SCHOOL OF GRADUATE STUDY

DEPARTMENT OF MATHEMATICS

A Project Seminar on Boolean Near Rings in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

Compiled By: - Meresa Kebede

Supervisor: – Dr. K. Venkateswarlu

June 2011G.C.

Addis Ababa, Ethiopia

Declaration

I declare that this project/thesis has been composed by me and that no part of the project/thesis has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

Meresa Kebede

____

Author's Signature

Permission

This is certify that this project/thesis/is compiled/a record of the research work done/ by Mr. Meresa Kebede in the department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project/thesis can be submitted for evaluation by examiners and eventual defense.

Dr. K. Venkateswarlu ______________________

Advisor's Signature

Acknowledgment

First of all I would like to thank my supervisor Dr. K. Venkateswarlu for his interesting seminars, guidance and patient advice.

Last but not least, I am grateful to my brother Mr. Michael Nigussie for his significant support and assistance.

Abstract

In this paper we sketch the connection of Boolean algebra and Boolean rings and concept of Boolean near rings with identity to construct a class of Boolean near rings called special Boolean near ring for which it fails to be Boolean rings. Finally, we shall see that the concept of distributively generated Boolean near ring and at last we proof that every distributively generated Boolean near ring is a Boolean ring.

List of Symbols and Notation

Symbols	its meaning
$\in$	is an element of
$\subseteq$	is a subset of
$\cap$	the intersection of
$\cup$	is a union of
N	Near ring
B	Special Boolean ring
$\wedge$	join
$\vee$	meet
$'$	complement
$\circ$	composition
$\mathbb{N}$	the set of Natural number
$\mathbb{Z}$	the set of integer number
$\mathbb{Q}$	the set of Rational number
$\mathbb{R}$	the set of Real number

Table of contents

Contents Page No.

Introduction

In this paper we will see that the concept of Boolean near ring from the wide concepts of ring theory. In particular, this paper contains three chapters. The first chapter is meant a preliminary. It discusses the definitions of near ring and some important definitions of useful terms of near rings. It also give a brief discussion of the elementary facts relating to rings in which every element is idempotent called Boolean ring and introduce the algebraic properties of Boolean ring.

In chapter two, we present the connection of Boolean ring with Boolean algebra by considering a recent set of postulates for Boolean algebras due to Huntington. Huntington shall be denote " $\vee$ " which corresponds to logical addition and to the formation of the union for classes and a unary operation denoted by the prime " $'$ " corresponds to logical negation and to the formation of the complement for classes.

Finally, in chapter three, we introduce the concept of Boolean near ring with identity to construct a class of Boolean near rings, called special Boolean near ring, and determine left ideals, factor near rings of special Boolean near rings. Also we verifies that the proof of every distributively generated Boolean near ring is a Boolean ring. At the end a Boolean near ring is isomorphic to a particular collection of functions which form a Boolean near ring with respect to the operations of addition and composition of mappings.

CHAPTER ONE

PRELIMINARIES RESULTS

1.1. The Elementary Concepts of Near Rings

Definition1.1.1. A ring $(R, +,\cdot)$ is a set R together with two binary operation " $+$ " and " $\cdot$ ", which we call addition and multiplication, defined on R such that the following axioms are satisfied:

1. $(R, +)$ is abelian group.
2. Multiplication is associative. That is, $(R,\cdot)$ is a semi group.
3. For all $a, b, c \in R$, the left distributive law, $a \cdot (b + c) = a \cdot b + a \cdot c$ and the right distributive law, $(a + b) \cdot c = a \cdot c + b \cdot c$ hold.

Remark1.1.2. We are well aware that axioms (1), (2) and (3) for a ring hold in any subset of the complex number that is a group under the operation of addition and that is closed under the operation of multiplication. For instance, $(\mathbb{Z}, +,\cdot)$, $(\mathbb{Q}, +,\cdot)$, $(\mathbb{R}, +,\cdot)$ and $(\mathbb{C}, +,\cdot)$ are rings.

Definition1.1.3. Let R and S be any rings, a map $f: R \to S$ is a ring homomorphism if the following two conditions are satisfied for all $a, b \in R$.

1. $f(a + b) = f(a) + f(b)$, and
2. $f(ab) = f(a)f(b)$.

Now we introduce the concept of near ring in the following manners:

Definition1.1.4. A near ring $(\mathrm{N}, +,\cdot)$ is an algebraic structure together with two binary operations addition " $+$ " and multiplication " $\cdot$ " such that

1. $(\mathrm{N}, +)$ is a group not necessarily abelian,
2. $(\mathrm{N},\cdot)$ is a semi group, that is multiplication is associative and
3. $n_1 \cdot (n_2 + n_3) = n_1 \cdot n_2 + n_1 \cdot n_3$, for every $n_1, n_2, n_3 \in \mathrm{N}$ (left distributive law). Hence, this type of near ring will be termed as *left* near ring.

Remarks1.1.5:

1. Similarly one can define *right* near ring by redefining condition (3) in definition 1.1.4 by $(n_1 + n_2) \cdot n_3 = n_1 \cdot n_3 + n_2 \cdot n_3$.

2. In this paper a near ring mean that only a left near ring unless otherwise stated.
3. It is customary to denote multiplication in a (near) ring by juxtaposition, using ab in place of $a \cdot b$.

Example1.1.6. Every ring is a near ring.

Example1.1.7. Let $\mathbb{Z}$ be the set of integers. Now define multiplication " $\cdot$ " on $\mathbb{Z}$ by $a \cdot b = b$ for all $a, b \in \mathbb{Z}$.

a. Clearly $(\mathbb{Z}, +)$ is an abelian group.
b. $(\mathbb{Z}, \cdot)$ is a semi group. That is multiplication is associative.
c. $a(b + c) = b + c$ (by definition)
$= ab + ac$, for all $a, b, c \in \mathbb{Z}$.

Therefore, $(\mathbb{Z}, +, \cdot)$ is a (left) near ring.

Definition1.1.8. Let N and N' be near rings. A mapping $f: N \to N'$ is a near ring homomorphism if the following two conditions are satisfied:

a. $f(n_1 + n_2) = f(n_1) + f(n_2)$
b. $f(n_1 n_2) = f(n_1) f(n_2)$.

Remarks1.1.9. Let N be a near ring. Then,

1. If $(N, +)$ is abelian, then we call N an abelian near ring.
2. If $(N, \cdot)$ is commutative, then we call N a commutative near ring.

Definition1.1.10. A non empty subset I of a near ring N is said to be an ideal if

a. $(I, +)$ is a Normal subgroup of (N, +)
b. N$I \subseteq I$ (i.e. $ni \in I$ for all $i \in I$ and $n \in N$, that is, I is a left ideal), and
c. $(n + i)m - nm \in I$ if $n, m \in$ N and $i \in I$.

Proposition1.1.11. The two sided ideals of a near ring N are defined to be the kernels of homomorphisms of N.

