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Addis Ababa University
School of Graduate Studies

Department of Mathematics

Almost distributive lattice
and
Pseudo-complementation on almost distributive lattice

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Declaration Letter

I, Goitom Telele, declare that this project has been compiled by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

Goitom Telele

Addis Ababa University
January 25, 2011

Permission Letter

This is to certify that this project is compiled by Mr Goitom Telele in the department of Mathematics, College of Mathematics and Computational Sciences, Addis Ababa University, under my supervision.

K.venkateswarlu (Prof.)

Addis Ababa University

January 25, 2011

Mathematical Symbols and Abbreviations

Notations	**Meaning**
1. ADL	*almost distributive lattice.*
2. GADL	. *Generalized almost distributive lattice.*
3. POSET	*partial ordered set.*
4. $(P \leq)$	*A set P together with partial ordered relation* $\leq$.
5. *Lub*	*least upper bound (supemum)*
6. glb	*greatest lower bound(infinimum)*
7. L1-L7	*the properties of almost distributive lattice*
8. P1-P3	*the properties Pseudo-complementation on ADL*
9. *G1-G6*	*the properties of generalized almost distributive lattice*
10. Iff	if and only if

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ABSTRACT

The purpose of this study is an attempt to explore the basic concept of almost distributive lattice with 0, Generalized almost distributive lattice and pseudo-complementation on almost distributive lattice with 0 and conceptual difference between lattice, almost distributive lattice with 0 , generalized almost distributive lattice and pseudo-complementation on almost distributive lattice with 0 are described. Basic definition, example lemmas and theorems are given. And it is proved that the pseudo-complementation * on almost distributive lattice with 0 is equationally defined.

Key words: poset, lattice, almost distributive lattice, Generalized almost distributive lattice ,pseudo-complementation on almost distributive lattice.

Introduction

The concept of ADL was introduced by U.M. swamy and G.C.Rao as an algebra $(L,\vee,\wedge)$ of type (2,2,0) which satisfies all properties of distributive lattice except the commutative of $\vee$ *and* $\wedge$ and the right distributive of $\vee$ *over* $\wedge$. It was observed that any one of these three properties converts almost distributive lattice in to a distributive lattice. Latter difficult classes of almost distributive lattices with 0, like pseudo-complementation on almost distributive lattice with 0, Stone almost distributive lattice and other concepts were characterized.

In this paper the basic concepts of almost distributive lattice with 0, Generalized almost distributive lattice and pseudo-complementation on almost distributive lattice with 0 is introduced with basic definition and examples, we tried to show the basic difference between them. We proved basic properties, lemmas, theorems and equivalent conditions. Those lemmas, theorems and equivalent conditions are also proved in step ways. At last we proved the pseudo –complementation * on almost distributive lattice with 0 is equatinally defined.

Preliminaries

We recall certain definitions, examples, and results. Certain concepts of lattice & poset are given. We begin with the following.

Definition: let P be a non –empty set, and let $\leq$ be a binary relation on P. then $\leq$ is called a partial order relation if it satisfies the following condition.

For all a,b and c in P

1) $a \leq a$for all a in P i.e $\leq$ is reflexive
2) $a \leq b$ and $b \leq a \Rightarrow a = b$ i.e $\leq$ is anti symmetry
3) $a \leq b \,\&\, b \leq c \Rightarrow a \leq c$ *i.e* $\leq$ *is transitive*

Definition: A set P together with a partial ordered relation $\leq$ is called a partial ordered set. Simply a poset. i.e. $(P\leq)$ is a poset.

Definition: let $(P \leq)$ be a poset

1) P is said bounded below if $\exists\, x \in P$ such that $x \leq a$ for all a in P
2) P is said bounded above if $\exists\, x \in P$ Such that $a \leq x$ for all a in P
3) P is said bounded if it is bounded below and above

Definition: let $(P \leq)$ be a poset and let $A \subseteq P$

1) An element x in P is called upper bound of A if $a \leq x \;\forall\, a \in A$
2) An element x in P is called lower bound of A if $x \leq a \;\forall\, a \in A$
3) An element x in P is called least upper bound (lub) or supremum of A if
   I) x is upper bound for A

II) If t is another upper bound for A then x ≤ t

4) An element x in P is called greatest lower bound (glb) or infinimum of A if

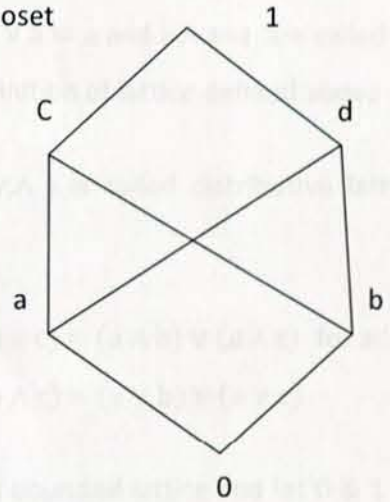
I) x is lower bound for A

II) If t is another lower bound for A then t ≤ x

Definition: a poset P is called a lattice if every pair of elements a,b in P has both supermum and infimum.

Remark: Every lattice is a poset but a poset need not be a lattice.

Example: let see the poset



Is a poset but not a lattice since inf {c,d} doesn't exist.

Definition: A system (L,V,∧) is called a lattice if the following condition holds for any a,b in L

I) L is non-empty

(II) a V b = b V a

$a \wedge b = b \wedge a$

i.e. $\vee$ and $\wedge$ are commutative

(III) $a \vee (b \vee c) = (a \vee b) \vee c$

$a \wedge (b \wedge c) = (a \wedge b) \wedge c$

i.e. $\vee$ and $\wedge$ are associative.

(Vi) $a \vee (a \wedge b) = a$

$a \wedge (a \vee b) = a$

I.e. absorption law holds.

Remark: I) In a lattice $a \vee a = a$ and $a \wedge a = a$ are called idempotent.

II) The two definition of lattice defined above are equivalent.

Definition: a lattice $(L, \vee, \wedge)$ is called distributive lattice if any one of the following condition holds.

I) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all a,b and c in L.

II) $a \vee (b \wedge c) = (a \vee b) \vee (a \vee c)$

Definition : let $(L, \vee, \wedge)$ is bounded lattice and let 0 & 1 be lower and upper bound of L respectively then

I) we say a is a complement of b if $a \wedge b = 0$ and $a \vee b = 1$

II) a complemented lattice is a bounded lattice in which every element has a complement.

