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Office of Graduate Program

Faculty of Computer and Mathematical Sciences

Department of Mathematics

A Graduate project report

On

K-TREES AND CATALAN IDENTITIES

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A Project submitted to the department of Mathematics in partial fulfillment of the requirements for the Degree of Master of Science in Mathematics.

Stream: *Combinatorics*

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Declaration

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

Jemal Mohammed

Permission

This is to certify that this project is compiled by Jemal Mohammed in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

_______________________________ _________________________

Melkamu Zeleke (Prof.) *Seyoum Getu (Dr.)*

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I) Acknowledgement

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II) Summary

Ordered trees are trees with a distinguished vertex called the *root* where the children of each internal vertex are linearly ordered. K-trees generalize ordered trees in the sense that ordered trees are 2-trees in which edges between nodes are drawn as double edges. A class of numbers are introduced which unify many well-known counting coefficients, such as the Catalan numbers, the Fine numbers and the Central Binomial numbers and also their generating functions are computed.

The Generalized Catalan numbers count the number of homogeneous ordered k-trees consisting of n k-cycles. We can prove the 17 most useful Catalan generating function identities by simple algebraic manipulations. In this project also we use ordered trees and k-trees to obtain generating function identities involving generalizations of Catalan numbers, Central Binomial numbers, and Fine numbers.

We give some examples to show possible applications of these identities, like the Fibonacci polynomials, which is the generalization of Fibonacci numbers, the higher derivative of Central Binomial numbers, enumerating edges of odd degree and odd out degree and also show that the ratio of generalized Fine numbers to Catalan numbers is asymptotic to $\frac{2k}{(k+1)^2}, \text{ for } k \geq 2$.

III) Preliminaries

Let n and r be non-negative integers. The Binomial coefficient $\binom{n}{r}$ is defined as

$\binom{n}{r} = \frac{n!}{r!(n-r)!}$, where $0 \le r \le n$. If $r > n$ then $\binom{n}{r} = 0$.

Theorem 1. (The Binomial Theorem)

Let x and y be arbitrary real numbers, and n an arbitrary non-negative integer then

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

Corollary 2. Let x be any real number. Then $(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$

Partitions

A family of subsets S_i of a set S is a partition of S if:

- Each S_i is non-empty.
- The subsets are pair-wise disjoint; i.e. $S_i \cap S_j = \phi$ if $i \neq j$; and
- The union of the subsets S_i is S; i.e. $\bigcup_{i=I} S_i = S$, where I denote an index set.

Each subset S_i is a *block* of the partition.

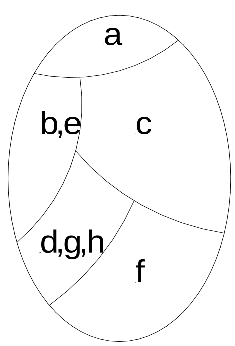


Figure 1: A partition of S

For example, consider the set S = **{a, b, c, d, e, f, g, h}.**

Then **P = {{a}, {b, e}, {c}, {d, g, h}, {f}}** is a partition of S.

The subsets $B_1 = \{a\}, B_2 = \{b,e\}, B_3 = \{c\}, B_4 = \{d,g,h\}, and\ B_5 = \{f\}$ are the blocks of the partitioning.

Non-crossing partitions

A non-crossing partition of π of the set S = {1, 2, 3,…, n} is a partition $\{B_1, B_2, ..., B_k\}$ of S such that if $a < b < c < d$ and $a, c \in B_i\ and\ b, d \in B_j, then\ i = j$.

For example, let n=8.Then $\pi = \{\{1,2,5\},\{3,4\},\{6,8\},\{7\}\}$ is a non-crossing partition of S. But $\{\{1,2,5\},\{3,7\},\{4\},\{6,8\}\}$ is crossing, since $3 < 6 < 7 < 8$, but the blocks $\{3,7\}\ and\ \{6,8\}$ are not equal.

- The ordinary generating function for the infinite sequence $(a_0, a_1, a_2, a_3, ...)$ is the power series

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

- If the generating functions of a sequence $\{a_n\}_{n\geq 0}$ *is* $g(z)$, then the generating functions of the sequence $\{na_n\}_{n\geq 0}$ *is* $zg'(z)$.
- A permutation on n-symbols is a one to one mapping of the S= {1, 2, 3, … n} onto itself. If σ is a permutation on n-symbols, then σ is completely determined by its value $\sigma(1), \sigma(2), ..., \sigma(n)$. An element $\sigma \in s_n$, where s_n is called the symmetric group on n symbols, is called a cycle of order **r** if there exists **r** symbols $i_1, i_2, ..., i_r$ such that $\sigma(i_1) = i_2, \sigma(i_2) = i_3, ..., \sigma(i_{r-1}) = i_r, , \sigma(i_r) = i_1$ and $\sigma(j) = j, \forall j \neq i_1, i_2, ..., i_r$.

The Lagrange inversion formula

Let $f(u)\ and\ \Phi(u)$ be formal power series in u, with $\Phi(0) = 0$. Then there is a unique formal power series $u = u(t)$ that satisfies $u = z\Phi(u)$.Further, the value $f(u(t))$ of f at that root $u = u(t)$, when expanded in a power series in t about $t = 0$ satisfies

$$[t^n]\{f(u(t))\} = \frac{1}{n}[u^{n-1}]\left\{f'(u)\left(\Phi(u)\right)^n\right\}.$$

A graph G consists of a non-empty set V(G) of vertices and a list E(G) of unordered pairs of these elements called edges. A graph G is *connected* if there is a path between every two distinct vertices. A *cycle* is a path with the same end point; it contains no repeated vertices. A graph is *acyclic* if it contains no cycles. A connected, acyclic graph is a *tree*.

A *path,* P, of length n from (x_0, y_0) *to* (x, y) with step set S is a sequence of points in the plane, $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) = (x, y)$ such that all $(x_{i+1} - x_i, y_{i+1} - y_i) \in S$ These points are called *vertices*. We define the height of a vertex, to be the ordinate of that point. We say that a path, P, is positive if each of its vertices has nonnegative height.

Snake-oil Method

This method is used to solve recurrence relations and find an exact formula, if possible, using generating functions.

i) Make sure that the set of values of the free variable (say n) for which the recurrence relation is true, is clearly delineated.

ii) Give a name such as $f(x)$, $A(x)$, etc., to the generating function that you will look for.

iii) Multiply both sides of the recurrence relation by x^n, and sum over all values of n for which the recurrence holds.

iv) Express both sides of the equation explicitly in terms of your generating function.

v) Solve the resulting equation for the unknown generating function.

vi) ***(optional)*** An exact formula for a_n can be obtained by expanding the generating function into a power series.

Riordan Array

A Riordan array is a couple of formal power series $D = (d(t), h(t))$; if both $d(t), h(t) \in f_0$, then the Riordan array is called proper. The Riordan array can be identified with the infinite, lower triangular array $(d_{n,k})_{n,k \in N}$ defined by:

$$d_{n,k} = [t^n] d(t) (t.h(t))^k, \quad where \ d(t) = \sum_{k=0}^{\infty} d_k t^k \ , \ h(t) = \sum_{k=0}^{\infty} h_k t^k \ .$$

A Dyck path of length 2n is a path in two-space from (0, 0) to (2n, 0) which uses only steps (1, 1) (north-east) and (1, -1) (south-east).Further a Dyck path does not go below the x-axis. A peak on a Dyck path is a node that is immediately preceded by a north-east step and is immediately followed by a south-east step. The height of a peak is the y-coordinate of the right end point of its up step.

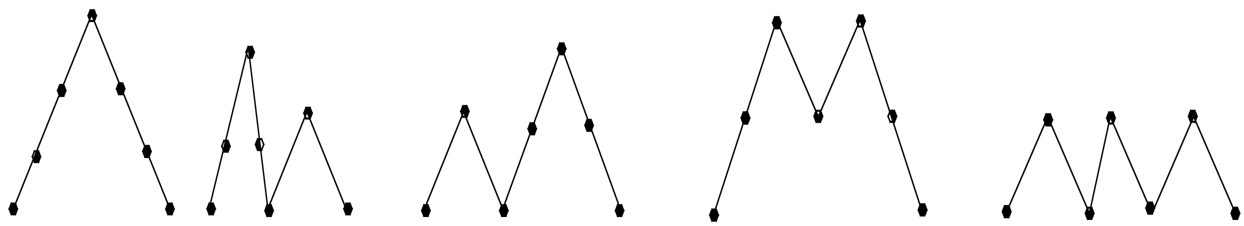


Figure 2: The five Dyck paths of length 6.

A Dyck path consists of either the empty path or it consists of an up step, followed by a Dyck path at height 1, followed by a down step, followed by a Dyck path at height o. A Fine path is a Dyck path with no peaks at level one.[10]

Notations:

- P_n : the set of legal sequences of n open and n close parentheses. A parenthetic expression is called *legal* if each open parenthesis has a matching close parenthesis.
- I_n : the set of dominating sequences of n+1 non-negative integers, $(a_i)_0^n$ such that $\sum_{i=0}^{n} a_i = n$ and $\sum_{i=0}^{n} a_i \geq i \;\; \forall i, 0 \leq i \leq n$.
- L_n : the set of admissible paths from (0,0) to (n, n) in an n x n lattice. A path is admissible if it does not pass below the diagonal y=x.
- B_n : the set of full binary trees with n internal nodes.[2]

Section 1. Introduction

This project presents ordered trees and k-trees in order to obtain identities involving the generating functions $C(z)=\frac{1-\sqrt{1-4z}}{2z}=\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}z^n$ of the Catalan numbers, $B(z)=\frac{1}{\sqrt{1-4z}}=\sum_{n=0}^{\infty}\binom{2n}{n}z^n$ of the Central Binomial numbers, and $F(z)=\frac{1}{z}\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}$ of Fine numbers and also we obtain similar identities involving the generating function C = C(z) of k-trees, B = B(z) of the analog of Central binomial numbers, and F = F(z) of generalized Fine numbers.

The project starts with some preliminaries to clarify basic concepts central to understand the relation underlying ordered trees, k-trees and Catalan identities. The second section provides an introduction to the concept of ordered trees by defining what it means and showing the relationships between Catalan numbers, Central Binomial numbers and Fine numbers with their generating functions.

In the third section we show the most interesting applications of Catalan identities; whereas the fourth section is intended to use k-trees, define the generalization of the Catalan identities and their relationships. Similar to the third section, the fifth section entails additional applications of the Catalan identities; and asymptotic results are also discussed as well.

Section 2. Ordered Trees and Catalan Identities

2.1) Ordered Trees

Ordered trees are trees with a distinguished vertex called the *root* where the children of each internal vertex are linearly ordered. An ordered tree can be drawn on a plane without self -intersections with the root placed on the top and branches sorted from left to right by their order.

The basic terminology of rooted trees reflects that of a family tree. Let T be a tree with root v_0. Let $\{v_0, v_1, v_2, ..., v_n\}$ be the path from v_0 *to* v_n , then:

- The *degree* of a vertex is the number of edges meeting at the vertex.
- The level of **x** is its distance (the number of edges separating it from the root of an ordered tree).
- A node of degree zero is called a *leaf.* Otherwise it is called an internal node. The root is the only node at level 0.
- The *subtree* rooted at v consists of v , its descendents, and all its edges.
- The tree with no edges is called the *empty tree*.

Below is an example of an ordered tree, T, consisting of a root labeled **a,** and sub trees $T_1 = \{b, e, f, h\}, T_2 = \{c\}$ *and* $T_3 = \{d, g, i, j\}$

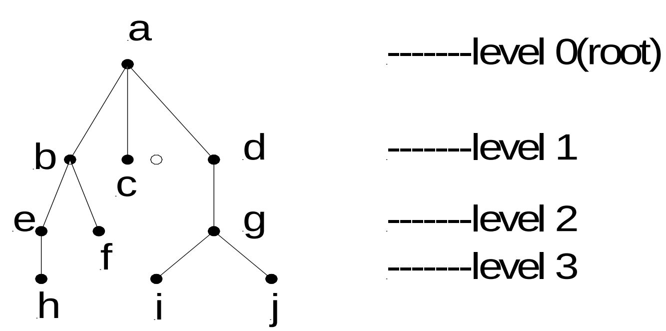


Figure 3: An ordered tree with 9 edges

Leaves: c, f, h, i, j

$$\text{internal nodes}\begin{cases} \text{of degree } 1: d, e \\ \text{of degree } 2: b, g \\ \text{of degree } 3: a \end{cases}$$

Level 0 (root): a

Level 1: b, c, d

Level 2: e, f, g

Level 3: h, i, j

Two (unlabelled) ordered trees are isomorphic iff they are isomorphic graphs and the isomorphism preserves the root and the order of branches.

