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On

Dimensions of Affine Varieties

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By

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Declaration

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

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Permission

This is to certify that this project is compiled by Hailegebriel Tsegay in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

Dr. Tilahun Abebaw

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Abstract

In this Project, we introduced the affine varieties, which are curves and surfaces (and higher-dimensional objects) defined by Polynomial Equations. By considering ideals in the polynomial ring $K[x_1, \dots, x_n]$, we also understood the dimension of an affine variety V and the affine Hilbert Function of an ideal I, which is a function on the non-negative integers s. Given an affine variety V which is the union of a finite number of linear subspaces of the affine space K^n, we defined the dimension of V in terms of the dimensions of the subspaces. We also defined the dimension of V in terms of the degree of the affine Hilbert Polynomial of the corresponding ideal $I = I(V) \subseteq K[x_1, \dots, x_n]$. Finally, we stated several basic properties of dimension over an infinite field K using the degree of affine Hilbert Polynomials.

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List of Mathematical Notations

Notation	Meaning
$\mathbb{R}$	The set of real numbers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{Z}$	The set of integers
$\mathbb{Z}^n{}_{\geq 0}$	The affine space of integers greater than or equal to zero
$\cup$	Union
$\cap$	Intersection
$\subseteq$	Subset of
$\subset$	Proper subset of
$\supset$	Contained in
$\subsetneq$	Not subset of
$\forall$	For all
$\in$	An element of
$\notin$	Not an element of
$\neq$	Different from
$:$	Such that
$\emptyset$	Empty set
$\blacksquare$	End of the proof

Introduction

Intuitively we all know that the dimension of something is supposed to be. A point is 0-dimensional, a line 1-dimensional, a plane 2-dimensional and the space we live 3-dimensional. The dimension gives the number of free parameters and the number of coordinates needed to specify a point. Also a curved line has dimension 1 and a curved surface like that of a sphere in 3-space has dimension 2. This intuitive notation has been made precise in many branches of Mathematics, and we have dimension concepts in linear algebra, in ring theory, in topology and so on. Sometimes, several of these concepts apply to the same object, and there is no guarantee at all that the same value is found for each definition. It is a game in topology to construct topological spaces that have dimensions $0, 1, 2$ for three different concepts of dimension. In linear algebra, the dimension of a vector space W over a field K is the cardinality of basis. One can show that all bases have the same cardinality. The dimension depends on the field K. For instance, a field of complex numbers has dimension 1 over itself, but dimension 2 over the field of real numbers.

Having these concepts, we are interested here to study the dimension of an affine variety in algebraic geometry. Moreover, given an affine variety V defined by the polynomials $f_1, \dots, f_s$ in the polynomial ring $K[x_1, \dots, x_n]$, the main Objective of this Project is to find the dimension of V. Thus, this paper is organized in two parts, namely: Chapter One and Chapter Two. The first Chapter contains the Preliminary part, which will introduce some of the basic themes of the Project. It consists of seven subtopics, namely: Polynomials and Affine Space, Affine Varieties, Ideals, Hilbert Basis Theorem and Groebner Basis, Radical Ideals and the Ideal-Variety Correspondence, Irreducible Affine Varieties and Prime Ideals, and Coordinate rings and Quotients of Polynomial Rings. The main part of this Project, Chapter Two, also contains four Subtopics; the Affine Variety of Monomial Ideals and Dimension, the

Complement of a Monomial Ideal, the Hilbert Functions and the Dimension of an Affine Variety, and Elementary Properties of Dimension.

CHPTER ONE
Preliminaries

In this Chapter, we will introduce some of the basic themes of the project. We will also develop some important definitions with examples and we will prove some basic propositions and Theorems that will help in our discussion later. The results discussed here will also play an important role in Chapter Two, which is the main part of this project.

1.1 Polynomials and Affine Spaces

We all know that what a field is. But, to remember, the basic intuition is that afield is a set where one can define addition, subtraction, multiplication, and division with the usual properties. We are also certainly familiar with polynomials in one and two variables. In this Section, we will study polynomials in n variables with coefficients in a field K and see how to relate these polynomials with affine space. We start by defining monomials.

Definition 1.1.1: A monomial in n variables $x_1, \dots, x_n$ is a product of the form $x_1^{\alpha_1} \dots x_n^{\alpha_n}$, where all of the exponents $\alpha_1, \dots, \alpha_n$ are non-negative integers. The total degree of this monomial is the sum $\alpha_1 + \cdots + \alpha_n$. If we let $\alpha = (\alpha_1, \dots, \alpha_n)$, then we set $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ then, we also set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ to denote the total degree of the monomial x^α.

Now, we are ready to define polynomials in n variables with coefficients in an arbitrary field K.

Definition 1.1.2: Let K be an arbitrary field. Then, a polynomial f in $x_1, \dots, x_n$ with coefficients in K is a finite linear combination (with coefficients in K) of monomials denoted as $f = \sum_\alpha a_\alpha x^\alpha$, where $a_\alpha \in K$ and the sum is over a finite number of integers $\alpha = (\alpha_1, \dots, \alpha_n)$. The degree of f is the maximum of the total degree $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and the set of all polynomials in $x_1, \dots, x_n$ with coefficients in K is denoted by $K[x_1, \dots, x_n]$.

Example 1.1.3: $f = 4xy^2z + 4z^4 - 5x^3 + 7x^2z^2$ is a polynomial in $K[x, y, z]$.

Here, one can show that under addition and multiplication, $K[x_1, \dots, x_n]$ satisfies all of the field axioms except the existence of multiplicative inverse. Such a mathematical structure is called a commutative ring. For this reason, we will refer to $K[x_1, \dots, x_n]$ as a polynomial ring.

Remark: A field K is said to be algebraically closed if any non-constant polynomial f in $K[x_1, \dots, x_n]$ has a root in K.

Example 1.1.4: $\mathbb{C}$ is algebraically closed field.

Definition 1.1.5: Let d be the degree of the polynomial $f = \sum_\alpha a_\alpha x^\alpha$. Then, f is called homogeneous, or a form, of degree d, if all coefficients a_α are zero except possible those belonging to monomials of degree d.

Let us define now the n-dimensional affine space over any field K and see the relation with polynomials.

Definition 1.1.6: Given a field K and a positive integer n, we define the n-dimensional affine space over K to be the set

$$K^n = \{(\alpha_1, \dots, \alpha_n) : \alpha_1, \dots, \alpha_n \in K\}.$$

Example 1.1.7: Consider $K = \mathbb{R}$. Then, we get the familiar space $\mathbb{R}^n$ from linear algebra. In general, we call $K^1 = K$ the affine line and K^2 the affine plane.

To see how polynomials relate to affine spaces, consider a polynomial $f = \sum_\alpha a_\alpha x^\alpha$ in $K[x_1, \dots, x_n]$. Then, this polynomial gives a function $f : K^n \to K$ defined as follows. Given $(\alpha_1, \dots, \alpha_n) \in K^n$, replace every x_i by a_i in the expression for f. Then, since all of the coefficients also lie in K, this operation gives an element $f(\alpha_1, \dots, \alpha_n) \in K$.

Now, we can construct the monomial $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ from the n-tuple of exponents $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n{}_{\geq 0}$. This observation establishes a one-to-one correspondence between the monomials in $K[x_1, \dots, x_n]$ and $\mathbb{Z}^n{}_{\geq 0}$. Therefore, any ordering $>$ we establish on the space $\mathbb{Z}^n{}_{\geq 0}$ will give us an ordering on monomials; if $\alpha > \beta$ according to this ordering, we will also say that $x^\alpha > x^\beta$.

Let us see now how to order monomials and then polynomials in $K[x_1, \dots, x_n]$. When we work on ordering of monomials in $K[x_1, \dots, x_n]$, we can use different techniques to order the given monomials. But, in our case we will use the one mostly used technique, called Graded Lexico Graphic Order (or simply grlex order).

Definition1.1.8 (grlex Order): Let $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n{}_{\geq 0}$, then we say that $\alpha >_{grlex} \beta$ if $|\alpha| = \sum_{i=1}^n \alpha_i > |\beta| = \sum_{i=1}^n \beta_i$ (or $|\alpha| = |\beta|$ and in the difference $\alpha - \beta \in \mathbb{Z}^n$, the left-most non-zero entry is positive).

Example1.1.9: $(1,2,4) >_{grlex} (1,1,5)$, since $|(1,2,4)| = |(1,1,5)| = 7$ and $(1,2,4) - (1,1,5) = (0, 1, -1)$.

Remark: If $f = \sum_\alpha a_\alpha x^\alpha$ is a polynomial in $K[x_1, \dots, x_n]$ and we have selected a monomial ordering $>$, then we can order the monomials of f easily with respect to $>$.

Example 1.1.10: Consider $f = 4xy^2z + 4z^4 - 5x^3 + 7x^2z^2$ as in example 1.1.3 above. Then, with respect to the grlex order we would have

$$f = 7x^2z^2 + 4xy^2z - 5x^3 + 4z^4.$$

Definition1.1.11: Let $f = \sum_\alpha a_\alpha x^\alpha$ be a non-zero polynomial in $K[x_1, \dots, x_n]$ and suppose $>$ be a grlex monomial ordering. Then,

a) The multidegree of f is $multideg(f) = \max(\alpha \in \mathbb{Z}^n{}_{\geq 0}: a_\alpha \neq 0)$, (The maximum is taken with respect to $>$)

b) The leading coefficient of f is $LC(F) = a_{multideg\ (f)} \in K$

c) The leading monomial of f is $LM(f) = x^{multideg\ (f)}$, (with coefficient one).

d) The leading term of f is $LT(f) = LC(f) \cdot LM(f)$.

Lemma 1.1.12: [1] Fix a positive integer s. Then, the number of monomials of total degree less than or equal to s in $K[x_1, \dots, x_m]$ is the binomial coefficient $\binom{m+s}{s}$.

1.2 Affine Varieties

In this Section, the geometry we are interested in concerns affine varieties, which are curves and surfaces (and higher dimensional objects) defined by polynomial equations in the polynomial ring $K[x_1, \dots, x_n]$.

Definition 1.2.1: Let K be a field and let $f_1, \dots, f_s$ be polynomials in $K[x_1, \dots, x_n]$. Then, we set $V(f_1, \dots, f_s) = \{(\alpha_1, \dots, \alpha_n) \in K^n : f_i(\alpha_1, \dots, \alpha_n) = 0\ \forall\ 1 \leq i \leq s\}$. We call $V(f_1, \dots, f_s)$ the affine variety defined by $f_1, \dots, f_s$.

Thus, an affine variety $V(f_1, \dots, f_s) \subseteq K^n$ is the set of all solutions of the system of equations $f_1(x_1, \dots, x_n) = f_2(x_1, \dots, x_n) = \cdots = f_s(x_1, \dots, x_n) = 0$. We will use the letters V, W etc to denote affine varieties.

Example 1.2.2: $V(x^2 - 4, y^2 - 1)$ is an affine variety.

Because $V(x^2 - 4, y^2 - 1) = V(f_1, f_2) = \{(\pm 2, \pm 1)\}$.

In general, graphs of polynomial functions are affine varieties.

Definition 1.2.3: If $f \in K[x_1, \dots, x_n]$ is a polynomial of degree one, then the affine variety $V(f)$ is called a hyper plane in K^n.

Now, let us see some of the basic properties of affine varieties.

