Coherently Driven Squeezed States
with
Thermal Noise

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Abstract

Applying the master equation for a single-mode light driven by a single-mode coherent light and damped by a single-mode thermal reservoir, we have constructed the Fokker-Planck equation for the Q-function. On solving the resulting Fokker-Planck equation employing the method of evaluating the *propagator method* discussed in Ref. [1], we have obtained the Q-function for a single-mode DSV driven by a single-mode coherent light and damped by a single-mode thermal reservoir. Finally, we have calculated using this Q-function the quadrature fluctuations, the mean photon number, the variance of the photon number and the photon number distribution. We have also carried out the same analysis for a two-mode DSV driven by a two-mode coherent light and damped by a two-mode thermal reservoir.

Contents

1. Introduction

In recent years, there has been a considerable interest to investigate the effects of damping on the statistical and squeezing properties of a single-mode displaced squeezed vacuum (DSV) [2-4]. Milburn and Walls [2] have derived the Q-function for DSV coupled to thermal reservoir and discussed the influence of damping on squeezing. They have also analyzed the attenuation of the oscillations in the photon number distribution of a DSV coupled to ordinary vacuum [3]. Moreover, Schramm and Grabert [4] presented a more general treatment of DSV coupled to thermal reservoir based on functional-integral method.

In addition, several researchers have analyzed the statistical and the squeezing properties of light [5-18], employing the Fokker-Planck equation [8-16] for the complex P-function [8-10], for the positive P-function [12-14] or for the Q-function [11, 15-17] and applying the Langevin equations [5, 6, 16, 18] for the annihilation and creation operators. Daniel and Fesseha [15] have investigated the statistical and squeezing properties of a Degenerate Parametric Oscillator (DPO) coupled to a vacuum reservoir by solving the Fokker-Planck equation for the Q-function employing the *propagator method* developed by Fesseha [1]. Misrak [16] and Alem [17] have also applied this method for analyzing the statistical and the squeezing properties of a Degenerate Parametric Oscillator (DPO) and a Nondegenerate Parametric Oscillator (NDPO) coupled to squeezed vacuum reservoirs.

In this research work, we seek to analyze the statistical and squeezing properties of a single-mode as well as a two-mode DSV driven by a coherent light and damped by a thermal reservoir employing the propagator method discussed in Ref. [1]. In this analysis we consider a cavity with a single-port mirror and the driving mode is treated classically, with its amplitude assumed to be constant.

This thesis is organized in two main parts. The first part deals with the statistical and squeezing properties of a single-mode DSV driven by a single-mode coherent light and damped by a single-mode thermal reservoir. The second part is devoted to the statistical and the squeezing properties of a two-mode DSV driven by a two-mode coherent light and damped by a two-mode thermal reservoir. In both parts, we first present a systematic derivation of the Fokker-Planck equation for the Q-function from the pertinent master equation. Then by transforming the Fokker-Planck equation into a Schrödinger-type equation, we determine the Q-function applying the *propagator method* discussed in Ref.[1]. Furthermore, we obtain the Q-function for a DSV by applying some properties of the coherent state and the squeeze operator. Using the resulting Q-function for a DSV driven by a coherent light and damped by a thermal light, we determine for each case the quadrature variances, the mean photon number, the variance of the photon number, and the photon number distribution. We also consider the special cases of these results at the initial time and at steady state. Finally, we present a brief discussion of the main results of this thesis.

2. A Single-mode Squeezed State with Thermal Noise

We seek to analyze the statistical and squeezing properties of a single-mode displaced squeezed vacuum (DSV) driven by a single-mode coherent light and damped by a single-mode thermal reservoir.

2.1. The Q-function

We wish here to calculate the Q-function for the system under consideration by solving the Fokker-Planck equation employing the *propagator method* [1]. With the driving mode treated classically, a single-mode light driven by a single-mode coherent light and damped by a single-mode thermal reservoir is describable in the interaction picture by the Hamiltonian

$$\hat{\mathcal{H}} = i\hbar\varepsilon(\hat{a}^\dagger - \hat{a}) + \hat{a}^\dagger\Gamma + \hat{a}\Gamma^\dagger,$$

where $\hat{a}$ is the annihilation operator for the cavity mode, ε (assumed to be real and constant) is a parameter proportional to the amplitude of the driving mode and Γ is the reservoir operator. Now using standard techniques [5-7, 16-18], the equation of evolution for the reduced density operator can be written as

$$\frac{\partial\hat{\rho}}{\partial t} = \frac{i}{\hbar}[\hat{\rho}, \hat{\mathcal{H}}'] + \frac{\gamma}{2}\overline{n}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) + \frac{\gamma}{2}(\overline{n}+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}), \tag{2.1}$$

in which

$$\hat{\mathcal{H}}' = i\hbar\varepsilon(\hat{a}^\dagger - \hat{a})$$

is the Hamiltonian describing the interaction between the driving and the cavity modes, γ is the cavity damping rate and $\overline{n}$ is the mean photon number of the thermal reservoir. In view of the fact that

$$[\hat{\rho}, \hat{\mathcal{H}}'] = i\hbar\varepsilon(\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a} - \hat{a}^\dagger\hat{\rho} + \hat{a}\hat{\rho}),$$

Eq.(2.1) can be rewritten in the form

$$\begin{aligned}\frac{\partial\hat{\rho}}{\partial t} &= -\varepsilon(\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a} - \hat{a}^\dagger\hat{\rho} + \hat{a}\hat{\rho}) + \frac{\gamma}{2}\overline{n}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) \\ &+\frac{\gamma}{2}(\overline{n}+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}).\end{aligned} \tag{2.2}$$

This represents the master equation for a single-mode light driven by a single-mode coherent light and damped by a single-mode thermal reservoir.

Next assuming that the density operator $\hat{\rho}$ has been put in the normal order, we wish to derive the Fokker-Planck equation for the Q-function. To this end, using the definition of the Q-function in terms of the density operator $\hat{\rho}$

$$Q(\alpha,t)=\frac{1}{\pi}\langle\alpha\mid\hat{\rho}\mid\alpha\rangle,$$

one can easily see that

$$\frac{\partial}{\partial t}Q(\alpha,t)=\frac{1}{\pi}\langle\alpha\mid\frac{\partial\hat{\rho}}{\partial t}\mid\alpha\rangle. \tag{2.3a}$$

Upon substituting (2.2) into (2.3a), one obtains

$$\begin{aligned}\frac{\partial}{\partial t}Q(\alpha,t)=&\frac{\varepsilon}{\pi}\left(\langle\alpha\mid\hat{a}^\dagger\hat{\rho}\mid\alpha\rangle-\langle\alpha\mid\hat{\rho}\hat{a}^\dagger\mid\alpha\rangle+\langle\alpha\mid\hat{\rho}\hat{a}\mid\alpha\rangle+\langle\alpha\mid\hat{a}\hat{\rho}\mid\alpha\rangle\right)\\&+\frac{\gamma\overline{n}}{2\pi}\left(2\langle\alpha\mid\hat{a}^\dagger\hat{\rho}\hat{a}\mid\alpha\rangle-\langle\alpha\mid\hat{a}\hat{a}^\dagger\hat{\rho}\mid\alpha\rangle-\langle\alpha\mid\hat{\rho}\hat{a}\hat{a}^\dagger\mid\alpha\rangle\right)\\&+\frac{\gamma}{2\pi}(\overline{n}+1)\left(2\langle\alpha\mid\hat{a}\hat{\rho}\hat{a}^\dagger\mid\alpha\rangle-\langle\alpha\mid\hat{a}^\dagger\hat{a}\hat{\rho}\mid\alpha\rangle-\langle\alpha\mid\hat{\rho}\hat{a}^\dagger\hat{a}\mid\alpha\rangle\right).\end{aligned} \tag{2.3b}$$

Now applying relations (A1-7) discussed in the Appendix for n=m=1, we find

$$\langle\alpha\mid\hat{a}^\dagger\hat{\rho}\mid\alpha\rangle=\alpha^*\langle\alpha\mid\hat{\rho}\mid\alpha\rangle, \tag{2.4a}$$

$$\langle\alpha\mid\hat{\rho}\hat{a}\mid\alpha\rangle=\alpha\langle\alpha\mid\hat{\rho}\mid\alpha\rangle, \tag{2.4b}$$

$$\langle\alpha\mid\hat{\rho}\hat{a}^\dagger\mid\alpha\rangle=\left(\frac{\partial}{\partial\alpha}+\alpha^*\right)\langle\alpha\mid\hat{\rho}\mid\alpha\rangle, \tag{2.4c}$$

$$\langle\alpha\mid\hat{a}\hat{\rho}\mid\alpha\rangle=\left(\frac{\partial}{\partial\alpha^*}+\alpha\right)\langle\alpha\mid\hat{\rho}\mid\alpha\rangle, \tag{2.4d}$$

$$\langle\alpha\mid\hat{a}^\dagger\hat{\rho}\hat{a}\mid\alpha\rangle=\alpha^*\alpha\langle\alpha\mid\hat{\rho}\mid\alpha\rangle, \tag{2.4e}$$

$$\langle\alpha\mid\hat{a}\hat{a}^\dagger\hat{\rho}\mid\alpha\rangle=\left(\frac{\partial}{\partial\alpha^*}\alpha^*+\alpha^*\alpha\right)\langle\alpha\mid\hat{\rho}\mid\alpha\rangle, \tag{2.4f}$$

$$\langle\alpha\mid\hat{\rho}\hat{a}\hat{a}^\dagger\mid\alpha\rangle=\left(\frac{\partial}{\partial\alpha}\alpha+\alpha^*\alpha\right)\langle\alpha\mid\hat{\rho}\mid\alpha\rangle, \tag{2.4g}$$

$$\langle\alpha \mid \hat{a}\hat{\rho}\hat{a}^{\dagger} \mid \alpha\rangle = \left(\frac{\partial^2}{\partial\alpha^*\partial\alpha} + \frac{\partial}{\partial\alpha^*}\alpha^* + \frac{\partial}{\partial\alpha}\alpha + \alpha^*\alpha - 1\right)\langle\alpha \mid \hat{\rho} \mid \alpha\rangle, \tag{2.4h}$$

$$\langle\alpha \mid \hat{a}^{\dagger}\hat{a}\hat{\rho} \mid \alpha\rangle = \left(\frac{\partial}{\partial\alpha^*}\alpha^* + \alpha^*\alpha - 1\right)\langle\alpha \mid \hat{\rho} \mid \alpha\rangle, \tag{2.4i}$$

$$\langle\alpha \mid \hat{\rho}\hat{a}^{\dagger}\hat{a} \mid \alpha\rangle = \left(\frac{\partial}{\partial\alpha}\alpha + \alpha^*\alpha - 1\right)\langle\alpha \mid \hat{\rho} \mid \alpha\rangle. \tag{2.4j}$$

On substituting relations (2.4) into (2.3b), the Fokker-Planck equation takes the form

$$\frac{\partial}{\partial t}Q(\alpha,t) = \left[\gamma(\overline{n}+1)\frac{\partial^2}{\partial\alpha^*\partial\alpha} + \frac{\gamma}{2}\frac{\partial}{\partial\alpha^*}\left(\alpha^* - \frac{2\varepsilon}{\gamma}\right) + \frac{\gamma}{2}\frac{\partial}{\partial\alpha}\left(\alpha - \frac{2\varepsilon}{\gamma}\right)\right]Q(\alpha,t).$$

Furthermore, introducing a new variable defined by

$$\beta = \alpha - \frac{2\varepsilon}{\gamma}, \tag{2.5}$$

this equation can be rewritten as

$$\frac{\partial}{\partial t}Q(\beta,t) = \left[\gamma(\overline{n}+1)\frac{\partial^2}{\partial\beta^*\partial\beta} + \frac{\gamma}{2}\frac{\partial}{\partial\beta^*}\beta^* + \frac{\gamma}{2}\frac{\partial}{\partial\beta}\beta\right]Q(\alpha,t). \tag{2.6}$$

In order to obtain the solution for this differential equation using the *propagator method*, it proves to be convenient to introduce Cartesian coordinates defined by

$$\beta = x + iy, \tag{2.7}$$

so that Eq.(2.6) takes the form

$$\frac{\partial}{\partial t}Q(x,y,t) = \left[\frac{\gamma}{4}(\overline{n}+1)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{\gamma}{2}\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right)\right]Q(x,y,t). \tag{2.8}$$

Upon replacing

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x, y, Q(x,y,t)\right) \text{ by } \left(\frac{i}{\hbar}\hat{p}_x, \frac{i}{\hbar}\hat{p}_y, \hat{x}, \hat{y}, \mid Q(t)\rangle\right),$$

Eq.(2.8) can be put in a Schrödinger type-equation

$$i\hbar\frac{\partial}{\partial t} \mid Q(t)\rangle = -\left[\frac{i\gamma}{4\hbar}(\overline{n}+1)\left(\hat{p}_x^2 + \hat{p}_y^2\right) + \frac{\gamma}{2}\left(\hat{p}_x\hat{x} + \hat{p}_y\hat{y}\right)\right] \mid Q(t)\rangle. \tag{2.9}$$

A formal solution for this equation is expressible as

$$\mid Q(t)\rangle = \hat{U} \mid Q(0)\rangle, \tag{2.10a}$$

where

$$\hat{U} = \exp\left(-\frac{i\hat{H}}{\hbar}t\right) \tag{2.10b}$$

and

$$\hat{H} = -\frac{i\gamma}{4\hbar}(\overline{n}+1)\left(\hat{p}_x^2 + \hat{p}_y^2\right) - \frac{\gamma}{2}\left(\hat{p}_x\hat{x} + \hat{p}_y\hat{y}\right). \tag{2.10c}$$

Next multiplying (2.10a) on the left by $\langle x, y \mid$, we have

$$Q(x,y,t) = \langle x, y \mid \hat{U} \mid Q(0)\rangle, \tag{2.11}$$

in which

$$Q(x,y,t) = \langle x, y \mid Q(t)\rangle.$$

Using the completeness relation for two-dimensional position eigenstates

$$I = \int_{-\infty}^{\infty} dx'dy' \mid x', y'\rangle\langle x', y' \mid,$$

one can express (2.11) as

$$Q(x,y,t) = \int_{-\infty}^{\infty} dx'dy'\, Q(x,y,t \mid x', y', 0)\, Q_0(x', y', 0), \tag{2.12a}$$

where

$$Q(x,y,t \mid x', y', 0) = \langle x, y \mid \hat{U} \mid x', y'\rangle \tag{2.12b}$$

is the Q-function propagator and

$$Q_0(x', y', 0), = \langle x', y' \mid Q(0)\rangle \tag{2.12c}$$

is the initial Q-function. According to Fesseha [1], the propagator associated with a quadratic Hamiltonian

$$\hat{H} = \sum_{j=1}^{N}\left(a_j\hat{p}_j^2 + b_j(t)\hat{p}_j\hat{x}_j + c_j(t)\hat{x}_j^2\right) \tag{2.13}$$

is expressible as

$$K(x_1, x_2, ..., x_N, t \mid x_1', x_2', ..., x_N', 0) = \prod_{j=1}^{N}\left[\frac{i}{2\pi\hbar}\frac{\partial^2 S_c}{\partial x_j'\partial x_j}\right]^{\frac{1}{2}}$$
$$\times \exp\left\{\frac{iS_c}{\hbar} - \xi\int_0^t dt' \sum_{j=1}^{N} b_j(t')\right\}, \tag{2.14}$$

where S_c stands for the classical action, ξ is a constant parametrizing operator ordering and is equal to $\frac{1}{2}$ for the antistandard form, a_j is a constant different from zero. Employing

(2.13) for $j = 1, 2$ in (2.10c) with $(\hat{x}_1, \hat{x}_2, \hat{p}_1, \hat{p}_2) = (\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y)$, we see that $b_x(t) = b_y(t) = -\frac{\gamma}{2}$. Thus the Q-function propagator corresponding to the Hamiltonian (2.10c) has the form

$$Q(x, y, t \mid x', y', 0) = \left[\frac{i}{2\pi\hbar}\frac{\partial^2 S_c}{\partial x'\partial x}\right]^{\frac{1}{2}} \left[\frac{i}{2\pi\hbar}\frac{\partial^2 S_c}{\partial y'\partial y}\right]^{\frac{1}{2}} \exp\left\{\frac{iS_c}{\hbar} + \frac{\gamma}{2}t\right\}. \tag{2.15}$$

We shall now proceed to calculate the classical action. To this end, we note that the Lagrangian corresponding to the quantum Hamiltonian (2.13) is given by

$$L = \sum_{j=1}^{N} \dot{x}_j p_j - H(x_1, x_2, ..., x_N, p_1, p_2, ..., p_N). \tag{2.16}$$

Applying the Hamilton's equations

$$\dot{x}_j = \frac{\partial H}{\partial p_j}, \tag{2.17}$$

along with the classical Hamiltonian

$$H = -\frac{i\gamma}{4\hbar}(\overline{n} + 1)\left(p_x^2 + p_y^2\right) - \frac{\gamma}{2}\left(p_x x + p_y y\right), \tag{2.18}$$

one can easily verify that

$$p_x = \frac{i\hbar}{\overline{n} + 1}\left(\frac{2}{\gamma}\dot{x} + x\right),$$
$$p_y = \frac{i\hbar}{\overline{n} + 1}\left(\frac{2}{\gamma}\dot{y} + y\right).$$

Hence the Lagrangian (2.16) in this case takes the form

$$L(\dot{x}, x, \dot{y}, y, t) = \frac{i\hbar}{\gamma(\overline{n} + 1)}\left[\left(\dot{x} + \frac{\gamma}{2}x\right)^2 + \left(\dot{y} + \frac{\gamma}{2}y\right)^2\right]. \tag{2.19}$$

Now employing the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} - \frac{\partial L}{\partial x_j} = 0 \tag{2.20}$$

along with(2.19), one readily gets

$$\ddot{x} - \frac{\gamma^2}{4}x = 0,$$
$$\ddot{y} - \frac{\gamma^2}{4}y = 0.$$

The solution of these equations satisfying the boundary conditions $x(0) = x'$, $y(0) = y'$, $x(T) = x''$ and $y(T) = y''$ can be written in the form

$$x(t) = Ae^{\frac{\gamma}{2}t} + Be^{-\frac{\gamma}{2}t}, \tag{2.21a}$$

$$y(t) = Ce^{\frac{\gamma}{2}t} + De^{-\frac{\gamma}{2}t}, \tag{2.21b}$$

in which

$$A = -\frac{\left(x' - x''e^{\frac{\gamma}{2}T}\right)}{e^{\gamma T} - 1}, \tag{2.22a}$$

$$C = -\frac{\left(y' - y''e^{\frac{\gamma}{2}T}\right)}{e^{\gamma T} - 1}. \tag{2.22b}$$

On substituting Eqs. (2.21) and their time derivatives into (2.19) and taking into account (2.22), there follows

$$L = \frac{i\hbar\gamma}{\overline{n}+1}e^{\gamma t}\left[\frac{x'^2 - 2x'x''e^{\frac{\gamma}{2}T} + x''^2e^{\gamma T} + y'^2 - 2y'y''e^{\frac{\gamma}{2}T} + y''^2e^{\gamma T}}{(1-e^{\gamma T})^2}\right],$$

so that the classical action

$$S_c = \int_0^T L(t)\,dt$$

takes the form

$$S_c = \frac{i\hbar}{\overline{n}+1}\left[\frac{x'^2 - 2x'xe^{\frac{\gamma}{2}t} + x^2e^{\gamma t} + y'^2 - 2y'ye^{\frac{\gamma}{2}t} + y^2e^{\gamma t}}{e^{\gamma t} - 1}\right], \tag{2.23}$$

where (x'', y'', T) has been replaced by (x, y, t). Now combination of (2.15) and (2.23) leads to

$$Q(x,y,t \mid x',y',0) = \left[\frac{1}{\pi(\overline{n}+1)(1-e^{-\gamma t})}\right] \times \exp\left[\frac{-x'^2e^{-\gamma t} + 2xe^{-\frac{\gamma}{2}t}x' - x^2 - y'^2e^{-\gamma t} + 2ye^{-\frac{\gamma}{2}t}y' - y^2}{(\overline{n}+1)(1-e^{-\gamma t})}\right]. \tag{2.24}$$

This represents the Q-function propagator for a single-mode light driven by a single-mode coherent light and damped by a single-mode thermal reservoir.