Proof:- Let K be the kernel of a ring homomorphism $f : N \to N$. Then,

a) Let $k \in K$ and $n \in N$, then $f(nkn^{-1}) = f(n)f(k)f(n^{-1}) = e$. Thus,
$nkn^{-1} \in K$ implies $nKn^{-1} \subseteq K$.
Hence, $(K, +)$ is a normal subgroup of $(N, +)$.

b) Since near rings satisfy the left distributive law, namely, $n0 = 0$.Thus,

$$n(0+0) = n0 + n0$$
$$= 0, \text{ for any } n \in N \text{ and } 0 \text{ is the additive identity of } N.$$

Therefore, $NK \subseteq K$.

c) $f[(n_1 + k)n_2 - n_1 n_2] = f[(n_1 + k)n_2] - (n_1 n_2)$

$$= f(n_1 + k)f(n_2) - f(n_1 n_2)$$
$$= [f(n_1) + f(k)]f(n_2) - f(n_1 n_2),$$
$$= f(n_1)f(n_2) - f(n_1)f(n_2)$$
$$= 0.$$

Thus, $(n_1 + k)n_2 - n_1 n_2 \in K$ for every $n_1, n_2 \in N$ and $k \in K$.

Therefore, K is an ideal of a near ring N. ■

Definition1.1.12. Let N be a left near ring. If $xy = 0$ implies that $yx = 0$, then N is said to be zero commutative and if for all $x \in N$, $0x = 0$, N is called zero symmetric.

Definitio1.1.13. An ideal P of a near ring N is called completely prime if $ab \in P$ implies $a \in P$ or $b \in P$.

Definition1.1.14. An element a of the near ring N will be called central if $xa = ax$ for all x in N.

Defintion1.1.15. An element a of a near ring N is called

1. Left zero divisor if $ab = 0$, then $a = 0$ for some non zero element $b \in N$.
2. Right zero divisor if $ba = 0$, then $a = 0$ for some non zero element $b \in N$.

In particular, a is zero divisor if it is a left or right zero divisor.

3. Nilpotent if $a^n = 0$ for some positive integer n.

Definition1.1.16. Let N be a near ring. Then N is said to be a direct sum of left(right) ideals $A, B \in N$ such that $N = A + B$ and $A \cap B = 0$. Thus, A and B are called direct summands of N.

Remark1.1.17. we denote $N = A \oplus B$ and for every n in N can be uniquely written as $n = a + b$ where $a \in A$ and $b \in B$.

1.2. Algebraic Properties of Boolean Rings

In this section we shall consider the elementary facts relating to rings in which every element is idempotent called Boolean ring. Now we lay down some of the algebraic properties of Boolean rings:

Definition1.2.1. A ring R in which every element is an idempotent, that is, satisfying $a^2 = a$ for all $a \in R$ is called a Boolean ring.

Proposition1.2.2. A s ub r ing of a B oolean ring i s B oolean. F urthermore, e very homomorphic image of a Boolean ring is also Boolean.

***Proof*:-**

a. Let A be a sub ring of the Boolean ring B. Thus,

For every $a \in A$, then a is an element of B. Since B is Boolean ring, hence a is idempotent, that is $a^2 = a$ for all $a \in A$.

Therefore, A is Boolean ring.

b. Let $f(A)$ be a ho momorphic i mage of B where $f: A \to B$ is a r ing epimorphism. Let $a \in f(A) \subseteq B$, then $a = f(b)$ for some $b \in B$ since f is Surjective map.

Hence, $a^2 = a \cdot a = f(b)f(b)$

$= f(b^2)$, since f is a homomorphism.

$= f(b)$, since $b^2 = b$.

$= a$.

Which implies that $a^2 = a$, for all $a \in f(A)$.

Thus, every element of $f(A)$ is idempotent.

Therefore, $f(A)$ is Boolean ring. ■

Lemma1.2.3. Let B be a Boolean ring. Then, characteristic of B is 2.

Proof: - Let $b \in$ B. Then, $b^2 = b$ since B is Boolean ring,

Thus, we have that $(b + b)^2 = b + b$,

which implies that $b^2 + b^2 + b^2 + b^2 = b + b$ and by the fact that $b^2 = b$

we have, $b + b + b + b = b + b$.

Hence, $b + b = 0$.

Therefore, characteristic of B is 2. ■

In particular, it follows then that $b = -b$ for all $b \in$ B.

Lemma1.2.4. If B is Boolean ring, then B is commutative.

Proof: - Let $a, b \in$ B. we want to show that $ab = ba$.

Since B is Boolean ring, then we have $a^2 = a$ and $b^2 = b$, for all $a, b \in B$.

Now, $a + b = (a + b)^2$

$= a^2 + ab + ba + b^2$, since $a^2 = a$ and $b^2 = b$, we get

$= a + ab + ba + b$, By adding $-a$ and $-b$ both sides, we obtain

$0 = ab + ba$.

Hence, $ab = -ba$. Using lemma1.2.3, we have $b = -b$ for every $b \in$ B.

Therefore, $ab = ba$. ■

Proposition1.2.5. Let B be a field. If B is Boolean ring, then B $\cong Z_2$.

Proof:- Let $b \in$ B and $b \neq 0$. Then,

$b^2 = b$, for every $b \in B$.

Implies that, $b^2 - b = 0$.

Also implies that, $b(b - 1) = 0$, thus $b = 0$ or $b - 1 = 0$.

However, since B is a field, b^{-1} exists. Thus, $b = 1$.

Therefore, B $= \{0,1\} \cong$ Z_2. ■

Theorem1.2.6. If B is a finite Boolean ring, then B has 2^n elements for some positive integer n.

Example1.2.7. Let $\wp(X)$ is the power set of a set X define addition and multiplication by $A + B = (A \backslash B) \cup (B \backslash A) = A \vartriangle B$ and $A \cdot B = A \cap B$ respectively. Hence, $\wp(X)$ is a Boolean ring for any set $A, B \in \wp(X)$ and $|\wp(X)| = 2^n$.

Theorem1.2.8. If I is an ideal of a Boolean ring B then B/I is also Boolean ring.

Proof: - Define a map f : B → B/I by $f(b) = b + I$, for all $b \in B$.

1. Let $a, b \in B$, then
   a) $f(a + b) = (a + b) + I$
      $= (a + I) + (b + I)$
      $= f(a) + f(b)$
   b) $f(ab) = ab + I$

$$= (a + I)(b + I)$$
$$= f(a)f(b)$$

Hence, f is a homomorphism.

2. Clearly f is an epimorphism.
3. Since the homomorphic image of a Boolean ring B is Boolean ring (by proposition 1.2.1), that is

$$B/I = f(B) = \{f(b) : b \in B\} = \{b + I : b \in B\} \text{ is Boolean ring.}$$

4. Also we have $(b + I) + (b + I) = (b + b) + I = 0 + I$ and

$$(b + I) \cdot (b + I) = (b \cdot b) + I = b + I.$$

Therefore, B/I is a Boolean ring. ■

Definition1.2.9. If B is a Boolean ring, the set $B/I = \{b + I : b \in \mathrm{B}\}$ is called a quotient Boolean ring of B.

Definition1.2.10. If B and B′ are Boolean rings and $f : \mathrm{B} \to \mathrm{B}'$ is a homomorphism of B into B′, the set $\{ b \in \mathrm{B} : f(b) = 0' where\ 0 \in \mathrm{B}$ is the additive identity$\}$ is called kernel of the homomorphism f denoted by $Ker\ f$.

Theorem1.2.11. If B and B′ are Boolean rings and $f : \mathrm{B} \to \mathrm{B}'$ is a homomorphism of B into B′ then $Ker\ f$ is an ideal of B.