III) we say that L is relatively complemented if for each $a \leq b$ in L, the interval $[a,b]$ is complemented.

Definition: an element m of a lattice L is called

I) Minimal element if $a \in L$ $a \leq m$ then $m = a$

II) Maximal element if $a \in L$ $m \leq a$ then $m = a$

Chapter one

Almost distributive lattice

Definition : A system (L,∨,∧,0) of type (2,2,0) is called an almost distributive lattice with 0 or simply ADL with 0 if for any a,b and c in L it satisfies the following condition

L1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

L2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

L3) $(a \vee b) \wedge b = b$

L4) $(a \vee b) \wedge a = a$

L5) $a \vee (a \wedge b) = a$

$L6)$ $0 \wedge a = 0$

$L7)$ $0 \vee a = a$

Example 1: let L be a non-empty set and let ∨ and ∧ be defined by the following tables

∨	0	a	b	c
0	0	a	b	C
a	a	a	a	A
b	b	b	b	B
C	c	a	b	C

∧	0	a	b	c
0	0	0	0	0
a	0	a	b	b
b	0	a	b	c
c	0	c	c	c

Then the system (L,∨,∧ ,o) is ADL with 0

Example 2: let L be a non-empty set and let ∨ and ∧ are defined as follows for a,b and a_0 in L

$$a \vee b = \begin{cases} a \text{ if } a \neq a_0 \\ b \text{ if } a = a_0 \end{cases} \qquad \text{and} \qquad a \wedge b = \begin{cases} b \text{ if } a \neq a_0 \\ a_0 \text{ if } a = a_0 \end{cases}$$

This is called a discrete ADL with 0 and this is also a relatively complemented in this case a_0 is considered as 0 element. Hence system (L,V,$\wedge$,a_0) is ADL as case $a_0 = 0$, the zero element.

Remark: the system (L,V,$\wedge$) is not a lattice. If we see from example1

since $(a \wedge b) \wedge c = b \wedge c = c$ and $a \wedge (b \wedge c) = a \wedge c = b$ Hence; b$\neq$ c i.e. the associative property does not hold hence it is not a lattice .from these fact we can say that an ADL with 0 may not be a lattice.

Lemma 1.1:

let L be ADL with zero and a $\in$L.Then the following holds.

1) $a \wedge 0=0$
2) $a \wedge a=a$
3) $a \vee a=a$
4) $a \vee 0=a$

Proof:

1) Consider $a \wedge 0$

$$a \wedge 0 = (0 \vee a) \wedge 0 \qquad \text{by (L7)}$$
$$= 0 \qquad \text{by (L4)}$$

Hence, $a \wedge 0 = 0$

2) Consider $a \wedge a$

$$a \wedge a = (0 \vee a) \wedge a \qquad \text{by (L7)}$$
$$= a \qquad \text{by (L3)}$$

Hence, $a \wedge a = a$

3) Consider $a \vee a$

$$a \vee a = a \vee (a \wedge a) \qquad \text{by Lemma 1.1(2)}$$

$$= a \qquad \text{by (L5)}$$

Hence, $a \vee a = a$

4) Consider $a \vee 0$

$$a \vee 0 = a \vee (a \wedge 0) \qquad \text{by Lemma 1.1(1)}$$
$$= a \qquad \text{by (L5)}$$

Hence, $a \vee 0 = a$

Definition : Let L be an ADL with 0. For any a and b in L defines $a \leq b$ iff $a \wedge b = a$ or equivalentlya $\vee\, b = b$ then $\leq$ is a partial ordering on L.

Lemma 1.2:

let L be ADL .Then for a and b in L the following holds

1) $a \wedge (a \vee b) = a$
2) $(a \wedge b) \vee b = b$
3) $a \vee (b \wedge a) = a$

Proof: 1) consider $a \wedge (a \vee b)$

$$a \wedge (a \vee b) = (a \wedge a) \vee (a \wedge b) \qquad \text{by(L2)}$$
$$= a \vee (a \wedge b) \qquad \text{by lemma 1.1(2)}$$
$$= a \qquad \text{by (L5)}$$

Hence, $a \wedge (a \vee b) = a$

2) Consider$(a \wedge b) \vee b$

$$(a \wedge b) \vee b = (a \wedge b) \vee (b \wedge b) \qquad \text{by lemma 1.1 (2)}$$
$$= (a \vee b) \wedge b \qquad \text{by (L1)}$$
$$= b \qquad \text{by (L3)}$$

Hence , $(a \wedge b) \vee b = b$

3) Consider $a \vee (b \wedge a)$

$$a \vee (b \wedge a) = (a \wedge a) \vee (b \wedge a) \qquad \text{by lemma 1.1 (2)}$$

$= (a \vee b) \wedge a$ by (L1)

$= a$ by (L4)

Hence, $a \vee (b \wedge a) = a$

Corollary 1.1:

Let L be an ADL let $a, b \in L$ then

1) $a \vee b = a$ if and only if $a \wedge b = b$
2) $a \vee b = b$ if and only if $a \wedge b = a$

Proof: 1)

($\Longrightarrow$) Assume $a \vee b = a$

Now $a \wedge b = (a \vee b) \wedge b$ by assumption

$= b$ by(L3)

Hence, $a \wedge b = b$

($\Longleftarrow$) Assume $a \wedge b = b$

Now $a \vee b = a \vee (a \wedge b)$ by assumption

$= a$ by (L5)

Hence, $a \vee b = a$.

2) ($\Longrightarrow$) Assume $a \vee b = b$

Now $a \wedge b = a \wedge (a \vee b)$ by assumption

$= a$ by Lemma1.2(1)

Hence, $a \wedge b = a$

($\Longleftarrow$) Assume $a \wedge b = a$

Now $a \vee b = (a \wedge b) \vee b$ by assumption

$= b$ by lemma 1.2 (2)

Hence , $a \vee b = b$

Lemma 1.3:

let L be an ADL with 0, let $m \in L$ then the following are equivalent.

