144

A DEGENERATE PARAMETRIC OSCILLATOR COUPLED TO A SQUEEZED VACUUM

A Thesis

Submitted to the

School of Graduate Studies

Addis Ababa University

In Partial Fulfillment

of the Requirements for the

Degree of Master of Science in Physics

by

Misrak Getahun

May 1998

Addis Ababa

Acknowledgement

I do not know how to express my thanks to my advisor and instructor, Dr. Fesseha Kassahun, head, Department of physics, AAU. He was not only an advisor but also a true father. It is only because of him that I could obtain a fund from ICTP, complete two years of study and realize this thesis.

I would like also to thank Ato Alem Mebratu, all the Physics Staffs of AAU and my instructors, my families and intimates.

Abstract

The intracavity quadrature fluctuations and photon statistics for the signal mode produced by a degenerate parametric oscillator coupled to a squeezed vacuum reservoir are calculated employing the Q-function. The Q-function is obtained by solving the Fokker-Planck equation applying the propagator method developed by Fesseha [1]. We also calculate the squeezing spectra of the output quadrature operators using the quantum Langevin equations.

Contents

1 Introduction

Some properties of light which cannot be explained by classical theories have generated a considerable interest in the past few years. One of these properties involves squeezing in which the fluctuations in one of quadratures is reduced below the quantum limit (vacuum level) at the expense of increased fluctuations in the other quadrature without violation of the uncertainty relation. Other nonclassical properties of light are sub-Poissonian statistics in which the photon number distribution is narrower than the Poissonian one and photon antibunching in which the second-order correlation function $g^2(\tau)$ is greater than $g^2(0)$.

Because of the quantum noise reduction achievable below the vacuum level in a squeezed light, it has potential applications in high-performance of optical communication [2], highly sensitive interferometer [3-8] and in gravitational-wave detection [9-11]. Different authors have predicted theoretically that a squeezed light can be generated in various physical processes. It has been shown that squeezing should be realizable in parametric oscillator [12-15], four-wave mixing [16-19] and resonance fluorescence [20-22].

A degenerate parametric oscillator is a prototype source of squeezed light. In a degenerate parametric oscillator a strong pump mode of frequency 2ω interacts with a nonlinear medium inside a cavity and gives rise to a signal mode of frequency ω. A theoretical analysis of the quantum fluctuations and photon statistics of the signal mode have been made by a number of authors [11,23-34]. Mulburn and Walls [23] have shown that the best squeezing attainable in the signal mode of the degenerate parametric oscillator is a reduction of intracavity quadrature fluctuations by a factor of 2 at steady state and threshold. This has been also confirmed by Lugiato and Strini [24].

A limit of 50% squeezing of the intracavity signal mode in the degenerate parametric oscillator arises from the leakage through the partially transmitting mirror, so that the vacuum fluctuations amplify the fluctuations in the cavity. However, Javed Anwar and M. Suhail Zubairy [25] have shown that the squeezing of the intracavity signal mode can be increased beyond a 50% limit by coupling the degenerate parametric oscillator to a squeezed vacuum reservoir.

The squeezing spectrum of the output signal mode of a parametric oscillator operating below and above threshold have been calculated by several authors [11,26,27,33] applying different methods. It has been found that a complete suppression of the noise at resonance ($\omega = 0$) in one quadrature of the output signal mode produced in a degenerate parametric oscillator is possible [27,33]. Experimentally, a squeezing amounting to a noise reduction of 60% below the vacuum level has been achieved in a near degenerate parametric oscillator operating below threshold [14,15].

In quantum optics, one can analyze the dynamics of a physical system applying the schrödinger formalism which leads to Fokker-Planck equation or the Heisenberg formalism which leads to the Langevin equations. In the schrödinger formalism a distribution function such as the Glauber-Sudarshan P-function or Q-function can be used in the analysis of the statistical or squeezing properties of a quantum optical system.

The main objective of this thesis is to calculate the intracavity quadrature fluctuations and photon statistics for the signal mode produced by a degenerate parametric oscillator coupled to a squeezed vacuum applying the Q-function. We also calculate the squeezing spectra for the output signal mode applying the quantum Langevin equations. The Q-function is obtained by solving the Fokker-Planck equation employing the propagator method developed by Fesseha [1].

We would like to point out that in the parametric oscillator considered in this thesis the pump mode is treated classically and its amplitude is assumed to be constant. In addition, we would like to stress that our analysis holds for a single-port mirror through which the parametric oscillator is coupled to a squeezed vacuum.

The organization of this thesis is as follows: In chapter two we first study the properties of a squeezed vacuum and then derive the Master equation as well as the quantum Langevin equations for a degenerate parametric oscillator coupled to a squeezed vacuum reservoir. In chapter three we derive the Fokker-Planck equation for the Q-function and obtain the solution of the resulting equation applying the propagator method. In chapter four the variances of the intracavity quadrature operators are calculated using the Q-function. In addition, we calculate the squeezing spectra of the output quadrature operators employing the quantum Langevin equations. In chapter five we obtain the photon number distribution, the mean and variance of the photon number, and the second-order correlation function applying the Q-function. Finally, in chapter six we present a brief discussion of our results.

2 Quantum Dynamics with a Squeezed Vacuum

The dynamics of a quantum system can be studied using the master equation or Langevin equations. In this chapter we first study the properties of a squeezing vacuum and then derive the master equation as well as the quantum Langevin equations for a degenerate parametric oscillator coupled to a squeezed vacuum reservoir. We also establish the relation between input and output operators.

2.1 A squeezed vacuum

Here we wish to determine the Q-function, the variances of the quadrature operators and the photon statistics for a squeezed vacuum. A single-mode squeezed vacuum is defined by

$$|r\rangle = \hat{S}(r)|0\rangle, \tag{2.1a}$$

where

$$\hat{S}(r) = e^{\frac{1}{2}r(\hat{a}^2-\hat{a}^{\dagger 2})} \tag{2.1b}$$

is the unitary squeeze operator with the squeeze parameter r taken for convenience to be real and positive and $\hat{a}$ ($\hat{a}^\dagger$) is the annihilation (creation) operator. The Q-function for this state can be expressed in terms of the antinormally ordered characteristic function as

$$Q(\alpha^*, \alpha, r) = \frac{1}{\pi}\int \frac{d^2z}{\pi}\phi_A(z^*, z, r)e^{z^*\alpha - z\alpha^*}, \tag{2.2}$$

in which

$$\phi_A(z^*, z, r) = Tr\left(|r\rangle\langle r|e^{-z^*\hat{a}}e^{z\hat{a}^\dagger}\right). \tag{2.3a}$$

On account of (2.1a) and $\hat{S}(r)\hat{S}^\dagger(r) = I$, we see that

$$\phi_A(z^*, z, r) = \langle 0|\hat{S}^\dagger(r)e^{-z^*\hat{a}}\hat{S}(r)\hat{S}^\dagger(r)e^{z\hat{a}^\dagger}\hat{S}^\dagger(r)|0\rangle. \tag{2.3b}$$

Upon expanding the first exponential function in power series, we have

$$\hat{S}^\dagger(r)e^{-z^*\hat{a}}\hat{S}(r) = \sum_{l=0}^{\infty}\frac{(-z^*)^l}{l!}\hat{S}^\dagger(r)\hat{a}^l\hat{S}(r). \tag{2.4a}$$

Inserting the identity $\hat{S}(r)\hat{S}^\dagger(r)$ between any pair of annihilation operators in (2.4a), one finds

$$\hat{S}^\dagger(r)e^{-z^*\hat{a}}\hat{S}(r) = \sum_{l=0}^{\infty}\frac{(-z^*\hat{a}(r))^l}{l!} = e^{-z^*\hat{a}(r)}, \tag{2.4b}$$

where

$$\hat{a}(r) = \hat{S}^\dagger(r)\hat{a}\hat{S}(r). \tag{2.4c}$$

Similarly one can easily verify that

$$\hat{S}^\dagger(r)e^{z\hat{a}^\dagger}\hat{S}(r) = e^{z\hat{a}^\dagger(r)}, \tag{2.5a}$$

in which

$$\hat{a}^\dagger(r) = \hat{S}^\dagger(r)\hat{a}^\dagger\hat{S}(r). \tag{2.5b}$$

In view of these results, Eq. (2.3b) can be written in the form

$$\phi_A(z^*, z, r) = \langle 0|e^{-z^*\hat{a}(r)}e^{z\hat{a}^\dagger(r)}|0\rangle. \tag{2.6}$$

We note that the first derivative of (2.4c) with respect to r is

$$\frac{d}{dr}\hat{a}(r) = \hat{S}^\dagger(r)\frac{1}{2}[\hat{a}, \hat{a}^2 - \hat{a}^{\dagger 2}]\hat{S}(r) = -\hat{a}^\dagger(r) \tag{2.7a}$$

and hence

$$\frac{d}{dr}\hat{a}^\dagger(r) = -\hat{a}(r). \tag{2.7b}$$

In order to decouple these equations, we differentiate (2.7a) once again with respect to r. Thus we have

$$\frac{d^2}{dr^2}\hat{a}(r) = \hat{a}(r).$$

The solution of this equation can be put in the form

$$\hat{a}(r) = Ae^r + Be^{-r}. \tag{2.8}$$

A and B can be obtained applying the condition $r = 0$. Hence we see that

$$\hat{a}(r)\big|_{r=0} = A + B = \hat{a},$$

$$\left.\frac{d\hat{a}(r)}{dr}\right|_{r=0} = A - B = -\hat{a}^\dagger.$$

It then follows that

$$A = \frac{1}{2}(\hat{a} - \hat{a}^\dagger),$$

$$B = \frac{1}{2}(\hat{a} + \hat{a}^\dagger),$$

so that substitution of these into (2.8), we get

$$\hat{a}(r) = \hat{a}\, cosh(r) - \hat{a}^\dagger sinh(r). \tag{2.9a}$$

From this we easily see that

$$\hat{a}^\dagger(r) = \hat{a}^\dagger cosh(r) - \hat{a}\, sinh(r). \tag{2.9b}$$

On account of (2.9) and the relation [11]

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}, \tag{2.10}$$

Eq. (2.7) can be put in the form

$$\phi_A(z^*, z, r) = e^{-\frac{1}{2}z^* z} \langle 0| e^{\hat{a}^\dagger (z\, cosh(r) + z^* sinh(r)) - \hat{a}(z^* cosh(r) + z\, sinh(r))} |0\rangle.$$

Employing once more (2.10) we can put the operators in normally ordered form. Hence the characteristic function turns out to be

$$\phi_A(z^*, z, r) = e^{-z^* z\, cosh^2(r) - \frac{1}{2} sinh(r) cosh(r) \left(z^{*2} + z^2\right)}. \tag{2.11}$$

Using (2.11) in (2.2) and carrying out the integration with the help of the relation

$$\int e^{(-a\alpha^*\alpha + B\alpha^2 + C\alpha^{*2} + b\alpha^* + c\alpha)} d\alpha = \frac{\pi}{\sqrt{a^2 - 4BC}} exp\left[\frac{abc + Bb^2 + Cc^2}{a^2 - 4BC}\right], \quad a > 0 \tag{2.12}$$

we find

$$Q(\alpha^*, \alpha, r) = \frac{sech(r)}{\pi} exp\left[-\alpha^*\alpha - \frac{1}{2} tanh(r)\left(\alpha^{*2} + \alpha^2\right)\right]. \tag{2.13}$$

This represents the Q-function for a squeezed vacuum state.

Next we seek to calculate the variances of the quadrature operators applying the Q-function (2.13). The quadrature operators for a single-mode light are defined by

$$\hat{a}_1 = \hat{a}^\dagger + \hat{a}, \tag{2.14a}$$

$$\hat{a}_2 = i(\hat{a}^\dagger - \hat{a}). \tag{2.14b}$$

The variances of these Harmitian operators can be put in the form

$$(\Delta \hat{a}_1)^2 = 1 + 2\langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{a}^{\dagger 2}\rangle + \langle \hat{a}^2\rangle - 2\langle \hat{a}^\dagger\rangle\langle \hat{a}\rangle - \langle \hat{a}^\dagger\rangle^2 - \langle \hat{a}\rangle^2, \tag{2.15a}$$

$$(\Delta\hat{a}_2)^2 = 1 + 2\langle\hat{a}^\dagger a\rangle - \langle\hat{a}^{\dagger 2}\rangle - \langle\hat{a}^2\rangle - 2\langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle + \langle\hat{a}^\dagger\rangle^2 + \langle\hat{a}\rangle^2. \qquad (2.15b)$$

The expectation value of the annihilation operator $\hat{a}$ can be expressed as

$$\langle\hat{a}\rangle = \int_{-\infty}^{\infty} d^2\alpha Q(\alpha^*, \alpha, r)\alpha = sech(r)\int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}exp\left[-\alpha^*\alpha - \frac{1}{2}tanh(r)\left(\alpha^{*2}+\alpha^2\right)\right]\alpha.$$

This can be written as

$$\langle\hat{a}\rangle = sech(r)\frac{\partial}{\partial b}\int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}exp\left[-\alpha^*\alpha - \frac{1}{2}tanh(r)(\alpha^{*2}+\alpha^2) + b\alpha\right]\Bigg|_{b=0},$$

so that on carrying out the integration using the relation (2.12), we obtain

$$\langle\hat{a}\rangle = sech(r)\,\frac{\partial}{\partial b}cosh(r)\Bigg|_{b=0} = 0. \qquad (2.16a)$$

It can be shown in a similar manner that

$$\langle\hat{a}^\dagger\rangle = 0 \qquad (2.16b)$$

Furthermore, we see that

$$\langle\hat{a}^2\rangle = sech(r)\int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}exp\left[-\alpha^*\alpha - \frac{1}{2}tanh(r)(\alpha^{*2}+\alpha^2)\right]\alpha^2$$

or

$$\langle\hat{a}^2\rangle = sech(r)\frac{\partial}{\partial b}\int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}exp\left[-\alpha^*\alpha + b\alpha^2 + c\alpha^{*2}\right],$$

in which $b = c = -\frac{1}{2}tanh(r)$. It then follows that

$$\langle\hat{a}^2\rangle = -sinh(r)cosh(r) \equiv -M. \qquad (2.16c)$$

One can easily establish in a similar fashion that

$$\langle\hat{a}^{\dagger 2}\rangle = -M. \qquad (2.16d)$$

The mean photon number can be written as

$$\langle\hat{a}^\dagger\hat{a}\rangle = sech(r)\left(-\frac{\partial}{\partial b} - 1\right)\int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}exp\left[-b\alpha^*\alpha - \frac{1}{2}tanh(r)\left(\alpha^{*2}+\alpha^2\right)\right]\Bigg|_{b=1}.$$

It then follows that

$$\langle\hat{a}^\dagger\hat{a}\rangle = sinh^2(r) \equiv N. \qquad (2.16e)$$

Therefore, combination of (2.15) and (2.16) yields

$$(\Delta\hat{a}_1)^2 = (2N - 2M + 1) = e^{-2r}, \qquad (2.17a)$$

$$(\Delta \hat{a}_2)^2 = (2N + 2M + 1) = e^{2r}. \tag{2.17b}$$

We note that the squeezing occurs in the first quadrature.

