential equations that have a wide range of applications in a variety of fields such as fluid mechanics, exothermic chemical reactions (see[14]) and autocatalytic reactions to mention but a few.

The aim of this paper is to prove the existence of solutions of the problem (1.1) in smooth domains in the presence of well-ordered lower and upper solutions by imposing some requirements on the nonlinearity of f. In this regard we investigate the existence, uniqueness, and multiplicity of positive solutions as well as the existence of weak solutions of the nonlinear elliptic boundary-value problem:

$$\begin{cases} -\mathcal{L}u = f(\xi, u) & in \quad \Omega \\ \mathcal{B}u = c(\xi) & on \quad \partial\Omega \end{cases} \tag{1.1}$$

and positive global solutions for

$$\begin{cases} u_t - \triangle u = f(\xi, t, u) & in \quad \mathcal{D} \\ \mathcal{B}u = c(\xi, t) & on \quad \partial\mathcal{D} \\ u(\xi, 0) = u_0(\xi) & in \quad \Omega \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$,

$$\mathcal{D} = \Omega \times (0, T] \times \Re \;\; ; \;\; \partial\mathcal{D} = \partial\Omega \times (0, T] \times \Re$$

$\mathcal{L}$ is a linear second order uniformly elliptic operator and $\mathcal{B}$ is a linear first order boundary operator.

The following is a breakdown of how this document is structured. Preliminary concepts will be discussed in Chapter 2. The maximum principle, which is an essential tool and that plays a key role in this paper is discussed in chapter 3. In chapter 4 the existence, uniqueness and multiplicity of positive solutions as well as the existence of a weak solution for problem (1.1) and global positive solutions for (1.2) are investigated using strictly upper and strictly lower solutions along with iterative sequence method and the Amann theorem. The topics covered in this paper are pulled from existing work and are not claimed to be original.

Chapter 2

Preliminaries

2.1 Partial Differential Equations and Classification of PDEs

2.1.1 Partial Differential Equations

Definition 2.1. *A partial differential equation is an expression involving unknown function of two or more variables and its partial derivatives. Assume that Ω is an open subset of $\mathbb{R}^n$, a type of expression*

$$F(\mathcal{D}^k u, \mathcal{D}^{k-1} u, ..., \mathcal{D}u, u, x) = 0, \ x \in \Omega \tag{2.1}$$

is called a k^{th}-order partial differential equation, where

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times ... \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

and

$$u : \Omega \to \mathbb{R}$$

is the unknown.

2.1.2 Classification of PDEs

PDEs can be divided into families of similar equations based on a variety of features. Order and linearity are the two most common properties.

Order : The order of a PDE is the order of the highest derivative in the equation.

Linearity : A PDE is linear if it is linear in the unknown function and its derivatives.

1. The partial differential equation (2.1) is said to be *linear* if it has the form

$$\sum_{|\alpha|\leq k} a_\alpha(x)D^\alpha u = f(x)$$

   Here, $\alpha \in N_0^n$. This linear PDE is homogeneous if $f \equiv 0$ otherwise it is nonhomogeneous.

2. The PDE (2.1) is *semilinear* if it has the form

$$\sum_{|\alpha|\leq k} a_\alpha(x)D^\alpha u + a_0(D^{k-1}u, ..., Du, u, x) = 0$$

3. The PDE (2.1) is *quasilinear* if it has the form

$$\sum_{|\alpha|\leq k} a_\alpha(D^{k-1}u, ..., Du, u, x)D^\alpha u + a_0(D^{k-1}u, ..., Du, u, x) = 0$$

4. The PDE (2.1) is *almost linear* if it is nonlinear only with the unknown function.

5. The PDE (2.1) is *fully nonlinear* if it depends nonlinearly upon the highest order derivatives.

2.2 Initial and Boundary Value Problems

Oftentimes PDEs contain several solutions, in that case we employ auxiliary conditions to single out one of them. We try to define the conditions in such a way that a unique solution may be specified. The physical phenomena which motivates these conditions are divided into two, namely initial and boundary conditions. These auxiliary conditions (initial and boundary conditions) are critical in that the solution will not be unique without them and no numerical method will operate without them.

2.2.1 Initial Conditions

The values of the unknown function u and an appropriate derivatives of u at the initial point are known as the initial conditions, commonly known as Cauchy conditions. That is, one of the variables is interpreted as temporal variable t and conditions are imposed at some moment ($u\big|_{t=t_0} = u_0$).

2.2.2 Boundary Conditions

Boundary conditions are prescribed on the boundary of the bounded domain Ω. The three types of boundary conditions that are most popular are mentioned hereunder,

1. **Dirichlet conditions** are the prescribed values of u at each point along the boundary, $\partial\Omega$. For instance with the Laplace equation,

$$\begin{cases} \Delta u = 0 & in \quad \Omega \\ u = f & on \quad \partial\Omega \end{cases}$$

2. **Neumann conditions** are the values of the normal derivative of u prescribed at each point along the boundary $\partial\Omega$. For the Laplace equation we have,

$$\begin{cases} \Delta u = 0 & in \quad \Omega \\ \frac{\partial u}{\partial n} = g & on \quad \partial\Omega \end{cases}$$

Here $\frac{\partial u}{\partial n} = n.\nabla u$ is the normal derivative where n is the unit outward normal to the boundary surface.

3. **Robin conditions(mixed conditions)** are the values of a linear combination of u and its normal derivative prescribed at each point along the boundary $\partial\Omega$. With Laplace equation it has the following form,

$$\begin{cases} \Delta u = 0 & in \quad \Omega \\ \alpha u + \beta \frac{\partial u}{\partial n} = g & on \quad \partial\Omega \end{cases}$$

2.3 Function Spaces

Many differential equations of physics are related to linear differential operators. These operators, like matrices, act on vector spaces as linear mappings. The elements of the vector spaces are functions, and the spaces are infinite dimensional, which are new feature. A function space is a collection of functions $\mathcal{F}$ having some structure. In this section we will go over some fundamental information about function spaces.

2.3.1 Hilbert and Banach Spaces

Normed Spaces

Definition 2.2. *Suppose V is a vector space. A norm is a mapping* $\|.\| : V \to [0, \infty)$ *that satisfies*

1. $\|v\| \geq 0$ *for any* $v \in V$*, and* $\|v\| = 0$ *if and only if* $v = 0$*;*
2. $\|\lambda v\| = |\lambda|\|v\|$ *for any* $v \in V$ *and* $\lambda \in \mathbb{R}$*;*
3. $\|u + v\| \leq \|u\| + \|v\|$ *for any* $u, v \in V$*.*

The space V equipped with the norm $\|.\|$ *is called a normed space.*

A sequence $x_1, x_2, ...$ in a normed space v is a Cauchy sequence if $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. If every Cauchy sequence converges to a limit in V then the space is complete.

Definition 2.3. *A Banach space is a complete normed space.*

Inner Product Spaces

Definition 2.4. *An inner product is a mapping* $\langle ., . \rangle : V \times V \to \mathbb{R}$ *that meets the following conditions for any* $u, v, w \in V$ *and* $\lambda \in \mathbb{R}$*:*

1. $\langle u, u \rangle \geq 0$ *and* $\langle u, u \rangle = 0$ *if and only if* $u = 0$
2. $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$
3. $\langle u, \lambda v \rangle = \lambda \langle u, v \rangle$
4. $\langle u, v \rangle = \overline{\langle v, u \rangle}$

The space V equipped with the inner product $\langle ., . \rangle$ *is called an inner product space.*

An inner product defines a norm $\|u\| = \sqrt{\langle u, u \rangle}$. The Cauchy-Schwartz inequality

$$|\langle u, v \rangle| \leq \|u\|\|v\|.$$

follows. Hilbert spaces are particularly important among inner product spaces.

Definition 2.5. *A Hilbert space is a complete inner product space.*

2.3.2 L^p Spaces

In the study of $L^p(\Omega)$ spaces, we look for functions that are equal a.e. on Ω.

Definition 2.6. *For $1 \leq p < \infty$, the space $L^p(\Omega)$ consists of the Lebesgue measurable functions $f : \Omega \to \mathbb{R}$ such that*

$$\int_\Omega |f(x)|^p dx < \infty,$$

and $L^\infty(\Omega)$ consists of the essentially bounded functions.

Suppose $\mathcal{F}$ is a collection of functions taking Ω into $\mathbb{R}$. We define the L^p norm on $\mathcal{F}$

$$\begin{cases} \|f\|_p = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}} & for \quad 1 \leq p < \infty \\ \|f\|_\infty = \sup\limits_{x\in\Omega} |f(x)| & for \quad p = \infty \end{cases}$$

Hence, $L^p(\Omega)$ space is defined as follows:

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} : \|f\|_p < \infty\}$$

Each L^p represents Banach space. The following are some useful inequalities:

- Cauchy-Schwartz $(\int f(x)g(x)dx)^2 \leq \int f^2(x)dx \int g^2(x)dx$
- Minkowski $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ where $p > 1$
- Hölder $\|fg\|_1 \leq \|f\|_p\|g\|_q$ where $\frac{1}{p} + \frac{1}{q} = 1$

2.3.3 Hölder Spaces

Definition 2.7. *A function v defined on Ω is said to be Lipschitz continuous if the inequality*

$$|v(x) - v(y)| \leq c\|x-y\| \qquad \forall x, y \in \Omega.$$

holds for some positive constant $c < 1$.

Here the standard Euclidean distance between x and y is denoted by $\|x-y\|$. The Lipschitz constant of v, written as $Lip(v)$, is the smallest possible constant in the preceding inequality. The relation

$$Lip(v) = \sup \left\{ \frac{|v(x) - v(y)|}{\|x-y\|} \middle| x, y \in \Omega, x \neq y \right\}.$$

characterizes the Lipschitz constant.

Definition 2.8. *A function v is said to be Hölder continuous with exponent $\beta \in (0,1]$ if for some constant c,*

$$|v(x) - v(y)| \leq c\|x - y\|^{\beta} \qquad for\ x, y \in \Omega.$$

Definition 2.9. *The Hölder Space $C^{0,\beta}(\overline{\Omega})$ is defined as the subspace of $C(\overline{\Omega})$ containing Hölder continuous functions with the exponent β and the norm*

$$\|v\|_{C^{0,\beta}(\overline{\Omega})} = \|v\|_{C(\overline{\Omega})} + \sup\left\{\frac{|v(x) - v(y)|}{\|x - y\|^{\beta}}\middle| x, y \in \Omega, x \neq y\right\}$$

The space $C^{0,\beta}(\overline{\Omega})$ becomes a Banach space. When $\beta = 1$, the Hölder Spaces $C^{0,1}(\overline{\Omega})$ contains all Lipschitz continuous functions. We define the Hölder Spaces in the same way for $m \in \mathbb{Z}_{+}$ and $\beta \in (0,1]$ as

$$C^{m,\beta}(\overline{\Omega}) = \|v\|_{C^{m}(\overline{\Omega})} + \sum \sup\left\{D^{\alpha}\frac{|v(x) - D^{\alpha}v(y)|}{\|x - y\|^{\beta}}\middle| x, y \in \Omega, x \neq y\right\}$$

2.3.4 Sobolev Spaces

Weak Derivatives

Definition 2.10. *Let $1 \leq p \leq \infty$. A function $v : \Omega \subseteq \mathbb{R}^{n} \to \mathbb{R}$ is called locally p-integrable, $v \in L^{p}_{loc}(\Omega)$, if for every $x \in \Omega$, there is an open neighborhood Ω' of x such that $\Omega' \subset \Omega$ and $v \in L^{p}(\Omega')$.*

In particular, given any domain Ω, the set of locally integrable functions is denoted by

$$L^{1}_{loc}(\Omega) = \{v : v \in L^{1}(\Omega'),\ \Omega' \subset\subset\ \Omega\}$$

Definition 2.11. *Suppose $u, v \in L^{1}_{loc}$ and α is a multi-index. A function v is said to be the α^{th} weak partial derivative of u, denoted*

$$D^{\alpha}u = v,$$

provided

$$\int_{\Omega} uD^{\alpha}\varphi dx = -(1)^{|\alpha|}\int_{\Omega} v\varphi dx$$

for all test functions $\varphi \in C^{\infty}_{0}(\Omega)$.