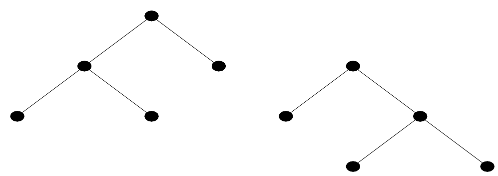


Figure 4: Non-isomorphic ordered trees

2. 2) Enumeration of Ordered Trees by number of edges

Let T_n $(n \geq 0)$ denote the set of ordered trees with n edges. Ordered trees may be defined recursively as follows: if $t_1, t_2, ...t_m$ are ordered trees, $m \geq 0$, then

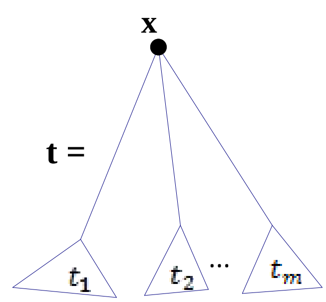
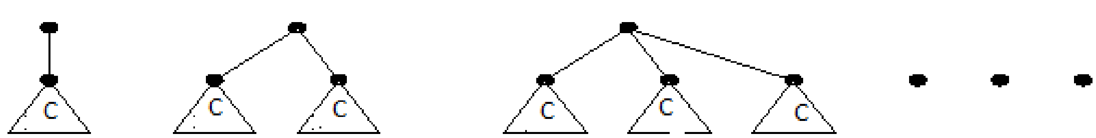


Figure 5:

is an ordered tree. The trees $t_1, t_2, ...t_m$ are *subtrees* of the distinigushed node called the *root* of t connecting them. The roots of the subtrees are children of the root of the tree.[2]

Let T(z) be the generating function of all ordered trees, according to size (i.e. z marks edges). Every nonempty tree is of the form shown in figure 6, where the $T_i s$ are trees, possibly empty. [3]



$$\Rightarrow 1 + zT + z^2T^2 + z^3T^3 + ...$$

Figure 6:

The enumeration of ordered trees on n edges can perform as follows.

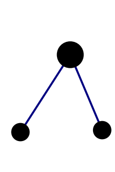
Let $C_n =$ number of ordered trees on n edges.

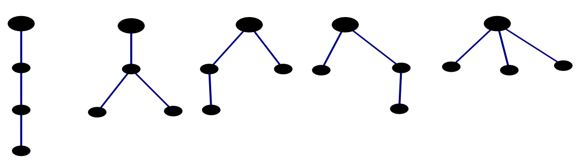
Let $C(z) = \sum_{n=0}^{\infty} C_n z^n$ be the generating function of C_n

The next illustration shows the relation between ordered rooted trees and Catalan numbers.

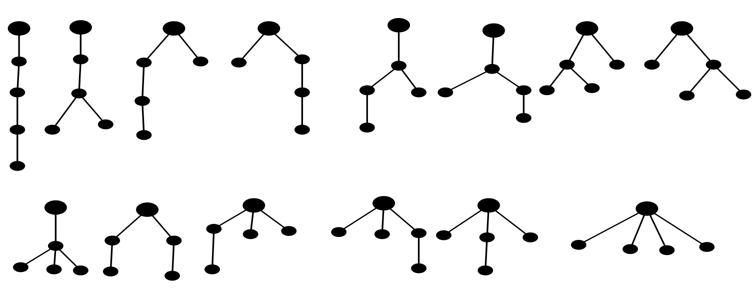
$T_0 =$  $\Rightarrow C_0 = 1$

$T_1 =$  $\Rightarrow C_1 = 1$

$T_2 =$  $\Rightarrow C_2 = 2$.

$T_3 =$ 

$\Rightarrow C_3 = 5$

$T_4 =$ 

$\Rightarrow C_4 = 14$.

Figure 7: The various possible ordered rooted trees with $0 \le n \le 4$ edges

Using this illustration, we conjecture that there are C_n ordered rooted trees with n edges.

Theorem 3: The Number of ordered trees on n edges is given by $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Proof:

Take any ordered tree t. Then partition it into a trivial tree (a tree with no edge) or non-trivial tree. The trivial tree contributes 1 to the sum. On decomposing the non-trivial part into left and right children the following recursion is obtained.

Figure 8: Generating ordered trees from another.

Therefore the generating function C (z) satisfies the following recursion.

$$C(z) = 1 + zC(z)^2$$

$$as\ u(z) = z(u(z)+1)^2,\ where\ u(z) = C(z) - 1$$

$$let\ \Phi(z) = (1+z)^2$$

$$by\ Lagrange\ inversion\ formula,$$

$$u_n = \frac{1}{n}[z^{n-1}](\Phi(z))^n$$

$$= \frac{1}{n}[z^{n-1}](1+z)^{2n}$$

$$= \frac{1}{n}\binom{2n}{n-1} \qquad \text{... by the binomial theorem}$$

$$u_n = \frac{1}{n+1}\binom{2n}{n}$$

$$but\ u_n \leftrightarrow C_n$$

$$so\ C_n = \frac{1}{n+1}\binom{2n}{n}$$ ∎

Therefore the number of ordered trees on n edges is C_n and hence the proof is complete.

2. 3) Enumeration of Ordered Trees by number of edges and leaves

The Cycle lemma

A sequence $p_1 p_2 ... p_t$ of m open parenthesis and n close parenthesis is called k-dominating, for positive integer k, if for every position $i, 1 \le i \le m+n$, the number of open parenthesis in $p_1 p_2 ... p_i$ is more than k times the number of close parenthesis.

Examples:

1) The sequence (((((())(() is 2-dominating.

2) The sequence ((((())(() is 1- dominating but not 2-dominating.

3) The sequence)(((())((*and* (())(()((are not 1-dominating.

Cycle lemma (Dvoretzky and Motzkin)

For a ny s equence $p_1 p_2 ... p_{m+n}$ of m o pen p arenthesis an d n cl ose p arenthesis, $m \ge kn$ there exists exactly $m - kn (out\ of\ m+n)$ cyclic permutations

$$p_j p_{j+1} ... p_{m+n} p_1 ... p_{j-1},\ 1 \le j \le m+n$$

that are k-dominating.

Examples:

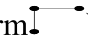
1) Of the 6 cyclic permutations of the sequence)((()(only two are 2-dominating (or legal prefixes): i.e. ((()() and (()()(

2) Of the 9 cyclic permutations of the sequence)((((())((of 6 open parenthesis and 3 close parenthesis, on ly 3 a re 2-dominating (o r legal p refixes): i. e. (()((((()), ((((()(() *and* ((())(()(.

Characterization lemma

1) The number of leaves in a (not edgeless) tree t

= the number of () patterns in p (t), where p (t) is legal parenthetic

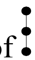
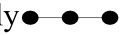
= the number of corners (i.e. path segments of the form ) in l(t),where l(t) is an admissible lattice path.

= the number of left leaves in b (t), where b (t) is a full binary tree.

= the number of right leaves in b' (t), where b' (t) is reflection of b (t).

2) The number of internal nodes in a (not edgeless) tree t

= One more than the number of **(((**or equivalently**)))** in p (t),

= One more than the number of  (or equivalently ) path segments in l(t)

= the number of right leaves in b (t).

= the number of left leaves in b' (t)

3) The number of nodes of degree d in a tree t

= the number of occurrences of d in i (t), where i (t) is dominating sequence

= the number of vertical path segments of length exactly d in l'(t), where l' (t) is reflection of l (t).

4) The number of nodes of degree d in all the tree in T_n

= the number of occurrences of d in all the sequences in I_n .

= the number of occurrences of runs of exactly d**(**'s (or equivalently d**)**'s) in all the expressions in P_n .[2]

Theorem 4. (Narayana)

The number $l_n(k)$ of ordered trees with n edges and k leaves is

$$l_n(k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} = \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1} = \frac{1}{n+1}\binom{n-1}{k-1}\binom{n+1}{k}$$

Proof:

The number of trees in T_n with k leaves is equal to the number of legal parenthetic expressions in P_n with k occurrences of **(). (**by Characterization lemma**),** which in turn equals the number of legal prefixes with n+1 **(**'s, n **)**'s, and k **()**'s. By the cycle lemma, of the k possible cyclic arrangements of a parenthetic expression beginning with **(**, ending with), and having n+1 **(**'s, n **)**'s, and k **()**'s, exactly one is a legal prefix. The total number of such expressions (legal or not) is $\binom{n}{k-1}\binom{n-1}{k-1}$

(the number of ways to partition both the ('s and the)'s into k nonempty runs);

$$\text{Thus} \quad l_n(k) = \frac{1}{k}\binom{n}{k-1}\binom{n-1}{k-1}$$ □

Examples:

1) Of the 5 ordered tree with 3 edges, $l_3(1)=1$ have 1 leaf, $l_3(2)=3$ have 2 leaves, and $l_3(3)=1$ have 3 leaves.

2) Of the 14 ordered tree with 4 edges, $l_4(1)=1$ have 1 leaf, $l_4(2)=6$ have 2 leaves, $l_4(3)=6$ have 3 leaves and $l_4(4)=1$ have 4 leaves.

2.4) Catalan Identities

2.4.1) Central Binomial Numbers

The central Binomial coefficients $\binom{2n}{n}$ are centrally located in even numbered rows in Pascal's triangle, as figure 9 shows.

$$\begin{array}{ccccccccccccc}
&&&&&&1&&&&&& \\
&&&&&1&&1&&&&& \\
&&&&1&&2&&1&&&& \\
&&&1&&3&&3&&1&&& \\
&&1&&4&&6&&4&&1&& \\
&1&&5&&10&&10&&5&&1& \\
1&&6&&15&&20&&15&&6&&1
\end{array}$$

Figure 9: Pascal's triangle

Then the first few terms are 1, 2, 6, 20, 70, 243,…

A Recurrence relation for the central Binomial coefficients

We know that

$$[z^n]C^sB = \binom{kn+s}{n}$$

$$[z^n]2zC^{2-1}B = 2[z^{n-1}]C^{2-1}B \qquad \ldots since \ \ from\ identity\ \ 2, B(z) = 1 + kzB(z)C^{k-1}(z)$$

$$= 2\binom{2(n-1)+2-1}{n-1}$$

$$= 2\binom{2n-1}{n-1}$$

$$= \binom{2n}{n}$$

This is the n^{th} central binomial coefficients.

Next, we will derive a recurrence relation for the central binomial coefficients

Let $B_n = \binom{2n}{n}$ then

$$B_{n+1} = \binom{2n+2}{n+1}$$

$$= \frac{(2n+2)!}{(n+1)!(n+1)!}$$

$$= \frac{2(2n+1)}{n+1}\frac{(2n)!}{n!n!}$$

$$= \frac{2(2n+1)}{n+1}B_n$$

Thus we have the recurrence relation

$$(n+1)B_{n+1} = 2(2n+1)B_n \text{ , } where\ \ B_0 = 1$$

A Generating function for the central binomial coefficients

$$Let\ \ B(z) = \sum_{n=0}^{\infty} B_n z^n$$

$$= B_0 + \sum_{n=1}^{\infty} B_n z^n$$

$$= B_0 + \sum_{n=0}^{\infty} B_{n+1} z^{n+1}$$

Differentiating both sides with respect to z,

$$B'(z) = \sum_{n=0}^{\infty} (n+1)B_{n+1} z^n$$

$$zB'(z) = \sum_{n=0}^{\infty}(n+1)B_{n+1}z^{n+1}$$
$$= \sum_{n=1}^{\infty} nB_n z^n$$
$$= \sum_{n=0}^{\infty} nB_n z^n$$

using the above recurrence relation

$$B'(z) = \sum_{n=0}^{\infty}(n+1)B_{n+1}z^n$$
$$= \sum_{n=0}^{\infty} 2(2n+1)B_n z^n$$
$$= 2\left(2\sum_{n=0}^{\infty} nB_n z^n + \sum_{n=0}^{\infty} B_n z^n\right)$$
$$= 2\left(2zB'(x) + B(z)\right)$$
$$= 4zB'(z) + 2B(z))$$

$$(1-4z)B'(z) = 2B(z)$$

$$so \quad \frac{B'(z)}{B(z)} = \frac{2}{1-4z}$$

Integrating both sides with respect to z, gives

$$\ln B(z) = \ln\frac{1}{\sqrt{1-4z}} + \ln k, \text{where } k \text{ is a constant}$$

$$B(z) = \frac{k}{\sqrt{1-4z}}$$

$$\text{since } B_0 = 1, k = 1$$

$$Thus,\ B(z) = \frac{1}{\sqrt{1-4z}}$$ □

Hence the central binomial function, $B(z)$, is the generating function for the central binomial coefficients, $\binom{2n}{n}$ is $\frac{1}{\sqrt{1-4z}}$.

2.4.2) Catalan Numbers

The Catalan function, C(z), is the generating function for the Catalan numbers, 1, 1, 2, 5, 14, 42,…,

$$C(z) = \sum_{n=0}^{\infty} C_n z^n = \frac{1-\sqrt{1-4z}}{2z}$$

Generating function for Catalan numbers

Let us derive the generating function C(z) for Catalan numbers using integral calculus and the generating function B(z).

$$Let \;\; B(z)=\sum_{n=0}^{\infty}\binom{2n}{n}z^n$$

$$=\frac{1}{\sqrt{1-4z}}$$

$$=\sum_{n=0}^{\infty}B_n z^n \;\; \textit{be the generating function for the Central Binomial coefficients.}$$

integrating the power series with respect to z, term by term,

$$\int\frac{dz}{\sqrt{1-4z}}=\sum_{n=0}^{\infty}\frac{1}{n+1}B_n z^{n+1}+k, \textit{where } k \textit{ is a constant}$$

$$-\frac{1}{2}\sqrt{1-4z}=\sum_{n=0}^{\infty}C_n z^{n+1}+k \qquad \ldots since \sum_{n=0}^{\infty}\frac{1}{n+1}B_n z^n$$

$$-\frac{1}{2}\sqrt{1-4z}=zC(z)+k$$

$$when \; z=0, this \; yields \; k=-\tfrac{1}{2} \qquad \ldots since \; C(0)=1$$

$$thus \;\; -\frac{1}{2}\sqrt{1-4z}=zC(z)-\tfrac{1}{2}$$

$$\Rightarrow \; C(z)=\frac{1-\sqrt{1-4z}}{2z}$$

□

A Recursive formula for C_n

Next, let us show a recursive formula for C_n using the generating function

$$\text{we have } (1\text{-}4z)^{\frac{1}{2}}=1+\frac{1}{2}(-4z)-\frac{1}{2^3}\frac{1}{2}\binom{2}{1}(-4z)^2+\frac{1}{2^5}\frac{1}{3}\binom{4}{2}(-4z)^3-\ldots$$

$$=1-2z-\frac{2}{2}\binom{2}{1}z^2-\frac{2}{3}\binom{4}{2}z^3-\frac{2}{4}\binom{6}{3}z^4-\ldots$$

$$we \; know \; that \; C(z)=\frac{1-\sqrt{1-4z}}{2z}$$

$$=\frac{1}{2z}\left(2z+\frac{2}{2}\binom{2}{1}z^2+\frac{2}{3}\binom{4}{2}z^3+\frac{2}{4}\binom{6}{3}z^4+\ldots\right)$$

$$= 1 + \frac{1}{2}\binom{2}{1}z + \frac{1}{3}\binom{4}{2}z^2 + \frac{1}{4}\binom{6}{3}z^3 + \ldots$$

since $C(z)$ is the generating function for $C_0, C_1, C_2, C_3, \ldots$

then $C_0 = 1,\ C_1 = \frac{1}{2}\binom{2}{1} = 1,\ C_2 = \frac{1}{3}\binom{4}{2} = 2,\ C_3 = \frac{1}{4}\binom{6}{3}, \ldots,$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

let $C_n = \frac{1}{n+1}\binom{2n}{n}$ *then* $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$

$$\frac{C_n}{C_{n-1}} = \frac{(2n)!(n-1)!}{(n+1)(n)!(2n-2)!} = \frac{4n-2}{n+1}$$

and also the recursive formula $C_n = \frac{4n-2}{n+1}C_{n-1}, n \geq 1$ can be employed to derive the explicit formula $C_n = \frac{1}{n+1}\binom{2n}{n}$

That is

$$C_n = \frac{4n-2}{n+1}C_{n-1}$$

$$= \frac{(4n-2)(4n-6)}{(n+1)n}C_{n-2}$$

$$= \frac{(4n-2)(4n-6)(4n-10)}{(n+1)n(n-1)}C_{n-3}$$

.