We next proceed to obtain the photon number distribution. The photon number distribution for a single-mode light defined by

$$P(n) = \langle n|\hat{\rho}(\hat{a}^\dagger, \hat{a})|n\rangle$$

is expressed in terms of the Q-function as [33]

$$P(n) = \frac{\pi}{n!} \frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n} Q(\alpha^*, \alpha, r) e^{\alpha^*\alpha}\Big|_{\alpha^*=\alpha=0}. \tag{2.18}$$

Employing the Q-function (2.13), we have

$$P(n) = \frac{sech(r)}{n!} \frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n} exp\left[-\frac{1}{2}tanh(r)\left(\alpha^2 + \alpha^{*2}\right)\right]\Big|_{\alpha^*=\alpha=0},$$

so that upon expanding the exponential function in power series

$$e^{(-1/2tanh(r)\alpha^{*2})} = \sum_k \frac{\left(-\frac{1}{2}tanh(r)\right)^k}{k!}\alpha^{*2k},$$

$$e^{(-1/2tanh(r)\alpha^{2})} = \sum_l \frac{\left(-\frac{1}{2}tanh(r)\right)^l}{l!}\alpha^{2l},$$

the above expression can be put in the form

$$P(n) = \frac{sech(r)}{n!} \sum_{k,l}^{\infty} \frac{\left(-\frac{1}{2}tanh(r)\right)^{k+l}}{k!l!} \frac{\partial^n}{\partial\alpha^{*n}}\alpha^{*2k}\frac{\partial^n}{\partial\alpha^n}\alpha^{2l}\Big|_{\alpha=\alpha^*=0}.$$

Using the relation

$$\frac{\partial}{\partial x^n}x^m = \frac{m!}{(m-n)!}x^{m-n},$$

we have

$$P(n) = \frac{sech(r)}{n!} \sum_{k,l}^{\infty} \frac{\left(-\frac{1}{2}tanh(r)\right)^{k+l}}{k!l!} \frac{(2k)!}{(2k-n)!}\frac{(2l)!}{(2l-n)!}\alpha^{*2k-n}\alpha^{2l-n}\Big|_{\alpha^*=\alpha=0}.$$

We note that

$$\alpha^{*2k-n}\alpha^{2l-n}\Big|_{\alpha^*=\alpha=0} = \delta_{2k,n}\delta_{2l,n}$$

and hence applying the properties of Kronecker delta function, we obtain

$$P(n) = \frac{1}{2^n n!}\frac{tanh^n(r)}{cosh(r)}H_n^2(0), \tag{2.19}$$

where $H_n(0) = (-1)^{\frac{n}{2}} \frac{n!}{(n/2)!}$ is a Hermite polynomial and n is even. This is the photon number distribution for a squeezed vacuum.

Finally, we calculate the variance of the photon number for a squeezed vacuum. The variance of the photon number

$$(\Delta N)^2 = \langle (\hat{a}^\dagger \hat{a})^2 \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2 \tag{2.20}$$

can be written in the form

$$(\Delta N)^2 = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2 - 3\langle \hat{a}^\dagger \hat{a} \rangle - 2. \tag{2.21}$$

Using the Q-function (2.13), we have

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = sech(r) \frac{\partial}{\partial b} \frac{\partial}{\partial c} \int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi} exp\left[-\alpha^* \alpha + b\alpha^2 + c\alpha^{*2}\right] \Big|_{b=c=-1/2tanh(r)}$$

and carrying out the integration applying (2.12), we get

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = sech(r) \frac{\partial}{\partial b} \frac{\partial}{\partial c} \frac{1}{\sqrt{1-4bc}} \Big|_{b=c=-1/2tanh((r)} .$$

Therefore

$$\langle a^2 a^{\dagger 2} \rangle = 3cosh^2(r)sinh^2(r) + 2cosh^2(r). \tag{2.22}$$

Thus combination of (2.16e), (2.21) and (2.22) leads to

$$(\Delta N)^2 = 2sinh^2(r)cosh^2(r). \tag{2.23a}$$

This can be rewritten in terms of the mean photon number as

$$(\Delta N)^2 = 2N(N+1). \tag{2.23b}$$

2.2 The Master Equation

We consider a system coupled to a squeezed vacuum reservoir described in the interaction picture by the Hamiltonian

$$\hat{H} = \hat{H}_S + \hat{H}_{SR},$$

where $\hat{H}_S$ describes the interaction Hamiltonian for the system and $\hat{H}_{SR}$ describes the weak interaction between the system and reservoir. Let $\hat{\chi}(t)$ be the total density operator of the system plus the reservoir in the interaction picture. Then the equation of evolution of the density operator is

$$\frac{d}{dt}\hat{\chi}(t) = \frac{1}{i\hbar}\left[\hat{H}_S(t) + \hat{H}_{SR}(t), \hat{\chi}(t)\right]. \tag{2.24}$$

The reduced density operator of the system is defined by

$$\hat{\rho}(t) = Tr_R(\hat{\chi}(t)), \tag{2.25}$$

in which Tr_R indicates the trace over the reservoir variables only. We assume that initially the system and reservoir are uncorrelated, so that

$$\hat{\chi}(0) = \hat{\rho}(0)R, \tag{2.26}$$

where R is the density operator of the reservoir. A formal solution of (2.24) can be written as

$$\hat{\chi}(t) = \hat{\chi}(0) + \frac{1}{i\hbar}\int_0^t \left[\hat{H}_S(t') + \hat{H}_{SR}(t'), \hat{\chi}(t')\right] dt'.$$

On substituting $\hat{\chi}(t)$ back into (2.24), we have

$$\frac{d}{dt}\hat{\chi}(t) = \frac{1}{i\hbar}\left[\hat{H}_S(t), \hat{\chi}(t)\right] + \frac{1}{i\hbar}\left[\hat{H}_{SR}(t), \hat{\chi}(0)\right] - \frac{1}{\hbar^2}\int_0^t \left[\hat{H}_{SR}(t), \left[\hat{H}_S(t'), \hat{\chi}(t')\right]\right] dt'$$
$$-\frac{1}{\hbar^2}\int_0^t \left[\hat{H}_{SR}(t), \left[\hat{H}_{SR}(t'), \hat{\chi}(t')\right]\right] dt'.$$

In view of (2.25) along with the above expression, we see that

$$\frac{d}{dt}\hat{\rho}(t) = \frac{1}{i\hbar}\left[\hat{H}_S(t), \hat{\rho}(t)\right] + \frac{1}{i\hbar}Tr_R\left[\hat{H}_{SR}(t), \hat{\rho}(0)R\right] - \frac{1}{\hbar^2}\int_0^t Tr_R\left[\hat{H}_{SR}(t), \left[\hat{H}_S(t'), \hat{\chi}(t')\right]\right] dt'$$
$$-\frac{1}{\hbar^2}\int_0^t Tr_R\left[\hat{H}_{SR}(t), \left[\hat{H}_{SR}(t'), \hat{\chi}(t')\right]\right] dt', \tag{2.27}$$

where

$$\left[\hat{H}_S(t), \hat{\rho}(t)\right] = Tr_R\left[\hat{H}_S(t), \hat{\chi}(t)\right].$$

Since the reservoir which is so large compared to the system, is not substantially affected by the interaction, one can apply the Born approximation

$$\hat{\chi}(t') = \hat{\rho}(t')R \tag{2.28}$$

and using the Markoff approximation, replacing $\hat{\rho}(t')$ by $\hat{\rho}(t)$, we have

$$\frac{d}{dt}\hat{\rho}(t) = \frac{1}{i\hbar}\left[\hat{H}_S(t), \hat{\rho}(t)\right] + \frac{1}{i\hbar}Tr_R\left[\hat{H}_{SR}(t), \hat{\rho}(0)R\right] - \frac{1}{\hbar^2}\int_0^t Tr_R\left[\hat{H}_{SR}(t), \left[\hat{H}_S(t'), \hat{\rho}(t)\right] R\right] dt'$$

$$-\frac{1}{\hbar^2}\int_0^t Tr_R\left[\hat{H}_{SR}(t), \left[\hat{H}_{SR}(t'), \hat{\rho}(t)R\right]\right] dt. \tag{2.29}$$

We note that $\hat{\rho}(t)$ is not a function of the reservoir mode operators, so that one can write

$$Tr_R\left[\hat{H}_{SR}(t), \hat{\rho}(0)R\right] = \left[\langle\hat{H}_{SR}(t)\rangle_R, \hat{\rho}(0)\right], \tag{2.30a}$$

$$Tr_R\left[\hat{H}_{SR}(t), \left[\hat{H}_S(t'), \hat{\rho}(t)\right] R\right] = \left[\langle\hat{H}_{SR}(t)\rangle_R, \left[\hat{H}_S(t'), \hat{\rho}(t)\right]\right]. \tag{2.30b}$$

We now consider a cavity mode coupled to a squeezed vacuum reservoir described by

$$\hat{H}_{SR}(t) = i\hbar\sum_k \lambda_k\left(\hat{a}^\dagger\hat{b}_k e^{i(\omega_0-\omega_k)t} - \hat{a}\hat{b}_k^\dagger e^{-i(\omega_0-\omega_k)t}\right), \tag{2.31}$$

where $\hat{a}$ and $\hat{b}_k$ are the annihilation operators for the cavity mode and the reservoir mode, respectively. Hence we see that

$$\langle\hat{H}_{SR}(t)\rangle_R = i\hbar\sum_k \lambda_k\left(\hat{a}^\dagger\langle\hat{b}_k\rangle_R e^{i(\omega_0-\omega_k)t} - \hat{a}\langle\hat{b}_k^\dagger\rangle_R e^{-i(\omega_0-\omega_k)t}\right).$$

Since for a squeezed vacuum reservoir

$$\langle\hat{b}_k\rangle_R = \langle\hat{b}_k^\dagger\rangle_R = 0,$$

we have

$$Tr_R\left[\hat{H}_{SR}(t), \hat{\rho}(0)R\right] = 0, \tag{2.32a}$$

$$Tr_R\left[\hat{H}_{SR}(t), \left[\hat{H}_S(t'), \hat{\rho}(t)\right] R\right] = 0. \tag{2.32b}$$

Therefore, on account of these results, expression (2.29) reduces to

$$\frac{d}{dt}\hat{\rho}(t) = \frac{1}{i\hbar}\left[\hat{H}_S(t), \hat{\rho}(t)\right] - \frac{1}{\hbar^2}\int_0^t Tr_R\left(R\hat{H}_{SR}(t)\hat{H}_{SR}(t')\right) dt'\hat{\rho}(t)$$

$$-\frac{1}{\hbar^2}\hat{\rho}(t)\int_0^t Tr_R\left(R\hat{H}_{SR}(t')\hat{H}_{SR}(t)\right) dt' + \frac{1}{\hbar^2}\int_0^t Tr_R\left(\hat{H}_{SR}(t)\hat{\rho}(t)R\hat{H}_{SR}(t')\right) dt'$$

$$+\frac{1}{\hbar^2}\int_0^t Tr_R\left(\hat{H}_{SR}(t')\hat{\rho}(t)R\hat{H}_{SR}(t)\right) dt'. \tag{2.33}$$

Now using the Hamiltonian (2.31), we can write

$$-\frac{1}{\hbar^2}\int_0^t Tr_R\left(R\hat{H}_{SR}(t)\hat{H}_{SR}(t')\right) dt' = I_1\hat{a}\hat{a}^\dagger + I_2\hat{a}^\dagger\hat{a} + I_3\hat{a}^2 + I_4\hat{a}^{\dagger 2}, \tag{2.34}$$

where

$$I_1 = -\sum_{j,k} \lambda_j \lambda_k \int_0^t \langle \hat{b}_j^\dagger \hat{b}_k \rangle_R e^{-i(\omega_0-\omega_j)t+i(\omega_0-\omega_k)t'} dt', \tag{2.35a}$$

$$I_2 = -\sum_{j,k} \lambda_j \lambda_k \int_0^t \langle \hat{b}_j \hat{b}_k^\dagger \rangle_R e^{i(\omega_0-\omega_j)t-i(\omega_0-\omega_k)t'} dt', \tag{2.35b}$$

$$I_3 = \sum_{j,k} \lambda_j \lambda_k \int_0^t \langle \hat{b}_j^\dagger \hat{b}_k^\dagger \rangle_R e^{-i(\omega_0-\omega_j)t-i(\omega_0-\omega_k)t'} dt', \tag{2.35c}$$

$$I_4 = \sum_{j,k} \lambda_j \lambda_k \int_0^t \langle \hat{b}_j \hat{b}_k \rangle_R e^{i(\omega_0-\omega_j)t+i(\omega_0-\omega_k)t'} dt'. \tag{2.35d}$$

Using the relations [34]

$$\langle \hat{b}_j^\dagger \hat{b}_k \rangle = N\delta_{j,k}, \tag{2.36a}$$

$$\langle \hat{b}_j \hat{b}_k^\dagger \rangle = (N+1)\delta_{j,k}, \tag{2.36b}$$

$$\langle \hat{b}_j \hat{b}_k \rangle = \langle \hat{b}_j^\dagger \hat{b}_k^\dagger \rangle = -M\delta_{j,2k_0-k}, \tag{2.36c}$$

we have

$$I_1 = -N \sum_k^\infty \lambda_k^2 \int_0^t e^{-i(\omega_0-\omega_k)(t-t')} dt'. \tag{2.37}$$

Now changing the summation into integration, we have

$$\sum_k \lambda_k^2 \int_0^t e^{-i(\omega_0-\omega_k)(t-t')} dt' = \int_0^\infty d\omega g(\omega) \lambda^2(\omega) \int_0^t e^{-i(\omega_0-\omega)(t-t')} dt',$$

where $g(\omega)$ is the density of states. Setting $t - t' = \tau$, we see that

$$\sum_k \lambda_k^2 \int_0^t e^{-i(\omega_0-\omega_k)(t-t')} dt' = \int_0^\infty d\omega g(\omega) \lambda^2(\omega) \int_0^\infty e^{-i(\omega_0-\omega)\tau} d\tau.$$