Lemma 1.2.4: If $V, W \subseteq K^n$ are affine varieties, then

a) $V \cap W$ is an affine variety

b) $V \cup W$ is an affine variety.

Proof: Suppose that $V = V(f_1, \dots, f_s)$ and $W = V(g_1, \dots, g_t)$.

Claim: a) $V \cap W = V(f_1, \dots, f_s, g_1, \dots, g_t)$,

b) $V \cup W = V(f_i\, g_j : 1 \leq i \leq s, 1 \leq j \leq t)$.

(a): This equality is trivial to prove since being in $V \cap W$ means that both $f_1, \dots, f_s$ and $g_1, \dots, g_t$ vanish, which is the same as $f_1, \dots, f_s, g_1, \dots, g_t$ vanishing.

(b): If $(a_1, \dots, a_n) \in V$, then all of the f_i's vanish at this point, which implies that all of the $f_i\, g_j$ *'s* also vanish at $(a_1, \dots, a_n)$.

Thus, $V \subseteq V(f_i\, g_j)$, and $W \subseteq V(fi\ gj)$ follows similarly.

This proves that $V \cup W \subseteq V(fi\ gj)$.

Going to the other way, choose $(a_1, \dots, a_n) \in V(fi\ gj)$. If this lies in V, then we are done, and if not, then $f_{i_0}(a_1, \dots, a_n) \neq 0$ for some i_0. Since $f_{i_0} g_j$ vanishes at $(a_1, \dots, a_n)$ for all j, the g_j's must vanish at this point, proving that $(a_1, \dots, a_n) \in W$.

This shows that $V(f_i\, g_j) \subseteq V \cup W$.

Therefore, $V \cup W = V(f_i\, g_j : 1 \leq i \leq s, 1 \leq j \leq t)$. ∎

This Lemma implies that finite intersections and unions of affine varieties are also affine varieties.

Example 1.2.5: Consider the union of the (x, y)-plane and the z-axis in the affine 3-space. Then, by the above formula, we have $V(z) \cup V(x, y) = V(zx, zy)$.

Next, we will see how ideals relate to affine varieties.

1.3 Ideals

The goal of this Section is to introduce some naturally occurring ideals and to see how ideals relate to affine varieties. The real importance of ideals is that they will give us a language for computing with affine varieties. The first natural example of an ideal is the ideal generated by a finite number of polynomials.

Definition 1.3.1: A subset I of $K[x_1, \dots, x_n]$ is said to be an ideal if it satisfies the following conditions.

a) $0 \in I$

b) If $f, g \in I$ then, $f + g \in I$

c) If $f \in I$ and $h \in K[x_1, \dots, x_n]$, then $fh \in I$.

Example 1.3.2: Let $f_1, \dots, f_s$ be polynomials in $K[x_1, \dots, x_n]$, then the set $\langle f_1, \dots, f_s \rangle = \{\sum_{i=1}^{s} h_i f_i \ : \ h_1, \dots, h_s \in K[x_1, \dots, x_n]\}$ is an ideal of $K[x_1, \dots, x_n]$, called the ideal generated by $f_1, \dots, f_s$.

Definition 1.3.3: An ideal I in $K[x_0, x_1, \dots, x_n]$ is said to be homogeneous if for each $f \in I$, the homogeneous components of f_i of f are in I as well.

Lemma 1.3.4: Let $V \subseteq K^n$ be an affine variety. Then, the subset $I(V) = \{f \in K[x_1, \dots, x_n] \ : \ f(\alpha_1, \dots, \alpha_n) = 0 \ \forall \ (\alpha_1, \dots, \alpha_n) \in V\}$ of $K[x_1, \dots, x_n]$ is an ideal called ideal of V.

Proof: We have $0 \in I(V)$ since the zero polynomial vanishes on all of the K^n, and so, in particularly it vanishes on V.

Let $f, \ g \in I(V)$ and $h \in K[x_1, \dots, x_n]$.

Claim: a) $f + g \in I(V)$

b) $fh \in I(V)$

a) Suppose $(a_1, \dots, a_n)$ be an arbitrary points of V.

Then $(f + g)(a_1, \dots, a_n) = f(a_1, \dots, a_n) + g(a_1, \dots, a_n) = 0 + 0 = 0$

Hence, $f + g \in I(V)$.

b) $(fh)(a_1, \dots, a_n) = f(a_1, \dots, a_n)h(a_1, \dots, a_n) = 0 . 0 = 0$

Hence, $fh \in I(V)$.

Therefore, $I(V)$ is an ideal of V. ■

Proposition1.3.5: Let V, W be affine varieties in K^n, then

a) $V \subseteq W$ if and only if $I(V) \supset I(W)$

b) $V = W$ if and only if $I(V) = I(W)$.

Proof: [1], [3]

Definition1.3.6: An ideal $I \subseteq K[x_1, \dots, x_n]$ is a monomial ideal if there is a subset $A \subseteq \mathbb{Z}^n{}_{\geq 0}$ (possibly infinite) such that I consists of all polynomials which are finite sums of the form $\sum_{\alpha \in A} h_\alpha x^\alpha$, where $h_\alpha \in K[x_1, \dots, x_n]$. In this case, we write

$$I = \langle x^\alpha : \alpha \in A \rangle.$$

Example1.3.7: The ideal $I = \langle x^4 y^2, x^3 y^4, x^2 y^5 \rangle \subseteq K[x_1, \dots, x_n]$ is a monomial ideal.

Lemma 1.3.8: Let the ideal I be a monomial ideal and let $f \in K[x_1, \dots, x_n]$. Then, the following statements are equivalent.

a) $f \in I$

b) Every term of f lies in I

c) f is a K-linear combination of the monomials in I.

Proof: [1]

1.4 The Hilbert Basis Theorem and Groebner Basis

The method of Groebner bases is used in several powerful computer algebra systems to study specific polynomial ideals that arise in applications. Our treatment will lead to ideal basis and the key idea we will use is that once we choose a monomial ordering $>$, each polynomial $f \in k[x_1, \dots, x_n]$ has a unique leading term $LT(f)$.

Definition 1.4.1: Let $I \subseteq K[x_1, \dots, x_n]$ be an ideal other than $\{0\}$. Then,

a) We denote the set of leading terms of elements of I by $LT(I)$. Thus,
$LT(I) = \{cx^{\alpha} : \text{there exists } f \in I \text{ with } LT(f) = cx^{\alpha}\}$

b) The ideal generated by the elements of $LT(I)$ is denoted by $\langle LT(I)\rangle$.

Example 1.4.2: Let $I = \langle f_1, f_2\rangle$, where $f_1 = x^3 - 2xy$ and $f_2 = x^2y - 2y^2 + x$.

Then, $xf_2 - yf_1 = x(x^2y - 2y^2 + x) - y(x^3 - 2xy)$

$= x^2 \in I.$

This implies, $x^2 = LT(x^2) \in \langle LT(I)\rangle$.

However, x^2 is not divisible by $LT(f_1) = x^3$ or $LT(f_2) = x^2y$, so that $x^2 \notin \langle LT(f_1), LT(f_2)\rangle$

Proposition 1.4.3: Let $I \subseteq K[x_1, \dots, x_n]$ be an ideal.

Then, $\langle LT(I)\rangle$ is a monomial ideal.

Proof: Let $g \in I - \{0\}$. Then, the leading monomial $LM(g)$ generates the monomial $\langle LM(I) : g \in I - \{0\}\rangle$. Since $LM(g)$ and $LT(g)$ defer by a non-zero constant, this ideal equals $\langle LT(I) : g \in I - \{0\}\rangle = \langle LT(I)\rangle$. Thus, $\langle LT(I)\rangle$ is a monomial ideal. ■

Theorem 1.4.4 (Hilbert Basis Theorem): Every ideal $I \subseteq K[x_1, \dots, x_n]$ has a finite generating set. That is, $I = \langle g_1, \dots, g_t\rangle$ for some $g_1, \dots, g_t \in I$.

Proof: [1], [4]

This Theorem shows that in fact, it also makes sense to speak of the affine varieties defined by an ideal $I \subseteq K[x_1, \dots, x_n]$.

Theorem 1.4.5: Let $I \subseteq K[x_0, x_1, \dots, x_n]$ be an ideal, then the following statements are equivalent:

a) I is a homogeneous ideal of $K[x_0, x_1, \dots, x_n]$

b) $I = \langle f_1, \dots, f_s\rangle$, where $f_1, \dots, f_s$ are homogeneous polynomials.

Proof: [1]

Definition 1.4.6: Fix a monomial ordering $>$ to be a grlex order. Then, a finite subset $G = \{g_1, \dots, g_t\}$ of an ideal I is said to be a Groebner basis (standard basis) of I if $\langle LT(g_1), \dots, LT(g_t)\rangle = \langle LT(I)\rangle$. Equivalently, the set $\{g_1, \dots, g_t\} \subseteq I$ is a Groebner basis of I if and only if the leading term of any element of I is divisible by one of the $LT(g_i)$.

Corollary 1.4.7: Fix a grlex monomial order $>$. Then, every ideal $I \subseteq K[x_1, \dots, x_n]$ other than $\{0\}$ has a Groebner basis. Furthermore, any Groebner basis for an ideal I is a basis of I.

Proof: [1]

Definition 1.4.8: Let $I \subseteq K[x_1, \dots, x_n]$ be an ideal. Then, we denote $V(I)$ to be the set

$$V(I) = \{(\alpha_1, \dots, \alpha_n) \in K^n : f(\alpha_1, \dots, \alpha_n) = 0, \ \forall f \in I\}.$$

Proposition 1.4.9: $V(I)$ is an affine variety. In particular, if $I = \langle f_1, \dots, f_s\rangle$,

then $V(I) = V(f_1, \dots, f_s)$.

Proof: By Hilbert Basis Theorem, we have $I = \langle f_1, \dots, f_s\rangle$.

Claim: $V(I) = V(f_1, \dots, f_s)$

To proof the forward direction, let $f_i \in I$.
If $f(\alpha_1, \dots, \alpha_n) = 0 \ \forall f \in I$, then $f_i(\alpha_1, \dots, \alpha_n) = 0 \in V(f_1, \dots, f_s)$.
This implies, $V(I) \subseteq V(f_1, \dots, f_s)$.

Conversely, let $(\alpha_1, \dots, \alpha_n) \in V(f_1, \dots, f_s)$ and let $f \in I$.
Then, since $I = \langle f_1, \dots, f_s\rangle$, we can write $f = \sum_{i=1}^{s} h_i f_i$ for some $h_i \in K[x_1, \dots, x_n]$. Then, $f(\alpha_1, \dots, \alpha_n) = \sum_{i=1}^{s} h_i(\alpha_1, \dots, \alpha_n) f_i(\alpha_1, \dots, \alpha_n)$

$$= \sum_{i=1}^{s} h_i(\alpha_1, \dots, \alpha_n). 0$$

$$= 0.$$

This implies, $V(f_1, \dots, f_s) \subseteq V(I)$.

Therefore, $V(I) = V(f_1, \dots, f_s)$. ■

The most important consequence of this Proposition is that affine varieties are determined by ideals.

Definition 1.4.10: Let $f, g \in K[x_1, \dots, x_n]$ be non-zero polynomials.