1) m is maximal with respect '≤'
2) $m \vee x = m$ for all $x \in L$
3) $m \wedge x = x$ for all $x \in L$

Proof:

(1)$\Longrightarrow$ (2) let m is maximal with respect to'≤' we want to show for all $x \in L$ $m \vee x = m$

$m \leq m \vee X$,Since m is maximal with respect to'≤' then by definition of maximality

$m \vee x = m$

Hence (2) holds

(2) $\Longrightarrow$ (3) Let for all $x \in L$ $m \vee x = m$ we want to show $m \wedge x = x$ for all $x \in L$

$m \wedge x = (m \vee x) \wedge x$ by assumpion

$= x$ by (L3)

Hence, $m \wedge x = x$

(3) $\Longrightarrow$ (1)

let $m \wedge x = x$ *for all* $x \in L$ we want show that m *is maximal with respect* '≤'

$x \leq m$ *for all* $m \in L$ by assumption

m *is maximal* by definition of maximality for any $x \in L$

Hence ,(1) holds

Theorem 1.1:

let L be ADL with 0.The following holds.

1) $(a \wedge b) \vee a = a$ iff $a \wedge (b \vee a) = a$
2) $(b \wedge a) \vee b = b$ iff $b \wedge (a \vee b) = b$
3) $(a \wedge b) = (b \wedge a)$ iff $(a \wedge b) \vee a = a$
4) $(a \vee b) = (b \vee a)$ iff $a \wedge (b \vee a) = a$
5) $(a \vee b) = (b \vee a)$ iff the supremum of a and b exist in Land is $a \vee b$

Proof:

(1)

($\Longrightarrow$)let $(a \wedge b) \vee a = a$ we want show that : $a \wedge (b \vee a) = a$

Consider

$a \wedge (b \vee a)$ then $a \wedge (b \vee a) = (a \wedge b) \vee (a \wedge a)$ by (L2)

$= (a \wedge b) \vee a$ by lemma 1.1(2)

$= a$ by assumption

Hence, $a \wedge (b \vee a) = a$

($\Longleftarrow$) Let $a \wedge (b \vee a) = a$ we want show that $(a \wedge b) \vee a = a$

Consider

$(a \wedge b) \vee a$ then $(a \wedge b) \vee a = (a \wedge b) \vee (a \wedge a)$ by lemma 1.1(2)

$= a \wedge (b \vee a)$ by (L2)

$= a$ by assumption

Hence, $(a \wedge b) \vee a = a$

(2)

$(\Longrightarrow)$let $(b\wedge a)\vee b = b$ w.s.t $b\wedge(a\vee b)$

Consider

$b\wedge(a\vee b)$ then $b\wedge(a\vee b) = (b\wedge a)\vee(b\wedge b)$ by (L2)

$= (b\wedge a)\vee b$ by lemma 1.1(2)

$= b$ by assumption

Hence , $b\wedge(a\vee b) = b$

$(\Longleftarrow)$ let $b\wedge(a\vee b) = b$ we want show that $(b\wedge a)\vee b = b$

Consider

$(b\wedge a)\vee b$ then $(b\wedge a)\vee b = (b\wedge a)\vee(b\wedge b)$ by lemma 1.1(2)

$= b\wedge(a\vee b)$ by (L2)

$= b$ by assumption

Hence, $(b\wedge a)\vee b = b$

(3)

$(\Longrightarrow)$ let$(a\wedge b) = (b\wedge a)$ we want show that $(a\wedge b)\vee a = a$

Consider

$(a\wedge b)\vee a$ then $(a\wedge b)\vee a = (a\wedge b)\vee(a\wedge a)$ by lemma 1.1(2)

$= (b\wedge a)\vee(a\wedge a)$ by assumption

$= (b\wedge a)\vee a$ by (L2)

$= a$ by (L3)

Hence, $(a\wedge b)\vee a = a$

$(\Longleftarrow)$ let $(a\wedge b)\vee a =$ we want show that $(a\wedge b) = (b\wedge a)$

Consider

$b\wedge a$ then $b\wedge a = b\wedge[(a\wedge b)\vee a]$ by assumption

$= [b\wedge(a\wedge b)]\vee(b\wedge a)$ by (L2)

$= (a\wedge b)\vee[a\wedge(b\wedge a)]$ as $b\wedge(a\wedge b) = a\wedge b$

$= a\wedge[b\vee(b\wedge a)]$ by (L1)

$= a \wedge b$ by (L5)

Hence, $(a \wedge b) = (b \wedge a)$

(4)

($\Rightarrow$) let $(a \vee b) = (b \vee a)$ we want show that $a \wedge (b \vee a) = a$

Consider

$a \wedge (b \vee a)$ then $a \wedge (b \vee a) = a \wedge (a \vee b)$ by assumption

$= a$ by lemma 1.2(1)

Hence, $a \wedge (b \vee a) = a$

($\Leftarrow$) let $a \wedge (b \vee a) = a$ we want show that $(a \vee b) = (b \vee a)$

Consider

$a \vee b$ then $a \vee b = [a \wedge (b \vee a)] \vee [b \wedge (b \vee a)]$ by assumption & *lemma* 1.2(1)

$= (a \vee b) \wedge (b \vee a)$ by (L2)

$= [(a \vee b) \wedge b] \vee [(a \vee b) \wedge a]$ by (L1)

$= b \vee a$ by (L3)& (L4)

Hence, $(a \vee b) = (b \vee a)$

(5)

($\Rightarrow$) let

$(a \vee b) = (b \vee a)$ we want show that the supremum of a and b exist in Land is $a \vee b$

for any a and b in L $a \leq a \vee b$ and $b \leq a \vee b$

Hence $a \vee b$ is upper bound for a and b let t be other upper bound for a and b

We went to show $a \vee b \leq t$

$(a \vee b) \wedge t = (a \wedge t) \vee (b \wedge t)$ by (L1)

$= a \vee b$ because by assumption t is upper bound for a and b

$a \vee b \leq t$ hence $a \vee b$ exist and is least upper boud for a and b

Lemma1.4:

let Lbe an ADL with 0 for any a , b &$c \in L$

$(a \vee b) \wedge c = (b \vee a) \wedge c$

Proof:

for $a, b \& c \in L$ we have $(a \wedge c) \leq c$ and $(b \wedge c) \leq c$

$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ by (L1)

$= a \vee b$ as $a \leq c$ and $b \leq c$

Similarly

$(b \vee a) \wedge c = (b \wedge c) \vee (a \wedge c)$ by (L1)

$= b \wedge a$ as $a \leq c$ and $b \leq c$

Then if the supreumum of a and b exist and is equal to $a \vee b$ then $a \vee b = b \vee a$

Hence, $(a \vee b) \wedge c = (b \vee a) \wedge c$

Lemma 1.5:

The binary operation $\wedge$ is assocative

Proof:

$(a \wedge b) \wedge c = (a \wedge b) \wedge \left[c \vee \left(a \wedge (b \wedge c)\right)\right]$ by corolary 1.1

$= [(a \wedge b) \wedge c] \vee \left[(a \wedge b) \wedge \left(a \wedge (b \wedge c)\right)\right]$ by (L2)

$= (a \wedge b) \vee [a \wedge (b \wedge c)$ as $a \wedge b \leq c \,\&(a \wedge b) \geq a \wedge (b \wedge c)$

$= a \wedge [b \vee (b \wedge c)]$ by (L2)

$= a \wedge b$ by (L2)

$= a \wedge (b \wedge c)$ by corolary 1.1

Hence, $\wedge$ is assocative.