.

.

$$= \frac{(4n-2)(4n-6)(4n-10)\ldots 6.2}{(n+1)n\ldots 3.2}C_0$$

$$= \frac{(2n-1)(2n-3)(2n-5)\ldots 3.1}{(n+1)!}2^n$$

$$= \frac{(2n)!2^n}{(n+1)!2^n n!}$$

$$= \frac{(2n)!}{(n+1)!n!}$$

$$= \frac{1}{n+1}\binom{2n}{n}$$

It yields the desired explicit formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$ □.

2.4.3) Fine Numbers

The Fine function, $F(z)$,

$$F(z) = \sum_{n=0}^{\infty} f_n z^n = \frac{1-\sqrt{1-4z}}{z(3-\sqrt{1-4z})}$$

is the generating function for the Fine numbers $1, 0, 1, 2, 6, 18, 57, 186, \ldots$

by applying Mathematica, we can find the terms of the Fine numbers

$$in[1] = f[z\_] = \frac{1-\sqrt{1-4*z}}{z*(3-\sqrt{1-4*z})}$$

$$out[1] = \frac{1-\sqrt{1-4z}}{(3-\sqrt{1-4z})z}$$

$$in[2] = Series[f[z], \{z, 0, 10\}]$$

$$out[2] = 1 + z^2 + 2z^3 + 6z^4 + 18z^5 + 57z^6 + 186z^7 + 622z^8 + 2120z^9 + 7338z^{10} + 0[z]^{11}$$

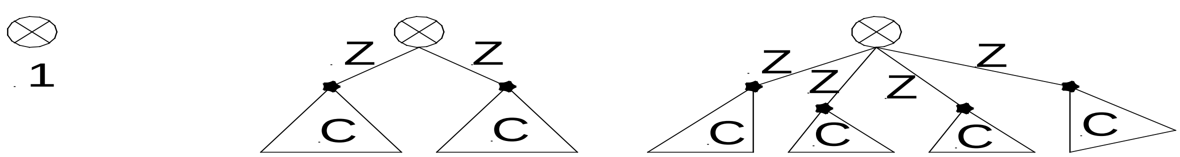


Figure 10: Recursive construction of Fine trees

$$then \quad F = 1 + z^2C^2 + z^4C^4 + \ldots$$

$$= \frac{1}{1-z^2C^2}$$

2.4.4) Identities Involving C(z), B(z) and F(z)

1) $C = 1 + zC^2 = \dfrac{1}{1 - zC}$

proof :

$$now\ C = \frac{1 - \sqrt{1-4z}}{2z}$$

$$then\ 1 + zC^2 = 1 + z\left(\frac{1 - 2\sqrt{1-4z} + 1 - 4z}{4z^2}\right)$$

$$= \frac{4z + 2 - 2\sqrt{1-4z} - 4z}{4z}$$

$$= \frac{1 - \sqrt{1-4z}}{2z} = C$$

$$so\quad C = 1 + zC^2$$

$$\frac{C-1}{C} = zC$$

$$1 - \frac{1}{C} = zC$$

$$1 - zC = \frac{1}{C}$$

$$C = \frac{1}{1 - zC}$$ □

2) B=1+2zCB

proof:

$$1+2zCB = 1+2z\left(\frac{1-\sqrt{1-4z}}{2z}\right)\left(\frac{1}{\sqrt{1-4z}}\right)$$

$$= 1 + \left(\frac{1-\sqrt{1-4z}}{\sqrt{1-4z}}\right)$$

$$= \frac{1}{\sqrt{1-4z}}$$

$$= B$$ □

3) $B = \frac{C}{1 - zC^2} = \frac{C}{2 - C}$

proof :

$$\frac{C}{1 - zC^2} = \frac{C}{1 - (C - 1)} \quad by\ identity\ 1$$

$$= \frac{C}{2 - C}$$

$from\ identity\ 2,\ B = \frac{1}{1 - 2zC}$

$$\frac{C}{2 - C} = \frac{1/(1 - zC)}{2 - \left(1/(1 - zC)\right)} \qquad ...by\ identity\ 1$$

$$= \frac{1}{1 - 2zC}$$

$$\Rightarrow B = \frac{C}{1 - zC^2} = \frac{C}{2 - C}$$

□

4) $C^2 = \frac{1}{1 - 2z}(1 + z^2C^4)$

proof :

$from\ identity\ 1, we\ get\ C = 1 + zC^2$

$$C^2(1 - 2z) = (1 + zC^2)^2(1 - 2z)$$
$$= 1 + 2zC^2 + z^2C^4 - 2z - 4z^2C^2 - 2z^3C^4$$
$$= 1 + 2zC^2 + z^2C^4 - 2z - 4z(zC^2) - 2z(zC^2)^2$$
$$= 1 + 2zC^2 + z^2C^4 - 2z - 4z(C - 1) - 2z(C - 1)^2 \quad ...since\ \ zC^2 = C - 1$$
$$= 1 + 2zC^2 + z^2C^4 - 2z - 4zC + 4z - 2zC^2 + 4zC - 2z$$
$$= 1 + z^2C^4$$

□

5) $C' = BC^2$

proof :

$$now\ \ C = \frac{1-\sqrt{1-4z}}{2z}\ \ then\ \ C' = \frac{1-\sqrt{1-4z}-2z}{2z^2\sqrt{1-4z}}$$

$$BC^2 = \frac{1}{\sqrt{1-4z}}\left(\frac{1-2\sqrt{1-4z}+1-4z}{4z^2}\right)$$

$$= \frac{2-2\sqrt{1-4z}-4z}{4z^2\sqrt{1-4z}}$$

$$= C' \qquad \square$$

6) $B = (zC)' = C + zC'$

proof :

$$(zC)' = \left(z\frac{1-\sqrt{1-4z}}{2z}\right)'$$

$$= \frac{1}{2}\left(\frac{-\frac{1}{2}(-4)}{\sqrt{1-4z}}\right)$$

$$= \frac{1}{\sqrt{1-4z}} = B$$

$$and\ \ C' = \frac{1-\sqrt{1-4z}-2z}{2z^2\sqrt{1-4z}}$$

$$zC' + C = \frac{1-\sqrt{1-4z}-2z}{2z\sqrt{1-4z}} + \frac{1-\sqrt{1-4z}}{2z}$$

$$= \frac{1}{\sqrt{1-4z}} = B \qquad \square$$

7) $1+zBC=\dfrac{B}{C}$

proof :

$$C+zBC^2=C+zC' \quad \text{by identity } 5$$
$$=B \quad \text{by identity } 6$$

8) $C(1+B)=2B$

proof :

from identity 7, we have $B=C(1+zBC)$

then $2B=2C(1+zBC)$

$$=C(2+2zBC)$$
$$=C(1+1+2zBC)$$
$$=C(1+B) \quad (\text{by identity } 2)$$

9) $F=\dfrac{C}{1+zC},\ C=\dfrac{F}{1-zF},\ zFC=C-F$

proof :

from figure 10, we have $F=\dfrac{1}{1-z^2C^2}$

$$=\frac{1}{(1-zC)(1+zC)}$$
$$=\frac{C}{1+zC} \quad (\text{by identity } 1)$$

then $F(1+zC)=C$

$zCF=C-F$

$F=C(1-zF)$

$$C=\frac{F}{1-zF}$$

10) $F = \dfrac{1}{1-z^2C^2} = \dfrac{2}{1+2z+\sqrt{1-4z}}$

proof :

from figure 10, *we obtain* $F = \dfrac{1}{1-z^2C^2}$

$$\frac{1}{1-z^2C^2} = \frac{1}{1-z^2\left(\left(1-\sqrt{1-4z}\right)/2z\right)^2}$$

$$= \frac{4}{2+2\sqrt{1-4z})+4z}$$

$$= \frac{2}{1+\sqrt{1-4z}+2z}$$ □

11) $1+C = (z+2)F$

proof :

$$1+C = 1+\frac{1}{1-zC} \qquad \textit{by identity } 1$$

$$= \frac{2-zC}{1-zC}\ \frac{1+zC}{1+zC}$$

$$= \frac{2+2zC-zC-z^2C^2}{1-z^2C^2}$$

$$= F(2+zC-z^2C^2) \qquad (\textit{by identity } 10)$$

$$= F(2+z(C-zC^2))$$

$$= F(2+z.1) \qquad (\textit{by identity } 1)$$

$$= (2+z)F$$ □

12) $z(z+2)F^2-(1+2z)F+1=0$

proof :

$$\text{from identity 9, we get } F=\frac{C}{1+zC}$$

$$z(z+2)F^2-(1+2z)F+1=z(z+2)\left(\frac{C}{1+zC}\right)^2-(1+2z)\frac{C}{1+zC}+1$$

$$=\frac{z^2C^2+2zC^2-C-2zC-zC^2-2z^2C^2}{(1+zC)^2}+1$$

$$=\frac{-z^2C^2+zC^2-C-2zC}{(1+zC)^2}+1$$

$$=\frac{-z^2C^2-1-2zC}{(1+zC)^2}+1 \qquad (\text{by identity } 1)$$

$$=\frac{-(z^2C^2+1+2zC)}{(1+zC)^2}+1$$

$$=-1+1$$

$$=0$$

□

13) $BC=F+3zBCF$

proof :

$$F+3zBCF=F(1+3zBC)$$

$$=\frac{1+3zBC}{1-z^2C^2} \qquad (\text{by identity } 10)$$

$$=\frac{1+2zBC+zBC}{1-z^2C^2}$$

$$=\frac{B+zBC}{1-z^2C^2} \qquad (\text{by identity } 2)$$

$$=\frac{B(1+zC)}{(1-zC)(1+zC)}$$

$$=\frac{B}{1-zC}$$

$$=BC \qquad (\text{by identity } 1)$$

□

14) $F + 2BC = 3BF$

proof :

by identity 13, *we have* $BC = F + 3zBCF$

then $2BC = 2F(1+3zBC)$

$$= 2F(1+2zBC+zBC)$$

$$= 2F(B+zBC) \qquad (by\ identity\ 2)$$

$$= 2BF(1+zC)$$

$$F + 2BC = F + 2BF(1+zC)$$

$$= F(1+2B(1+zC))$$

$$= F(1+2B+2BzC)$$

$$= F(B+2B) \qquad (by\ identity\ 2)$$

$$= 3BF$$ ∎

15) $\frac{1}{1-4z} = BC + z(BC)^2$

proof :

$$BC + z(BC)^2 = BC(1+zBC)$$

$$= BC(1+\frac{B-1}{2}) \qquad by\ identity\ 2$$

$$= BC(\frac{1+B}{2})$$

$$= \frac{B}{2}C(1+B)$$

$$= \frac{B}{2}2B \qquad by\ identity\ 8$$

$$= B^2$$

we know that the generating function of $B = \frac{1}{\sqrt{1-4z}}$

then $B^2 = \frac{1}{1-4z}$

Hence $\frac{1}{1-4z} = BC + z(BC)^2$ ∎

16) $B' = 2B^3$

proof :

$$from\ identity\ 2, B = \frac{1}{1-2zC}\quad then\ B' = \frac{2(zC)'}{(1-2zC)^2}$$

$$= 2B.B^3\ by\ identity\ 1\&6 \qquad \square$$

17) $F' = 2zBCF^2$

proof :

$$from\ identity\ 10,\ F = \frac{1}{1-z^2C^2}\quad then\ F' = \frac{(z^2C^2)'}{(1-z^2C^2)^2}$$

$$= F^2\left(2zC^2 + 2z^2CC'\right)$$

$$= F^2\left(2zC^2 + 2z^2CBC^2\right) \qquad (by\ identity\ 5)$$

$$= F^2zC^2(2+2zCB)$$

$$= F^2zC^2(1+1+2zBC)$$

$$= F^2zC^2(1+B) \qquad (by\ identity\ 2)$$

$$= F^2zCC(1+B)$$

$$= F^2zC(2B) \qquad (by\ identity\ 8)$$

$$= 2zBCF^2 \qquad \square$$

Section 3. Some Applications of Catalan Identities

Fibonacci polynomials

Fibonacci pol ynomials a re a p olynomial s equence w hich c an be considered as a generalization of the Fibonacci n umbers. T he F ibonacci num bers a re r ecovered by evaluating the Fibonacci polynomials at x=1.