If $multideg(f) = \alpha$ and $multideg(g) = \beta$, then let $\gamma = (\gamma_1, \dots, \gamma_n)$, where $\gamma_i = \max(\alpha_i, \beta_i)$ for each i. Then, we define the S-polynomial of f and g as the combination

$$S(f, g) = \frac{x^\gamma}{LT(f)} f - \frac{x^\gamma}{LT(g)} g.$$

Example 1.4.11: Consider the polynomials $f = x^3y^2 - x^2y^3 + x$ and $g = 3x^4y + y^2$ in $\mathbb{R}[x, y]$ with the grlex order. Then, $\gamma = (4, 2)$ and

$$S(f, g) = \frac{x^4y^2}{x^3y^2} f - \frac{x^4y^2}{3x^4y} g = xf - \left(\frac{1}{3}\right) yg = -x^3y^3 + x^2 - \left(\frac{1}{3}\right) y^3$$

Theorem 1.4.12: Let I be an ideal in $K[x_1, \dots, x_n]$. Then, a basis $G = \{g_1, \dots, g_t\}$ for I is a Groebner basis for I if and only if for all pairs $i \neq j$, the remainder on division of $S(g_i, g_j)$ by G (listed in some order) is zero.

Proof: [1]

Example 1.4.13: Consider the ring $\mathbb{R}[x, y]$ with grlex order, and let

$$I = \langle f_1, f_2 \rangle = \langle x^3 - 2xy, x^2y + 2y^2 + x \rangle.$$

Then, $\langle f_1, f_2 \rangle$ is not a Groebner basis for I since $LT\big(S(f_1, f_2)\big) = -x^2 \notin \langle LT(f_1), LT(f_2) \rangle$ as in example 1.4.2.

To produce a Groebner basis, one natural idea is extending the original generating set to a Groebner basis by adding more polynomials in I. In one sense, this adds nothing new, and even introduces an element of redundancy.

So we have $S(f_1, f_2) = -x^2 \in I$, and its remainder on division by $F = (f_1, f_2)$ is $-x^2$, which is non-zero. Hence, we should introduce that remainder in our generating set as a new generator $f_3 = -x^2$.

If we set $F = (f_1, f_2, f_3)$, we use Theorem 1.4.12 to test if this new set is a Groebner basis for I.

We compute: $S(f_1, f_2) = f_3$, so

$$\frac{S(f_1, f_2)}{F} = 0,$$

$$S(f_1, f_3) = (x^3 - 2xy) - x(-x^2) = -2xy, \text{ but}$$

$$\frac{S(f_1, f_3)}{F} = 0 - 2xy \neq 0$$

Hence, we must add $f_4 = -2xy$ to our generating set.

If we set $F = (f_1, f_2, f_3, f_4)$, then

$$\frac{S(f_1, f_2)}{F} = \frac{S(f_1, f_3)}{F} = 0\ ,$$

$$S(f_1, f_4) = y(x^3 - 2xy) - \left(-\frac{1}{2}\right)x^2(-2xy) = 2xy^2 = yf_4, \text{ so}$$

$$\frac{S(f_1, f_4)}{F} = 0,$$

$$S(f_2, f_3) = (x^2y - 2y^2 + x) - (-y)(-x^2) = -2y^2 + x, \text{ but}$$

$$\frac{S(f_2, f_3)}{F} = 0 - 2y^2 + x \neq 0$$

Thus, we must also add $f_5 = -2y^2 + x$ to our generating set.

Setting $F = (f_1, f_2, f_3, f_4, f_5)$, one can compute $\frac{S(f_i, f_j)}{F}$ for all $1 \leq i \leq j \leq 5$. Then, by Theorem 1.4.12, it follows that a grlex Groebner basis for I is given by

$$\{f_1, f_2, f_3, f_4, f_5\} = \{x^3 - 2xy, x^2y - 2y^2 + x, -x^2, -2xy, = -2y^2 + x\}.$$

Theorem 1.4.14 (Hilbert's Nullstellensatz Theorem): Let K be an algebraically closed field. If $f, f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$ are such that $f \in I(V(f_1, \ldots, f_s))$, then there exists an integer $m \geq 1$ such that $f^m \in \langle f_1, \ldots, f_s \rangle$ (And conversely).

Proof: [1]

1.5 Radical Ideals and the Ideal-Variety Correspondence

To further explore the relation between ideals and affine varieties, it is natural to recast Hilbert's Nullstellensatz in terms of ideals. Can we characterize the sorts of ideals that appear as the ideals of an affine variety? That is can we identify those ideals that consists of all polynomials which vanish on some affine variety V? To answer this question, we have a key observation in the following simple Lemma.

Lemma 1.5.1: Let V be an affine variety. If $f^m \in I(V)$, then $f \in I(V)$.

Proof: Let $x \in V$, and suppose $f^m \in I(V)$.

Then, $(f(x))^m = 0$. But, this can be happen if and only if $f(x) = 0$ since K is a field.

Now, since x was arbitrary, we must have $f \in I(V)$. ■

Thus, an ideal consisting of all polynomials which vanish on an affine variety V has the property that if some power of a polynomial belongs to the ideal, then the polynomial itself must belong to the ideal. This leads to the following definition.

Definition 1.5.2: An ideal I is said to be radical ideal if $f^m \in I$ for some integer $m \geq 1$ implies that $f \in I$.

Remark: Rephrasing Lemma 1.5.1 in terms of radical ideals shows that $I(V)$ is a radical ideal.

Definition 1.5.3: Let $I \subseteq K[x_1, \dots, x_n]$ be an ideal. Then, the radical of I, denoted by $\sqrt{I}$ is the set $\{f : f^m \in I \text{ for some integer } m \geq 1\}$.

Remark: We always have $f \in \sqrt{I}$ since $f \in I$ implies that $f^1 \in I$ and, hence, $f \in \sqrt{I}$ by Definition 1.5.2.

Lemma 1.5.4: If I is an ideal in $K[x_1, \dots, x_n]$, then $\sqrt{I}$ is an ideal in $K[x_1, \dots, x_n]$ containing I. Furthermore, $\sqrt{I}$ is a radical ideal.

Proof: [1]

Theorem 1.5.5 (The Strong Nullstellensatz Theorem): Let K be an algebraically closed field. If I is an ideal in $K[x_1, \dots, x_n]$, then $I(V(I)) = \sqrt{I}$.

Proof: To prove the forward direction, suppose that $f \in I(V(I))$.
Then, by definition, f vanishes on $V(I)$.
By Hilbert's Nullstellensatz, there exists an integer $m \geq 1$ such that $f^m \in I$. This means that $f \in \sqrt{I}$.
Thus, since f was arbitrary, we have $I(V(I)) \subseteq \sqrt{I}$.

Conversely, suppose that $f \in \sqrt{I}$.
This implies, $f^m \in I$ for some integer $m \geq 1$.
Hence, f^m vanishes on $V(I)$ and, hence, f vanishes on $V(I)$.
Thus, $f \in I(V(I))$.
This implies$\sqrt{I} \subseteq I(V(I))$.
Therefore, $I(V(I)) = \sqrt{I}$. ■

The most important consequence of the Strong Nullstellensatz Theorem is that it allows us to setup a dictionary between geometry and algebra. The basis of the dictionary is contained in the following Theorem.

Theorem 1.5.6 (The Ideal-Variety Correspondence Theorem): Let K be an arbitrary field. Then, the maps:

$$\text{Affine varieties} \xrightarrow{I} \text{Ideals}$$

$$\text{Ideals} \xrightarrow{V} \text{Affine varieties}$$

are inclusion reversing. That is, if $I_1 \subseteq I_2$, then $V(I_1) \supset V(I_2)$ and, similarly, if $V_1 \subseteq V_2$ are affine varieties, then $I(V_1) \supset I(V_2)$. Furthermore, for any affine variety V, we have $V(I(V)) = V$, so that I is always one-to-one.

Proof: [1]

Theorem 1.5.7: If I and J are ideals in $K[x_1, \dots, x_n]$, then

a) $V(IJ) = V(I) \cup V(J)$, where the product IJ is an ideal generated by fg, for $f \in I$, and $g \in J$

b) $V(I \cap J) = V(I) \cup V(J)$.

Proof (a): To prove the forward direction, suppose that $x \in V(IJ)$.

Then, $g(x).h(x) = 0 \;\; \forall g \in I \; and \;\; \forall h \in J$.

If $g(x) = 0 \;\; \forall g \in I$, then $x \in V(I)$.

If $g(x) \neq 0 \;\; \forall g \in I$, then we must have $h(x) = 0 \;\; \forall h \in J$.

In either case, we have $x \in V(I) \cup V(J)$.

This implies, $V(IJ) \subseteq V(I) \cup V(J)$.

Conversely, suppose that $x \in V(I) \cup V(J)$.

Then, either $g(x) = 0 \;\; \forall g \in I$ or $h(x) = 0 \;\; \forall h \in J$.

Thus, $g(x)h(x) = 0 \;\; \forall g \in I \; and \;\; \forall h \in J$.

This implies, $f(x) = 0 \;\; \forall f \in IJ$.

Hence, $x \in V(IJ)$ implies that $V(I) \cup V(J) \subseteq V(IJ)$.

Therefore, $V(IJ) = V(I) \cup V(J)$.

b): We know that $IJ \subseteq I \cap J$. Then, we have $V(I \cap J) \subseteq V(IJ)$ (by Proposition 1.3.5). But, $V(IJ) = V(I) \cup V(J)$, (by (a).

This implies that $V(I \cap J) \subseteq V(I) \cup V(J)$.

Conversely, let $x \in V(I) \cup V(J)$. Then, $x \in V(I)$ or $x \in V(J)$.

This means, $f(x) = 0 \;\; \forall f \in I$ or $f(x) = 0 \;\; \forall f \in J$.

Thus, certainly, $f(x) = 0 \;\; \forall f \in \; I \cap J$. Hence, $x \in V(I \cap J)$.

This implies, $V(I) \cup V(J) \subseteq V(I \cap J)$.

Then, $V(I \cap J) = V(I) \cup V(J)$. ∎

1.6 Irreducible Affine Varieties and Prime Ideals

In Section 2, we have already seen that the union of two affine varieties is also an affine variety and we considered $V(xz, yz)$, which is the union of a line and a plane.

Intuitively, it is natural to think of the line and the plane as more fundamental than $V(xz, yz)$. Intuition also tells us that a line or a plane is irreducible in some sense; they do not obviously seem to be a union of finitely many simpler affine varieties. In this Section, we will formalize this notation.

Definition 1.6.1: An affine variety $V \subseteq K^n$ is said to be irreducible if whenever V is written in the form $V = V_1 \cup V_2$, where V_1 and V_2 are affine varieties, then either $V_1 = V$ or $V_2 = V$.

Example 1.6.2: The twisted cubic $V(y^2 - x^2, z - x^3)$ in $\mathbb{R}^3$ appears to be irreducible.

Definition 1.6.3: An ideal $I \subseteq K[x_1, \dots, x_n]$ is said to be prime ideal if f, g in $K[x_1, \dots, x_n]$ and $fg \in I$, then $f \in I$ or $g \in I$.

Proposition 1.6.4: Let $V \subseteq K^n$ be an affine variety. Then, V is irreducible if and only if $I(V)$ is a prime ideal.

Proof: To prove the forward direction, suppose that V is irreducible.

We want to show $I(V)$ is prime ideal.

