Addis Ababa University
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Graduate Seminar Report

On

Classical Ramsey Numbers

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Preface

This seminar paper is on classical Ramsey numbers.

Ramsey's theorem states that if $p,q \geq 2$ are integers, then there is a positive integer n such that if we color the edges of k_n using two colors, red and blue, it is impossible to color the edges of the k_n without forming either a red k_p or a blue k_q. In short we formulate it as $k_n \rightarrow k_p, k_q$ (read as k_n arrows k_p, k_q). The smallest value of such n is denoted by $r(p,q)$, known as the Ramsey number.

Ramsey died too young to obtain further results. Till now very little is known about specific values of Ramsey numbers.

In this paper, I raised the question "Why is it impossible to color the edges of $k_{r(p,q)}$ without forming either a red k_p or a blue k_q ?" and tried to answer it. Also a trial has been made to change the question "What is the smallest value of n for which $k_n \rightarrow k_p, k_q$?" to two equivalent forms.

Finally I would like to express my heart felt gratitude to my teacher and advisor Dr.Seyoum Getu for giving me valuable reference books and reading this paper as well as for his critical comments.

Desta Legesse.

Abstract

This paper has two parts. In the first part the question "Why is it impossible to color the edges of $k_{r(p,q)}$ without forming either a red k_p or a blue k_q ?" is answered while in the second the question "What is the smallest value of n for which $k_n \to k_p, k_q$?" is changed to equivalent forms.

Introduction

A graph G is composed of a finite set **V** of elements called vertices and a set **E** of lines joining pairs of distinct vertices called edges. We denote the graph whose vertex set is **V** and whose edge set is **E** by **G= (V,E)**.

By a k_n we mean a graph with n vertices and all edges joining these vertices with each other.

<u>eg.</u>

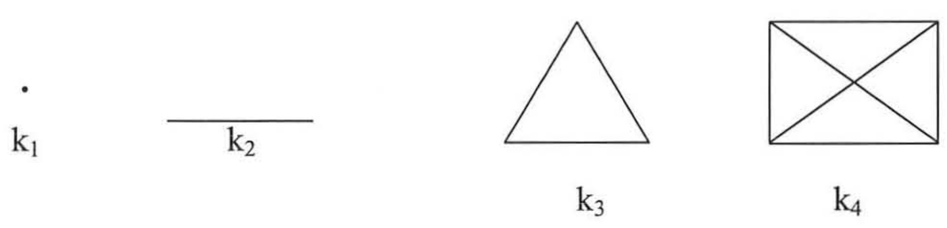


Fig.1

Ramsey's theorem states that if $p,q \geq 2$ are integers, then there is a positive integer n such that if we color the edges of k_n using two colors, red and blue, it is impossible to color the edges of the k_n without forming either a red k_p or a blue k_q. In short we formulate it as $k_n \to k_p,k_q$ (read as k_n arrows k_p,k_q). The smallest value of such n is denoted by r(p,q),known as the Ramsey number.

A famous example for the two color Ramsey theorem is k_6 which arrows k_3,k_3.

To prove $k_6 \to k_3,k_3$, let's put 6 points on a plane and call one of them *v*. There are 5 edges joining *v* to the remaining 5 points. Let's color them red or blue. At least 3 of them will have the same color, red, say. Consider the 3 vertices at the other ends of these 3 red edges:

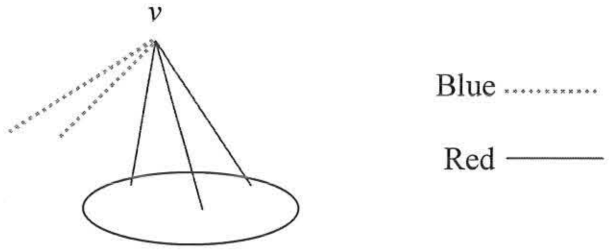


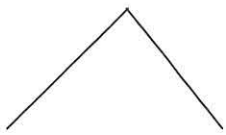
Fig.2

If any of the edges joining these 3 vertices with each other is red, then we have a red triangle. On the other hand if there is no red edge, we get a blue triangle.

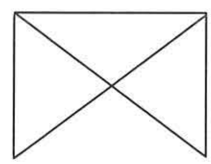
A complete subgraph with all its edges having the same color is called a *monochromatic sub graph* or a *monochromatic clique*.

The word *coloring* refers to assigning either of the colors red or blue for the edges and is used in a different sense from edge coloring; i.e. adjacent edges can have the same color.