Next we proceed to calculate the initial Q-function (2.12c). Since the quantum system under consideration is initially in a DSV state, we need to evaluate the Q-function for a DSV. We recall that a single-mode DSV is defined as

$$\mid \lambda, r\rangle = e^{\lambda\hat{a}^\dagger - \lambda^*\hat{a}}e^{\frac{r}{2}\left(\hat{a}^2 - \hat{a}^{\dagger 2}\right)} \mid 0\rangle,$$

where the squeeze parameter r is taken for convenience to be real and positive, λ is a complex parameter, and $\hat{a}$ is the annihilation operator. The density operator for this system is then expressible as

$$\hat{\rho} = e^{\lambda\hat{a}^\dagger - \lambda^*\hat{a}}e^{\frac{r}{2}\left(\hat{a}^2 - \hat{a}^{\dagger 2}\right)} \mid 0\rangle\langle 0 \mid e^{-\frac{r}{2}\left(\hat{a}^2 - (\hat{a}^\dagger)^2\right)}e^{-(\lambda\hat{a}^\dagger - \lambda^*\hat{a})},$$

so that the Q-function takes the form

$$Q(\alpha^*, \alpha, r) = \frac{1}{\pi}\langle\alpha \mid e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}} e^{\frac{r}{2}\left(\hat{a}^2 - \hat{a}^{\dagger 2}\right)} \mid 0\rangle\langle 0 \mid e^{-\frac{r}{2}\left(\hat{a}^2 - \hat{a}^{\dagger 2}\right)} e^{\lambda^* \hat{a} - \lambda \hat{a}^\dagger} \mid \alpha\rangle, \tag{2.25}$$

Applying the Backer-Hausdorff identity

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]}, \tag{2.26}$$

which holds for $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$, we have

$$e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}} = e^{-\frac{1}{2}\lambda^* \lambda} e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}} \tag{2.27a}$$

and

$$e^{\lambda^* \hat{a} - \lambda \hat{a}^\dagger} = e^{-\frac{1}{2}\lambda^* \lambda} e^{-\lambda \hat{a}^\dagger} e^{\lambda^* \hat{a}}. \tag{2.27b}$$

Thus introducing (2.27) into (2.25), we obtain

$$Q(\alpha^*, \alpha, r) = \frac{1}{\pi} e^{-\lambda^* \lambda} \langle\alpha \mid e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}} \mid r\rangle\langle r \mid e^{-\lambda \hat{a}^\dagger} e^{\lambda^* \hat{a}} \mid \alpha\rangle,$$

in which

$$\mid r\rangle = e^{\frac{r}{2}\left(\hat{a}^2 - \hat{a}^{\dagger 2}\right)} \mid 0\rangle$$

is a single-mode squeezed vacuum state. On applying the property of the coherent state (A1) discussed in the Appendix, this expression can be rewritten as

$$Q(\alpha^*, \alpha, r) = \frac{1}{\pi} \exp\left[-\lambda^* \lambda + \lambda^* \alpha + \lambda \alpha^*\right] \langle\alpha \mid e^{-\lambda^* \hat{a}} \mid r\rangle\langle r \mid e^{-\lambda \hat{a}^\dagger} \mid \alpha\rangle. \tag{2.28}$$

Next we proceed to evaluate $\langle\alpha \mid e^{-\lambda^* \hat{a}} \mid r\rangle$. Inserting the completeness relation for single-mode coherent states

$$I = \int_{-\infty}^{\infty} \frac{d^2\beta}{\pi} \mid \beta\rangle\langle\beta \mid$$

into this expression, we see that

$$\langle\alpha \mid e^{-\lambda^* \hat{a}} \mid r\rangle = \int_{-\infty}^{\infty} \frac{d^2\beta}{\pi} e^{-\lambda^* \beta} \langle\alpha \mid \beta\rangle\langle\beta \mid r\rangle, \tag{2.29}$$

so that employing the relation

$$\langle\alpha \mid \beta\rangle = e^{\alpha^* \beta - \frac{1}{2}(\alpha^* \alpha + \beta^* \beta)}, \tag{2.30}$$

we have

$$\langle\alpha \mid e^{-\lambda^* \hat{a}} \mid r\rangle = e^{-\frac{1}{2}\alpha^* \alpha} \int_{-\infty}^{\infty} \frac{d^2\beta}{\pi} \langle\beta \mid r\rangle \exp\left[-\frac{1}{2}\beta^* \beta + (\alpha^* - \lambda^*)\beta\right]. \tag{2.31}$$

Now we proceed to determine the explicit form of the function $\langle\beta \mid r\rangle$. To this end, we note that

$$\langle\beta \mid r\rangle = \frac{1}{\beta^*}\langle\beta \mid \hat{a}^\dagger \mathbf{S}(r) \mid 0\rangle,$$

where

$$\mathbf{S}(r) = e^{\frac{r}{2}(\hat{a}^2 - (\hat{a}^\dagger)^2)}$$

is the single-mode squeeze operator. Inserting the identity operator

$$I = \mathbf{S}(r)\mathbf{S}^\dagger(r)$$

in the above expression, we get

$$\langle\beta \mid r\rangle = \frac{1}{\beta^*}\langle\beta \mid \mathbf{S}(r)\mathbf{S}^\dagger(r)\hat{a}^\dagger \mathbf{S}(r) \mid 0\rangle,$$

so that taking into account the relation [5, 6, 21]

$$\mathbf{S}^\dagger(r)\hat{a}^\dagger\mathbf{S}(r) = \hat{a}^\dagger\cosh(r) - \hat{a}\sinh(r),$$

we have

$$\beta^*\langle\beta \mid r\rangle = \cosh(r)\langle\beta \mid \mathbf{S}(r)\hat{a}^\dagger \mid 0\rangle.$$

Again inserting in this expression the identity operator

$$I = \mathbf{S}^\dagger(r)\mathbf{S}(r),$$

we note that

$$\beta^*\langle\beta \mid r\rangle = \cosh(r)\langle\beta \mid \mathbf{S}(r)\hat{a}^\dagger\mathbf{S}^\dagger(r)\mathbf{S}(r) \mid 0\rangle.$$

and using the relation

$$\mathbf{S}(r)\hat{a}^\dagger\mathbf{S}^\dagger(r) = \hat{a}^\dagger\cosh(r) + \hat{a}\sinh(r),$$

we see that

$$\beta^*\langle\beta \mid r\rangle = \cosh^2(r)\langle\beta \mid \hat{a}^\dagger \mid r\rangle + \cosh(r)\sinh(r)\langle\beta \mid \hat{a} \mid r\rangle.$$

In addition, applying (A1) and (A4) discussed in the Appendix, one can rewrite this as

$$\frac{\partial}{\partial\beta^*}\langle\beta \mid r\rangle = -\left(\frac{1}{2}\beta + \beta^*\tanh(r)\right)\langle\beta \mid r\rangle.$$

The solution for this equation is

$$\langle \beta \mid r \rangle = K(\beta, r) \exp\left\{-\frac{1}{2}\beta^*\beta - \frac{1}{2}\beta^{*2}\tanh(r)\right\}, \tag{2.32a}$$

where

$$K(\beta, r) = \langle \beta \mid r \rangle\Big|_{\beta^*=0}. \tag{2.32b}$$

But

$$\langle \beta \mid r \rangle\Big|_{\beta^*=0} = \langle 0 \mid e^{\beta^*\hat{a} - \beta\hat{a}^\dagger}|r\rangle\Big|_{\beta^*=0} = \langle 0 \mid e^{-\beta\hat{a}^\dagger}|r\rangle = \langle 0 \mid r \rangle. \tag{2.33}$$

Hence from (2.32b) and (2.33), we see that

$$K(\beta, r) = \langle 0 \mid r \rangle = K(r)$$

and (2.32a) can be rewritten in the form

$$\langle \beta \mid r \rangle = K(r) \exp\left\{-\frac{1}{2}\beta^*\beta + -\frac{1}{2}\beta^{*2}\tanh(r)\right\}.$$

Applying the normalization condition for $\langle \beta \mid r \rangle$, we get

$$\mid K(r) \mid^2 \int_{-\infty}^{\infty} \frac{d^2\beta}{\pi} \exp\left\{-\beta^*\beta - \frac{1}{2}(\beta^{*2} + \beta^2)\tanh(r)\right\} = 1,$$

so that upon carrying out the integration using the relation

$$\int_{-\infty}^{\infty} \frac{d^2\beta}{\pi} \exp\left[-A\beta^*\beta + B\beta^{*2} + C\beta^2 + b\beta + d\beta^*\right]$$
$$= \frac{1}{\sqrt{A^2 - 4BC}} \exp\left\{\frac{Abd + Bb^2 + Cd^2}{A^2 - 4BC}\right\}, \tag{2.34}$$

one easily obtains

$$K(r) = \sqrt{\mathrm{sech}(r)}.$$

Hence expression (2.32a) takes the form

$$\langle \beta \mid r \rangle = \sqrt{\mathrm{sech}(r)}\, \exp\left\{-\frac{1}{2}\beta^*\beta - \frac{1}{2}\beta^{*2}\tanh(r)\right\}. \tag{2.35}$$

On substituting (2.35) into (2.31), there follows

$$\langle \alpha \mid e^{-\lambda^*\hat{a}} \mid r \rangle = \sqrt{\mathrm{sech}(r)}\, e^{-\frac{1}{2}\alpha^*\alpha} \int_{-\infty}^{\infty} \frac{d^2\beta}{\pi} \exp\left\{-\beta^*\beta - \frac{1}{2}\beta^{*2}\tanh(r) + (\alpha^* - \lambda^*)\beta\right\}$$

and upon carrying out the integration with the help of (2.34), we readily get

$$\langle \alpha \mid e^{-\lambda^*\hat{a}} \mid r \rangle = \sqrt{\mathrm{sech}(r)}\, \exp\left\{-\frac{1}{2}\alpha^*\alpha - \frac{1}{2}(\alpha^* - \lambda^*)^2\tanh(r)\right\}. \tag{2.36}$$

Taking into account (2.36) and its complex conjugate along with (2.28), we obtain the well-known Q-function for a single-mode DSV

$$Q(\alpha^*, \alpha, r) = \frac{\mathrm{sech}(r)}{\pi} \exp\left\{ - \mid \alpha - \lambda \mid^2 - \frac{1}{2}\left[(\alpha^* - \lambda^*)^2 + (\alpha - \lambda)^2\right] \tanh(r) \right\}. \quad (2.37)$$

In order to calculate the Q-function (2.12a) using (2.24) and (2.37), we see from (2.5) and (2.7) that

$$\alpha' = \frac{2\varepsilon}{\gamma} + x' + iy',$$

so that taking into account this relation along with (2.37), one can easily find

$$Q(x', y', r) = \frac{\mathrm{sech}(r)}{\pi} \exp\left[- \frac{e^r}{\cosh(r)} \left(x'^2 - \left(\lambda^* + \lambda - \frac{4\varepsilon}{\gamma}\right) x' + \left(\frac{2\varepsilon}{\gamma} - \frac{1}{2}(\lambda^* + \lambda) \right)^2 \right) \right.$$
$$\left. - \frac{e^{-r}}{\cosh(r)} \left(y'^2 - i(\lambda^* - \lambda) - \left(\frac{\lambda^* - \lambda}{2}\right)^2 \right) \right].$$

Now substituting this and (2.24) into (2.12a), we obtain

$$Q(x, y, t) = \left[\frac{\mathrm{sech}(r)}{\pi^2 (\overline{n} + 1)(1 - e^{-\gamma t})} \right] \exp\left[\frac{-x^1 - y^2}{(\overline{n} + 1)(1 - e^{-\gamma t})} - \frac{e^r}{\cosh(r)} \left(\frac{2\varepsilon}{\gamma} - \left(\frac{\lambda^* + \lambda}{2}\right) \right)^2 \right.$$
$$\left. + \frac{e^{-r}}{\cosh(r)} \left(\frac{\lambda^* - \lambda}{2}\right)^2 \right]$$
$$\times \int_{-\infty}^{\infty} dx' \exp\left\{ - \left(\frac{1}{(\overline{n} + 1)(e^{\gamma t} - 1)} + \frac{e^r}{\cosh(r)} \right) x'^2 \right.$$
$$\left. + \left(\frac{2x\, e^{\frac{\gamma t}{2}}}{(\overline{n} + 1)(e^{\gamma t} - 1)} - \frac{e^r}{\cosh(r)} \left(\frac{4\varepsilon}{\gamma} - (\lambda^* + \lambda) \right) \right) x' \right\}$$
$$\times \int_{-\infty}^{\infty} dy' \exp\left\{ - \left(\frac{1}{(\overline{n} + 1)(e^{\gamma t} - 1)} + \frac{e^{-r}}{\cosh(r)} \right) y'^2 \right.$$
$$\left. + \left(\frac{2y\, e^{\frac{\gamma t}{2}}}{(\overline{n} + 1)(e^{\gamma t} - 1)} + \frac{ie^{-r}}{\cosh(r)}(\lambda^* - \lambda) \right) y' \right\},$$

so that on carrying out the integration and taking into account (2.7) along with (2.5), the Q-function is found to be

$$Q(\alpha^*, \alpha, t) = \frac{G(t)}{\pi} \exp\left[- A(t)\alpha^*\alpha + B(t)(\alpha^{*2} + \alpha^2) + C(t)\alpha + C^*(t)\alpha^* \right], \quad (2.38)$$

where

$$A(t) = \frac{(\overline{n} + 1)(1 - e^{-\gamma t})\,\mathrm{sech}^2 r + e^{-\gamma t}}{(\overline{n} + 1)^2 (1 - e^{-\gamma t})^2 \mathrm{sech}^2 r + 2(\overline{n} + 1)e^{-\gamma t}(1 - e^{-\gamma t}) + e^{-2\gamma t}}, \quad (2.39a)$$

$$B(t) = \frac{-\frac{1}{2}e^{-\gamma t}\tanh r}{(\overline{n}+1)^2\left(1-e^{-\gamma t}\right)^2\text{sech}^2 r + 2(\overline{n}+1)e^{-\gamma t}\left(1-e^{-\gamma t}\right)+e^{-2\gamma t}}, \tag{2.39b}$$

$$C(t) = \frac{2\varepsilon}{\gamma}(A-2B)\left(1-e^{\frac{-\gamma t}{2}}\right) + (A\lambda^* - 2B\lambda)e^{\frac{-\gamma t}{2}}, \tag{2.39c}$$

$$G(t) = \sqrt{A^2-4B^2}\exp\left[\frac{-ACC^* - B\left(C^2+C^{*2}\right)}{A^2-4B^2}\right]. \tag{2.39d}$$

Expression (2.38) represents the Q-function for a coherently driven single-mode DSV damped by a single-mode thermal noise.

2.2. Quadrature Fluctuations

The squeezing properties of a single-mode radiation is described by two quadrature operators defined by

$$\hat{a}_1 = \hat{a}^\dagger + \hat{a}, \tag{2.40a}$$

$$\hat{a}_2 = i(\hat{a}^\dagger - \hat{a}). \tag{2.40b}$$

These two operators $\hat{a}_1$ and $\hat{a}_2$ obey the commutation relation

$$[\hat{a}_1, \hat{a}_2] = 2i$$

and the variances of $\hat{a}_1$ and $\hat{a}_2$ are expressible as

$$\Delta\hat{a}_1^2 = \langle\hat{a}_1, \hat{a}_1\rangle = \langle\hat{a}_1^2\rangle - \langle\hat{a}_1\rangle^2, \tag{2.41a}$$

$$\Delta\hat{a}_2^2 = \langle\hat{a}_2, \hat{a}_2\rangle = \langle\hat{a}_2^2\rangle - \langle\hat{a}_2\rangle^2, \tag{2.41b}$$

where

$$\langle\hat{a}_1\rangle = \langle\hat{a}^\dagger\rangle + \langle\hat{a}\rangle, \tag{2.41c}$$

$$\langle\hat{a}_2\rangle = i\left(\langle\hat{a}^\dagger\rangle - \langle\hat{a}\rangle\right), \tag{2.41d}$$

$$\langle\hat{a}_1^2\rangle = -1 + 2\langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^{\dagger 2}\rangle + \langle\hat{a}^2\rangle, \tag{2.41e}$$

$$\langle\hat{a}_2^2\rangle = -1 + 2\langle\hat{a}\hat{a}^\dagger\rangle - \langle\hat{a}^{\dagger 2}\rangle - \langle\hat{a}^2\rangle. \tag{2.41f}$$

We recall that the mean value of an operator $\hat{A}(\hat{a}^\dagger, \hat{a}, t)$ is expressible in terms of the Q-function as

$$\langle\hat{A}\rangle = \int_{-\infty}^{\infty} d^2\alpha\, Q(\alpha^*, \alpha, t)A_a(\alpha^*, \alpha, t), \tag{2.42}$$

in which $A_a(\alpha^*, \alpha, t)$ is the c-number equivalent of the operator $\hat{A}(\hat{a}^\dagger, \hat{a}, t)$ for the antinormal ordering. Applying (2.42), one can rewrite relations (2.41) in terms of the Q-function (2.38) in the form

$$\langle \hat{a}_1 \rangle = \frac{G}{\pi} \int_{-\infty}^{\infty} d^2\alpha (\alpha^* + \alpha) \exp\left[-A\alpha^*\alpha + B\left(\alpha^{*2} + \alpha^2\right) + C\alpha + C^*\alpha^*\right], \tag{2.43a}$$

$$\langle \hat{a}_2 \rangle = \frac{iG}{\pi} \int_{-\infty}^{\infty} d^2\alpha (\alpha^* - \alpha) \exp\left[-A\alpha^*\alpha + B\left(\alpha^{*2} + \alpha^2\right) + C\alpha + C^*\alpha^*\right], \tag{2.43b}$$

$$\langle \hat{a}_1^2 \rangle = -1 + \frac{G}{\pi} \int_{-\infty}^{\infty} d^2\alpha (\alpha^* + \alpha)^2 \exp\left[-A\alpha^*\alpha + B\left(\alpha^{*2} + \alpha^2\right) + C\alpha + C^*\alpha^*\right], \tag{2.43c}$$

$$\langle \hat{a}_2^2 \rangle = -1 - \frac{G}{\pi} \int_{-\infty}^{\infty} d^2\alpha (\alpha^* - \alpha)^2 \exp\left[-A\alpha^*\alpha + B\left(\alpha^{*2} + \alpha^2\right) + C\alpha + C^*\alpha^*\right]. \tag{2.43d}$$

Employing the relation

$$\alpha \exp[a\alpha + b\beta] = \frac{\partial}{\partial a} \exp[a\alpha + b\beta], \tag{2.44}$$

we can express (2.43) in the form

$$\langle \hat{a}_1 \rangle = \frac{G}{\pi} \left(\frac{\partial}{\partial c^*} + \frac{\partial}{\partial c}\right) \int_{-\infty}^{\infty} d^2\alpha \exp\left[-A\alpha^*\alpha + B\left(\alpha^{*2} + \alpha^2\right) + C\alpha + C^*\alpha^*\right], \tag{2.45a}$$

$$\langle \hat{a}_2 \rangle = \frac{iG}{\pi} \left(\frac{\partial}{\partial c^*} - \frac{\partial}{\partial c}\right) \int_{-\infty}^{\infty} d^2\alpha \exp\left[-A\alpha^*\alpha + B\left(\alpha^{*2} + \alpha^2\right) + C\alpha + C^*\alpha^*\right], \tag{2.45b}$$

$$\langle \hat{a}_1^2 \rangle = -1 + \frac{G}{\pi} \left(\frac{\partial}{\partial c^*} + \frac{\partial}{\partial c}\right)^2 \int_{-\infty}^{\infty} d^2\alpha \exp\left[-A\alpha^*\alpha + B\left(\alpha^{*2} + \alpha^2\right) + C\alpha + C^*\alpha^*\right], \tag{2.45c}$$

$$\langle \hat{a}_2^2 \rangle = -1 + \frac{G}{\pi} \left(\frac{\partial}{\partial c^*} - \frac{\partial}{\partial c}\right)^2 \int_{-\infty}^{\infty} d^2\alpha \exp\left[-A\alpha^*\alpha + B\left(\alpha^{*2} + \alpha^2\right) + C\alpha + C^*\alpha^*\right]. \tag{2.45d}$$

On carrying out the integration employing relation (2.34) and then performing the differentiation, one can easily obtain

$$\langle \hat{a}_1 \rangle = \frac{C^* + C}{A - 2B}, \tag{2.46a}$$

$$\langle \hat{a}_2 \rangle = \frac{-i(C^* - C)}{A + 2B}, \tag{2.46b}$$

$$\langle \hat{a}_1^2 \rangle = -1 + \langle \hat{a}_1 \rangle^2 + \frac{2}{A - 2B}, \tag{2.46c}$$

$$\langle \hat{a}_2^2 \rangle = -1 + \langle \hat{a}_2 \rangle^2 + \frac{2}{A + 2B}, \tag{2.46d}$$

from which follows

$$\Delta \hat{a}_1^2 = -1 + \frac{2}{A - 2B},$$

$$\Delta \hat{a}_2^2 = -1 + \frac{2}{A + 2B},$$

so that substitution of the explicit values of A(t) and B(t) from (2.39) leads to

$$\Delta \hat{a}_1^2 = (1 + 2\overline{n}) \left(1 - e^{-\gamma t}\right) + e^{-\gamma t} \left(\cosh 2r - \sinh 2r\right), \tag{2.47a}$$

$$\Delta \hat{a}_2^2 = (1 + 2\overline{n}) \left(1 - e^{-\gamma t}\right) + e^{-\gamma t} \left(\cosh 2r + \sinh 2r\right). \tag{2.47b}$$

This result is in complete agreement with the result of Milburn and Walls [2], for a squeezed vacuum coupled to a thermal reservoir.