Lemma 2.1. *[11](Generalized Variational Lemma) Let $v \in L^{1}_{loc}(\Omega)$ with Ω a non-empty open set in $\mathbb{R}^{n}$. If*

$$\int_{\Omega} v(x)\varphi dx = 0 \qquad \forall\varphi \in C^{\infty}_{0}, v \in L^{1}_{loc}$$

then $v = 0$ a.e. on Ω.

Lemma 2.2. *[13](Uniqueness of weak derivatives) A weak α^{th} partial derivatives of u, if it exists, is uniquely defined up to a set of measure zero.*

Proof. Assume that $v, \tilde{v} \in L^1_{loc}(\Omega)$ are both weak α^{th} partial derivatives of u i.e.

$$\int_\Omega u D^\alpha \varphi dx = -(1)^{|\alpha|} \int_\Omega v \varphi dx = -(1)^{|\alpha|} \int_\Omega \tilde{v} \varphi dx$$

for every $\varphi \in C_0^\infty(\Omega)$. This implies that $\int_\Omega (v - \tilde{v}) \varphi dx = 0$ for every $\varphi \in C_0^\infty(\Omega)$. whence, $v - \tilde{v} = 0$ a.e. □

Sobolev Spaces

Definition 2.12. *Assume that Ω is an open subset of $\mathbb{R}^n$. The sobolev space $W^{k,p}(\Omega)$ contains functions $u \in L^p(\Omega)$ such that for every multi-index α with $|\alpha| \leq k$, the weak derivative $D^\alpha u$ exists and $D^\alpha u \in L^p(\Omega)$. That is,*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \ \forall \ \alpha, \ |\alpha| \leq k\}$$

Definition 2.13. *If $u \in W^{k,p}(\Omega)$, the norm is defined as*

$$\|u\|_{W^{k,p}(\Omega)} := \Big(\int_\Omega \sum_{|\alpha| \leq k} |D^\alpha u|^p dx \Big)^{\frac{1}{p}},$$

and

$$\|u\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \leq k} ess \sup_\Omega |D^\alpha u|.$$

when $p = 2$, we write $H^k \equiv W^{k,2}(\Omega)$.

Elementary Properties

Theorem 2.1. *[13](Properties of weak derivatives) Assume that $u, v \in W^{k,p}(\Omega)$ and $|\alpha| \leq k$. Then*

1. *$D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$.*
2. *$D^\beta(D^\alpha u) = D^\alpha(D^\beta u)$ for all multi-indices α, β with $|\alpha| + |\beta| \leq k$.*
3. *For every $\lambda, \mu \in \mathbb{R}, \lambda u + \mu v \in W^{k,p}(\Omega)$ and*

$$D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v.$$

4. *If $\Omega' \subset \Omega$ is open, then $u \in W^{k,p}(\Omega')$.*

5. *If $\eta \in C_0^\infty(\Omega)$, then $\eta u \in W^{k,p}(\Omega)$ and*

$$D^\alpha(\eta u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \eta D^{\alpha-\beta} u \quad (Leibniz's formula)$$

where

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}, \ \alpha! = \alpha_1!...\alpha_n!$$

and $\beta \leq \alpha$ means that $\beta_j \leq \alpha_j$ for every j=1,...,n.

Proof. (1) This is a direct result of the concept of weak derivatives.
(2) Suppose $\varphi \in C_0^\infty(\Omega)$. Then $D^\beta\varphi \in C_0^\infty(\Omega)$. Thus

$$\begin{aligned}
(-1)^{|\beta|}\int_\Omega D^\beta(D^\alpha u)\varphi dx &= \int_\Omega D^\alpha u D^\beta \varphi dx \\
&= (-1)^{|\alpha|}\int_\Omega u D^{\alpha+\beta}\varphi dx \\
&= (-1)^{|\alpha|}(-1)^{|\alpha+\beta|}\int_\Omega D^{\alpha+\beta} u\varphi dx
\end{aligned}$$

for all test functions $\varphi \in C_0^\infty(\Omega)$. Take note of this

$$\begin{aligned}
|\alpha| + |\alpha+\beta| &= \alpha_1 + ... + \alpha_n + (\alpha_1+\beta_1) + ... + (\alpha_n+\beta_n) \\
&= 2(\alpha_1 + ... + \alpha_n) + \beta_1 + ... + \beta_n \\
&= 2|\alpha| + |\beta|.
\end{aligned}$$

Because $2|\alpha|$ is an even number, the preceding estimate, together with the uniqueness results Lemma (2.2) and Lemma(2.1) leads to the conclusion that

$$D^\beta(D^\alpha u) = D^{\alpha+\beta}u.$$

(3) and (4) are obvious.

(5) First we will look at the case $|\alpha| = 1$. Let $\varphi \in C_0^\infty(\Omega)$. By the concept of weak derivative and Leibniz's rule for differentiable functions

$$\begin{aligned}
\int_\Omega \eta u D^\alpha \varphi dx &= \int_\Omega (u D^\alpha(\eta\varphi) - u(D^\alpha \eta)\varphi) dx \\
&= -\int_\Omega (\eta D^\alpha u + u D^\alpha \eta)\varphi dx
\end{aligned}$$

for all $\varphi \in C_0^\infty(\Omega)$.

Hence, $D^\alpha(\eta u) = \eta D^\alpha u + u D^\alpha \eta$ as required.

Next assume $n < k$ and the Leibniz's formula holds true for all $|\alpha| \leq n$ and all functions η. Take the case $|\alpha| = n+1$. Then $\alpha = \beta + \gamma$ for some $|\beta| = n, |\gamma| = 1$. Then for $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned}
\int_\Omega \eta u D^\alpha \varphi dx &= \int_\Omega \eta u D^\beta (D^\gamma \varphi) dx \\
&= (-1)^{|\beta|} \int_\Omega \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} D^\sigma \eta D^{\beta-\sigma} u D^\gamma \varphi dx \\
&= (-1)^{|\beta|+|\gamma|} \int_\Omega \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} D^\gamma (D^\sigma \eta D^{\beta-\sigma} u) \varphi dx \\
&= (-1)^{|\alpha|} \int_\Omega \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} [D^\rho \eta D^{\alpha-\rho} u + D^\sigma \eta D^{\alpha-\sigma} u] \varphi dx \; ; \; \rho = \sigma + \gamma \\
&= (-1)^{|\alpha|} \int_\Omega \left[\sum_{\sigma \leq \alpha} \binom{\alpha}{\sigma} D^\sigma \eta D^{\alpha-\sigma} u \right] \varphi dx
\end{aligned}$$

The last line is in view of the basic property,

$$\binom{\beta}{\sigma - \gamma} + \binom{\beta}{\sigma} = \binom{\alpha}{\sigma}.$$

□

Completeness of Sobolev Spaces

The fact that Sobolev spaces are complete is one of their most useful properties. As a result, Sobolev spaces are closed under limits of Cauchy sequences.

A sequence $\{u_i\}$ of functions $u_i \in W^{k,p}(\Omega), i = 1, 2, ...$ converges in $W^{k,p}(\Omega)$ to a function $u \in W^{k,p}(\Omega)$, if for each $\varepsilon > 0$ there exists i_ε such that

$$\|u_i - u\|_{W^{k,p}(\Omega)} < \varepsilon \quad when \quad i \geq i_\varepsilon.$$

Equivalently,

$$\lim_{i \to \infty} \|u_i - u\|_{W^{k,p}(\Omega)} = 0.$$

A sequence $\{u_i\}$ is a Cauchy sequence in $W^{k,p}(\Omega)$, if for every $\varepsilon > 0$ there exists i_ε such that

$$\|u_i - u_j\|_{W^{k,p}(\Omega)} < \varepsilon \quad when \quad i, j \geq i_\varepsilon.$$

Theorem 2.2. *[13](**Completeness**). The Sobolev space $W^{k,p}(\Omega), 1 \leq p < \infty$, k=1,2,... is a Banach space.*

Proof. We need to look into the following points.

(1) $\|.\|_{W^{k,p}(\Omega)}$ is a norm.

(2) $W^{k,p}(\Omega)$ is complete.

1, Let us demonstrate that $\|.\|_{W^{k,p}(\Omega)}$ is a norm.
(i) $\|u\|_{W^{k,p}(\Omega)} = 0 \Leftrightarrow u = 0$ a.e in Ω.
$(\Rightarrow)$
$\|u\|_{W^{k,p}(\Omega)} = 0 \implies \|u\|_{L^p(\Omega)} = 0, \implies u = 0$ a.e in Ω.
$(\Leftarrow)$
$u = 0$ a.e in Ω implies

$$\int_\Omega D^\alpha u\varphi dx = (-1)^{|\alpha|}\int_\Omega uD^\alpha\varphi dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. This implies that $D^\alpha u = 0$ a.e in Ω for all $\alpha, |\alpha| \leq k$.

(ii) $\|\lambda u\|_{W^{k,p}(\Omega)} = |\lambda|\|u\|_{W^{k,p}(\Omega)}, \lambda \in \mathbb{R}$. Obvious.

(iii) The triangle inequality for $1 \leq p < \infty$ follows from the simple inequality $(a+b)^\alpha \leq a^\alpha + b^\alpha, a, b \geq 0, 0 < \alpha \leq 1$, and the Minkowski's inequality implies,

$$\begin{aligned}
\|u+v\|_{W^{k,p}(\Omega)} &= \Big(\sum_{|\alpha|\leq k}\|D^\alpha u + D^\alpha v\|^p_{L^p(\Omega)}\Big)^{\frac{1}{p}} \\
&\leq \Big(\sum_{|\alpha|\leq k}(\|D^\alpha u\|_{L^p(\Omega)} + \|D^\alpha v\|_{L^p(\Omega)})^p\Big)^{\frac{1}{p}} \\
&\leq \Big(\sum_{|\alpha|\leq k}\|D^\alpha u\|^p_{L^p(\Omega)}\Big)^{\frac{1}{p}} + \Big((\sum_{|\alpha|\leq k}\|D^\alpha v\|^p_{L^p(\Omega)}\Big)^{\frac{1}{p}} \\
&= \|u\|_{W^{k,p}(\Omega)} + \|v\|_{W^{k,p}(\Omega)}.
\end{aligned}$$

2. Assume that $\{u_i\}$ is a Cauchy sequence in $W^{k,p}(\Omega)$. Because

$$\|D^\alpha u_i - D^\alpha u_j\|_{L^p(\Omega)} \leq \|u_i - u_j\|_{W^{k,p}(\Omega)}, \ |\alpha| \leq k,$$

consequently, $(D^\alpha u_i)$ is a Cauchy sequence in $L^p(\Omega), |\alpha| \leq k$. The completeness of $L^p(\Omega)$ implies that there exists $u_\alpha \in L^p(\Omega)$ such that $D^\alpha u_i \to u_\alpha$ in $L^p(\Omega)$. Specifically, $u_i \to u_{(0,\dots,0)} = u$ in $L^p(\Omega)$.

3. We demonstrate that $D^\alpha u = u_\alpha, |\alpha| \leq k$. We would like to argue

$$\begin{aligned}
\int_\Omega u D^\alpha \varphi dx &= \lim_{i\to\infty} \int_\Omega u_i D^\alpha \varphi dx \\
&= \lim_{i\to\infty} (-1)^{|\alpha|} \int_\Omega D^\alpha u_i \varphi dx \\
&= (-1)^{|\alpha|} \int_\Omega u_\alpha \varphi dx
\end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega)$. The definition of the weak derivative was applied on the second line. After that, we will illustrate how to put the first and last equalities together.

For $1 < p < \infty$ let $\varphi \in C_0^\infty(\Omega)$. By Hölder's inequality we have

$$\begin{aligned}
\left| \int_\Omega u_i D^\alpha \varphi dx - \int_\Omega u D^\alpha \varphi dx \right| &= \left| \int_\Omega (u_i - u) D^\alpha \varphi dx \right| \\
&\leq \|u_i - u\|_{L^p(\Omega)} \|D^\alpha \varphi\|_{L^{p'}(\Omega)} \to 0.
\end{aligned}$$

As a result, the first inequality is obtained. In the same way, the last inequality follows since,

$$\left| \int_\Omega D^\alpha u_i \varphi dx - \int_\Omega u_\alpha \varphi dx \right| \leq \|D^\alpha u_i - u_\alpha\|_{L^p(\Omega)} \|\varphi\|_{L^{p'}(\Omega)} \to 0.$$

In a similar manner we can prove for $p = 1$ and $p = \infty$.