Since $exp\,[\pm i(\omega_0 - \omega)\tau]$ is a rapidly oscillating function of τ, we have extended the upper limit of time integration to infinity. In addition, recalling that

$$\int_0^\infty e^{\pm i(\omega_0-\omega)\tau} d\tau \approx \pi\delta(\omega_0 - \omega),$$

we find

$$\sum_k \lambda_k^2 \int_0^t e^{-i(\omega_0-\omega_k)(t-t')} dt' = \int_0^\infty d\omega g(\omega) \lambda^2(\omega) \delta(\omega_0 - \omega) = \frac{\gamma}{2}, \tag{2.38}$$

where

$$\gamma = 2\pi g(\omega_0) \lambda^2(\omega_0) \tag{2.39}$$

is defined to be the cavity damping rate. In view of (2.38), expression (2.37) takes the form

$$I_1 = -\frac{\gamma}{2}N. \tag{2.40a}$$

Proceeding in a similar way, one can readily establish that

$$I_2 = -\frac{\gamma}{2}(N+1), \tag{2.40b}$$

$$I_3 = I_4 = -\frac{\gamma}{2}M. \tag{2.40c}$$

Now combination of (2.40) and (2.34) leads to

$$-\frac{1}{\hbar^2}\int_0^t Tr_R\left(R\hat{H}_{SR}(t)\hat{H}_{SR}(t')\right)dt' = -\frac{\gamma}{2}\left((N+1)\hat{a}^\dagger\hat{a} + N\hat{a}\hat{a}^\dagger + M\hat{a}^2 + M\hat{a}^{\dagger 2}\right). \tag{2.41a}$$

We also note that

$$-\frac{1}{\hbar^2}\int_0^t Tr_R\left(R\hat{H}_{SR}(t')\hat{H}_{SR}(t)\right)dt' = -\frac{1}{\hbar^2}\int_0^t Tr_R\left(R\hat{H}_{SR}(t)\hat{H}_{SR}(t')\right)dt'. \tag{2.41b}$$

Moreover, employing (2.31) one can write

$$\frac{1}{\hbar^2}\int_0^t Tr_R\left(\hat{H}_{SR}(t')\hat{\rho}(t)R\hat{H}_{SR}(t)\right)dt' = -\left(I_1\hat{a}^\dagger\hat{\rho}\hat{a} + I_2\hat{a}\hat{\rho}\hat{a}^\dagger + I_3\hat{a}\hat{\rho}\hat{a} + I_4\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\right),$$

so that applying the results described by (2.40), we obtain

$$\frac{1}{\hbar^2}\int_0^t Tr_R\left(\hat{H}_{SR}(t')\hat{\rho}(t)R\hat{H}_{SR}(t)\right)dt' = \frac{\gamma}{2}\left((N+1)\hat{a}\hat{\rho}\hat{a}^\dagger + N\hat{a}^\dagger\hat{\rho}\hat{a} + M\hat{a}\hat{\rho}\hat{a} + M\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\right). \tag{2.41c}$$

It is easy to see that

$$\frac{1}{\hbar^2}\int_0^t Tr_R\left(\hat{H}_{SR}(t)\hat{\rho}(t)R\hat{H}_{SR}(t')\right)dt' = \frac{1}{\hbar^2}\int_0^t Tr_R\left(\hat{H}_{SR}(t')\hat{\rho}(t)R\hat{H}_{SR}(t)\right)dt'. \tag{2.41d}$$

On account (2.41), the master equation (2.33) becomes

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) = \frac{1}{i\hbar}\left[\hat{H}_S, \hat{\rho}(t)\right] &+ \frac{\gamma}{2}(N+1)\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right) + \frac{\gamma}{2}N\left(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\right)\\ &+\frac{\gamma}{2}M\left(2\hat{a}\hat{\rho}\hat{a} - \hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2 + 2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2}\right)\end{aligned} \tag{2.42a}$$

or, this can be rewritten in the form

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) = \frac{1}{i\hbar}\left[\hat{H}_S, \hat{\rho}(t)\right] &+ \frac{\gamma}{2}(N+1)\left([\hat{a}, \hat{\rho}\hat{a}^\dagger] + [\hat{a}\hat{\rho}, \hat{a}^\dagger]\right) + \frac{\gamma}{2}N\left([\hat{a}^\dagger, \hat{\rho}\hat{a}] + [\hat{a}^\dagger\hat{\rho}, \hat{a}]\right)\\ &+\frac{\gamma}{2}M\left([\hat{a}\hat{\rho}, \hat{a}] + [\hat{a}, \hat{\rho}\hat{a}] + [\hat{a}^\dagger, \hat{\rho}\hat{a}^\dagger] + [\hat{a}^\dagger\hat{\rho}, \hat{a}^\dagger]\right).\end{aligned} \tag{2.42b}$$

This is the equation of evolution of the reduced density operator for a cavity mode coupled to a squeezed vacuum reservoir. The effects of the reservoir are incorporated by the parameters N and M.

With the pump mode treated classically, a degenerate parametric oscillator is describable in the interaction picture by the Hamiltonian

$$\hat{H}_S = \frac{i\hbar\kappa\beta}{2}\left(\hat{a}^2 - \hat{a}^{\dagger 2}\right), \tag{2.43}$$

in which κ is the coupling constant and β is the amplitude of the pump mode. Hence the master equation (2.42) for the degenerate parametric oscillator coupled to a squeezed vacuum can be written in the form

$$\frac{d}{dt}\hat{\rho}(t) = \frac{\kappa\beta}{2}\left(\hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2 - \hat{a}^{\dagger 2}\hat{\rho} + \hat{\rho}\hat{a}^{\dagger 2}\right) + \frac{\gamma}{2}(N+1)\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right)$$
$$+\frac{\gamma}{2}N\left(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\right) + \frac{\gamma}{2}M\left(2\hat{a}\hat{\rho}\hat{a} - \hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2 + 2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2}\right). \tag{2.44}$$

2.3 The Quantum Langevin Equations

The dynamics of a system coupled to a reservoir can also be described using Langevin equations. We seek here to derive the quantum Langevin equation for the operator $\hat{a}$. To this end, we note that this operator evolves in time according to

$$\frac{d}{dt}\hat{a}(t) = \frac{1}{i\hbar}\left[\hat{a}, \hat{H}_S(t) + \hat{H}_{SR}(t)\right], \tag{2.45}$$

where $\hat{H}_S(t)$ describes the interaction in the cavity and $\hat{H}_{SR}(t)$ represents the interaction between the cavity mode and the reservoir.

Upon substituting the Hamiltonian (2.31) into (2.45) and using the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, one finds

$$\frac{d}{dt}\hat{a}(t) = \frac{1}{i\hbar}\left[\hat{a}, \hat{H}_S(t)\right] + \sum_k \lambda_k \hat{b}_k(t) e^{i(\omega_0 - \omega_k)t}. \tag{2.46}$$

On account of the Hamiltonian (2.31) the time evolution of the annihilation operator for the reservoir mode is

$$\frac{d}{dt}\hat{b}_k(t) = \sum_j \lambda_j \left(\hat{a}^\dagger[\hat{b}_k, \hat{b}_j]e^{i(\omega_0-\omega_j)t} - \hat{a}[\hat{b}_k, \hat{b}_j^\dagger]e^{-i(\omega_0-\omega_j)t}\right).$$

Using the commutation relations

$$\left[\hat{b}_j, \hat{b}_k^\dagger\right] = \delta_{j,k}, \tag{2.47a}$$

$$\left[\hat{b}_j, \hat{b}_k\right] = \left[\hat{b}_j^\dagger, \hat{b}_k^\dagger\right] = 0, \tag{2.47b}$$

one can easily show that

$$\frac{d}{dt}\hat{b}_k(t) = -\lambda_k \hat{a}(t) e^{-i(\omega_0 - \omega_k)t}. \tag{2.48}$$

The solution of this equation can be written in the form

$$\hat{b}_k(t) = \hat{b}_k(0) - \lambda_k \int_0^t \hat{a}(t') e^{-i(\omega_0 - \omega_k)t'} dt', \tag{2.49}$$

so that upon substituting this into (2.46), there follows

$$\frac{d}{dt}\hat{a}(t) = \frac{1}{i\hbar}\left[\hat{a}, \hat{H}_S(t)\right] - \hat{a}(t)\sum_k \lambda_k^2 \int_0^t e^{i(\omega_0 - \omega_k)(t-t')} dt' + \hat{F}(t), \tag{2.50}$$

in which we have replaced $\hat{a}(t')$ by $\hat{a}(t)$ (Markoff approximation) and

$$\hat{F}(t) = \sum_k \lambda_k \hat{b}_k(0) e^{i(\omega_0 - \omega_k)t} \tag{2.51}$$

is the noise operator. In view of (2.38), the Langevin equation (2.50) takes the form

$$\frac{d}{dt}\hat{a}(t) = \frac{1}{i\hbar}\left[\hat{a}, \hat{H}_S(t)\right] - \frac{\gamma}{2}\hat{a}(t) + \hat{F}(t). \tag{2.52}$$

Now employing the Hamiltonian given by (2.43), we have

$$\frac{1}{i\hbar}[\hat{a}, \hat{H}_S(t)] = -\kappa\beta\hat{a}^\dagger(t),$$

so that (2.52) turns out to be

$$\frac{d}{dt}\hat{a}(t) = -\frac{\gamma}{2}\hat{a}(t) - \kappa\beta\hat{a}^\dagger(t) + \hat{F}(t). \tag{2.53}$$

This represents the Langevin equation for $\hat{a}(t)$ for the signal mode produced by the degenerate parametric oscillator coupled to a squeezed vacuum reservoir.

Next we seek to study the properties of the noise operator $\hat{F}(t)$. Employing (2.51) and the commutation relation (2.47), it can be easily verified that

$$\left[\hat{F}(t), \hat{F}^\dagger(t')\right] = \sum_k \lambda_k^2 e^{-i(\omega_0 - \omega_k)(t-t')}. \tag{2.54}$$

Changing the summation into integration, we have

$$\sum_k \lambda_k^2 e^{-i(\omega_0-\omega_k)(t-t')} = \int_0^\infty d\omega g(\omega)\lambda^2(\omega)e^{-i(\omega_0-\omega)(t-t')}.$$

We now replace $g(\omega)\lambda^2(\omega)$ by $g(\omega_0)\lambda^2(\omega_0)$ and setting $\omega - \omega_0 = \omega'$, we see that

$$\sum_k \lambda_k^2 e^{-i(\omega_0-\omega_k)(t-t')} = g(\omega_0)\lambda^2(\omega_0)\int_{-\omega_0}^\infty e^{i\omega'(t-t')}d\omega'.$$

Since $exp\,[\pm i\omega'(t-t')]$ is a rapidly oscillating function of ω', we extended the lower limit of integration to infinity. In addition, using the relation

$$\int_{-\infty}^\infty e^{\pm i\omega'(t-t')}d\omega' \approx 2\pi\delta(t-t'),$$

we find

$$\sum_k \lambda_k^2 e^{-i(\omega_0-\omega_k)(t-t')} = \gamma\delta(t-t'). \tag{2.55}$$

Hence the commutation relation for the noise operator turns out to be

$$\left[\hat{F}(t), \hat{F}^\dagger(t')\right] = \gamma\delta(t-t'), \tag{2.56a}$$

where

$$\gamma = 2\pi g(\omega_0)\lambda^2(\omega_0).$$

Using once more the relation (2.47), one can readily establish that

$$\left[\hat{F}(t), \hat{F}(t')\right] = \left[\hat{F}^\dagger(t), \hat{F}^\dagger(t')\right] = 0. \tag{2.56b}$$

In view of (2.16a and b), we find that

$$\langle\hat{F}(t)\rangle = \langle\hat{F}^\dagger(t)\rangle = 0. \tag{2.57}$$

In addition, we have

$$\langle\hat{F}^\dagger(t)\hat{F}(t')\rangle = \sum_{j,k}\lambda_j\lambda_k\langle\hat{b}_j^\dagger(0)\hat{b}_k(0)\rangle e^{-i(\omega_0-\omega_j)t+i(\omega_0-\omega_k)t'}. \tag{2.58}$$

Applying (2.36a), we see that

$$\langle\hat{F}^\dagger(t)\hat{F}(t')\rangle = N\sum_k \lambda_k^2 e^{-i(\omega_0-\omega_k)(t-t')}, \tag{2.59}$$

so that on account of (2.55), Eq. (2.59) becomes

$$\langle \hat{F}^\dagger(t)\hat{F}(t')\rangle = \gamma N\delta(t-t'). \tag{2.60a}$$

One can also verify in a similar manner that

$$\langle \hat{F}(t)\hat{F}^\dagger(t')\rangle = \gamma(N+1)\delta(t-t'), \tag{2.60b}$$

$$\langle \hat{F}(t)\hat{F}(t')\rangle = \langle \hat{F}^\dagger(t)\hat{F}^\dagger(t')\rangle = -\gamma M\delta(t-t'). \tag{2.60c}$$

We note that the properties of the noise operator described by (2.57-2.60) hold for a squeezed vacuum reservoir. For a thermal and ordinary vacuum reservoirs one can obtain the properties of the noise operator, from (2.57-2.60) by proper choice of the parameters N and M.

2.4 The Input-Output Relation

The quantum Langevin equations provide a description of the internal dynamics of a system driven by the input field (a_{in}). Here we seek to establish the relation between input and output mode operators. The solution of (2.48) can be put in two different forms depending on the boundary conditions. For $t=\tau$, we have

$$\hat{b}_k(t) = \hat{b}_k(\tau) + \lambda_k \int_t^\tau \hat{a}(t')e^{-i(\omega_0-\omega_k)t'}dt', \tag{2.61}$$

so that upon substituting this into (2.46), we see that

$$\frac{d}{dt}\hat{a}(t) = \frac{1}{i\hbar}\left[\hat{a},\hat{H}_S(t)\right] + \sum_k \lambda_k^2 \int_t^\tau \hat{a}(t')e^{i(\omega_0-\omega_k)(t-t')}dt' + \sum_k \lambda_k\hat{b}_k(\tau)e^{i(\omega_0-\omega_k)t}. \tag{2.62}$$

Applying the Markoff approximation, replacing $\hat{a}(t')$ by $\hat{a}(t)$, and setting $t'-t=\tau'$, one can write that

$$\frac{d}{dt}\hat{a}(t) = \frac{1}{i\hbar}\left[\hat{a},\hat{H}_S(t)\right] + \hat{a}(t)\sum_k \lambda_k^2 \int_0^{\tau-t} e^{-i(\omega_0-\omega_k)\tau'}d\tau' - \sqrt{\gamma}\hat{a}_{out}(t), \tag{2.63}$$

where

$$\hat{a}_{out}(t) = -\frac{1}{\sqrt{\gamma}}\sum_k \lambda_k\hat{b}_k(\tau)e^{i(\omega_0-\omega_k)t}.$$

is defined to be the output mode. In view of (2.38), one can see that

$$\frac{d}{dt}\hat{a}(t) = \frac{1}{i\hbar}\left[\hat{a},\hat{H}_S(t)\right] + \frac{\gamma}{2}\hat{a}(t) - \sqrt{\gamma}\hat{a}_{out}(t). \tag{2.64}$$

Furthermore, applying the definition

$$\hat{a}_{in}(t) = \frac{1}{\sqrt{\gamma}}\hat{F}(t), \tag{2.65}$$

Eq. (2.53), which holds for $t = 0$, can be rewritten as

$$\frac{d}{dt}\hat{a}(t) = \frac{1}{i\hbar}\left[\hat{a}, \hat{H}_S(t)\right] - \frac{\gamma}{2}\hat{a}(t) + \sqrt{\gamma}\hat{a}_{in}(t), \tag{2.66}$$

Therefore, the relation between the input and output operators can be obtained by subtracting (2.66) from (2.64):

$$\hat{a}_{out}(t) = \sqrt{\gamma}\hat{a}(t) - \hat{a}_{in}(t). \tag{2.67}$$

The properties of the input operator can be obtained from the properties of the noise operator. Thus in view of the definition of the input operator (2.65) and (2.56), we have

$$\left[\hat{a}_{in}(t), \hat{a}^{\dagger}_{in}(t')\right] = \delta(t - t'), \tag{2.68a}$$

$$[\hat{a}_{in}(t), \hat{a}_{in}(t')] = \left[\hat{a}^{\dagger}_{in}(t), \hat{a}^{\dagger}_{in}(t')\right] = 0. \tag{2.68b}$$

In addition, replacing $\hat{F}(t)$ by $\sqrt{\gamma}a_{in}(t)$ in (2.57) and (2.60), one gets

$$\langle\hat{a}_{in}(t)\rangle = \langle\hat{a}^{\dagger}_{in}(t)\rangle = 0, \tag{2.69a}$$

$$\langle\hat{a}^{\dagger}_{in}(t)\hat{a}_{in}(t')\rangle = N\delta(t - t'), \tag{2.69b}$$

$$\langle\hat{a}_{in}(t)\hat{a}^{\dagger}_{in}(t')\rangle = (N + 1)\delta(t - t'), \tag{2.69c}$$

$$\langle\hat{a}_{in}(t)\hat{a}_{in}(t')\rangle = \langle\hat{a}^{\dagger}_{in}(t)\hat{a}^{\dagger}_{in}(t')\rangle = -M\delta(t - t'). \tag{2.69d}$$

3 The Q-function

In section (2.2) we have derived the equation of evolution of the density operator for the signal mode produced by the degenerate parametric oscillator coupled to a squeezed vacuum reservoir. Here we wish to derive the Fokker-Planck equation for the Q-function and then obtain the solution of the resulting Fokker-Planck equation applying the propagator method developed by Fesseha [1].