A single-mode radiation is said to be in a squeezed state if either

$$\Delta \hat{a}_1 \leq 1 \leq \Delta \hat{a}_2$$

or

$$\Delta \hat{a}_1 \geq 1 \geq \Delta \hat{a}_2$$

without violating the uncertainty principle [5-13, 16, 17]

$$\Delta \hat{a}_1 \Delta \hat{a}_2 \geq 1.$$

A light mode for which

$$\Delta \hat{a}_1 \Delta \hat{a}_2 = 1$$

is said to be in a minimum uncertainty state. As we can see from (2.47)

$$\Delta \hat{a}_2 \geq 1, \text{ for } t \geq 0$$

and

$$\Delta \hat{a}_1 \begin{cases} \leq 1, & \text{for } 0 \leq t \leq \frac{1}{\gamma} \ln \left[1 + \frac{1}{2\overline{n}} \left(1 - e^{-2r}\right)\right] \\ \geq 1, & \text{otherwise} \end{cases}.$$

Therefore, the cavity mode under consideration is in a squeezed state for

$$0 \leq t \leq \frac{1}{\gamma} \ln \left[1 + \frac{1}{2\overline{n}} \left(1 - e^{-2r}\right)\right].$$

We will consider now certain special cases of the quadrature variances described by (2.47.)

A. We see that for $t = 0$

$$\Delta\hat{a}_1^2 = e^{-2r},$$

$$\Delta\hat{a}_2^2 = e^{2r},$$

which are the known results for the quadrature fluctuations of a single-mode DSV.

B. At steady state, we have

$$\Delta\hat{a}_1^2 = \Delta\hat{a}_2^2 = 2\bar{n} + 1,$$

which represent the quadrature variances for the superposition of a single-mode coherent and thermal light.

2.3. The Mean and Variance of the Photon Number

Here we wish to determine the mean and variance of the photon number.

a. The Mean Photon Number

The mean photon number is defined in terms of the annihilation and creation operators as

$$\langle\hat{n}\rangle = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle. \tag{2.48}$$

Now employing (2.42), one can express the mean photon number as

$$\langle\hat{n}\rangle = -1 + \frac{G}{\pi}\int_{-\infty}^{\infty} d^2\alpha(\alpha^*\alpha) \exp\left[-A\alpha^*\alpha + B\left(\alpha^{*2} + \alpha^2\right) + C\alpha + C^*\alpha^*\right],$$

and using (2.44) this equation can be rewritten in the form

$$\langle\hat{n}\rangle = -1 + \frac{G}{\pi}\frac{\partial^2}{\partial C\partial C^*}\int_{-\infty}^{\infty} d^2\alpha \exp\left[-A\alpha^*\alpha + B\left(\alpha^{*2} + \alpha^2\right) + C\alpha + C^*\alpha^*\right].$$

Upon carrying out the integration with the aid of (2.34) and then performing the differentiation, one can easily obtain

$$\langle\hat{n}\rangle = \left|\frac{AC^* + 2BC}{A^2 - 4B^2}\right|^2 + \frac{A}{A^2 - 4B^2} - 1. \tag{2.49a}$$

Taking into account (2.39) this expression can be put in the form

$$\langle\hat{n}\rangle = \frac{4\varepsilon^2}{\gamma^2}\left(1-e^{-\frac{\gamma t}{2}}\right)^2 + \frac{2\varepsilon}{\gamma}(\lambda^* + \lambda)e^{-\frac{\gamma t}{2}}\left(1-e^{-\frac{\gamma t}{2}}\right)$$
$$+\overline{n}\left(1-e^{-\gamma t}\right) + \left(|\lambda|^2 + \sinh^2 r\right)e^{-\gamma t}. \tag{2.49b}$$

b. The Variance of the Photon Number

The variance of the photon number is also defined by

$$\Delta\hat{n}^2 = \langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2$$

or, one can write this as

$$\Delta\hat{n}^2 = \langle\hat{a}^2(t)\hat{a}^{\dagger 2}(t)\rangle - \langle\hat{n}\rangle^2 - 3\langle\hat{n}\rangle - 2. \tag{2.50}$$

On applying (2.42), we have

$$\langle\hat{a}^2(t)\hat{a}^{\dagger 2}(t)\rangle = \frac{G}{\pi}\int_{-\infty}^{\infty} d^2\alpha(\alpha^*\alpha)^2 \exp\left[-A\alpha^*\alpha + B\left(\alpha^{*2}+\alpha^2\right) + C\alpha + C^*\alpha^*\right],$$

so that in view of (2.34) and (2.44), this takes the form

$$\langle\hat{a}^2(t)\hat{a}^{\dagger 2}(t)\rangle = \frac{G}{\sqrt{A^2-4B^2}}\frac{\partial^4}{\partial C^2\partial C^{*2}}\exp\left[\frac{AC^*C + B\left(C^{*2}+C^2\right)}{A^2-4B^2}\right].$$

On differentiating and taking into account (2.49a), we get

$$\langle\hat{a}^2(t)\hat{a}^{\dagger 2}(t)\rangle = 2 + \langle\hat{n}\rangle^2 + 2\langle\hat{n}\rangle\left(1+\frac{1}{A-2B}\right) - \left(1-\frac{1}{A-2B}\right)^2$$
$$+\frac{2B}{A^2-4B^2}\left(\frac{C-C^*}{A+2B}\right)^2 + 2\left(\frac{2B}{A^2-4B^2}\right)^2.$$

Consequently, the variance of the photon number becomes

$$\Delta\hat{n}^2 = \langle\hat{n}\rangle\left(\frac{2}{A-2B}-1\right) - \frac{1}{4}\left(\frac{2}{A-2B}-2\right)^2$$
$$+\frac{2B(t)}{A^2-4B^2}\left(\frac{C-C^*}{A+2B}\right)^2 + 2\left(\frac{2B}{A^2-4B^2}\right)^2, \tag{2.51a}$$

the explicit form of which is

$$\Delta\hat{n}^2 = \langle\hat{n}\rangle\left[1 - e^{-\gamma t}\left(1 - e^{-2r}\right) + 2\overline{n}\left(1 - e^{-\gamma t}\right)\right]$$
$$+2e^{-2\gamma t}\cosh^2(r)\sinh^2(r) - (\lambda^* - \lambda)^2 e^{-2\gamma t}\cosh(r)\sinh(r)$$
$$-\left(\frac{1}{2}e^{-\gamma t}\left(1 - e^{-2r}\right) - \overline{n}\left(1 - e^{-\gamma t}\right)\right)^2. \qquad (2.51b)$$

Now we consider some special cases of Eq.(2.51b).

A. We note that at $t = 0$

$$\langle\hat{n}\rangle = |\lambda|^2 + \sinh^2(r) \qquad (2.52)$$

and hence (2.51b) reduces to

$$\Delta\hat{n}^2 = \langle\hat{n}\rangle e^{-2r} + 2\cosh^2 r\sinh^2 r - (\lambda^* - \lambda)^2\cosh(r)\sinh(r) - \frac{1}{4}\left(1 - e^{-2r}\right)^2. \qquad (2.53a)$$

Writing λ in polar coordinates as

$$\lambda = |\lambda|e^{i\phi},$$

we see that

$$\lambda^{*2} + \lambda^2 = 2|\lambda|^2\cos 2\phi$$

or

$$\lambda^{*2} + \lambda^2 = 2\cos 2\phi\left(\langle\hat{n}\rangle - \sinh^2(r)\right).$$

Hence (2.53a) can be rewritten in the form

$$\Delta\hat{n}^2 = \langle\hat{n}\rangle\left[\cosh(2r) - \cos(2\phi)\sinh(2r)\right] + \sinh^2 r\left[\cos(2\phi)\sinh(2r) + 1\right] \qquad (2.53b)$$

This is the variance of the photon number for a single-mode DSV.

If $\lambda = 0$ in (2.53b), we see that

$$\Delta\hat{n}^2 = (1 + \cosh(2r))\sinh^2 r = 2\left(\langle\hat{n}\rangle^2 + \langle\hat{n}\rangle\right).$$

This shows that the photon statistics for a single-mode squeezed vacuum (SV) is super-Poissonian.

Upon setting $r = 0$ in (2.53b), we obtain the variance of the photon number for a single-mode coherent light

$$\Delta\hat{n}^2 = \langle\hat{n}\rangle.$$

B. We note that at steady state

$$\langle \hat{n} \rangle = \overline{n} + \frac{4\varepsilon^2}{\gamma^2} \tag{2.54a}$$

and hence (2.51b) takes the form

$$\Delta \hat{n}^2 = \langle \hat{n} \rangle + \overline{n}^2 + \frac{8\varepsilon^2}{\gamma^2}\overline{n}. \tag{2.54b}$$

This shows that the photon statistics for the superposition of a single-mode coherent and thermal light is super-Poissonian.

By setting $\overline{n} = 0$ in (2.54b), we get the result which holds for coherent light

$$\Delta \hat{n}^2 = \langle \hat{n} \rangle. \tag{2.55a}$$

Moreover, by setting $\varepsilon = 0$, we get the variance of the photon number for thermal light

$$\Delta \hat{n}^2 = \overline{n} + \overline{n}^2. \tag{2.55b}$$

2.4. The Photon Number Distribution

The photon number distribution for a single-mode light is expressible as [11, 15, 16]

$$P(n,t) = \frac{\pi}{n!}\frac{\partial^{2n}}{\partial \alpha^{*n} \partial \alpha^n}\left(Q(\alpha^*, \alpha, t)e^{\alpha^* \alpha}\right)\Big|_{\alpha^* = \alpha = 0}. \tag{2.56a}$$

Taking into account (2.38) along with (2.56a), we obtain

$$P(n,t) = \frac{G}{n!}\frac{\partial^{2n}}{\partial \alpha^{*n} \partial \alpha^n}\ \exp\left(D\alpha^*\alpha + B\left(\alpha^{*2} + \alpha^2\right) + C^*\alpha^* + C\alpha\right]\Big|_{\alpha^* = \alpha = 0}, \tag{2.56b}$$

where

$$D(t) = 1 - A(t).$$

By expanding the exponential function in power series, we have

$$P(n,t) = \frac{G(t)}{n!}\sum_{s,m,l,k,t}\frac{D^s B^{m+l} C^k (C^*)^t}{s!m!l!k!t!}\frac{\partial^n}{\partial \alpha^{*n}}\alpha^{*^{s+2m+t}}\frac{\partial^n}{\partial \alpha^n}\alpha^{s+2l+k}\Big|_{\alpha^* = \alpha = 0}. \tag{2.56c}$$

On performing the differentiation applying the relation

$$\frac{\partial^n}{\partial x^n}x^m = \frac{m!}{(m-n)!}x^{m-n}, \tag{2.57}$$

expression (2.56c) takes the form

$$P(n,t) = \frac{G(t)}{n!} \sum_{s,m,l,k,t} \frac{D^s B^{m+l} C^k (C^*)^t}{s!m!l!k!t!} \frac{(s+2m+t)!}{(s+2m+t-n)!}$$

$$+ \frac{(s+2l+k)!}{(s+2l+k-n)!} \alpha^{*^{s+2m+t}} \alpha^{s+2l+k} \Big|_{\alpha^*=\alpha=0},$$

so that on using the condition $\alpha^* = \alpha = 0$, this expression becomes

$$P(n,t) = \frac{G(t)}{n!} \sum_{s,m,l,k,t} \frac{D^s B^{m+l} C^k (C^*)^t}{s!m!l!k!t!} \frac{(s+2m+t)!}{(s+2m+t-n)!} \frac{(s+2l+k)!}{(s+2l+k-n)!} \delta_{s+2m+t,n} \delta_{s+2l+k,n}. \tag{2.58a}$$

We note that

$$s = n - k - 2l,$$

$$t = 2l - 2m + k.$$

In view of these results, we see that

$$P(n,t) = n!G(t) \sum_{k=2m-2l}^{n-2l} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{D^{n-k-2l} B^{m+l} C^k (C^*)^{2l-2m+k}}{(n-k-2l)!(2l-2m+k)!l!m!k!}. \tag{2.58b}$$

This represents the photon number distribution for a single-mode DSV driven by a single-mode coherent light and damped by a single-mode thermal reservoir.

Next we consider some special cases of interest for the result described by (2.58b). A. At $t = 0$, we see from (2.39) that

$$A(0) = 1, \tag{2.59a}$$

$$B(0) = -\frac{1}{2} \tanh(r), \tag{2.59b}$$

$$C(0) = \lambda^* + \lambda \tanh(r), \tag{2.59c}$$

$$G(0) = \operatorname{sech} r \exp\left[-\lambda^* \lambda - \frac{1}{2} \left(\lambda^{*^2} + \lambda^2 \right) \tanh(r) \right]. \tag{2.59d}$$

Hence using the result $D(0) = 0$, we have

$$P(n,0) = n!G(t) \sum_{k=2m-2l}^{n-2l} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{B^{m+l} C^k (C^*)^{2l-2m+k}}{(n-k-2l)!(2l-2m+k)!l!m!k!} \delta_{k+2l,n}$$

and on account of the fact that $k = n - 2l$, we note

$$P(n,0) = n!G(t) \sum_{m=0}^{[n]} \sum_{l=0}^{[n]} \frac{B^{m+l} C^{n-2l} (C^*)^{n-2m}}{(n-2m)!l!m!(n-2l)!},$$

in which $[n] = \frac{n}{2}$ for even n and $[n] = \frac{n-1}{2}$ for odd n. In view of (2.59), this expression takes the form

$$P(n,0) = \frac{\operatorname{sech}(r)}{n!} \exp\left[-\lambda^*\lambda - \frac{1}{2}\left(\lambda^{*^2} + \lambda^2\right)\tanh(r)\right]\left(\frac{\tanh(r)}{2}\right)^n$$

$$\times \sum_{m=o}^{\frac{n}{2}} \frac{(-1)^m n! \left(\frac{\lambda^*+\lambda\tanh(r)}{\sqrt{\frac{1}{2}\tanh(r)}}\right)^{n-2m}}{m!(n-2m)!} \sum_{l=o}^{\frac{n}{2}} \frac{(-1)^l n! \left(\frac{\lambda+\lambda^*\tanh(r)}{\sqrt{\frac{1}{2}\tanh(r)}}\right)^{n-2l}}{l!(n-2l)!},$$

so that on applying the Hermite polynomial defined by

$$H_n(x) = \sum_{m=o}^{\frac{n}{2}} \frac{(-1)^m n!}{m!(n-2m)!}(2x)^{n-2m},$$

we obtain [5,25]

$$P(n,0) = \frac{\tanh^n(r)}{n!2^n\cosh(r)} \exp\left[-\lambda^*\lambda - \frac{1}{2}\left(\lambda^{*^2} + \lambda^2\right)\tanh(r)\right]\left|H_n\left(\frac{\lambda+\lambda^*\tanh(r)}{\sqrt{2\tanh(r)}}\right)\right|^2,$$

which is the well-known photon number distribution for a single-mode DSV.

B. We note that at steady state (2.39) reduces to

$$A(\infty) = 1/(\overline{n}+1), \tag{2.60a}$$

$$B(\infty) = 0, \tag{2.60b}$$

$$C(\infty) = 2\varepsilon/\gamma(\overline{n}+1), \tag{2.60c}$$

$$G(\infty) = [1/(\overline{n}+1)]\exp\left\{-4\varepsilon^2/\gamma^2(\overline{n}+1)\right\}, \tag{2.60d}$$

so that (2.58b) takes the form

$$P(n) = n!G \sum_{k=2m-2l}^{n-2l} \sum_{m=0}^{\infty}\sum_{l=0}^{\infty} \frac{D^{n-k-2l}C^k\left(C^*\right)^{2l-2m+k}}{(n-k-2l)!(2l-2m+k)!l!m!k!}\delta_{m,-l}$$

and then taking into account $m = -l = 0$, we see that

$$P(n) = G\sum_{k=0}^{n} \frac{D^{n-k}C'^{2k}}{(n-k)!(k!)^2}n!. \tag{2.61}$$

Applying the Laguere polynomial defined by

$$\mathcal{L}_n(x) = \sum_{k=0}^{n} \frac{(-1)^k x^k}{(n-k)!(k!)^2}n!, \tag{2.62}$$

we can rewrite (2.61) as

$$P(n) = D^n G \mathcal{L}_n \left(\frac{-C^2}{D} \right)$$

and taking into account (2.60), we get

$$P(n) = \frac{\overline{n}^n}{(\overline{n}+1)^{n+1}} \exp\left[\frac{-4\varepsilon^2}{\gamma^2(\overline{n}+1)} \right] \mathcal{L}_n \left(\frac{-4\varepsilon^2}{\gamma^2 \overline{n}(\overline{n}+1)} \right). \tag{2.63}$$

This expression represents the photon number distribution for the superposition of a single-mode coherent and thermal light. This result has been established by Mollow and Glauber [20] and Lachs [26].

If $\overline{n} = 0$ in (2.60), then we see that $A(\infty) = 1$ and $D(\infty) = 0$. Hence (2.61) takes the form

$$P(n) = G \sum_{k=0}^{n} \frac{C^{2k}}{(n-k)!(k!)^2} n! \delta_{n,k}$$

and in view of the result $n = k$, there follows

$$P(n) = \frac{G}{n!} C^{2n},$$

so that on taking into account (2.60) for $\overline{n} = 0$, we get

$$P(n) = \frac{(4\varepsilon^2/\gamma^2)^n}{n!} \exp\left[\frac{-4\varepsilon^2}{\gamma^2} \right]. \tag{2.64}$$

This distribution function referred to as the Poissonian distribution, represents the well-known photon number distribution for a single-mode coherent light.