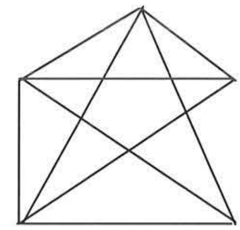
A k_n with one side missing is one which becomes a k_n when the missing side is filled. Clearly a k_1 & a k_2 with one side missing are vacuous sets.



A k_3 with one side missing



A k_4 with one side missing



A k_5 with one side missing

Fig.3

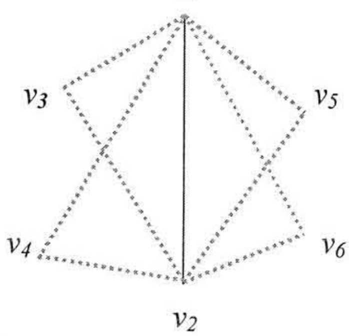
The main result

We have shown that $k_6 \rightarrow k_3, k_3$. But why is it impossible to color the edges of k_6 with out forming a monochromatic k_3 of either color?

In a k_6 there are ${}_6C_3$=20 triangles. If these triangles were drawn separately, they would have 20×3=60 edges. But a k_6 has only ${}_6C_2$= 15 edges. Hence an edge is shared by $^{60}\!/_{15}$ = 4 triangles in a k_6.

Now let's try to form a k_6 whose edges colored with red and blue but doesn't consist of a monochromatic k_3 of either color. What problem shall we encounter?

Let's start with a red edge, v_1v_2. As shown below, it is shared by 4 triangles. Let all triangles on v_1v_2 have blue sides, except v_1v_2.



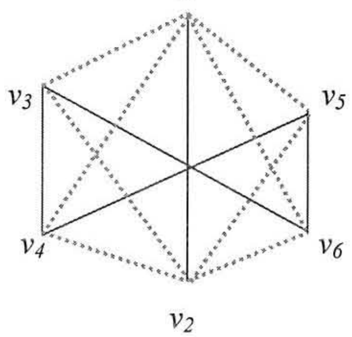


Fig.4

Since our trial is 'not to form' either a red k_3 or a blue k_3, color $v_4 v_5$, $v_3 v_4$, $v_5 v_6$ and $v_3 v_6$ red , as shown above.

Observe that $v_4 v_6$ is shared by a blue k_3 with one side missing (it becomes a k_3 when the missing side is filled) and a red k_3 with one side missing. If we color $v_4 v_6$ red, we get a red k_3 ($v_3 v_4 v_6$ & $v_5 v_4 v_6$). Otherwise a blue k_3($v_1 v_4 v_6$ & $v_2 v_4 v_6$) will be obtained. The same is true about $v_3 v_5$.

Let's do some further trials to avoid the formation of a monochromatic k_3 in a k_6.

Again let's start with a red edge, $v_1 v_2$. Suppose the 4 triangles on it are colored in the way shown in Fig.4 below.

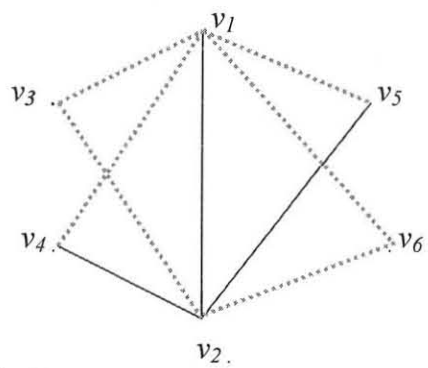


Fig.5

Consider $v_4 v_5$. It is shared by a blue k_3 with one side missing ($v_4 v_1 v_5$) and a red k_3 with one side missing ($v_4 v_2 v_5$).

Shall we color $v_4 v_5$ red or blue? In both cases the formation of a monochromatic clique is inevitable.

In our third trial, let $v_1 v_2$ be our starting line, red as usual.

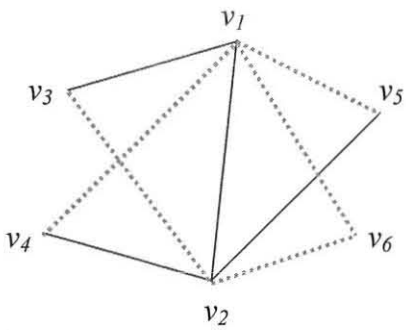


Fig.6

Also in this case, our common side for the blue k_3 ($v_4 v_1 v_5$) and the red k_3 ($v_4 v_2 v_5$) is $v_4 v_5$.