3.1 The Fokker-Planck Equation

The Fokker-Planck equation for the Q-function corresponding to (2.44) can be obtained by putting all terms in normal order. Using the relation

$$\hat{a}^n f(\hat{a}^\dagger, \hat{a}) = \sum_{l=0}^{n} \frac{n!}{l!(n-l)!} \frac{\partial^l}{\partial \hat{a}^{\dagger l}} \left(f(\hat{a}^\dagger, \hat{a}) \hat{a}^{n-l} \right), \tag{3.1a}$$

$$f(\hat{a}^\dagger, \hat{a}) \hat{a}^{\dagger n} = \sum_{l=0}^{n} \frac{n!}{l!(n-l)!} \frac{\partial^l}{\partial \hat{a}^l} \left(\hat{a}^{\dagger n-l} f(\hat{a}^\dagger, \hat{a}) \right), \tag{3.1b}$$

one finds

$$\hat{a}^2 \hat{\rho} = \hat{\rho} \hat{a}^2 + 2\frac{\partial}{\partial \hat{a}^\dagger}(\hat{\rho}\hat{a}) + \frac{\partial^2}{\partial \hat{a}^{\dagger 2}} \hat{\rho}, \tag{3.2a}$$

$$\hat{\rho} \hat{a}^{\dagger 2} = \hat{a}^{\dagger 2} \hat{\rho} + 2\frac{\partial}{\partial \hat{a}}(\hat{a}^\dagger \hat{\rho}) + \frac{\partial^2}{\partial \hat{a}^2} \hat{\rho}. \tag{3.2b}$$

It then follows that

$$\hat{a}^2 \hat{\rho} - \hat{\rho}\hat{a}^2 - \hat{a}^{\dagger 2}\hat{\rho} + \hat{\rho}\hat{a}^{\dagger 2} = \frac{\partial^2}{\partial \hat{a}^{\dagger 2}}\hat{\rho} + 2\frac{\partial}{\partial \hat{a}^\dagger}(\hat{\rho}\hat{a}) + 2\frac{\partial}{\partial \hat{a}}(\hat{a}^\dagger \hat{\rho}) + \frac{\partial^2}{\partial \hat{a}^2}\hat{\rho}. \tag{3.3}$$

Applying (3.1), one can also show that

$$\hat{a}^\dagger(\hat{a}\hat{\rho}) = \hat{a}\hat{\rho}\hat{a}^\dagger - \frac{\partial}{\partial \hat{a}}(\hat{a}\hat{\rho}) = \hat{a}\hat{\rho}\hat{a}^\dagger - \frac{\partial}{\partial \hat{a}}(\hat{\rho}\hat{a}) - \frac{\partial^2}{\partial \hat{a}^\dagger \partial \hat{a}}\hat{\rho},$$

$$(\hat{\rho}\hat{a}^\dagger)\hat{a} = \hat{a}\hat{\rho}\hat{a}^\dagger - \frac{\partial}{\partial \hat{a}^\dagger}(\hat{\rho}\hat{a}^\dagger) = \hat{a}\hat{\rho}\hat{a}^\dagger - \frac{\partial}{\partial \hat{a}^\dagger}(\hat{a}^\dagger \hat{\rho}) - \frac{\partial^2}{\partial \hat{a}^\dagger \partial \hat{a}}\hat{\rho}.$$

It then turns out that

$$2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger \hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger \hat{a} = 2\frac{\partial^2}{\partial \hat{a}^\dagger \partial \hat{a}}\hat{\rho} + \frac{\partial}{\partial \hat{a}}(\hat{\rho}\hat{a}) + \frac{\partial}{\partial \hat{a}^\dagger}(\hat{a}^\dagger \hat{\rho}). \tag{3.4}$$

Employing once more (3.1), one easily finds

$$\hat{a}(\hat{a}^\dagger\hat{\rho}) = \hat{a}^\dagger\hat{\rho}\hat{a} + \frac{\partial}{\partial\hat{a}^\dagger}(\hat{a}^\dagger\hat{\rho}),$$

$$(\hat{\rho}\hat{a})\hat{a}^\dagger = \hat{a}^\dagger\hat{\rho}\hat{a} + \frac{\partial}{\partial\hat{a}}(\hat{\rho}\hat{a}),$$

so that

$$2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger = -\left(\frac{\partial}{\partial\hat{a}}(\hat{\rho}\hat{a}) + \frac{\partial}{\partial\hat{a}^\dagger}(\hat{a}^\dagger\hat{\rho})\right). \tag{3.5}$$

Moreover, in view of (3.2) and the relations

$$\hat{a}\hat{\rho}\hat{a} = \hat{\rho}\hat{a}^2 + \frac{\partial}{\partial\hat{a}^\dagger}(\hat{\rho}\hat{a}),$$

$$\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger = \hat{a}^{\dagger 2}\hat{\rho} + \frac{\partial}{\partial\hat{a}}(\hat{a}^\dagger\hat{\rho}),$$

we get

$$2\hat{a}\hat{\rho}\hat{a} - \hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2 + 2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2} = -\left(\frac{\partial^2}{\partial\hat{a}^2}\hat{\rho} + \frac{\partial^2}{\partial\hat{a}^{\dagger 2}}\hat{\rho}\right). \tag{3.6}$$

On account of the results (3.3-3.6), Eq. (2.44) takes the form

$$\frac{d}{dt}\hat{\rho} = \frac{\kappa\beta}{2}\left(\frac{\partial^2}{\partial\hat{a}^{\dagger 2}}\hat{\rho} + 2\frac{\partial}{\partial\hat{a}^\dagger}(\hat{\rho}\hat{a}) + 2\frac{\partial}{\partial\hat{a}}(\hat{a}^\dagger\hat{\rho}) + \frac{\partial^2}{\partial\hat{a}^2}\hat{\rho}\right) + \frac{\gamma}{2}\left(\frac{\partial}{\partial\hat{a}}(\hat{\rho}\hat{a}) + \frac{\partial}{\partial\hat{a}^\dagger}(\hat{a}^\dagger\hat{\rho})\right)$$
$$+\frac{\gamma}{2}\left(2(N+1)\frac{\partial^2}{\partial\hat{a}^\dagger\partial\hat{a}}\hat{\rho} - M\left(\frac{\partial^2}{\partial\hat{a}^{\dagger 2}}\hat{\rho} + \frac{\partial^2}{\partial\hat{a}^2}\hat{\rho}\right)\right), \tag{3.7}$$

where $\hat{\rho} = \hat{\rho}(\hat{a}^\dagger, \hat{a}, t)$ is assumed to be in normal order. Hence the Fokker-Planck equation for the Q-function for the degenerate parametric oscillator coupled to a squeezed vacuum is

$$\frac{d}{dt}Q(\alpha^*, \alpha, t) = \frac{\kappa\beta}{2}\left(\frac{\partial^2}{\partial\alpha^{*2}} + 2\frac{\partial}{\partial\alpha^*}\alpha + 2\frac{\partial}{\partial\alpha}\alpha^* + \frac{\partial^2}{\partial\alpha^2}\right)Q + \frac{\gamma}{2}\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right)Q$$
$$+\frac{\gamma}{2}\left(2(N+1)\frac{\partial^2}{\partial\alpha^*\partial\alpha} - M\left(\frac{\partial^2}{\partial\alpha^{*2}} + \frac{\partial^2}{\partial\alpha^2}\right)\right)Q. \tag{3.8}$$

3.2 The Q-function

We now proceed to obtain the solution of (3.8) employing the propagator method. Introducing Cartesian coordinates defined by

$$\alpha = x + iy, \tag{3.9a}$$

we see that

$$\frac{\partial}{\partial \alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \tag{3.9b}$$

Using (3.9), one can easily show that

$$\frac{\partial^2}{\partial \alpha^* \partial \alpha} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

$$\frac{\partial^2}{\partial \alpha^2} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} - 2i\frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y^2}\right),$$

$$\frac{\partial}{\partial \alpha}\alpha = \frac{1}{2}\left(\frac{\partial}{\partial x}x + i\frac{\partial}{\partial x}y - i\frac{\partial}{\partial y}x + \frac{\partial}{\partial y}y\right),$$

$$\frac{\partial}{\partial \alpha^*}\alpha = \frac{1}{2}\left(\frac{\partial}{\partial x}x + i\frac{\partial}{\partial x}y + i\frac{\partial}{\partial y}x - \frac{\partial}{\partial y}y\right).$$

Upon substituting these relations and their complex conjugates into (3.8), the Fokker-Planck equation becomes

$$\frac{d}{dt}Q(x,y,t) = \left(\frac{\kappa\beta + \gamma(N - M + 1)}{4}\frac{\partial^2}{\partial x^2} - \frac{\kappa\beta - \gamma(N + M + 1)}{4}\frac{\partial^2}{\partial y^2}\right.$$
$$\left. + \frac{2\kappa\beta + \gamma}{2}\frac{\partial}{\partial x}x - \frac{2\kappa\beta - \gamma}{2}\frac{\partial}{\partial y}y\right) Q(x,y,t). \tag{3.10}$$

In order to solve this differential equation using the propagator method, we need to transform it into a schrödinger-type equation. This can be achieved replacing $(x, y, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, Q(x,y,t))$ by $(\hat{x}, \hat{y}, i\hat{p}_x, i\hat{p}_y, \mid Q(t)\rangle)$. One then finds

$$i\frac{d}{dt}\mid Q(t)\rangle = \left(-\frac{i}{4}\eta_1\hat{p}_x^2 + \frac{i}{4}\eta_2\hat{p}_y^2 - \lambda_1\hat{p}_x\hat{x} + \lambda_2\hat{p}_y\hat{y}\right)\mid Q(t)\rangle. \tag{3.11}$$

where

$$\eta_1 = \kappa\beta + \gamma(N - M + 1), \tag{3.12a}$$

$$\eta_2 = \kappa\beta - \gamma(N + M + 1), \tag{3.12b}$$

$$\lambda_1 = \kappa\beta + \frac{\gamma}{2}, \tag{3.12c}$$

$$\lambda_2 = \kappa\beta - \frac{\gamma}{2}. \tag{3.12d}$$

A formal solution of (3.11) can be put in the form

$$\mid Q(t)\rangle = e^{-i\hat{H}t}\mid Q(0)\rangle, \tag{3.13}$$

in which

$$\hat{H} = -\frac{i}{4}\eta_1\hat{p}_x^2 + \frac{i}{4}\eta_2\hat{p}_y^2 - \lambda_1\hat{p}_x\hat{x} + \lambda_2\hat{p}_y\hat{y}, \tag{3.14}$$

is the "quantum Hamiltonian". On multiplying (3.13) by $\langle x, y \mid$ and inserting the completeness relation

$$I = \int dx'dy' \mid x', y'\rangle\langle x', y' \mid,$$

we see that

$$Q(x, y, t) = \int dx'dy' Q(x, y, t|x', y', 0)Q(x', y', 0), \tag{3.15}$$

where

$$Q(x', y', 0) = \langle x', y' \mid Q(0)\rangle$$

is the initial Q-function and

$$Q(x, y, t|x', y', 0) = \langle x, y \mid e^{-i\hat{H}t} \mid x', y'\rangle$$

is the Q-function propagator.

According to Fesseha [1] the propagator associated with a quadratic Hamiltonian

$$\hat{H} = a\hat{p}_x^2 + a'\hat{p}_y^2 + b(t)\hat{p}_x\hat{x} + b'(t)\hat{p}_y\hat{y} + c(t)\hat{x}^2 + c'(t)\hat{y}^2 \tag{3.16}$$

is expressible in the form

$$K(x, y, t|x', y', 0) = \left[\frac{i}{2\pi}\frac{\partial^2 S_c}{\partial x'\partial x}\right]^{\frac{1}{2}} \left[\frac{i}{2\pi}\frac{\partial^2 S_c}{\partial y'\partial y}\right]^{\frac{1}{2}} exp\left[-\xi \int_0^t [b(t') + b'(t')]dt' + iS_c\right], \tag{3.17}$$

where S_c is the "classical action", ξ is a constant parameterizing operator ordering, a and a' are constants different from zero.