Furthermore, setting $\varepsilon = 0$ in (2.60), we have $C(\infty) = 0$. Then (2.61) becomes

$$P(n) = G \sum_{k=0}^{n} \frac{n! D^{n-k}}{(n-k)!(k!)^2} \delta_k,$$

so that using the result $k = 0$, we obtain

$$P(n) = G D^n.$$

Finally, on taking into account (2.60) for $\varepsilon = 0$, we have

$$P(n) = \frac{\overline{n}^n}{(\overline{n}+1)^{n+1}}, \tag{2.65}$$

which is the well-known photon number distribution for a single-mode thermal light.

3. A Two-mode Squeezed State with Thermal Noise

We wish here to analyze the statistical and squeezing properties of a two-mode DSV driven by a two-mode coherent light and damped by a two-mode thermal reservoir.

3.1. The Q-function

With the driving mode treated classically, a two-mode light driven by a two-mode coherent light and damped by a two-mode thermal reservoir is describable in the interaction picture by the Hamiltonian

$$\hat{\mathcal{H}} = i\hbar\varepsilon_a(\hat{a}^\dagger - \hat{a}) + i\hbar\varepsilon_b(\hat{b}^\dagger - \hat{b}) + \hat{a}^\dagger\Gamma_a + \hat{a}\Gamma_a^\dagger + \hat{b}^\dagger\Gamma_b + \hat{b}\Gamma_b^\dagger,$$

where $\hat{a}(\hat{b})$ is the annihilation operator for the cavity mode a (mode b), $\varepsilon_a(\varepsilon_b)$ (assumed to be real and constant) is a parameter proportional to the amplitude of the driving mode a (mode b), Γ_a and Γ_b are the reservoir operators. It can be verified applying the standard techniques [5-7, 16-18] that the equation of evolution for the density operator of this system has the form

$$\begin{aligned}\frac{\partial\hat{\rho}}{\partial t} = {} & \frac{i}{\hbar}[\hat{\rho}, \hat{\mathcal{H}}'] + \frac{\gamma_a}{2}\overline{n}_a(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) \\ & + \frac{\gamma_a}{2}(\overline{n}_a + 1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) \\ & + \frac{\gamma_b}{2}\overline{n}_b(2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^\dagger) \\ & + \frac{\gamma_b}{2}(\overline{n}_b + 1)(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}),\end{aligned} \tag{3.1}$$

where

$$\hat{\mathcal{H}}' = i\hbar\varepsilon_a(\hat{a}^\dagger - \hat{a}) + i\hbar\varepsilon_b(\hat{b}^\dagger - \hat{b})$$

is the Hamiltonian describing the interaction between the driving and the cavity modes, (γ_a, γ_b) are the cavity damping rates and $(\overline{n}_a, \overline{n}_b)$ are the mean photon numbers of the two-mode thermal reservoir. Taking into account the commutation relation

$$[\hat{\rho}, \hat{\mathcal{H}}'] = i\hbar\varepsilon_a(\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a} - \hat{a}^\dagger\hat{\rho} + \hat{a}\hat{\rho}) + i\hbar\varepsilon_b(\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b} - \hat{b}^\dagger\hat{\rho} + \hat{b}\hat{\rho}),$$

Eqs.(3.1) can be rewritten as

$$\frac{\partial\hat{\rho}}{\partial t} = -\varepsilon_a(\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a} - \hat{a}^\dagger\hat{\rho} + \hat{a}\hat{\rho}) - \varepsilon_b(\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b} - \hat{b}^\dagger\hat{\rho} + \hat{b}\hat{\rho})$$
$$+\frac{\gamma_a}{2}\overline{n}_a(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) + \frac{\gamma_a}{2}(\overline{n}_a + 1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a})$$
$$+\frac{\gamma_b}{2}\overline{n}_b(2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{a}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^\dagger) + \frac{\gamma_b}{2}(\overline{n}_b + 1)(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}). \qquad (3.2)$$

This is the master equation for a two-mode light driven by a two-mode coherent light and damped by a two-mode thermal reservoir.

We now proceed to obtain the Fokker-Planck equation for the Q-function. To this end, we assume that the density operator $\hat{\rho}(\hat{a}^\dagger, \hat{a}, \hat{b}^\dagger, \hat{b}, t)$ has been put in the normal order. Taking into account the definition of the Q-function in terms of the density operator $\hat{\rho}(\hat{a}^\dagger, \hat{a}, \hat{b}^\dagger, \hat{b}, t)$ for a two-mode light

$$Q(\alpha, \beta, t) = \frac{1}{\pi}\langle\alpha, \beta \mid \hat{\rho} \mid \beta, \alpha\rangle,$$

along with (3.2), the time evolution of the Q-function is expressible as

$$\frac{\partial}{\partial t}Q(\alpha,\beta,t) = \frac{\varepsilon_a}{\pi^2}\left[\langle\alpha,\beta \mid \hat{a}^\dagger\hat{\rho} \mid \beta,\alpha\rangle - \langle\alpha,\beta \mid \hat{\rho}\hat{a}^\dagger \mid \beta,\alpha\rangle\right.$$
$$\left.+\langle\alpha,\beta \mid \hat{\rho}\hat{a} \mid \beta,\alpha\rangle + \langle\alpha,\beta \mid \hat{a}\hat{\rho} \mid \beta,\alpha\rangle\right]$$
$$+\frac{\varepsilon_b}{\pi^2}\left[\langle\alpha,\beta \mid \hat{b}^\dagger\hat{\rho} \mid \beta,\alpha\rangle - \langle\alpha,\beta \mid \hat{\rho}\hat{b}^\dagger \mid \beta,\alpha\rangle\right.$$
$$\left.+\langle\alpha,\beta \mid \hat{\rho}\hat{b} \mid \beta,\alpha\rangle + \langle\alpha,\beta \mid \hat{b}\hat{\rho} \mid \beta,\alpha\rangle\right]$$
$$+\frac{\gamma_a\overline{n}_a}{2\pi^2}\left[2\langle\alpha,\beta \mid \hat{a}^\dagger\hat{\rho}\hat{a} \mid \beta,\alpha\rangle - \langle\alpha,\beta \mid \hat{a}\hat{a}^\dagger\hat{\rho} \mid \beta,\alpha\rangle\right.$$
$$\left.-\langle\alpha,\beta \mid \hat{\rho}\hat{a}\hat{a}^\dagger \mid \beta,\alpha\right]$$
$$+\frac{\gamma_b\overline{n}_b}{2\pi^2}\left[2\langle\alpha,\beta \mid \hat{b}^\dagger\hat{\rho}\hat{b} \mid \beta,\alpha\rangle - \langle\alpha,\beta \mid \hat{b}\hat{b}^\dagger\hat{\rho} \mid \beta,\alpha\rangle\right.$$
$$\left.-\langle\alpha,\beta \mid \hat{\rho}\hat{b}\hat{b}^\dagger \mid \beta,\alpha\right]$$
$$+\frac{\gamma_a}{2\pi^2}(\overline{n}_a+1)\left[2\langle\alpha,\beta \mid \hat{a}\hat{\rho}\hat{a}^\dagger \mid \beta,\alpha\rangle - \langle\alpha,\beta \mid \hat{a}^\dagger\hat{a}\hat{\rho} \mid \beta,\alpha\rangle\right.$$
$$\left.-\langle\beta,\alpha \mid \hat{\rho}\hat{a}^\dagger\hat{a} \mid \beta,\alpha\rangle\right]$$
$$+\frac{\gamma_b}{2\pi^2}(\overline{n}_b+1)\left[2\langle\alpha,\beta \mid \hat{b}\hat{\rho}\hat{b}^\dagger \mid \beta,\alpha\rangle - \langle\alpha,\beta \mid \hat{b}^\dagger\hat{b}\hat{\rho} \mid \beta,\alpha\rangle\right.$$
$$\left.-\langle\beta,\alpha \mid \hat{\rho}\hat{b}^\dagger\hat{b} \mid \beta,\alpha\rangle\right]. \qquad (3.3a)$$

On applying relations (2.4) for a two-mode system Eq.(3.3a) reduces to

$$\frac{\partial}{\partial t}Q(\alpha,\beta,t) = \Bigg[\gamma_a(\overline{n}_a+1)\frac{\partial^2}{\partial\alpha^*\partial\alpha} + \gamma_b(\overline{n}_b+1)\frac{\partial^2}{\partial\beta^*\partial\beta}$$
$$+\frac{\gamma_a}{2}\frac{\partial}{\partial\alpha^*}\left(\alpha^* - \frac{2\varepsilon_a}{\gamma_a}\right) + \frac{\gamma_a}{2}\frac{\partial}{\partial\alpha}\left(\alpha - \frac{2\varepsilon_a}{\gamma_a}\right)$$
$$+\frac{\gamma_b}{2}\frac{\partial}{\partial\beta^*}\left(\beta^* - \frac{2\varepsilon_b}{\gamma_b}\right) + \frac{\gamma_b}{2}\frac{\partial}{\partial\beta}\left(\beta - \frac{2\varepsilon_b}{\gamma_b}\right)\Bigg] Q(\alpha,\beta,t). \qquad (3.3b)$$

This represents the Fokker-Planck equation for a two-mode DSV driven by a two-mode coherent light and damped by a two-mode thermal reservoir. In addition, introducing new variables defined as

$$\alpha_1 = \alpha - \frac{2\varepsilon_a}{\gamma_a} \qquad (3.4a)$$

and

$$\beta_1 = \beta - \frac{2\varepsilon_b}{\gamma_b}, \qquad (3.4b)$$

one can rewrite Eq.(3.3b) as

$$\frac{\partial}{\partial t}Q(\alpha_1,\beta_1,t) = \Bigg[\gamma_a(\overline{n}_a+1)\frac{\partial^2}{\partial\alpha_1^*\partial\alpha_1} + \gamma_b(\overline{n}_b+1)\frac{\partial^2}{\partial\beta_1^*\partial\beta_1} + \frac{\gamma_a}{2}\frac{\partial}{\partial\alpha_1^*}\alpha_1^*$$
$$+\frac{\gamma_a}{2}\frac{\partial}{\partial\alpha_1}\alpha_1 + \frac{\gamma_b}{2}\frac{\partial}{\partial\beta_1^*}\beta_1^* + \frac{\gamma_b}{2}\frac{\partial}{\partial\beta_1}\beta_1\Bigg] Q(\alpha_1,\beta_1,t). \qquad (3.5)$$

In order to solve (3.5) applying the *propagator method* [1], it proves to be convenient to introduce Cartesian coordinates defined by

$$\alpha_1 = x + iy, \qquad (3.6a)$$

$$\beta_1 = u + iv, \qquad (3.6b)$$

so that Eq.(3.5) takes the form

$$\frac{\partial}{\partial t}Q(x,y,,u,v,t) = \Bigg[\frac{\gamma_a}{4}(\overline{n}_a+1)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + \frac{\gamma_b}{4}(\overline{n}_b+1)\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)$$
$$+\frac{\gamma_a}{2}\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right) + \frac{\gamma_b}{2}\left(\frac{\partial}{\partial u}u + \frac{\partial}{\partial v}v\right)\Bigg] Q(x,y,,u,v,t). \quad (3.7)$$

Upon replacing

$$\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial u},\frac{\partial}{\partial v},x,y,u,v,Q(x,y,u,v,t)\right) \text{ by } \left(\frac{i}{\hbar}\hat{p}_x,\frac{i}{\hbar}\hat{p}_y,\frac{i}{\hbar}\hat{p}_u,\frac{i}{\hbar}\hat{p}_v,\hat{x},\hat{y},\hat{u},\hat{v},\mid Q(t)\rangle\right),$$

Eq.(3.7) can be put in a Schrödinger type-equation

$$i\hbar\frac{\partial}{\partial t} \mid Q(t)\rangle = \hat{H} \mid Q(t)\rangle, \tag{3.8a}$$

where

$$\hat{H} = -\frac{i\gamma_a}{4\hbar}(\overline{n}_a + 1)\left(\hat{p}_x^2 + \hat{p}_y^2\right) - \frac{\gamma_a}{2}\left(\hat{p}_x\hat{x} + \hat{p}_y\hat{y}\right)$$
$$-\frac{i\gamma_b}{4\hbar}(\overline{n}_b + 1)\left(\hat{p}_u^2 + \hat{p}_y^2\right) - \frac{\gamma_b}{2}\left(\hat{p}_u\hat{u} + \hat{p}_v\hat{v}\right) \tag{3.8b}$$

is the quantum Hamiltonian. A formal solution for (3.8a) is expressible as

$$\mid Q(t)\rangle = \hat{U} \mid Q(0)\rangle, \tag{3.9a}$$

in which

$$\hat{U} = \exp\left(-\frac{i\hat{H}}{\hbar}t\right). \tag{3.9b}$$

On multiplying (3.9a) on the left by $\langle x, y, u, v \mid$, we obtain

$$Q(x, y, , u, v, t) = \langle x, y, u, v \mid \hat{U} \mid Q(0)\rangle, \tag{3.10a}$$

where

$$Q(x, y, u, v, t) = \langle x, y, u, v \mid Q(t)\rangle. \tag{3.10b}$$

Employing the completeness relation

$$I = \int_{-\infty}^{\infty} dx'dy'du'dv' \mid x', y', u', v'\rangle\langle x', y', u', v' \mid,$$

one can rewrite (3.10a) as

$$Q(x, y, u, v, t) = \int_{-\infty}^{\infty} dx'dy'du'dv'\, Q(x, y, u, v, t \mid x', y', u', v', 0)\, Q_0(x', y', u', v', 0), \tag{3.11a}$$

in which

$$Q(x, y, u, v, t \mid x', y', u', v', 0) = \langle x, y, u, v \mid \hat{U} \mid x', y', u', v'\rangle \tag{3.11b}$$

is the Q-function propagator and

$$Q_0(x', y', u', v', 0), = \langle x', y', u', v' \mid Q(0)\rangle \tag{3.11c}$$

is the initial Q-function. Now comparison of (2.13) and (3.8b) shows that $b_x(t) = b_y(t) = -\frac{\gamma_a}{2}$ and $b_u(t) = b_v(t) = -\frac{\gamma_b}{2}$. Hence the Q-function propagator (2.14) associated with the Hamiltonian (3.8b) takes the form

$$Q(x,y,u,v,t \mid x',y',u',v',0) = \left[\frac{i}{2\pi\hbar}\frac{\partial^2 S_c}{\partial x'\partial x}\right]^{\frac{1}{2}} \left[\frac{i}{2\pi\hbar}\frac{\partial^2 S_c}{\partial y'\partial y}\right]^{\frac{1}{2}} \left[\frac{i}{2\pi\hbar}\frac{\partial^2 S_c}{\partial u'\partial u}\right]^{\frac{1}{2}}$$

$$\times \left[\frac{i}{2\pi\hbar}\frac{\partial^2 S_c}{\partial v'\partial v}\right]^{\frac{1}{2}} \exp\left\{\frac{iS_c}{\hbar} + \frac{\gamma_a+\gamma_b}{2}t\right\}. \tag{3.12}$$

The classical Hamiltonian corresponding to the quantum Hamiltonian (3.8b) is expressible as

$$\begin{aligned} H = & -\frac{i\gamma_a}{4\hbar}(\overline{n}_a+1)\left(p_x^2+p_y^2\right) - \frac{\gamma_a}{2}\left(p_x x + p_y y\right) \\ & -\frac{i\gamma_b}{4\hbar}(\overline{n}_b+1)\left(p_u^2+p_y^2\right) - \frac{\gamma_b}{2}\left(p_u u + p_v v\right) \end{aligned} \tag{3.13}$$

and upon applying Hamilton's equations (2.17), one can easily obtain

$$p_x = \frac{i\hbar}{\overline{n}_a+1}\left(\frac{2}{\gamma_a}\dot{x} + x\right), \tag{3.14a}$$

$$p_y = \frac{i\hbar}{\overline{n}_a+1}\left(\frac{2}{\gamma_a}\dot{y} + y\right), \tag{3.14b}$$

$$p_u = \frac{i\hbar}{\overline{n}_b+1}\left(\frac{2}{\gamma_b}\dot{u} + u\right), \tag{3.14c}$$

$$p_v = \frac{i\hbar}{\overline{n}_b+1}\left(\frac{2}{\gamma_b}\dot{v} + v\right). \tag{3.14d}$$

Taking into account (3.13) and (3.14) along with (2.16), one can easily verify that

$$\begin{aligned} L = & \frac{i\hbar}{\gamma_a(\overline{n}_a+1)}\left[\left(\dot{x}+\frac{\gamma_a}{2}x\right)^2 + \left(\dot{y}+\frac{\gamma_a}{2}y\right)^2\right] \\ & +\frac{i\hbar}{\gamma_b(\overline{n}_b+1)}\left[\left(\dot{u}+\frac{\gamma_b}{2}u\right)^2 + \left(\dot{v}+\frac{\gamma_b}{2}v\right)^2\right], \end{aligned} \tag{3.15}$$

so that in view of (2.20) and (3.15), we have

$$\begin{aligned} \ddot{x} - \frac{\gamma_a^2}{4}x &= 0, \\ \ddot{y} - \frac{\gamma_a^2}{4}y &= 0, \\ \ddot{u} - \frac{\gamma_b^2}{4}u &= 0, \\ \ddot{v} - \frac{\gamma_b^2}{4}v &= 0. \end{aligned}$$

The solution for this differential equation can be put in the form

$$x(t) = Ae^{\frac{\gamma_a}{2}t} + Be^{-\frac{\gamma_a}{2}t}, \tag{3.16a}$$

$$y(t) = Ce^{\frac{\gamma_a}{2}t} + De^{-\frac{\gamma_a}{2}t}, \tag{3.16b}$$

$$u(t) = Ge^{\frac{\gamma_b}{2}t} + Fe^{-\frac{\gamma_b}{2}t}, \tag{3.16c}$$

$$v(t) = He^{\frac{\gamma_b}{2}t} + Ke^{-\frac{\gamma_b}{2}t}, \tag{3.16d}$$

so that when these results are inserted into (3.15), there follows

$$L = \frac{i\hbar\gamma_a}{(\overline{n}_a + 1)} e^{\gamma_a t}\left(A^2 + C^2\right) + \frac{i\hbar\gamma_b}{(\overline{n}_b + 1)} e^{\gamma_b t}\left(G^2 + H^2\right)$$

and hence the classical action

$$S_c = \int_0^T L(t)dt$$

takes the form

$$S_c = \frac{i\hbar}{(\overline{n}_a + 1)}\left(e^{\gamma_a T} - 1\right)\left(A^2 + C^2\right) + \frac{i\hbar}{(\overline{n}_b + 1)}\left(e^{\gamma_b T} - 1\right)\left(G^2 + H^2\right). \tag{3.17}$$

Applying the boundary conditions $x(0) = x'$, $x(T) = x''$, $y(0) = y'$, $y(T) = y''$, $u(0) = u'$, $u(T) = u''$, $v(0) = v'$ and $v(T) = v''$ to (3.16), one can easily get

$$A = -\frac{\left(x' - x'' e^{\frac{\gamma_a}{2}T}\right)}{e^{\gamma_a T} - 1}, \tag{3.18a}$$

$$C = -\frac{\left(y' - y'' e^{\frac{\gamma_a}{2}T}\right)}{e^{\gamma_a T} - 1}, \tag{3.18b}$$

$$G = -\frac{\left(u' - u'' e^{\frac{\gamma_b}{2}T}\right)}{e^{\gamma_b T} - 1}, \tag{3.18c}$$

$$H = -\frac{\left(v' - v'' e^{\frac{\gamma_b}{2}T}\right)}{e^{\gamma_b T} - 1}. \tag{3.18d}$$

Hence combination of (3.17) and (3.18) leads to

$$S_c = \frac{i\hbar}{\overline{n}_a + 1}\left[\frac{x'^2 - 2x'xe^{\frac{\gamma_a}{2}t} + x^2 e^{\gamma_a t} + y'^2 - 2y'ye^{\frac{\gamma_a}{2}t} + y^2 e^{\gamma_a t}}{e^{\gamma_a t} - 1}\right]$$
$$+ \frac{i\hbar}{\overline{n}_b + 1}\left[\frac{u'^2 - 2u'ue^{\frac{\gamma_b}{2}t} + u^2 e^{\gamma_b t} + v'^2 - 2v've^{\frac{\gamma_b}{2}t} + v^2 e^{\gamma_b t}}{e^{\gamma_b t} - 1}\right], \tag{3.19}$$

where we have replaced (x'', y'', u'', v'', T) by (x, y, u, v, t). Now taking into account (3.19) along with (3.12), one readily obtains

$$Q(x,y,u,v,t\,|\,x',y',u',v',0)=\left[\frac{1}{\pi^2(\overline{n}_a+1)(\overline{n}_b+1)(1-e^{-\gamma_a t})(1-e^{-\gamma_b t})}\right]$$

$$\times\exp\left\{-\frac{\left(x'e^{-\frac{\gamma_a}{2}t}-x\right)^2+\left(y'e^{-\frac{\gamma_a}{2}t}-y\right)^2}{(\overline{n}_a+1)(1-e^{-\gamma_a t})}\right.$$

$$\left.-\frac{\left(u'e^{-\frac{\gamma_b}{2}t}-u\right)^2+\left(v'e^{-\frac{\gamma_b}{2}t}-v\right)^2}{(\overline{n}_b+1)(1-e^{-\gamma_b t})}\right\}. \tag{3.20}$$

This expression represents the Q-function propagator for a two-mode light driven by a two-mode coherent light and damped by a two-mode thermal reservoir.