As a result the weak derivatives $D^\alpha u$ exist and $D^\alpha u = u_\alpha, |\alpha| \leq k$. As we also know that

$$D^\alpha u_i \to u_\alpha, |\alpha| \leq k,$$

we have come to the conclusion that

$$\|u_i - u\|_{W^{k,p}(\Omega)} \to 0.$$

Therefore, $u_i \to u$ in $W^{k,p}(\Omega)$. □

Corollary 2.1. *[11] The Sobolev space $H^k(\Omega)$ is a Hilbert space with the inner product*

$$\langle u, v \rangle = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u D^\alpha v dx \qquad u, v \in H^k(\Omega).$$

Remark 2.1. *If p = 2, we usually write*

$$H^k(\Omega) = W^{k,2}(\Omega).$$

Sobolev embedding theorems

Sobolev Embedding theorems are what make the Sobolev spaces interesting and significant. The Sobolev spaces $W^{k,p}(\Omega)$ are defined using weak derivatives. The smoothness using weak derivatives is weaker than that using the classical derivatives.

Definition 2.14. *Let V and W be two Banach spaces with $V \subseteq W$. We say the space V is continuously embedded in W and write $V \hookrightarrow W$, if*

$$\|v\|_W \leq c\|v\|_V \quad \forall v \in V. \tag{2.2}$$

If (2.2) holds and each bounded sequence in V has a convergent subsequence in W, then we say the space V is compactly embedded in W and write $V \hookrightarrow\hookrightarrow W$,

Here is the general sobolev embedding theorem for the proof see [13]

Theorem 2.3. *[15](General Sobolev embedding) Suppose $1 \leq p \leq \infty, k \in \mathbb{Z}_+$ and Ω be a bounded Lipschitz domain in $\mathbb{R}^n$.*
Case 1, $kp > n$

$$W^{k,p}(\Omega) \hookrightarrow C(\overline{\Omega})$$

Case 2, $kp = n$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega),\ for\ all\ q \in [1, \infty)$$

Furthermore

$$W^{n,1}(\Omega) \hookrightarrow C(\overline{\Omega})$$

Case 3, $kp < n$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega),\ with\ \frac{1}{q} = \frac{1}{p} - \frac{k}{p}.$$

Chapter 3

The Maximum Principle

Maximum principles are tools employed to study properties of solutions for second order elliptic as well as parabolic partial differential equations, both linear and nonlinear. The maximum principle for second order elliptic and parabolic partial differential equations is discussed in this chapter.

Definition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $n \in \mathbb{N}$, where the continuous coefficient functions a_{ij}, b_i, c : $\Omega \to \mathbb{R}$ in $C^0(\Omega)$ for i,j = 1,2,...,n are defined. Furthermore, the matrix $A(x) = \left(a_{ij}\right)_{i,j=1}^n$ be symmetric for all $x \in \Omega$. The linear partial differential operator of the second order $\mathcal{L} : C^2(\Omega) \to C^0(\Omega)$ is defined as:*

$$\mathcal{L}u := -\sum_{i,j=1}^{n} a_{ij}u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu, \qquad x \in \Omega \tag{3.1}$$

Definition 3.2. *If A(x) is an $n \times n$ symmetric positive semidefinite matrix (i.e. all the eigenvalues of A(x) are real and nonnegative), then for every $x \in \Omega$, the operator $\mathcal{L}$ is called degenerate elliptic operator.*

Definition 3.3. *Let $\mathcal{L}$ be a degenerate elliptic operator. Then $\mathcal{L}$ is uniformly elliptic in Ω if there exists $\theta > 0$ such that*

$$\xi^t a(x)\xi \geq \theta|\xi|^2 \qquad \forall x \in \Omega \;\forall \xi \in \mathbb{R}^n$$

Definition 3.4. *Let $\mathcal{D} := \Omega \times (0, T)$ for some $T > 0$. A parabolic operator is operator of the form*

$$\mathcal{P} := u_t - \sum_{i,j=1}^{n} a_{ij}u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu$$

where coefficients satisfy elliptic conditions.

We can also write $\mathcal{P}$ as: $\mathcal{P} = u_t + \mathcal{L}$

Remark 3.1. *1. Symmetric : $a_{ij}(x) = a_{ji}(x)$ for all i,j and $x \in \Omega$*

2. *Boundeness of the coefficients : There exists a constant k with*

$$|a_{ij}(x)|, |b_i(x)|, |c(x)| \leq k \qquad for\ all\ i, j\ and\ x \in \Omega$$

3.1 Maximum Principle for elliptic Operators

3.1.1 Weak Maximum Principle

We begin by determining the conditions under which a function must attain its maximum (or minimum) on the boundary. The set $\Omega \subset \mathbb{R}^n$ is always assumed to be open and bounded.

Theorem 3.1. *[13](Weak maximum principle) Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$; c = 0 in Ω*

(i) If

$$\mathcal{L}u \leq 0 \quad in \quad \Omega, \tag{3.2}$$

then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

(ii) If

$$\mathcal{L}u \geq 0 \quad in \quad \Omega, \tag{3.3}$$

then

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$$

Remark 3.2. *A lower solution is a function that satisfies (3.2). As a result, we can say that a lower solution attains its maximum on $\partial\Omega$. Similarly, if (3.3) is true, u is an upper solution and attains its minimum on $\partial\Omega$.*

Proof. We only need to prove item (i), since if u is an upper solution, then -u is a lower solution, and $\min u = -\max(u)$.
Case 1, Assume that we have strict inequality

$$\mathcal{L}u < 0 \quad in \quad \Omega \tag{3.4}$$

If $\mathcal{L}u < 0$ in Ω, then u cannot achieve its maximum anywhere in Ω. Suppose it did , say at the point $x_0 \in \Omega$ with

$$u(x_0) = \max_{\overline{\Omega}} u \tag{3.5}$$

Then all first derivatives of u vanish at this point i.e,

$$Du(x_0) = 0 \tag{3.6}$$

However, the matrix of the second partial derivative is negative semi-definite at its maximum, and we infer that

$$D^2u(x_0) \leq 0. \tag{3.7}$$

which is a contradiction.

Case 2, Since the matrix $A = \left(a_{ij}(x_0)\right)$ is symmetric and positive definite, there exists an orthogonal matrix $O = \left(a_{ij}\right)$ so that

$$OAO^T = diag(d_1, ..., d_n), \qquad OO^T = I \tag{3.8}$$

with $d_k > 0$ (k = 1,2,...,n). Write $y = x_0 + O(x - x_0)$. Then $x - x_0 = O^T(y - x_0)$, and so

$$u_{x_i} = \sum_{k=1}^{n} u_{y_k} o_{ki}, \quad u_{x_i x_j} = \sum_{k,l=1}^{n} u_{y_k y_l} o_{ki} o_{lj} \quad (i, j = 1, ..., n).$$

Hence at the point x_0,

$$\begin{aligned}
\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} &= \sum_{k,l=1}^{n} \sum_{i,j=1}^{n} a_{ij} u_{y_k y_l} o_{ki} o_{lj} \\
&= \sum_{k=1}^{n} d_k u_{y_k y_k} \\
&\leq 0
\end{aligned} \tag{3.9}$$

since d_k and $u_{y_k y_k}(x_0) \leq 0$ (k=1,2,...,n), according to (3.7).

Thus at x_0

$$\mathcal{L}u = -\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} \geq 0,$$

in the light of (3.6) and (3.9). So (3.4) and (3.5) are incomplete, and we have a contradiction. Thus for a strict lower solution the maximum must occur on $\partial\Omega$.

Case 3, For the general case, consider the function

$$u_\varepsilon = u(x) + \varepsilon e^{\gamma x_1} \qquad x \in \Omega$$

where $\gamma > 0$ will be selected below and $\varepsilon > 0$. Recall that the uniform ellipticity condition implies $a_{ij}(x) \geq \theta$ (i = 1,...,n). Therefore

$$\begin{aligned}
\mathcal{L}u_\varepsilon &= \mathcal{L}u + \varepsilon \mathcal{L}(e^{\gamma x_1}) \\
&= e^{\gamma x_1}[-\gamma^2 a_{11} + \gamma b_1] \\
&\leq e^{\gamma x_1}[-\gamma^2 \theta + \|\mathbf{b}\|_{L^\infty} \gamma] \\
&< 0 \text{ in } \Omega
\end{aligned}$$

We now choose γ sufficiently large. Then according to case 1 and 2 above we have

$$\max_{\overline{\Omega}} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon$$

Then letting $\varepsilon \to 0$ we get $\max_{\overline{\Omega}} u_0 = \max_{\partial\Omega} u_0$ this implies that u has a maximum at the boundary.

□

Theorem 3.2. *[13](Weak maximum principle for c > 0)*
Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and c > 0 in Ω,

(i) If

$$\mathcal{L}u \leq 0 \quad in\, \Omega,$$

then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ \tag{3.10}$$

(ii) If

$$\mathcal{L}u \geq 0 \quad in\, \Omega,$$

then

$$\min_{\overline{\Omega}} u \geq -\min_{\partial\Omega} u^- \tag{3.11}$$

Remark 3.3. *So in particular , if $\mathcal{L}u = 0$ in Ω, then*

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|. \tag{3.12}$$

Proof. Suppose u is a lower solution. If $u \leq 0$ throughout Ω, the theorem is trivially true. Let $V := \{x \in \Omega : u > 0\}$., then

$$\begin{aligned} Ku &:= \mathcal{L}u - cu \\ &\leq -cu \\ &\leq 0 \qquad in\ V \end{aligned}$$

Because the operator K does not include a zeroth-order term, Theorem 3.1 indicates that

$$\max_{\overline{V}} u = \max_{\partial V} u = \max_{\partial\Omega} u^+.$$

This yields (3.10) in the condition where $V \neq \emptyset$. Otherwise $u \leq 0$ throughout Ω, and (3.10) likewise follows.

□

3.1.2 Strong Maximum Principle

The weak maximum principle states that at the boundary, u reaches its maximum. The Theorem does not rule out the possibility that some of these points are interior because u can assume its maximum at multiple positions. We then show that a subsolution u cannot reach its maximum at an interior point of a connected region unless u is constant, which significantly strengthens the preceding arguments. This is the strong maximum principle, which is based on a boundary maximum point analysis of the outer normal derivative $\frac{\partial u}{\partial v}$. Let's start with the Hopf's Lemma before moving on to the proof of the strong maximum principle.

Lemma 3.1. *[13](Hopf's Lemma) Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and c = 0 in Ω, Assume further*

$$\mathcal{L}u \leq 0 \ in\ \Omega$$

and there exists a point $x_0 \in \partial\Omega$ such that

$$u(x_0) > u(x)\ for\ all\ x \in \Omega \tag{3.13}$$

Assume finally that Ω fulfills the interior ball condition at x_0; that is, there exists an open ball $B \subset \Omega$ with $x_0 \in \partial B$.

(i) Then

$$\frac{\partial u}{\partial v}(x_0) > 0,$$

where v is the outward unit normal to B at x_0.

(ii) If

$$c \geq 0 \; in \; \Omega$$

the same result applies if

$$u(x_0) \geq 0.$$

The purpose of (i) is the strict inequality i.e, $\frac{\partial u}{\partial v}(x_0) > 0$ is self evident. It is important to note that if $\partial\Omega$ is C^2, the interior ball condition holds automatically.