Definition: The Fibonacci polynomials are defined recursively by,

$$F_1(x)=1, F_2(x)=x, and\ F_{n+1}(x)=xF_n(x)+F_{n-1}(x) \quad for\ n\geq 2.$$

The first few Fibonacci polynomials are

$$F_1(x)=1,\ F_2(x)=x,\ F_3(x)=x^2+1,\ F_4(x)=x^3+2x,\ F_5(x)=x^4+3x^2+1,\ F_6(x)=x^5+4x^3+3x$$

Then the degree of F_n *is* $n-1$.

1) Prove that $\sum_{n=0}^{\infty} x^n F_{n+1}(4/x)=\sum_{k=0}^{\infty}\binom{2n+2}{n-2k}F_{2k+1}(x)$

Proof:

First let us show that $\sum_{n=0}^{\infty} F_{n+1}(x)z^n=\dfrac{1}{1-xz-z^2}$, $\sum_{n=0}^{\infty} x^n F_{n+1}(4/x)z^n=\dfrac{1}{1-4z-x^2z^2}$

and $\sum_{k\geq 0} F_{2k+1}(x)z^k=\dfrac{1-z}{1-(2+x^2)z+z^2}$.

$$Let\ \sum_{n=0}^{\infty} F_{n+1}(x)z^n=F(x,z)\ \ then\ F(x,z)=F_1(x)+\sum_{n=1}^{\infty} F_{n+1}(x)z^n$$

$$=F_1(x)+F_2(x)z+\sum_{n=2}^{\infty} F_{n+1}(x)z^n$$

by snake-oil method

$$\sum_{n=2}^{\infty} F_{n+1}(x)z^n=\sum_{n=2}^{\infty} xF_n(x)z^n+\sum_{n=2}^{\infty} F_{n-1}(x)z^n$$

$$F(x,z)-1-xz=xz\sum_{n=2}^{\infty} F_n(x)z^{n-1}+z^2\sum_{n=2}^{\infty} F_{n-1}(x)z^{n-2}$$

$$=xz\sum_{n=1}^{\infty} F_{n+1}(x)z^n+z^2\sum_{n=0}^{\infty} F_{n+1}(x)z^n$$

$$=xz(F(x,z)-1)+z^2F(x,z)$$

$$\Rightarrow\ F(x,z)=\frac{1}{1-xz-z^2}$$

and

$$let\ \sum_{n=0}^{\infty} x^n F_{n+1}(4/x)z^n = G(x,z)\ \ then\ \ G(x,z) = F_1(4/x) + \sum_{n=1}^{\infty} x^n F_{n+1}(4/x)z^n$$

$$= F_1(4/x) + xF_2(4/x)z + \sum_{n=2}^{\infty} x^n F_{n+1}(4/x)z^n$$

From the above definition, we obtain

$$x^n\, F_{n+1}(4/x) = x^n\, 4/x\, F_n(4/x) + x^n F_{n-1}(4/x)$$

Then by snake-oil method

$$\sum_{n=2}^{\infty} x^n\, F_{n+1}(4/x)z^n = \sum_{n=2}^{\infty} x^n\, 4/x\, F_n(4/x)z^n + \sum_{n=2}^{\infty} x^n\, F_{n-1}(4/x)z^n$$

$$G(x,z) - F_1(4/x) - xzF_2(4/x) = \frac{4xz}{x}\sum_{n=2}^{\infty} x^{n-1}F_n(4/x)z^{n-1} + x^2z^2\sum_{n=2}^{\infty} x^{n-2}\, F_{n-1}(4/x)z^{n-2}$$

$$G(x,z) - 1 - \frac{4xz}{x} = 4z\sum_{n=1}^{\infty} x^n F_{n+1}(4/x)z^n + x^2z^2\sum_{n=0}^{\infty} x^n\, F_{n+1}(4/x)z^n$$

$$G(x,z) - 1 - 4z = 4z\left(G(x,z) - F_1(4/x)\right) + x^2z^2G(x,z)$$

$$\Rightarrow\ G(x,z) = \frac{1}{1-4z-x^2z^2}$$

and

$$let\ \sum_{k\ge 0} F_{2k+1}(x)z^k = H(x,z), then\ H(x,z) = F_1(x) + \sum_{k\ge 1} F_{2k+1}(x)z^k$$

$$= F_1(x) + F_3(x)z + \sum_{k\ge 2} F_{2k+1}(x)z^k$$

by snake − oil method we obtain,

$$\sum_{k\ge 2} F_{2k+1}(x)z^k = \sum_{k\ge 2} xF_{2k}(x)z^k + \sum_{k\ge 2} F_{2k-1}(x)z^k$$

$$H(x,z) - 1 - (x^2+1)z = xz\sum_{k\ge 2} F_{2k}(x)z^{k-1} + z\sum_{k\ge 2} F_{2k-1}(x)z^{k-1} \qquad ...\sin ce\ F_3(x) = x^2+1$$

$$= xz\sum_{k\ge 1} F_{2k+2}z^k + z\sum_{k\ge 1} F_{2k+1}z^k$$

$$= xz\left(F_4(x)z + F_6(x)z^2 + F_8(x)z^3 + ...\right) + z\left(H(x,z) - 1\right)$$

$$= xz\left(xF_3(x)z + F_2(x)z + xF_5(x)z^2 + F_4(x)z^2 + xF_7(x)z^3 + F_6(x)z^3 + ...\right)$$

$$+ z\left(H(x,z) - 1\right)$$

$$H(x,z)(1-z) - (1+x^2z) = xz(xF_3(x)z + F_2(x)z + xF_5(x)z^2 + xF_3(x)z^2 + F_2(x)z^2$$

$$+ xF_7(x)z^3 + xF_5(x)z^3 + F_4(x)z^3 + ...)$$

$$= xz(xF_3(x)z + F_2(x)z + xF_5(x)z^2 + xF_3(x)z^2 + F_2(x)z^2 + xF_7(x)z^3 + xF_5(x)z^3 + xF_3(x)z^3 + F_2(x)z^3 + \ldots)$$

$$= xz(F_2(x)z + F_2(x)z^2 + F_2(x)z^3 + \ldots + xF_3(x)z + xF_3(x)z^2 + xF_3(x)z^3 + \ldots + xF_5(x)z^2 + xF_5(x)z^3 + \ldots + xF_7(x)z^3 + \ldots)$$

$$= xz\left(\frac{z}{1-z}F_2(x) + \frac{xz}{1-z}F_3(x) + \frac{xz^2}{1-z}F_5(x) + \frac{xz^3}{1-z}F_7(x) + \ldots\right)$$

$$= xz\left(\frac{zx}{1-z} + \frac{xz}{1-z}\sum_{k\geq 1} F_{2k+1}(x)z^{k-1}\right) \qquad \ldots \text{since } F_2(x) = x$$

$$= xz\left(\frac{zx}{1-z} + \frac{xz}{1-z}\frac{1}{z}\sum_{k\geq 1} F_{2k+1}(x)z^{k}\right)$$

$$= xz\left(\frac{zx}{1-z} + \frac{x}{1-z}\left(H(x)-1\right)\right)$$

$$\text{so } H(x,z) = \frac{1-z}{1-2z+z^2-x^2z}$$

$$\text{Therefore } \sum_{k\geq 0} F_{2k+1}(x)z^k = \frac{1-z}{1-(2+x^2)z+z^2} \qquad \ldots(*)$$

$$\text{Let } A = \left(a_{n,k}\right)_{n,k\geq 0} = \binom{2n+2}{n-2k} = \begin{bmatrix} 1 & 0 & 0 & \ldots \\ 4 & 0 & 0 & \ldots \\ 15 & 1 & 0 & \ldots \\ 56 & 8 & 0 & \ldots \\ 210 & 45 & 1 & \ldots \\ 792 & 220 & 12 & \ldots \\ 3003 & 1001 & 91 & \ldots \\ 11440 & 4368 & 560 & \ldots \\ & \ldots & & \end{bmatrix}$$

Recall that: The generating function of

$$C = 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + \ldots$$

$$\Rightarrow C' = 1 + 4z + 15z^2 + 56z^3 + 210z^4 + \ldots, C^2 = 1 + 2z + 5z^2 + 14z^3 + 42z^4 + \ldots,$$

$$C^2.C^2 = C^4 = 1 + 4z + 14z^2 + 48z^3 + 165z^4 + \ldots$$

by identity 5, *we get* $C' = BC^2$

so the generating function of the first column of $A = \left(a_{n,k}\right)_{n,k\geq 0}$ *is* $C' = BC^2$.

By Riordan array,

$$d(z)=BC^2 \quad and \quad zd(z)h(z)=z^2+8z^3+45z^4+220z^5+1001z^6+...$$
$$=z^2\left(1+8z+45z^2+220z^3+1001z^4+...\right)$$
$$=z^2\left(1+4z+15z^2+56z^3+210z^4+...\right)\left(1+4z+14z^2+48z^3+165z^4+...\right)$$
$$=z^2BC^2C^4$$
$$\Rightarrow h(z)=zC^4$$

$$and\ d(z)\left(zh(z)\right)^n=BC^2\left(z.zC^4\right)^n=z^{2n}BC^{4n}$$

then the generating functions of the columns of the matrix $A=(a_{n,k})_{n,k\geq 0}$ *with* $a_{n,k}=\binom{2n+2}{n-2k}$ *are* $BC^2, z^2BC^6, z^4BC^{10}, z^6BC^{14}$,....

thus the identities can be written in matrix form as

$$\begin{bmatrix} F_1(\frac{4}{x}) \\ xF_2(\frac{4}{x}) \\ x^2F_3(\frac{4}{x}) \\ x^3F_4(\frac{4}{x}) \\ . \\ . \\ . \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & ... \\ 4 & 0 & 0 & ... \\ 15 & 1 & 0 & ... \\ 56 & 8 & 0 & ... \\ 210 & 45 & 1 & ... \\ 792 & 220 & 12 & ... \\ 3003 & 1001 & 91 & ... \\ 11440 & 4368 & 560 & ... \\ & ... & & \end{bmatrix} \begin{bmatrix} F_1(x) \\ F_3(x) \\ F_5(x) \\ F_7(x) \\ . \\ . \\ . \end{bmatrix}$$

introducing generating functions in z in the column yields.

$$\sum_{n=0}^{\infty} x^n F_{n+1}(\tfrac{4}{x})z^n = \begin{bmatrix} BC^2 & z^2BC^6 & z^4BC^{10} & z^6BC^{14} & ... \end{bmatrix} \begin{bmatrix} F_1(x) \\ F_3(x) \\ F_5(x) \\ F_7(x) \\ . \\ . \end{bmatrix} \qquad ...(**)$$

so the right hand side of (**) becomes

$$\sum_{n=1}^{\infty}(z^{2n-2}BC^{4n-2})F_{2n-1}(x)=\sum_{n=0}^{\infty}(z^{2n}BC^{2+4n})F_{2n+1}(x)$$
$$=BC^2\sum_{n=0}^{\infty}F_{2n+1}(x)(z^2C^4)^n$$

$$= BC^2 \frac{1 - z^2C^4}{1 - (2 + x^2)z^2C^4 + (z^2C^4)^2} \qquad ...by\ (*)$$

$$= BC^2 \frac{1 - z^2C^4}{(1 - z^2C^4)^2 - x^2z^2C^4}$$

$$from\ \ identity\ \ 3\ \ we\ get\quad 1 - zC^2 = \frac{C}{B} \qquad ...(***)$$

$$= C\sqrt{1-4z}$$

$$rationalize\ \ the\ \ numerator\ of\ \ the\ left\ hand\ side\ of\ \ (***)$$

$$that\ \ is\quad (1 - zC^2)\frac{1 + zC^2}{1 + zC^2} = C\sqrt{1-4z}$$

$$1 - z^2C^4 = C\sqrt{1-4z}(1 + zC^2)$$

$$= C^2\sqrt{1-4z} \qquad by\ \ identity\ 1$$

$$\sum_{n=0}^{\infty}(z^{2n}BC^{2+4n})F_{2n+1}(x) = BC^2 \frac{C^2\sqrt{1-4z}}{(C^2\sqrt{1-4z})^2 - x^2z^2C^4}$$

$$= B\frac{\sqrt{1-4z}}{(1-4z) - x^2z^2}$$

$$= \frac{1}{\sqrt{1-4z}}\frac{\sqrt{1-4z}}{(1-4z) - x^2z^2}$$

$$= \frac{1}{1 - 4z - x^2z^2}$$

$$Therefore \qquad \sum_{n=0}^{\infty} x^n F_{n+1}(\tfrac{4}{x}) = \sum_{k=0}^{\infty}\binom{2n+2}{n-2k}F_{2k+1}(x) \qquad \square$$

2) Show that $[z^n]\, zBF = \frac{2}{3}\binom{2n-1}{n} + \frac{1}{3}F_{n-1}$, where the $(F_n)_{n\geq 0}$ are the Fine numbers.