Now we proceed to calculate the initial Q-function (3.11c). At $t=0$ we have a two-mode DSV in the cavity. From the definition of a two-mode DSV state

$$|\,\eta,\lambda,r\rangle=e^{\lambda\hat{a}^\dagger-\lambda^*\hat{a}}e^{\eta\hat{b}^\dagger-\eta^*\hat{a}}e^{r(\hat{a}\hat{b}-\hat{a}^\dagger\hat{b}^\dagger)}\,|\,0,0\rangle,$$

the density operator $\hat{\rho}$ is expressible as

$$\hat{\rho}=e^{\lambda\hat{a}^\dagger-\lambda^*\hat{a}}e^{\eta\hat{b}^\dagger-\eta^*\hat{a}}e^{r(\hat{a}\hat{b}-\hat{a}^\dagger\hat{b}^\dagger)}\,|\,0,0\rangle\langle 0,0\,|\,e^{-r(\hat{a}\hat{b}-\hat{a}^\dagger\hat{b}^\dagger)}e^{-(\lambda\hat{a}^\dagger-\lambda^*\hat{a})}e^{-(\eta\hat{b}^\dagger-\eta^*\hat{a})}, \tag{3.21}$$

where $\hat{a}(\hat{b})$ is the annihilation operator, r is the squeeze parameter (taken to be real and positive) and λ and η are complex parameters. Applying the definition of the Q-function for a two-mode DSV

$$Q(\alpha,\beta,r)=\frac{1}{\pi^2}\langle\alpha,\beta\,|\,\hat{\rho}\,|\,\beta,\alpha\rangle,$$

we see that

$$Q(\alpha,\beta,r)=\frac{1}{\pi^2}\langle\alpha,\beta\,|\,e^{\lambda\hat{a}^\dagger-\lambda^*\hat{a}}e^{\eta\hat{b}^\dagger-\eta^*\hat{a}}\,|\,r\rangle\langle r\,|\,e^{-(\lambda\hat{a}^\dagger-\lambda^*\hat{a})}e^{-(\eta\hat{b}^\dagger-\eta^*\hat{a})}\,|\,\beta,\alpha\rangle,$$

where

$$|\,r\rangle=e^{r(\hat{a}\hat{b}-\hat{a}^\dagger\hat{b}^\dagger)}\,|\,0,0\rangle$$

is a two-mode squeezed vacuum state and

$$\mathrm{S}_{ab}(r)=e^{r\left(\hat{a}\hat{b}-\hat{a}^\dagger\hat{b}^\dagger\right)}$$

is a two-mode squeeze operator. Taking into consideration the Bakker-Hausdorff identity (2.26) and (B1) discussed in the Appendix, one can rewrite the above Q-function as

$$Q(\alpha,\beta,r) = \frac{1}{\pi^2} \exp\left[-\lambda^*\lambda - \eta^*\eta + \lambda\alpha^* + \lambda^*\alpha + \eta\beta^* + \eta^*\beta\right]$$
$$\times\langle\alpha,\beta \mid e^{-\lambda^*\hat{a}}e^{-\eta^*\hat{b}} \mid r\rangle\langle r \mid e^{-\lambda\hat{a}^\dagger}e^{-\eta\hat{b}^\dagger} \mid \beta,\alpha\rangle. \tag{3.22}$$

Now we proceed to determine the explicit form of $\langle\alpha,\beta \mid e^{-\lambda^*\hat{a}}e^{-\eta^*\hat{b}} \mid r\rangle$. Using the completeness relation for two-mode coherent states

$$I = \int_{-\infty}^{\infty} \frac{d^2\alpha_1}{\pi}\frac{d^2\beta_1}{\pi} \mid \beta_1,\alpha_1\rangle\langle\alpha_1,\beta_1 \mid,$$

we can write

$$\langle\alpha,\beta \mid e^{-\lambda^*\hat{a}}e^{-\eta^*\hat{b}} \mid r\rangle = \int_{-\infty}^{\infty} \frac{d^2\alpha_1}{\pi}\frac{d^2\beta_1}{\pi} \exp\left[-\lambda^*\alpha_1 - \eta^*\beta_1\right]\langle\alpha,\beta \mid \beta_1,\alpha_1\rangle\langle\alpha_1,\beta_1 \mid r\rangle, \tag{3.23}$$

so that taking into account relation (2.30), one easily obtains

$$\langle\alpha,\beta \mid e^{-\lambda^*\hat{a}}e^{-\eta^*\hat{b}} \mid r\rangle = \exp\left[-\frac{1}{2}\alpha^*\alpha - \frac{1}{2}\beta^*\beta\right]$$
$$\times\int_{-\infty}^{\infty} \frac{d^2\alpha_1}{\pi}\frac{d^2\beta_1}{\pi}\langle\alpha_1,\beta_1 \mid r\rangle \exp\left\{-\frac{1}{2}\alpha_1^*\alpha_1 + (\alpha^* - \lambda^*)\,\alpha_1 -\frac{1}{2}\beta_1^*\beta_1 + (\beta^* - \eta^*)\,\beta_1\right\}. \tag{3.24}$$

Next we wish to evaluate $\langle\alpha_1,\beta_1 \mid r\rangle$. We note that

$$\langle\alpha_1,\beta_1 \mid r\rangle = \frac{1}{\alpha_1^*}\langle\beta_1,\alpha_1 \mid \hat{a}^\dagger \mathrm{S}_{ab}(r) \mid 0,0\rangle.$$

Inserting the identity operator

$$I = \mathrm{S}_{ab}(r)\mathrm{S}_{ab}^\dagger(r),$$

we get

$$\langle\alpha_1,\beta_1 \mid r\rangle = \frac{1}{\alpha_1^*}\langle\alpha_1,\beta_1 \mid \mathrm{S}_{ab}(r)\mathrm{S}_{ab}^\dagger(r)\hat{a}^\dagger\mathrm{S}_{ab}(r) \mid 0,0\rangle,$$

so that taking into account the relation [5]

$$\mathrm{S}_{ab}^\dagger(r)\hat{a}^\dagger\mathrm{S}_{ab}(r) = \hat{a}^\dagger\cosh(r) - \hat{b}\sinh(r),$$

we see that

$$\alpha_1^*\langle\alpha_1,\beta_1 \mid r\rangle = \cosh(r)\langle\alpha_1,\beta_1 \mid \mathrm{S}_{ab}(r)\hat{a}^\dagger \mid 0,0\rangle.$$

On applying the the identity operator

$$I = \mathrm{S}_{ab}^\dagger(r)\mathrm{S}_{ab}(r),$$

in this relation, we have

$$\alpha_1^*\langle\alpha_1,\beta_1 \mid r\rangle = \cosh(r)\langle\alpha_1,\beta_1 \mid \mathbf{S}_{ab}(r)\hat{a}^\dagger\mathbf{S}_{ab}^\dagger(r)\mathbf{S}_{ab}(r) \mid 0\rangle$$

and using the relation

$$\mathbf{S}_{ab}(r)\hat{a}^\dagger\mathbf{S}_{ab}^\dagger(r) = \hat{a}^\dagger\cosh(r) + \hat{b}\sinh(r),$$

we readily get

$$\alpha_1^*\langle\alpha_1,\beta_1 \mid r\rangle = \alpha_1^*\cosh^2(r)\langle\alpha_1,\beta_1 \mid r\rangle + \cosh(r)\sinh(r)\langle\alpha_1,\beta_1 \mid \hat{b} \mid r\rangle.$$

Thus employing relation (A4) discussed in the Appendix, one can rewrite this expression as

$$\frac{\partial}{\partial\beta_1^*}\langle\alpha_1,\beta_1 \mid r\rangle + \left(\frac{1}{2}\beta_1 + \alpha_1^*\tanh(r)\right)\langle\alpha_1,\beta_1 \mid r\rangle = 0 \qquad (3.25).$$

The solution of this equation has the form

$$\langle\alpha_1,\beta_1 \mid r\rangle = K(\alpha_1^*,\alpha_1,\beta_1,r)\exp\left\{-\frac{1}{2}\beta_1^*\beta_1 - \beta_1^*\alpha_1\tanh(r)\right\}, \qquad (3.26a)$$

in which

$$K(\alpha_1^*,\alpha_1,\beta_1,r) = \langle\alpha_1,\beta_1 \mid r\rangle\Big|_{\beta_1^*=0}. \qquad (3.26b)$$

In view of the fact that

$$\langle\alpha_1,\beta_1|r\rangle\Big|_{\beta_1^*=0} = \langle\alpha_1,0 \mid e^{\beta_1^*\hat{b}-\beta_1\hat{b}^\dagger}|r\rangle\Big|_{\beta_1^*=0} = \langle\alpha_1,0 \mid e^{-\beta_1\hat{b}^\dagger}|r\rangle = \langle\alpha_1,0 \mid r\rangle,$$

expression (3.26b) takes the form

$$K(\alpha_1^*,\alpha_1,\beta_1,r) = \langle\alpha_1,0 \mid r\rangle = K(\alpha_1^*,\alpha_1,r) \qquad (3.26c)$$

Taking into account (3.26a) and (3.26c), we have

$$\langle\alpha_1,\beta_1 \mid r\rangle = K(\alpha_1^*,\alpha_1,r)\exp\left\{-\frac{1}{2}\beta_1^*\beta_1 - \beta_1^*\alpha_1^*\tanh(r)\right\}. \qquad (3.27a)$$

Following the same procedure, one can also verify that

$$\langle\alpha_1,\beta_1 \mid r\rangle = K(\beta_1^*,\beta_1,r)\exp\left\{-\frac{1}{2}\alpha_1^*\alpha_1 - \beta_1^*\alpha_1^*\tanh(r)\right\}, \qquad (3.27b)$$

so that comparison of (2.7a) and (2.7b) leads to

$$K(\beta_1^*,\beta_1,r)e^{-\frac{1}{2}\alpha_1^*\alpha_1} = K(\alpha_1^*,\alpha_1,r)e^{-\frac{1}{2}\beta_1^*\beta_1}.$$

Now upon dividing both sides of this equation by $\exp\left[-\frac{1}{2}\alpha_1^*\alpha_1 - \frac{1}{2}\beta_1^*\beta_1\right]$, we can write

$$\frac{K(\beta_1^*, \beta_1, r)}{e^{-\frac{1}{2}\beta_1^*\beta_1}} = \frac{K(\alpha_1^*, \alpha_1, r)}{e^{-\frac{1}{2}\alpha_1^*\alpha_1}} = K(r),$$

from which follows

$$K(\beta_1^*, \beta_1, r) = K(r)e^{-\frac{1}{2}\beta_1^*\beta_1} \tag{3.28a}$$

and

$$K(\alpha_1^*, \alpha_1, r) = K(r)e^{-\frac{1}{2}\alpha_1^*\alpha_1}. \tag{3.28b}$$

Substitution of (3.28) into (3.27) gives

$$\langle \alpha_1, \beta_1 \mid r \rangle = K(r) \exp\left\{-\frac{1}{2}\alpha_1^*\alpha_1 - \frac{1}{2}\beta_1^*\beta_1 - \beta_1^*\alpha_1^* \tanh(r)\right\},$$

so that applying the normalization condition for $\langle \alpha_1, \beta_1 \mid r \rangle$, we have

$$|K(r)|^2 \int_{-\infty}^{\infty} \frac{d^2\alpha_1}{\pi} \frac{d^2\beta_1}{\pi} \exp\left[-\alpha_1^*\alpha_1 - \beta_1^*\beta_1 - (\beta_1^*\alpha_1^* + \beta_1\alpha_1)\tanh(r)\right] = 1$$

and on carrying out the integration applying (2.34), there follows

$$K(r) = \operatorname{sech}(r).$$

Hence expression (3.26a) takes the form

$$\langle \alpha_1, \beta_1 \mid r \rangle = \operatorname{sech}(r) \exp\left\{-\frac{1}{2}\alpha_1^*\alpha_1 - \frac{1}{2}\beta_1^*\beta_1 - \beta_1^*\alpha_1^* \tanh(r)\right\}, \tag{3.29}$$

so that when (3.29) is inserted into (3.24), we get

$$\begin{aligned}\langle \alpha, \beta \mid e^{-\lambda^*\hat{a}} e^{-\eta^*\hat{b}} \mid r \rangle = \operatorname{sech}(r) &\exp\left\{-\frac{1}{2}\alpha^*\alpha - \frac{1}{2}\beta^*\beta\right\} \\ &\times \int_{-\infty}^{\infty} \frac{d^2\alpha_1}{\pi}\frac{d^2\beta_1}{\pi} \exp\left[-\alpha_1^*\alpha_1 - \beta_1^*\beta_1 + (\alpha^* - \lambda^*)\alpha_1\right. \\ &\qquad \left. + (\beta^* - \eta^*)\beta_1 - \alpha_1^*\beta_1^* \tanh(r)\right]\end{aligned}$$

and on carrying out the integration, one finds

$$\langle \alpha, \beta \mid e^{-\lambda^*\hat{a}} e^{-\eta^*\hat{b}} \mid r \rangle = \operatorname{sech}(r) \exp\left\{-\frac{1}{2}\alpha^*\alpha - \frac{1}{2}\beta^*\beta - (\alpha^* - \lambda^*)(\beta^* - \eta^*)\tanh(r)\right\}. \tag{3.30}$$

On substituting (3.30) along with its complex conjugate into (3.22), we obtain the well-known result for the Q-function of a two-mode DSV

$$Q(\alpha,\alpha^*,\beta,\beta^*,r) = \frac{\mathrm{sech}^2(r)}{\pi^2}\exp\Big\{ - |\alpha-\lambda|^2 - |\beta-\eta|^2 - (\alpha-\lambda)(\beta-\eta)\tanh(r)$$
$$- (\alpha^*-\lambda^*)(\beta^*-\eta^*)\tanh(r)\Big\}. \tag{3.31}$$

Next to calculate the Q-function (3.11a) using (3.20) and (3.31), we see from (3.4) and (3.6) that

$$\alpha' = \frac{2\varepsilon_a}{\gamma_a} + x' + iy',$$
$$\beta' = \frac{2\varepsilon_b}{\gamma b} + u' + iv',$$

so that (3.31) can be rewritten in terms of x', y', u', v' as

$$\begin{aligned} Q(x',y',u',v',r) = &\left[\frac{\mathrm{sech}^2(r)}{\pi^2}\right]\exp\Bigg\{-\left(\frac{\lambda^*+\lambda}{2} - \frac{2\varepsilon_a}{\gamma_a}\right)^2 - \left(\frac{\eta^*+\eta}{2} - \frac{2\varepsilon_b}{\gamma_b}\right)^2 \\ &+\left(\frac{\lambda^*-\lambda}{2}\right)^2 + \left(\frac{\eta^*-\eta}{2}\right)^2 \\ &-\frac{1}{2}\left(\lambda^*+\lambda - \frac{4\varepsilon_a}{\gamma_a}\right)\left(\eta^*+\eta-\frac{4\varepsilon_b}{\gamma_b}\right)\tanh r \\ &-\frac{1}{2}(\lambda^*-\lambda)(\eta^*-\eta)\tanh(r)\Bigg\} \\ \times\exp\Bigg\{ &- x'^2 + \left[\lambda^*+\lambda-\frac{4\varepsilon_a}{\gamma_a} + \left(\eta^*+\eta-\frac{4\varepsilon_b}{\gamma_b}\right)\tanh r\right]x' \\ &-u'^2 + \left[\eta^*+\eta-\frac{4\varepsilon_b}{\gamma_b} + \left(\lambda^*+\lambda-\frac{4\varepsilon_a}{\gamma_a}\right)\tanh r\right]u' \\ &-y'^2 + i\left[\lambda^*-\lambda-(\eta^*-\eta)\tanh r\right]y' \\ &-v'^2 + i\left[\eta^*-\eta-(\lambda^*-\lambda)\tanh(r)\right]v' \\ &-2(x'u'+y'v')\tanh(r)\Bigg\}. \end{aligned} \tag{3.32}$$

Taking into account (3.11a), (3.20) and (3.32), we obtain

$$
\begin{aligned}
Q(x,y,u,v,t) = &\left[\frac{\operatorname{sech}^2(r)}{\pi^4(\overline{n}_a+1)(\overline{n}_b+1)(1-e^{-\gamma_a t})(1-e^{-\gamma_b t})}\right] \\
&\times \exp\left[\frac{-x^2-y^2}{(\overline{n}_a+1)(1-e^{-\gamma_a t})} + \frac{-u^2-v^2}{(\overline{n}_b+1)(1-e^{-\gamma_b t})}\right. \\
&\qquad - \left(\lambda^*\lambda + \eta^*\eta - \frac{2\varepsilon_a}{\gamma_a}(\lambda^*+\lambda) - \frac{2\varepsilon_b}{\gamma_b}(\eta^*+\eta) + \frac{4\varepsilon_a^2}{\gamma_a^2} + \frac{4\varepsilon_b^2}{\gamma_b^2}\right) \\
&\qquad \left. - \left(\eta^*\lambda^* + \eta\lambda - \frac{2\varepsilon_b}{\gamma_b}(\lambda^*+\lambda) - \frac{2\varepsilon_a}{\gamma_a}(\eta^*+\eta) + \frac{8\varepsilon_a\varepsilon_b}{\gamma_a\gamma_b}\right)\tanh(r)\right] \\
&\times \int_{-\infty}^{\infty} dx' \exp\left\{ -\frac{\overline{n}_a(1-e^{-\gamma_a t})+1}{(\overline{n}_a+1)(1-e^{-\gamma_a t})}x'^2 \right. \\
&\qquad\qquad + \left[\frac{2xe^{-\frac{\gamma_a}{2}t}}{(\overline{n}_a+1)(1-e^{-\gamma_a t})} + \left(\lambda^*+\lambda-\frac{4\varepsilon_a}{\gamma_a}\right)\right. \\
&\qquad\qquad\quad \left.\left. + \left(\eta^*+\eta-\frac{4\varepsilon_b}{\gamma_b}\right)\tanh(r)\right]x'\right\} \\
&\times \int_{-\infty}^{\infty} dy' \exp\left\{ -\frac{\overline{n}_a(1-e^{-\gamma_a t})+1}{(\overline{n}_a+1)(1-e^{-\gamma_a t})}y'^2 \right. \\
&\qquad\qquad + \left[\frac{2ye^{-\frac{\gamma_a}{2}t}}{(\overline{n}_a+1)(1-e^{-\gamma_a t})} + i(\lambda^*-\lambda)\right. \\
&\qquad\qquad\quad \left.\left. - i(\eta^*-\eta)\tanh(r)\right]y'\right\} \\
&\times \int_{-\infty}^{\infty} du' \exp\left\{ -\frac{\overline{n}_b(1-e^{-\gamma_b t})+1}{(\overline{n}_b+1)(1-e^{-\gamma_b t})}u'^2 \right. \\
&\qquad\qquad + \left[\frac{2ue^{-\frac{\gamma_b}{2}t}}{(\overline{n}_b+1)(1-e^{-\gamma_b t})} + \left(\eta^*+\eta-\frac{4\varepsilon_b}{\gamma_b}\right)\right. \\
&\qquad\qquad\quad \left.\left. + \left(\lambda^*+\lambda-\frac{4\varepsilon_a}{\gamma_a}\right)\tanh(r) - 2x'\tanh(r)\right]u'\right\} \\
&\times \int_{-\infty}^{\infty} dv' \exp\left\{ -\frac{\overline{n}_b(1-e^{-\gamma_b t})+1}{(\overline{n}_b+1)(1-e^{-\gamma_b t})}v'^2 \right. \\
&\qquad\qquad + \left[\frac{2ve^{-\frac{\gamma_b}{2}t}}{(\overline{n}_b+1(1-e^{-\gamma_b t})} + i(\eta^*-\eta)\right. \\
&\qquad\qquad\quad \left.\left. - i(\lambda^*-\lambda)\tanh(r) - 2y'\tanh(r)\right]v'\right\}.
\end{aligned}
$$