Proof. 1, Suppose $c \geq 0$. We may as well further suppose $B = B_0(0, r)$ for some radius $r > 0$. Define

$$v(x) := e^{-\lambda|x|^2} - e^{-\lambda r^2} \qquad x \in B(0, r)$$

for $\lambda > 0$ as selected below. We then compute using the uniform ellipticity condition.

$$\begin{aligned} \mathcal{L}v &= -\sum_{i,j=1}^{n} a_{ij} v_{x_i x_j} + \sum_{i=1}^{n} b_i v_{x_i} + cv \\ &= e^{-\lambda|x|^2} \sum_{i,j=1}^{n} a_{ij}(-4\lambda^2 x_i x_j + 2\lambda\delta_{ij}) \\ &\quad -e^{-\lambda|x|^2} \sum_{i=1}^{n} b_i 2\lambda x_i + c(e^{-\lambda|x|^2} - e^{-\lambda r^2}) \\ &\leq e^{-\lambda|x|^2}(-4\theta\lambda^2|x|^2 + 2\lambda tr\mathbf{A} + 2\lambda|\mathbf{b}||x| + c), \end{aligned}$$

for $\mathbf{A} = \left(a_{ij}\right)$, $b = (b_1, ..., b_n)$. Next, consider the open annular region

$$R := B_0(0, r) - B(0, \frac{r}{2}).$$

We have,

$$\mathcal{L}v \;\leq\; e^{-\lambda|x|^2}(-\theta\lambda^2 r^2 + 2\lambda tr\mathbf{A} + 2\lambda|\mathbf{b}|r + c) \leq 0 \tag{3.14}$$

in R, provided $\lambda > 0$ is fixed large enough.

2, Because of (3.13) there exists a constant $\epsilon > 0$ sufficiently small that

$$u(x_0) \geq u + \epsilon v \qquad x \in \partial B(0, \frac{r}{2}). \tag{3.15}$$

Also, keep in mind

$$u(x_0) \geq u - \epsilon v \qquad x \in \partial B(0, r) \tag{3.16}$$

Since $v \equiv o$ on $\partial B(0, r)$.

3, From (3.14) we see

$$\mathcal{L}(u + \epsilon v - u(x_0)) \leq -cu(x_0) \leq 0 \, in \, R,$$

and from (3.15) and (3.16) we observe

$$u + \epsilon v - u(x_0) \leq 0 \, on \, \partial R.$$

Because of weak maximum principle, Theorem (3.2), $u + \epsilon v - u(x_0) \leq 0$ in R. But $u(x_0) + \epsilon v(x_0) - u(x_0) = 0$, and hence,

$$\frac{\partial u}{\partial v}(x_0) + \epsilon \frac{\partial u}{\partial v}(x_0) \geq 0.$$

As a result,

$$\frac{\partial u}{\partial v}(x_0) \geq -\epsilon \frac{\partial u}{\partial v}(x_0) = -\frac{\epsilon}{r} \mathcal{L}v(x_0).x_0 = 2\lambda \epsilon r e^{-\lambda r^2} > 0.$$

as required. □

We are now in a position to prove the strong maximum principle.

Theorem 3.3. *[13](Strong maximum principle). Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and c = 0 in Ω. Suppose also Ω is connected, open and bounded.*

(i) if

$$\mathcal{L}u \leq 0 \; in \; \Omega$$

and u attains its maximum over $\overline{\Omega}$ at an interior point, then u is constant in Ω.

(ii) Similarly, if

$$\mathcal{L}u \geq 0 \; in \; \Omega$$

and u attains its minimum over $\overline{\Omega}$ at an interior point, then u is constant in Ω.

Proof. Assume that $M := \max_{\overline{\Omega}} u$ and $C := \{x \in \Omega : u(x) = M\}$. Then we prove that if $C \neq \emptyset$ then $C = \Omega$ (so u is constant). set

$$V := \{x \in \Omega : u < M\}.$$

V is open in Ω. We assert that $\partial V = \emptyset$, so $V = \overline{V}$ in Ω. This implies that V is closed and open. So $V = \emptyset$ since Ω is connected and $\Omega \setminus V \neq \emptyset$. Assume by contradiction that $\partial V \cap \Omega \neq \emptyset$. This implies that there exist $y \in V$ satisfying $dist(y, C) < dist(y, \partial\Omega) = r$. So $B(y,r) \subset V$ and $\partial B(y,r) \cap \partial V \neq \emptyset$, whereas $\partial B(y,r) \cap \partial\Omega = \emptyset$. Let $x_0 \in \partial B(y,r) \cap \partial V$. So, there exists $r' < r$ and $y' \in [x, y]$ (y′ is in the segment connecting y,x) such that $B(y', r') \subset V$ and $\partial B(y', r') \cap \partial V = \{x_0\}$. So $u(z) < M$ and since $x_0 \in \Omega, Du(x_0) = 0$. However, if Hopf Lemma were applied to the set B(y′,r′) we would have $\frac{\partial u}{\partial v}(x_0) > 0$. But this is a contradiction since u attains its maximum at $x_0 \in \Omega$, we have $Du(x_0) = 0$. □

Theorem 3.4. *[13](Strong maximum principle with $c \geq 0$) Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $c \geq 0$ in Ω. Suppose also Ω is connected.*

(i) If

$$\mathcal{L}u \leq 0 \ in \ \Omega$$

and u attains a nonnegative maximum over $\overline{\Omega}$ at an interior point, then u is constant in Ω.

(ii) Similarly, if

$$\mathcal{L}u \geq 0 \ in \ \Omega$$

and u attains a non positive minimum over $\overline{\Omega}$ at an interior point, then u is constant in Ω

We can prove this theorem in the same way we proved the last one, except we will use statement (ii) from Hopf's Lemma.

Remark 3.4. *In the weak maximum principle a lower solution attains its maximum on the boundary, However, in the strong maximum principle a lower solution can not attain its maximum at an interior point unless it is constant.*

3.2 Maximum Principle for Parabolic Operators

3.2.1 Weak Maximum Principle

In this section we will consider parabolic operator of the form $u_t + \mathcal{L}u + cu$.

Definition 3.5 (parabolic boundary). *Let $\mathcal{D} = \Omega \times (0, T)$. Then the parabolic boundary is*

$$\partial^* \mathcal{D} = \partial\Omega \times [0, T] \cup \overline{\Omega} \times \{0\}.$$

We assume the solutions to the parabolic problem are classical, in the sense that belong to the following set

$$C^{1,2}(\mathcal{D}) = \{u : D \to \mathrm{R} | u(.,t) \in C^1(\Omega), u(x,.) \in C^2(0,T) \;\; \forall x \in \Omega, \; t \in (0,T)\}.$$

Theorem 3.5. *[9](weak maximum principle for parabolic operator) Let Ω be a bounded open set and $u \in C(\overline{\mathcal{D}}) \cap C^{1,2}(\mathcal{D})$ such that $u_t + \mathcal{L}u + cu \leq 0$. Assume moreover that $c \geq 0$.*

- *If $c \equiv 0$, then $\max_{\overline{\mathcal{D}}} u = \max_{\partial^* \mathcal{D}} u$,*
- *If $c \not\equiv 0$, then $\max_{\overline{\mathcal{D}}} u \leq \max_{\partial^* \mathcal{D}} u^+$, where $u^+(y) = \max(u(y), 0)$.*

The proof this theorem goes like the elliptic case.

3.2.2 Strong Maximum Principle

In this section we will consider uniformly parabolic operators according to the following definition.

Definition 3.6. *Let $u_t + \mathcal{L}u$ be a parabolic operator. Then it is uniformly parabolic in $\mathcal{D}$ if there exists $\theta > 0$ such that*

$$\xi^t a(x,t) \xi \geq \theta |\xi|^2 \qquad \forall (x,t) \in \mathcal{D} \;\; \forall \xi \in \mathbb{R}^n.$$

Remark 3.5. *Note that uniformly parabolic operator is a degenerate elliptic operator (not uniformly elliptic!).*

Also for parabolic operators, there is a strong maximum principle, that we are not going to prove (see[13])

Theorem 3.6. *[9](strong maximum principle) Let Ω be a connected set and $u \in C(\overline{\mathcal{D}}) \cap C^{1,2}(\mathcal{D})$ such that $u_t + \mathcal{L}u + cu \leq 0$. Assume moreover that $c \geq 0$.*

- *If $c \equiv 0$, and there exists $(x,t) \in \mathcal{D} such that M = \max_{\overline{\mathcal{D}}} u = u(x,t)$, then $u(y,s) \equiv M$ for all $y \in \overline{\Omega}$ and all $s \in [0,t]$;*
- *If $c \not\equiv 0$, and there exists $(x,t) \in \mathcal{D} such that M = \max_{\overline{\mathcal{D}}} u = u(x,t) \geq 0$, then $u(y,s) \equiv M$ for all $y \in \overline{\Omega} and all s \in [0,t]$.*

Remark 3.6. *We can state this maximum principle as follows: if u attains a maximum (a nonnegative maximum if $c \not\equiv 0$) at some interior point, then u is constant.*

3.3 Comparison Principle

3.3.1 Weak Comparison Principle

Corollary 3.1. *(Weak Comparison Principle) Let $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ such that*

$$\begin{cases} \mathcal{L}u \leq \mathcal{L}v & in \quad \Omega \\ u \leq v & in \quad \partial\Omega \end{cases}$$

Then, $u \leq v$ in Ω.

Proof. Let $w = u - v$. Then $\mathcal{L}w \geq 0$ in Ω and $w \leq 0$ on $\partial\Omega$, then by theorem 3.1 and 3.2 above we have

$$\begin{aligned} \max |u| &\leq \max(\max_{\bar{\Omega}} w, -\min_{\bar{\Omega}} w) \\ &\leq \max(\max_{\partial\Omega} w^+, -\min_{\partial\Omega} w^-) \\ &\leq \max_{\partial\Omega} |w| \end{aligned}$$

Then applying the above result to w with $w = 0$ on $\partial\Omega$ we have that $w \equiv 0$ in Ω. Hence,

$$\begin{aligned} u - v &\leq w \\ u - v &\leq 0 \\ u &\leq v. \end{aligned}$$

□

3.3.2 Strong Comparison Principle

Corollary 3.2. *(strong comparison principle) Let $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ and $c \leq 0$ such that*

$$\begin{cases} \mathcal{L}u \leq \mathcal{L}v & in \quad \Omega \\ u \leq v & in \quad \partial\Omega \end{cases}$$

Then, either $u \equiv v$ or $u < v$ in Ω.

Proof. suppose $w = u - v$. then $\mathcal{L}w \leq 0$ in Ω and $w \leq 0$ on $\partial\Omega$. If there exists $x \in \Omega$ such that $w(x) = 0$, then by the strong maximum principle, w is constant, so $w \equiv 0$. If such x does not exist, then $w < 0$ in Ω. □

3.4 Liouville type results

We prove a Liouville type theorem for upper solutions of uniformly elliptic operators using strong and weak maximum principle for uniformly elliptic operators. So, in this section $\mathcal{L}$ will be a uniformly elliptic operator in $\mathbb{R}^n$. Furthermore, we assume that $\mathcal{L}$ has an upper solution that blows up at infinity.

Proposition 3.1. *(Liouville type result) Suppose that $\mathcal{L}$ is uniformly elliptic and assume there exists $M > 0$ and $w \in C^2(R^n \backslash B(0, M))$ such that*

- $\mathcal{L}w(x) \geq 0 \;\; for \;\; every \;\; |x| > M$
- $\lim\limits_{|x| \to +\infty} w(x) = +\infty.$

holds. Let $u \in C^2(\mathbb{R}^n)$ such that $\mathcal{L}u \leq 0$ in $\mathbb{R}^n$ and $u(x) \leq C$ for every $x \in \mathbb{R}^n$. Then u is constant.

Proof. Suppose u is bounded from above and $\varepsilon > 0$. Define $v_\varepsilon = u - \varepsilon w$ for $|x| > 2M$. Then $v_\varepsilon \in C^2\{x \in \mathbb{R}^n, |x| \geq 2M\}$ and $\lim_{|x| \to +\infty} v_\varepsilon(x) = -\infty$ and $\mathcal{L}v_\varepsilon = \mathcal{L}u - \varepsilon \mathcal{L}w \leq 0$ for every $|x| > 2M$. Define $C_\varepsilon = \max_{|x|=2M} v_\varepsilon(x)$. So, since $\lim_{|x| \to +\infty} v_\varepsilon(x) = -\infty$, there exists $K_\varepsilon > 2M$ such that $v_\varepsilon(x) < C_\varepsilon$ for every $|x| \geq K_\varepsilon$.