Corollary 1.2:

let L be ADL with 0 then $a \wedge b = b \wedge a$ when ever $a \leq b$

Proof: we have when $a \leq b$ then by corollary 1.1

$a \wedge b = a \quad \& \quad a \vee b = b \quad or \quad b \wedge a = a \quad \& \quad b \vee a = b$

$a \wedge b = a = b \wedge a$ hence $a \wedge b = b \wedge a$ when ever $a \leq b$

Theorem 1.2:

let L be a distributive ADL with 1 then $a \wedge 1 = a$ iff L is bounded for $a \in L$

Proof: suppose $a \wedge 1 = a$ we want show that Lis bounded

Consider

$1 \vee a$ then $1 \vee a = 1 \vee (a \wedge 1)$ by assumption

$= (1 \vee a) \wedge (1 \vee 1)$ by (L2)

$= (1 \vee a) \wedge 1$ by lemma 1.1(3)

$= 1$ by lemma 1.2(3)

Also consider

$0 \wedge a$ then $0 \wedge a = 0 \wedge (a \wedge 1)$ by assumption

$= (0 \wedge a) \wedge 1$ by lemma 1.5

$= 0 \wedge 1$ by (L6)

$= 0$ by(L6)

Hence, L is bounded

Conversely assume L is bounded we want show that $a \wedge 1 = a$

Consider

$a \wedge 1$ then $a \wedge 1 = a \wedge (1 \vee a)$ as Lis bdd with 1

$= (a \wedge 1) \vee (a \vee a)$ by (L2)

$= (a \wedge 1) \vee a$ by lemma1.1(2)

$= a$ by lmma 1.2(1)

Hence, $a \wedge 1 = a$

Definition: A system $(G, \vee, \wedge)$ is called a generalized almost distributive lattice in short GADL if it satisfies the following axioms.

G1) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

G2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

G3) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

G4) $a \wedge (a \vee b) = a$

G5) $(a \vee b) \wedge a = a$

G6) $(a \wedge b) \vee b = b$

Example A: let G={a,b,c} define the binary operation ∨ & ∧ on G as follows .

∨	a	b	c
a	a	b	a
b	b	b	b
C	c	c	c

∧	a	b	c
a	a	a	c
b	a	b	c
c	a	a	c

Hence, the system (G, ∨,∧) is a generalized almost distributive lattice.

Example B: let G={a,b,c} define the binary operation ∨ & ∧ on G as follows .

∨	a	b	c
a	a	a	a
b	a	b	b
C	c	c	c

∧	a	b	c
a	a	a	c
b	b	b	c
C	b	b	c

Hence, the system (G, ∨,∧) is a generalized almost distributive lattice.

Remark: A generalized almost distributive lattice need not be an almost distributive lattice. If we see example A is not almost distributive lattice for$(c \vee b) \wedge b \neq b$, but it is GADL. From such condition we see that GADL need not be ADL.

Lemma 1.6:

for any a,b in G where G is generalized almost distributive lattice we have the following

1) a∨a=0

2) $a \wedge a = a$

3) $a \vee (a \wedge b) = a$

4) $a \vee (a \wedge b) = a$

proof: 1) $a \vee a = ((a \vee a) \wedge a) \vee a$ by (G5)

$= a$ by(G6)

2) $a \wedge a = a \wedge (a \vee a)$ by lemma 1.6(1)

$= a$ by (G4)

3) $a \vee (a \wedge b) = (a \vee a) \wedge (a \vee b)$ by (G3)

$= a \wedge (a \vee b)$ by lemma 1.6(1)

$= a$ by(G4)

4) $a \vee (b \wedge a) = (a \vee b) \wedge (a \vee a)$ by(G3)

$= (a \vee b) \wedge a$ by lemma 1.6(1)

$= a$ by (G5)

Lemma 1.7:

for any a,b in G where G is generalized almost distributive lattice we have the following

.1) if $a \wedge b = b$ then $a \vee b = a$

2) $a \vee b = b$ if and only if $a \wedge b = a$

proof: let a,b $\in$G

1) suppose $a \wedge b = b$ then $a \vee b = a \vee (a \wedge b)$ by supposition

$= a$ by lemma 1.6(3)

2) suppose $a \vee b = b$ then $a \wedge b = a \wedge (a \vee b)$ by supposition

$= a$ by (G4)

conversely let $a \wedge b = a$, $a \vee b = (a \wedge b) \vee b$ by supposition

$= b$ by(G6)

Remark: In generalized almost distributive lattice the converse of lemma 1.7(1) does not hold. In example A above we observe thatc $\vee$ b = but $c \wedge b = a$.

conversely let $a \wedge b = a$, $a \vee b = (a \wedge b) \vee b$ by supposition

$= b$ by(G6)

Remark: In generalized almost distributive lattice the converse of lemma 1.7(1) does not hold. In example A above we observe that$c \vee b = but\ c \wedge b = a$.

Chapter two

Pseudo –complementation on almost distributive lattices

Definition: $let\ (L,\vee,\wedge,0) beADL\ with\ 0\ then\ the\ unary\ opration$ a ⟶ a^* $is\ called$ a pseudo-complementation on L if it satisfies the following conditions.

(p1) $a \wedge b = 0 \Longrightarrow a^* \wedge b = b$

(p2) $a \wedge a^* = 0$

(p3) $(a \vee b)^* = a^* \wedge b^*$

Example 2.1: let X be discrete ADL with 0 and with at least two elements say 1,2 other than 0 then $(X^3,\vee,\wedge,0)\ is$ ADL where ∨ & ∧ are defined coordinates .now for any X∈ X^3we write $|X|$ for the number non-zero entries in X .defined * on X^3as follows ,for any X∈ X^3 i=1,2,3

$$X_i^* = \begin{cases} 0 & if\ X_i \neq 0 \\ 1\ if\ X_i = 0\ and\ |X| = 1\ and 0^* = (2,2,2) \\ 2 & if\ X_{i=0} and\ |x| > 1 \end{cases}$$

Then $(X^3,\vee,\wedge,0)\ is$ ADLwith (0, 0, 0) as o element .in this case $(X^3,\vee,\wedge,0)$ satisfies (p1) and (p2) but not (p3) .if we take a=(1,0,0) and b=(0,1,0)then a^* =(0,1,1) and b^* =(1,0,1) but if we see a∨b=(1,1,0) and $(a \vee b)^*$=(0,0,2) then (p3) is not satisfied.