Let $fg \in I(V)$ and, set $V_1 = V \cap V(f)$ and $V_2 = V \cap V(g)$.
Since the intersection of two affine varieties is also affine variety by Lemma 1.2.4, then V_1 and V_2 are affine varieties.

Claim 1: $V = V_1 \cup V_2$.

Since $V_1 = V \cap V(f) \subseteq V$ and $V_2 = V \cap V(g) \subseteq V$, then $V_1 \cup V_2 \subseteq V$.
Now, let $(a_1, \dots, a_n) = x \in V$, then $(fg)(x) = 0$.
This implies, $x \in V(fg)$, so that $x \in V \cap \mathrm{V}(fg)$.
By Lemma 1.2.4, we have $V(fg) = V(f) \cup V(g)$.
Then, $x \in V \cap (V(f) \cup V(g))$.
This implies, $x \in (V \cap V(f)) \cup (V \cap V(g))$,
And hence, $x \in V_1 \cup V_2$.

Thus, $V \subseteq V_1 \cup V_2$.

Therefore, $V = V_1 \cup V_2$.

Claim 2: Either $f \in I(V)$ or $g \in I(V)$.

Since V is irreducible, either $V = V_1$ or $V = V_2$.

Without loss of generality, say $V = V_2 = V \cap V(g)$.

This implies, g vanishes on V and hence, $g \in I(V)$.

Therefore, $I(V)$ is prime ideal.

Conversely, assume that $I(V)$ is prime ideal and let $V = V_1 \cup V_2$

Suppose that $V \neq V_1$.

Claim: $I(V) = I(V_2)$

We have $I(V) \subseteq I(V_2)$ since $V_2 \subseteq V$.

It remains to show $I(V_2) \subseteq I(V)$.

Since $V_1 \subsetneq V$, we have $I(V) \subsetneq I(V_1)$.

Now, pick $f \in I(V_1) - I(V)$ and take any $g \in I(V_2)$.

Then, since $V = V_1 \cup V_2$, fg vanishes on V and hence, $fg \in I(V)$.

But, $I(V)$ is prime and $f \notin I(V)$.

Thus, $g \in I(V)$ and then $I(V_2) \subseteq I(V)$.

Hence, $I(V) = I(V_2)$

Then, by part (b) of Proposition 1.3.5, we have $V = V_2$

Therefore, V is irreducible. ■

1.7 Coordinate Rings and Quotients of Polynomial Rings

In this Section, we will see how to construct the coordinate ring $K[V]$, which is a special case of what is called the quotient of a polynomial ring $K[x_1, \dots, x_n]$ modulo an ideal I. From the word "quotient", we may guess that the issue is to define a division operation, but this is not the case. Instead, forming the quotient will indicate the sort of "lumping together" of polynomials that we will mention when describing the elements of the coordinate ring $K[V]$.

Definition 1.7.1: Let $V \subseteq K^m$ and $W \subseteq K^n$ be affine varieties. Then, a function $\phi : V \longrightarrow W$ is said to be a polynomial mapping if there exist polynomials $f_1, \ldots, f_n \in K[x_1, \ldots, x_m]$ such that $\phi(a_1, \ldots, a_m) = (f_1(a_1, \ldots, a_m), \ldots, f_n(a_1, \ldots, a_m)) \ \forall \ (a_1, \ldots, a_m) \in V$. We say that the n-tuple of polynomials $(f_1, \ldots, f_n) \in (K[x_1, \ldots, x_m])^n$ represents ϕ. The collection of polynomial functions $\phi : V \longrightarrow K$ is denoted by $K[V]$.

Now, since K is a field, we can define a sum and product function for any pair of functions ψ, $\phi : V \longrightarrow K$ by adding and multiplying images.
For each $p \in V$, we have $(\phi + \psi)(p) = \phi(p) + \psi(p)$,

$$(\phi . \psi)(p) = \phi(p) . \psi(p).$$

Furthermore, if we pick specific representatives f, $g \in K[x_1, \ldots, x_m]$ for ϕ, ψ respectively, then by definition, the polynomial sum $f + g$ represents $\phi + \psi$ and the polynomial product $f . g$ represents $\phi . \psi$. It follows that $\phi + \psi$ and $\phi . \psi$ are polynomial functions on V.

Thus, we see that $K[V]$ has sum and product operations constructed using the sum and product operations in $K[x_1, \ldots, x_m]$. All of the usual properties of sum and products of polynomials also hold for functions in $K[V]$. Thus, $K[V]$ is a commutative ring and we call it the coordinate ring of V.

Proposition 1.7.2: Let $V \subseteq K^n$ be an affine variety. Then, the following statements are equivalent.

a) V is irreducible
b) $I(V)$ is a prime ideal
c) $K[V]$ is an integral domain.

Proof: [1]

Definition 1.7.3: Let $I \subseteq K[x_1, \dots, x_n]$ be an ideal, and let $g \in K[x_1, \dots, x_n]$. Then, we say that f and g are congruent modulo I, written as $f \equiv g(modI)$ if $f - g \in I$.

Example 1.7.4: Consider $I = \langle x^2 - y^2, x + y^3 + 1\rangle \subseteq K[x, y]$. And let $f = x^4 - y^4 + x$ and $g = x + x^5 + x^4y^3 + x^4$.

Then, $f \equiv g(modI)$.

That is, $f - g = x^4 - y^4 + x - (x + x^5 + x^4y^3 + x^4)$

$$= x^4 - y^4 + x - x - x^5 - x^4y^3 - x^4$$

$$= x^4 - y^4 - x^5 - x^4y^3 - x^4$$

$$= (x^2 + y^2)(x^2 - y^2) - (x^4)(x + y^3 + 1) \in I.$$

Therefore, $f \equiv g(modI)$.

Definition 1.7.5: The quotient of $K[x_1, \dots, x_n]$ modulo I, written as $K[x_1, \dots, x_n]/I$ is the set of equivalence classes for congruence modulo I such that

$$K[x_1, \dots, x_n]/I = \{[f] : f \in K[x_1, \dots, x_n]\}.$$

Since $K[x_1, \dots, x_n]$ is a ring, given any two classes $[f], [g] \in K[x_1, \dots, x_n]/I$, we can define the sum and product operations on classes by using the corresponding operations on elements of $K[x_1, \dots, x_n]$. That is, $[f] + [g] = [f + g]$ (Sum in $K[x_1, \dots, x_n]$)

$[f].[g] = [f.g]$ (Product in $K[x_1, \dots, x_n]$)

Theorem 1.7.6: Let $V \subseteq K^n$ be an affine variety. Then $K[V] \cong K[x_1, \dots, x_n]/I(V)$.

Proof: [2]

CHAPTER TWO
Dimension of Affine Varieties

The most important invariant of a linear subspace of affine space is its dimension. In Chapter One, we have seen numerous examples of affine varieties which have clearly defined dimension, at least from a naive point of view. Also, we have seen that if $V(f_1, \ldots, f_s) \neq \emptyset$, then the equations $f_1 = \cdots = f_s = 0$ have a common solution. If $V(f_1, \ldots, f_s)$ is finite, we can find all of the solutions explicitly. We also show that this notation accords well with what we would expect intuitively. The hardest is the one concerning dimension, for it involves some sophisticated concepts. In this Chapter, we will carefully define the dimension of any affine variety and show how to compute it.

2.1 Affine Variety of Monomial Ideals and Dimension

In this Section, given a monomial ideal $I \subseteq K[x_1, \ldots, x_n]$, we will compute the dimension of the affine variety defined by such an ideal.

To begin, consider the monomial ideal $I = \langle y^2z^3, x^5z^4, x^2yz^2 \rangle \subseteq K[x, y, z]$. (1)

Now, let H_x be the plane defined by $x = 0$ (so $V(x) = H_x$).

Similarly, let H_y be the plane defined by $y = 0$

H_z be the plane defined by $z = 0$

H_{xy} be the line defined by $x = y = 0$.

Then, since $V(I) = V(f_1, \ldots, f_s)$ by Proposition 1.4.8, we have

$V(I) = V(y^2z^3, x^5z^4, x^2yz^2)$

This implies, $V(I) = V(y^2z^3) \cap V(x^5z^4) \cap V(x^2yz^2)$

$= (H_y \cup H_z) \cap (H_x \cup H_z) \cap (H_x \cup H_y \cup H_z)$

We observe that the plane H_z belongs to each of the three terms, and hence, to their intersection.

Therefore, by collecting terms not contained in H_z, we get

$V(I) = H_y \cap H_x \cap (H_x \cup H_y) = H_{xy}$ (by the distributive property of intersections over unions).

Thus, $V(I)$ is the union of the (x, y)-plane H_z and the z-axis H_{xy}.

We know that the dimension of a union of finitely many vector subspaces of K^n is the largest of the dimensions of the subspaces.

Hence, since the dimension of H_{xy} is two as a vector space, the dimension of $V(I)$ is two as an affine variety.

Now, let us see what the dimension affine variety of general monomial ideal looks like.

Definition 2.1.1: In K^n, a vector subspace defined by setting some subset of the variables $x_1, \dots, x_n$ equals to zero is called a coordinate subspace.

Proposition 2.1.2: The affine variety of a monomial ideal in $K[x_1, \dots, x_n]$ is a finite union of coordinate subspaces of K^n.

Proof: First, note that if $x_{i_1}^{\alpha_1} \dots x_{i_r}^{\alpha_r}$ is a monomial in $K[x_1, \dots, x_n]$ with $\alpha_j \geq 1$ for $1 \leq j \leq r$, then $V\left(x_{i_1}^{\alpha_1} \dots x_{i_r}^{\alpha_r}\right) = H_{x_{i_1}} \cup \dots \cup H_{x_{i_r}}$, where $H_{x_k} = V(x_k)$.

Thus, the affine variety defined by a monomial is a union of coordinate hyper planes. Since a monomial ideal is generated by a finite collection of monomials, the affine variety corresponding to a monomial ideal is a finite intersection of unions of coordinate hyper planes.

By the distributive property of intersections over unions, any finite intersection of unions of coordinate hyper planes can be rewritten as a finite union of intersections of coordinate hyper planes (as (1) above). But, the intersection of any collection of coordinate hyper planes is a coordinate subspace. ■

Remark: When we write the affine variety of monomial ideal I as a union of finitely many coordinate subspaces, we can omit a subspace if it is contained in another in the union.

Thus, we can write $V(I)$ as a union of coordinate subspaces;

$$V(I) = V_1 \cup \dots \cup V_p, \text{ where } V_i \neq V_j \text{ for } i \neq j.$$

Having this in mind, we have the following formal definition of dimension of an affine variety V.

Definition 2.1.3: Let V be an affine variety, which is the union of finite number of linear subspaces of K^n. Then, the dimension of V, denoted by $\dim V$, is the largest of the dimensions of the subspaces.

Example 2.1.4: The dimension of the union of two planes and a line is two, and the dimension of the union of three lines is one.

Now, to compute the dimension of the affine variety corresponding to a monomial ideal I, we find the maximum of the dimensions of the coordinate subspaces contained in $V(I)$.

Let $I = \langle m_1, \dots, m_t \rangle$ be a proper ideal generated by the monomials m_j. Then, to compute $\dim V(I)$, we need to pick out the component of $V(I) = \bigcap_{j=1}^{t} V(m_j)$ of largest dimension. If we can find a collection of variables $x_{i_1}, \dots, x_{i_r}$ such that at least one of these variables appears in each m_j, then the coordinate subspaces, defined by the equations $x_{i_1} = \cdots = x_{i_r} = 0$, is contained in $V(I)$.