We wish to obtain the Q-function propagator employing (3.17). We see from (3.14) that $b(t') = -\lambda_1$, $b'(t') = \lambda_2$ and $\xi = \frac{1}{2}$ for the antistandard form of ordering. Thus the Q-function propagator associated with the "Hamiltonian" (3.14) can be written as

$$Q(x, y, t|x', y', 0) = \left[\frac{i}{2\pi}\frac{\partial^2 S_c}{\partial x'\partial x}\right]^{1/2} \left[\frac{i}{2\pi}\frac{\partial^2 S_c}{\partial y'\partial y}\right]^{1/2} e^{\left(\frac{\lambda_1 - \lambda_2}{2}\right)t + iS_c}. \tag{3.18}$$

We now proceed to determine the classical action. The "Hamiltonian function" associated with the "quantum Hamiltonian" (3.14) is

$$H = -\frac{i}{4}\eta_1 p_x^2 + \frac{i}{4}\eta_2 p_y^2 - \lambda_1 p_x x + \lambda_2 p_y y.$$

Using Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p_x},$$

$$\dot{y} = \frac{\partial H}{\partial p_y}.$$

one easily finds

$$p_x = \frac{2i}{\eta_1}(\dot{x} + \lambda_1 x), \tag{3.19a}$$

$$p_y = -\frac{2i}{\eta_2}(\dot{y} - \lambda_2 y), \tag{3.19b}$$

so that the Lagrangian

$$L = \dot{x}p_x + \dot{y}p_y - H(x, y, p_x, p_y, t),$$

takes the form

$$L = \frac{i}{\eta_1}(\dot{x} + \lambda_1 x)^2 - \frac{i}{\eta_2}(\dot{y} - \lambda_2 y)^2. \tag{3.20}$$

Now applying the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0,$$

we get

$$\ddot{x} - \lambda_1^2 x = 0,$$

$$\ddot{y} - \lambda_2^2 y = 0.$$

The solution of these equations can be written as

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{-\lambda_1 t}, \tag{3.21a}$$

$$y(t) = d_1 e^{\lambda_2 t} + d_2 e^{-\lambda_2 t}. \tag{3.21b}$$

Substitution of (3.21) and their derivatives with respect to t into (3.20) gives

$$L = \frac{4i\lambda_1^2}{\eta_1}c_1^2 e^{2\lambda_1 t} - \frac{4i\lambda_2^2}{\eta_2}d_2^2 e^{-2\lambda_2 t}, \tag{3.22}$$

so that the classical action takes the form

$$S_c = \int_0^\tau L dt = \frac{2i\lambda_1}{\eta_1}c_1^2\left(e^{2\lambda_1\tau} - 1\right) + \frac{2i\lambda_2}{\eta_2}d_2^2\left(e^{-2\lambda_2\tau} - 1\right). \tag{3.23}$$

Upon setting $x(0) = x'$, $x(\tau) = x''$, $y(0) = y'$ and $y(\tau) = y''$, one easily obtains from (3.21)

$$c_1 = -\frac{x' - x'' e^{\lambda_1\tau}}{e^{2\lambda_1\tau} - 1}, \tag{3.24a}$$

$$d_2 = -\frac{y' - y'' e^{-\lambda_2\tau}}{e^{-2\lambda_2\tau} - 1}. \tag{3.24b}$$

Hence combination of (3.23) and (3.24) leads to

$$S_c = \frac{2i\lambda_1}{\eta_1}\left(\frac{x'^2 - 2xx'e^{\lambda_1 t} + x^2 e^{2\lambda_1 t}}{e^{2\lambda_1 t} - 1}\right) + \frac{2i\lambda_2}{\eta_2}\left(\frac{y'^2 - 2yy'e^{-\lambda_2 t} + y^2 e^{-2\lambda_2 t}}{e^{-2\lambda_2 t} - 1}\right), \tag{3.25}$$

where (x'', y'', τ) has been replaced by (x, y, t). It then follows that

$$\frac{\partial^2 S_c}{\partial x' \partial x} = -\frac{4i\lambda_1 e^{\lambda_1 t}}{\eta_1 \left(e^{2\lambda_1 t} - 1\right)}, \tag{3.26a}$$

$$\frac{\partial^2 S_c}{\partial y' \partial y} = -\frac{4i\lambda_2 e^{-\lambda_2 t}}{\eta_2 \left(\epsilon^{-2\lambda_2 t} - 1\right)}. \tag{3.26b}$$

Therefore, on account of (3.25) and (3.26), the Q-function propagator takes the form

$$\begin{aligned} Q(x, y, t|x', y', 0) = \frac{1}{\pi}&\left[\frac{2\lambda_1}{\eta_1 \left(1 - e^{-2\lambda_1 t}\right)} \frac{2\lambda_2}{\eta_2 \left(1 - e^{2\lambda_2 t}\right)}\right]^{1/2} \\ &\times exp\left[-\frac{2\lambda_1}{\eta_1 \left(1 - e^{-2\lambda_1 t}\right)} x^2 - \frac{2\lambda_2}{\eta_2 \left(1 - e^{2\lambda_2 t}\right)} y^2\right] \\ &\times exp\left[-\frac{2\lambda_1 e^{-2\lambda_1 t}}{\eta_1 \left(1 - e^{-2\lambda_1 t}\right)} x'^2 + \frac{4\lambda_1 x e^{-\lambda_1 t}}{\eta_1 \left(1 - e^{-2\lambda_1 t}\right)} x'\right. \\ &\left.- \frac{2\lambda_2 e^{2\lambda_2 t}}{\eta_2 \left(1 - e^{2\lambda_2 t}\right)} y'^2 + \frac{4\lambda_2 y e^{\lambda_2 t}}{\eta_2 \left(1 - e^{2\lambda_2 t}\right)} y'\right]. \end{aligned} \tag{3.27}$$

Although we are interested in the case for which the signal mode is initially in a vacuum state, we also need the Q-function satisfying the initial condition

$$Q(x', y', 0) = \frac{1}{\pi} exp\left[-(x' - x_0)^2 - (y' - y_0)^2\right]. \tag{3.28}$$

Thus substituting (3.27) and (3.28) into (3.15), we have

$$\begin{aligned} Q(x, y, t) = \frac{1}{\pi^2}&\left[\frac{2\lambda_1}{\eta_1 \left(1 - e^{-2\lambda_1 t}\right)} \frac{2\lambda_2}{\eta_2 \left(1 - e^{2\lambda_2 t}\right)}\right]^{1/2} \\ &\times exp\left[-\frac{2\lambda_1}{\eta_1 \left(1 - e^{-2\lambda_1 t}\right)} x^2 - x_0^2 - \frac{2\lambda_2}{\eta_2 \left(1 - e^{2\lambda_2 t}\right)} y^2 - y_0^2\right] \\ &\times \int dx' exp\left[-\left(\frac{2\lambda_1 e^{-2\lambda_1 t}}{\eta_1 \left(1 - e^{-2\lambda_1 t}\right)} + 1\right) x'^2 + 2\left(\frac{2\lambda_1 e^{-\lambda_1 t}}{\eta_1 \left(1 - e^{-2\lambda_1 t}\right)} x + x_0\right) x'\right] \\ &\times \int dy' exp\left[-\left(\frac{2\lambda_2 e^{2\lambda_2 t}}{\eta_2 \left(1 - e^{2\lambda_2 t}\right)} + 1\right) y'^2 + 2\left(\frac{2\lambda_2 e^{\lambda_2 t}}{\eta_2 \left(1 - e^{2\lambda_2 t}\right)} y + y_0\right) y'\right]. \end{aligned}$$

On carrying out the integration using the relation

$$\int_{-\infty}^{\infty} e^{-ax^2 + bx} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}, \quad a > 0 \tag{3.29}$$

there follows the result

$$Q(x,y.t)=\frac{1}{\pi}\left[\frac{2\lambda_1}{2\lambda_1+\eta_1\left(1-e^{-2\lambda_1 t}\right)}\frac{2\lambda_2}{2\lambda_2+\eta_2\left(1-e^{2\lambda_2 t}\right)}\right]^{1/2}$$

$$\times exp\left[-\frac{2\lambda_1}{2\lambda_1+\eta_1\left(1-e^{-2\lambda_1 t}\right)}\left(x-x_0e^{-\lambda_1 t}\right)^2 -\frac{2\lambda_2}{2\lambda_2+\eta_2\left(1-e^{2\lambda_2 t}\right)}\left(y-y_0e^{\lambda_2 t}\right)^2\right]. \tag{3.30}$$

In view of (3.9a), we note that

$$x=\frac{1}{2}(\alpha+\alpha^*),$$

$$y=-\frac{i}{2}(\alpha-\alpha^*).$$

Hence Eq. (3.30) can be rewritten as

$$Q(\alpha^*,\alpha,t)=Aexp\left[-D\alpha^*\alpha+\frac{1}{2}C\left(\alpha^{*2}+\alpha^2\right)+B\alpha^*+B^*\alpha\right], \tag{3.31}$$

where

$$A(t)=\frac{\sqrt{D^2-C^2}}{\pi}exp\left[-D\alpha_0^*\alpha_0+\frac{1}{2}C\left(\alpha_0^{*2}+\alpha_0^2\right)\right], \tag{3.32a}$$

$$B(t)=\frac{1}{2}\left[(D-C)(\alpha_0^*+\alpha_0)\epsilon^{-\lambda_1 t}+(D+C)(\alpha_0^*-\alpha_0)e^{\lambda_2 t}\right], \tag{3.32b}$$

$$C(t)=-\frac{1}{2}\left[\frac{1}{\epsilon^{-2\lambda_1 t}+\frac{\eta_1}{2\lambda_1}\left(1-\epsilon^{-2\lambda_1 t}\right)}-\frac{1}{\epsilon^{2\lambda_2 t}+\frac{\eta_2}{2\lambda_2}\left(1-\epsilon^{2\lambda_2 t}\right)}\right]. \tag{3.32c}$$

$$D(t)=\frac{1}{2}\left[\frac{1}{\epsilon^{-2\lambda_1 t}+\frac{\eta_1}{2\lambda_1}\left(1-e^{-2\lambda_1 t}\right)}+\frac{1}{\epsilon^{2\lambda_2 t}+\frac{\eta_2}{2\lambda_2}\left(1-\epsilon^{2\lambda_2 t}\right)}\right]. \tag{3.32d}$$

This represents the Q-function for the signal mode produced by the degenerate parametric oscillator coupled to a squeezed vacuum. We are interested in the case for which the signal mode is initially in a vacuum state. Hence setting $\alpha_0^*=\alpha_0=0$ results in

$$Q(\alpha^*,\alpha,t)=\frac{\sqrt{D^2-C^2}}{\pi}exp\left[-D\alpha^*\alpha+\frac{1}{2}C\left(\alpha^{*2}+\alpha^2\right)\right]. \tag{3.33}$$

4 Quadrature Fluctuations

In order to study the squeezing properties of a light mode, we need to calculate the variances of the quadrature operators. In this chapter we seek to calculate for the signal mode the variances of the quadrature operators employing the Q-function and the squeezing spectrum applying the quantum Langevin equations.

4.1 The Variances of Quadrature Operators

The variances of quadrature operators $\hat{a}_j$ defined by (2.14) can be written as

$$(\Delta\hat{a}_j)^2 = \langle\hat{a}_j, \hat{a}_j\rangle, \tag{4.1}$$

where

$$\langle A, B\rangle = \langle AB\rangle - \langle A\rangle\langle B\rangle \tag{4.2}$$

and $j = 1, 2$. Applying (3.33), one can write that

$$\langle\hat{a}_1\rangle = \sqrt{D^2 - C^2}\int\frac{d^2\alpha}{\pi}(\alpha^* + \alpha)exp\left[-D\alpha^*\alpha + \frac{1}{2}C\left(\alpha^2 + \alpha^{*2}\right)\right]$$

or, this can be rewritten as

$$\langle\hat{a}_1\rangle = \sqrt{D^2 - C^2}\,\frac{\partial}{\partial b}\int\frac{d^2\alpha}{\pi}exp\left[-D\alpha^*\alpha + \frac{1}{2}C\left(\alpha^2 + \alpha^{*2}\right) + b(\alpha + \alpha^*)\right]\bigg|_{b=0}.$$

On carrying out the integration applying (2.12), we get

$$\langle\hat{a}_1\rangle = \frac{\partial}{\partial b}e^{\frac{b^2}{D-C}}\bigg|_{b=0} = 0 \tag{4.3a}$$

Similarly, it is easy to show

$$\langle\hat{a}_2\rangle = 0 \tag{4.3b}$$

and hence expressions (4.1) reduce to

$$(\Delta\hat{a}_1)^2 = \langle(\hat{a}^\dagger + \hat{a})^2\rangle, \tag{4.4a}$$

$$(\Delta\hat{a}_2)^2 = -\langle(\hat{a}^\dagger - \hat{a})^2\rangle. \tag{4.4b}$$

Using the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, Eq. (4.4) can be put in the form

$$(\Delta \hat{a}_1)^2 = \langle 2\hat{a}\hat{a}^\dagger + \hat{a}^{\dagger 2} + \hat{a}^2 - 1 \rangle. \tag{4.5a}$$

$$(\Delta \hat{a}_2)^2 = \langle 2\hat{a}\hat{a}^\dagger - \hat{a}^{\dagger 2} - \hat{a}^2 - 1 \rangle. \tag{4.5b}$$

Furthermore, employing the Q-function (3.33), one can write that

$$(\Delta \hat{a}_1)^2 = \sqrt{D^2 - C^2}\left(2\frac{\partial}{\partial C} - 2\frac{\partial}{\partial D} - 1\right)\int_0^\infty \frac{d^2\alpha}{\pi} exp\left[-D\alpha^*\alpha + \frac{1}{2}C(\alpha^{*2} + \alpha^2)\right].$$

On carrying out the integration with the help of (2.12) and then performing the differentiation, we obtain

$$(\Delta \hat{a}_1)^2 = \frac{2}{D - C} - 1, \tag{4.6}$$

so that taking into account (3.32) along with (3.12), Eq. (4.6) can be put in the form

$$(\Delta \hat{a}_1)^2 = e^{-(\gamma + 2\kappa\beta)t} + \frac{\gamma(2N - 2M + 1)}{\gamma + 2\kappa\beta}\left(1 - e^{-(\gamma+2\kappa\beta)t}\right). \tag{4.7a}$$

Following a similar procedure, one easily finds

$$(\Delta \hat{a}_2)^2 = e^{-(\gamma - 2\kappa\beta)t} + \frac{\gamma(2N + 2M + 1)}{\gamma - 2\kappa\beta}\left(1 - e^{-(\gamma-2\kappa\beta)t}\right). \tag{4.7b}$$

In view of (2.17), we have

$$(\Delta \hat{a}_1)^2 = e^{-(\gamma + 2\kappa\beta)t} + \frac{\gamma e^{-2r}}{\gamma + 2\kappa\beta}\left(1 - e^{-(\gamma+2\kappa\beta)t}\right). \tag{4.8a}$$

$$(\Delta \hat{a}_2)^2 = e^{-(\gamma - 2\kappa\beta)t} + \frac{\gamma e^{2r}}{\gamma - 2\kappa\beta}\left(1 - e^{-(\gamma-2\kappa\beta)t}\right). \tag{4.8b}$$

It is easy to see that the squeezing occurs in the first quadrature. We also note that the signal mode will be in steady state for $\kappa\beta < \gamma/2$ (below threshold). Hence at steady state, the variances take the form

$$(\Delta \hat{a}_1)^2 = \frac{\gamma e^{-2r}}{\gamma + 2\kappa\beta}, \tag{4.8c}$$

$$(\Delta \hat{a}_2)^2 = \frac{\gamma e^{2r}}{\gamma - 2\kappa\beta} \tag{4.8d}$$

and at threshold ($\gamma = 2\kappa\beta$), we see from (4.8c and d) that

$$(\Delta \hat{a}_1)^2 = \frac{1}{2}e^{-2r}, \tag{4.8e}$$

$$(\Delta \hat{a}_2)^2 \to \infty. \tag{4.8f}$$

Moreover, expression (4.8e) shows that the variance of the first quadrature operator at steady and threshold is the product of the variance when the degenerate parametric oscillator is coupled to ordinary vacuum and the variance associated with the squeezed vacuum. We note that one effect of the squeezed vacuum reservoir is to increase the degree of squeezing of the signal mode. This result is the same as the one obtained by Anwar and Zubairy [25]. In the absence of parametric interaction ($\kappa = 0$), Eq. (4.8a and b) reduce to

$$(\Delta \hat{a}_1)^2 = e^{-\gamma t} + e^{-2r}\left(1 - e^{-\gamma t}\right), \tag{4.9a}$$

$$(\Delta \hat{a}_2)^2 = e^{-\gamma t} + e^{2r}\left(1 - e^{-\gamma t}\right). \tag{4.9b}$$

At steady state these expression reduce to (2.17) which are the variances of the quadrature operators for a squeezed vacuum.