Furthermore, by weak maximum principle in the set $\{x \in \mathbb{R}^n : 2M < |x| < K_\varepsilon\}$ we have that

$$\max_{\{x \in \mathbb{R}^n : 2M \leq |x| \leq K_\varepsilon\}} v_\varepsilon(x) = \max_{\{x \in \mathbb{R}^n : |x| = 2M or |x| = K_\varepsilon\}} v_\varepsilon(x). \tag{3.17}$$

since $v_\varepsilon(x) < C_\varepsilon$ for every $|x| \geq K_\varepsilon$, we obtain from (3.17) that for every $|y| \geq 2M$

$$\begin{aligned} v_\varepsilon(y) &= u(y) - \varepsilon w(y) \\ &\leq \max_{\{x \in \mathbb{R}^n : |x| = 2M\}} v_\varepsilon(x) \\ &\leq \max_{\{x \in \mathbb{R}^n : |x| = 2M\}} u(x) - \varepsilon \min_{\{x \in \mathbb{R}^n : |x| = 2M\}} w(x). \end{aligned} \tag{3.18}$$

Letting $\varepsilon \to 0$ in (3.18), we obtain

$$u(y) \leq \max_{\{x \in \mathbb{R}^n : |x| = 2M\}} u(x) \forall |y| > 2M.$$

Moreover by weak maximum principle applied in $B(0, 2M)$, we get that

$$u(y) \leq \max_{\{x \in \mathbb{R}^n : |x| = 2M\}} u(x) \forall |y| < 2M.$$

Putting together the last two inequalities we get

$$u(y) \leq \max_{\{x \in \mathbb{R}^n : |x| = 2M\}} u(x) \quad \forall y \in \mathbb{R}^n.$$

As a result u attains a maximum in some point in $\partial B(0, 2M)$, so u is constant according to strong maximum principle. $\square$

Chapter 4

The Method of Lower and Upper Solution

4.1 Multiplicity of Positive Solutions of Nonlinear Elliptic PDEs

4.1.1 Definitions

Definition 4.1. *(i) A function $\underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ is a lower solution of (1.1) if $\underline{u}$ fulfils the following inequality*

$$\begin{cases} -\mathcal{L}\underline{u} \geq f(\xi, \underline{u}) & in \quad \Omega \\ \mathcal{B}\underline{u} \leq c & on \quad \partial\Omega \end{cases} \tag{4.1}$$

(ii) A function $\overline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ is an upper solution of (1.1) if $\overline{u}$ fulfils the inequality,

$$\begin{cases} -\mathcal{L}\overline{u} \leq f(\xi, \overline{u}) & in \quad \Omega \\ \mathcal{B}\overline{u} \geq c & on \quad \partial\Omega \end{cases} \tag{4.2}$$

Remark 4.1. *1. If u is a lower and an upper solution of (1.1) at the same time, then u is a solution.*

2. The pair $\underline{u}, \overline{u}$ are said to be ordered if $\underline{u} \leq \overline{u}$ on $\overline{\Omega}$.

Here and elsewhere $\Omega \subset \mathbb{R}^n$ is a bounded domain.

Definition 4.2. *1, A map $\underline{u}$ is said to be a strict lower solution of (1.1) if,*

(i) $\underline{u}$ is a lower solution of (1.1) and

(ii) $u > \underline{u}$ for all solutions of (1.1), such that $u(t) \geq \underline{u}(t)$ holds for all $t \in [0, T]$.

2, A map $\bar{u}$ is said to be a strict upper solution of (1.1) if

(i) $\bar{u}$ is an upper solution of (1.1) and

(ii) $u < \bar{u}$ for all solutions of (1.1) with $u(t) \leq \bar{u}(t)$ hold for all $t \in [0, T]$.

Assume that a pair of ordered upper and lower solutions $\overline{u}, \underline{u}$ of (1.1) exists and define p as

$$P = \{u \in C(\overline{\Omega}) : \underline{u} \leq u \leq \overline{u}\}.$$

(A.1) Suppose that f is a monotone nondecreasing function and $c > 0$ such that

(i) for any $u_1, u_2 \in C^2(\Omega, \mathbb{R})$ and $\xi \in \Omega$, f satisfies

$$u_1 < u_2 \Rightarrow f(\xi, u_1) < f(\xi, u_2). \tag{4.3}$$

(ii) for $\underline{u}, \overline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ with $\underline{u} \leq \overline{u}$ on $\overline{\Omega}$, and

$$\min \underline{u} \leq u_1 \leq u_2 \leq \max \overline{u},$$

assume that there exists $\sigma > 0$ such that the following inequality holds,

$$f(\xi, u_1) - f(\xi, u_2) > -\sigma(u_1 - u_2), \tag{4.4}$$

(iii) the inequalities $f(\xi, 0) \geq 0, c(\xi', 0) \geq 0, u_0(\xi) \geq 0$ hold for every $\xi \in \Omega, \xi' \in \partial\Omega$.

To demonstrate the existence of solutions we begin with either

$$u^{(0)} = \underline{u} \quad or \quad u^{(0)} = \overline{u},$$

as initial iteration and then build a sequence $\{u^{(n)}\}$ using the following linear iteration process:

$$\begin{cases} -(\mathcal{L} - \sigma)u^{(n)} = f(\xi, u^{(n-1)}) + \sigma u^{(n-1)} & in \quad \Omega, \\ \mathcal{B}u^{(n)} = c & on \quad \partial\Omega. \end{cases}$$

Then, we obtain an existence theorem of the maximal and minimal solutions in the subsection that follows.

4.1.2 Existence and Multiplicity Results

Theorem 4.1. *[18] Suppose that the assumption (A.1) holds. Let $\underline{u}, \overline{u}$ be lower and upper solutions of (1.1) respectively such that $\underline{u} \leq \overline{u}$ and $f(\xi, u)$ be a smooth map* *with* $\min \underline{u} \leq u \leq \max \overline{u}$. *Then, there exists two nonnegative solutions* u_1 *and* u_2 *of problem (1.1) with*

$$\underline{u} \leq u_1 \leq u_2 \leq \overline{u}.$$

Proof. By (A.1) clearly $\underline{u} = 0$ is a lower solution of (1.1).
Define

$$T : C(\bar{\Omega}) \to C^2(\Omega) \cap C(\bar{\Omega})$$

as $w = Tu$, and

$$\begin{cases} -(\mathcal{L} - \sigma)w = f(\xi, u) + \sigma u & in \quad \Omega \\ \mathcal{B}w = c & on \quad \partial\Omega. \end{cases}$$

by Schauder estimation (regularity theorem) for elliptic equations (Leray and Schauder define a completely continuous mapping from a metric space A into a metric space B as a continuous mapping on A which takes bounded subsets of A into relatively compact ones of B.), T is completely continuous operator because it takes C^α into $C^{2+\alpha}$.

In addition, the operator T is also monotone increasing i.e,

$$u_1 < u_2 \Rightarrow Tu_1 < Tu_2,$$

provided that u_1 and u_2 are restricted to the set $\min \underline{u} \leq u_1, u_2 \leq \max \overline{u}$. In fact for $u_1 \leq u_2$ we have,

$$\begin{cases} -(\mathcal{L} - \sigma)(Tu_2 - Tu_1) = f(\xi, u_2) - f(\xi, u_1) + \sigma(u_2 - u_1) & in \quad \Omega \\ \mathcal{B}(Tu_2 - Tu_1) = 0 & on \quad \partial\Omega. \end{cases} \tag{4.5}$$

And hence,

$$\begin{cases} -\mathcal{L}(Tu_2 - Tu_1) \geq -\sigma(u_2 - u_1) + \sigma(u_2 - u_1) & in \quad \Omega \\ \mathcal{B}(Tu_2 - Tu_1) = 0, & on \quad \partial\Omega. \end{cases}$$

Therefore,

$$\begin{cases} -\mathcal{L}(Tu_2 - Tu_1) \geq 0 & in \quad \Omega \\ \mathcal{B}(Tu_2 - Tu_1) = 0 & on \quad \partial\Omega. \end{cases}$$

Thus, $Tu_1 < Tu_2$ *in* Ω.

Putting $u_0 = \underline{u}$ or $u_0 = \overline{u}$, one can generate the sequence

$$\{u_n\} = \{Tu_{n-1}\}$$

as

$$\begin{cases} -(\mathcal{L}-\sigma)u_n = f(\xi, u_{n-1}(\xi)) + \sigma u_{n-1} & in \quad \Omega, \\ \mathcal{B}u_k = c & on \quad \partial\Omega. \end{cases}$$

We set $\{\underline{u}^{(n)}\}$ when $u^{(0)} = \underline{u}$, and $\{\bar{u}^{(n)}\}$ when $u^{(0)} = \overline{u}$. Then, by the continuity of T the sequence $\{\bar{u}^{(n)}\}$ and $\{\underline{u}^{(n)}\}$ converges monotonically to $\bar{u}_{\max}$ and $\underline{u}_{\min}$ respectively. Hence,

$$\bar{u} = \bar{u}_{\max} \text{ and } \underline{u} = \underline{u}_{\min}$$

are two fixed point of T. This completes the proof. □

Corollary 4.1. *[18] Suppose $\bar{u}_{max}$ and $\underline{u}_{min}$ are solutions of (1.1). If w is a solution of (1.1) with $\underline{u} \leq w \leq \bar{u}$, then it satisfies the inequalities $\underline{u}_{min} \leq w \leq \bar{u}_{max}$.*

Proof. By theorem (4.1), we have $w = Tw$ and $\bar{u}_1 = T\bar{u}$, since $w \leq \overline{u}$ we have $w < \bar{u}_1$. By induction $w < \bar{u}_1^{(n)}$ for every n. Hence, $w \leq \bar{u}_{\max}$. In the same way we can show that $w \geq \underline{u}_{\min}$. Therefore, $\underline{u}_{\min} \leq w \leq \bar{u}_{\max}$. □

Theorem 4.2. *[18] Suppose (A.1) holds. Then (1.1) has positive local solution $u^+(\xi)$.*

Proof. Theorem (4.1) denotes existence of lower and upper solution. Then the existence of at least one positive local solution $u^+(\xi)$ of (1.1) comes from theorem 4.1. □

The monotone iterative approach for elliptic boundary value problems is based on the positivity lemma, which is crucial in nonlinear elliptic boundary value problems. For the sake of illustration of the uniqueness of positive solutions, a lemma is introduced here.

Lemma 4.1. *Assume that $g, \underline{u}, \overline{u}$ are nonnegative bounded functions, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying the following inequalities:*

$$\begin{cases} -\mathcal{L}u + gu \geq 0 & in \quad \Omega, \\ \mathcal{B}u \geq 0 & on \quad \partial\Omega. \end{cases}$$

Then, $u \geq 0$ in $\overline{\Omega}$. Moreover, $u > 0$ in Ω, unless $u = 0$.

Now we will proceed on the basis of the following assumption:

(A.2) Suppose $u_1, u_2 \in$ P with $u_1 \leq u_2$, $f_1(\xi) \in \Omega$ is a bounded nonnegative map and the map $f(\xi, u)$ satisfies the inequality

$$f(\xi, u_1) - f(\xi, u_2) \geq -f_1(\xi)(u_1 - u_2) \text{ in } \Omega.$$

The uniqueness result of positive solutions for problem(1.1) follows from the next theorem.

Theorem 4.3. *[18] Assume that $\underline{u}(\xi), \overline{u}(\xi)$ are lower solution and upper solution of problem (1.1) respectively. If in (A.1) (i) and (iii) are satisfied and (A.2) hold true, then problem (1.1) has a unique positive solution in P.*

Proof. We only need to proof the uniqueness of positive solution since the existence of positive solution is already established in theorem (4.1).

Let $u_1, u_2 \in P$ be two positive solutions with $u_1 \leq u_2$. If we set $w = u_1 - u_2$, then $w \leq 0$ and by (A.2) and hence

$$\begin{cases} -\mathcal{L}w = f(\xi, u_1) - f(\xi, u_2) \geq 0 & in \quad \Omega \\ \mathcal{B}w = c(\xi) - c(\xi) = 0 & on \quad \partial\Omega \end{cases}$$

We have w = 0 in $\partial\Omega$ so by using Lemma (4.1) we have w = 0 in Ω . Hence $u_1 - u_2 = 0$. Therefore, $u_1 = u_2$. □

We then use Amann Theorem [4] to demonstrate the existence of multiple positive solutions. Let $\underline{u}_1, \underline{u}_2$ be two lower solutions and $\overline{u}_1, \overline{u}_2$ be two upper solutions of problem (1.1).