Proof: - Since $Ker\ f = \{b \in \mathrm{B} | f(b) = 0'\}$ where $0' \in \mathrm{B}'$ is the additive identity, is a non empty and subset of B is clear. Now we want to show that

a. Let $a, b \in Ker\ f$, that is $f(a) = 0' = f(y)$. Then,

$$f(a + b) = f(a) + f(b) = 0' + 0' = 0'$$

Thus, $a + b \in Ker\ f$.

b. Let $a \in Ker\ f$ and $c \in B$. Then,

$$f(cx) = f(c)f(a) = f(c) \cdot 0' = 0' \text{ and}$$
$$f(ac) = f(a)f(c) = 0' \cdot f(c) = 0'.$$

Hence, $ca \in kerf$, whenever $a, b \in Ker\ f$ and $c \in \mathrm{B}$.

Therefore, $kerf$ is an ideal of B. ■

Remark1.2.12. A ring R is said to be a p-ring if p is a fixed prime and $x^p = x$, $px = 0$ for each x in R. Thus a Boolean ring is a 2-ring.

CHAPTER TWO

BOOLEAN RINGS AND BOOLEAN ALGEBRAS

Now we see that the concept of Boolean algebra by defining the formal definition of a lattice to see the connection of Boolean rings and Boolean algebra.

Definition2.1. Let X be a p artial o rder s et (poset), th en X is s aid t o be bounde d provided that there are elements $0,1$ of X such that $0 \leq x \leq 1$ for all $x \in X$.

Defintion2.2. A poset X is called a lattice provided $sup\{x, y\}$ and $inf\{x, y\}$ exists and are unique for all $x, y \in X$ and denote $x \vee y$ for $sup\{x, y\}$ and $x \wedge y$ for $inf\{x, y\}$.

Definition2.3. A la ttice X is s aid to b e d istributive p rovided f or a ll x, y and z be elements of X, we have $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Definition2.4. A B oolean al gebra B is a b ounded d istributive la ttice w ith u niary operation $': B \to B$ such that $a \vee a' = 1$ and $a \wedge a' = 0$.

Proposition2.5. Two bi nary ope rations $\wedge$ and $\vee$ on a set S are the i nfimum and supremum operations of a lattice if and only if the identities

i. Idempotence: $x \wedge x = x, x \vee x = x$;

ii. Commutativity: $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$;

iii. Associativity: $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$;

iv. Absorption laws: $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$ hold for all $x, y, z \in S$.

Theorem2.6. Let B be a Boolean ring. Let define $x \vee y = x + y + x \cdot y$, $x' = 1 + x$ and $x \wedge y = x \cdot y$. Then B is a Boolean algebra.

Proof:- Let $x \in B$,

- $x \wedge x' = x \wedge (1 + x) = x(1 + x)$

  $= x + x^2$, since $x^2 = x$ we have

  $= 0$; and

- $x \vee x' = x + (1 + x) + x(1 + x)$

  $= x + 1 + x$, since $x + x = 0$. Then, we have

  $= 1$.

Therefore, B is a Boolean algebra. ∎

Theorem2.7. Let B be a Boolean algebra. For any x and y be elements of B, now define $x + y = (x \wedge y') \vee (x' \wedge y)$ and $x \cdot y = x \wedge y$. Then B is a Boolean ring.

Proof:-

i. $x + 0 = (x' \wedge 0) \vee (x \wedge 0') = 0 \vee x = x$;

ii. $x + x = (x \wedge x') \vee (x' \wedge x) = 0 \vee 0 = 0$;

iii. Associability of $+$ follows from that of $\vee$ and $\wedge$. So B is a group.

iv. B is commutative $x + y = (x \wedge y') \vee (x' \wedge y) = (y \wedge x') \vee (y' \wedge x) = y + x$.

v. Also, $' \cdot '$ will distribute as B is a distributive lattice.

vi. Also $x \cdot 1 = x \wedge 1 = x$, thus B is a ring with identity.

vii. For any $x \in B$, we have $x \cdot x = x \wedge x = x$.

Therefore, B is a Boolean ring. ■

Theorem2.8. If A is a Boolean ring with unit e, the introduction of a binary operation "$\vee$" and a unary operation "$'$" defined by $a \vee b = a + b + ab$ and $a' = a + e$ converts A into an algebraic system B in which

1) $a \vee b = b \vee a$;
2) $a \vee (b \vee c) = (a \vee b) \vee c$;
3) $(a' \vee b')' \vee (a' \vee b)' = a$; *and*

the old operations being expressed in terms of the new through the equations

4) $a + b = ab' \vee a'b = (a' \vee b'')' \vee (a'' \vee b')'$;
5) $ab = (a' \vee b')'$.

On the other hand, if B is an algebraic system obeying the equations 1, 2 and 3, then B is a Boolean algebra; and the introduction of new operations through the equations 4 and 5 converts B into a Boolean ring A with zero $0 = e' = (a \vee a')'$ and a unit $e = a \vee a'$.

Proof:- Let A is any Boolean ring, either with or without unit, then for any $a, b \in A$. we have

1. $a \vee b = a + b + ab$ (by definition of $\vee$)

 $= a + b + ba,$ Since $ab = ba$.

 $= b \vee a.$

2. $a \vee (b \vee c) = a \vee (b + c + bc)$ (by definition of $a \vee b$)

 $= a + (b + c + bc) + a(b + c + bc)$

$= a + b + c + bc + ab + ac + abc$

$= [a + b + ab] + c + bc + ac + abc$

$= [a + b + ab] + c + [ac + bc + abc]$

$= [a + b + ab] + c + c[a + b + ab]$

$= (a \vee b) \vee c.$

Now if a Boolean ring B has a unit e, then using $a' = a + e$, we obtain

- ❖ $a'' = (a + e) + e = a + (e + e) = a + 0 = a$; since $e + e = 0$.
- ❖ $(a \vee b)' = (a \vee b) + e$ (by $a' = a + e$)

  $= (a + b + ab) + e$ (by $a \vee b = a + b + ab$)

  $= (a + e)(b + e)$

  $= a'b'.$

Thus, we prove the following

3. $(a' \vee b')' \vee (a' \vee b)' = a''b'' \vee a''b'$ (by $(a \vee b)' = a'b'$)

   $= ab \vee ab'$ ($a'' = a$ and $b'' = b$)

   $= ab + ab' + (ab)(ab')$ (by $a \vee b = a + b + ab$)

   $= ab + a(b + e) + ab(b + e)$ (by $b' = b + e$)

   $= ab + ab + a + ab + ab$ (since $2(ab + ab) = 0$)

   $= a,$

4. $(a' \vee b'')' \vee (a'' \vee b')' = (a' \vee b)' \vee (a \vee b')'$, since $b'' = b$ and $a'' = a$

   $= a''b' \vee a'b''$ (by $(a \vee b)' = a'b'$)

   $= ab' \vee a'b$ (by $a'' = a$ and $b'' = b$)

   $= ab' + a'b + (ab')(a'b)$ (by $a \vee b = a + b + ab$)

   $= a(b + e) + (a + e)b + ab(a + e)(b + e)$

   $= ab + a + ab + b + (a + a)(b + b)$

   $= a + b.$

5. $(a' \vee b')' = a''b'' = ab.$

Therefore, the introduction of the new operations $\vee$ and $'$ converts a Boolean ring A into a Boolean algebra B.