Example 2.2: Let L be ADL with 0 with at least two elements defined
$a^* = 0 for\ all\ in\ L\ then$ L satisfies (p2) & (p3) but not (p1). If we take $0 \neq b \in L\ then\ 0 \wedge b = 0\ and\ 0^* \wedge b = 0 \neq b$ hence (p1) is not satisfied.

Example 2.3: let L be bounded distributive lattice with bounds 0≠1 now for any a ∈ L defined a^* $= 1\ for\ all\ a \in L$.then L satisfies (p1) & (p3) but not (p2).

Remark: (p1), (p2)&(p3) are independent.

Example 2.3: let L=$\{a, b, c, 0\}$ be a non-empty set and let $\vee$ and $\wedge$ is defined by the following tables

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
C	c	a	b	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	b
b	0	a	b	c
c	0	c	c	c

And defined $x^* = 0\ if x \neq 0\ and\ 0^* = \mathrm{a}\ then\ (L\ \vee, \wedge, 0)$ is ADL with 0 and x$\longrightarrow x^*$ is a pseudo-complementation on L. Remember that $(L,\vee,\wedge)$ is not a lattice.

Proof: for all a & bin L

1) $x \wedge 0 = 0\ for\ all\ x\ in\ L\ and\ x^* \wedge 0 = 0$
2) $x \wedge x^* = 0\ for\ all\ x\ in\ L\ as\ x^* = 0 for\ x \neq 0\ and\ 0 \wedge 0^* = 0 \wedge \mathrm{a} = 0$
3) $(\mathrm{a} \vee \mathrm{b})^* = \mathrm{a}^* \wedge b^* that\ is\ for\ any\ values\ in\ the\ table\ this\ proprty\ holds$

Hence, $x \longrightarrow x^*$ is a pseudo-complementation on L.

Example 2.4: let (R, +, 0) be a commutative regular ring. to each $a \in \mathrm{L}$ let a^0 be the unique idempotent element in R such that $\mathrm{aR} = \mathrm{a}^0\mathrm{R}$. and defined for any a and b in R

1) $\mathrm{a} \wedge \mathrm{b} = \mathrm{a}^0\mathrm{b}$
2) $\mathrm{a} \vee \mathrm{b} = \mathrm{a} + (1\_\mathrm{a}^0)\mathrm{b}$
3) $\mathrm{a}^* = 1\_\mathrm{a}^0$

Then (R,V,Λ ,0) Is ADL with 0. And * is a pseudo-complementation on R

Proof: let $\mathrm{a} \wedge \mathrm{b} = o\ we\ want\ to\ show$: $\mathrm{a}^* \wedge \mathrm{b} = \mathrm{b}$

But by the property of a & b defined in the problem $a \wedge b = a^0 b$ hence by our assumption $\mathrm{a} \wedge \mathrm{b} = 0 \Longrightarrow \mathrm{a}^0\mathrm{b} = 0$ then consider $\mathrm{a}^* \wedge \mathrm{b}$

$a^* \wedge b = (1 - a^0) \wedge b$ as $a^* = (1 - a^0)$

$= (1 - a^0)^0\, b$ as $a \wedge b = a^0 b$

$= (1 - a^0)b$ as $aR = a^0R$

$= b - a^0 b$ by commutative ring

$= b$ as from assumption $a^0 b = 0$

Hence, $a \wedge b = o \implies a^* \wedge b = b$

Also for any $a \in L$ we want to show : $a \wedge a^* = 0$

Consider $a \wedge a^*$then $a \wedge a^* = a \wedge (1 - a^0)$ by defination

$= a^0(1_{-}\, a^0)$ by definition

$= a^0 \text{-}\, a^0$ a^0 is idempotent

$= 0$

Hence, $a \wedge a^* = 0$ for any $a \in L$

In similar way $(a \vee b)^* = a^* \wedge b^*$ as a result * is a pseudo-complementation on R.

Remark: in case of distributive lattice with 0 it is well known that an element a^* satisfying the property (p1) & (p2) is unique if it exists and (p3) is a consequence of (p1) & (p2) and hence, there can be at most one pseudo-complementation .However; in ADL with 0 there can be several pseudo-complementation consider the following example.

Example 2.5: let $(X \vee, \wedge, 0)$ be a discrete ADL with 0 .for any x≠0in X define

$$a^* = \begin{cases} o \text{ if } a \neq 0 \\ x \text{ if} a = 0 \end{cases}$$

Then * is a pseudo-complementation on X here with each x≠ 0 in X we obtain a pseudo-complementation on $(X \vee, \wedge, 0)$

Proof:

(1) let$a \wedge b = 0$ *we want show that* $a^* \wedge b = b$ since it is a discrete ADL

$a \wedge b = 0$ If $a = 0$ then

$a^* \wedge b = (0^*) \wedge b$ as $a = 0$

$= a \wedge b$ by function define

$= b$ B/c it is discrete let $a \wedge b = b$ if $a \neq 0$

(2) *for* $a \in X$ we want show $a \wedge a^* = 0$

Consider $a \wedge a^*$ case 1: if $a = 0$ $a \wedge a^* = 0 \wedge 0^*$ as $a = 0$

$= 0 \wedge x$ by the function defined

$= 0$ b/c it is discrete $a \wedge b = 0$ if $a = 0$

Case 2: if $a \neq 0$ $a \wedge a^* = a \wedge 0$ by the function defined

$= 0$ by definition of ADL

(3) let a &b $\in X$ *we want show that* $(a \vee b)^* = a^* \wedge b^*$

Consider $(a \vee b)^*$ case 1: if $a \neq 0$

$(a \vee b)^* = a^*$ by definition of discerte as $a \vee b = a$

$= 0$ since $a \neq 0$

case 2: if $a = 0$ $(a \vee b)^* = b^*$ as it is discerte $a \vee b = b$

$= 0$ as $b \neq 0$

from case 1 $a^* = 0$ & $b^* = 0$ $a^* \wedge b^* = 0 \wedge 0 = 0$

from case 2 $a^* = x$ as $a = 0$ & $b^* = 0$ as $b \neq 0$ $a^* \wedge b^* = x \wedge 0 = 0$

Hence, $(a \vee b)^* = a^* \wedge b^*$

Therefore $(X\ \vee, \wedge, 0)$ be a discrete ADL with 0 and * is a pseudo-complementation on X.