In all cases that we considered, an edge is shared by a triangle whose two sides are red and another triangle whose two sides are blue. Hence we get either a red k_3 or a blue k_3 .The following theorem generalizes the result.

Theorem 1 When we color all the edges of a $k_{r(p,q)}$, $p,q>2$ in two colors, red and blue, at least one edge is shared by a red k_p with one side missing and a blue k_q with one side missing.(The missing side being their common side).

Proof Suppose this is not the case, i.e an edge is not shared by a red k_p with one side missing and a blue k_q with one side missing.

In other words to the maximum we get a red k_p with one side missing and a blue k_{q-1} with one side missing or red k_{p-1} with one side missing and a blue k_q with one side missing on a single edge.

In the first case the missing side of the red k_p and the blue k_{q-1} (the common side) can be colored blue and we get a blue k_{q-1} while in the latter case this common side can be colored red giving a red k_{p-1}.

Hence there is neither a red k_p nor a blue k_q in the $k_{r(p,q)}$ which is a contradiction.

□

This theorem can be extended to multicolor Ramsey numbers and the proof is analogous.

Generalization of Theorem 1 When we color all the edges of a $k_{r(n_1,n_2,\ldots\ldots\ldots,n_k)}$, $n_1,n_2,\ldots..,n_k$ >2 with k-different colors at least one edge is shared by a k_{n_i} of ith color with one side missing and a k_{n_j} of jth color with one side missing where $1 \le i \le k$, $1 \le j \le k$ and $i \ne j$.

Proof Suppose this is not the case. So we have to the maximum a k_{n_i} of ith color with one side missing and a $k_{n_{j-1}}$ of jth color with one side missing or a $k_{n_{i-1}}$ of ith color with one side missing and a k_{n_j} of jth color with one side missing on a single edge; where $1 \le i \le k$, $1 \le j \le k$ and $i \ne j$.

In the first case, the missing side of the k_{n_i} of ith color and the $k_{n_{j-1}}$ of jth color can be colored with the jth color and we get a $k_{n_{j-1}}$ of jth color. In the latter case the missing side of the $k_{n_{i-1}}$ of the ith color and k_{n_j} of jth color can be colored with the ith color giving a $k_{n_{i-1}}$ of ith color.

Thus there is no a k_{n_i} of ith color, where $1 \le i \le k$ in our $k_{r(n_1,n_2,\ldots\ldots\ldots,n_k)}$ which is again a contradiction. □

What is the implication of Theorem 1? We can avoid the formation of either a red k_p or a blue k_q in a k_n unless at least one edge is shared by a red k_p with one side missing and a blue k_q with one side missing (the missing side being their common side).

Let v_iv_j be the common edge. Let it be colored red; i.e we have a red k_p. Let's try to avoid the formation of the red k_p by coloring one of its edges other than v_iv_j, v_kv_l, blue. But v_kv_l is also shared by a red k_p with one side missing and a blue k_q with one side missing. Hence if we color v_kv_l blue, a blue k_q will be formed. Now let's try to color one of the edges of the blue k_q, v_mv_n red so that we can avoid monochromatic subgraph in our k_n. But v_mv_n is in turn shared by a red k_p with one side missing and a blue k_q with one side missing. This fact leads us to the following corollary.

Corollary 1 When we color all the edges of a $k_{r(p,q)}$,p,q >2 in two colors, red and blue, if a monochromatic subgraph is formed, then every edge of it is shared by a red k_p with one side missing and a blue k_q with one side missing whose edges are in turn shared by a red k_p with one side missing and a blue k_q with one side missing.

Now let's try to change the question, "What is the smallest value of n for which $k_n \rightarrow k_p,k_q$?", to an equivalent form.

A k_6 can be colored red and blue in the following manner:

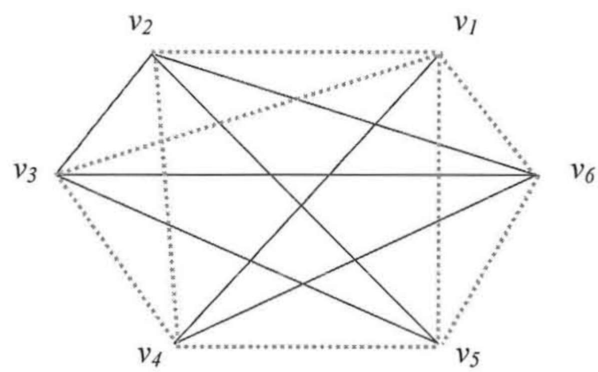


Fig.7

Consider a 6×6 chessboard. Let a square in the (i,j)-position of this chessboard (i[th] row and j[th] column) be colored the same color as the edge v_iv_j.