Moreover, $aa' = (a' \vee a'')' = e' = 0$, then we shall see that the following important properties:

1) $a + b = ab' \vee a'b$

$= ba' \vee b'a$

$= b + a.$

2) $a + (b + c) = a + [bc' \vee b'c]$

$= a(bc' \vee b'c)' \vee a'(bc' \vee b'c)$

$= a(bc')'(b'c)' \vee a'bc' \vee a'b'c$

$= a(b' \vee c)(b \vee c') \vee a'bc' \vee a'b'c$

$= ab'c' \vee abc \vee a'bc' \vee a'b'c$

$= ca'b' \vee cab \vee c'ab' \vee c'a'b = c + (a + b)$

$= (a + b) + c.$

3) $ab + ac = ab(ac)' \vee (ab)'(ac)$

$= ab(a' \vee c') \vee (a' \vee b')(ac)$

$= abc' \vee ab'c$

$= a(b'c \vee bc')$

$= a(b + c).$

4) $a + a = aa' \vee a'a = 0$; and

5) $a + 0 = a0' \vee a'0$ (by theorem2.8(4))

$= a0'$, since $0' = 0 + e$. Then we have,

$= a.$

6) $aa = (a' \vee a')' = a'' = a.$

To complete the proof of the theorem we must show finally the define equations are valid. Since we have

$a + e = ae' \vee a'e$

$= a0 \vee a'$ since $e' = 0$

$= a'$; and

$a \vee b = (a \vee b)''$

$= (a'b')'$

$= (a + e)(b + e) + e$

$= ab + a + b + e + e$

$= a + b + ab.$

Hence, the desired results are established. ∎

Theorem2.9. If A is a Boolean ring with unit e, then the replacement of the operation $+$ by a new operation $\vee$ defined by $ab = a + b + ab$ converts a Boolean ring A in to an algebraic system B with the properties

1. $a \vee b = b \vee a$;
2. $a(b \vee c) = ab \vee ac$;
3. $(a \vee b)c = ac \vee bc$;
4. There exist an element 0 such that $a \vee 0 = a$ for every a;
5. If there exist an element 0 with the property (4), then there exist at least one such element 0 to which corresponds a fixed element e such that the equations $x \vee a = e$, $xa = 0$ have a solution for every element a;
6. $a \vee a = a$;
7. $aa = a$;
8. Where the old operation "$+$" is defined in terms of the new by the relation, $a + b$ is a unique solution of the simultaneous equations $x \vee ab = a \vee b$ and $x(ab) = 0$.

Proof:- Let A is a Boolean ring with unit e, then

1. $a \vee b = a + b + ab$ (by definition)

   $= a + b + ba$, Since $ab = ba$.

   $= b \vee a$.

2. $a(b \vee c) = a(b + c + bc)$

   $= ab + ac + abc$

   $= ab + ac + a^2bc$, Since $a = a^2$,

   $= ab + ac + (ab)(ac)$

   $= ab \vee ac$;

3. $(a \vee b)c = (a + b + ab)c$

   $= ac + bc + abc$

   $= ac + bc + abc^2$, since $c^2 = c$

   $= ac + bc + (ac)(bc)$

   $= ac \vee bc$.

4. $a \vee 0 = a + 0 + a0 = a$;
5. Let take $x = a + e$, where e is the unit in B,

Thus, $(a+e)\vee a = (a+e)+a+(a+e)a$

$= a+e+a+aa+ea$

$= a+e+a+a+a$, since $aa = a$ and $2a = 0$.

$= e$.

Hence, $x \vee a = e$.

And, $(a+e)a = aa+ea = a+a = 0$

Hence, $xa = 0$.

6. $a \vee a = a+a+aa$

$= a+a+a;$ since $aa = a$ and $2a = 0$,

$= a$.

7. $aa = a$ is obvious.

8. Let $x = a+b$, then we want to show that

➢ $x(ab) = (a+b)ab$

$= aab + bab$, since $aa = a$, $ba = ab$ and $bb = b$.

$= ab + ab$, since $char(A) = 2$,

$= 0$.

➢ $x \vee ab = (a+b) \vee ab$

$= (a+b) + ab + (a+b)ab$

$= a+b+ab$, since $(a+b)ab = 0$,

$= a \vee b$. ■

Remarks2.10. By introducing the above properties, we obtain:

i. $(a+b)ab = 0$;

ii. $(a+b) \vee ab = a \vee b$; and

iii. $(a \vee b)(a \vee c) = a \vee bc$.

Definition2.11. Let B be a Boolean ring with unity e. Then the element a is said to be less than or to be contained in the element b or , in symbols $a < b$, whenever any of the following equivalent relations are satisfied for every $a, b \in B$.

1. $ab = a$
2. $a \vee b = b$
3. $ab' = a$

4. $a' \vee b = e$.

Now we show that the above equivalent relations whether holds true or not.

$(1 \Rightarrow 2)$: Suppose $ab = a$, then

$$a \vee b = a + b + ab$$

$$= a + b + a, \text{ since } ab = a.$$

$$= b, \text{ since } a + a = 0.$$

$(2 \Rightarrow 3)$: Suppose $a \vee b = b$, then

$$ab' = a(b + e)$$

$$= a(a + b + ab + e), \text{ since } b = a + b + ab = a \vee b.$$

$$= aa + ab + aab + ae$$

$$= a + ab + ab + a, \text{ since } aa = a \text{ and } ae = a.$$

$$= 0, \text{ since } a + a = 0 \text{ and } ab + ba = 0.$$

$(3 \Rightarrow 4)$: Suppose $ab' = 0$, then

$$a' \vee b = (a + e) \vee b$$

$$= (a + e) + b + (a + e)b$$

$$= a + e + b + ab + b$$

$$= a + e + a, \text{ since } b + b = 0 \text{ and } ab = 0.$$

$$= e, \text{ since } a + a = 0.$$

$(4 \Rightarrow 1)$: Suppose $a' \vee b = e$, then

$$ab = a(a \vee b)$$

$$= a \vee ab$$

$$= a + ab + aab$$

$$= a + ab + ab, \text{ since } ab = b.$$

$$= a, \text{ since } ab + ab = 0.$$

■

Theorem2.12. Let B be a Boolean ring. Then for $a, b, c, d \in B$, the following holds:

a. $a < b$ and $b < c$ imply $a < c$;

b. $0 < a$ and $a < e$ for every a when the Boolean ring B has a unit e;

c. *If* $a < c$ and $b < d$ imply $ab < cd$ and $a \vee b < c \vee d$;

d. $bc = 0$ implies that $ac = 0$ if and only if $a < b$.

Proof:-

a. We know that, $a < b$ implies that $ab = a$ and $b < c$ implies that $bc = b$, thus together implies that

$$ac = (ab)c = a(bc) = ab = a.$$

Hence, $a < c$.

b. This is trivial, since $0 < a$ implies $0a = 0$ and $a < e$ implies that $ae = a$.

c. Let u s u se t hese r elations $ac = a$ and $bd = d$, f or $a < c$ and $b < d$ respectively, then we obtain

$$(ab)(cd) = (abcd)$$
$$= (ac)(bd), \text{ since } bc = cb.$$
$$= ab$$

Hence, $ab < cd$; And

$$(a \vee b)(c \vee d) = (a + b + ab)(c + d + cd)$$
$$= ac + ad + acd + bc + bd + bcd + abc + abd + abcd$$
$$= a + ad + ad + bc + b + bc + ab + ab + ab$$
$$= a + b + ab$$
$$= a \vee b$$

Hence, $(a \vee b) < (c \vee d)$.

d. Suppose that $a < b$, implies that $ab = a$. Then together $ab = a$ and $bc = 0$ gives

$$ac = (ab)c$$
$$= a(bc), \text{ since } bc = 0, \text{ then}$$
$$= 0$$

Conversely, suppose if $ac = 0$ where $bc = 0$.