Theorem 4.4. *[4] Suppose that E is a Banach space, $K \subset E$ is a normal solid cone. Assume that there are $\underline{u}_1, \underline{u}_2, \overline{u}_1, \overline{u}_2 \in E$ such that $\underline{u}_1 < \overline{u}_1 < \underline{u}_2 < \overline{u}_2$. Then the operator $\mathcal{L} : [\underline{u}_1, \overline{u}_2] \to E$ satisfying*

$$\underline{u}_1 \leq \mathcal{L}\underline{u}_1 \ , \ \mathcal{L}\overline{u}_1 \leq \overline{u}_1 \ , \ \underline{u}_2 < \mathcal{L}\underline{u}_2 \ , \ \mathcal{L}\overline{u}_2 \leq \overline{u}_2$$

has at least three fixed points u_1, u_2, u_3 such that

$$\underline{u}_1 < u_1 < \overline{u}_1 \ , \ \underline{u}_2 < u_2 \leq \overline{u}_2 \ , \ \underline{u}_2 \nleq u_3 \nleq \overline{u}_1.$$

Theorem 4.5. *[18] Assume that (A.1) holds. Suppose that $\underline{u}_1, \underline{u}_2$ are lower solutions and $\overline{u}_1, \overline{u}_2$ are upper solutions of (1.1) such that $\underline{u}_2, \overline{u}_1$ are strict with $\underline{u}_1 < \overline{u}_1 < \underline{u}_2 < \overline{u}_2$. In this case (1.1) has at least three solutions u_1, u_2, u_3 such that*

$$\underline{u}_1 \leq \mathcal{L}\underline{u}_1 \ , \ \mathcal{L}\overline{u}_1 \leq \overline{u}_1 \ , \ \underline{u}_2 < \mathcal{L}\underline{u}_2, \mathcal{L}\overline{u}_2 \leq \overline{u}_2$$

.

Proof. We shall show that $\mathcal{L}$ is strongly increasing operator. Equivalently saying for all $u_1, u_2, u_3 \in [\underline{u}_2, \overline{u}_2]$ with $u_1 < u_2, u_1(\xi) \leq u_2(\xi)$ and $u_1(\xi) \neq u_2(\xi)$. As a result of (A.1), we have

$$f(\xi, u_1) - f(\xi, u_2) \geq 0 \, for \, all \xi \in \Omega$$

Since $u_1(\xi) \neq u_2(\xi)$ there exists a neighborhood $\Omega' \subset \Omega$ such that $u_1(\xi) \leq u_2(\xi)$ for $\xi \in \Omega'$. Thus by $(A.1)$ we have for all $\xi \in \Omega$

$$f(\xi, u_1) - f(\xi, u_2) \geq 0 \, for \, all \xi \in \Omega'. \tag{4.6}$$

By using (4.6),we have

$$-(\mathcal{L} - \sigma)(Tu_2 - Tu_1) = f(\xi, u_1) - f(\xi, u_2) + \sigma(u_2 - u_1)$$

$$\geq -\sigma(u_2 - u_1) + \sigma(u_2 - u_1) = 0 \, for \, all \, \xi \in \Omega$$

Thus, by strong maximum principle, $Tu_2 < Tu_1$ in Ω and we have come to the conclusion that T is strongly increasing operator. Next, we will show that $\underline{u}_1 \leq T\underline{u}_1$. Consider the following problem:

$$\begin{cases} -(\mathcal{L} - \sigma)(T\underline{u}_1 - \underline{u}_1) = f(\xi, \underline{u}_1) + \sigma \underline{u}_1 & in \quad \Omega \\ \mathcal{B}(T\underline{u}_1 - \underline{u}_1) = c & on \quad \partial\Omega. \end{cases}$$

As $\underline{u}_1$ a lower solution of (1.1) we have

$$\begin{aligned} -(\mathcal{L} - \sigma)(T\underline{u}_1 - \underline{u}_1) &= -(\mathcal{L} - \sigma)T\underline{u}_1 + (\mathcal{L} - \sigma)\underline{u}_1 \\ &= -(\mathcal{L} - \sigma)T\underline{u}_1 + \mathcal{L}\underline{u}_1 - \sigma\underline{u}_1 \\ &= -(\mathcal{L} - \sigma)T\underline{u}_1 - \sigma\underline{u}_1 + \mathcal{L}\underline{u}_1 \\ &\geq f(\xi, \underline{u}_1) - f(\xi, \underline{u}_1) + \sigma\underline{u}_1 - \sigma\underline{u}_1 \geq 0 \end{aligned}$$

Hence, $-(\mathcal{L} - \sigma)(T\underline{u}_1 - \underline{u}_1) \geq 0$ and we get $T\underline{u}_1 \geq \underline{u}_1$ by strong maximum principle. As $\underline{u}_1$ a lower solution of (1.1) we have

$$\begin{aligned} \mathcal{B}(T\underline{u}_1 - \underline{u}_1) &= \mathcal{B}(T\underline{u}_1) - \mathcal{B}(\underline{u}_1) \\ &\geq \mathcal{B}(T\underline{u}_1) - c \\ &= c - c = 0 \end{aligned}$$

Thus, $\mathcal{B}(T\underline{u}_1 - \underline{u}_1) \geq 0$, i.e by strong maximum principle we conclude that $T\underline{u}_1 \geq \underline{u}_1$. Correspondingly, we have $T\underline{u}_2 \geq \underline{u}_2$. We know that $T\underline{u}_1 \neq \underline{u}_1$. since $\underline{u}_2$ is a lower solution of (1.1), it is strict solution of (1.1). Thus $\underline{u}_2 < \mathcal{L}\underline{u}_2$.
In a similar manner we get

$$\mathcal{L}\overline{u}_1 \leq \overline{u}_1 \, , \, \mathcal{L}\overline{u}_2 \leq \overline{u}_2$$

By Theorem (4.4), T has at least three fixed points u_1, u_2, u_3 with

$$\underline{u}_1 < u_1 < \overline{u}_1 \,,\; \underline{u}_2 < u_2 \leq \overline{u}_2 \,,\; \underline{u}_2 \nleq u_3 \nleq \overline{u}_1$$

□

In the Corollary that follows and the discussions thereafter,

$$\mathcal{D} = \Omega \times (0, T]$$

Corollary 4.2. *[18] Suppose that (A.1) holds. Let $\underline{u}_1, \underline{u}_2$ be strict lower solutions and $\overline{u}_1$ be strict upper solution of (1.1) such that $\underline{u}_1 < \overline{u}_1 < \underline{u}_2$. Then (1.1) has at least three solutions u_1, u_2, u_3 such that*

$$\underline{u}_1 \leq \mathcal{L}\underline{u}_1 \,,\; \mathcal{L}\overline{u}_1 \leq \overline{u}_1 \,,\; \underline{u}_2 < \mathcal{L}\underline{u}_2 \,,\; \mathcal{L}\overline{u}_2 \leq \overline{u}_2.$$

Proof. Assume that there are two upper solutions $\overline{u}_1, \overline{u}_2$ satisfying

$$\underline{u}_1(\xi) \leq \overline{u}_1(\xi) < \overline{u}_2(\xi) < \underline{u}_2(\xi) \; for \; \xi \in \Omega \cup \partial\mathcal{D}$$

Let $\overline{u}_1$ and $\overline{u}_2$ be upper solutions such that $\overline{u}_1(\xi) < \overline{u}_2(\xi)$ for $\xi \in \Omega$ and

$$\frac{\partial \overline{u}_1}{\partial \nu}(\xi) > \frac{\partial \overline{u}_2}{\partial \nu}(\xi) \; for \; \xi \in \partial\Omega.$$

As a result, we only need to show that

$$\overline{u}_2(\xi) < \underline{u}_2(\xi) \; for \; \xi \in \Omega$$

Keep in mind:

$$\underline{u}_2(\xi) - \overline{u}_1(\xi) = 0 \; for \; \xi \in \partial\Omega$$

$$\underline{u}_2(\xi) - \overline{u}_1(\xi) > 0 \; for \; \xi \in \Omega \cup \partial\mathcal{D}$$

Provided that

$$\frac{\partial \underline{u}_2}{\partial \nu}(\xi) - \frac{\partial \overline{u}_1}{\partial \nu}(\xi) < 0 \; for \; \xi \in \partial\Omega.$$

Then by strong maximum principle we have

$$\underline{u}_2(\xi) - \overline{u}_1(\xi) > 0 \; for \; \xi \in \Omega$$

Assume there is a $\xi_0 \in \partial\Omega$ with

$$\frac{\partial \underline{u}_2}{\partial \nu}(\xi_0) - \frac{\partial \overline{u}_1}{\partial \nu}(\xi_0) = 0$$

Let $w = \underline{u}_2 - \overline{u}_1$. since $\frac{\partial w}{\partial \nu}(\xi_0) > 0$, we find an open ball B_{ξ_0} such that $\frac{\partial w}{\partial \nu}(\xi) > 0$ for all $y \in B_{\xi_0} \cap \bar{\Omega}$. since $w(\xi) = 0$ for $\xi \in \partial\Omega, w(\xi) < 0$ for $\xi \in B_{\xi_0} \cap \Omega$ that implies:

$$0 < \underline{u}_2(\xi) - \overline{u}_1(\xi) < 0 \quad for \quad \xi \in \mathcal{B}_{\xi_0} \cap \Omega$$

this is a contradiction. Hence we are done! □

4.2 Positive global solutions of nonlinear elliptic BVPs

In this section we return to problem (1.2) and analyze existence and uniqueness of the positive global solutions. Consider the case where $\mathcal{B}u$ is either

$$\begin{cases} \mathcal{B}u(\xi,t) = u(\xi,t) & on \quad \partial\mathcal{D} \\ \quad or & \\ \mathcal{B}u(\xi,t) = \lambda u_v(\xi,t) + \mu u(\xi,t) & on \quad \partial\mathcal{D} \\ u(\xi,0) = u_0(\xi) & in \quad \Omega \end{cases}$$

Where the initial non-negative smooth map $u_0(\xi)$ satisfies $u_0(\xi) = 0$ on $\partial\Omega$.

From now on, we will use

$$\mathcal{M}u = u_t - \mathcal{L}u, \quad \text{where} \quad \Delta u = \mathcal{L}u$$

4.2.1 Definitions

Definition 4.3. *(i) A function $\underline{u}(\xi,t) \in C(\overline{\mathcal{D}}) \cap C^{1,2}(\mathcal{D})$ is a lower solution of (1.2) if $\underline{u}$ satisfies the following inequalities:*

$$\begin{cases} \mathcal{M}\underline{u} \geq f(\xi,t,\underline{u}) & in \quad \mathcal{D} \\ \mathcal{B}\underline{u} \leq c(\xi,t) & on \quad \partial\mathcal{D} \\ u(\xi,0) \leq u_0(\xi) & in \quad \Omega \end{cases} \tag{4.7}$$

(ii) A function $\overline{u}(\xi,t) \in C(\overline{\mathcal{D}}) \cap C^{1,2}(\mathcal{D})$ is an upper solution of (1.2) if $\overline{u}$ satisfies the following inequalities:

$$\begin{cases} \mathcal{M}\overline{u} \leq f(\xi,t,\overline{u}) & in \quad \mathcal{D} \\ \mathcal{B}\overline{u} \geq c(\xi,t) & on \quad \partial\mathcal{D} \\ u(\xi,0) \geq u_0(\xi) & in \quad \Omega \end{cases} \tag{4.8}$$

Clearly, any solution of (1.2) is an upper solution as well as a lower solution.