Now, for all $1 \leq j \leq t$, let $M_j = \{k \in \{1, \dots, n\} : x_k \text{ divides the monomial } m_j\}$ be the set of subscripts of variables occurring with positive exponents in m_j, and,
let $M = \{J \subseteq \{1, \dots, n\} : J \cap M_j \neq \emptyset \text{ for all } 1 \leq j \leq t\}$ consists of all subsets of $\{1, \dots, n\}$ which have non-empty intersection with every set M_j.

Note that $M \neq \emptyset$, because $\{1, \ldots, n\} \in M$.

Now, if we let $|J|$ denote the number of elements in J, then we have the following proposition.

Proposition 2.1.5: With the above notation,

$$dim\, V(I) = n - \min(|J| : J \in M).$$

Proof: Let $J = \{i_1, \ldots, i_r\}$ be an element of M such that $|J| = r$ is minimal in M.

Since each monomial m_j contains some power of some x_{i_k}, where $1 \leq k \leq r$, the coordinate subspace $W = V(x_{i_1}, \ldots, x_{i_r})$ is contained in $V(I)$. Then, the dimension of W is $n - r$.

Hence, by Definition 2.1.3, dim $V(I)$ is at least $n - r = n - |J|$.

Assume dim $V(I) > n - r$.

Then, for some $s < r$, there is a coordinate subspace $W' = V(x_{k_1}, \ldots, x_{k_s})$ contained in $V(I)$.

This implies, each monomial m_j vanishes on W', and in particular, it vanishes at the point $p \in W'$ whose k_i'th coordinate is 0 for all $1 \leq i \leq s$, and whose other coordinates are 1.

Hence, at least one of the x_{k_i} must divide m_j.

This implies, we have $J' = \{k_1, \ldots, k_s\} \in M$.

It follows that $|J'| = s < r$ which contradicts to the minimulity of r.

Thus, dim $V(I) = n - r = n - \min(|J| : J \in M)$. ■

Example 2.1.6: Consider $I = \langle x_2^2 x_3^3, x_1^5 x_3^4, x_1^2 x_2 x_3^2 \rangle$,

(Note that x_1, x_2, x_3 act as x, y, z respectively).

Then, $\langle x_2^2 x_3^3, x_1^5 x_3^4, x_1^2 x_2 x_3^2 \rangle = \langle m_1, m_2, m_3 \rangle$

Where $m_1 = x_2^2 x_3^3$, $m_2 = x_1^5 x_3^4$ and $m_3 = x_1^2 x_2 x_3^2$.

This implies, we have $M_1 = \{2,3\}$, $M_2 = \{1,3\}$ and $M_3 = \{1,2,3\}$.

Then, $M = \{\{1,2,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{3\}\}$.

Now, using Proposition 2.1.5, we have

$dim\, V(I) = n - \min(|J| : J \in M) = 3 - 1 = 2$

Remark: If some variable, say x_i, appears in every monomial in a set of generators for a proper monomial ideal I, then $\dim \mathrm{V}(I) = n - 1$, since $J = \{i\} \in M$.

Example 2.1.7: In Example 2.1.6 above, the variable x_3 appears in every monomial in the set of generators for the ideal I.

Then, $\dim \mathrm{V}(I) = n - 1 = 3 - 1 = 2$ since $J = \{3\} \in M$.

2.2 The Complement of a Monomial Ideal

In this Section, we will see the monomials not contained in a monomial ideal $I \subseteq K[x_1, \dots, x_n]$. Since there may be infinitely many such monomials, we should find a formula for the number of monomials $x^\alpha \notin I$ which have total degree less than some bound. The results here will play an important role in defining the dimension of an arbitrary affine variety.

To begin, let us introduce some new notation. For each monomial ideal I, we let $C(I) = \{\alpha \in \mathbb{Z}^n_{\geq 0} : x^\alpha \notin I\}$ be the set of exponents of monomials not in I. We also set $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, \dots, 0)$, …, $e_n = (0, \dots, 0, 1)$.

Further, we define the coordinate subspace of $\mathbb{Z}^n_{\geq 0}$ determined by $e_{i_1}, \dots, e_{i_r}$, where $i_1 < \dots < i_r$ to be the set

$[e_{i_1}, \dots, e_{i_r}] = \{a_1 e_{i_1} +, \dots, + a_r e_{i_r} : a_j \in \mathbb{Z}_{\geq 0}\ \forall\ 1 \leq j \leq r\}$. Then, we say that $[e_{i_1}, \dots, e_{i_r}]$ an r-dimensional coordinate subspace of $\mathbb{Z}^n_{\geq 0}$.

Definition 2.2.1: Any subset of $\mathbb{Z}^n_{\geq 0}$ is said to be a translate of a coordinate subspace $[e_{i_1}, \dots, e_{i_r}]$ if it is of the form

$$\alpha + [e_{i_1}, \dots, e_{i_r}] = \{\alpha + \beta : \beta \in [e_{i_1}, \dots, e_{i_r}]\},$$

Where $\alpha = \sum_{i \neq \{i_1, \dots, i_r\}} a_i e_i$ for all $a_i \geq 0$.

Note that this restriction on α means that we are translating by a vector perpendicular to $[e_{i_1}, \dots, e_{i_r}]$.

Example 2.2.2: The set $\{(1, l) : l \in \mathbb{Z}_{\geq 0}\} = e_1 + [e_2]$ is a translate of the subspace $[e_2]$ in the plane $\mathbb{Z}^2_{\geq 0}$ of exponents.

Remark: There is a direct correspondence between the coordinate subspace in $V(I)$ and the coordinate subspace of $\mathbb{Z}^n_{\geq 0}$ contained in $C(I)$.

Proposition 2.2.3: Let $I \subset K[x_1, \dots, x_n]$ be a proper monomial ideal. Then,

a) The coordinate subspace $V(x_i : i \notin \{i_1, \dots, i_r\})$ is contained in $V(I)$ if and only if $[e_{i_1}, \dots, e_{i_r}] \subseteq C(I)$.
b) The dimension of $V(I)$ is the dimension of the largest coordinate subspace in $C(I)$.

Proof (a): To prove the forward direction, suppose that $V(x_i : i \notin \{i_1, \dots, i_r\})$ is contained in $V(I)$. But, first note that $W = V(x_i : i \notin \{i_1, \dots, i_r\})$ contains the point p whose i_j'th coordinate is 1 for all $1 \leq j \leq r$ and whose other coordinates are zero.

For any $\alpha \in [e_{i_1}, \dots, e_{i_r}]$, the monomial x^α can be written in the form $x^\alpha = x_{i_1}^{\alpha_{i_1}} \dots x_{i_r}^{\alpha_{i_r}}$. Then, $x^\alpha = 1$ at p, so that $x^\alpha \notin I$, since $p \in W \subseteq V(I)$ by hypothesis.

Hence, $\alpha \in C(I)$.

This implies, $[e_{i_1}, \dots, e_{i_r}] \subseteq C(I)$.

Conversely, suppose that $[e_{i_1}, \dots, e_{i_r}] \subseteq C(I)$.

Since I is proper, every monomial in I contains at least one variable other than $x_{i_1}, \dots, x_{i_r}$. This means that every monomial in I vanishes on any point $(a_1, \dots, a_n) \in K^n$ for which $a_i = 0$ when $i \notin \{i_1, \dots, i_r\}$. So every monomial in I vanishes on the coordinate subspace

$$V(x_i : i \notin \{i_1, \dots, i_r\}).$$

Hence, $V(x_i : i \notin \{i_1, \dots, i_r\})$ is contained in $V(I)$.

(b): Note that the coordinate subspace $V(x_i : i \notin \{i_1, \dots, i_r\})$ has dimension r. Now, from part (a), the dimensions of the coordinate subspaces of K^n contained in $V(I)$ and the coordinate subspaces of $\mathbb{Z}^n_{\geq 0}$ contained in $C(I)$ are the same. Then, by Definition 2.1.3, $dim\, V(I)$ is the maximum of the dimensions of the coordinate subspaces of K^n contained in $V(I)$.

Therefore, $dim\, V(I)$ is the dimension of the largest coordinate subspace in $C(I)$. ■

Now, fix a non-negative integer s, our next goal is to find a formula for the number of monomials of total degree less than or equal to s in the complement of a monomial ideal $I \subseteq K[x_1, \dots, x_n]$. To do this, we have the following important proposition.

Proposition 2.2.4: [1] If $I \subseteq K[x_1, \dots, x_n]$ is a monomial ideal with $dimV(I) = d$, then for all s sufficiently large, the number of points in $C(I)$ of total degree less than or equal to s is a polynomial of degree d in s which can be written in the form:

$$\sum_{i=0}^{d} a_i \binom{s}{d-i}, \text{ where } a_i \in \mathbb{Z} \text{ for all } 0 \leq i \leq d \text{ and } a_i > 0.$$

2.3 The Hilbert Functions and the Dimension of an Affine Variety

In this Section, we will define the Hilbert Function of an ideal I and use it to define the dimension of an affine variety V, in terms of the number of monomials not contained in the ideal I. That is, we will need to consider the number of monomials of total degree less than or equal to some non-negative integer s.

In Chapter One of our discussion, we defined the quotient of a ring modulo an ideal I. And here, there is an analogous operation on vector spaces which we will use to make our work precise.

Remark: Given a vector space V and a subspace W of V, the relation on V defined by $v_1 \sim v_2$ if $v_1 - v_2 \in W$ is an equivalence relation. The set of equivalence classes of $\sim$ is denoted by V/W, so that

$$V/W = \{[v] : v \in V\}.$$

Then, the operations $[v_1] + [v_2] = [v_1 + v_2]$ and $a[v] = [av]$, where $v_1, v_2, v \in V$ and $a \in K$ are well-defined and make into K-vector space, called the quotient space of V modulo W. When V is finite-dimensional, we can compute the dimension of V/W as follows.

Proposition 2.3.1: Let W be a subspace of a finite-dimensional vector space V. Then,

a. W and V/W are also finite-dimensional vector spaces, and

b. $dimV = dimW + dim\ (V/W)$.

Proof (a): If V is a finite-dimensional vector space, it is a standard fact from linear algebra that W is also finite-dimensional. It follows that V/W is also finite-dimensional.

(b): Suppose that V is a finite-dimensional. Let $v_1, \ldots, v_m$ be a basis of W, so that $dimW = m$. In V, the vectors $v_1, \ldots, v_m$ are linearly independent and hence, can be extended to a basis $v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+n}$ of V.

Thus, $dimV = m + n$.

Claim: $[v_{m+1}], \dots, [v_{m+n}]$ form a basis of V/W.

To see $[v_{m+1}], \dots, [v_{m+n}]$ span V/W, take $[v] \in V/W$. If we write v in the form $v = \sum_{i=1}^{m+n} a_i v_i$, then $v \sim a_{m+1} v_{m+1} + \dots + a_{m+n} v_{m+n}$, since their difference is $a_1 v_1 + \dots + a_m v_m \in W$.

It follows that in V/W, we have $[v] = [a_{m+1} v_{m+1} + \dots + a_{m+n} v_{m+n}]$

$$= a_{m+1}[v_{m+1}] + \dots + a_{m+n}[v_{m+n}].$$

Thus, $[v_{m+1}], \dots, [v_{m+n}]$ span V/W.