Lemma 2.1:

Let L be ADL with 0 and * be a pseudo-complementation on L then for any a , b in L we have the following.

1) 0^* *is maximal*
2) If a is maximal then $a^* = 0$
3) $0^{**} = 0$
4) $a^* \wedge a = 0$
5) $a^{**} \wedge a = a$

6) $a^* = a^{***}$

Proof: (1)

for any $a \in L$ $0 \wedge a = 0$ by (L6)

$0^* \wedge a = a$ by definition Pseudo-complement From the above 0^* is maximal by definition of maximality

(2) suppose a is maximal if $a \leq a \vee a^*$ and hence $a = a \vee a^*$ because a is maximal it hold Then $a^* = (a \vee a^*)^* = a^* \wedge a^{**}$ by definition of pseudo-complementation

$= 0$

Hence $a^* = 0$

(3) From lemma 2.1(1) we proved that 0^* is maximal & from lemma 2.1(2) we also proved that the complement of a maximal element is zero then from these two concepts we conclude that $0^{**} = 0$

(4) From the concept of ADL $a \wedge b = 0$ iff $b \wedge a = 0$ then since it is a pseudo-complementation we have that $a \wedge a^* = 0$ then from the two concepts we get that $a^* \wedge a = 0$

(5) Since it is pseudo-complementation we have that $a \wedge a^* = 0$ and from lemma 2.1(4) we have $a^* \wedge a = 0$ then from this $a^{**} \wedge a = a$

(6) From lemma 2.1(5) we have $a^{**} \wedge a = a$ then $a^{**} = a^{**} \vee a$ by corollary 1.1

$a^{***} = (a^{**} \vee a)^*$ Complementing the equation

$= a^{***} \wedge a^*$ by (P3)

$= a^*$ by lemma 2.1(5)

Lemma 2.2:

Let L be ADL with o and * pseudo-complementation on L, Then for any $a, b \in L$ we have the following,

(1) $a^* = 0$ if and only if a^{**} is maximal
(2) $a^* \leq 0^*$
(3) $a^* \wedge b^* = b^* \wedge a^*$
(4) $a \leq b \Rightarrow b^* \leq a^*$
(5) $a^* \leq (a \wedge b)^*$ and $b^* \leq (a \wedge b)^*$
(6) $a^* \leq b^* \Leftrightarrow b^{**} \leq a^{**}$
(7) $a = 0 \Leftrightarrow a^{**} = 0$

Proof: (1)

($\Longrightarrow$) assume $a^* = 0$ *we want show that* a^{**} is maximal

$(0)^* = 0$ by lemma 2.1(1)

$(a)^{**}$ *is maximal* from the above step& assumption $a^* = 0$

($\Longleftarrow$) assume $(a)^{**} =$ *is maximal* *we went show that* $a^* = 0$

Since $(a)^{**} =$ *is maximal* then $((a)^{**})^* = 0$ by lemma 2.1(2)

$a^* = 0$ by lemma 2.1(6)

(2) $a = a \vee 0$ by lemma 1.1(4)

$a^* = (a \vee o)^*$ complementing both side of equation

$a^* = a^* \wedge 0^*$ by (p3)

$a^* \leq 0^*$ as it is Poset

(3) $a^* \leq 0^*, b^* \leq 0^*$ by lemma 2.2(2)

Therefore, $a^* \wedge b^* = b^* \wedge a^*$

(4) Assume $a \leq b$ *we want show that* $b^* \leq a^*$

we know $a \leq b \Rightarrow b = a \vee b$ by defination

$b^* = (a \vee b)^*$ complementing both side

$= a^* \wedge b^*$ by (p3)

$= b^* \wedge a^*$ by lemma2.2(3)

$b^* = b^* \wedge a^*$

Hence, $b^* \leq a^*$ as it is a poset

(5) From the definition of ADL(L4)

$a = a \vee (a \wedge b)$ by (L4)

$a^* = \left(a \vee (a \wedge b)\right)^*$ complementing both side

$a^* = a^* \wedge (a \wedge b)^*$ by (p3)

$a^* \leq (a \wedge b)^*$ as it is poset

and since $a \wedge b \leq b$ and $a \wedge b \leq a$

$b^* \leq (a \wedge b)^*$ by lemma 2.2(4)

$a^* \leq (a \wedge b)^*$ and $b^* \leq (a \wedge b)^*$. by lemma 2.2(4)

Hence, $a^* \leq (a \wedge b)^*$ and $b^* \leq (a \wedge b)^*$.

(6) $a^* \leq b^* \Leftrightarrow b^{**} \leq a^{**}$

($\Rightarrow$) Assume $a^* \leq$ we want show $b^{**} \leq a^{**}$

$a^* \leq b^* \Rightarrow b^{**} \leq a^{**}$ by lemma 2.2(4)

Conversely, let $b^{**} \leq a^{**}$ *we want show that* $a^* \leq b^*$

Since $b^{**} \leq a^{**} \Rightarrow a^{***} \leq b^{***}$ by lemma 2.2(4)

$a^* = a^{***}$ by lemma 2.1(6)

$a^{***} \leq b^{***}$ by above step

$a^* \leq b^*$ by lemma 2.1(6)

(7) $a = 0 \Leftrightarrow a^{**} = 0$

($\Rightarrow$) Let $a = 0$ *we want show that* $a^{**} = 0$

we have $0^{**} = 0$. by Lemma 2.1(3)

. $a^{**} = 0$ by assumption $a = 0$

Hence, $a^{**} = 0$

($\Leftarrow$) Assume $a^{**} = 0$ *we want show that* $a = 0$

From the definition of ADL we have $0 = 0 \wedge a$

$0 = 0 \wedge a$ by (L6)

$= a^{**} \wedge a$ by the assumption

$= a$ by Lemma 2.1(5)

Hence, $a = 0$.

Lemma 2.3:

Let L be an ADL with 0 and $*$ a pseudo- complementation on L, then for any $a, b \in L$ the following are equivalent.

