# A SEMINAR REPORT ON THE MASTER OF SCIENCE IN MATHEMATICS.

SOME REMARKS ON SOLUTION OF OPTIMUM/BEST ROUTE NETWORK AND DETERMINATION OF NUMBER OF SERVICES TO BE OPERATED ONTO THESE ROUTES:

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JUNE: 2004

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## PREFACE

This paper is produced as a requirement in partial fulfillment of the Master of Science degree in mathematics at Addis Ababa University. It compiles all presentation I made on the examiners board.

Today the aggregate of subjects called optimization is a captivating merge of heuristic and rigor theory and experiment truly it can be studied as a branch of pure mathematics,

Yet has a wide range of applications in almost any branch of science and technology.

The paper attempts to forward some issues on solution of best route network and determination of number of services to be operated onto these routes

The paper comprises of three chapters organized in order of dependence one on the other.

The first chapter introduces basic definitions and examples on network and the need for central terminal that seem simple although they are crucial importance as serving as the basis of optimum route network. The second chapter deals with the solution of best route network.

The third chapter where the theme of the paper commences makes a survey of determination of number of services to be operated onto the best routes. Optimization is often characterized as the sciences of determining the 'best' solutions to certain mathematically defined problems which are mostly models of physical reality. It also involves the investigation of optimal conditions for problems.

I want to express my deepest gratitude to my advisor and instructor D.r Mobin Ahmad who cultivated me not only academically but also in other social aspects. I strongly appreciate his excellent behavior both in the class and advising at his office. His constructive comments, advises and provision of materials like books, journals on social affairs are extremely invaluable. Having D.r Mobin Ahmad as my advisor and instructor really made me happy. Also I am grateful to all my friends for their moral and material support.

ADEM MOHAMMED

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# Chapter I. central terminals and route network

## 1.1. Introduction; Why we need central terminals?

The need for location of central terminal is of vital importance for finding the best routing system in a network. A terminus does not facilitate in evolving a route grid with high utility. However, by acting as a junction of various routes ,it provides opportunities to the indirect demand for changing routes. The idea of a terminal in a network ,as such, is in contradiction to the idea of optimal route network. This can further be explained by the fact that each route has to pass at least one terminal. This results in the lengthening of routes as compared to when we have direct routes. Having a terminus is highly advantageous from the management point of view, since it facilitates smoother operation of the transport company, help to keep track of buses and acts as a center for assigning different buses to various routes depending upon the demand. Furthermore, a terminus acts as a center for submitting tariffs, a central station for providing necessary

facilities for staff and passengers like facilities of inquiry,
drinking water, canteen, medicine and general store etc.
While selecting a location for terminus one has to see that it offers best prospects for spreading routes spanning network in all directions and that preferably lesser number of passengers have to change routes. The later consideration implies that a terminal should be on routes having higher utilities. A terminus should offer better scope for generating circuit routes through it.

The difficulty arising in terminus location can be visualized if one notes that the problems of optional location of terminus and an optional route network are highly interrelated. Routes can not be determined with out having a predetermined termini locations and a set of termini locations cannot be evaluated without having optional route legend. Before selecting a terminus, and route network in turn we do not know what routes will have to be added in the future and what terminals will be connected