Claim: $ab = a$ implies that $a < b$.

Let $c = a + ab$. Thus, $bc = b(a + ab)$

$$= ba + bab, \text{ since } ba = ab \text{ and } bb = b, \text{ then}$$
$$= ab + ab$$
$$= 0.$$

Hence, this enables us to conclude that

$$a + ab = a(a + ab) = ac = 0$$

Thus, $ab = -a$. But, $a = -a$.

Hence, $ab = a$. Therefore, $a < b$. ■

CHAPTER THREE
BOOLEAN NEAR RINGS

3.1. Special Boolean Near Rings

In this topics we introduce the concept of Boolean near ring with identity and we construct a class of Boolean near rings, called special, and determine left ideals, factor near rings which are Boolean rings for these special Boolean near rings.

Definition3.1.1. A near ring $(\mathrm{N}, +,\cdot)$ is Boolean if there exist a Boolean ring $(\mathrm{B}, +,\wedge)$ with identity such that $'\cdot'$ is defined in terms of $+,\ \wedge$ and, for every $b \in \mathrm{B}$ such that $b \cdot b = b$.

Remark3.1.2. Recall that a near ring $(\mathrm{N}, +,\cdot)$ is said to be Boolean if $x^2 = x$ for all $x \in \mathrm{N}$. If $(\mathrm{R}, +,\cdot)$ is an Boolean ring, then for all $a, b \in \mathrm{R}$, we have $a + a = 0$ and $a \cdot b = b \cdot a$. The following examples show that this is not the case for all Boolean near rings.

Example3.1.3. Given a non trivial group$(\mathrm{N}, +)$, define multiplication by $a \cdot b = b$, for all $a, b \in N$. Then,

1. It is clear that $(N, +)$ is a group (given);
2. $a(bc) = bc = c$ and $(ab)c = c$.
   Thus, $a(bc) = (ab)c$. Hence, $(N,\cdot)$ is a semigroup.
3. $a(b + c) = b + c$
   $= ab + ac.$

Hence, $(N, +,\cdot)$ is a near ring.

4. let $a \in N$, thus $a^2 = a \cdot a = a$.

Therefore, $(N, +,\cdot)$ is a Boolean near ring.

However, $a \cdot b = b \neq a = b \cdot a$, thus $(\mathrm{N}, +,\cdot)$ is a Boolean near ring for which $'\cdot'$ is not commutative and $a + a = 2a \neq 0$, thus which $(\mathrm{N}, +,\cdot)$ need not be of characteristic two.

Defintion3.1.4. In a Boolean ring $(\mathrm{B}, +,\wedge)$ with identity 1 one can define complementation by $a' = a + 1$ and $a \vee b = (a' \wedge b')'$.

Now let us consider the following theorem which clarifies Boolean near rings that are not Boolean near rings.

Theorem3.1.5. let $(\mathrm{B},+,\wedge)$ be a Boolean ring with the identity. Fix $x \in \mathrm{B}$ and define a multiplication on B by $a \cdot b = (a \vee x) \wedge b$. Then $(\mathrm{B},+,\cdot)$ is a Boolean near ring which is a Boolean ring if and only if $x = 0$.

Proof: - Clearly $(B,+)$ is a group and let $a, b, c \in \mathrm{B}$, then

i. $a \cdot (b \cdot c) = a \cdot [(b \vee x) \wedge c]$

$= (a \vee x) \wedge [(b \vee x) \wedge c]$; and

$(a \cdot b) \cdot c = [(a \vee x) \wedge b] \cdot c$

$= \big[[(a \vee x) \wedge b] \vee x\big] \wedge c$

$= [(a \vee x) \wedge (b \vee x)] \wedge c.$

Thus, $a \cdot (b \cdot c) = (a \cdot b) \cdot c.$

Hence, $(B,\cdot)$ is a semi group (multiplication is associative).

ii. $a \cdot (b + c) = (a \vee x) \wedge (b + c)$

$= (a \vee x) \wedge b + (a \vee x) \wedge c$

$= (a \cdot b) + (a \cdot c).$

Hence, multiplication is left distributive over addition.

Therefore, $(B,+,\cdot)$ is a (left) near ring.

iii. Now $b \cdot b = (b \vee x) \wedge b$

$= b \vee bx$

$= b$ since by theorem 2.5 $b \vee bx = b + bx + bbx = b.$

Therefore, $(\mathrm{B},+,\cdot)$ is a Boolean near ring.

iv. If $x = 0$, then $a \cdot b = (a \vee 0) \wedge b = a \wedge b$. Thus, $(\mathrm{B},+,\cdot) = (\mathrm{B},+,\wedge)$.

Hence, $(B,+,\cdot)$ is a Boolean ring.

Conversely, suppose $(B,+,\cdot)$ is a Boolean ring.

Now $(x + x) \cdot x = 0 \cdot x$

$= (0 \vee x) \wedge x$ since $0 \vee x = x$ and $x \wedge x = x,$

$= x$ …………………………………….. (1)

But, the left side of the equation (1) will give

$(x \cdot x) + (x \cdot x) = x^2 + x^2 = x + x = 0.$

Hence, $(\mathrm{B},+,\cdot)$ is not a ring if $x \neq 0.$ ■

Remarks3.1.6:

1. Boolean near rings of type defined in the theorem3.1.5 will be called *Special Boolean near ring.*
2. Now onwards the remainder of this topics will be devoted in discussing topics for Special Boolean near rings $(B,+,\cdot)$.

Definition3.1.7. Let $(B,+,\cdot)$ be a special Boolean near rings and let $t \in \mathrm{B}$, now define $P(t) = \{a \in \mathrm{B} : a \wedge t = a\}$. If $\mathrm{S} \subseteq \mathrm{B}$ and $t \in \mathrm{B}$, also define $S(t) = \{s \wedge t : s \in S\}$.

Definition3.1.8. Let $(\mathrm{B},+,\cdot)$ be a special Boolean near ring. Then, the maximal sub Z-ring of B is defined by $B_Z = \{b : b \wedge x = b\} = P(x)$ and the maximal sub C-ring of B is defined by $B_C = \{b : b \wedge x = 0\} = P(x')$.

Proposition3.1.9. Let $(\mathrm{B},+,\cdot)$ be a special Boolean near ring and let L be a left ideal of B. Then $L = L(x) \oplus L(x')$, be a direct sum of left ideals, and if $a_1, b_1 \in L(x)$ and $a_2, b_2 \in L(x')$, then $(a_1 + b_1) \cdot (a_2 + b_2) = (a_1 \cdot a_2) + (b_1 \cdot b_2)$.

Proof:- Since $L \subseteq B$, then for every $x \in B$, first we want to show that $L(x) \subseteq L$ and $L(x') \subseteq L$. Thus,

i. $L(x) = x \wedge a$

$= x \cdot a \in L$. Hence, $L(x) \subseteq L$.

ii. $L(x') = x' \wedge a$

$= (1 + x) \wedge a$

$= a + x \wedge a \in L$, since $x \wedge a \in L$.

Thus, $L(x') \subseteq L$ for every $a \in L$.

Next we want to show that $L(x)$ and $L(x')$ are left ideals of B.

iii. $L(x) \subseteq P(x) = \{b \in \mathrm{B} : x \cdot b = b, \text{for all } x \in \mathrm{B}\}$. Thus, $\mathrm{B}L(x) \subseteq L(x)$.