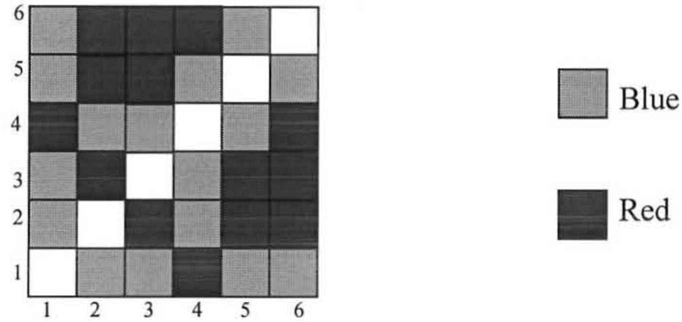


Fig.8

The squares on the diagonal that runs from lower left to upper right are not colored. The colors of the squares above the diagonal are symmetric to the colors of the squares below the diagonal. This is because squares that are equidistant from the diagonal represent the same edge. For instance, since v_1v_2 is blue, the squares in the position (1, 2) and (2, 1) will be colored blue.

Now let's consider only the lower part (the part below the diagonal) of the chess board. It represents sufficiently the edges of our k_6. The position of a square is read with the first coordinate larger than the second coordinate. If we are asked to tell the color of the edge v_iv_j for j > i, first we write it as v_jv_i and go to the square in the (j,i) position.(The square in the column denoted by i and row denoted by j). For example, the edge v_1v_2 is represented by the square at position (2, 1).

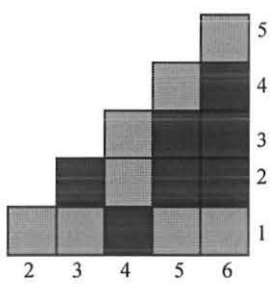


Fig. 9

Now an edge is represented by a single square and hence we have $5+4+3+2+1=15={}_6C_2$ squares.

Which squares do represent the red k_3 in our k_6? The squares in the positions

(5,3) (6,3)

(3, 2),(5,2) and (3,2),(6,2).

To conclude the result, consider a k_n whose edges colored red or blue in any way. Let a square in the (i,j),i>j, position of a lower part of $n\times n$-chessboard be colored the same color as the edge v_iv_j in the k_n. Pick a vertex of the k_n, say v_1. n-1 edges can be drawn from it and these edges are represented by n-1 squares in the 1^{st} row of our chessboard. Take another vertex, v_2 say. We can draw n-2 edges (since v_1v_2 has already been counted and they will be represented by the n-2 squares in the 2^{nd} row). Proceeding this way the $(n-1)^{th}$ vertex can be connected to only one vertex (the n^{th}) and that edge is represented by a single square in the $(n-1)^{th}$ row.

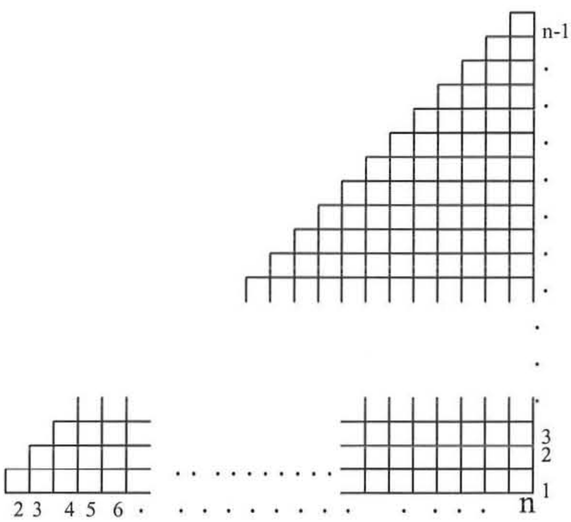


Fig.10

What feature does a red k_p in our k_n display, if we have one? Let's denote the vertices of our k_n by $v_1, v_2, \ldots\ldots, v_n$. Suppose $v_{i_1}, v_{i_2}, \ldots\ldots, v_{i_p}$ be the vertices of our k_p where $1 \le i_r \le n$ for $1 \le r \le p$. Picking a vertex v_{i_1} we connect it to all vertices of the k_p other than itself, i.e we have all combinations $v_{i_1}v_{i_2}, v_{i_1}v_{i_3}, \ldots\ldots, v_{i_1}v_{i_p}$. Picking v_{i_2} we get $v_{i_2}v_{i_3}, v_{i_2}v_{i_4}, \ldots\ldots\ldots, v_{i_2}v_{i_p}$. Finally picking $v_{i_{p-1}}$ we get the edge $v_{i_{p-1}}v_{i_p}$.