We now proceed to consider some special cases. We note that for thermal reservoir $N = \overline{n}$ and $M = 0$, where $\overline{n}$ is the mean photon number for the thermal reservoir. Then for this case (4.7) takes the form

$$(\Delta \hat{a}_1)^2 = e^{-(\gamma+2\kappa\beta)t} + \frac{\gamma(2\overline{n}+1)}{\gamma+2\kappa\beta}\left(1 - e^{-(\gamma+2\kappa\beta)t}\right), \tag{4.10a}$$

$$(\Delta \hat{a}_2)^2 = e^{-(\gamma-2\kappa\beta)t} + \frac{\gamma(2\overline{n}+1)}{\gamma-2\kappa\beta}\left(1 - e^{-(\gamma-2\kappa\beta)t}\right). \tag{4.10b}$$

At steady state, these expressions reduce to

$$(\Delta \hat{a}_1)^2 = \frac{\gamma(2\overline{n}+1)}{\gamma+2\kappa\beta}, \tag{4.10c}$$

$$(\Delta \hat{a}_2)^2 = \frac{\gamma(2\overline{n}+1)}{\gamma-2\kappa\beta} \tag{4.10d}$$

and at threshold, we see that

$$(\Delta \hat{a}_1)^2 = \frac{1}{2}(2\overline{n}+1), \tag{4.10e}$$

$$(\Delta \hat{a}_2)^2 \to \infty. \tag{4.10f}$$

Expressions (4.10) represent the variances of the quadrature operators for the signal mode produced by the degenerate parametric oscillator coupled to a thermal reservoir. It can be easily seen that $(\Delta \hat{a}_1)^2 < 1$ for $\overline{n} < \kappa\beta/\gamma$. Hence the signal mode would be in squeezed state if the mean photon number of the thermal reservoir is less than $\kappa\beta/\gamma$.

In the case of an ordinary vacuum reservoir ($N = 0$ and $M = 0$), Eq. (4.7) turns out to be

$$(\Delta\hat{a}_1)^2 = e^{-(\gamma+2\kappa\beta)t} + \frac{\gamma}{\gamma+2\kappa\beta}\left(1 - e^{-(\gamma+2\kappa\beta)t}\right), \tag{4.11a}$$

$$(\Delta\hat{a}_2)^2 = e^{-(\gamma-2\kappa\beta)t} + \frac{\gamma}{\gamma-2\kappa\beta}\left(1 - e^{-(\gamma-2\kappa\beta)t}\right). \tag{4.11b}$$

These represent the variances of the quadrature operators for the signal mode produced by the degenerate parametric oscillator. One can see from (4.11) that the signal mode is in a squeezed state and the squeezing occurs in the first quadrature. At steady state and threshold the fluctuations in the first intracavity quadrature turns out to be one-half. This results in a 50% reduction of the noise [23,33].

Moreover, in the absence of damping ($\gamma = 0$), Eq. (4.7) reduce to

$$(\Delta\hat{a}_1)^2 = e^{-2\kappa\beta t}, \tag{4.12a}$$

$$(\Delta\hat{a}_2)^2 = e^{2\kappa\beta t}. \tag{4.12b}$$

These are the variances of the quadrature operators for the signal mode produced by a degenerate parametric amplifier. We thus observe that the signal mode produced by this system is in a squeezed state.

4.2 Squeezing Spectrum

We next wish to determine the squeezing spectrum for the signal mode applying the quantum Langevin equations. The squeezing spectrum for a signal mode is defined by

$$S_j^{out}(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau}\langle\hat{a}_j^{out}(t+\tau), \hat{a}_j^{out}(t)\rangle_{ss}, \tag{4.13}$$

where $a_j^{out}(t)$ represent the output quadrature operators. To calculate the squeezing spectrum (4.13), we first evaluate the two-time correlation function for the output quadrature operators. We see that the sum of (2.67) and its adjoint gives

$$\hat{a}_1^{out}(t) = \sqrt{\gamma}\hat{a}_1(t) - \hat{a}_1^{in}(t), \tag{4.14a}$$

where $\hat{a}_1(t)$ and $\hat{a}_1^{in}(t)$ are the first intracavity and input quadrature operators, respectively. The relation for the second quadrature operators can be obtained by subtracting (2.67) from its adjoint and multiplying by i:

$$\hat{a}_2^{out}(t) = \sqrt{\gamma}\hat{a}_2(t) - \hat{a}_2^{in}(t). \tag{4.14b}$$

In view of (2.69a) and (4.3), we have

$$\langle \hat{a}_1^{out}(t)\rangle = 0, \tag{4.15}$$

so that using this result, one can write

$$\langle \hat{a}_1^{out}(t+\tau), \hat{a}_1^{out}(t)\rangle = \gamma\langle \hat{a}_1(t+\tau)\hat{a}_1(t)\rangle - \sqrt{\gamma}\langle \hat{a}_1(t+\tau)\hat{a}_1^{in}(t)\rangle$$
$$-\sqrt{\gamma}\langle \hat{a}_1^{in}(t+\tau)\hat{a}_1(t)\rangle + \langle \hat{a}_1^{in}(t+\tau)\hat{a}_1^{in}(t)\rangle. \tag{4.16}$$

We now proceed to obtain the explicit form of the two time correlation functions involved in (4.16). On account of (2.65), the Langevin equation (2.53) can be rewritten as

$$\frac{d}{dt}\hat{a}(t) = -\frac{\gamma}{2}\hat{a}(t) - \kappa\beta\hat{a}^\dagger(t) + \sqrt{\gamma}\hat{a}_{in}(t). \tag{4.17}$$

In order to decouple this equation, we add (4.17) with its adjoint and get

$$\frac{d}{dt}\hat{a}_1(t) = -\left(\frac{\gamma}{2} + \kappa\beta\right)\hat{a}_1(t) + \sqrt{\gamma}\hat{a}_1^{in}(t), \tag{4.18}$$

where

$$\hat{a}_1^{in}(t) = \frac{1}{\sqrt{\gamma}}\left(\hat{F}(t) + \hat{F}^\dagger(t)\right). \tag{4.19}$$

A formal solution of (4.18) can be put in the form

$$\hat{a}_1(t) = \hat{a}_1(0)e^{-(\frac{\gamma}{2}+\kappa\beta)t} + \sqrt{\gamma}\int_0^t \hat{a}_1^{in}(t')e^{-(\frac{\gamma}{2}+\kappa\beta)(t-t')}dt' \tag{4.20}$$

and hence, one can write that

$$\hat{a}_1(t+\tau) = \hat{a}_1(0)e^{-(\frac{\gamma}{2}+\kappa\beta)(t+\tau)} + \sqrt{\gamma}\int_0^{t+\tau} \hat{a}_1^{in}(t')e^{-(\frac{\gamma}{2}+\kappa\beta)(t+\tau-t')}dt'. \tag{4.21}$$

It then follows

$$\langle \hat{a}_1(t+\tau)\hat{a}_1(t)\rangle = \langle \hat{a}_1(0)\hat{a}_1(0)\rangle e^{-(\frac{\gamma}{2}+\kappa\beta)(2t+\tau)} + \sqrt{\gamma}\int_0^t \langle \hat{a}_1^{in}(t')\hat{a}_1(0)\rangle e^{-(\frac{\gamma}{2}+\kappa\beta)(2t+\tau-t')}dt'$$
$$+\sqrt{\gamma}\int_0^{t+\tau} \langle \hat{a}_1(0)\hat{a}_1^{in}(t')\rangle e^{-(\frac{\gamma}{2}+\kappa\beta)(2t+\tau-t')}dt'$$
$$+\gamma\int_0^{t+\tau} dt' \int_0^t dt'' \langle \hat{a}_1^{in}(t')\hat{a}_1^{in}(t'')\rangle e^{-(\frac{\gamma}{2}+\kappa\beta)(2t+\tau-t'-t'')}. \tag{4.22}$$

On account of (4.19), we have

$$\langle \hat{a}_1^{in}(t')\hat{a}_1(0)\rangle = \frac{1}{\sqrt{\gamma}}\left(\langle \hat{F}(t')\hat{a}_1(0)\rangle + \langle \hat{F}^\dagger(t')\hat{a}_1(0)\rangle\right). \tag{4.23}$$

From (2.51) we note that

$$\langle \hat{F}(t')\hat{a}_1(0)\rangle = \sum_k \lambda_k \langle \hat{b}_k(0)\hat{a}_1(0)\rangle e^{i(\omega_0-\omega_k)t'}.$$

Since $\hat{b}_k(0)$ and $\hat{a}_1(0)$ are not correlated, one can write

$$\langle \hat{b}_k(0)\hat{a}_1(0)\rangle = Tr_R(\hat{b}_k(0))Tr_S(\hat{a}_1(0)).$$

We recall that the expectation value of the annihilation (or the creation) operator for a squeezed vacuum mode is zero, so that

$$\langle \hat{F}(t')\hat{a}_1(0)\rangle = 0.$$

We also see that

$$\langle \hat{F}^\dagger(t')\hat{a}_1(0)\rangle = 0.$$

Hence

$$\langle \hat{a}_1^{in}(t')\hat{a}_1(0)\rangle = 0, \tag{4.24a}$$

similarly, one can easily show that

$$\langle \hat{a}_1(0)\hat{a}_1^{in}(t')\rangle = 0. \tag{4.24b}$$

We note that

$$\langle \hat{a}_1^{in}(t')\hat{a}_1^{in}(t'')\rangle = \langle \hat{a}_{in}^\dagger(t')\hat{a}_{in}^\dagger(t'')\rangle + \langle \hat{a}_{in}^\dagger(t')\hat{a}_{in}(t'')\rangle + \langle \hat{a}_{in}(t')\hat{a}_{in}^\dagger(t'')\rangle + \langle \hat{a}_{in}(t')\hat{a}_{in}(t'')\rangle.$$

On account of (2.69), we see that

$$\langle \hat{a}_1^{in}(t')\hat{a}_1^{in}(t'')\rangle = (2N - 2M + 1)\delta(t' - t''). \tag{4.25}$$

Substitution of (4.24) and (4.25) into (4.22) and then applying the property of the Dirac delta function

$$\int_a^b f(x')\delta(x - x')dx' = \begin{cases} f(x) & \text{for } a < x < b \\ 0 & \text{otherwise} \end{cases}, \tag{4.26}$$

we get

$$\langle \hat{a}_1(t+\tau)\hat{a}_1(t)\rangle = \left[e^{-(\gamma+2\kappa\beta)t} + \gamma(2N - 2M + 1)\int_0^t e^{-(\gamma+2\kappa\beta)(t-t')}dt'\right] e^{-(\frac{\gamma}{2}+\kappa\beta)\tau},$$

where we have used the fact that $\langle \hat{a}_1(0)\hat{a}_1(0)\rangle = 1$ for the signal mode initially in vacuum state. Then carrying out the integration, we obtain

$$\langle \hat{a}_1(t+\tau)\hat{a}_1(t)\rangle = \left[e^{-(\gamma+2\kappa\beta)t} + \frac{\gamma(2N-2M+1)}{\gamma+2\kappa\beta}\left(1-e^{-(\gamma+2\kappa\beta)t}\right)\right]e^{-(\frac{\gamma}{2}+\kappa\beta)\tau}. \quad (4.27)$$

On multiplying (4.21) by $\hat{a}_1^{in}(t)$ on the right and in view of (4.24b) and (4.25), we have

$$\langle \hat{a}_1(t+\tau)\hat{a}_1^{in}(t)\rangle = \sqrt{\gamma}(2N-2M+1)\int_0^t e^{-(\frac{\gamma}{2}+\kappa\beta)(t+\tau-t')}\delta(t-t')dt'.$$

Then applying (4.26), we get

$$\langle \hat{a}_1(t+\tau)\hat{a}_1^{in}(t)\rangle = \sqrt{\gamma}(2N-2M+1)e^{-(\frac{\gamma}{2}+\kappa\beta)\tau}. \quad (4.28)$$

Furthermore, multiplying (4.20) by $\hat{a}_1^{in}(t+\tau)$ on the left along with (4.24a) and (4.25), we see that

$$\langle \hat{a}_1^{in}(t+\tau)\hat{a}_1(t)\rangle = \sqrt{\gamma}(2N-2M+1)\int_0^t e^{-(\frac{\gamma}{2}+\kappa\beta)(t-t')}\delta(t'+\tau-t')dt',$$

so that using the relation (4.26), we obtain

$$\langle \hat{a}_1^{in}(t+\tau)\hat{a}_1(t)\rangle = 0. \quad (4.29)$$

From (4.25), we see that

$$\langle \hat{a}_1^{in}(t+\tau)\hat{a}_1^{in}(t)\rangle = (2N-2M+1)\delta(\tau). \quad (4.30)$$

Now using (4.27-4.30) in (4.15), one finds the two-time correlation function to be

$$\begin{aligned}\langle \hat{a}_1^{out}(t+\tau)\hat{a}_1^{out}(t)\rangle &= (2N-2M+1)\delta(\tau)\\ &+\gamma\left[e^{-(\gamma+2\kappa\beta)t} - \frac{(2N-2M+1)}{\gamma+2\kappa\beta}\left(2\kappa\beta+\gamma e^{-(\gamma+2\kappa\beta)t}\right)\right]e^{-(\frac{\gamma}{2}+\kappa\beta)\tau}.\end{aligned} \quad (4.31a)$$

Following a similar fashion, it can be readily verified that

$$\begin{aligned}\langle \hat{a}_2^{out}(t+\tau)\hat{a}_2^{out}(t)\rangle &= (2N+2M+1)\delta(\tau)\\ &+\gamma\left[e^{-(\gamma-2\kappa\beta)t} + \frac{(2N+2M+1)}{\gamma-2\kappa\beta}\left(2\kappa\beta-\gamma e^{-(\gamma-2\kappa\beta)t}\right)\right]e^{-(\frac{\gamma}{2}-\kappa\beta)\tau}.\end{aligned} \quad (4.31b)$$