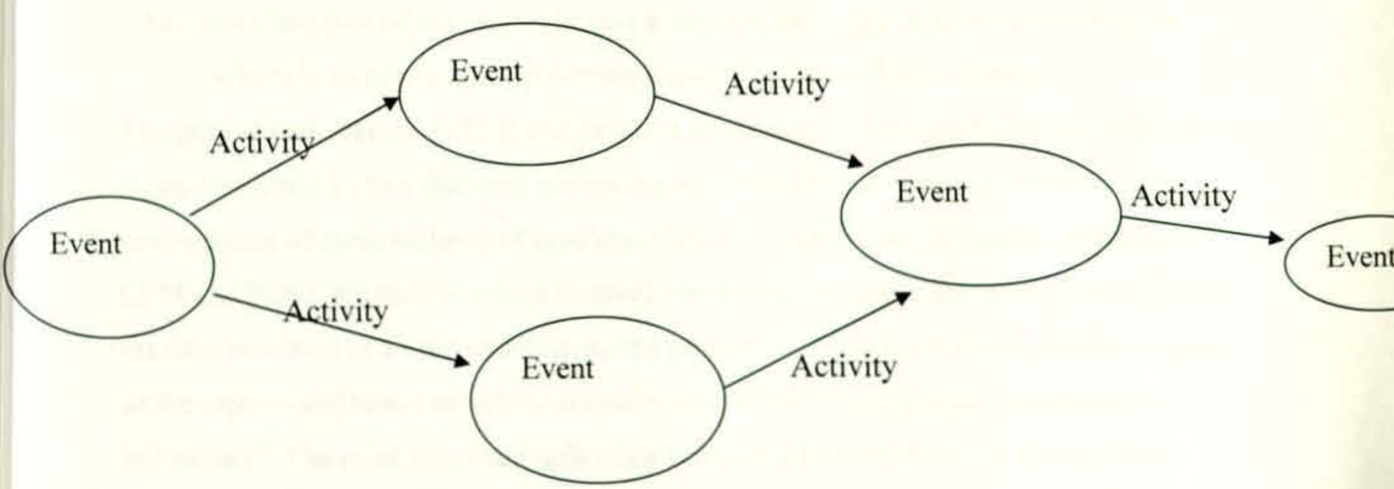
### 1.3. What is a network?

**Definition:** A network(or linear graph)consists of a set of arcs(or edges links)connecting various pairs of the nodes. Each arc has a specified orientation (direction). Consequently the network is said to be directed. A simple notation serves to describe a directed network: Number the points 1,2,...,p and designate an arc stating at node i and ending at node j by (i,j).Assume there is only one arc (i,j)and call an item traversing (i,j) a unite of flow along that arc. The network is bipartite if the nodes can be partitioned into two groups $G_1$ and $G_2$.

### 1.4. Elements of a network(network representation of a project)

It is always possible to break up the entire project into a number of distinct, well defined jobs or tasks(called activities).The beginning or end of each such activity constitutes an event of the project.

In the network diagram, an activity is represented by arrows while events are represented, usually, by circles, as shown in the figure 1.1.




Therefore the basic elements of a project network are

Activities and Event

Fig. 1.1

## 1.5. Techniques of project management

CPM (critical path method )and PERT (programme evaluation and review technique) are techniques of project management useful in the basic managerial functions of planning, scheduling and controlling a project

. The planning part of a project involves the breaking of the project into several smaller tasks or jobs that must be executed in order to complete the venture of the project. The gross requirements of manpower, material and equipment are determined in this phase. The estimates of costs and durations for the various jobs are also made in the planning stage.

The scheduling of a project deals with the fixing of the various jobs or tasks in the time order in which they have to be performed. The manpower and material requirements needed at this stage are found. The expected times of completion of each of the jobs are also calculated at this stage.

The controlling part of the project begins with a review of the difference between the schedule and the actual performance after the project has been stated.

The critical path method (CPM) and program evaluation and review technique (PERT) Were developed by two different groups during 1956-1958 almost simultaneous. The development of these techniques revolutionilized the entire field of project management. CPM and PERT are basically time oriented methods in the sense that both of them lead to the determination of a time schedule for the project. Both the methods are similar in most of the aspects and hence are often (collectively) reffered to as "project management techniques". The most important difference between CPM and PERT is that the time estimates for the different activities or jobs are assumed to be deterministic in CPM, while they are taken as probabilistic in PERT. It is important to note that neither CPM nor PERT is a decision making technique. They will only aid the decision-maker especially by calling attention to places where decisions are necessary.