It remains to show that $[v_{m+1}], \dots, [v_{m+n}]$ are linearly independent in V/W.

Since $v_1, \dots, v_m, v_{m+1}, \dots, v_{m+n}$ are linearly independent in V such that $v_1, \dots, v_m$ are linearly independent in W.

It follows that $[v_{m+1}], \dots, [v_{m+n}]$ are linearly independent in V/W.

Hence, $[v_{m+1}], \dots, [v_{m+n}]$ form a basis of V/W, and $dim\,(V/W) = n$.

Therefore, $dimV = m + n$

$$= dimW + dim\,(V/W).$$ ■

Our next goal is to define the affine Hilbert Function of an ideal I in $K[x_1, \dots, x_n]$. But, first we have the following remark.

Remark: By considering as a vector space over a field K, the polynomial ring $K[x_1, \dots, x_n]$ has infinite dimension, and the same is true for any non-zero ideal.

So, to get something finite-dimensional, let us restrict ourselves to polynomials of total degree less than or equal to some positive integer s. Let $K[x_1, \dots, x_n]_{\leq s}$ denote the set of polynomials of total degree less than or equal to s. By Lemma 1.1.12 of Chapter One, it follows that $K[x_1, \dots, x_n]_{\leq s}$ is a vector space of dimension $\binom{n+s}{s}$.

Then, given an ideal $I \subseteq K[x_1, \dots, x_n]$, let $I_{\leq s} = I \cap K[x_1, \dots, x_n]_{\leq s}$ denote the set of polynomials in I of total degree less than or equal to s.

Remark: $I_{\leq s}$ is a vector subspace of $K[x_1, \dots, x_n]_{\leq s}$.

Definition 2.3.2: Let I be an ideal in $K[x_1, \dots, x_n]$. Then, the affine Hilbert Function of I is a function on the non-negative integers s defined by

$$aHF_I(s) = dim(K[x_1, \dots, x_n]_{\leq s}/I_{\leq s}) = dimK[x_1, \dots, x_n]_{\leq s} - dimI_{\leq s},$$

where the last equality is by proposition 2.3.1

Proposition 2.3.3: Let I be a proper monomial ideal in $K[x_1, \dots, x_n]$. Then,

a. For all $s \geq 0$, $aHF_I(s)$ is the number of monomials not in I of total degree $\leq s$
b. For all s sufficiently large, the affine Hilbert Function of I is given by a polynomial function
$$aHF_I(s) = \sum_{i=0}^{d} b_i \binom{s}{d-i}, \text{ where } b_i \in \mathbb{Z} \text{ and } b_0 > 0$$
c. The degree of the polynomial in part (b) is the maximum of the dimensions of the coordinate subspaces contained in $V(I)$.

Proof (a): First note that $\{x^\alpha : |\alpha| \leq s\}$ is a basis of $K[x_1, \dots, x_n]_{\leq s}$. Further, Lemma 1.3.8 of Chapter One shows that $\{x^\alpha : |\alpha| \leq s, \text{ where } x^\alpha \in I\}$ is a basis of $I_{\leq s}$. Consequently, the monomials in $\{x^\alpha : |\alpha| \leq s, \text{where } x^\alpha \notin I\}$ are exactly what we add to a basis of $I_{\leq s}$ to get a basis $\{x^\alpha : |\alpha| \leq s\}$ of $K[x_1, \dots, x_n]_{\leq s}$. It follows from the proof of Proposition 2.2.3 that $\{[x^\alpha] : |\alpha| \leq s, \text{ where } x^\alpha \notin I\}$ is a basis of the quotient space $K[x_1, \dots, x_n]_{\leq s}/I_{\leq s}$. This shows that $\dim(K[x_1, \dots, x_n]_{\leq s}/I_{\leq s})$ is the number of monomials not in I of total degree less than or equal to s. But,

$$\dim(K[x_1, \dots, x_n]_{\leq s}/I_{\leq s}) = aHF_I(s) \text{ by Definition 2.3.2.}$$

(b): From part (a), we have $aHF_I(s)$ equals to the number of monomials not in I of total degree $\leq s$. But, for s sufficiently large, Proposition 2.2.4 says that the number of monomials not in I of total degree less than or equal to s is a polynomial of degree d in s, and we can write this in the form $\sum_{i=0}^{d} b_i \binom{s}{d-i}$, where $b_i \in \mathbb{Z}$ and $b_0 > 0$.

Thus, $aHF_I(s) = \sum_{i=0}^{d} b_i \binom{s}{d-i}$.

(c): By definition, we know that $dimV(I)$ is the maximum of the dimensions of the coordinate subspaces contained in $V(I)$. But, $dimV(I) = d$ by Proposition 2.2.4 ∎

We are ready now to link the ideals of Section two to arbitrary ideals in $K[x_1, \dots, x_n]$. The key ingredient here is the following observation. As we discussed in Chapter One, we say that a monomial order $>$ on $K[x_1, \dots, x_n]$ is a grlex order if $x^{\alpha} > x^{\beta}$ whenever $|\alpha| > |\beta|$. Having this in mind, we have the following proposition.

Proposition 2.3.4: Let $I \subseteq K[x_1, \dots, x_n]$ be an ideal and let $>$ be a grlex order on $K[x_1, \dots, x_n]$. Then, the monomial ideal $\langle LT(I) \rangle$ has the same affine Hilbert Function as I.

Proof: Fix a non-negative integer s and consider the leading monomials $LM(f)$ of all elements of $f \in I_{\leq s}$.

Then, there are only finitely many such monomials, so that

$$\{LM(f) : f \in I_{\leq s}\} = \{LM(f_1), \dots, LM(f_m)\} \quad (1)$$

for some polynomials $f_1, \dots, f_m \in I_{\leq s}$.

By rearranging and deleting duplicates, we can assume that

$$LM(f_1) > \cdots > LM(f_m).$$

Claim 1: $f_1, \dots, f_m$ are basis of $I_{\leq s}$ as a vector space over K.

Now, consider a non-trivial linear combination $a_1 f_1 + \cdots + a_m f_m$ and choose the smallest i such that $a_i \neq 0$.

Then, given how we ordered the leading monomials, there is nothing to cancel $a_i LT(f_i)$, so the linear combination is non-zero.
Hence, $f_1, \dots, f_m$ are linearly independent.

Next, let $W = [f_1, \dots, f_m] \subseteq I_{\leq s}$ be the subspace spanned by $f_1, \dots, f_m$. If $W \neq I_{\leq s}$, pick $f \in I_{\leq s} - W$ with $LM(f)$ minimal.

By (1), we have $LM(f) = LM(f_i)$ for some i, and, hence, $LT(f) = \lambda LT(f_i)$ for some $\lambda \in$ K.

Then, $f - \lambda f_i \in I_{\leq s}$ has a smaller leading monomial, so that $f - \lambda f_i \in$ W by the minimality of $LM(f)$.
This implies, $f \in$ W, which is contradiction.
It follows that $W = [f_1, \dots, f_m] = I_{\leq s}$.
Thus, $f_1, \dots, f_m$ are basis of $I_{\leq s}$.

Now, we know that the monomial ideal $\langle LT(I) \rangle$ is generated by the leading terms (or leading monomials) of elements of I.
Hence, $LM(f_i) \in \langle LT(I) \rangle_{\leq s}$ since $f \in I_{\leq s}$.

Claim 2: $LM(f_1), \dots, LM(f_m)$ are basis of $\langle LT(I) \rangle_{\leq s}$.

Arguing as in claim (1) above, $LM(f_1), \dots, LM(f_m)$ are linearly independent. So, it remains to show that $[LM(f_1), \dots, LM(f_m)] = \langle LT(I) \rangle_{\leq s}$.
By Lemma 1.3.8 of Chapter One, it suffices to show that
$\{LM(f_1), \dots, LM(f_m)\} = \{LM(f) : f \in I,$
where $LM(f)$ has total degree $\leq s\}$ (2)

Now, to relate this to (1), note that $>$ is a grlex order.
This implies that for any non-zero polynomial $f \in K[x_1, \dots, x_n]$, $LM(f)$ has the same total degree as f.
In particular, if $LM(f)$ has total degree less than or equal to s, then so does f, which means (2) follows immediately from (1).
That is, $LM(f_1), \dots, LM(f_m)$ are basis of $\langle LT(I) \rangle_{\leq s}$.
Thus, $I_{\leq s}$ and $\langle LT(I) \rangle_{\leq s}$ have the same dimension since they both have basis consisting of m elements.
Then, the dimension formula of Proposition 2.3.1 implies that

$$
\begin{aligned}
aHF_I(s) &= \dim(K[x_1, \dots, x_n]_{\leq s}/I_{\leq s}) \\
&= \dim(K[x_1, \dots, x_n]_{\leq s}/\langle LT(I)\rangle_{\leq s}) \\
&= aHF_{\langle LT(I)\rangle_{\leq s}}(s)
\end{aligned}
\qquad \blacksquare
$$

Remark: If we combine Propositions 2.3.3 and 2.3.4, it follows immediately that if I is any ideal in $K[x_1, \dots, x_n]$ and s is sufficiently large, the affine Hilbert Function of I can be written as

$aHF_I(s) = \sum_{i=0}^{d} b_i \binom{s}{d-i}$, where the b_i are integers and b_0 is positive.

This leads to the following definition.

Definition 2.3.5: The polynomial which equals to $aHF_I(s)$ for sufficiently large s is called the affine Hilbert Polynomial of I and is denoted by $aHP_I(s)$.

By definition, the affine Hilbert Function of an ideal I coincides with the affine Hilbert Polynomial of I when s is sufficiently large. We next compare the degrees of the affine Hilbert Polynomial for I and $\sqrt{I}$.

Remark: Given an ideal $I \subseteq K[x_1, \dots, x_n]$ and an affine variety V, then $\sqrt{I}$ is monomial and $V(I) = V(\sqrt{I})$.

Proposition 2.3.6: If $I \subseteq K[x_1, \dots, x_n]$ is an ideal, then the affine Hilbert Polynomial of I and $\sqrt{I}$ have the same degree.

Proof: First, recall that for a monomial ideal I, the degree of the affine Hilbert Polynomial is the dimension of the largest coordinate subspace of K^n contained in $V(I)$. Then, since $\sqrt{I}$ is monomial and $V(I) = V(\sqrt{I})$, it follows that aHP_I and $aHP_{\sqrt{I}}$ have the same degree.

Now, let I be an arbitrary ideal in $K[x_1, \dots, x_n]$ and $>$ be a grlex order on $K[x_1, \dots, x_n]$.

Claim: $\langle LT(I)\rangle \subseteq \langle LT(\sqrt{I})\rangle \subseteq \sqrt{\langle LT(I)\rangle}$ (3)

When we discuss about radical ideals in Chapter One, we know

that $I \subseteq \sqrt{I}$. This implies that $\langle LT(I)\rangle \subseteq \langle LT(\sqrt{I})\rangle$.

It remains to show that $\langle LT(\sqrt{I})\rangle \subseteq \sqrt{\langle LT(I)\rangle}$.

To show this, let x^{α} be a monomial in $LT(\sqrt{I})$.

This means that there is a polynomial $f \in \sqrt{I}$ such that $LT(f) = x^{\alpha}$.