At steady state expressions (4.31) reduce to

$$\langle \hat{a}_1^{out}(t+\tau)\hat{a}_1^{out}(t)\rangle_{ss} = (2N-2M+1)\left[\delta(\tau) - \frac{2\kappa\beta\gamma}{\gamma+2\kappa\beta}e^{-(\frac{\gamma}{2}+\kappa\beta)\tau}\right], \quad (4.32a)$$

$$\langle \hat{a}_2^{out}(t+\tau)\hat{a}_2^{out}(t)\rangle_{ss} = (2N+2M+1)\left[\delta(\tau) + \frac{2\kappa\beta\gamma}{\gamma - 2\kappa\beta} e^{-(\frac{\gamma}{2}-\kappa\beta)\tau}\right]. \tag{4.32b}$$

On account of the result (4.32a) along with (2.17), the squeezing spectrum (4.13) for the first quadrature can be put in the form

$$S_1^{out}(\omega) = e^{-2r} - \frac{2\kappa\beta\gamma}{\gamma + 2\kappa\beta} e^{-2r} \int_{-\infty}^{\infty} e^{-((\frac{\gamma}{2}+\kappa\beta)+i\omega)\tau} d\tau. \tag{4.33}$$

We note that

$$\int_{-\infty}^{\infty} e^{-((\frac{\gamma}{2}+\kappa\beta)+i\omega)\tau} d\tau = \int_{-\infty}^{0} e^{-((\frac{\gamma}{2}+\kappa\beta)+i\omega)\tau} d\tau + \int_{0}^{\infty} e^{-((\frac{\gamma}{2}+\kappa\beta+i\omega)\tau} d\tau.$$

Then applying the stationarity property in the first integral, we have

$$\int_{-\infty}^{\infty} e^{-((\frac{\gamma}{2}+\kappa\beta)+i\omega)\tau} d\tau = \int_{-\infty}^{0} e^{((\frac{\gamma}{2}+\kappa\beta)-i\omega)\tau} d\tau + \int_{0}^{\infty} e^{-((\frac{\gamma}{2}+\kappa\beta)+i\omega)\tau} d\tau,$$

from which follows

$$\int_{-\infty}^{\infty} e^{-((\frac{\gamma}{2}+\kappa\beta)+i\omega)\tau} d\tau = \frac{\gamma + 2\kappa\beta}{(\frac{\gamma}{2}+\kappa\beta)^2 + \omega^2}. \tag{4.34}$$

Hence using this result in (4.33), we obtain

$$S_1^{out}(\omega) = \left[1 - \frac{2\kappa\beta\gamma}{(\frac{\gamma}{2}+\kappa\beta)^2 + \omega^2}\right] e^{-2r}. \tag{4.35a}$$

One can show in a similar way that the squeezing spectrum for the second quadrature operator has the form

$$S_2^{out}(\omega) = \left[1 + \frac{2\kappa\beta\gamma}{(\frac{\gamma}{2}-\kappa\beta)^2 + \omega^2}\right] e^{2r}. \tag{4.35b}$$

Expressions (4.35) represent the squeezing spectra for the signal mode produced by the degenerate parametric oscillator coupled to a squeezed vacuum. From (4.35a) we see that the squeezing spectrum is a Lorentzian with a half width $\frac{\gamma}{2} + \kappa\beta$. At threshold the expressions given by (4.35) reduce to

$$S_1^{out}(\omega) = \frac{\omega^2}{\gamma^2 + \omega^2} e^{-2r}, \tag{4.36a}$$

$$S_2^{out}(\omega) = \frac{\gamma^2 + \omega^2}{\omega^2} e^{2r}. \tag{4.36b}$$

We note that the effect of the squeezed vacuum reservoir is to increase the degree of squeezing. We also notice that for $\omega = 0$ (the cavity resonance), there is a complete suppression of the noise in the first quadrature while the noise in the second quadrature tends to infinity. Finally, we observe that for $r = 0$, expression (4.36) reduce to the result obtained by Collet and Gardiner [27], and Daniel and Fesseha [33].

5 The Photon Statistics

The statistical properties of a light mode is described in terms of the mean and the variance of the photon number, the photon number distribution and the second-order correlation function. The relation between the mean and variance of the photon number can be used to classify the photon statistics as Poissonian, super-Poissonian and sub-Poissonian. In addition, the bunching and antibunching phenomena of the photons can be described using the second-order correlation function.

Here we wish to calculate the photon number distribution, the mean and variance of the photon number as well as the second-order correlation function for the signal mode employing the Q-function.

5.1 The Photon Number Distribution

Applying (2.18) along with (3.33), the photon number distribution for the signal mode can be expressed as

$$P(n) = \frac{\sqrt{D^2 - C^2}}{n!} \frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n} exp\left[(1 - D(t))\alpha^*\alpha + \frac{1}{2}C(t)(\alpha^{*2} + \alpha^2)\right]\bigg|_{\alpha^*=\alpha=0} . \tag{5.1}$$

Now expanding the exponential function in a power series, we have

$$exp[(1 - D)\alpha^*\alpha] = \sum_k^\infty \frac{(1 - D)^k}{k!}\alpha^{*k}\alpha^k,$$

$$exp\left(\frac{1}{2}C\alpha^2\right) = \sum_l^\infty \frac{1}{2^l}\frac{C^l}{l!}\alpha^{2l},$$

$$exp\left(\frac{1}{2}C\alpha^{*2}\right) = \sum_m^\infty \frac{1}{2^m}\frac{C^m}{m!}\alpha^{*2m},$$

and hence (5.1) takes the form

$$P(n) = \frac{\sqrt{D^2 - C^2}}{n!} \sum_{k,l,m}^\infty \frac{1}{2^{l+m}} \frac{(1 - D)^k C^{l+m}}{k!l!m!} \frac{\partial^n}{\partial\alpha^n}\alpha^{k+2l} \frac{\partial^n}{\partial\alpha^{*n}}\alpha^{*k+2m}\bigg|_{\alpha^*=\alpha=0} . \tag{5.2}$$

On account of the relation

$$\frac{\partial^n}{\partial x^n}x^m = \frac{m!}{(m - n)!}x^{m-n},$$

Eq. (5.2) can be put in the form

$$P(n) = \frac{\sqrt{D^2 - C^2}}{n!} \sum_{k,l,m}^{\infty} \frac{1}{2^{l+m}} \frac{(1-D)^k C^{l+m}}{k!l!m!} \frac{(k+2l)!}{(k+2l-n)!} \frac{(k+2m)!}{(k+2m-n)!} \alpha^{k+2l-n} \alpha^{*k+2m-n} \Big|_{\alpha^*=\alpha=0} \tag{5.3}$$

We note that

$$\alpha^{k+2l-n} \alpha^{*k+2m-n} \Big|_{\alpha^*=\alpha=0} = \delta_{k,n-2l} \delta_{k,n-2m}$$

from which follows

$$l = m,$$

$$k = n - 2l.$$

Hence expression (5.3) reduces to

$$P(n) = \frac{\sqrt{D^2 - C^2}}{n!} \sum_{l=0}^{\infty} \frac{(n!)^2}{2^{2l}(l!)^2(n-2l)!} C^{2l}(1-D)^{n-2l}. \tag{5.4}$$

Since $(n - 2l)!$ is defined for nonnegative integers, we see that

$$\frac{n}{2} \geq l.$$

In view of this result, we have

$$P(n) = \frac{\sqrt{D^2 - C^2}}{n!} \sum_{l=0}^{[n]} \frac{(n!)^2}{2^{2l}(l!)^2(n-2l)!} C^{2l}(1-D)^{n-2l}, \tag{5.5}$$

where $[n] = \frac{n}{2}$ for even n and $[n] = \frac{n-1}{2}$ for odd n. This expression represents the photon number distribution for the signal mode.

In the absence of the cavity damping ($\gamma = 0$), one can easily obtain from (3.32)

$$C(t) = -tanh(\kappa\beta t),$$

$$D(t) = 1.$$

Then on substituting these values into (5.5), the photon number distribution turns out to be

$$P(n) = \begin{cases} 0 & \text{(for odd } n) \\ \frac{1}{2^n n!} \frac{tanh^n(\kappa\beta t)}{cosh(\kappa\beta t)} H_n^2(0) & \text{(for even } n) \end{cases}, \tag{5.6}$$

where $H_n(0)$ is the Hermite polynomial. This shows that the probability of finding odd number of signal photons is zero due to the fact that the signal photons are always generated in pairs. Furthermore, at steady state we get from (3.32)

$$C = -\frac{2s(N+1)+(M-s)}{(N+1)^2-(M-s)^2},$$

$$D = \frac{(N+1)+2s(M-s)}{(N+1)^2-(M-s)^2},$$

where $s = \kappa\beta/\gamma$. Hence on account of these results, the photon number distribution (5.5) can be written as

$$P(n) = (1-4s^2)^{\frac{1}{2}} \frac{[N(N+1)-M^2+s^2]^n}{[(N+1)^2-(M-s)^2]^{n+\frac{1}{2}}} \sum_{l=0}^{[n]} \frac{(n!)}{2^{2l}(l!)^2(n-2l)!} \left[\frac{(M-s)+2s(N+1)}{N(N+1)-M^2+s^2}\right]^{2l}. \tag{5.7}$$

For the case in which the coupling constant κ is zero, we see that (5.7) reduces to the form

$$P(n) = \frac{1}{[(N+1)^2-M^2]^{n+1/2}} \sum_{l=0}^{[n]} \frac{n!}{2^{2l}(l!)^2(n-2l)!} M^{2l}\left[N(N+1)-M^2\right]^{n-2l}. \tag{5.8}$$

This represents the photon number distribution for the reservoir. We recall that for a squeezed vacuum reservoir $N = sinh^2(r)$ and $M = sinh(r)cosh(r)$. Then in view of this, Eq. (5.8) reduces to (2.19). For the case of thermal reservoir where $N = \overline{n}$ and $M = 0$, expression (5.8) turns out to be

$$P(n) = \frac{\overline{n}^n}{(\overline{n}+1)^{n+1}}. \tag{5.9}$$

Furthermore, for the case $N = M = 0$, it is readily seen that

$$P(n) = \delta_{n,0} \tag{5.10}$$

which is the photon number distribution for ordinary vacuum reservoir.

5.2 The Mean and Variance of the Photon Number

We now seek to calculate the mean and the variance of the photon number for the signal mode. Using the Q-function (3.33), the mean photon number for the signal mode can be expressed as

$$\overline{n}(t) = \sqrt{D^2-C^2}\left(-\frac{\partial}{\partial D}-1\right)\int_0^\infty \frac{d^2\alpha}{\pi} exp\left[-D\alpha^*\alpha + \frac{1}{2}C(\alpha^2+\alpha^{*2})\right]. \tag{5.11}$$

Upon carrying out the integration with the help of (2.12) and then the differentiation, we obtain

$$\overline{n}(t) = \frac{D}{D^2 - C^2} - 1. \tag{5.12a}$$

Using the explicit form of C and D, the mean photon number of the signal mode can be put in the form

$$\overline{n}(t) = \frac{1}{4}\left[\left(\frac{\gamma(2N - 2M + 1)}{\gamma + 2\kappa\beta} - 1\right)\left(1 - e^{-(\gamma+2\kappa\beta)t}\right)\right.$$
$$\left. + \left(\frac{\gamma(2N + 2M + 1)}{\gamma - 2\kappa\beta} - 1\right)\left(1 - e^{-(\gamma-2\kappa\beta)t}\right)\right]. \tag{5.12b}$$

At steady state this expression reduces to

$$\overline{n}_{ss}(t) = \frac{1}{4}\left[\frac{\gamma(2N - 2M + 1)}{\gamma + 2\kappa\beta} + \frac{\gamma(2N + 2M + 1)}{\gamma - 2\kappa\beta} - 2\right]. \tag{5.13}$$

In the absence of the parametric interaction ($\kappa = 0$), we see from (5.13) that

$$\overline{n}_{ss}(t) = N, \tag{5.14}$$

which represents the mean photon number for the reservoir.

Next we seek to calculate the variance of the photon number for the signal mode. We note that the variance of the photon number can be expressed as

$$(\Delta n(t))^2 = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - \overline{n}^2(t) - 3\overline{n}(t) - 2, \tag{5.15}$$

where $\overline{n}(t) = \langle \hat{a}^\dagger \hat{a} \rangle$ is the mean photon number for the signal mode. Applying the Q-function (3.33), we have

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \sqrt{D^2 - C^2} \int_0^\infty \frac{d^2\alpha}{\pi} \alpha^2 \alpha^{*2} exp\left[-D\alpha^*\alpha + \frac{1}{2}C(\alpha^2 + \alpha^{*2})\right].$$

This can be put in the form

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \sqrt{D^2 - C^2} \frac{\partial^2}{\partial D^2} \int_0^\infty \frac{d^2\alpha}{\pi} exp\left[-D\alpha^*\alpha + \frac{1}{2}C(\alpha^2 + \alpha^{*2})\right],$$

so that carrying out the integration applying (2.12), we get

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \sqrt{D^2 - C^2} \frac{\partial^2}{\partial D^2} \left(\frac{1}{\sqrt{D^2 - C^2}}\right),$$

from which follows

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{2D^2}{(D^2 - C^2)^2} + \frac{C^2}{(D^2 - C^2)^2}. \tag{5.16a}$$

This can be rewritten in terms of the mean photon number as

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \left(2 + \frac{C^2}{D^2}\right)(\overline{n}+1)^2. \tag{5.16b}$$

Hence on using (5.16) in (5.15), the variance for the signal mode becomes

$$(\Delta n(t))^2 = \overline{n}(t)(\overline{n}(t)+1) + \left(\frac{C}{D}(\overline{n}(t)+1)\right)^2. \tag{5.17}$$

We note from this result that $(\Delta n(t))^2 > \overline{n}(t)$, which shows the signal mode has super-Poissonian statistics. In the absence of parametric interaction and at steady state, one can easily obtain from (3.32)

$$C = -\frac{M}{(N+1)^2 - M^2},$$

$$D = \frac{N+1}{(N+1)^2 - M^2}.$$

Using this along with (5.14), one can put (5.17) in the form

$$(\Delta n(t))^2_{ss} = N(N+1) + M^2 \tag{5.18}$$

This represents the variance of the photon number for the reservoir. For the case of a squeezed vacuum reservoir, we note that $N(N+1) = M^2$. Then (5.18) turns out to be the same as (2.23).