Let $s \wedge x, t \wedge x \in L(x)$. Then $(s \wedge x) + (t \wedge x) = (s + t) \wedge x \in L(x)$.

Hence, $L(x)$ is a left ideal of B.

iv. Also let $s \wedge x' \in L(x')$ and $b \in \mathrm{B}$, then

$b \cdot (s \wedge x') = (b \vee x) \wedge (s \wedge x')$

$= [b \wedge (s \wedge x')] \vee [x \wedge (s \wedge x')]$

$= [(b \wedge s) \wedge x'] \vee [s \wedge x \wedge x']$, since $x \wedge x' = 0$, then $s \wedge x \wedge x' = 0$.

$= (b \wedge s) \wedge x' \in L(x')$.

Thus, $L(x')$ is a left ideal of B.

v. Certainly $L = L(x) + L(x')$ and $L(x) \cap L(x^{'}) = \{0\}$.

Finally, let $a_1, b_1 \in L(x)$, $a_2, b_2 \in L(x')$ and assume that $a_1 \wedge a_2^{'} = a_1$, $a_1^{'} \wedge a_2 = a_2$, $b_1 \wedge b_2^{'} = b_1$ and $b_1^{'} \wedge b_2 = b_2$

Hence we have that

$$\begin{aligned}
(a_1 + a_2) \cdot (b_1 + b_2) &= [(a_1 \wedge a'_2) \vee (a'_1 \wedge a_2)] \cdot [(b_1 \wedge b'_2) \vee (b'_1 \wedge b_2)] \\
&= [a_1 \vee a_2 \vee x] \wedge [b_1 \vee b_2] \\
&= (a_1 \wedge b_1) \vee (x \wedge b_1) \vee (a_2 \wedge b_2) \\
&= (a_1 \wedge b_1) \vee b_1 \vee (a_2 \wedge b_2) \\
&= b_1 \wedge (a_2 \wedge b_2) \\
&= b_1 \wedge (a_2 \wedge b_2) \vee (x \wedge b_2) \\
&= b_1 \vee (a_2 \vee x) \wedge b_2] \\
&= \{b_1 \wedge [(a^{'}{}_2 \wedge x^{'}) \vee b^{'}{}_2]\} \vee \{b^{'}{}_1 \wedge [(a_2 \vee x) \wedge b_2]\} \\
&= b_1 + [(a_2 \vee x) \wedge b_2] \\
&= [(a_1 \vee x) \wedge b_1] + [(a_2 \vee x) \wedge b_2] \\
&= (a_1 \cdot b_1) + (a_2 \cdot b_2).
\end{aligned}$$

This completes the proof. ■

Lemma3.1.10. If L is a left ideal of $(\mathrm{B}, +, \cdot)$ with $L \subseteq P(x')$, then $P(a) \subseteq L$ for any $a \in L$.

***Proof*:-** Let L be the left ideal of B and $a \in L$. Then,

Thus, $b \cdot a \in L$ for every $b \in$ B.

But, $b \cdot a = (b \vee x) \wedge a$

$= (b \wedge a) \vee (x \wedge a)$

$= b \wedge a$, since $a \in L \subseteq P(x')$, that is $x \wedge a = 0$ by definition 3.1.8

Hence, $b \wedge a \in L$ for all $b \in$ B.

Therefore, $P(a) \subseteq L$.

Corollary3.1.11. If $a \in P(x')$, then $P(a)$ is a left ideal of B.

***Proof*:-** If $t \in P(a)$, then $b \cdot t = b \wedge t \in P(a)$, for all $b \in$ B.

Hence, $P(a)$ is a left ideal of B.

Lemma3.1.12. If $k \in P(x)$ and $a, b \in B$, then $(a + k) \cdot b + a \cdot b = 0$.

Proof:- Suppose $k \in P(x)$. Then $x' \in P(k')$ so that $k^{'} \vee x = 1$.

Now $(a + k) = (a \wedge k') \vee (a' \wedge k)$

So that, $(a + k) \vee x = [(a \wedge k^{'}) \vee (a^{'} \wedge k)] \vee x$

$$= [(a \vee x) \wedge (k' \vee x)] \vee [(a' \vee x) \wedge (k \vee x)]$$

$$= (a \vee x) \vee [(a' \wedge x) \vee x]$$

$$= (a \vee x).$$

Thus, $(a + k) \cdot b + a \cdot b = a \cdot b + a \cdot b = 0$. ■

Lemma3.1.13. If $k \in P(x')$ and $a, b \in$ B, then $(a + k) \cdot b + a \cdot b = k \wedge b$.

Proof:- using $k \wedge x^{'} = k, k^{'} \wedge x = x$ and $k \wedge x = 0$, we have :

$(a + k) \cdot b + a \cdot b = [(a + k) + a] \cdot b$

$$= \{[(a \wedge k') \vee (a' \wedge k)] \vee x + (a \vee x)\} \wedge b$$

$$= [\{[x \vee (a \wedge k') \vee (k \wedge a')] \wedge (a' \wedge x')\} \vee \{x^{'} \wedge (a \wedge k^{'})^{'} \wedge (k \wedge a^{'})^{'} \wedge (a \vee x)\}] \wedge b$$

$$= [\{[(k \wedge a^{'} \wedge x^{'}) \vee \{[(x^{'} \wedge a^{'}) \vee (x^{'} \wedge k)] \wedge [(k^{'} \wedge x) \vee a]\}] \wedge b$$

$$= [(k \wedge a') \vee \{[(x \vee a)' \vee k] \wedge (x \vee a)\}] \wedge b$$

$$= [(k \wedge a') \vee (k \wedge a)] \wedge b$$

$$= k \wedge b.$$ ■

Lemma3.1.14. If $a, b \in$ B, then $(a + b) \cdot c + a \cdot c + b \cdot c = x \wedge c$.

Proof:- Since $(a + b) \cdot c + a \cdot c + b \cdot c = [(a + b) + a + b] \cdot c$

$$= \{[(a + b) \vee x] + (a \vee x) + (b \vee x)\} \wedge c$$

Then, we have that

$(a + b) \vee x + (a \vee x) + (b \vee x) = (a + b) \vee x + \{[(a \vee x) \wedge b^{'} \wedge x^{'}] \vee [a^{'} \wedge x^{'} \wedge (b \vee x)]\}$

$$= (a + b) \vee x + \{[(a \wedge b') \vee (a' \wedge b)] \wedge x'\}$$

$$= (a + b) \vee x + (a + b) \wedge x'$$

$$= \{[(a + b) \vee x] \wedge [(a + b) \wedge x^{'}]^{'}\} \vee \{[(a + b) \vee x]^{'} \wedge [(a + b) \wedge x']\}$$

$$= \{[(a + b) \wedge (a + b)^{'}] \vee x\} \vee \{(a + b)^{'} \wedge x^{'} \wedge (a + b)\}$$

$$= (0 \vee x) \vee (0 \wedge x')$$

$$= x.$$

Hence, $(a + b) \cdot c + a \cdot c + b \cdot c = x \wedge c$. ■

Theorem3.1.15. Let I be an ideal of $(\mathrm{B},+,\cdot)$. Then B/I ia a Boolean ring if and only if $P(x) \subseteq I$.