(1) $a \wedge b = 0$

(2) $a^{**} \wedge b = 0$

(3) $a^{**} \wedge b^{**} = 0$

(4) $a \wedge b^{**} = 0$

(5) Proof: ($1 \Longrightarrow 2$)

Assume $a \wedge b = 0$. *we want show that* $a^{**} \wedge b = 0$

From assumption $a \wedge b = 0 \quad \Longrightarrow \quad a^{*} \wedge b = b$ by (p1)

Then consider $a^{**} \wedge b$

$a^{**} \wedge b = a^{**} \wedge a^{*} \wedge b$ by substitution

$= 0 \wedge b$ by Lemma 2.1(5)

$= 0$ by (L6)

Hence, $a^{**} \wedge b = 0$.

($2 \Longrightarrow 3$) Assume $a^{**} \wedge b = 0$ *we want show that* $a^{**} \wedge b^{**} = 0$

Consider $a^{**} \wedge b = 0$

$a^{**} \wedge b = 0 \Rightarrow b \wedge a^{**} = 0$ since $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$

$b^{*} \wedge a^{**} = a^{**}$ by (p1)

Then consider $a^{**} \wedge b^{**}$

$a^{**} \wedge b^{**} = b^{*} \wedge a^{**} \wedge b^{**}$ Substitution

$= a^{**} \wedge b^{*} \wedge b^{**}$ $a \wedge b \wedge c = b \wedge a \wedge c$

$= a^{**} \wedge 0$ by Lemma 2.1(5)

$= 0$ by Lemma 1.1(1)

(3⟹4) Assume $a^{**} \wedge b^{**} = 0$ *we want show that* $a \wedge b^{**} = 0$

Consider $a^{**} \wedge b^{**} = 0$

$a^{**} \wedge b^{**} = 0$ assumption

$a \wedge a^{**} \wedge b^{**} = 0$ as $a \wedge a^{**} = a^{**}$

$a^{**} \wedge a \wedge b^{**} = 0$ $a \wedge b \wedge c = b \wedge a \wedge c$

$a \wedge b^{**} = 0$ by Lemma 2.1(5)

Hence, $a \wedge b^{**} = 0$.

(4⟹1) Assume $a \wedge b^{**} = 0$ *we want show that* $a \wedge b = 0$

Consider $a \wedge b$

$a \wedge b = a \wedge b^{**} \wedge b$ by Lemma 2.1(5)

$= 0 \wedge b$ by assumption

$= 0$ by (L6)

Hence, $a \wedge b = 0$.

Lemma 2.4:

Let L be an ADL with 0 and let * be a pseudo-complementation on L.Then for anya, $b \in L$ following holds.

1) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$
2) $(a \wedge b)^{*} = (b \wedge a)^{*}$
3) $(a \vee b)^{*} = (b \vee a)^{*}$

Proof:

(1) let a, b *in L then*

$a^{**} \wedge b^{**} \wedge (a \wedge b)^{**} = a^{**} \wedge$ by (p3)

$= a^{**} \wedge (a \wedge b)^{**}$ by lemma 2.2(6)

$= [a^* \vee (a \wedge b)^*]^*$ by (p3)

$= (a \wedge b)^{**}$ by lemma2.2(6)

$(a \wedge b)^{**} \leq a^{**} \wedge b^{**}$ ……………………………………………… σ

and consider $a \wedge b \wedge (a \wedge b)^*$

then $a \wedge b \wedge (a \wedge b)^* = 0$ by (p2)

$a^{**} \wedge b \wedge (a \wedge b)^* = 0$ by lemma 2.3 ($1 \Longrightarrow 2$)

$b \wedge a^{**} \wedge (a \wedge b)^* = 0$ as $a \wedge b \wedge c = b \wedge a \wedge c$

$b^{**} \wedge a^{**} \wedge (a \wedge b)^* = 0$ by lemma 2.3 ($1 \Longrightarrow 2$)

$(a \wedge b)^* \wedge a^{**} \wedge b^{**} = 0$ as $a \wedge b = 0$ iff $b \wedge a = 0$

$(a \wedge b)^{**} \wedge a^{**} \wedge b^{**} = a^{**} \wedge b^{**}$ by (p1) then

$a^{**} \wedge b^{**} \leq (a \wedge b)^{**}$ ……………………………………………… ω

hence; from σ and ω we conclude that $a^{**} \wedge b^{**} = (a \wedge b)^{**}$

(2) consider $(a \wedge b)^*$

$(a \wedge b)^* = (a \wedge b)^{***}$ by lemma 2.1(6)

$= (a^{**} \wedge b^{**})^*$ by lemma 2.4(1)

$= (b^{**} \wedge a^{**})^*$ by lemma2.4(4)

$= (b \wedge a)^{***}$ by lemma 2.4(1)

$= (b \wedge a)^*$ by lemma 2.1(6)

hence, $(a \wedge b)^* = (b \wedge a)^*$

(3) consider $(a \vee b)^*$

$(a \vee b)^* = a^* \wedge b^*$ by (p3)

$= (b^* \wedge a^*)$ by lemma 2.2(4)

$= (b \vee a)^*$ by (p3)

hence, $(a \vee b)^* = (b \vee a)^*$

Theorem 2.1:

let $(L, \vee, \wedge, 0)$ be ADL with 0. Then a unary operation $*: L \rightarrow L$ is a pseudo-complementation on L if and only if the following equations hold.

1) $a \wedge a^* = 0$

2) $a^{**} \vee a = a^{**}$

3) $(a \vee b)^* = a^* \wedge b^*$

4) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

5) $0^* \wedge a = a$

Proof: $(\Longrightarrow)$

Assume * is a pseudo-complementation on L. *we want show that* property 1→5 in the theorem satisfies. i.e

(1) $a \wedge a^* = 0$

(2) $a^{**} \vee a = a^{**}$

(3) $(a \vee b)^* = a^* \wedge b^*$

(4) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

(5) $0^* \wedge a = a$

Since * is a pseudo-complementation on L. then by definition of pseudo-complementation we have the following

(p1) $a \wedge b = 0 \Longrightarrow a^* \wedge b = b$

(p2) $a \wedge a^* = 0$

(p3) $(a \vee b)^* = a^* \wedge b^*$

Then from the definition of pseudo-complementation 1&3 are satisfied

$0 \wedge a = 0$ by (L6)

$0^* \wedge a = a$ by (p1)

Hence,5 holds.