Upon carrying out the integration and taking into account (3.4) and (3.6), the Q-function is found to be

$$Q(\alpha,\alpha^*,\beta,\beta^*,t) = \frac{\mathcal{K}(t)}{\pi^2}\exp\left[-A_1(t)\alpha^*\alpha + C_1(t)\alpha + C_1^*\alpha^* - A_2(t)\beta^*\beta\right.$$

$$\left.+C_2(t)\beta + C_2^*(t)\beta^* + D(t)(\alpha\beta+\alpha^*\beta^*)\right], \qquad (3.33)$$

where

$$A_1(t) = \frac{(\overline{n}_b+1)(1-e^{-\gamma_b t})\mathrm{sech}^2(r) + e^{-\gamma_b t}}{F(t)}, \qquad (3.34a)$$

$$A_2(t) = \frac{(\overline{n}_a+1)(1-e^{-\gamma_a t})\mathrm{sech}^2(r) + e^{-\gamma_a t}}{F(t)}, \qquad (3.34b)$$

$$D(t) = \frac{-e^{-\frac{1}{2}(\gamma_a+\gamma_b)t}\tanh(r)}{F(t)}, \qquad (3.34c)$$

$$F(t) = (\overline{n}_a+1)(\overline{n}_b+1)(1-e^{-\gamma_a t})(1-e^{-\gamma_b t})\mathrm{sech}^2(r) + e^{-(\gamma_a+\gamma_b)t}$$

$$+(\overline{n}_b+1)e^{-\gamma_a t}(1-e^{-\gamma_b t}) + (\overline{n}_a+1)e^{-\gamma_b t}(1-e^{-\gamma_a t}), \qquad (3.34d)$$

$$C_1(t) = \frac{2\varepsilon_a}{\gamma_a}A_1\left(1-e^{-\frac{1}{2}\gamma_a t}\right) - \frac{2\varepsilon_b}{\gamma_b}D\left(1-e^{-\frac{1}{2}\gamma_b t}\right)$$

$$+A_1\lambda^* e^{-\frac{1}{2}\gamma_a t} - D\eta e^{-\frac{1}{2}\gamma_b t}, \qquad (3.34e)$$

$$C_2(t) = \frac{2\varepsilon_b}{\gamma_b}A_2\left(1-e^{-\frac{1}{2}\gamma_b t}\right) - \frac{2\varepsilon_a}{\gamma_a}D\left(1-e^{-\frac{1}{2}\gamma_a t}\right)$$

$$+A_2\eta^* e^{-\frac{1}{2}\gamma_b t} - D\lambda e^{-\frac{1}{2}\gamma_a t}, \qquad (3.34f)$$

$$\mathcal{K}(t) = \left[A_1A_2 - D^2\right]\exp\left[\frac{-A_1C_2C_2^* - A_2C_1C_1^* - D\left(C_1C_2 + C_1^*C_2^*\right)}{A_1A_2-D^2}\right]. \qquad (3.34g)$$

3.2. Quadrature Fluctuations

The squeezing properties of a two-mode light are described by two quadrature operators defined as [5].

$$\hat{C}_1 = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a} + \hat{b}^\dagger + \hat{b}\right),$$

$$\hat{C}_2 = \frac{i}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a} + \hat{b}^\dagger - \hat{b}\right),$$

so that these two quadrature operators satisfy the Commutation relation

$$[\hat{C}_1, \hat{C}_2] = 2i.$$

The variances of these two quadratures are given by

$$\Delta\hat{C}_1^2 = \langle\hat{C}_1^2\rangle - \langle\hat{C}_1\rangle^2, \tag{3.35a}$$

$$\Delta\hat{C}_2^2 = \langle\hat{C}_2^2\rangle - \langle\hat{C}_2\rangle^2, \tag{3.35b}$$

in which

$$\langle\hat{C}_1\rangle = \frac{1}{\sqrt{2}}\left(\langle\hat{a}^\dagger\rangle + \langle\hat{a}\rangle + \langle\hat{b}^\dagger\rangle + \langle\hat{b}\rangle\right),$$

$$\langle\hat{C}_2\rangle = \frac{i}{\sqrt{2}}\left(\langle\hat{a}^\dagger\rangle - \langle\hat{a}\rangle - \langle\hat{b}^\dagger\rangle - \langle\hat{b}\rangle\right),$$

$$\langle\hat{C}_1^2\rangle = -1 + \frac{1}{2}\left(\langle\hat{a}^{\dagger^2}\rangle + \langle\hat{a}^2\rangle + \langle\hat{b}^{\dagger^2}\rangle + \langle\hat{b}^2\rangle\right) + \langle\hat{a}\hat{a}^\dagger\rangle$$
$$+\langle\hat{b}\hat{b}^\dagger\rangle + \langle\hat{a}\hat{b}\rangle + \langle\hat{a}\hat{b}^\dagger\rangle + \langle\hat{a}^\dagger\hat{b}\rangle + \langle\hat{a}^\dagger\hat{b}^\dagger\rangle,$$

$$\langle\hat{C}_2^2\rangle = -1 - \frac{1}{2}\left(\langle\hat{a}^{\dagger^2}\rangle + \langle\hat{a}^2\rangle + \langle\hat{b}^{\dagger^2}\rangle + \langle\hat{b}^2\rangle\right) + \langle\hat{a}\hat{a}^\dagger\rangle$$
$$+\langle\hat{b}\hat{b}^\dagger\rangle - \langle\hat{a}\hat{b}\rangle + \langle\hat{a}\hat{b}^\dagger\rangle + \langle\hat{a}^\dagger\hat{b}\rangle - \langle\hat{a}^\dagger\hat{b}^\dagger\rangle.$$

We recall that the mean value of an operator $\hat{A}(\hat{a}^\dagger, \hat{b}^\dagger, \hat{a}, \hat{b}, t)$ is expressible in terms of the Q-function in the form

$$\langle\hat{A}\rangle = \int_{-\infty}^{\infty} d^2\alpha d^2\beta\, Q(\alpha, \alpha^*, \beta, \beta^*, t) A_a(\alpha, \alpha^*, \beta, \beta^*, t), \tag{3.36}$$

where $A_a(\alpha, \alpha^*, \beta, \beta^*, t)$ is the c-number equivalent of the operator $\hat{A}(\hat{a}^\dagger, \hat{b}^\dagger, \hat{a}, \hat{b}, t)$ for the antinormal ordering. Then taking into account (3.33), the above expressions can be put in the form

$$\langle\hat{C}_1\rangle = \frac{\mathcal{K}}{\sqrt{2}}\int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}(\alpha + \alpha^* + \beta + \beta^*)$$
$$\times\exp[-A_1\alpha^*\alpha + C_1\alpha + C_1^*\alpha^* - A_2\beta^*\beta + C_2^*\beta^*$$
$$+C_2\beta + D(\alpha\beta + \alpha^*\beta^*)],$$

$$\langle\hat{C}_2\rangle = \frac{i\mathcal{K}}{\sqrt{2}}\int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}(\alpha^* - \alpha + \beta^* - \beta)$$
$$\times\exp[-A_1\alpha^*\alpha + C_1\alpha + C_1^*\alpha^* - A_2\beta^*\beta + C_2^*\beta^*$$
$$+C_2\beta + D(\alpha\beta + \alpha^*\beta^*)],$$

$$\langle \hat{C}_1^2 \rangle = -1 + \frac{\mathcal{K}}{2}\int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\,(\alpha+\alpha^*+\beta+\beta^*)^2$$
$$\times\exp[-A_1\alpha^*\alpha + C_1\alpha + C_1^*\alpha^* - A_2\beta^*\beta$$
$$+C_2^*\beta^* + C_2\beta + D(\alpha\beta+\alpha^*\beta^*)],$$
$$\langle \hat{C}_2^2 \rangle = -1 - \frac{\mathcal{K}}{2}\int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\,(\alpha^*-\alpha+\beta^*-\beta)^2$$
$$\times\exp[-A_1\alpha^*\alpha + C_1\alpha + C_1^*\alpha^* - A_2\beta^*\beta$$
$$+C_2^*\beta^* + C_2\beta + D(\alpha\beta+\alpha^*\beta^*)],$$

so that employing relation (2.44), we have

$$\langle \hat{C}_1 \rangle = \frac{\mathcal{K}}{\sqrt{2}}\left(\frac{\partial}{\partial C_1^*} + \frac{\partial}{\partial C_1} + \frac{\partial}{\partial C_2^*} + \frac{\partial}{\partial C_2}\right)$$
$$\times\int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\exp[-A_1\alpha^*\alpha + C_1\alpha + C_1^*\alpha^* - A_2\beta^*\beta + C_2\beta$$
$$+C_2^*\beta^* + D(\alpha\beta+\alpha^*\beta^*)],$$
$$\langle \hat{C}_2 \rangle = \frac{i\mathcal{K}}{\sqrt{2}}\left(\frac{\partial}{\partial C_1^*} - \frac{\partial}{\partial C_1} + \frac{\partial}{\partial C_2^*} - \frac{\partial}{\partial C_2}\right)$$
$$\times\int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\exp[-A_1\alpha^*\alpha + C_1\alpha + C_1^*\alpha^* - A_2\beta^*\beta + C_2\beta$$
$$+C_2^*\beta^* + D(\alpha\beta+\alpha^*\beta^*)],$$
$$\langle \hat{C}_1^2 \rangle = -1 + \frac{\mathcal{K}}{2}\left(\frac{\partial}{\partial C_1^*} + \frac{\partial}{\partial C_1} + \frac{\partial}{\partial C_2^*} + \frac{\partial}{\partial C_2}\right)^2$$
$$\times\int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\exp[-A_1\alpha^*\alpha + C_1\alpha + C_1^*\alpha^* - A_2\beta^*\beta$$
$$+C_2\beta + C_2^*\beta^* + D(\alpha\beta+\alpha^*\beta^*)],$$
$$\langle \hat{C}_2^2 \rangle = -1 - \frac{\mathcal{K}}{2}\left(\frac{\partial}{\partial C_1^*} - \frac{\partial}{\partial C_1} + \frac{\partial}{\partial C_2^*} - \frac{\partial}{\partial C_2}\right)^2$$
$$\times\int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\exp[-A_1\alpha^*\alpha + C_1\alpha + C_1^*\alpha^* - A_2\beta^*\beta$$
$$+C_2\beta + C_2^*\beta^* + D(\alpha\beta+\alpha^*\beta^*)].$$

Now upon carrying out the integration applying (2.34), we get

$$\langle \hat{C}_1 \rangle = \frac{\mathcal{K}}{\sqrt{2}\,(A_1A_2-D^2)}\left\{\left(\frac{\partial}{\partial C_1^*} + \frac{\partial}{\partial C_1} + \frac{\partial}{\partial C_2^*} + \frac{\partial}{\partial C_2}\right)\right.$$
$$\left.\times\exp\left[\frac{A_1C_2C_2^* + A_2C_1C_1^* + D(C_1C_2 + C_1^*C_2^*)}{A_1A_2-D^2}\right]\right\},$$

$$\langle \hat{C}_2 \rangle = \frac{i\mathcal{K}}{\sqrt{2}\,(A_1A_2 - D^2)} \Bigg\{ \left(\frac{\partial}{\partial C_1^*} - \frac{\partial}{\partial C_1} + \frac{\partial}{\partial C_2^*} - \frac{\partial}{\partial C_2} \right) \\ \times \exp\left[\frac{A_1C_2C_2^* + A_2C_1C_1^* + D(C_1C_2 + C_1^*C_2^*)}{A_1A_2 - D^2} \right] \Bigg\},$$

$$\langle \hat{C}_1^2 \rangle = -1 + \frac{\mathcal{K}}{2\,(A_1A_2 - D^2)} \Bigg\{ \left(\frac{\partial}{\partial C_1^*} + \frac{\partial}{\partial C_1} + \frac{\partial}{\partial C_2^*} + \frac{\partial}{\partial C_2} \right)^2 \\ \times \exp\left[\frac{A_1C_2C_2^* + A_2C_1C_1^* + D(C_1C_2 + C_1^*C_2^*)}{A_1A_2 - D^2} \right] \Bigg\},$$

$$\langle \hat{C}_2^2 \rangle = -1 - \frac{\mathcal{K}}{2\,(A_1A_2 - D^2)} \Bigg\{ \left(\frac{\partial}{\partial C_1^*} - \frac{\partial}{\partial C_1} + \frac{\partial}{\partial C_2^*} - \frac{\partial}{\partial C_2} \right)^2 \\ \times \exp\left[\frac{A_1C_2C_2^* + A_2C_1C_1^* + D(C_1C_2 + C_1^*C_2^*)}{A_1A_2 - D^2} \right] \Bigg\},$$

so that upon differentiation, we have

$$\langle \hat{C}_1 \rangle = \frac{1}{\sqrt{2}} \left(\frac{(A_1 + D)(C_2 + C_2^*) + (A_2 + D)(C_1 + C_1^*)}{A_1A_2 - D^2} \right), \tag{3.37a}$$

$$\langle \hat{C}_2 \rangle = \frac{i}{\sqrt{2}} \left(\frac{(A_1 - D)(C_2 - C_2^*) + (A_2 - D)(C_1 - C_1^*)}{A_1A_2 - D^2} \right), \tag{3.37b}$$

$$\langle \hat{C}_1^2 \rangle = -1 + \langle \hat{C}_1 \rangle^2 + \frac{A_1 + A_2 + 2D}{A_1A_2 - D^2}, \tag{3.37c}$$

$$\langle \hat{C}_2^2 \rangle = -1 + \langle \hat{C}_2 \rangle^2 + \frac{A_1 + A_2 - 2D}{A_1A_2 - D^2}, \tag{3.37d}$$

from which follows

$$\Delta \hat{C}_1^2 = \frac{A_1 + A_2 + 2D}{A_1A_2 - D^2} - 1, \tag{3.38a}$$

$$\Delta \hat{C}_2^2 = \frac{A_1 + A_2 - 2D}{A_1A_2 - D^2} - 1. \tag{3.38b}$$

On substituting the explicit values of $A_1(t), A_2(t)$and$D(t)$ from (3.33) into (3.38), we readily get

$$\Delta\hat{C}_1^2 = \frac{1}{2}(1+2\overline{n}_a)\left(1-e^{-\gamma_a t}\right) + \frac{1}{2}(1+2\overline{n}_b)\left(1-e^{-\gamma_b t}\right)$$
$$+\frac{1}{2}\left(e^{-\gamma_a t}+e^{-\gamma_b t}\right)\cosh(2r) - e^{-\frac{1}{2}(\gamma_a+\gamma_b)t}\sinh(2r), \tag{3.39a}$$
$$\Delta\hat{C}_2^2 = \frac{1}{2}(1+2\overline{n}_a)\left(1-e^{-\gamma_a t}\right) + \frac{1}{2}(1+2\overline{n}_b)\left(1-e^{-\gamma_b t}\right)$$
$$+\frac{1}{2}\left(e^{-\gamma_a t}+e^{-\gamma_b t}\right)\cosh(2r) + e^{-\frac{1}{2}(\gamma_a+\gamma_b)t}\sinh(2r). \tag{3.39b}$$

A two-mode light is said to be in a squeezed state if either

$$\Delta\hat{C}_1 \leq 1 \leq \Delta\hat{C}_2$$

or

$$\Delta\hat{C}_1 \geq 1 \geq \Delta\hat{C}_2$$

without violating the uncertainty principle

$$\Delta\hat{C}_1\Delta\hat{C}_2 \geq 1.$$

A light mode with

$$\Delta\hat{C}_1\Delta\hat{C}_2 = 1$$

is said to be in a minimum uncertainty state. From (3.39), we see that

$$\Delta\hat{C}_2 \geq 1, \text{ for } t \geq 0$$

and for $\gamma_a = \gamma_b = \gamma$,

$$\Delta\hat{C}_1 \begin{cases} \leq 1, & \text{for } 0 \leq t < \frac{1}{\gamma}\ln\left[1+\frac{1-e^{-2r}}{\overline{n}_a+\overline{n}_b}\right] \\ > 1, & \text{otherwise} \end{cases} .$$

Therefore, the light mode under consideration is in a squeezed state for the time interval

$$0 \leq t < \frac{1}{\gamma}\ln\left[1+\frac{1-e^{-2r}}{\overline{n}_a+\overline{n}_b}\right].$$

Next we consider some special cases of the result described by (3.39).

A. At the initial time ($t=0$), expression (3.39) reduces to

$$\Delta\hat{C}_1^2 = e^{-2r},$$

$$\Delta\hat{C}_2^2 = e^{2r},$$

which is the well known result for the quadrature variances of a two-mode DSV.

B. At steady state, we get

$$\Delta\hat{C}_1^2 = \Delta\hat{C}_2^2 = \bar{n}_a + \bar{n}_b + 1,$$

which indicates that the superposition of a two-mode coherent light and a two-mode thermal light is not in a squeezed state.