Also, we know that $f^r \in I$ for some $r \geq 0$.

It follows that $x^{r\alpha} = LT(f^r) \in \langle LT(I)\rangle$.

Thus, $x^{\alpha} \in \sqrt{\langle LT(I)\rangle}$, and, hence, $\langle LT(\sqrt{I})\rangle \subseteq \sqrt{\langle LT(I)\rangle}$.

If I_1 and I_2 are any ideals in $K[x_1, \dots, x_n]$ such that $I_1 \subseteq I_2$, then we have

$$degaHP_{I_2} \leq degaHP_{I_1}.$$

If we apply this fact to (3), we obtain

$$degaHP_{\sqrt{\langle LT(I)\rangle}} \leq degaHP_{\langle LT(\sqrt{I})\rangle} \leq degaHP_{\langle LT(I)\rangle}.$$

By the result for monomial ideals, the two outer terms here are equal.

Hence, we conclude that $aHP_{\langle LT(I)\rangle}$ and $aHP_{\langle LT(\sqrt{I})\rangle}$ have the same degree

and the same is true for aHP_I and $aHP_{\sqrt{I}}$ by Proposition 2.3.4. ■

Now, recall that $V(I) = V(\sqrt{I})$ for all ideals. Thus, the degree of the affine Hilbert Polynomial is the same for a large collection of ideals defining the same variety. Moreover, the degree of the affine Hilbert Polynomial is the same as our intuitive notation of the dimension of the affine variety of monomial ideals. So it should no surprise that in the general case, we define dimension in terms of the degree of the affine Hilbert Polynomial. We will always assume that the field K is infinite.

Definition 2.3.7: The dimension of an affine variety $V \subseteq K^n$, denoted by $dimV$, is the degree of the affine Hilbert Polynomial of the corresponding ideal

$$I = I(V) \subseteq K[x_1, \dots, x_n].$$

Example 2.3.8: Consider the twisted cubic $V = V(y - x^2, z - x^3) \subseteq \mathbb{R}^3$.

We know that $I = I(V) = \langle y - x^2, z - x^3\rangle \subseteq \mathbb{R}[x, y, z]$.

Now, using the grlex order, we have a Groebner basis for I given by $\{y^3 - z^2, x^2 - y, xy - z, xy - y^2\}$, so that $\langle LT(I) \rangle = \langle y^3,\ x^2,\ xy,\ xz \rangle$.

Then, $dimV = degaHP_I = \ degaHP_{\langle LT(I) \rangle}$.

But, $degaHP_{\langle LT(I) \rangle}$ is equals to the maximum dimension of a coordinate subspace in $V(\langle LT(I) \rangle)$ by Propositions 2.3.3 and 2.3.4.

Since $V(\langle LT(I) \rangle) = V(y^3,\ x^2,\ xy,\ xz) = V(x, y) \subseteq \mathbb{R}^3$, we conclude that $dimV = 1$.

This agrees with our intuition that the twisted cubic should be 1-dimensional since it is a curve in $\mathbb{R}^3$.

Remark: One drawback of definition 2.3.7 is that to find the dimension of an affine variety V, we need to know $I(V)$, which, in general, is difficult to compute. It should be much nicer if $dimV$ is the degree of aHP_I, where I is an arbitrary ideal defining V. Unfortunately, this is not true in general.

Example 2.3.9: Consider $I = \langle x^2 + y^2 \rangle \subseteq \mathbb{R}[x, y]$, then we have

$$degaHP_I(s) = 1.$$

But, we see that $V = V(I) = \{(0,0)\}$ has dimension zero.

Thus, $dimV(I) \neq degaHP_I$.

When the field K is algebraically closed, these difficulties go away. More precisely, we have the following theorem that tells us how to compute the dimension in terms of any defining ideal.

Theorem 2.3.10 (The Dimension Theorem): Let $V = V(I)$ be an affine variety, where $I \subseteq K[x_1, \dots, x_n]$ is an ideal. If K is algebraically closed, then

a) $dimV = degaHP_I$.

b) If $>$ is a grlex order on $K[x_1, \dots, x_n]$, then
$dimV = degaHP_{\langle LT(I) \rangle}$

= the maximum dimension of coordinate subspace in $V(\langle LT(I)\rangle)$.

Proof (a): Suppose $V = V(I)$ is an affine variety and $I \subseteq K[x_1, \dots, x_n]$ is an ideal. Then, since K is algebraically closed, the Strong Nullstellensatz implies that $I(V) = I(V(I)) = \sqrt{I}$. Then,

$dimV = degaHP_{I(V)} = degaHP_{I(V(I))} = degaHP_{\sqrt{I}}$.

But, by Proposition 2.3.6, we have $degaHP_{\sqrt{I}} = degaHP_I$.

This implies, $dimV = degaHP_{\sqrt{I}} = degaHP_I$.

(b): Suppose $>$ is a grlex order on $K[x_1, \dots, x_n]$.

From (a), we have $dimV = degaHP_I$.

Then, by Proposition 2.3.4, we have $degaHP_I = degaHP_{\langle LT(I)\rangle}$.

Thus, $dimV = degaHP_{\langle LT(I)\rangle}$.

Now, part (c) of Proposition 2.3.3 implies that $degaHP_I$ is the maximum of the dimensions of the coordinate subspaces contained in $V(I)$.

Thus, $dimV$ is equals to the maximum of the dimensions of the coordinate subspaces contained in $V(\langle LT(I)\rangle)$. ■

Remark: Over algebraically closed field, to compute the dimension of $V = V(I)$, one can proceed as follows.

1. Compute a Groebner basis for I using grlex order
2. Compute the maximal dimension d of coordinate subspaces contained in $V(\langle LT(I)\rangle)$.
3. Use The Dimension Theorem to get $dimV = d$.

Example 2.3.11: See Example 2.3.8 above.

2.4 Elementary Properties of Dimension

Using Definition 2.3.7, we can now state several basic properties of dimension. As we discussed in the preceding Section, we assume that the field K is infinite. We first observe now the following proposition.

Proposition 2.4.1: Let V_1 and V_2 be affine varieties. If $V_1 \subseteq V_2$, then $dimV_1 \leq dimV_2$.

Proof: Assume that $V_1 \subseteq V_2$.

Claim: $dimV_1 \leq dimV_2$

Now, let $dimV_1 = n$ and

$dimV_2 = m$

By Definition 2.1.3, we know that the dimension of an affine variety V is the largest of the dimensions of the subspaces. This implies that m is the maximum of the dimensions of the subspaces in V_2 and n is the maximum of the dimensions of the subspaces in V_1.

It follows that $n \leq m$.

Thus, $dimV_1 \leq dimV_2$. ∎

We next will study the relation between the dimension of an affine variety and the number of defining equations. We begin with the case where the affine variety V is defined by a single equation. So we have the following proposition.

Proposition 2.4.2: Let K be an algebraically closed field, and $f \in K[x_0, x_1, \ldots, x_n]$ be a non-constant homogeneous polynomial. Then, the dimension of the affine variety V in K^n defined by f is given by $dimV(f) = n - 1$.

Proof: Suppose $f \in K[x_0, x_1, \ldots, x_n]$ is a non-constant homogeneous polynomial, and let $>$ be a grlex order on $K[x_0, x_1 \ldots, x_n]$.

Then, since K is algebraically closed, The Dimension Theorem says that the dimension of $V(f)$ is the maximum dimension of a coordinate

subspace contained in $V(\langle LT(I)\rangle)$, where $I = \langle f\rangle$. Then, $\langle LT(I)\rangle =$
$\langle LT(f)\rangle$ and since $LT(f)$ is a non-constant monomial, the affine variety
$V(LT(f))$ is a union of subspaces of K^n of dimension $n-1$.
It follows that $dimV(I) = n-1$.
Therefore, $dimV(f) = n-1$. ■

Thus, if K is algebraically closed field, a hyper surface $V(f)$ in K^n always has dimension $n-1$.

Remark: This result is not valid if K is not algebraically closed.

Example 2.4.3: Let $I = \langle x^2+y^2\rangle$ in $\mathbb{R}[x,y]$.

In Section 2.3, we have seen that $V(f) = \{(0,0)\}$ has dimension zero.
Yet, Proposition 2.4.2 would predict that the dimension was one.
In fact, over a non-algebraically closed field, the affine variety in K^n defined by a single polynomial can have any dimension between 0 and 1.

The following theorem establishes the analogue statement of Proposition 2.4.2 when the affine variety $V(f)$ is replaced by an arbitrary affine variety V.

Remark: If I is an ideal and f is a polynomial, then $V(I+\langle f\rangle) = V(I)\cap V(\langle f\rangle)$.

Theorem 2.4.4: Let K be an algebraically closed field and let I be a homogeneous ideal in $K[x_0, x_1 \dots, x_n]$. If f is any non-constant homogeneous polynomial, then
$dimV(I) \geq \dim V(I+\langle f\rangle) \geq dimV(I) - 1$.

Proof: To compute $dimV(I+\langle f\rangle)$, we will need to compute the affine Hilbert Polynomials aHP_I and $aHP_{(I+\langle f\rangle)}$.
We first note that since $I \subseteq I+\langle f\rangle$, then $degaHP_I \geq degaHP_{I+\langle f\rangle}$.
This implies, $dimV(I) \geq \dim(I+\langle f\rangle)$ by the Dimension Theorem.

It remains to show that $\dim(I + \langle f \rangle) \geq dimV(I) - 1$.

To do this, suppose that f has total degree $r > 0$ and fix a total degree $s \geq r$. Consider the map $\pi : K[x_0, x_1, \ldots, x_n]_s / I_s \longrightarrow K[x_0, x_1, \ldots, x_n]_s / (I + \langle f \rangle)_s$ defined by $\pi([g]) = [g]$, where $[g] \in K[x_0, x_1, \ldots, x_n]_s / I_s$, so that π is well-defined linear map and onto. To investigate its kernel, we will use the map $\alpha_f : K[x_0, x_1, \ldots, x_n]_{s-r} / I_{s-r} \longrightarrow K[x_0, x_1, \ldots, x_n]_s / I_s$ defined by $\alpha_f([h]) = [fh]$, where $[h] \in K[x_0, x_1, \ldots, x_n]_{s-r} / I_{s-r}$, so that α_f is also well-defined linear map.

Claim 1: Kernel of π is exactly the image of α_f, that is

$$\alpha_f(K[x_0, x_1, \ldots, x_n]_{s-r} / I_{s-r}) = \{[g] : \pi([g]) = [0] \in K[x_0, x_1, \ldots, x_n]_s / (I + \langle f \rangle)_s\}. \qquad (1)$$

To prove this, note that if $h \in K[x_0, x_1, \ldots, x_n]_{s-r}$, then $fh \in (I + \langle f \rangle)_s$, and hence, $\pi([fh]) = [0] \in K[x_0, x_1, \ldots, x_n]_s / (I + \langle f \rangle)_s$.