First let us find the generating function counting all blocks of odd size

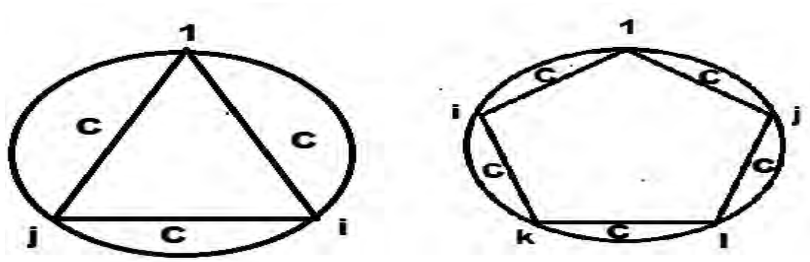


Figure 11.

If we fix 1 in a block of size 3, i.e. where i and j are another two elements in a block, then we need the rest also non-crossing partitions. Since we have z^3 for three elements and C^3 for the remaining partitions.

Then the generating function for the total number of non-crossing partitions, where 1 is in a block of size 3 is $z^3 C^3$. But we may have also any element instead of 1 in a block of size 3 and similar argument as above gives us the same generating function holds for either 1 or 2 or … n is in a block of size 3.

Thus the generating function for the total number of blocks of size 3 is given by $\frac{1}{3}z(z^3C^3)'$. Similarly we can find the generating function for the total number of blocks of size 5 is given by $\frac{1}{5}z(z^5C^5)'$.

So the generating function counting all blocks of odd size is

$$\begin{aligned}
&z(zC)'+z\frac{((zC)^3)'}{3}+z\frac{((zC)^5)'}{5}+\ldots\\
&=z[(zC)'](1+(zC)^2+(zC)^4+\ldots)\\
&=z[(zC)']\frac{1}{1-z^2C^2}\\
&=zBF \qquad \textit{by identity } 6 \textit{ and } 10
\end{aligned}$$

$$\textit{from identity } 14, \textit{we have } (k+1)BF=kBC+F$$

$$zBF=\frac{k}{k+1}zBC+\frac{1}{k+1}zF$$

$$\begin{aligned}
[z^n]zBF&=\frac{k}{k+1}[z^n]zBC+\frac{1}{k+1}[z^n]zF\\
&=\frac{k}{k+1}[z^{n-1}]BC+\frac{1}{k+1}[z^{n-1}]F\\
&=\frac{k}{k+1}\binom{k(n-1)+1}{n-1}+\frac{1}{k+1}F_{n-1}, \qquad \text{since } [z^n]BC^S=\binom{kn+s}{n}
\end{aligned}$$

$$= \frac{k}{k+1}\binom{kn-k+1}{n-1} + \frac{1}{k+1}F_{n-1}$$

set $k = 2$, *we get*

$$[z^n]zBF = \frac{2}{3}\binom{2n-1}{n-1} + \frac{1}{3}F_{n-1}$$ ∎

3) Show that the expected number of peaks over fine paths of length $2n$ *is* $\frac{n}{2}$.

Proof:

Let $\lambda(t,z)$ be the bivariate generating function for Dyck paths with a t marking each peak and z marking each up step.

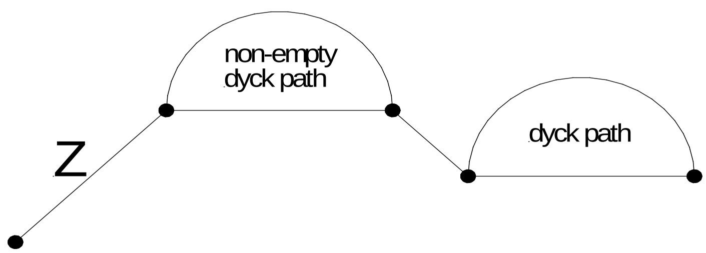


Figure 12.

$$\Rightarrow \lambda(t,z) = 1 + z(\lambda - 1 + t)\lambda$$

by implicit differentiation, we have

$$\frac{\partial\lambda}{\partial t} = \frac{\partial}{\partial t}(1 + z\lambda^2 - z\lambda + zt\lambda)$$

$$= 2z\lambda\frac{\partial\lambda}{\partial t} - z\frac{\partial\lambda}{\partial t} + z\lambda + zt\frac{\partial\lambda}{\partial t}$$

$$= \frac{\partial\lambda}{\partial t}(2z\lambda - z + zt) + z\lambda$$

$$\frac{\partial\lambda}{\partial t}(1 - 2z\lambda + z - zt) = z\lambda$$

$$\frac{\partial\lambda}{\partial t} = \frac{z\lambda(t,z)}{1 - 2z\lambda(t,z) + z - zt}$$

$$\left(\frac{\partial \lambda}{\partial t}\right)_{t=1} = \frac{z\lambda(1,z)}{1-2z\lambda(1,z)+z-z(1)}$$

$$= \frac{zC}{1-2zC}, \qquad \ldots \text{since } \lambda(1,z)=C$$

$$= \frac{zC}{1-2z(\frac{1-\sqrt{1-4z}}{2z})}$$

$$= \frac{zC}{1-(1-\sqrt{1-4z})}$$

$$= \frac{zC}{\sqrt{1-4z})}$$

$$= zCB \qquad \ldots(*)$$

And let $T(t,z)$ be the bivariate generating function for Fine paths with a t marking each peak and z marking each up step.

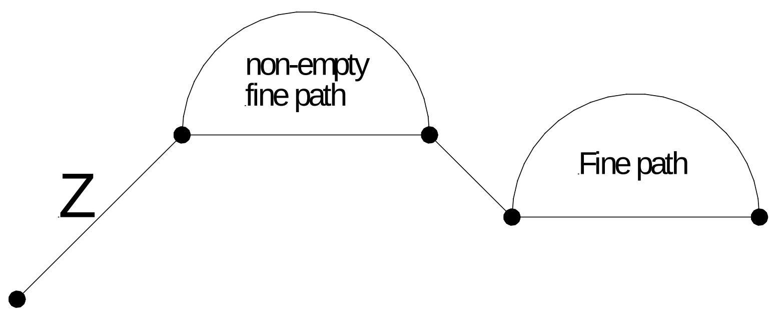


Figure 13.

$$\Rightarrow \; T(t,z) = 1 + z(\lambda - 1)T$$

by implicit differentiation,we have

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}(1 + z\lambda T - zT)$$

$$= zT\frac{\partial \lambda}{\partial t} + z\lambda\frac{\partial T}{\partial t} - z\frac{\partial T}{\partial t}$$

$$\frac{\partial T}{\partial t}(1 - z\lambda + z) = zT\frac{\partial \lambda}{\partial t}$$

$$\frac{\partial T}{\partial t} = \frac{zT(t,z)}{1 - z\lambda(t,z) + z}\frac{\partial \lambda(t,z)}{\partial t}$$

$$\left(\frac{\partial T}{\partial t}\right)_{t=1} = \frac{zT(1,z)}{1-z\lambda(1,z)+z}\left(\frac{\partial\lambda(t,z)}{\partial t}\right)_{t=1}$$

$$= \frac{zF}{1-zC+z}zBC \qquad \text{...since } T(1,z)=F,\ \lambda(1,z)=C \text{ and by } *$$

$$= \frac{z^2BC\,F}{1-z(C-1)}$$

$$= \frac{z^2BC\,F}{1-z(zC^2)} \qquad \text{...by identity 1}$$

$$= z^2BC\,F.F \qquad \text{...by identity 10}$$

$$= z^2BC\,F^2$$

$$= \frac{z}{2}2zBC\,F^2 \qquad \text{...(**)}$$

from identity 10, $F = \dfrac{1}{1-z^2C^2} \;\Rightarrow\; F' = \dfrac{2zC(C+zC')}{(1-z^2C^2)^2} = 2zCBF^2$

$$(**) \text{ becomes } \left(\frac{\partial T}{\partial t}\right)_{t=1} = \frac{z}{2}F'$$

$$[z^n]\left(\frac{\partial T}{\partial t}\right)_{t=1} = \frac{1}{2}[z^{n-1}]F'$$

$$= \frac{1}{2}n\,F_n$$

Hence the expected number of peaks over fine paths of length $2n = \dfrac{[z^n]\left(\frac{\partial T}{\partial t}\right)_{t=1}}{F_n} = \dfrac{n}{2}$

4) Prove that $\frac{d^n}{dz^n}B=\frac{(2n)!}{n!}B^{2n+1}$

proof :

Method I)

Let us applying the principles of mathematical induction

for $n=1$

$\frac{d}{dz}B=B'=2B^3$ ...*by identity* 16

and right hand side becomes $\frac{(2.1)!}{1!}B^{2.1+1}=2B^3$

$\Rightarrow$ *it is true for* $n=1$

Assume it is true for $n=k$, *that is* $\frac{d^k}{dz^k}B=\frac{(2k)!}{k!}B^{2k+1}$

we want to show that it is true for $n=k+1$, *that is* $\frac{d^{k+1}}{dz^{k+1}}B=\frac{(2(k+1))!}{(k+1)!}B^{2(k+1)+1}$

now $\frac{d^{k+1}}{dz^{k+1}}B=\frac{d}{dz}\left(\frac{d^k}{dz^k}B\right)$

$$=\frac{d}{dz}\left(\frac{(2k)!}{k!}B^{2k+1}\right) \quad \ldots \quad \textit{from the assumption}$$

$$=\frac{d}{dz}\left(\frac{(2k)!}{k!}\left(\frac{1}{\sqrt{1-4z}}\right)^{2k+1}\right)$$

$$=\frac{(2k)!}{k!}\frac{d}{dz}\left(\left(\frac{1}{\sqrt{1-4z}}\right)^{2k+1}\right)$$

$$=\frac{(2k)!}{k!}\frac{2(2k+1)}{\left(\sqrt{1-4z}\right)^{2k+3}}$$

$$=\frac{2(2k+1)!}{k!\left(\sqrt{1-4z}\right)^{2(k+1)+1}}\frac{k+1}{k+1}$$

$$=\frac{(2k+2)!}{(k+1)!}B^{2(k+1)+1}$$

So it is true for n= k+1

$$\textit{Hence} \quad \frac{d^n}{dz^n}B=\frac{(2n)!}{n!}B^{2n+1}$$

Method II)

We know that $B=\frac{1}{\sqrt{1-4z}}$

$$\frac{d}{dz}B=\frac{2}{\left(\sqrt{1-4z}\right)^3}=2B^3=2!B^{2.1+1}$$

$$=\frac{(2.1)!}{1!}B^{2.1+1}$$

$$\frac{d^2}{dz^2}B=\frac{d}{dz}\left(\frac{d}{dz}B\right)=\frac{d}{dz}\left(\frac{2}{\left(\sqrt{1-4z}\right)^3}\right)=\frac{4.3}{\left(\sqrt{1-4z}\right)^5}=4.3B^5=\frac{(4)!}{2!}B^{2.2+1}$$

$$=\frac{(2.2)!}{2!}B^{2.2+1}$$

$$\frac{d^3}{dz^3}B=\frac{d}{dz}\left(\frac{d^2}{dz^2}B\right)=\frac{d}{dz}\left(\frac{4.3}{\left(\sqrt{1-4z}\right)^5}\right)=\frac{6.5.4}{\left(\sqrt{1-4z}\right)^7}=6.5.4B^7=\frac{(6)!}{3!}B^{2.3+1}$$

$$=\frac{(2.3)!}{3!}B^{2.3+1}$$

.

.

.

$$\frac{d^n}{dz^n}B=\frac{(2.n)!}{n!}B^{2.n+1}$$

Therefore the proof is complete.

Section 4. k-Trees and Generalized Catalan Identities

4.1) k-Trees

The graph consisting of two nodes joined by an edge is a 2-tree, and a 2-tree with n+1 nodes is any graph obtained by joining a new node to any two nodes already joined in a 2-tree with n nodes [9]. k-trees generalize ordered trees in the sense that ordered trees are 2-trees in which edges between nodes are drawn as double edges.

A k-tree is constructed from a single distinguished k-cycle, an elementary k-sided cycle, by repeatedly gluing other k-cycles to existing ones along an edge. More than one cycle can be glued to a non-terminal or internal edge. If K is any nonempty subset of {2, 3, 4,…},then a k-tree is obtained in similar way using k-cycles with k ε K.

An edge of the distinguished k-cycle is designated as a root and all children of the internal edges are shown in order in the plane containing the distinguished k-cycle. If an internal e dge ha s more t han one c hild, w e gl ue t he c ycles t o t he i nternal e dge w ith diminishing size. If all cycles of a k-tree have the same number of sides, say k, we refer to it as a homogeneous k-tree. [6]

For example, we obtain the three 3-trees below using two 3-cycles(i.e. n = 2 and k = 3)

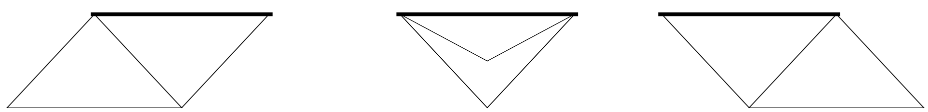


Figure 14: The three 3-trees consisting of two 3-cycles

Then the number of 3-trees consisting of 2 triangles is $C_{2,3} = 3$.

and we obtain the twelve 3-trees shown below using three 3-cycles(i.e. n = 3 and k= 3)

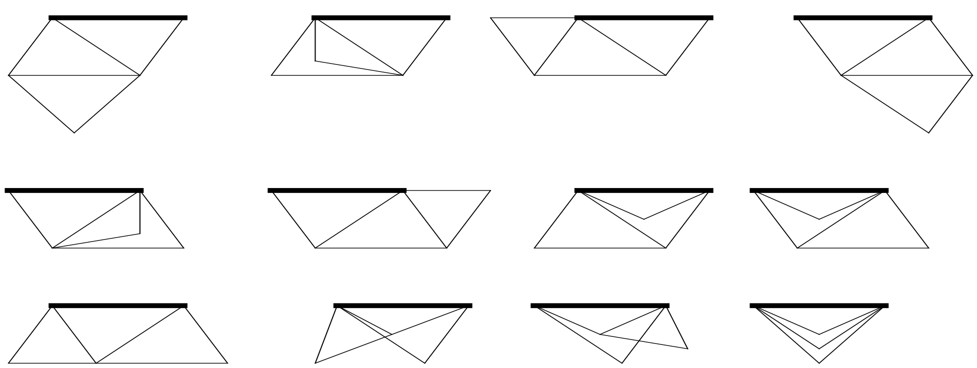


Figure 15: the twelve 3-trees on three cycles.