3.3. The Mean and Variance of the Photon Number

Here we wish to determine the mean and variance of the photon number for a two-mode DSV driven by a two-mode coherent light and damped by a two-mode thermal reservoir.

a. The Mean Photon Number

The mean photon number for a two-mode light is defined by

$$\langle\hat{n}\rangle = \langle\hat{n}_a\rangle + \langle\hat{n}_b\rangle, \tag{3.40}$$

where

$$\hat{n}_a = \hat{a}^\dagger(t)\hat{a}(t) \tag{3.41a}$$

and

$$\hat{n}_b = \hat{b}^\dagger(t)\hat{b}(t) \tag{3.41b}$$

are the photon number operators for mode a and mode b. Taking into account (3.36), one can write the mean photon number of mode a in terms of the Q-function (3.33) in the form

$$\begin{aligned}\langle\hat{n}_a\rangle = -1 + \mathcal{K}(t)\int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}(\alpha\alpha^*)\\ \times\exp[-A_1\alpha^*\alpha + C_1\alpha + C_1^*\alpha^* - A_2\beta^*\beta\\ + C_2\beta + C_2^*\beta^* + D(\alpha\beta + \alpha^*\beta^*)]\end{aligned} \tag{3.42a}$$

and employing (2.44), one can express this as

$$\langle \hat{n}_a \rangle = -1 + \mathcal{K}(t) \frac{\partial^2}{\partial C_1^* \partial C_1} \int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi}$$
$$\times \exp[-A_1 \alpha^* \alpha + C_1 \alpha + C_1^* \alpha^* - A_2 \beta^* \beta$$
$$+ C_2 \beta + C_2^* \beta^* + D(t)(\alpha\beta + \alpha^* \beta^*)]. \qquad (3.42b)$$

Upon performing the integration applying (2.34), we have

$$\langle \hat{n}_a \rangle = -1 + \frac{\mathcal{K}}{(A_1 A_2 - D^2)} \frac{\partial^2}{\partial C_1^* \partial C_1} \exp\left[\frac{A_1 C_2 C_2^* + A_2 C_1 C_1^* + D\left(C_1 C_2 + C_1^* C_2^*\right)}{A_1 A_2 - D^2} \right],$$

so that on carrying out the differentiation, there follows

$$\langle \hat{n}_a \rangle = \left| \frac{A_2 C_1^* + D C_2}{A_1 A_2 - D^2} \right|^2 + \frac{A_2}{A_1 A_2 - D^2} - 1. \qquad (3.43)$$

Following a similar procedure, one can also show that

$$\langle \hat{n}_b \rangle = \left| \frac{A_1 C_2^* + D C_1}{A_1 A_2 - D^2} \right|^2 + \frac{A_1}{A_1 A_2 - D^2} - 1. \qquad (3.44)$$

Taking into account (3.34), one easily obtains

$$\langle \hat{n}_a \rangle = \frac{4\varepsilon_a^2}{\gamma_a^2} \left(1 - e^{-\frac{\gamma_a}{2}t}\right)^2 + \frac{2\varepsilon_a}{\gamma_a} (\lambda^* + \lambda) e^{-\frac{\gamma_a}{2}t} \left(1 - e^{-\frac{\gamma_a}{2}t}\right)$$
$$+ \bar{n}_a \left(1 - e^{-\gamma_a t}\right) + \left(|\lambda|^2 + \sinh^2(r)\right) e^{-\gamma_a t} \qquad (3.45)$$

and

$$\langle \hat{n}_b \rangle = \frac{4\varepsilon_b^2}{\gamma_b^2} \left(1 - e^{-\frac{\gamma_b}{2}t}\right)^2 + \frac{2\varepsilon_b}{\gamma_b} (\eta^* + \eta) e^{-\frac{\gamma_b}{2}t} \left(1 - e^{-\frac{\gamma_b}{2}t}\right)$$
$$+ \bar{n}_b \left(1 - e^{-\gamma_b t}\right) + \left(|\eta|^2 + \sinh^2(r)\right) e^{-\gamma_b t}. \qquad (3.46)$$

Substituting (3.45) and (3.46) into (3.40) the mean photon number becomes

$$\langle \hat{n} \rangle = \frac{4\varepsilon_a^2}{\gamma_a^2} \left(1 - e^{-\frac{\gamma_a}{2}t}\right)^2 + \frac{2\varepsilon_a}{\gamma_a} (\lambda^* + \lambda) e^{-\frac{\gamma_a}{2}t} \left(1 - e^{-\frac{\gamma_a}{2}t}\right)$$
$$+ \frac{4\varepsilon_b^2}{\gamma_b^2} \left(1 - e^{-\frac{\gamma_b}{2}t}\right)^2 + \frac{2\varepsilon_b}{\gamma_b} (\eta^* + \eta) e^{-\frac{\gamma_b}{2}t} \left(1 - e^{-\frac{\gamma_b}{2}t}\right)$$
$$+ \bar{n}_a \left(1 - e^{-\gamma_a t}\right) + \left(|\lambda|^2 + \sinh^2(r)\right) e^{-\gamma_a t}$$
$$+ \bar{n}_b \left(1 - e^{-\gamma_b t}\right) + \left(|\eta|^2 + \sinh^2(r)\right) e^{-\gamma_b t}. \qquad (3.47)$$

b. The Variance of the Photon Number

The variance of the photon number for a two-mode light is expressible as

$$\Delta\hat{n}^2 = \Delta\hat{n}_a^2 + \Delta\hat{n}_b^2 + 2\left(\langle\hat{n}_a\hat{n}_b\rangle - \langle\hat{n}_a\rangle\langle\hat{n}_b\rangle\right). \tag{3.48}$$

We now proceed to calculate the variances of the photon number for mode a and mode b. One can write $\langle\hat{n}_a^2\rangle$ in terms of $\hat{a}$ and $\hat{a}^\dagger$ as

$$\langle\hat{n}_a^2\rangle = 1 + \langle\hat{a}^2\hat{a}^{\dagger 2}\rangle - 3\langle\hat{a}\hat{a}^\dagger\rangle \tag{3.49}$$

and employing the Q-function (3.33), one can write this as

$$\langle\hat{n}_a^2\rangle = 1 + \mathcal{K}(t)\int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\left(\alpha^2\alpha^{*2} - 3\alpha\alpha^*\right)$$
$$\times\exp[-A_1\alpha^*\alpha + C_1\alpha + C_1^*\alpha^* - A_2\beta^*\beta$$
$$+C_2\beta + C_2^*\beta^* + D(\alpha\beta + \alpha^*\beta^*)],$$

so that applying (2.44) and upon carrying out the integration, we get

$$\langle\hat{n}_a^2\rangle = 1 + \frac{\mathcal{K}(t)}{(A_1A_2 - D^2)}\left(\frac{\partial^2}{\partial C_1^2}\frac{\partial^2}{\partial C_1^{*2}} - 3\frac{\partial}{\partial C_1}\frac{\partial}{\partial C_1^*}\right)$$
$$\times\exp\left[\frac{A_1C_2C_2^* + A_2C_1C_1^* + D\left(C_1C_2 + C_1^*C_2^*\right)}{A_1A_2 - D^2}\right]. \tag{3.50}$$

Upon performing the integration and taking into account (3.43), one readily obtains

$$\langle\hat{n}_a^2\rangle = -1 + \langle\hat{n}_a\rangle^2 - \langle\hat{n}_a\rangle + 2\langle\hat{n}_a\rangle\frac{A_2}{A_1A_2 - D^2} + 2\frac{A_2}{A_1A_2 - D^2} - \left(\frac{A_2}{A_1A_2 - D^2}\right)^2.$$

It then follows that

$$\Delta\hat{n}_a^2 = -1 - \langle\hat{n}_a\rangle\left(1 - 2\frac{A_2}{A_1A_2 - D^2}\right) + 2\frac{A_2}{A_1A_2 - D^2} - \left(\frac{A_2}{A_1A_2 - D^2}\right)^2. \tag{3.51}$$

Taking into account (3.34), one easily finds

$$\Delta\hat{n}_a^2 = \langle\hat{n}_a\rangle\left[2\overline{n}_a\left(1 - e^{-\gamma_a t}\right) + 2e^{-\gamma_a t}\sinh^2 r + 1\right] - \left[\overline{n}_a\left(1 - e^{-\gamma_a t}\right) + e^{-\gamma_a t}\sinh^2 r\right]^2. \tag{3.52}$$

Following a similar procedure, we obtain

$$\Delta\hat{n}_b^2 = \langle\hat{n}_b\rangle\left[2\overline{n}_b\left(1 - e^{-\gamma_b t}\right) + 2e^{-\gamma_b t}\sinh^2 r + 1\right] - \left[\overline{n}_b\left(1 - e^{-\gamma_b t}\right) + e^{-\gamma_b t}\sinh^2 r\right]^2 \tag{3.53}$$

and

$$\langle\hat{n}_a\hat{n}_b\rangle = \langle\hat{n}_a\rangle\langle\hat{n}_b\rangle - \left[e^{-\frac{1}{2}(\gamma_a+\gamma_b)t}\cosh(r)\sinh(r)\right]$$

$$\times\left[-e^{-\frac{1}{2}(\gamma_a+\gamma_b)t}\cosh(r)\sinh(r) + \frac{8\varepsilon_a\varepsilon_b}{\gamma_a\gamma_b}\left(1-e^{-\frac{\gamma_a}{2}t}\right)\left(1-e^{-\frac{\gamma_b}{2}t}\right)\right.$$

$$+(\lambda^*\eta^*+\lambda\eta)e^{-\frac{1}{2}(\gamma_a+\gamma_b)t} + \frac{2\varepsilon_a}{\gamma_a}(\eta^*+\eta)e^{-\frac{\gamma_b}{2}t}\left(1-e^{-\frac{\gamma_a}{2}t}\right)$$

$$\left.+\frac{2\varepsilon_b}{\gamma_b}(\lambda^*+\lambda)e^{-\frac{\gamma_a}{2}t}\left(1-e^{-\frac{\gamma_b}{2}t}\right)\right]. \tag{3.54}$$

On account of (3.51), (3.52), (3.53) and (3.54), the variance of the photon number (3.48) takes the form

$$\Delta\hat{n}^2 = \Delta\hat{n}_a^2 + \Delta\hat{n}_b^2 - 2\left[\left(e^{-\frac{1}{2}(\gamma_a+\gamma_b)t}\cosh(r)\sinh(r)\right)\right]$$

$$\times\left[-e^{-\frac{1}{2}(\gamma_a+\gamma_b)t}\cosh(r)\sinh(r) + \frac{8\varepsilon_a\varepsilon_b}{\gamma_a\gamma_b}\left(1-e^{-\frac{\gamma_a}{2}t}\right)\left(1-e^{-\frac{\gamma_b}{2}t}\right)\right.$$

$$+(\lambda^*\eta^*+\lambda\eta)e^{-\frac{1}{2}(\gamma_a+\gamma_b)t} + \frac{2\varepsilon_a}{\gamma_a}(\eta^*+\eta)e^{-\frac{\gamma_b}{2}t}\left(1-e^{-\frac{\gamma_a}{2}t}\right)$$

$$\left.+\frac{2\varepsilon_b}{\gamma_b}(\lambda^*+\lambda)e^{-\frac{\gamma_a}{2}t}\left(1-e^{-\frac{\gamma_b}{2}t}\right)\right]. \tag{3.55}$$

Now we consider special cases of the result described by (3.55).

A. At $t=0$, we see that

$$\Delta\hat{n}_a^2 = \left(|\lambda|^2+\sinh^2 r\right)\cosh(2r) - \sinh^4 r, \tag{3.56}$$

$$\Delta\hat{n}_b^2 = \left(|\eta|^2+\sinh^2 r\right)\cosh(2r) - \sinh^4 r, \tag{3.57}$$

and hence

$$\Delta\hat{n}^2 = \Delta\hat{n}_a^2 + \Delta\hat{n}_b^2$$

$$-2\frac{\cosh(r)\sinh(r)}{\langle\hat{n}_a\rangle\langle\hat{n}_b\rangle}\left[\lambda^*\eta^*+\lambda\eta-\cosh(r)\sinh(r)\right]. \tag{3.58}$$

This represents the variance of the photon number for two-mode DSV.

When $\lambda=\eta=0$ in (3.56-58), we have

$$\Delta\hat{n}_a^2 = \sinh^4 r + \sinh^2 r = \langle\hat{n}_a\rangle^2 + \langle\hat{n}_a\rangle,$$

$$\Delta\hat{n}_b^2 = \sinh^4 r + \sinh^2 r = \langle\hat{n}_b\rangle^2 + \langle\hat{n}_b\rangle$$

and

$$\Delta\hat{n}^2 = \langle\hat{n}\rangle + \langle\hat{n}_a\rangle^2 + \langle\hat{n}_b\rangle^2 + 2\frac{\cosh^2 r}{\sinh^2 r}.$$

This is the variance of the photon number for a two-mode SV.

Upon setting $r = 0$ in (3.56-58), we obtain the result which holds for coherent light

$$\Delta\hat{n}_a^2 = |\lambda|^2 = \langle\hat{n}_a\rangle,$$

$$\Delta\hat{n}_b^2 = |\eta|^2 = \langle\hat{n}_b\rangle$$

and

$$\Delta\hat{n}^2 = \langle\hat{n}\rangle.$$

B. At steady state, we note that

$$\Delta\hat{n}_a^2 = \overline{n}_a^2 + \overline{n}_a + \frac{8\varepsilon_a^2}{\gamma_a^2}\overline{n}_a + \frac{4\varepsilon_a^2}{\gamma_a^2} = \langle\hat{n}_a\rangle + \overline{n}_a^2 + \frac{8\varepsilon_a^2}{\gamma_a^2}\overline{n}_a, \tag{3.59}$$

$$\Delta\hat{n}_b^2 = \overline{n}_b^2 + \overline{n}_b + \frac{8\varepsilon_b^2}{\gamma_b^2}\overline{n}_b + \frac{4\varepsilon_b^2}{\gamma_b^2} = \langle\hat{n}_b\rangle + \overline{n}_b^2 + \frac{8\varepsilon_b^2}{\gamma_b^2}\overline{n}_b \tag{3.60}$$

and

$$\Delta\hat{n}^2 = \langle\hat{n}\rangle + \overline{n}_a^2 + \overline{n}_b^2 + \frac{8\varepsilon_a^2}{\gamma_a^2}\overline{n}_a + \frac{8\varepsilon_b^2}{\gamma_b^2}\overline{n}_b. \tag{3.61}$$

This shows the photon statistics for the superposition of two-mode coherent and thermal light is super-Poissonian.

When $\varepsilon_a = \varepsilon_b = 0$ in (3.59-61), we get

$$\Delta\hat{n}_a^2 = \overline{n}_a^2 + \overline{n}_a,$$

$$\Delta\hat{n}_b^2 = \overline{n}_b^2 + \overline{n}_b$$

and

$$\Delta\hat{n}^2 = \overline{n}_a^2 + \overline{n}_b^2 + \langle\hat{n}\rangle,$$

in which

$$\langle\hat{n}\rangle = \overline{n}_a + \overline{n}_b.$$

This represents the variance of the photon number for two-mode thermal light.

Again when $\overline{n}_a = \overline{n}_b = 0$ in (3.59-61), we see hat

$$\Delta\hat{n}_a^2 = \frac{4\varepsilon_a^2}{\gamma_a^2} = \langle\hat{n}_a\rangle,$$

$$\Delta\hat{n}_b^2 = \frac{4\varepsilon_b^2}{\gamma_b^2} = \langle\hat{n}_b\rangle$$

and

$$\Delta\hat{n}^2 = \frac{4\varepsilon_a^2}{\gamma_a^2} + \frac{4\varepsilon_b^2}{\gamma_b^2} = \langle\hat{n}\rangle.$$

This is the variance of the photon number for a two-mode coherent light.

3.4. The Photon Number Distribution

We seek here to obtain the joint probability of observing m photons of mode a and n photons of mode b. The photon number distribution for a two-mode light is expressible in terms of the Q-function as [17]

$$P(m,n,t) = \frac{\pi^2}{n!m!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\frac{\partial^{2m}}{\partial\beta^{*m}\partial\beta^m}\left(Q(\alpha^*,\alpha,\beta^*,\beta,t)e^{\alpha^*\alpha+\beta^*\beta}\right)\Big|_{\alpha^*=\alpha=\beta^*=\beta=0},$$

so that taking into account (3.33), we obtain

$$\begin{aligned}P(m,n,t) = \frac{\mathcal{K}(t)}{n!m!}&\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\frac{\partial^{2m}}{\partial\beta^{*m}\partial\beta^m}\\ &\times\exp\left[P_1\alpha^*\alpha + C_1^*\alpha^* + C_1\alpha + P_2\beta^*\beta + C_2^*\beta^* + C_2\beta\right.\\ &\left.+D\left(\alpha^*\beta^* + \alpha^*\beta^*\right)\right]\Big|_{\alpha^*=\alpha=\beta^*=\beta=0},\end{aligned} \tag{3.62}$$

in which

$$P_1(t) = 1 - A_1(t), \tag{3.63}$$

$$P_2(t) = 1 - A_2(t). \tag{3.64}$$

Upon expanding the exponential function in (3.62) in power series, we have

$$\begin{aligned}P(m,n,t) = \frac{\mathcal{K}(t)}{n!m!}\sum_{s,p,t,q,k,l,i,j}&\frac{P_1^sC_1^pC_1^{*t}P_2^qC_2^kC_2^{*l}D^{i+j}}{s!p!t!q!k!l!i!j!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\frac{\partial^{2m}}{\partial\beta^{*m}\partial\beta^m}\\ &\times\left(\alpha^{*s+t+j}\alpha^{s+p+i}\beta^{*q+l+j}\beta^{q+k+i}\right)\Big|_{\alpha^*=\alpha=\beta^*=\beta=0}\end{aligned}$$

and on performing the differentiation using relation (2.57), one gets

$$P(m,n,t) = \frac{\mathcal{K}(t)}{n!m!} \sum_{s,p,t,q,k,l,i,j} \frac{P_1^s C_1^p C_1^{*t} P_2^q C_2^k C_2^{*l} D^{i+j}}{s!p!t!q!k!l!i!j!} \frac{(s+t+j)!}{(s+t+j-n)!} \frac{(s+p+i)!}{(s+p+i-n)!}$$

$$\times \frac{(q+k+i)!}{(q+k+i-m)!} \frac{(q+l+j)!}{(q+l+j-m)!}$$

$$\times \left(\alpha^{*s+t+j-n} \alpha^{s+p+i-n} \beta^{*q+l+j-m} \beta^{q+k+i-m} \right) \Big|_{\alpha^*=\alpha=\beta^*=\beta=0} .$$

Now application of the condition $\alpha^* = \alpha = \beta^* = \beta = 0$ leads to

$$P(m,n,t) = \frac{\mathcal{K}(t)}{n!m!} \sum_{s,p,t,q,k,l,i,j} \frac{P_1^s C_1^p C_1^{*t} P_2^q C_2^k C_2^{*l} D^{i+j}}{s!p!t!q!k!l!i!j!} \frac{(s+t+j)!}{(s+t+j-n)!} \frac{s+p+i)!}{(s+p+i-n)!}$$

$$\times \frac{(q+k+i)!}{(q+k+i-m)!} \frac{(q+l+j)!}{(q+l+j-m)!} \delta_{s+p+i,n} \delta_{s+t+j,n}$$

$$\times \delta_{q+k+i,m} \delta_{q+l+j,m} ,$$

so that on using the conditions

$$s+p+i-n=0,$$

$$s+t+j-n=0$$

$$q+k+i-m=0,$$

$$q+l+j-m=0,$$

we have

$$p = n-s-i,$$

$$t = n-s-j,$$

$$k = m-q-i,$$

$$l = m-q-j.$$

Taking into account this results, we get

$$P(m,n,t) = \frac{\mathcal{K}(t)}{n!m!} \sum_{s=0}^{n} \sum_{q=0}^{m} \sum_{i=0}^{[r]} \sum_{j=0}^{[r]} \frac{P_1^s C_1^{n-s-i} C_1^{*n-s-j} P_2^q C_2^{m-q-i}}{s!q!i!j!(n-s-i)!(n-s-j)!}$$

$$\times \frac{C_2^{*m-q-j} D^{i+j} (m!)^2 (n!)^2}{(m-q-i)!(m-q-j)!}, \tag{3.65}$$

where

$$[r] = min[m-q, n-s].$$

This expression represents the photon number distribution for a two-mode DSV driven by a two-mode coherent light and damped by a two-mode thermal light.

Next we wish to consider special cases of the result described by (3.65).