Conversely, if $g \in K[x_0, x_1, \ldots, x_n]_s$ and $\pi([g]) = [0]$, then $g \in (I + \langle f \rangle)_s$. This means, $g = g' + gh$ for some $g' \in I$.
If we write $g' = \sum_i g'_i$ and $h = \sum_i h_i$ as sums of homogeneous polynomials, where g'_i and h_i have total degree i, it follows that $g = g'_s + gh_{s-r}$ since f and g are homogeneous.
Since I is a homogeneous ideal, we have $g'_s \in I_s$, and it follows that $[g] = [gh_{s-r}] = \alpha_f([gh_{s-r}]) \in K[x_0, x_1, \ldots, x_n]_s / I_s$. This shows that $[g]$ is in the image of α_f as claimed.
Now, since π is onto and we know that its kernel by (1), the Dimension Theorem for linear mappings shows that

$$\dim(K[x_0, x_1, \ldots, x_n]_s / I_s) = \dim \alpha_f(K[x_0, x_1, \ldots, x_n]_{s-r} / I_{s-r}) + \dim(K[x_0, x_1, \ldots, x_n]_s / (I + \langle f \rangle)_s).$$

Then, certainly, $\dim \alpha_f(K[x_0, x_1, \ldots, x_n]_{s-r} / I_{s-r}) \leq$

$$\dim(K[x_0, x_1, \dots, x_n]_{s-r}/I_{s-r}). \qquad (2)$$

with equality if and only if α_f is one-to-one.

Hence, $\dim (K[x_0, x_1, \dots, x_n]_s/(I + \langle f \rangle)_s) \geq \dim(K[x_0, x_1, \dots, x_n]_s/I_s) - \dim(K[x_0, x_1, \dots, x_n]_{s-r}/I_{s-r})$.

In terms of affine Hilbert Functions, this tells us

$$aHF_{I+\langle f \rangle}(s) \geq aHF_I(s) - aHF_I(s-r), \text{ whenever } s \geq r.$$

Thus, if s is sufficiently large, we obtain the inequality

$$aHP_{I+\langle f \rangle}(s) \geq aHP_I(s) - aHP_I(s-r) \qquad (3)$$

for the Hilbert Polynomials.

Claim 2: $dimV(I + \langle f \rangle) \geq \dim \mathrm{V}(I) - 1$

Suppose that aHP_I has degree d.

Then, the polynomial on the right hand side of (3) has degree $d - 1$.

Thus, (3) shows that $aHP_{I+\langle f \rangle}(s)$ is greater than or equal to a polynomial of degree $d - 1$ for s sufficiently large.

This implies, $degaHP_{I+\langle f \rangle}(s) \geq d - 1$.

Now, since K is algebraically closed, we can conclude that $dimV(I + \langle f \rangle) \geq \dim \mathrm{V}(I) - 1$.

Therefore, $dimV(I) \geq dimV(I + \langle f \rangle) \geq \dim \mathrm{V}(I) - 1$. ∎

Remark: Theorem 2.4.4 can fail sometimes even when K is algebraically closed.

Example 2.4.5: Consider the ideal $I = \langle xz, yz \rangle \subseteq \mathbb{C}[x, y, z]$.

Then, we can easily see that in $\mathbb{C}^3$, we have $V(I) = V(z) \cup V(x, y)$, so that $V(I)$ is the union of the (x, y)-plane and the z-axis. In particular, $V(I)$ has dimension 2.

Now, let $f = z - 1 \in \mathbb{C}[x, y, z]$, then $V(f)$ is the plane $z = 1$. It follows that $V(I + \langle f \rangle) = V(I) \cap V(f)$ consists of the single point $(0, 0, 1)$.

We know that a single point has dimension 0, yet Theorem 2.4.4 would predict that $V(I + \langle f \rangle)$ had dimension at least 1. Hence, it fails. What goes wrong here is that the planes $z = 0$ and $z = 1$ are parallel and hence, do not meet in affine space.

By carefully analyzing the proof of Theorem 2.4.4, we can give a condition that

$$dimV(I + \langle f \rangle) = \dim \mathrm{V}(I) - 1.$$

Corollary 2.4.6: Let K be an algebraically closed field and $I \subseteq K[x_0, \ x_1, \dots, x_n]$ be a homogeneous ideal. If f is a non-constant homogeneous polynomial whose class in the quotient ring $K[x_0, \ x_1, \dots, x_n]/I$ is not a zero divisor, then

$$dimV(I + \langle f \rangle) = \dim \mathrm{V}(I) - 1.$$

Proof: As we observed from the proof of Theorem 2.4.4, the inequality (2) is an equality if the multiplication map α_f is one-to-one.

Claim: α_f is one-to-one

Suppose that $[h] \in K[x_0, x_1, \dots, x_n]_{s-r}/I_{s-r}$ is non-zero

This implies, $h \notin I_{s-r}$ and hence, $h \notin I$ since $I_{s-r} = I \cap K[x_0, x_1, \dots, x_n]_{s-r}$

Thus, $[h] \in K[x_0, \ x_1, \dots, x_n]/I$ is non-zero, so that $[f][h] = [fh]$ is non-zero in $K[x_0, \ x_1 \dots, x_n]$ by our assumption on f.

This shows that α_f is one-to-one.

Now, since (2) is an equality, the proof of Theorem 2.4.4 shows that we also get the equality

$$\dim K[x_0, x_1, \dots, x_n]_s/(I + \langle f \rangle)_s = \dim(K[x_0, x_1, \dots, x_n]_s/I_s) - \dim\,(K[x_0, x_1, \dots, x_n]_{s-r}/I_{s-r})$$

when $s \geq r$.

In terms of affine Hilbert Polynomial, this says that

$aHP_{I+\langle f \rangle}(s) = aHP_I(s) - aHP_I(s - r)$.

It follows that $dimV(I + \langle f \rangle) = \dim \mathrm{V}(I) - 1$. ■

Now, recall that from Lemma 1.2.4 of Chapter One, we have seen that if V and W are affine varieties, $V \cup W$ is also an affine variety which was the basic property of affine varieties. We next study that the dimension of the union of affine varieties.

Proposition 2.4.7: If V and W are affine varieties in K^n, then

$$dim(V \cup W) = max(dimV, dimW).$$

Proof: Suppose V and W are affine varieties in K^n.

Claim: $dim(V \cup W) \leq max(dimV, dimW)$.

Let $I = I(V)$ and $J = I(W)$, so that $dimV = degaHP_I$ and $dimW = degaHP_J$.

Then, by Theorem 1.5.7 (b) of Chapter One, we have $I(V \cup W) = I(V) \cap I(W) = I \cap J = IJ$.

It is more convenient to work with the product ideal IJ and we note that $IJ \subseteq I \cap J \subseteq \sqrt{IJ}$.

Then, we conclude that $degaHP_{IJ} \leq degaHP_{I \cap J} \leq degaHP_{\sqrt{IJ}}$.

But, Proposition 2.3.6 says that the outer terms are equal.

Hence, we conclude that $dim(V \cup W) = degaHP_{IJ}$.

Now, fix a grlex order $>$ on $K[x_1, \dots, x_n]$.

Then, by Propositions 2.3.3 and 2.3.4, it follows that $dimV$, $dimW$ and $dim(V \cup W)$ are given by the maximal dimension of a coordinate subspace contained in $V(\langle LT(I) \rangle)$, $V(\langle LT(J) \rangle)$ and $V(\langle LT(IJ) \rangle)$ respectively.

Now, we note that $\langle LT(IJ) \rangle \supset \langle LT(I) \rangle . \langle LT(J) \rangle$.

This implies, $V(\langle LT(IJ) \rangle) \subseteq V(\langle LT(I) \rangle) \cup V(\langle LT(J) \rangle)$ by Proposition 1.3.5.

Since K is infinite, every coordinate subspace is irreducible, and as a result, a coordinate subspace contained in $V(\langle LT(IJ)\rangle)$ lies in either $V(\langle LT(I)\rangle)$ or $V(\langle LT(J)\rangle)$.

This implies, $dim(V \cup W) \leq max(dimV, dimW)$.

Showing $max(dimV, dimW) \leq dim(V \cup W)$ follows immediately from Proposition 2.4.1.

Therefore, $dim(V \cup W) = max(dimV, dimW)$. ■

This proposition has the following useful corollary.

Corollary 2.4.8: The dimension of an affine variety V is the largest of the dimensions of its irreducible components.

Proof: Suppose $V = V_1 \cup \ldots \cup V_r$ is the decomposition of V into its irreducible components. Then, Proposition 2.4.7 and induction on r shows that

$$dimV = max\{dimV_1, \ldots, dimV_r\}.$$ ■

Remark: Corollary 2.4.8 allows us to reduce to the case of an irreducible affine variety when computing dimensions.

The following proposition shows that for irreducible affine varieties, the nation of dimension is especially well-behaved.

Proposition 2.4.9: Let K be an algebraically closed field and let $V \subseteq K^n$ be an irreducible affine variety. Then,

a) If $f \in K[x_1, \ldots, x_n]$ is a homogeneous polynomial which does not vanish on V, then

$$dim(V \cap V(f)) = dimV - 1$$

b) If $W \subset V$ is an affine variety, that is $W \neq V$, then

$$dimW < dimV.$$

Proof (a): By Proposition 1.7.2, we know that $I(V)$ is a prime ideal and the coordinate ring $K[V]$ is an integral domain.

Also we have $K[V]$ isomorphic to $K[x_0, x_1, \dots, x_n]/I(V)$ by Theorem 1.7.6. Now, since $f \notin I(V)$, the class of f is non-zero in $K[x_0, x_1, \dots, x_n]/I(V)$ and, hence, is not a zero divisor.

Then, by Corollary 2.4.6, we conclude that $dim(V \cap V(f)) = dimV - 1$.

(b): Suppose that $W \subset V$ is a proper subset of V.

Then, we can find $f \in I(W) - I(V)$.

Thus, $W \subseteq V \cap V(f)$.

It follows that from Proposition 2.4.1,

$dimW \leq dim\big(V \cap V(f)\big) = dimV - 1$.

But, we have from (a) that $dim\big(V \cap V(f)\big) = dimV - 1 < dimV$.

Hence, $dimW \leq dimV - 1$.

Therefore, $dimW < dimV$. ■

Part (a) of Proposition 2.4.9 asserts that when V is irreducible affine variety and the homogeneous polynomial $f \in K[x_1, \dots, x_n]$ does not vanish on V, then some component of $V \cap V(f)$ has dimension $dimV - 1$.

Conclusions

In this project, we have seen that if $V, W \subseteq K^n$ are affine varieties, then $V \cap W$ and $V \cup W$ are also affine varieties. In general, finite intersections and unions of affine varieties are also affine varieties. As many mathematicians say, the most important invariant of a linear subspace of affine space is its dimension and it involves some sophisticated concepts. In this paper, we understood that how to compute the dimension of affine varieties in different cases. As we have discussed earlier, the dimension of an affine variety V, which is the union of finite number of linear subspaces of the affine space K^n can be defined as the largest of the dimensions of the subspaces. In another case, the dimension of an affine variety $V \subseteq K^n$ is defined as the degree of the affine Hilbert Polynomial of the corresponding ideal $I = I(V) \subseteq K[x_1, \dots, x_n]$. But, one shortcoming in this case is that to find the dimension of an affine variety V, we need to know $I(V)$, which, in general, is difficult to compute. When the field K is algebraically closed, these difficulties go away as we discussed in the Dimension Theorem, which tells us how to compute the dimension in terms of any defining ideal.

Although defining the dimension of an affine variety as the degree of the affine Hilbert Polynomial is very useful for proving the properties of dimension, Hilbert Polynomials do not give the full story about dimension of affine varieties. In algebraic geometry, there are many ways or approaches to formulate the concept of dimension.

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