Connecting the subscripts we learn that our red k_p is represented by red squares in the rows $i_1, i_2, \ldots\ldots\ldots, i_{p-1}$ and in the columns $i_2, i_3, \ldots\ldots\ldots, i_{p-1}, i_p$, i.e we have $p-1, p-2, \ldots\ldots, 2, 1$ red squares in the $i_1^{th}, i_2^{th}, \ldots\ldots\ldots, i_{p-1}^{th}$ rows respectively.

Note that the difference between successive i_r's, $1 \le r \le p$ may be greater than 1.

So, equivalent form of the Ramsey problem is the following:

"What is the smallest value of n for which we can color the lower part of an $n \times n$-chessboard without forming either red squares in all positions:

$$\begin{array}{r}
(i_p,i_{p-1}) \\
. \\
. \\
. \\
(i_4,i_3),\ldots\ldots,(i_{p-1},i_3),(i_p,i_3) \\
(i_3,i_2),(i_4,i_2),\ldots\ldots,(i_{p-1},i_2),(i_p,i_2) \\
(i_2,i_1),(i_3,i_1),\ldots\ldots\ldots,(i_{p-1},i_1),(i_p,i_1)
\end{array}$$

or blue squares in all positions:

$$\begin{array}{r}
(h_q,h_{q-1}) \\
. \\
. \\
. \\
(h_4,h_3),\ldots\ldots,(h_{q-1},h_3),(h_q,h_3) \\
(h_3,h_2),(h_4,h_2),\ldots\ldots,(h_{q-1},h_2),(h_q,h_2) \\
(h_2,h_1),(h_3,h_1),\ldots\ldots\ldots,(h_{q-1},h_1),(h_q,h_1)
\end{array}$$

Where $1 \le i_r \le n$ for $1 \le r \le p$ and $1 \le h_t \le n$ for $1 \le t \le q$ are integers but not necessarily consecutive?"

The next equivalent form focuses on r(p,p). Before we go to that, note that in a k_n every edge is shared by ${}_{n-2}C_{p-2}$ k_p's. How? Let's fix two vertices. We should select p-2 additional vertices out of the remaining n-2 vertices of our k_n in order to form a k_p. Such selection is done in ${}_{n-2}C_{p-2}$ ways.

Now let's give a code of b_i, $1 \leq i < {}_nC_2$ for blue edges and a code of r_j, $1 \leq j < {}_nC_2$ for red edges. Assume the b_i used to represent the edge $v_l v_m$ is different from the b_k used to represent the edge $v_s v_t$, i.e.the b_i's and r_j's are distinguishable. So we have ${}_nC_2$ b_i's and r_j's.

In a k_n if neither a red k_p nor a blue k_p is going to be formed, there will be a maximum of ${}_pC_2 - 1$ red edges also a a maximum of ${}_pC_2 - 1$ blue edges in each k_p.

Here is the equivalent form:

We have ${}_nC_2$-set of b_i's and r_j's. We want to make ${}_nC_p$ sums. Each b_i and r_j being used in exactly ${}_{n-2}C_{p-2}$ sums and only once in a single sum, each sum having ${}_pC_2$ terms. In each sum the number of b_i's ranges from 1 to ${}_pC_2 - 1$, also the number of r_j's ranges from 1 to ${}_pC_2 - 1$.

What is the maximum value of n for which such sums are impossible? Clearly r(p,p)=n.

Final remarks

Of course, these equivalent forms are not only as difficult as the original Ramsey problem but also stated in longer statements.

Until now, only small numbers of exact values of the Ramsey numbers are known. Since the colors used are immaterial, note that r(p,q)=r(q,p). r(p,2)=p (because all edges of a k_p can be colored red or at least one of its edge can be colored blue).

The table below shows of Ramsey numbers whose exact values are known.

p \ q	3	4	5	6	7	8	9
3	6	9	14	18	23	28	36
4	9	18	25				

Table. 1

At the end of my paper; I want to put what Erdos said for I felt that it verifies how tiresome the trial to determine r(p,q) is.

"Assume an evil spirit orders us to compute r(5,5) or else he will destroy all mankind. It may then be best if all mathematicians and computers start working on the answer. If, however, he orders us to compute r(6,6), then we had better think about how to destroy him before he destroys us".

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