Proof:- Suppose B/I is a Boolean ring. Let $a+I, b+I$ and $c+I$ be elements of B/I . Then the right distributive law holds so that

$$[(a+I)+(b+I)](c+I) = (a+I)(c+I)+(b+I)(c+I) \qquad \text{……………… (2)}$$

Thus, $(a+b)\cdot c + I = a\cdot c + b\cdot c + I$.

Implies that $(a+b)\cdot c + a\cdot c + b\cdot c \in I$.

Hence, $(a+b)\cdot c + a\cdot c + b\cdot c = x \wedge c \in I$, by lemma 3.1.14.

Since c is arbitrary, we have $P(x) \subseteq I$.

Conversely, if $P(x) \subseteq I$, then equation (2) is valid if and only if

$(a+b)\cdot c + a\cdot c + b\cdot c \in I$. But, $(a+b)\cdot c + a\cdot c + b\cdot c \in I$, by lemma 3.1.14.

This completes the proof. ■

Remark3.1.16. The Boolean ring B/I in theorem 3.1.15 is called Quotient ring.

Lemma3.1.17. Let $(\mathrm{B},+,\wedge)$ be a Boolean ring with identity 1, and let A be an ideal of B. Then A is a direct summand if and only if $A = P(x)$ for some $x \in \mathrm{B}$.

Proof:- For $x \in \mathrm{B}$, we have $\mathrm{B} = P(x) \oplus P(x')$.

Conversely, suppose that $\mathrm{B} = A \oplus C$ where A and C are ideals.

Now $1 = x + x'$ for $x \in A$ and $x' \in C$.

Let $a \in A$, then $a = a \wedge 1$

$$= a \wedge (x + x')$$

$$= (a \wedge x) + (a \wedge x')$$

But, $a \wedge x' = 0$, since $x' \in C$.

Hence, $a = a \wedge x$ which implies that $a \in P(x)$.

Consequently, $A \subseteq P(x)$.

But $P(x) \subseteq A$, since $x \in A$.

Therefore, $A = P(x)$. ■

3.2. Distributively Generated Boolean Near Rings

Definition3.2.1. le t N be a right near ring a nd if N contains a m ultiplicative semigroup S whose elements generate $(N,+)$ and if it satisfy $(x+y)s = xs + ys$, for all $x, y \in N$ and $s \in S$. We say that N is a distributively generated near ring.

Definition3.2.2. A near ring N is called an N-system if

1. $xz = yz$ and $z \neq 0$ implies $x = y$;
2. There exists $e \neq 0$ in N such that $e^2 = e$, and
3. There exists n in N such that $n + n = e$.

Theorem3.2.3. (Neumann):- The additive group of an N-system is abelian.

Lemma3.2.4. If N is a Boolean near ring, then $xy = xyx$ for each $x, y \in N$.

Theorem3.2.5. If N is a Boolean near ring, then $xyz = xzy$ for each $x, y, z \in N$.

Proof:- Let $x, y, z \in N$. Then $y(x - xz)z = y(xz - xz) = y0$ multipling both sides by $x - xz$ we obtain

$$[(x - xz)y(x - xz)]z = (x - xz)y0 \qquad \text{……… (3)}$$

implies $(x - xz)yz = (x - xz)y0$, by Lemma 3.2.4 ……. (4)

Since $zyz = zy$, equation (4) becomes

$$xyz - xzy = xy0 - xzy0. \qquad \text{……. (5)}$$

Next, $z(x - xz)z = z(xz - xz) = z0$ and thus, by Lemma 3.2.4, we obtain

$z(x - xz) = z0$ and this gives $yz(x - xz) = yz0$. …..… (6)

Again, by Lemma 3.2.4, $yz(x - xz) = yz(x - xz)yz$ and thus,

$yz(x - xz)yz = yz(x - xz) = yz0$. (by equations(6),) ………… (7)

Now pre-multiplying both sides of by equations (7) by $x - xz$, we obtain

$[(x - xz)yz(x - xz)yz] = (x - xz)yz0$ and this gives, by idempotency, that is

$$(x - xz)yz = (x - xz)yz0. \qquad \text{…………………. (8)}$$

Again, using $zyz = zy$, we obtain from the preceding equation that

$xyz - xzyz = xyz0 - xzyz0$ which implies that

$xyz - xzy = xyz0 - xzy0$

But equation (5) shows $xyz - xzy = xy0 - xzy0$. ………… (9)

Then post multiplying by 0 we obtain $xyz0 - xzy0 = xy0 - xzy0$ and

This gives that $xyz0 = xy0$ for each $x, y, z \in N$.

Hence, by using the result just obtained, we have that

$xy0 = xxy0 = xx0 = x0$ for each $x, y \in N$.

Returning to the equation (9) $xyz - xzy = xy0 - xzy0$.

we conclude that $xyz - xzy = xy0 - xzy0 = x0 - x0 = 0$.

Therefore, $xyz = xzy$ for each $x, y, z \in N$. ■

Theorem3.2.6. Let N be a Boolean right near ring. For each $x, y \in N$, then there exists an $e \in N$ such that $ex = x$ and $ey = y$. Then, N is a Boolean ring.

Proof:- Since N is a Boolean near ring, we want to show that $(N, +)$ is abelian group and N is commutative under the operation of multiplication.

i. Let $x \in N$. Consider x and $x + x$. By assumption, there exists an idempotent $e \in N$ such that $ex = x$ and $e(x + x) = (x + x)$.

Thus, $x + x = ex + ex$

$$\begin{aligned}
&= (e + e)x \\
&= (e + e)^2 x \\
&= [e(e + e) + e(e + e)]x \\
&= e(e + e)x + e(e + e)x \\
&= e(ex + ex) + e(ex + ex) \\
&= e(x + x) + e(x + x) \\
&= (x + x) + (x + x).
\end{aligned}$$

Hence, $x + x = 0$. Thus, each non-zero elements of $(N, +)$ is of order 2.

Therefore, $(N, +)$ is an abelian group.

ii. Now, let $x, y \in N$. Then by assumption, there exists an idempotent $e \in N$ such that $ex = x$ and $ey = y$. Thus

$xy = exy$

$= eyx$, by lemma3.2.5

$= yx$, since $ey = y$.

Hence, multiplication is commutative.

Therefore, N is a Boolean ring. ■

Theorem3.2.7:- [1] If N is a distributively generated near ring and $(N,+)$ is abelian, then N commutative ring.

Theorem3.2.8. Every distributively generated Boolean right near ring N is a Boolean ring.

Proof:- Let N denote a distributively generated Boolean right near ring and suppose S is a multiplicative semigroup, that is $x(yz) = (xy)z$ for all $x, y, z \in S$ whose elements s generate $(N,+)$ and which satisfy $s(x+y) = sx + sy$ for each $x, y \in N$. Let $x \in N$ and $s_i \in S$ for $i \in \mathbb{N}$. Suppose that $s_i + s_i = 0$ with $s_i 0 = 0$, $x0 = 0$.

Hence, $s(x+x) = sx + sx$

$= (s+s)x$

$= 0x$, by zero symmetric we have

$= 0$

Next, let $x, y \in N$ and $y = s_1 + s_2 + \cdots + s_n$, where each $s_i \in S$.

Assume that $y \neq 0$. Then,

$$
\begin{aligned}
y(x+x) &= (s_1 + s_2 + \cdots + s_n)(x+x) \\
&= s_1(x+x) + s_2(x+x) + \cdots + s_n(x+x) \\
&= (s_1+s_1)x + (s_2+s_2)x + \cdots + (s_n+s_n)x \\
&= 0x + 0x + \cdots + 0x \\
&= 0.
\end{aligned}
$$

Then, $x + x = 0$.