4.2.2 Existence and uniqueness of solutions

We set $u \in C(\overline{\mathcal{D}})$ with $\underline{u} \leq u \leq \overline{u}$ in $\overline{\mathcal{D}}$ and choose σ as in (A.1) with

$$f(\xi, u_1) - f(\xi, u_2) + \sigma(u_1 - u_2) > 0,$$

on

$$\min \underline{u}(\xi, t) \leq u_i \leq \max \overline{u}(\xi, t)\ ,\ i = 1, 2$$

Defining $\bar{u}^{(1)}$ by

$$\begin{cases} \mathcal{M}\bar{u}^{(1)} + \sigma\bar{u}^{(1)} = f(\xi, t, \overline{u}) + \sigma\overline{u} & in \quad \mathcal{D} \\ \mathcal{B}\bar{u}^{(1)} = c & on \quad \partial\mathcal{D} \\ \bar{u}^1(\xi, 0) = u_0(\xi) & in \quad \Omega \end{cases} \tag{4.9}$$

Then, by the maximum principle for parabolic equation we have,

$$\bar{u}^{(1)}(\xi, t) < \overline{u}(\xi, t) \ in\ \mathcal{D}$$

Defining $\psi : \overline{u}(\xi, t) \mapsto \bar{u}^{(1)}(\xi, t)$ we have $\bar{u}^{(1)} = \psi\overline{u}$ is monotone operator in the sense of Collatz [12] and similarly doing $\bar{u}^{(1)} = \psi\underline{u}$ we get

$$\{\bar{u}^{(n)}\}, \{\underline{u}^{(n)}\}$$

with

$$\bar{u}^{(n)} = \psi\bar{u}^{(n-1)}$$

and

$$\underline{u}^{(n)} = \psi\underline{u}^{(n-1)}$$

in which

$$\bar{u}^{(1)} = \psi\overline{u} \quad , \quad \underline{u}^{(1)} = \psi\underline{u}.$$

then we have the following theorem.

Theorem 4.6. *[18] Suppose that (A.1) holds true and let $\underline{u}(\xi, t), \overline{u}(\xi, t) \in \overline{\mathcal{D}}$ are lower and upper solutions of (1.2) respectively. If there is a σ such that*

$$f(\xi, u_1) - f(\xi, u_2) + \sigma(u_1 - u_2) > 0,$$

where

$$\min_{\Omega} \underline{u} \leq u_i \leq \max_{\Omega} \overline{u} \quad i = 1, 2$$

then there exists a unique strong solution u of (1.2) with

$$\lim_{n\to\infty} \bar{u}^{(n)} = \psi u = u = \lim_{n\to\infty} \underline{u}^{(n)},$$

where $\{\bar{u}^{(n)}\}$ *is a decreasing and* $\{\underline{u}^{(n)}\}$ *is an increasing sequences.*

Next, we look at the case where c is not time dependent. If c is time independent, the following corollary follows immediately from Theorem 4.6.

Corollary 4.3. *[18] Let (A.1) hold and* $\underline{u}(\xi) \leq u(\xi) \leq \bar{u}(\xi)$ *be a solution of*

$$\begin{cases} -\mathcal{L}u = f(\xi, u(\xi)) & in \quad \Omega \\ \mathcal{B}u = c & on \quad \partial\Omega \end{cases}$$

where $\bar{u}(\xi)$ *and* $\underline{u}(\xi)$ *are an upper and lower solutions respectively, then we obtain a global regular solution* $u(\xi, t) \in (\underline{u}, \bar{u}(\xi))$ *for all* $t > 0$.

We are going to introduce two new identities now:

$$\begin{cases} -\mathcal{L}u = f(\xi, u) & on \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases} \tag{4.10}$$

and

$$\begin{cases} \mathcal{M}u = f(\xi, t, u) & in \quad \mathcal{D} \\ u = 0 & on \quad \partial\mathcal{D} \\ u(\xi, 0) = \bar{u}(\xi) & in \quad \Omega \end{cases} \tag{4.11}$$

Theorem 4.7. *[18] Let assumption (A.1) hold and let* $\bar{u}(\xi)$ *be an upper solution of (4.10). If* $u(\xi, t)$ *is a solution of (4.11) then,* $u_t \leq 0$*. i.e.* $u(\xi, t)$ *is nonincreasing on t.*

Proof. Suppose $\{u^{(n)}(\xi, t)\}$ in $\mathcal{D}$ is a sequence of maps by $u^{(0)}(\xi, t) = \bar{u}(\xi)$ and for $n \geq 1$,

$$\begin{cases} \mathcal{M}u^{(n)} + \sigma u^{(n)} = f(\xi, u^{(n-1)}) + \sigma u^{(n-1)} & in \quad \mathcal{D} \\ u^{(n)} = 0 & on \quad \partial\mathcal{D} \\ u^{(n)}(\xi, 0) = \bar{u}(\xi) & in \quad \Omega \end{cases} \tag{4.12}$$

Then the sequence (of functions) $\{u^{(n)}(\xi, t)\}$ is nondecreasing,

$$\bar{u}(\xi) \geq u^{(1)}(\xi, t) \geq ... \geq u^{(n-1)}(\xi, t) \geq u^{(n)}(\xi, t) \geq ... \tag{4.13}$$

Actually, we have that

$$\begin{cases} \mathcal{M}(u^{(1)} - \bar{u}) + \sigma(u^{(1)} - \bar{u}) = -[f(\xi, \bar{u} - \mathcal{L}\bar{u})] \geq 0 \\ \mathcal{B}(u^{(1)} - \bar{u}) = c(\xi, t) - \mathcal{B}\bar{u} \geq 0 \end{cases}$$

By strong maximum principle we conclude that $\bar{u} \geq u^{(1)}$. Moreover, we can easily prove by induction that $u^{(n-1)}(\xi, t) \geq u^{(n)}(\xi, t)$ for $n \in \mathbb{N}$, the inequality (4.13) comes into existence and we arrive at a conclusion $u^{(n)} \to u^* as \;\; n \to \infty$.

Hence, $u^*(\xi, t)$ is a solution of

$$\begin{cases} \mathcal{M}u^* = f(\xi, u^*) & in \quad \mathcal{D} \\ u^* = 0 & on \quad \partial\mathcal{D} \\ u^*(\xi, 0) = \bar{u}(\xi) & in \quad \Omega \end{cases}$$

Thus, since our solution is unique we have $u^*(\xi, t) = u(\xi, t) \in \mathcal{D}$. Differentiating (4.12) with respect to t we have

$$\begin{cases} \mathcal{M}(u^{(n)})_t + \sigma(u^{(n)})_t = fu^{(n-1)}(\xi, u^{(n-1)})u_t^{n-1} & in \quad \mathcal{D} \\ (u^{(n)})_t = 0 & on \quad \partial\mathcal{D} \end{cases}$$

Since $fu^{(n-1)}(\xi, u^{(n-1)})u_t^{n-1}$ is in $\mathcal{D}$ it is bounded.

Define, if $\delta > 0$

$$w_n = \frac{u^{(n)}(\xi, \delta) - u^{(n)}(\xi, 0)}{\delta}, \quad \xi \in \Omega$$

Consequently, by (4.12) and (4.13) we get $w_n \leq 0$. Hence, $(u^{(n)}(\xi, 0))_t \leq 0$ for $\xi \in \Omega$.

Moreover we have $(u^{(n)})_t \leq 0$.

In a similar manner as we did in theorem (4.1) we can get

$$u^{(n)} \mapsto u \in C^{1+\eta} \; on \; \Omega \text{ for } \; 0 < \eta < 1.$$

Therefore, $u_t(\xi, t) \leq 0$ in $\mathcal{D}$. We are done! $\square$

Assume $f(\xi,t,u)$ is a C^1 mapping for u and satisfies the inequalities:

$$\begin{cases} f(\xi,t,\eta_1) \geq 0 & in \quad \mathcal{D}\cap\mathbb{R}^+ \\ f(\xi,t,\eta_2) \leq 0 & on \quad \mathcal{D}\cap\mathbb{R}^+ \\ \eta_1\mu(\xi,t) \leq c(\xi,t) \leq \eta_2\mu(\xi,t) & on \quad \partial\mathcal{D}\cap\mathbb{R}^+ \end{cases} \tag{4.14}$$

Where η_1 and η_2 are nonnegative constants with $\eta_1 < \eta_2$.

Theorem 4.8. *[18] Suppose that (4.14) holds. If there exists two nonnegative constants η_1 and η_2 with $\eta_1 < \eta_2$ then, for all $u \in [\eta_1,\eta_2]$ (1.2) has a unique positive and bounded global solution.*

Proof. Let $\underline{u} = \eta_1, \overline{u} = \eta_2$ then, by (4.14) we have

$$\begin{cases} \mathcal{M}\underline{u} = 0 \geq f(\xi,t,\eta_2) = f(\xi,t,\overline{u}) & in \quad \mathcal{D}\times\mathbb{R}^+ \\ \mathcal{B}\overline{u} = \lambda\overline{u}_v + \mu\overline{u} = \eta_2\mu(\xi,t) \geq c(\xi,t) & on \quad \partial\mathcal{D}\times\mathbb{R}^+ \\ \overline{u} = \eta_2 & in \quad \Omega \end{cases}$$

This allows us to draw the conclusion that $\overline{u} = \eta_2$ is an upper solution when $u \leq \eta_2$. In the same manner when $u \geq \eta_1$ we have $\underline{u} = \eta_1$ is a lower solution. As a result, by theorem (4.6) we conclude the result. □

The uniqueness result of a positive global solution for problem (1.2) can now be expressed as follows. We assume for this reason that:

$(*)$ for every $\xi \in \Omega, \xi' \in \partial\Omega$: $f(\xi,t) \geq 0$, $c(\xi',t) \geq 0$ and $u_0(\xi) \geq 0$.

Theorem 4.9. *[18] Suppose that $(*)$, (4.3) and (4.4) hold. If there exists a mapping G with*

$$f_u(\xi,t,u)u \leq G(\xi,t)u, \; for\ every\ u \geq 0\ in\ \mathcal{D} \tag{4.15}$$

then, (1.2) has a unique global positive solution.

Proof. By using the mean-value theorem, (4.3) and (4.4) we have

$$\begin{aligned} f(\xi,t,u) &= f_u(\xi,t,\eta)u + f(\xi,t,0) && (4.16) \\ &\geq f_u(\xi,t,\eta)u \ in\ \mathcal{D} && (4.17) \end{aligned}$$

in which $\eta := \eta(\xi,t)$ is an intermediate value between u and 0. Using Lemma 3 and $\mathcal{M}$ we write

$$f_3(\xi,t) = -f_u(\xi,t,\eta)$$

Thus, $u = 0$ or $u > 0$ in $\mathcal{D}$. u is a positive because otherwise u = 0 only if all of the maps in $(*)$ is equal to 0, This implies that $u = \underline{u}$ is a lower solution of (1.2). Suppose w is a solution then,

$$\begin{cases} \mathcal{M}w = Gw + f(\xi, t, 0) & in \quad \mathcal{D} \\ \mathcal{B}w = c(\xi, t) & on \quad \partial\mathcal{D} \\ w(\xi, 0) = u_0(\xi) & in \quad \Omega \end{cases}$$

Thus, w is an upper positive solution of (1.2). In fact for $\overline{u} = w$ and applying the mean-value theorem again we have

$$f(\xi, t, \overline{u}) = f_{\overline{u}}(\xi, t, \eta)(\overline{u}) + f(\xi, t, 0)$$

where $\eta = \eta(\xi, t)$ is located between $\overline{u}$ and 0
In combination with $(*)$, (4.3),(4.4) and (4.15) we have

$$f(\xi, t, \overline{u}) \leq G(\xi, t)\overline{u} + f(\xi, t, 0) \; in \, \mathcal{D}$$

Thus,

$$\begin{cases} \mathcal{M}\overline{u} = G\overline{u} + f(\xi, t, 0) \geq f(\xi, t, \overline{u}) & in \quad \mathcal{D} \\ \mathcal{B}w = c(\xi, t) & on \quad \partial\mathcal{D} \\ w(\xi, 0) = u_0(\xi) & in \quad \Omega \end{cases}$$

that is $\overline{u}$ is an upper positive solution of (1.2). Therefore, by theorem (7) we deduce the unique positive global solution of (1.2). □

4.3 Lower and Upper Solutions in the Weak sense

4.3.1 Definitions

Definition 4.4. *(i) A weak lower solution of (1.1) is a nonnegative function $\underline{u} \in H_0^1(\Omega)$ that satisfies*

$$\int_\Omega \nabla\underline{u}\nabla\phi dx \leq \int_\Omega f(\underline{u})\phi dx \quad in \quad \Omega$$

for all $\phi \in H_0^1(\Omega)$.

(ii) A weak upper solution of (1.1) is a nonnegative function $\bar{u} \in H_0^1(\Omega)$ that satisfies

$$\int_\Omega \nabla\bar{u}\nabla\phi dx \geq \int_\Omega f(\bar{u})\phi dx \quad in \quad \Omega$$

for all $\phi \in H_0^1(\Omega)$.