5.3 The Second-Order Correlation Function

Here we want to evaluate the second-order correlation function for the signal mode. The second-order correlation function is defined by

$$g^{(2)}(\tau) = \frac{\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle}{\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle^2}. \tag{5.19}$$

Now we proceed to obtain the explicit form of $\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle$. We note that in the Schrödinger picture one can write that

$$\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle = Tr(\hat{\rho(t)}\hat{a}^\dagger\hat{a}^\dagger(\tau)\hat{a}(\tau)\hat{a}), \tag{5.20}$$

in which $\hat{a} = \hat{a}(0)$. Upon expanding the density operator in the normal order and introducing the completeness relation

$$I = \int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha|, \tag{5.21}$$

Eq. (5.20) can be written as

$$\langle \hat{a}^{\dagger}(t)\hat{a}^{\dagger}(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle = \int \frac{d^2\alpha}{\pi} \sum_{l,m} C_{l,m}(t) Tr(|\alpha\rangle\langle\alpha|\hat{a}^{\dagger l}\hat{a}^{m}\hat{a}^{\dagger}\hat{a}^{\dagger}(\tau)\hat{a}(\tau)\hat{a}). \quad (5.21)$$

Using the relation

$$|\alpha\rangle\langle\alpha|\hat{a}^{m} = \left(\alpha + \frac{\partial}{\partial\alpha^{*}}\right)^{m} |\alpha\rangle\langle\alpha| \quad (5.22)$$

and the cyclic property of the trace, one can put (5.21) in the form

$$\langle \hat{a}^{\dagger}(t)\hat{a}^{\dagger}(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle = \int d^2\alpha Q\left(\alpha^{*}, \alpha + \frac{\partial}{\partial\alpha^{*}}, t\right) \alpha^{*}\alpha Tr(|\alpha\rangle\langle\alpha| \hat{a}^{\dagger}(\tau)\hat{a}(\tau)), \quad (5.23)$$

where

$$Q\left(\alpha^{*}, \alpha + \frac{\partial}{\partial\alpha^{*}}, t\right) = \frac{1}{\pi}\sum_{l,m} C_{l,m}(t)\alpha^{*l}\left(\alpha + \frac{\partial}{\partial\alpha^{*}}\right)^{m}. \quad (5.24)$$

Furthermore, one can write that

$$Tr(\hat{\rho}(0)\hat{a}^{\dagger}(\tau)\hat{a}(\tau)) = Tr(\hat{\rho}(\tau)\hat{a}^{\dagger}(0)\hat{a}(0)), \quad (5.25)$$

in which $\hat{\rho}(0) = |\alpha\rangle\langle\alpha|$. We note that

$$Tr(\hat{\rho}(\tau)\hat{a}^{\dagger}(0)\hat{a}(0)) = \int d^2\lambda(\lambda^{*}\lambda - 1)Q(\lambda^{*}, \lambda, \tau), \quad (5.26)$$

where

$$Q(\lambda^{*}, \lambda, \tau) = A'(\tau) exp\left[-D'\lambda^{*}\lambda + \frac{1}{2}C'(\lambda^{*2} + \lambda^{2}) + B'^{*}\lambda + B'\lambda^{*}\right], \quad (5.27)$$

A', B', C' and D' are given by (3.32) with $(\alpha_0^{*}, \alpha_0, t)$ replaced by $(\alpha^{*}, \alpha, \tau)$. Hence employing (5.27) in (5.26), we have

$$Tr(\hat{\rho}(\tau)\hat{a}^{\dagger}\hat{a}) = \left(\frac{\partial^2}{\partial B'^{*}\partial B'} - 1\right) \int d^2\lambda A'(\tau) exp\left[-D'\lambda^{*}\lambda + \frac{1}{2}C'(\lambda^{*2} + \lambda^{2}) + B'^{*}\lambda + B'\lambda^{*}\right].$$

On carrying out the integration with the help of (2.12), we obtain

$$Tr(\hat{\rho}(\tau)\hat{a}^{\dagger}\hat{a}) = \left(\frac{\partial^2}{\partial B'^{*}\partial B'} - 1\right) \frac{A'(\tau)}{\sqrt{D'^2 - C'^2}} exp\left[\frac{D'B'^{*}B' + \frac{1}{2}C(B'^{*2} + B'^{2})}{D'^2 - C'^2}\right]$$

and performing the differentiation results in

$$Tr(\hat{\rho}(\tau)\hat{a}^{\dagger}\hat{a}) = \left[\frac{D'}{D'^2 - C'^2} - 1 + \frac{(D'B' + C'B'^{*})(D'B'^{*} + C'B')}{D'^2 - C'^2}\right]. \quad (5.28)$$

Using the explicit form of B', C' and D', one easily obtains

$$D'B' + C'B'^{*} = \frac{(D'^2 - C'^2)}{2}\left((\alpha^{*} + \alpha)e^{-2\lambda_1\tau} + (\alpha^{*} - \alpha)e^{2\lambda_2\tau}\right).$$

It then follows that

$$\frac{(D'B'+C'B'^*)(D'B'^*+C'B')}{D'^2-C'^2}=E(\tau)\alpha^*\alpha+\frac{1}{2}E'(\tau)(\alpha^{*2}+\alpha^2),$$

in which

$$E(\tau)=\frac{1}{2}\left(e^{-2\lambda_1\tau}+e^{2\lambda_2\tau}\right), \tag{5.29a}$$

$$E'(\tau)=\frac{1}{2}\left(e^{-2\lambda_1\tau}-e^{2\lambda_2\tau}\right). \tag{5.29b}$$

On account of this result , Eq. (5.28) takes the form

$$Tr(\hat{\rho}(\tau)\hat{a}^\dagger\hat{a})=E(\tau)\alpha^*\alpha+\frac{1}{2}E'(\tau)(\alpha^{*2}+\alpha^2)+\overline{n}(\tau). \tag{5.30}$$

where

$$\begin{aligned}\overline{n}(\tau)=\frac{1}{4}\Bigg[&\left(\frac{\gamma(2N-2M+1)}{\gamma+2\kappa\beta}-1\right)\left(1-e^{-(\gamma+2\kappa\beta)\tau}\right)\\ &+\left(\frac{\gamma(2N+2M+1)}{\gamma-2\kappa\beta}-1\right)\left(1-e^{-(\gamma-2\kappa\beta)\tau}\right)\Bigg].\end{aligned} \tag{5.31}$$

Upon substituting (5.30) into (5.23), we see that

$$\begin{aligned}\langle\hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle=&\int d^2\alpha Q\left(\alpha^*,\alpha+\frac{\partial}{\partial\alpha^*},t\right)\\ &\times\left(E(\tau)\alpha^{*2}\alpha^2+\frac{1}{2}E'(\tau)(\alpha^{*3}\alpha+\alpha^*\alpha^3)+\overline{n}(\tau)\alpha^*\alpha\right).\end{aligned} \tag{5.32}$$

We note that

$$\langle\hat{A}\rangle=Tr\left(\hat{\rho}(t)A(\hat{a}^\dagger,\hat{a})\right).$$

Hence expanding the density operator in normal order and inserting the completeness relation for coherent states, we have

$$\langle\hat{A}\rangle=\int\frac{d^2\alpha}{\pi}\sum_{l,m}C_{l,m}(t)Tr\left(|\alpha\rangle\langle\alpha|\hat{a}^{\dagger l}\hat{a}^m A(\hat{a}^\dagger,\hat{a})\right).$$

Using the relation (5.22), we see that

$$\langle\hat{A}\rangle=\int\frac{d^2\alpha}{\pi}Q\left(\alpha^*,\alpha+\frac{\partial}{\partial\alpha^*},t\right)A_n(\alpha^*,\alpha), \tag{5.33}$$

where $A_n(\alpha^*,\alpha)$ is the c-number equivalent of $A(\hat{a}^\dagger,\hat{a})$ for the normal ordering. Therefore, in view of (5.33) expression (5.32) can be written in the form

$$\langle\hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle=E(\tau)\langle\hat{a}^{\dagger2}\hat{a}^2\rangle+\frac{1}{2}E'(\tau)\langle\hat{a}^{\dagger3}\hat{a}+\hat{a}^\dagger\hat{a}^3\rangle+\overline{n}(\tau)\langle\hat{a}^\dagger\hat{a}\rangle. \tag{5.34}$$

From (5.40) we see that $g^{(2)}(\tau)$ has no local extermum points, so that $g^{(2)}(\tau)$ is either an increasing or a decreasing function of τ. Now upon setting $\tau = 0$, expression (5.40) takes the form

$$\frac{d}{d\tau}g^{(2)}(\tau)\bigg|_{\tau=0} = -\frac{\gamma}{\overline{n}^2(t)}\left[2\overline{n}^2(t) + \frac{C^2}{D^2}(\overline{n}(t)+1)^2 + \frac{6\kappa\beta}{\gamma}\frac{C}{D}\overline{n}(t)(\overline{n}(t)+1) - \frac{1}{4}\overline{n}(t)(u+v-2)\right]. \tag{5.42}$$

At steady state we have from (3.32) and (5.13)

$$C(t) = \frac{\frac{\gamma}{\gamma+2\kappa\beta}u - \frac{\gamma}{\gamma-2\kappa\beta}v}{\left(1+\frac{\gamma}{\gamma+2\kappa\beta}u\right)\left(1+\frac{\gamma}{\gamma-2\kappa\beta}v\right)}, \tag{5.43a}$$

$$D(t) = \frac{\frac{\gamma}{\gamma+2\kappa\beta}u + \frac{\gamma}{\gamma-2\kappa\beta}v + 2}{\left(1+\frac{\gamma}{\gamma+2\kappa\beta}u\right)\left(1+\frac{\gamma}{\gamma-2\kappa\beta}v\right)}, \tag{5.43b}$$

$$\overline{n}(t) = \frac{1}{4}\left(\frac{\gamma}{\gamma+2\kappa\beta}u + \frac{\gamma}{\gamma-2\kappa\beta}v - 2\right). \tag{5.43c}$$

Upon substituting these into (5.42), we get

$$\frac{d}{d\tau}g^{(2)}(\tau)\bigg|_{\tau=0} = -\frac{16\gamma}{\left(\frac{\gamma}{\gamma+2\kappa\beta}u + \frac{\gamma}{\gamma-2\kappa\beta}v - 2\right)^2}\left[\left(u - \frac{\gamma+2\kappa\beta}{\gamma}\right)^2 + \left(v - \frac{\gamma-2\kappa\beta}{\gamma}\right)^2\right]. \tag{5.44}$$

This shows that the slope $\left(\frac{d}{d\tau}g^{(2)}(\tau)\big|_{\tau=0}\right)$ is negative, so that $g^{(2)}(\tau)$ decreases as τ increases, $g^{(2)}(\tau) < g^{(2)}(0)$. Hence at steady state the signal mode produced by a degenerate parametric oscillator coupled to a squeezed vacuum exhibits photon bunching.

In the absence of damping ($\gamma = 0$), from (3.32) and (5.12) we get

$$C/D = -tanh(\kappa\beta t),$$

$$\overline{n}(t) = sinh^2(\kappa\beta t),$$

so that (5.39) takes the form

$$g^{(2)}(0) = 1 + \frac{cosh(\kappa\beta t)}{sinh^2(\kappa\beta t)}. \tag{5.45}$$

This is the second-order correlation function for the signal mode produced by a degenerate parametric amplifier. The squeezed light generated by this system exhibits photon bunching.

Moreover, one effect of the squeezed vacuum is to increase the mean photon number of the intracavity signal mode. Finally, we have determined the second-order correlation function and at steady state we have found $g^2(\tau)$ to be less than $g^2(0)$. This shows that the signal mode exhibits photon bunching. It also turns out that $g^2(0)$ is greater than unity at any time and hence the signal mode has a super-Possonian statistics.

References

[1] K. Fesseha, J. Math. phys. **33**, 2179 (1992).

[2] W. Voger and D. Welsch Quantum optics Lecture (Akademie Verlag,1994).

[3] B. Yurke, S. L. Mccall, and J. R. Kauder, phys. Rev. A **33**, 4033 (1986).

[4] P. Grangier, R. E. Slusher, B. Yurke and A. Laporta, Phys. Rev. Lett. **59**, 2153 (1987).

[5] J. Gea-Banacloche and G. Leuchs, J. Opt. Soc. Am. **B4**, 1667 (1987).

[6] B. Yurke, P. Grangier, and R. E. Slusher, J. Opt. Soc. Am. **B4**, 1677 (1987).

[7] V. Chickarmane, and S. V. Dhurandhar, phys. Rev. A **54**, 786 (1996).

[8] P. Grangier, R. E. Slusher, B. Yurke and A. LaPorta. phys. Rev. Lett. **59**, 2153 (1987).

[9] C. M. Caves, Phys. Rev. D **23**, 1693 (1981).

[10] A. F Pace, and M. J. Collett, phys. Rev. A **47**, 3173 (1993).

[11] D.F Walls and G. J Milburn, Quantum Optics (Springer-Verlag Berlin, 1995).

[12] A. Takahashi, Adv. Commun. Syst. **1**, 227 (1965).

[13] D. Stoler. phys. Rev D **1**, 3217 (1970); **4**, 1225 (1971).

[14] L. Wu. H. Kimble, J. Hail and H. Wu. Phys. Rev. Lett. **57**, 2520 (1986).

[15] L. A. Wu. M. Xiao, and H. J. Kimble, J. opt. soc. Am. **B4**, 1465 (1987).

[16] R. M. Shelby, M. D. Levenson, S. H. Perlmulter, R. G. DeVoe and D. F. Walls. Phys. Rev. Lett. **57**, 691 (1986).

[17] R. S. Bondurent, P. Kumar, J. H. Shapiro, and M. Maeda, Phys. Rev. A **30**, 343 (1984).

[18] R. E. Slusher, L. W. Hollberg, B. Yurke, J. C. Mertz and J. E. Valley, phys. Rev. Lett. **55**, 2409 (1985).

[19] M. D. Rein and D. F. Walls, phys. Rev. A **31**, 1622 (1985).

[20] D. F. Walls and P. Zoller, phys. Rev. Lett. **47**, 709 (1981).

[21] L. Mandel, phys. Rev. Lett. **49**, 136 (1982).

[22] W. Vogel and D. G. Welsch, phys. Rev. Lett. **54**, 1802 (1985).

[23] G. J. Milburn and D. F. Walls, Opt. Commun. **39**, 401 (1981).

[24] L. A. Lugiato and G. Strini, Opt. Commun. **41**, 67 (1982).

[25] J. Anwar and M. S. Zubairy, Phys. Rev. A **45**, 1804 (1992).

[26] M. J Collet and D. F. Walls, Phys. Rev. A **32**, 2887 (1985).

[27] M. J. Collet and C. W. Gardiner, Phys. Rev. A **30**, 1386 (1984).

[28] C. M. Savage and D. F. Walls, J. Opt. Soc. Am **B. 4**, 1514 (1987).

[29] R. Vyas and S. Singh, Phys. Rev. A **40**, 5147 (1989).

[30] W. Scheich and J. A. Wheeler, J. Opt. Soc. Am. **B4**, 1715 (1987).

[31] R. Vyas, and S. Singh, Phys. Rev. A **38**, 2423 (1988).

[32] C. M. Caves and D. D. Crouch, J. Opt. Soc. Am. **B4**, 1525 (1987).

[33] B. Daniel and K. Fesseha, Opt. Commu. (to be published).

[34] M. O. Sccully, and M. S. Zubairy, Quantum Optics (Cambridge university press, 1997) (see p. 254, Eq. 8.2.1).