Thus, each non-zero element in $(N,+)$ is of order 2.

Hence, $(N,+)$ is an abelian group and by theorem 3.2.7,

Therefore, N is a Boolean ring. ■

Remark3.2.9. Let R denote a Boolean ring. Let A and B denote a subrings of R such that $A \cap B = \{0\}$ and suppose $ab = 0$ for each $a \in A$ and $b \in B$.

Let take N to be the set of all mappings: $f : R \to R$ such that, $f(x) = ax + b$, for each $x \in R$, and where $a \in A$ and $b \in B$.

Now we want to show that $(N, +, \circ)$ is a Boolean near ring where $'+'$ and $'\circ'$ denote the ordinary addition and composition of mappings respectively. Let $f(x) = ax + b$ $g(x) = cx + d$ and $h(x) = mx + n$ for every $a, c, m \in A$, $b, d, n \in B$ and $x \in R$. Then,

1. $f(x) + g(x) = (a + c)x + (b + d) \in N$ and $(f \circ g)(x) = (ac)x + (ad + b)$. Thus, N is closed under these operations.
2. Clearly $(N, +)$ is an abelian group.
3. Also $(N, \circ)$ is semi group as composition is associative.
4. $f \circ (g + h) = f \circ g + f \circ h$.

Hence $(N, +, \circ)$ is a Boolean left near ring.

5. $(f \circ f)(x) = f(f(x))$

$= f(ax + b)$

$= a(ax + b) + b$

$= aax + ab + b$, since a sub ring of Boolean ring is Boolean, then $aa = a$ and $ab = 0$, we have

$= ax + b$

$= f(x)$.

Therefore, $(N, +, \circ)$ is Boolean ring.

Now, let A denote a Boolean ring and B denote an additive abelian group. Consider the group direct sum $A \oplus B$ of A and B. Define a multiplication in $A \oplus B$ by $(a_1, b_1)(a_2, b_1) = (a_1 a_2, b_1)$. Thus, it can be verified directly that $A \oplus B$ forms a Boolean right near ring with commutative addition and satisfies the identity $(x - y)0 = xy - yx$ and denote by $N(A, B)$.

Theorem3.2.10. Let N denote a Boolean right near ring in which the addition is commutative and suppose, for each $x, y \in N$, that

$$(x - y)0 = xy - yx. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

Then there exist a Boolean ring A and an abelian group B such that $N \cong N(A, B)$.

Proof:- Let $A = \{a \in N : a0 = 0\}$ and let $B = \{b \in N : b0 = b\}$.

i. Clearly, A and B are additive subgroups of N.

ii. For each $a_1, a_2 \in A$, by equation (10), we have

$a_1 a_2 - a_2 a_1 = (a_1 - a_2)0$

$= a_1 0 - a_2 0$

$= 0.$

Thus, $a_1 a_2 = a_2 a_1$.

Also, A is closed with respect to multiplication since

$$(a_1a_2)0 = a_1(a_20)$$
$$= a_10$$
$$= 0; \text{ for each } a_1, a_2 \in A.$$

Hence A is a Boolean ring.

iii. Furthermore, $A \cap B = \{0\}$, from the definitions of A and B along with Theorem 3.2.5, we obtain

$$ab = ab0$$
$$= a0b$$
$$= a0$$
$$= 0, \text{ for each } a \in A \text{ and } b \in B.$$

iv. Let $\phi : N \to N(A, B)$ denote a mapping defined by $\phi(x) = (x - x0, x0)$ for each $x \in N$. It is easy to see that ϕ is additive.

a) Let $x_1, x_2 \in N$. Assume that $x_1 = x_2$ then

$x_10 = x_20$ and $x_1 - x_10 = x_2 - x_20$. Thus

$(x_1 - x_10, x_10) = (x_2 - x_20, x_20)$ implies that

$\phi(x_1) = \phi(x_2)$.

Hence, ϕ is well defined.

b) $$\phi(x_1+x_2) = ((x_1+x_2) - (x_1+x_2)0, (x_1+x_2)0)$$
$$= ((x_1 - x_10) + (x_2 - x_20), x_10 + x_20)$$
$$= (x_1 - x_10, x_10) + (x_2 - x_20, x_20)$$
$$= \phi(x_1) + \phi(x_2).$$

c) To see that ϕ is also multiplicative, first let $x_1, x_2 \in N$. Using the identity equation (3), we obtain

$$x_1(x_2 - x_20) - (x_2 - x_20)x_1 = [x_1 - (x_2 - x_20)]0$$
$$= x_10 - (x_2 - x_20)0$$
$$= x_10 - x_20 + x_20$$
$$= x_10.$$

But, $x_1(x_2 - x_20) - (x_2 - x_20)x_1 = x_1(x_2 - x_20) - x_2x_1 - x_20x_1 = x_10$.
and by rearranging we obtain

$$x_1(x_2 - x_20) = x_10 + x_2x_1 + x_20$$

$= (x_1 - x_2)0 + x_2x_1$

$= x_1x_2 - x_2x_1 + x_2x_1$

$= x_1x_2.$

Also, by Theorem 3.2.5, $x_1x_20 = x_10x_2 = x_10$.

Thus, $\phi(x_1)\phi(x_2) = (x_1 - x_10, x_10)(x_2 - x_20, x_20)$

$= \big((x_1 - x_10)(x_2 - x_20), x_10\big)$

$= (x_1(x_2 - x_20) - x_10(x_2 - x_20), x_10)$

$= (x_1x_2 - x_10x_2, x_10)$

$= (x_1x_2 - x_10, x_10)$ Since, $x_10x_2 = x_10$.

$= (x_1x_2 - x_1x_20, x_1x_20)$ Since, $x_1x_20 = x_10$.

$= \phi(x_1x_2)$.

Hence, by (b) and (c), we say that ϕ is a ring homomorphism.

v. Suppose $\phi(x) = \phi(y)$. Let $x, y \in N$, we obtain

$$(x - x0, x0) = (y - y0, y0)$$

Which implies $x - x0 = y - y0$ and $x0 = y0$, we get $x = y$.

Thus, ϕ is a monomorphism.

vi. Now, for each $(a, b) \in N(A, B)$, let $c = a + b$. Then $c0 = (a + b)0 = a0 + b0 = 0 + b = b$ and $c - c0 = a + b - b = a$.

Thus, $\phi(c) = (c - c0, c0) = (a, b)$.

Hence, ϕ is an epimorphism.

Therefore, ϕ is an isomorphism and consequently, $N \cong N(A, B)$. ■

References

1. A. Frölich, ***Distributively generated near rings (I. ideal theory)***, Proc. London. Math. Soc (3) 8(1958), 76-94.
2. D. J. Hansen and Jiang Luh, ***Boolean Near Rings and Weak Commutativity***, J. Austral. Math. Soc. (Series A) 47 (1989), 103-107.
3. James R. Clay and Donald A, Lawver. ***Boolean Near Rings***, Canad. Math. Bull. Vol. 12, no. 3, 1969. 265-273.
4. M. H. Stone, ***The Theory of Representation for Boolean Algebras***. Transactions of the American Mathematical Society, Vol. 40, No. 1. (Jul., 1936), pp. 40-50.
5. W. B. Vasantha Kandasamy. ***Smarandache Near rings***. American Research Press.