(iii) A function $u \in H_0^1(\Omega)$ *is said to be a weak solution of (1.1) if it is both a weak upper and weak lower solution. i.e*

$$\int_\Omega \nabla u \nabla \phi dx = \int_\Omega f(u)\phi dx \quad in \quad \Omega$$

4.3.2 Existence of Weak Solution

Lemma 4.2. *[5] Suppose* $f \in L^2(\Omega)$*. Then there exists a unique solution* $v \in H_0^1(\Omega)$ *to problem (1.1).*

Proof. **Existance**

Suppose ϕ_1 be the positive eigenfunction associated with the first eigenvalue λ_1 of $(-\mathcal{L}, H_0^1(\Omega))$ with $|\phi_1|_2 = 1$. Keep in mind that

$$-\mathcal{L}(\varepsilon\phi_1) = \varepsilon\lambda_1\phi_1 \; in \; \Omega$$

so, for sufficiently small $\varepsilon > 0$ we have

$$\varepsilon\lambda_1\phi_1 \leq \varepsilon\phi_1 \; in \; \Omega$$

Thus,

$$-\mathcal{L}(\varepsilon\phi_1) \leq \varepsilon\phi_1 \; in \; \Omega$$

Let us consider e the solution of the problem

$$\begin{cases} \Delta e = 1 & in \quad \Omega \\ e = 0 & on \quad \partial\Omega \end{cases}$$

Note that γe is a weak upper solution of (1.1) i.e

$$-\mathcal{L}(\gamma e) \geq \gamma e$$

When γ is sufficiently large we may use the regularity of the functions ϕ_1 and e to deduce that

$$\varepsilon\phi_1 \leq \gamma e \; in \; \bar{\Omega} \; \varepsilon > 0$$

is small enough. Thus, there exist a solution $v \in H_0^1$ to problem (1.1).

Uniqueness

Suppose that $\tilde{u}$ be solutions (1.1). A simple calculation reveals that the functions are equal i.e.

$$\int_\Omega \nabla u \nabla \phi dx = \int_\Omega \nabla \tilde{u} \nabla \phi dx = \int_\Omega f(u)\phi dx$$

and so

$$\int_\Omega |\nabla u - \nabla \tilde{u}| \nabla \phi dx = 0$$

for each $v \in H_0^1(\Omega)$ we set $v = u - \tilde{u}$ and use (6) to deduce

$$\int_\Omega |\nabla u - \nabla \tilde{u}|^2 dx = 0$$

Thus, $u = \tilde{u}$ a.e in Ω □

Lemma 4.3. *[19] Let u and v be two nonnegative functions such that*

$$\begin{cases} -\Delta u \geq -\Delta v & in \quad \Omega \\ u = v = 0 & in \quad \partial\Omega \end{cases}$$

Then, $u \geq v$ a.e in Ω.

Proof. Multiplying both sides of the inequality above by u and v respectively and integrating we obtain

$$\|u\|^2 \geq < v, u >$$

and

$$< u, v > \geq \|v\|^2$$

from this inequalities we have

$$\|u\|^2 \geq < u, v > \geq \|v\|^2$$

which implies,

$$\|u\|^2 \geq \|v\|^2$$

and so

$$\|u\| \geq \|v\|$$

since u and v are nonnegative functions we have $u \geq v$. □

(A.3) $f \in C^1(0, \infty)$ is increasing function such that

$$\lim_{u \to +\infty} f(u) = +\infty.$$

(A.4) Moreover, $f \in C^1(0, \infty)$ satisfies

$$\lim_{u \to +\infty} \frac{f(u)}{u} = 0.$$

Theorem 4.10. *[18] Assume that (A.3) and (A.4) hold. Then problem (1.1) has a positive weak solution.*

Proof. Suppose ϕ_1 is the positive eigenfunction with $||\phi_1|| = 1$ associated with the first eigenvalue ν of $-\Delta$ with Dirichlet boundary conditions. Let $\delta > 0$ be such that $|\nabla\phi_1|^2 - \sigma\phi_1^2 > 0$ on

$$\bar{\Omega}_\delta = \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}.$$

Define u as

$$\underline{u} = \frac{1}{2}\phi_1^2.$$

We will show that $\underline{u}$ is a weak lower solution of (1.1).

Let $\phi \in H_0^1(\Omega)$ with $\phi > 0$ in Ω. We have that

$$\begin{aligned}
\int_{\bar{\Omega}_\delta} \nabla\underline{u}\nabla\phi dx &= \int_{\bar{\Omega}_\delta} \phi_1\nabla\phi_1\nabla\phi dx \\
&= \int_{\bar{\Omega}_\delta} \nabla\phi_1\nabla(\phi_1.\phi)dx - \int_{\bar{\Omega}_\delta} |\nabla\phi|^2\phi dx \\
&= \int_{\bar{\Omega}_\delta} (\sigma\phi_1^2 - |\nabla|^2)\phi dx.
\end{aligned}$$

Then we have $\nu\phi_1^2 - |\nabla\phi|^2 < 0$ since $|\nabla\phi|^2 - \nu\phi_1^2 > 0$ On $\bar{\Omega}_\delta$.

Thus,

$$\int_{\bar{\Omega}_\delta} \nabla\underline{u}\nabla\phi dx < 0.$$

By (A.3) we get $f(\underline{u}) > 0$

Then,

$$\int_{\bar{\Omega}_\delta} \nabla\underline{u}\nabla\phi dx \leq \int_{\bar{\Omega}_\delta} f(\underline{u})\phi dx. \tag{4.18}$$

Next we have $\phi_1 \geq d$ on $\Omega\backslash\bar{\Omega}_\delta$ for some $d > 0$. By (A.3) and by the definition of $\underline{u}$ it follows that:

$$\begin{aligned}\int_{\Omega\setminus\bar{\Omega}_\delta} f(\underline{u}\phi dx &\geq \int_{\Omega\setminus\bar{\Omega}_\delta} \nu\phi dx \\ &\geq \int_{\Omega\setminus\bar{\Omega}_\delta} (\nu\phi_1^2 - |\nabla\phi|^2)\phi dx \\ &= \int_{\bar{\Omega}_\delta} \nabla\underline{u}\nabla\phi dx \end{aligned} \tag{4.19}$$

From (4.18) and (4.19) we can conclude that for any $\phi \in H_0^1(\Omega)$,

$$\int_\Omega \nabla\underline{u}\nabla\phi dx \leq \int_\Omega f(\underline{u})\phi dx$$

i.e, $\underline{u}$ is a weak lower solution of problem (1.1).

Let us consider e the solution of the problem

$$\begin{cases} \Delta e = 1 & in \quad \Omega \\ e = 0 & on \quad \partial\Omega \end{cases} \tag{4.20}$$

We shall construct a weak upper solution of (1.1)

Let $\bar{u} = Ce$, where C is a positive real number which will be chosen later. We will verify that $\bar{u}$ is a weak upper solution. Let $\phi \in H_0^1(\Omega)$ with $\phi > 0$ in Ω. Then from (4.20) we get

$$\begin{aligned}\int_\Omega \nabla\bar{u}\nabla\phi dx &= C\int_\Omega \nabla e\nabla\phi dx \\ &= C\int_\Omega \phi dx.\end{aligned}$$

By (A.3) we can choose C sufficiently large so that

$$C \geq f(C\|e\|_\infty).$$

Therefore,

$$\begin{aligned}\int_\Omega \nabla \bar{u} \nabla \phi dx &\geq f(C\|e\|_\infty)\int_\Omega \phi dx \\ &\geq \int_\Omega f(C\|e\|_\infty)\phi dx \\ &\geq \int_\Omega f(\bar{u})\phi dx.\end{aligned}$$

which implies that for a large enough C $\bar{u}$ is a weak upper solution of (1.1). In order to obtain a weak solution of (1.1) we define the sequence

$$\{u_n\} \subset E = H_0^1(\Omega) \cap C(\Omega)$$

as $u_0 = \bar{u} \subset E = H_0^1(\Omega) \cap C(\Omega)$, and u_n is a unique solution of the problem

$$\begin{cases} -\Delta u_n = f(u_{n-1}) & in \quad \Omega \\ u = 0 & in \quad \partial\Omega. \end{cases} \tag{4.21}$$

If $u_{n-1} \in E$ is given then the right-hand-side of (4.21) is independent of u_n since $f(u_0) \in C(\Omega) \subset L^2(\Omega)$ and from Lemma (4.2). Then (4.21) with n = 1 has unique solution $u_1 \in H_0^1(\Omega)$. Thus, $u_1 \in C(\Omega)$. Consequently, we conclude that $u_1 \in E$.

In the same way we construct the following elements $u_n \in E$ of our sequence. From(4.21) and by the fact that u_0 is a weak upper solution of (1.1) we have,

$$\begin{cases} -\Delta u_0 \geq f(u_0) = -\Delta u_1 & in \quad \Omega \\ u = 0 & in \quad \partial\Omega \end{cases}$$

From this and lemma (4.3) we deduce that $u_0 \geq u_1$.

Since $u_0 \geq \underline{u}$ and by the monotonicity of f we have,

$$-\Delta u_1 = f(u_0) \geq f(\underline{u}) \geq -\Delta \underline{u}.$$

And hence, from this and lemma (4.3) we conclude that $u_1 \geq \underline{u}$.

Fuinally, from (4.21) and by the monotonicity of f, $u_1 \geq \underline{u}$ and $u_0 \geq u_1$.

Now, for u_2 we write

$$-\Delta u_1 = f(u_0) \geq f(u_1) = -\Delta u_2$$

and

$$-\Delta u_2 = f(u_1) \geq f(\underline{u}) \geq -\Delta \underline{u}.$$

Consequently, $u_1 \geq u_2$ and $u_2 \geq \underline{u}$.

Repeating this argument we get a bounded monotonic sequence $\{u_n\}$ satisfying

$$\bar{u} = u_0 \geq u_1 \geq ... \geq u_n \geq \underline{u} > 0$$

Since the function f is continuous and by the definition of the sequence $\{u_n\}$ there exists a constant $\mathcal{R}$ which is independent from n such that

$$|f(u_{n-1})| < \mathcal{R}. \tag{4.22}$$

Using (4.22), multiplying the first equation of (4.21) by u_n, integrating and using Hölder inequality and Sobolev's embedding we can show that

$$\begin{aligned}\int_\Omega |\nabla u_n|^2 dx &\leq \int_\Omega f(u_n) u_n dx \\ &\leq \mathcal{R} \int_\Omega |u_n| dx \\ &\leq \mathcal{R} \|u_n\|_{H_0^1(\Omega)}.\end{aligned}$$

Then,

$$\|u_n\|_{H_0^1(\Omega)} \leq \mathcal{R} \ \forall \ n, \tag{4.23}$$

where $\mathcal{R}$ is a constant and independent of from n. We may deduce from (4.23) that $\{u_n\}$ has a sequence which converges weakly in $H_0^1(\Omega, \mathbb{R})$ to u with $u \geq \underline{u} > 0$. Now if $n \to +\infty$ is true, we conclude that u is a positive weak solution of (1.1) and this completes the proof. □

Summary

The first, seemingly self-evident argument to make is that the method of upper and lower solutions not only indicates the existence but also localize the solution. In this paper, the method of lower and upper solution is used to investigate the existence of solutions of two types of nonlinear elliptic boundary value problems. Theorems regarding the existence, uniqueness and multiplicity of positive solutions of nonlinear elliptic boundary value problem were studied.

Secondly, this method which loosely speaking resembles the intermediate value theorem (and the squeezing theorem), roughly states that if we can find a couple of functions then we can find a solution localized between them, i.e. if $\underline{u}$ and $\overline{u}$ are lower and upper solutions respectively, of a differential equation then we can find a solution u such that

$$\underline{u} < u < \overline{u} \quad \forall t \in [0, T].$$

Finally, the method works but it is not advisable for linear problems since in this case the differential equation can be solved analytically and hence we can pick the desired solution by evoking the auxiliary conditions(initial/boundary) to get the integration constants. Therefore, the existence of solutions provided by this method informs us nothing new other than what we previously know. Nonlinear equations on the other hand are difficult to solve analytically. As a result, we must choose between a graphical intuition that a solution exists or an existence based on the method of upper and lower solutions.

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