Then the number of 3-trees consisting of 3 triangles is $C_{3,3} = 12$.

4.2) Generalized Catalan Identities

4.2.1) Generalized Catalan Numbers

Let the generating function of k-trees be

$$C(z)=\sum_{n=0}^{\infty} C_{n,k} z^n$$

Where $C_{n,k}$ = the number of k-trees with exactly n k-cycles.

If we begin with a distinguished k-cycle and construct an ordered k-tree recursively by attaching another k-cycle to one of the k edges of the distinguished k-cycle as shown below.

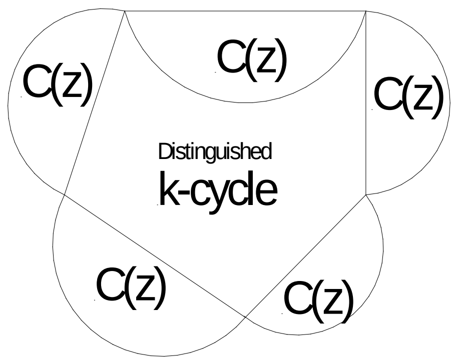


Figure 16: Recursive construction of k-trees

Then we obtain a functional relation

$$C(z)=1+zC^k(z)=\frac{1}{1-zC^{k-1}(z)}$$

where 1 counts the empty tree consisting of only the distinguished edge.

$$C(z)=1+zC^k(z) \;\; as \;\; u(z)=z\left(u(z)+1\right)^k, \; where \; u(z)=C(z)-1$$

$$\text{let } \Phi(z)=(1+z)^k$$

$$choosing \quad f(u(z))=\left(1+u(z)\right)^s=C^s(z)$$

$$by \;\; Lagrange \;\; inversion \;\; formula, we \;\; have$$

$$[z^n]C^s(z)=[z^n]\left(1+u(z)\right)^s=[z^n]f(u(z))$$

$$=\frac{1}{n}[u^{n-1}]f'(u(z))\left(\Phi(\mathrm{u})\right)^n$$

$$=\frac{1}{n}[u^{n-1}]s\left(1+u(z)\right)^{s-1}\left(1+u(z)\right)^{kn}$$

$$=\frac{s}{n}[u^{n-1}]\left(1+u(z)\right)^{kn+s-1}$$

$$=\frac{s}{n}\binom{kn+s-1}{n-1} \qquad \text{...by the binomial theorem}$$

$$=\frac{s}{kn+s}\binom{kn+s}{n}$$

Hence the generating function $\mathrm{C(z)}=\sum_{\mathrm{n=0}}^{\infty}\mathrm{C}_{\mathrm{n,k}}z^{n}$ satisfies the recurrence relation

$$\mathrm{C(z)=1+zC^{k}}(z)$$

and

$$[z^{n}]C^{s}(z)=\frac{s}{kn+s}\binom{kn+s}{n}$$

If we let s = 1, we obtain sequnces of numbers given by

$$\mathrm{C}_{\mathrm{n,k}}=\frac{1}{kn+1}\binom{kn+1}{n}=\frac{1}{(k-1)n+1}\binom{kn}{n}$$

This is called the Catalan numbers.

4.2.2) Analog of Central Binomial Numbers

Let

$$\mathrm{B(z)}=\sum_{\mathrm{n=0}}^{\infty}B_{\mathrm{n,k}}z^{n}$$

where $B_{\mathrm{n,k}}$ = the number of k-trees with n k cycles in which exactly one edge in the whole tree is colored red.

A functional relation

$$\mathrm{B(z)=1+kzB(z)C^{k-1}}(z)$$

can be obtained recursively as shown below.

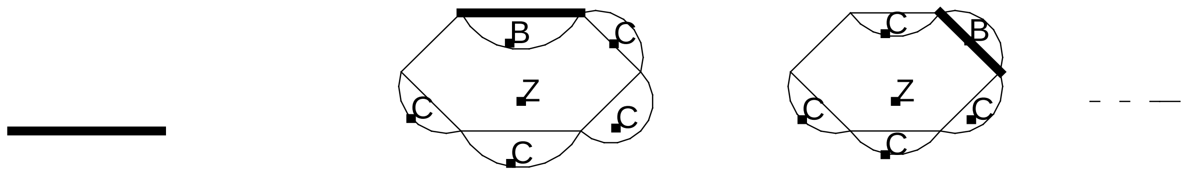


Figure 17: recursive construction of B (z)

- ❖ Show that $B_{n,k} = \binom{kn}{n}$, where $B_{n,k}$ is the total number of edges in k-trees with n k-cycles.

Proof**:**

Now by lemma 5,we have $[z^n]C^s B = \binom{kn+s}{n}$

$$[z^n]kzBC^{k-1} = k[z^{n-1}]BC^{k-1}$$

$$= k\binom{k(n-1)+k-1}{n-1}$$

$$= k\binom{kn-1}{n-1}$$

$$= \binom{kn}{n}$$

Hence the proof is complete.

4.2.3) Generalized Fine Numbers

$$Now\ \ C = \frac{1-\sqrt{1-4z}}{2z} \qquad and \qquad F = \frac{C}{1+zC},$$

$$then\ \ F = \frac{\frac{1-\sqrt{1-4z}}{2z}}{1+z(\frac{1-\sqrt{1-4z}}{2z})}$$

$$= \frac{1-\sqrt{1-4z}}{2z}\frac{2}{3-\sqrt{1-4z}}$$

$$= \frac{1-\sqrt{1-4z}}{z(3-\sqrt{1-4z})}, \textit{which is the generating function of the Fine numbers.}$$

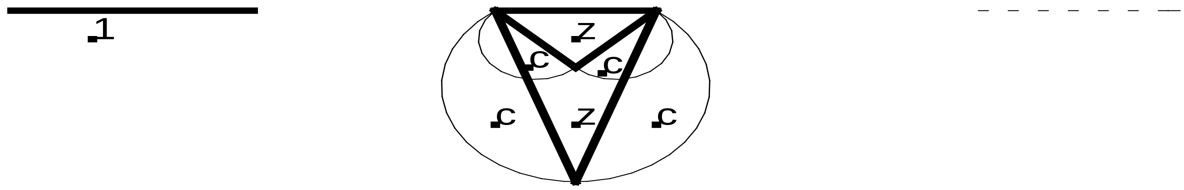


Figure 18: recursive construction of fine trees

The Fine numbers F_n are the coefficients of z^n in the Maclaurin series of

$$F(z) = \frac{1-\sqrt{1-4z}}{z(3-\sqrt{1-4z})}$$

The first few terms of the fine numbers are 1, 0, 1, 2, 6, 18, 57,…

Let $F_{n,k}$ be the number of k-trees on n k cycles with even number of cycles glued to the distinguished edge. A functional relation for its generating function

$$F(z) = \sum_{n=0}^{\infty} F_{n,k} z^n$$

is obtained recursively as shown above**.**

$$\begin{aligned} F(z) &= 1 + z^2C^{2(k-1)} + z^4C^{4(k-1)} + z^6C^{6(k-1)} + \ldots \\ &= \frac{1}{1-z^2C^{2(k-1)}} && \text{...by geometric series} \\ &= \frac{C}{1+zC^{k-1}} && \text{...by identity 4} \end{aligned}$$

4.3) Generalized Identities Involving C (z) , B (z) and F (z)

Using the functional relations,

1) $C = 1 + zC^k = \dfrac{1}{1-zC^{k-1}}$

proof:

from figure 16, we get $C = 1 + zC^k$

$$\Rightarrow \frac{C-1}{C} = zC^{k-1}$$

$$1 - \frac{1}{C} = zC^{k-1}$$

$$\Rightarrow C = \frac{1}{1-zC^{k-1}}$$

$2)\ B = 1 + kzBC^{k-1}$ ... *from figure* 17 □

$3)\ F = \dfrac{1}{1 - z^2 C^{2(k-1)}}$ ... *from figure* 18 □

$4)\ F = \dfrac{C}{1 + zC^{k-1}}; C = \dfrac{F}{1 - zFC^{k-2}}; C - F = zFC^{k-1}$

proof :

$$\text{from identity } 3, \text{we have } F = \frac{1}{1 - z^2 C^{2(k-1)}}$$

$$= \frac{1}{\left(1 + zC^{k-1}\right)\left(1 - zC^{k-1}\right)}$$

$$= \frac{C}{1 + zC^{k-1}} \qquad \text{...by identity } 1$$

$$\text{then } F + FzC^{k-1} = C$$

$$C - F = FzC^{k-1}$$

$$F = C(1 - FzC^{k-2})$$

$$C = \frac{F}{1 - zFC^{k-2}}$$ □

$5)\ B = \dfrac{C}{1 - (k-1)zC^k} = \dfrac{C}{k - (k-1)C}$

proof :

$$\text{from identity } 1, \text{we get } C - 1 = zC^k$$

$$\frac{C}{1 - (k-1)zC^k} = \frac{C}{1 - (k-1)(c-1)}$$

$$= \frac{C}{k + C - kC}$$

$$= \frac{C}{k - (k-1)C}$$

and from identity $2, B = 1 + kzBC^{k-1}$

$$\Rightarrow B = \frac{1}{1 - kzC^{k-1}}$$

$$\frac{C}{k-(k-1)C} = \frac{\left(1/(1-zC^{k-1})\right)}{k-(k-1)\left(1/(1-zC^{k-1})\right)} \qquad \text{...by identity } 1$$

$$= \frac{1}{1-kzC^{k-1}} = B \qquad \square$$

6) $C^2 = \dfrac{1}{1-2zC^{k-2}}(1+z^2C^{2k})$

proof :

from identity $1,$ *we get* $C = 1 + zC^k$

$$C^2(1-2zC^{k-2}) = (1+zC^k)^2(1-2zC^{k-2})$$

$$= 1 + 2zC^k + z^2C^{2k} - 2zC^{k-2} - 4z^2C^{2k-2} - 2z^3C^{3k-2}$$

$$= 1 + 2zC^k + z^2C^{2k} - 2zC^{k-2} - 4zC^{k-2}(zC^k) - 2zC^{k-2}(zC^k)^2$$

$$= 1 + 2zC^k + z^2C^{2k} - 2zC^{k-2} - 4zC^{k-2}(C-1) - 2zC^{k-2}(C-1)^2$$

$$= 1 + 2zC^k + z^2C^{2k} - 2zC^{k-2} - 4zC^{k-1} + 4zC^{k-2} - 2zC^k + 4zC^{k-1} - 2zC^{k-2}$$

$$= 1 + z^2C^{2k} \qquad \square$$

7) $C' = BC^k$

proof :

from identity $1,$ *we have* $C = 1 + zC^k$

$$C' = C^k + zkC^{k-1}C'$$

$$\Rightarrow C' = \frac{C^k}{1 - zkC^{k-1}}$$

$$= C^k B \qquad \text{...by identity } 2 \qquad \square$$

8) $(zC)' = C(1+zBC^{k-1}) = B(2C-B)$

proof :

$from\,identity\ 7\,we\,have\ C' = BC^k$

$(zC)' = zC' + C$

$\quad = zBC^k + C$

$\quad = C(1+zBC^{k-1})$

$and\ C + zBC^k = C + B(C-1) \qquad (by\ identity\ 1)$

$\quad = C(1+B) - B$

$and\ \ by\ identity\ \ 2,\ B = 1 + kzBC^{k-1}$

$\quad = 1 + zBC^{k-1} + (k-1)zBC^{k-1}$

$B - (k-1)zBC^{k-1} = 1 + zBC^{k-1}$

$B\left(1-(k-1)zC^{k-1}\right) = 1 + zBC^{k-1}$

$from\ identity\ 9,\ \dfrac{B}{C} = 1 + (k-1)zBC^{k-1}$

$1 - \dfrac{B}{C} = -(k-1)zBC^{k-1}$

$2 - \dfrac{B}{C} = 1 - (k-1)zBC^{k-1}$

$B(2 - \dfrac{B}{C}) = 1 + zBC^{k-1}$

$B(\dfrac{2C-B}{C}) = 1 + zBC^{k-1}$

$B(2C-B) = C(1+zBC^{k-1})$

$\quad = (zC)'$ ∎

9) $\dfrac{B}{C} = 1 + (k-1)zBC^{k-1}$

proof :