A. At $t=0$, we see from (3.34) that

$$A_1(0) = A_2(0) = 1, \tag{3.66a}$$

$$C_1(0) = \lambda^* + \eta \tanh(r), \tag{3.66b}$$

$$C_2(0) = \eta^* + \lambda \tanh(r), \tag{3.66c}$$

$$D(0) = -\tanh(r), \tag{3.66d}$$

$$\mathcal{K}(0) = \mathrm{sech}^2(r) \exp\left[-\lambda^*\lambda - \eta^*\eta - (\lambda^*\eta^* + \lambda\eta)\tanh(r)\right]. \tag{3.66e}$$

Hence using the result $P_1(0) = P_2(0) = 0$, one can rewrite (3.65) in the form

$$P(m,n,0) = \frac{\mathcal{K}(t)}{n!m!} \sum_{s=0}^{n} \sum_{q=0}^{m} \sum_{i=0}^{min[m-q,n-s]} \sum_{j=0}^{min[m-q,n-s]} \frac{C_1^{n-s-i} C_1^{*n-s-j} C_2^{m-q-i}}{s!q!i!j!(n-s-i)!(n-s-j)!} \delta_s$$
$$\times \frac{C_2^{*m-q-j} D^{i+j} (m!)^2 (n!)^2}{(m-q-i)!(m-q-j)!} \delta_q$$

and in view of the fact that $s = q = 0$, we have

$$P(m,n,0) = \frac{\mathcal{K}(0)}{n!m!} \sum_{i=0}^{min[m,n]} \sum_{j=0}^{min[m,n]} \frac{C_1^{n-i} C_1^{*n-j} C_2^{m-i} C_2^{*m-j} D^{i+j} (m!)^2 (n!)^2}{i!j!(n-i)!(n-j)!(m-i)!(m-j)!}. \tag{3.67}$$

Now setting

$$p = min[m,n]$$

and

$$q = max[m,n],$$

one can rewrite (3.67) as

$$P(m,n,0) = \frac{p!}{q!} \mathcal{K}(0) \mid C_1 \mid^{2n} \mid C_2 \mid^{2m} \sum_{i=0}^{p} \frac{(-1)^i \left(-\frac{D}{C_1 C_2}\right)^i q!}{i!(p-i)!(q-i)!} \sum_{j=0}^{p} \frac{(-1)^j \left(-\frac{D}{C_1^* C_2^*}\right)^j q!}{j!(p-j)!(q-j)!}.$$

Upon letting

$$k = p - i$$

and

$$l = p - j,$$

this can be rewritten as

$$P(m,n,0) = \frac{p!}{q!}\mathcal{K}(0)D^{2p}(0) \mid C_1 \mid^{2n-2p}\mid C_2 \mid^{2m-2p}$$
$$\times \sum_{k=0}^{p} \frac{(-1)^k (p+(q-p))!}{k!(p-k)!(q-p+k)!}\left(-\frac{C_1C_2}{D}\right)^k$$
$$\times \sum_{l=0}^{p} \frac{(-1)^l (p+(q-p))!}{l!(p-l)!(q-p+l)!}\left(-\frac{C_1^*C_2^*}{D}\right)^l,$$

so that on applying the general Laguere polynomial defined by

$$\mathcal{L}_n^{\alpha}(x) = \sum_{i=0}^{n} \frac{(-1)^i (n+\alpha)!}{i!(n-i)!(\alpha+i)!}x^i,$$

we have

$$P(m,n,0) = \frac{p!}{q!}\mathcal{K}(0)D^{2p}(0) \mid C_1 \mid^{2n-2p}\mid C_2 \mid^{2m-2p}\left|\mathcal{L}_p^{q-p}\left(-\frac{C_1C_2}{D}\right)\right|^2. \qquad (3.68a)$$

Thus on substituting (3.66) into (3.68a), we get

$$P(m,n,0) = \frac{p!}{q!}\frac{\tanh^{2p}(r)}{\cosh^2(r)} \mid \lambda + \eta^* \tanh(r) \mid^{2n-2p}\mid \eta + \lambda^* \tanh(r) \mid^{2m-2p}$$
$$\times \exp\left[-\lambda^*\lambda - \eta^*\eta - (\lambda\eta + \lambda^*\eta^*)\tanh(r)\right]$$
$$\times \left|\mathcal{L}_p^{q-p}\left(\lambda^*\lambda + \eta^*\eta + \lambda\eta\coth(r) + \lambda^*\eta^*\tanh(r)\right)\right|^2, \qquad (3.68b)$$

which is the well-known photon number distribution for a two-mode DSV [27].

B. At steady state, we have from (3.34) that

$$A_1(\infty) = 1/(\overline{n}_a + 1), \qquad (3.69a)$$

$$A_2(\infty) = 1/(\overline{n}_b + 1), \qquad (3.69b)$$

$$D(\infty) = 0, \qquad (3.69c)$$

$$C_1(\infty) = 2\varepsilon_a/\gamma_a(\overline{n}_a + 1), \qquad (3.69d)$$

$$C_2(\infty) = 2\varepsilon_b/\gamma_b(\overline{n}_b+1), \tag{3.69e}$$

$$\mathcal{K}(\infty) = \tfrac{1}{(\overline{n}_a+1))(\overline{n}_b+1)} \exp\left[-\tfrac{4\varepsilon_b^2}{\gamma_b^2(\overline{n}_b+1)} - \tfrac{4\varepsilon_a^2}{\gamma_a^2(\overline{n}_a+1)}\right] \tag{3.69f}$$

and (3.65) takes the form

$$P(m,n) = \frac{\mathcal{K}(t)}{n!m!}\sum_{s=0}^{n}\sum_{q=0}^{m}\sum_{i=0}^{min[m-q,n-s]}\sum_{j=0}^{min[m-q,n-s]}\frac{P_1^s C_1^{n-s-i} C_1^{*n-s-j} P_2^q C_2^{m-q-i}}{s!q!i!j!(n-s-i)!(n-s-j)!}$$

$$\times\frac{C_2^{*m-q-j}(m!)^2(n!)^2\delta_{i,-j}}{(m-q-i)!(m-q-j)!},$$

so that taking into account $i=-j=0$, we get

$$P(m,n) = \frac{\mathcal{K}(\infty)}{n!m!}\sum_{s=0}^{n}\sum_{q=0}^{m}\frac{P_1^s |C_1|^{2n-2s} P_2^q |C_2|^{2m-2q}(m!)^2(n!)^2}{s!q!\left((n-s)!\right)^2\left((m-q)!\right)^2}. \tag{3.70}$$

Upon setting

$$n-s=k$$

and

$$m-q=l,$$

one can rewrite expression (3.70) as

$$P(m,n) = \mathcal{K}(\infty)\sum_{k=0}^{n}\frac{P_1^{n-k}|C_1|^{2k}}{(n-k)!(k!)^2}n!\sum_{l=0}^{m}\frac{P_2^{m-l}|C_2|^{2l}}{(m-l)!(l!)^2}m!. \tag{3.71}$$

Taking into account the Laguere polynomial (2.62), we readily get

$$P(m,n) = \mathcal{K}P_1^n P_2^m \mathcal{L}_n\left(\frac{|C_1|^2}{P_1}\right)\mathcal{L}_m\left(\frac{|C_2|^2}{P_2}\right) \tag{3.72a}$$

and substitution of (3.69) gives

$$P(m,n) = \frac{\overline{n}_a^n}{(\overline{n}_a+1)^{n+1}}\frac{\overline{n}_b^m}{(\overline{n}_b+1)^{m+1}}\exp\left[-\frac{4\varepsilon_a^2}{\gamma_a^2(\overline{n}_a+1)} - \frac{4\varepsilon_b^2}{\gamma_b^2(\overline{n}_b+1)}\right]$$

$$\times\mathcal{L}_n\left(-\frac{4\varepsilon_a^2}{\gamma_a^2\overline{n}_a(\overline{n}_a+1)}\right)\mathcal{L}_m\left(-\frac{4\varepsilon_b^2}{\gamma_b^2\overline{n}_b(\overline{n}_b+1)}\right) \tag{3.72b}$$

or

$$P(m,n) = P(m)P(n), \tag{3.72c}$$

where

$$P(n) = \frac{\overline{n}_a^n}{(\overline{n}_a+1)^{n+1}}\exp\left[\frac{-4\varepsilon_a^2}{\gamma_a^2(\overline{n}_a+1)}\right]\mathcal{L}_n\left(\frac{-4\varepsilon_a^2}{\gamma_a^2\overline{n}_a(\overline{n}_a+1)}\right), \tag{3.73}$$

represents the probability of observing n photons of the superposition of mode a thermal and coherent light and

$$P(m) = \frac{\overline{n}_b^m}{(\overline{n}_b + 1)^{m+1}} \exp\left[\frac{-4\varepsilon_b^2}{\gamma_b^2(\overline{n}_b + 1)}\right] \mathcal{L}_m\left(\frac{-4\varepsilon_b^2}{\gamma_b^2\overline{n}_b(\overline{n}_b + 1)}\right) \tag{3.74}$$

is the probability of observing m photons of the superposition of mode b thermal and coherent light. Hence at steady state the photon number distribution turns out to be the product of the photon number distributions corresponding to two superpositions of single-mode coherent and thermal light.

If $\overline{n}_a = \overline{n}_b = 0$, we see that $P_1(\infty) = P_2(\infty) = 0$ and (3.71) takes the form

$$P(m,n) = \mathcal{K}\sum_{k=0}^{n} \frac{n!\,|C_1|^{2k}}{(n-k)!(k!)^2}\delta_{n,k} \sum_{l=0}^{m} \frac{m!\,|C_2|^{2l}}{(m-l)!(l!)^2}\delta_{m,l}.$$

Applying the result

$$k = n$$

and

$$l = m,$$

we get

$$P(m,n) = \mathcal{K}\frac{|C_1|^{2n}}{n!}\frac{|C_2|^{2m}}{m!},$$

the explicit form of which is

$$P(m,n) = \frac{(4\varepsilon_a^2/\gamma_a^2)^n}{n!}\frac{(4\varepsilon_b^2/\gamma_b^2)^m}{m!} \exp\left[-\frac{4\varepsilon_a^2}{\gamma_a^2} - \frac{4\varepsilon_b^2}{\gamma_b^2}\right]. \tag{3.75}$$

This represents the photon number distribution for a two-mode coherent light and is the product of two Poisson Distribution functions.

Finally, for $\varepsilon_a = \varepsilon_b = 0$, we see that $C_1(\infty) = C_2(\infty) = 0$ and (3.71) turns out to be

$$P(m,n) = \mathcal{K}\sum_{k=0}^{n} \frac{P_1^{n-k}n!}{(n-k)!(k!)^2}\delta_k \sum_{l=0}^{m} \frac{P_2^{m-l}m!}{(m-l)!(l!)^2}\delta_l$$

and in view of the fact that

$$k = l = 0,$$

we obtain

$$P(m,n) = \mathcal{K}P_1^n P_2^m,$$

so that taking into account (3.69) for $\varepsilon_a = \varepsilon_b = 0$, one can easily obtain

$$P(m,n) = \frac{\overline{n}_a^n}{(\overline{n}_a + 1)^{n+1}}\frac{\overline{n}_b^m}{(\overline{n}_b + 1)^{m+1}}. \tag{3.76}$$

This also represents the photon number distribution for a two-mode thermal light.

4. Conclusion

We have obtained the Fokker-Planck equation for the Q-function from the master equation of a single-mode light driven by a single-mode coherent light and damped by a single-mode thermal reservoir. We have solved the resulting Fokker-Planck equation employing the *propagator method* developed by Fesseha [1]. The main task of obtaining the Q-function propagator for a quadratic Hamiltonian of any order applying this method essentially reduces to the problem of determining the classical action.

The Q-function has been used to calculate the quadrature fluctuations, the mean photon number, the variance of the photon number and the photon number distribution for a single-mode DSV driven by a single-mode coherent light and damped by a single-mode thermal reservoir. We have also carried out the same analysis for a two-mode DSV driven by a two-mode coherent light and damped by a two-mode thermal reservoir.

It has been found that at steady state the quadrature variances in both the single and the two modes cases are the products of the quadrature variances associated with thermal and coherent light modes. We have also seen that squeezing occurs for very short times and as time increases the degree of squeezing decreases and eventually the light mode would not be in a squeezed state. Furthermore, we have seen that at steady state the photon statistics in both cases is super-Poissonian.

Finally, we would like to point out that at steady state the photon number distribution in the two-mode case is the product of the photon number distributions corresponding to two superpositions of single-mode coherent and thermal light.

Appendix

We recall that

$$\hat{a}^n \mid \alpha\rangle = \alpha^n \mid \alpha\rangle \tag{A1}$$

and

$$\mid \alpha\rangle = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} \mid 0\rangle.$$

Then employing the Bakker-Hausdorff identity (2.26), we see that

$$\mid \alpha\rangle = e^{-\frac{1}{2}\alpha^*\alpha} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} \mid 0\rangle$$

or

$$\mid \alpha\rangle = e^{-\frac{1}{2}\alpha^*\alpha} e^{\alpha\hat{a}^\dagger} \mid 0\rangle. \tag{A2}$$

Now applying (A2), we note that

$$\begin{aligned}
\hat{a}^{\dagger^n} \mid \alpha\rangle &= e^{-\frac{1}{2}\alpha^*\alpha} e^{\alpha\hat{a}^\dagger} \hat{a}^{\dagger^n} \mid 0\rangle \\
&= e^{-\frac{1}{2}\alpha^*\alpha} \frac{\partial^n}{\partial\alpha^n} e^{\alpha\hat{a}^\dagger} \mid 0\rangle \\
&= \frac{\partial}{\partial\alpha}\left(e^{-\frac{1}{2}\alpha^*\alpha} \frac{\partial^{n-1}}{\partial\alpha^{n-1}} e^{\alpha\hat{a}^\dagger} \mid 0\rangle\right) - \frac{\partial}{\partial\alpha}\left(e^{-\frac{1}{2}\alpha^*\alpha}\right) \frac{\partial^{n-1}}{\partial\alpha^{n-1}} e^{\alpha\hat{a}^\dagger} \mid 0\rangle \\
&= \left(\frac{\partial}{\partial\alpha} + \frac{1}{2}\alpha^*\right) e^{-\frac{1}{2}\alpha^*\alpha} \frac{\partial^{n-1}}{\partial\alpha^{n-1}} e^{\alpha\hat{a}^\dagger} \mid 0\rangle \\
&= \left(\frac{\partial}{\partial\alpha} + \frac{1}{2}\alpha^*\right)^2 e^{-\frac{1}{2}\alpha^*\alpha} \frac{\partial^{n-2}}{\partial\alpha^{n-2}} e^{\alpha\hat{a}^\dagger} \mid 0\rangle \\
&\;\;\vdots \\
&= \left(\frac{\partial}{\partial\alpha} + \frac{1}{2}\alpha^*\right)^k e^{-\frac{1}{2}\alpha^*\alpha} \frac{\partial^{n-k}}{\partial\alpha^{n-k}} e^{\alpha\hat{a}^\dagger} \mid 0\rangle \\
&\;\;\vdots \\
\hat{a}^{\dagger^n} \mid \alpha\rangle &= \left(\frac{\partial}{\partial\alpha} + \frac{1}{2}\alpha^*\right)^n e^{-\frac{1}{2}\alpha^*\alpha} e^{\alpha\hat{a}^\dagger} \mid 0\rangle.
\end{aligned}$$

It then follows that

$$\hat{a}^{\dagger^n} \mid \alpha\rangle = \left(\frac{\partial}{\partial\alpha} + \frac{1}{2}\alpha^*\right)^n \mid \alpha\rangle. \tag{A3}$$

Employing the same procedure, we get

$$\langle\alpha \mid \hat{a}^n = \left(\frac{\partial}{\partial\alpha^*} + \frac{1}{2}\alpha\right)^n \langle\alpha \mid . \tag{A4}$$

$$\begin{aligned}
\hat{a}^{\dagger n} \mid \alpha\rangle\langle\alpha \mid &= e^{-\alpha^*\alpha} e^{\alpha\hat{a}^\dagger} \hat{a}^{\dagger n} \mid 0\rangle\langle 0 \mid e^{\alpha^*\hat{a}} \\
&= e^{-\alpha^*\alpha} \frac{\partial^n}{\partial\alpha^n}\left(e^{\alpha\hat{a}^\dagger}\right) \mid 0\rangle\langle 0 \mid e^{\alpha^*\hat{a}} \\
&= \frac{\partial}{\partial\alpha}\left(e^{-\alpha^*\alpha} \frac{\partial^{n-1}}{\partial\alpha^{n-1}}(e^{\alpha\hat{a}^\dagger}) \mid 0\rangle\langle 0 \mid e^{\alpha^*\hat{a}}\right) \\
&\quad - \frac{\partial}{\partial\alpha}\left(e^{-\alpha^*\alpha}\right) \frac{\partial^{n-1}}{\partial\alpha^{n-1}}(e^{\alpha\hat{a}^\dagger}) \mid 0\rangle\langle 0 \mid e^{\alpha^*\hat{a}} \\
&= \left(\frac{\partial}{\partial\alpha} + \alpha^*\right) e^{-\alpha^*\alpha} \frac{\partial^{n-1}}{\partial\alpha^{n-1}}(e^{\alpha\hat{a}^\dagger}) \mid 0\rangle\langle 0 \mid e^{\alpha^*\hat{a}} \\
&= \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)^2 e^{-\alpha^*\alpha} \frac{\partial^{n-2}}{\partial\alpha^{n-2}}(e^{\alpha\hat{a}^\dagger}) \mid 0\rangle\langle 0 \mid e^{\alpha^*\hat{a}} \\
&\ \ \vdots \\
&= \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)^k e^{-\alpha^*\alpha} \frac{\partial^{n-k}}{\partial\alpha^{n-k}}(e^{\alpha\hat{a}^\dagger}) \mid 0\rangle\langle 0 \mid e^{\alpha^*\hat{a}} \\
&\ \ \vdots \\
\hat{a}^{\dagger n} \mid \alpha\rangle\langle\alpha \mid &= \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)^n e^{-\alpha^*\alpha} e^{\alpha\hat{a}^\dagger} \mid 0\rangle\langle 0 \mid e^{\alpha^*\hat{a}},
\end{aligned}$$

so that we have

$$\hat{a}^{\dagger n} \mid \alpha\rangle\langle\alpha \mid = \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)^n \mid \alpha\rangle\langle\alpha \mid . \tag{A5}$$

One can also establish that

$$\mid \alpha\rangle\langle\alpha \mid \hat{a}^n = \left(\frac{\partial}{\partial\alpha^*} + \alpha\right)^n \mid \alpha\rangle\langle\alpha \mid . \tag{A6}$$

With the help of (A5) and (A6), we can show that

$$\begin{aligned}
\hat{a}^{\dagger m} \mid \alpha\rangle\langle\alpha \mid \hat{a}^n &= \left(\frac{\partial}{\partial\alpha^*} + \alpha\right)^n \hat{a}^{\dagger m} \mid \alpha\rangle\langle\alpha \mid \\
&= \left(\frac{\partial}{\partial\alpha^*} + \alpha\right)^n \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)^m \mid \alpha\rangle\langle\alpha \mid,
\end{aligned}$$

so that

$$\hat{a}^{\dagger^m} \mid \alpha\rangle\langle\alpha \mid \hat{a}^n = \left(\frac{\partial}{\partial\alpha^*} + \alpha\right)^n \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)^m \mid \alpha\rangle\langle\alpha \mid . \qquad (A7)$$

Employing (A2) once more, we see that

$$\frac{\partial^n}{\partial\alpha^{*^n}} \mid \alpha\rangle = \frac{\partial^n}{\partial\alpha^{*^n}} e^{-\frac{1}{2}\alpha^*\alpha} e^{\alpha\hat{a}^\dagger} \mid 0\rangle$$

$$\frac{\partial^n}{\partial\alpha^{*^n}} \mid \alpha\rangle = \left(-\frac{1}{2}\right)^n \alpha^n e^{-\frac{1}{2}\alpha^*\alpha} e^{\alpha\hat{a}^\dagger} \mid 0\rangle.$$

Hence

$$\frac{\partial^n}{\partial\alpha^{*^n}} \mid \alpha\rangle = \left(-\frac{1}{2}\right)^n \alpha^n \mid \alpha\rangle. \qquad (A8)$$

Following the same procedure, one can easily verify that

$$\frac{\partial^n}{\partial\alpha^n} \langle\alpha \mid = \left(-\frac{1}{2}\right)^n \alpha^{*^n} \langle\alpha \mid . \qquad (A9)$$

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