$(a \wedge b)^{**} = a^{**} \wedge b^{**}$ by lemma 2.4(1)

Hence,4 holds.

$a^{**} \wedge a = a$ by lemma 2.1(5)

$a \leq a^{**}$ by definition of poset

$a^{**} \vee a = a^{**}$ by definition of poset

Hence,2 holds.

Conversely assume property 1→5 in the theorem satisfies. i.e.

(1) $a \wedge a^* = 0$
(2) $a^{**} \vee a = a^{**}$
(3) $(a \vee b)^* = a^* \wedge b^*$
(4) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$
(5) $0^* \wedge a = a$

we want show that * is a pseudo-complementation on L. i.e. the following holds.

(p1) $a \wedge b = 0 \Longrightarrow a^* \wedge b = b$
(p2) $a \wedge a^* = 0$
(p3) $(a \vee b)^* = a^* \wedge b^*$

(p2)&(p3) are true from property 1&3 of our assumption.

We want to show $a \wedge b = 0 \Longrightarrow a^* \wedge b = b$

let a&$b \in L$ *such that* $a \wedge b = 0$

then $b = b^{**} \wedge b$ by lemma 2.1(5)
$= 0^* \wedge b^{**} \wedge b$ by assumption (5)
$= (a^* \wedge a^{**})^* \wedge b^{**} \wedge b$ by lemma 2.1(5)
$= (a \vee a^*)^{**} \wedge b^{**} \wedge b$ by assumption (3)
$= [(a \vee a^*) \wedge b)]^{**} \wedge b$ by assumption (4)
$= [(a \wedge b) \vee (a^* \wedge b)]^{**} \wedge b$ by (L1)
$= [(0) \vee (a^* \wedge b)]^{**}$by assumption $a \wedge b = 0$
$= (a^* \wedge b)^{**} \wedge b$ by (L7)
$= a^{***} \wedge b^{**} \wedge b$ by assumption (4)
$= a^{***} \wedge b$ by lemma2.1(5)
$= a^* \wedge b$ by lemma2.1(6)

Hence, $a \wedge b = 0 \Longrightarrow a^* \wedge b = b$ then (p1) holds.

Therefore * is a pseudo-complementation on L.

Theorem 2.2:

let (L,V,∧ ,0) be ADL with 0. Then a unary operation $*: L \to L$ is a pseudo-complementation on L if and only if the following equations hold.

1) $a^* \wedge b = (a \wedge b)^* \wedge b$
2) $0^* \wedge a = a$
3) $0^{**} = 0$
4) $(a \vee b)^* = a^* \wedge b^*$

Proof:

(⟸) Assume that * satisfies the given equation 1→4 i.e.

1) $a^* \wedge b = (a \wedge b)^* \wedge b$
2) $0 \wedge a = a$
3) $0^{**} = 0$
4) $(a \vee b)^* = a^* \wedge b^*$

we want show that * is a pseudo-complementation on L i.e. the following holds.

(p1) $a \wedge b = 0 \Longrightarrow a^* \wedge b = b$

(p2 $a \wedge a^* = 0$

(p3) $(a \vee b)^* = a^* \wedge b^*$

let $a\&b \in L$ such that $a \wedge b = 0$ we need toshow $a^* \wedge b = b$
then consider $a^* \wedge b$

$a^* \wedge b = (a \wedge b)^* \wedge b$ by assumption (1)

$= 0^* \wedge b$ by our assumption we consider $a \wedge b = 0$

$= b$ by assumption (1)

Hence, $a \wedge b = 0 \Longrightarrow a^* \wedge b = b$ then (p1) holds.

Consider $a \wedge a^*$ then $a \wedge a^* = 0$ by lemma 2.1(4)

Hence,(2) holds. And (p3) holds from assumption (4)

Therefore * is a pseudo-complementation on L.

(⟹) Assume * is a pseudo-complementation on L i.e. the following holds.

(p1)$a \wedge b = 0 \Longrightarrow a^* \wedge b = b$

(p2 $a \wedge a^* = 0$

(p3) $(a \vee b)^* = a^* \wedge b^*$

we want show that * satisfies the given equation 1→4 i.e.

1) $a^* \wedge b = (a \wedge b)^* \wedge b$
2) $0 \wedge a = a$
3) $0^{**} = 0$
4) $(a \vee b)^* = a^* \wedge b^*$

Since * is a pseudo-complementation on L. then $a \wedge b = 0 \Longrightarrow a^* \wedge b = b$

then consider $(a \wedge b)^* \wedge b$

$(a \wedge b)^* \wedge b = 0^* \wedge b$ from assumption we let $a \wedge b = 0$

$= b$ by theorem 2.1

$a^* \wedge b = (a \wedge b)^* \wedge b$ above result and our assumption

Hence, (1)holds.

$0 \wedge a = 0$ by (L6)

$0^* \wedge a = a$ by (p1)

Hence, (2) holds.

$0^{**} = 0$ by lemma2.1(3)

Hence, (3) holds.

Same is true for (4) by our assumption. Hence; * satisfies the given equation 1→4.

Theorem 2.3:

let L be a relatively complemented ADL with 0 and with maximal element m_0. For any a in L define a^* to be the complement of a in $[0, a\vee m_0]$ then * is a pseudo-complementation on L.

Proof: let a&b in L if $a\wedge b = 0$ then we want show that $a^* \wedge b = b$

$a^* \wedge b = (a \vee a^*) \wedge b$ by definition of relative complement

$= (a \vee m_0) \wedge b$ by the defination given $a \vee a^* = a \vee m_0$

$= m_0 \wedge b$ as m_0 is maximal

$= b$ as m_0 is maximal

Hence, $a^* \wedge b = b$

since m_0 is maximal $a \wedge a^* = 0$ by definition of a^* then $(a \vee b) \wedge a^* \wedge b^* = 0$ now conside$(a \vee b) \vee (a^* \wedge b^*)$

$(a \vee b) \vee (a^* \wedge b^*) = ((a \vee b) \vee a^*) \wedge ((a \vee b) \vee b^*)$

$= \big(b \vee (a \vee a^*)\big) \wedge (a \vee (b \vee b^*))$ by lemma 1.4

$= \big(b \vee (a \vee m_0)\big) \wedge (a \vee (b \vee m_0))$ as $a \vee a^* = a \vee m_0$

$= (a \vee b) \vee m_0$ by lemma 1.1(4)

there fore $(a \vee b)^* = (a^* \wedge b^*)$

Hence,*is a pseudo-complementation on L.

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