$1 + (k-1)zBC^{k-1} = 1 + kzBC^{k-1} - zBC^{k-1}$

$\quad = B - zBC^{k-1} \qquad ...by\ identity\,2$

$\quad = B(1 - zC^{k-1})$

$\quad = \dfrac{B}{C} \qquad ...by\ identity\,1$ ∎

10) $kB = C(1+(k-1)B)$

proof :

$$\text{from identity } 5, \text{we have } B = \frac{C}{k-(k-1)C}$$

$$Bk - BC(k-1) = C$$

$$Bk = C(1+(k-1)B)$$ □

11) $1 + C = (zC^{k-2} + 2)F$

proof :

$$1 + C = 1 + \frac{1}{1 - zC^{k-1}} \quad \text{by identity } 1$$

$$= \frac{2 - zC^{k-1}}{1 - zC^{k-1}}\left(\frac{1 + zC^{k-1}}{1 + zC^{k-1}}\right)$$

$$= \frac{2 + 2zC^{k-1} - zC^{k-1} - z^2C^{2(k-1)}}{1 - z^2C^{2(k-1)}}$$

$$= (2 + zC^{k-1} - z^2C^{2(k-1)})F \quad \text{...by identity } 3$$

$$= (2 + zC^{k-2}(C - zC^k))F$$

$$= (2 + zC^{k-2}.1)F \quad \text{...by identity } 1$$

$$= (2 + zC^{k-2})F$$ □

12) $z\left(zC^{k-2} + 2\right)C^{k-2}F^2 - \left(1 + 2zC^{k-2}\right)F + 1 = 0$

proof :

$$\text{from identity } 4, \text{we get } F = \frac{C}{1 + zC^{k-1}}$$

$$z(zC^{k-2} + 2)C^{k-2}F^2 - (1 + 2zC^{k-2})F + 1$$

$$= z(zC^{k-2} + 2)C^{k-2}\left(\frac{C}{1 + zC^{k-1}}\right)^2 - (1 + 2zC^{k-2})\frac{C}{1 + zC^{k-1}} + 1$$

$$= \frac{z^2C^{2k-2} + 2zC^k - (C + 2zC^{k-1})(1 + zC^{k-1})}{(1 + zC^{k-1})^2} + 1$$

$$= \frac{z^2C^{2k-2} + 2zC^k - C - zC^k - 2zC^{k-1} - 2z^2C^{k-2}}{(1 + zC^{k-1})^2} + 1$$

$$= \frac{-z^2C^{2k-2} + zC^k - C - 2zC^{k-1}}{(1+zC^{k-1})^2} + 1$$

$$= \frac{-z^2C^{2k-2} - 1 - 2zC^{k-1}}{(1+zC^{k-1})^2} + 1$$

$$= \frac{-\left(z^2C^{2k-2} + 1 + 2zC^{k-1}\right)}{(1+zC^{k-1})^2} + 1$$

$$= -1 + 1$$

$$= 0$$ □

13) $BC = (k+1)zFBC^{k-1} + F$

Proof :

$$BC = F\left(1 + (k+1)zBC^{k-1}\right)$$

$$= \frac{(k+1)zBC^{k-1} + 1}{1 - z^2C^{2(k-1)}} \qquad \text{...by identity 3}$$

$$= \frac{kzBC^{k-1} + zBC^{k-1} + 1}{1 - z^2C^{2(k-1)}}$$

$$= \frac{B + zBC^{k-1}}{1 - z^2C^{2(k-1)}} \qquad \text{...by identity 2}$$

$$= \frac{B(1+zC^{k-1})}{(1-zC^{k-1})(1+zC^{k-1})}$$

$$= \frac{B}{1 - zC^{k-1}}$$

$$= BC \qquad \text{...by identity 1}$$ □

14) $F + kBC = (k+1)BF$

proof :

by identity 13, *we have* $BC = (k+1)zFBC^{k-1} + F$

$$kBC = k\left((k+1)zFBC^{k-1} + F\right)$$

$$= kF\left((k+1)zBC^{k-1} + 1\right)$$

$$= kF(kzBC^{k-1} + 1 + zBC^{k-1})$$

$$= kF(B + zBC^{k-1}) \qquad \text{...by identity 2}$$

$$= kFB(1 + zC^{k-1})$$

$$then\ F+kBC=F+kFB(1+zC^{k-1})$$
$$=F\left(1+kB(1+zC^{k-1})\right)$$
$$=F(1+kB+kBzC^{k-1})$$
$$=F(B+kB) \qquad ...by\ identity\ 2$$
$$=(k+1)BF \qquad \square$$

15) $B'=kB^3C^{k-2}$

proof :

from identity 2, we have $B=\dfrac{1}{1-kzC^{k-1}}$

$$\Rightarrow B'=\frac{1}{(1-kzC^{k-1})^2}k(C^{k-1}+z(k-1)C^{k-2}C')$$
$$=B^2kC^{k-2}\left(C+z(k-1)BC^k\right) \qquad ...\ by\ identity\ 7$$
$$=B^2kC^{k-2}C\left(1+zkBC^{k-1}-zBC^{k-1}\right)$$
$$=B^2kC^{k-2}C(B-zBC^{k-1}) \qquad ...by\ identity\ 2$$
$$=B^2kC^{k-2}CB(1-zC^{k-1})$$
$$=B^3kC^{k-2}C\frac{1}{C} \qquad ...by\ identity\ 1$$
$$=kB^3C^{k-2} \qquad \square$$

16) $F'=2zBC^{2k-3}F^2$

proof :

$$Now\ F=\frac{1}{1-z^2C^{2(k-1)}}\quad then\ F'=\frac{(z^2C^{2(k-1)})'}{(1-z^2C^{2(k-1)})^2}$$
$$=F^2\left(2zC^{2(k-1)}+2(k-1)z^2C^{2k-3}C'\right)$$
$$=F^2\left(2zC^{2(k-1)}+2(k-1)z^2C^{2k-3}BC^k\right) \qquad ...by\ identity\ 7$$
$$=F^2\left(2zC^{2(k-1)}+2kz^2C^{3k-3}B-2z^2C^{3k-3}B\right)$$
$$=F^2\,2zC^{2(k-1)}\left(1+kzC^{k-1}B-zC^{k-1}B\right)$$
$$=2F^2zC^{2(k-1)}B(1-zC^{k-1})$$
$$=2F^2zC^{2(k-1)}B\frac{1}{C} \qquad ...by\ identity\ 1$$
$$=2zBC^{2k-3}F^2 \qquad \square$$

Lemma 5.

$$[z^n]C^s(z)B(z) = \binom{kn+s}{n}$$

Proof:

Let us consider lattice paths from $(0,0)$ *to* $(n,(k-1)n+s)$ in the plane.Given any such path we can split it in to two subpaths at the point where the original path visits the line $y=(k-1)x$ for the last time as shown below.

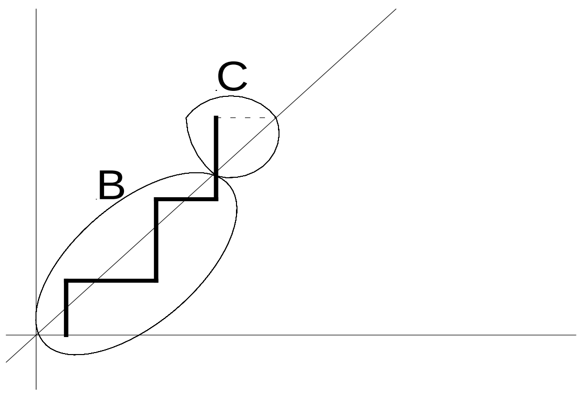


Figure 19: An example of partition of a path into B(z) and copies of C(z)

The first part is counted by B (z) [11] and the second part is counted by C(z) [7].Since the total number of paths from $(0,0)$ *to* (a,b) is $\binom{a+b}{b}$ and the path that is counted by C(z) can be further subdivided up to s Catalan paths. Hence the generating function of the Number of lattice paths from $(0,0)$ *to* $(n,(k-1)n+s)$ (0,0) is $C^s(z)B(z)$ and

$$[z^n]C^s(z)B(z) = \binom{kn+s}{n}$$ □

Lemma 6.

$$[z^n]C^s(z)F(z) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{s+2(k-1)i}{nk+s-2i} \binom{nk+s-2i}{n-2i}$$

Proof:

$$[z^n]C^s(z)F(z) = [z^n]C^s(z)\frac{1}{1-z^2C^{2(k-1)}} \qquad \text{...by identity 3}$$

$$= [z^n]\sum_{i=0}^{\infty} C^s(z)\left(z^2C^{2(k-1)}\right)^i \qquad \text{...since } \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$= [z^n]\sum_{i=0}^{\infty} z^{2i}C^{s+2(k-1)i}(z)$$

$$= [z^{n-2i}]\sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} C^{s+2(k-1)i}(z)$$

$$= \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{s+2(k-1)i}{(n-2i)k+s+2(k-1)i}\binom{(n-2i)k+s+2(k-1)i}{n-2i}$$

$$(\text{by the generalized catalan number})$$

$$= \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{s+2(k-1)i}{nk+s-2i}\binom{nk+s-2i}{n-2i}$$

so the proof is complete.

Section 5. Additional Applications and Asymptotic Results

5.1) Enumerating Edges of Odd Degree

Let $O_{n,k}$ be the total number of edges of odd degree among all k-trees with n k-cycles,

and let O(z) be the corresponding function, that is,

$$O(z) = \sum_{n=0}^{\infty} o_{n,k} z^n$$

Observation: All non-distinguished edges of odd degree in k-trees have even out degree.

I) the generating function for the total number of edges of odd degree at Level $l = 0$ among all k-trees is

$$zC^{k-1}(z) + z^3 C^{3(k-1)}(z) + \ldots$$

$$= \frac{zC^{k-1}(z)}{1 - z^2 C^{2(k-1)}(z)} \qquad \textit{by geometric series}$$

$$= zC^{k-1}(z)F(z) \qquad \textit{by identity } 3$$

II) the generating function for the total number of non-distinguished edges of odd degree at level 5 among all k-trees is obtained recursively as shown in figure 20.In addition to the five C_s shown in the figure, we can attach two C_s at each of the five non-terminal edges of the unique path. Hence, the generating function for the total number of non-distinguished edges of odd degree at level 5 among all k-trees is $2^5 z^5 C^{15}(z)F(z)$.

In general, the generating function for the total number of non-distinguished edges of odd degree at level $l \geq 1$ among all k-trees is

$$(k-1)^l z^l C^{kl}(z)F(z)$$

Hence the generating function for the total number of non-distinguished edges of odd degree among all k-trees is

$$\sum_{l=1}^{\infty} (k-1)^l z^l C^{kl}(z)F(z)$$

$$= \frac{(k-1)zC^k(z)}{1-(k-1)zC^k(z)} F(z)$$

$$= \frac{(k-1)zC^k(z)}{1-kzC^k(z)+zC^k(z)} F(z)$$

$$= \frac{(k-1)zC^k(z)}{C - kzC^k(z)} F(z) \qquad \text{...by identity 1}$$

$$= \frac{(k-1)zC^k(z)}{C(1-kzC^{k-1}(z))} F(z)$$

$$= z(k-1)C^{k-1}(z)B(z)F(z) \qquad \text{...by identity 1}$$

Combining I and II we obtain:

$$O(z) = zC^{k-1}(z)F(z) + z(k-1)C^{k-1}(z)B(z)F(z)$$

$$= zC^{k-1}(z)F(z)\big(1+(k-1)B(z)\big)$$

$$= zC^{k-2}(z)F(z)C\big(1+(k-1)B(z)\big)$$

$$= kzC^{k-2}(z)F(z)B(z) \qquad \text{... by identity 10}$$

$$= kzC^{k-2}(z)\left(\frac{F(z)+kB(z)C(z)}{k+1}\right) \qquad \text{... by identity 14}$$

$$= \frac{k}{k+1} zC^{k-2}(z)F(z) + \frac{k^2}{k+1} zB(z)C^{k-1}(z)$$

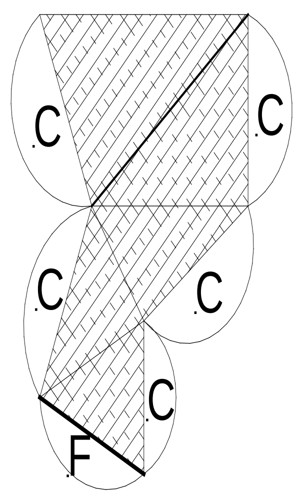


Figure 20: The unique path of 3-cycles to a terminal cycle at level 5.

Therefore, the total number $O_{n,k}$ of edges of odd degree among all k-trees with n k-cycles is:

$$O_{n,k} = [z^n]\left(\frac{k}{k+1} zC^{k-2}(z)F(z) + \frac{k^2}{k+1} zB(z)C^{k-1}(z)\right)$$

$$= [z^{n-1}]\left(\frac{k}{k+1} C^{k-2}(z)F(z) + \frac{k^2}{k+1} B(z)C^{k-1}(z)\right)$$

$$=\frac{k}{k+1}\left([z^{n-1}]C^{k-2}(z)F(z)\right)+\frac{k^2}{k+1}\left([z^{n-1}]B(z)C^{k-1}(z)\right)$$

$$=\frac{k}{k+1}\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\frac{k-2+2(k-1)i}{(n-1)k+k-2-2i}\binom{(n-1)k+k-2-2i}{n-1-2i}+\frac{k^2}{k+1}\binom{k(n-1)+k-1}{n-1}$$

$$(by\ lemma\ 5\ and\ lemma\ 6)$$

Example:

1) The total number of edges of odd degree among the twelve 3-trees Consisting of three 3-cycles in figure 16 is:

$$o_{3,3}=\frac{3}{4}\left(\frac{1}{7}\binom{7}{2}+\frac{5}{5}\binom{5}{0}\right)+\frac{9}{4}\binom{8}{2}$$

$$=66$$

5.2) Enumerating Edges of Odd out degree

Let $u_{n,k}$ be the total number of edges of odd out degree among all k-trees with n k-cycles and let

$$u(z)=\sum_{n=0}^{\infty}u_{n,k}z^n$$

be the corresponding generating function.

Then the total number of edges of out degree 1 among all k-trees with n k-cycles is

$$n[z^n]\left(zC^{k-1}(z)\right)$$

$$=[z^n]\left(z\left(zC^{k-1}(z)\right)'\right)\qquad let\ a_n\leftrightarrow g(z)\,then\ \ n\,a_n\leftrightarrow zg'(z)$$

Similarly, it can be shown that the total number of edges of out degree $2m+1(m\geq 0)$ among all k-trees with n k-cycles is:

$$\frac{n}{2m+1}[z^n]\left(z^{2m+1}C^{(2m+1)(k-1)}(z)\right)$$

$$=[z^n]\left(z\frac{\left(z^{2m+1}C^{(2m+1)(k-1)}(z)\right)'}{2m+1}\right)$$

$$= [z^n]\left(z\left(z^{2m}C^{(2m+1)(k-1)}(z) + z^{2m+1}(k-1)C^{(2m+1)(k-1)-1}(z)C'\right)\right)$$

$$= [z^n]\left(z\left(z^{2m}C^{(2m+1)(k-1)-1}(z)\left(C + z(k-1)BC^k\right)\right)\right) \qquad ...by\ identity\ 7$$

$$= [z^n]\left(z^{2m+1}C^{(2m+1)k-(2m+2)}(z)C\left(1 + z(k-1)BC^{k-1}\right)\right)$$

$$= [z^n]\left(z^{2m+1}C^{(2m+1)k-(2m+2)}(z)C\left(B - zBC^{k-1}\right)\right) \qquad ...by\ identity\ 2$$

$$= [z^n]\left(z^{2m+1}C^{(2m+1)k-(2m+2)}(z)CB\left(1 - zC^{k-1}\right)\right)$$

$$= [z^n]\left(z^{2m+1}C^{(2m+1)k-(2m+2)}(z)B(z)\right) \qquad ...by\ identity\ 1$$

Therefore,

$$U(z) = zB(z)C^{k-2}(z) + z^3(k-1)B(z)C^{3k-4}(z) + ...$$

$$= zB(z)C^{k-2}(z)\left(1 + z^2(k-1)C^{2k-2}(z) + ...\right)$$

$$= zB(z)C^{k-2}(z)\left(\frac{1}{1 - z^2C^{2k-2}(z)}\right) \qquad ...by\ the\ geometric\ series$$

$$= zB(z)C^{k-2}(z)F(z) \qquad ...\ by\ identity\ 3$$

$$= zC^{k-2}(z)B(z)F(z)$$

$$= zC^{k-2}(z)\left(\frac{F(z) + kB(z)C(z)}{k+1}\right) \qquad ...\ by\ identity\ 14$$

$$= \frac{1}{k+1}zC^{k-2}(z)F(z) + \frac{k}{k+1}zB(z)C^{k-1}(z)$$

Therefore, the total number $u_{n,k}$ of edges of odd out degree among all k-trees with n k-cycles is:

$$U_{n,k} = [z^n]\left(\frac{1}{k+1}zC^{k-2}(z)F(z) + \frac{k}{k+1}zB(z)C^{k-1}(z)\right)$$

$$= [z^{n-1}]\left(\frac{1}{k+1}C^{k-2}(z)F(z) + \frac{k}{k+1}B(z)C^{k-1}(z)\right)$$

$$= \frac{1}{k+1}\left([z^{n-1}]C^{k-2}(z)F(z)\right) + \frac{k}{k+1}\left([z^{n-1}]B(z)C^{k-1}(z)\right)$$

$$= \frac{1}{k+1}\sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor}\frac{k-2+2(k-1)i}{(n-1)k+k-2-2i}\binom{(n-1)k+k-2-2i}{n-1-2i} + \frac{k}{k+1}\binom{k(n-1)+k-1}{n-1}$$

$$(by\ lemma\ 5\ and\ lemma\ 6) \qquad \square$$

Example:

1) The total number of edges of odd out degree among the twelve 3-trees

Consisting of three 3-cycles in figure 16 is:

$$O_{3,3} = \frac{1}{4}\left(\frac{1}{7}\binom{7}{2} + \frac{5}{5}\binom{5}{0}\right) + \frac{3}{4}\binom{8}{2}$$
$$= 22$$

5.3) Asymptotic Results

Theorem 7.

$$\lim_{n\to\infty}\frac{F_n}{C_n} = \frac{4}{9}, \textit{where } C_n = \frac{1}{n+1}\binom{2n}{n} \textit{and } [z^n]F = F_n$$

proof :

$$\lim_{n\to\infty}\frac{C_{n-1}}{C_n} = \lim_{n\to\infty}\frac{\frac{1}{n}\binom{2(n-1)}{n-1}}{\frac{1}{n+1}\binom{2n}{n}}$$
$$= \lim_{n\to\infty}\frac{n+1}{4n-2}$$
$$= \frac{1}{4}$$

Recall that: The first few terms of the Fine numbers and the Catalan numbers are

$1,0,1,2,6,18,57,186,622,\ldots$ *and* $1,1,2,5,14,42,132,429,\ldots$ respectively, then

$$\frac{F_0}{C_0} = 1,\ \frac{F_1}{C_1} = 0,\ \frac{F_2}{C_2} = \frac{1}{2},\ \frac{F_3}{C_3} = \frac{2}{5},\ \frac{F_4}{C_4} = \frac{6}{14},$$
$$\frac{F_5}{C_5} = \frac{18}{42},\frac{F_6}{C_6} = \frac{57}{132},\ \frac{F_7}{C_7} = \frac{186}{429},\ \frac{F_8}{C_8} = \frac{622}{1430},\ldots$$

$$\textit{as } n \to \infty, \textit{the sequence} \left\{\frac{F_n}{C_n}\right\} \textit{is increasing and bounded}.$$

Then let $\lim_{n\to\infty}\frac{F_n}{C_n} = l\,(exists)$

Now by identity 11, we get $1+C=(z+2)F$

$$C = 2F + zF - 1$$

$$\textit{then by snake-oil method}$$

$$\sum_{n\geq 0} C_n z^n = 2\sum_{n\geq 0} F_n z^n + z\sum_{n\geq 0} F_n z^n - 1$$

$$C_0 + \sum_{n\geq 1} C_n z^n = 2F_0 + 2\sum_{n\geq 1} F_n z^n + \sum_{n\geq 0} F_n z^{n+1} - 1$$

$$1 + \sum_{n\geq 1} C_n z^n = 2 + 2\sum_{n\geq 1} F_n z^n + \sum_{n\geq 0} F_{n-1} z^n - 1$$

$$C_n = 2F_n + F_{n-1},\ \textit{for}\ n \geq 1$$

$$1 = \frac{C_n}{C_n} = \frac{2F_n}{C_n} + \frac{F_{n-1}}{C_n}$$

$$1 = \frac{2F_n}{C_n} + \frac{F_{n-1}}{C_n}\frac{C_{n-1}}{C_{n-1}}$$

$$\textit{thus}\ \lim_{n\to\infty} 1 = 2\lim_{n\to\infty}\frac{F_n}{C_n} + \lim_{n\to\infty}\frac{F_{n-1}}{C_{n-1}}\lim_{n\to\infty}\frac{C_{n-1}}{C_n}$$

$$1 = 2l + l/4$$

$$l = 4/9$$

$$\textit{Hence}\quad \lim_{n\to\infty}\frac{F_n}{C_n} = \frac{4}{9}$$

□

Theorem 8.

$$\lim_{n\to\infty}\frac{F_{n,k}}{C_{n,k}} = \frac{2k}{(k+1)^2},\ \textit{for}\ k \geq 2$$

proof :

From lemma 6 we have

$$[z^n]C^s(z)F(z) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{s+2(k-1)i}{nk+s-2i}\binom{nk+s-2i}{n-2i}$$

Letting $s=0$, we have $F_{n,k} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2(k-1)i}{nk-2i}\binom{nk-2i}{n-2i}$ *and* $C_{n,k} = \frac{1}{(k-1)n+1}\binom{nk}{n}$

From the generalized Catalan number, we have

$$[z^n]C^s(z) = \frac{s}{kn+s}\binom{kn+s}{n}$$

$$\lim_{n\to\infty}\frac{F_{n,k}}{C_{n,k}} = \lim_{n\to\infty}\left(\frac{1}{C_{n,k}}\frac{2(k-1)}{kn-2}\binom{kn-2}{n-2}+\frac{1}{C_{n,k}}\frac{4(k-1)}{kn-4}\binom{kn-4}{n-4}+\dots\right)$$

$$= \lim_{n\to\infty}\frac{1}{C_{n,k}}\left(\frac{2i(k-1)}{kn-2i}\binom{kn-2i}{n-2i}\right)$$

$$= \lim_{n\to\infty}\frac{(k-1)n+1}{\binom{kn}{n}}\left(\frac{2i(k-1)}{kn-2i}\binom{kn-2i}{n-2i}\right)$$

$$= 2i(k-1)\lim_{n\to\infty}\frac{\left(k-1+1/n\right)}{\binom{kn}{n}}\left(\frac{\left((1-1/n)\dots(1-\left(2i-1/n\right)\right)}{\left(k\left(k-1/n\right)\dots\left(k-2i/n\right)\right)}\right)$$

$$= 2i(k-1)\frac{k-1}{k^{2i+1}}$$

$$= 2i\frac{(k-1)^2}{k^{2i+1}}$$

$$\textit{then}\ \lim_{n\to\infty}\frac{F_{n,k}}{C_{n,k}} = \sum_{i=1}^{\infty}2i\frac{(k-1)^2}{k^{2i+1}}$$

$$= \frac{2(k-1)^2}{k}\sum_{i=1}^{\infty}\frac{i}{k^{2i}} \qquad \dots(*)$$

$$\sum_{i=1}^{\infty}\frac{i}{k^{2i}} = \frac{1}{k^2}+\frac{2}{k^4}+\frac{3}{k^6}+\frac{4}{k^8}+\dots$$

$$\textit{let}\ S = \frac{1}{k^2}+\frac{2}{k^4}+\frac{3}{k^6}+\frac{4}{k^8}+\dots\ \textit{then}\ \frac{S}{k^2} = \frac{1}{k^4}+\frac{2}{k^6}+\frac{3}{k^8}+\frac{4}{k^{10}}+\dots$$

$$\Rightarrow S = \frac{1}{k^2}+\frac{1}{k^4}+\frac{1}{k^4}+\frac{1}{k^6}+\frac{2}{k^6}+\frac{1}{k^8}+\frac{3}{k^8}+\dots$$

$$= \left(\frac{1}{k^2}+\frac{1}{k^4}+\frac{1}{k^6}+\frac{1}{k^8}+\dots\right)+\left(\frac{1}{k^4}+\frac{2}{k^6}+\frac{3}{k^8}+\dots\right)$$

$$= \left(\frac{1/k^2}{1-1/k^2}\right)+\frac{S}{k^2}$$

$$\Rightarrow S = \left(\frac{k}{k^2 - 1} \right)^2$$

*so * becomes* $\lim_{n\to\infty} \frac{F_{n,k}}{C_{n,k}} = \frac{2(k-1)^2}{k} \left(\frac{k}{k^2-1} \right)^2 = \frac{2k}{(k+1)^2}$

Therefore $\lim_{n\to\infty} \frac{F_{n,k}}{C_{n,k}} = \frac{2k}{(k+1)^2}$, *for* $k \geq 2$ ∎

Corollary 9.

$$\lim_{n\to\infty} \frac{F_{n+1,k}}{F_{n,k}} = \lim_{n\to\infty} \frac{C_{n+1,k}}{C_{n,k}} = \frac{k^k}{(k-1)^{k-1}}$$

Proof:

Now $F_{n,k} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2(k-1)i}{nk-2i} \binom{nk-2i}{n-2i}$ *then* $F_{n+1,k} = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{2(k-1)i}{(n+1)k-2i} \binom{(n+1)k-2i}{n+1-2i}$

$$\lim_{n\to\infty} \frac{F_{n+1,k}}{F_{n,k}} = \lim_{n\to\infty} \frac{\left(2(k-1)i/(n+1)k-2i\right)\binom{(n+1)k-2i}{n+1-2i}}{\left(2(k-1)i/nk-2i\right)\binom{nk-2i}{n-2i}}$$

$$= \lim_{n\to\infty} \frac{n^k \left(\left(k-\left(2i+1-k/n\right)\right)\left(k-\left(2i+1-(k-1)/n\right)\right)\ldots\left(k-\left(2i+1-k+(k+1)/n\right)\right)\right)}{n\left(1-(2i+1)/n\right)n^{k-1}\left(\left((k-1)+k-1/n\right)\left((k-1)+k-2/n\right)\ldots\left((k-1)+k-\left(k-1/n\right)\right)\right)}$$

$$= \lim_{n\to\infty} \frac{k^k}{(k-1)^{k-1}} = \frac{k^k}{(k-1)^{k-1}}$$

and $C_{n,k} = \frac{1}{(k-1)n+1} \binom{nk}{n}$ *then* $C_{n+1,k} = \frac{1}{(k-1)(n+1)+1} \binom{(n+1)k}{n+1}$

$$\lim_{n\to\infty}\frac{C_{n+1,k}}{C_{n,k}}=\lim_{n\to\infty}\frac{\binom{kn+k}{n+1}\Big/(kn+k-n)}{\binom{kn}{n}\Big/(kn-n+1)}$$

$$=\lim_{n\to\infty}\frac{n^k\left(\left(k+k/n\right)\left(k+(k-1)/n\right)\ldots\left(k+\left(k-(k-1)/n\right)\right)\right)}{n\left(1+1/n\right)n^{k-1}\left(\left((k-1)+k/n\right)\left((k-1)+(k-1)/n\right)\ldots\left((k-1)+\left(k-(k-2)\right)/n\right)\right)}$$

$$=\lim_{n\to\infty}\frac{k^k}{(k-1)^{k-1}}$$

$$=\frac{k^k}{(k-1)^{k-1}}$$

Hence $\lim_{n\to\infty}\frac{F_{n+1,k}}{F_{n,k}}=\lim_{n\to\infty}\frac{C_{n+1,k}}{C_{n,k}}=\frac{k^k}{(k-1)^{k-1}}$ □

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