The use of project network and the identification of the critical path and the activity slacks are the common features of both CPM and PERT. The primary difference between PERT and CPM is that PERT employs statistically related time estimates whereas CPM uses a single value time estimate for each of the activities. Thus PERT can be in projects where there is uncertainty regarding the durations of the various activities. This is the main reason for its wider use in research and development type of projects for which there is little or no prior experience. On the other hand, CPM is more useful for projects where background of experience is available to estimate the duration of various activities (operations) with some degree of certainty. This is the reason why CPM has been widely used in the construction (like houses, highways bridges ) and the manufacturing projects. PERT starts with the same network diagram as CPM but uses three time estimates to approximate the time for each activity. The three time estimates generally used are the optimistic time, the pessimistic time and the most likely time. These three times can be easily guessed by the person in charge of the project. An important feature of PERT is that the probability rules can be applied to find the chances of completing the projects by any given date.

# chapter II. Optimum Network

## 2.1. Finding the critical path.

Once the network of a project is complete the planner can estimate how long it will take to complete the project. Obviously the time for completing the entire project is not the sum of the individual operation times since some operation can be done simultaneously.

In fact the total time for completion of the project will be governed by a particular set of operations known as critical operations and the chain formed by these critical operation in the project network is called the critical path. This is the origin of the name critical path method.

Example 2.1. consider the following network.

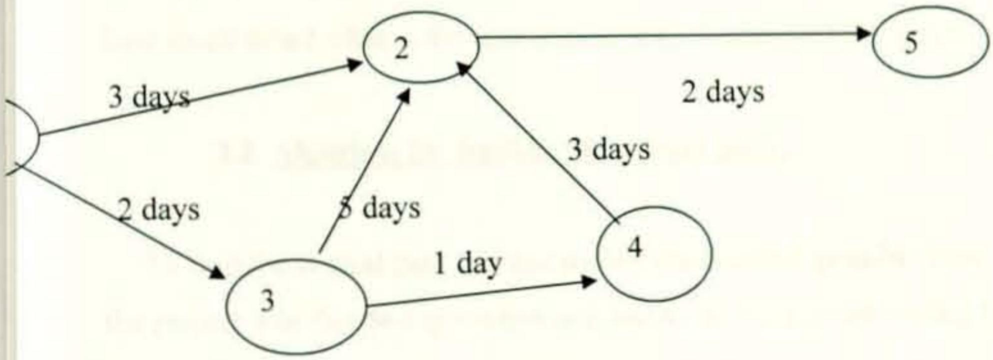



Fig. 2.1

The shortest possible time that will take for the completion of the project is determined by that series of activities which occupies the longest time in the project. The various series of activities (path) in the example are given below.

| Path defined by Nodes | Total time required (days) | Is the path critical? |
|---|---|---|
| 1-2-5 | 3+2=5 | NO |
| 1-3-2-5 | 2+5+2=9 | Yes |
| 1-3-4-2-5 | 2+1+3+2=8 | No |

It can be seen that the shortest possible time for completing this project is 9 days corresponding to the path 1-3-2-5.

Thus the activities 1,3;3,2;and 2,5 are called the critical activities, and the path 1-3-2-5 is called the critical path.

Once the critical path is found one can get meaningful answers to such questions as:

How long the project take?

Can the contract (or scheduled) completion date be met?

If there is a delay in one operation (activity) will it delay the entire project? if so by how much time? what is the economical way to speed up the project?

### 2.2. Algorism for finding the critical path.

To find the critical path one has to find the smallest possible time for completing the project. For this two quantities one known as the earliest stating time and the other known as the latest completion times are defined.

The critical path is determined in two phases. in the first phase we start our calculations from the start node and move towards the end node by making use of the earliest completion time. In the second phase we begin our calculations from the end node and move towards the start node by using the latest completion time.

We will first see the first phase calculations.

Let $Es_i$ denote the earliest starting time of all activities emanating from event (node) i.

Conventionally the stating time of the project is taken as zero that is $ES_1=0$. If $t_{i\,j}$ represents the duration of the activity (i,j) the earlier starting time Esj can be obtained as:

$ES_j=\max(ESi+t_{ij})$……….(2.1) for all incoming activities (i,j).

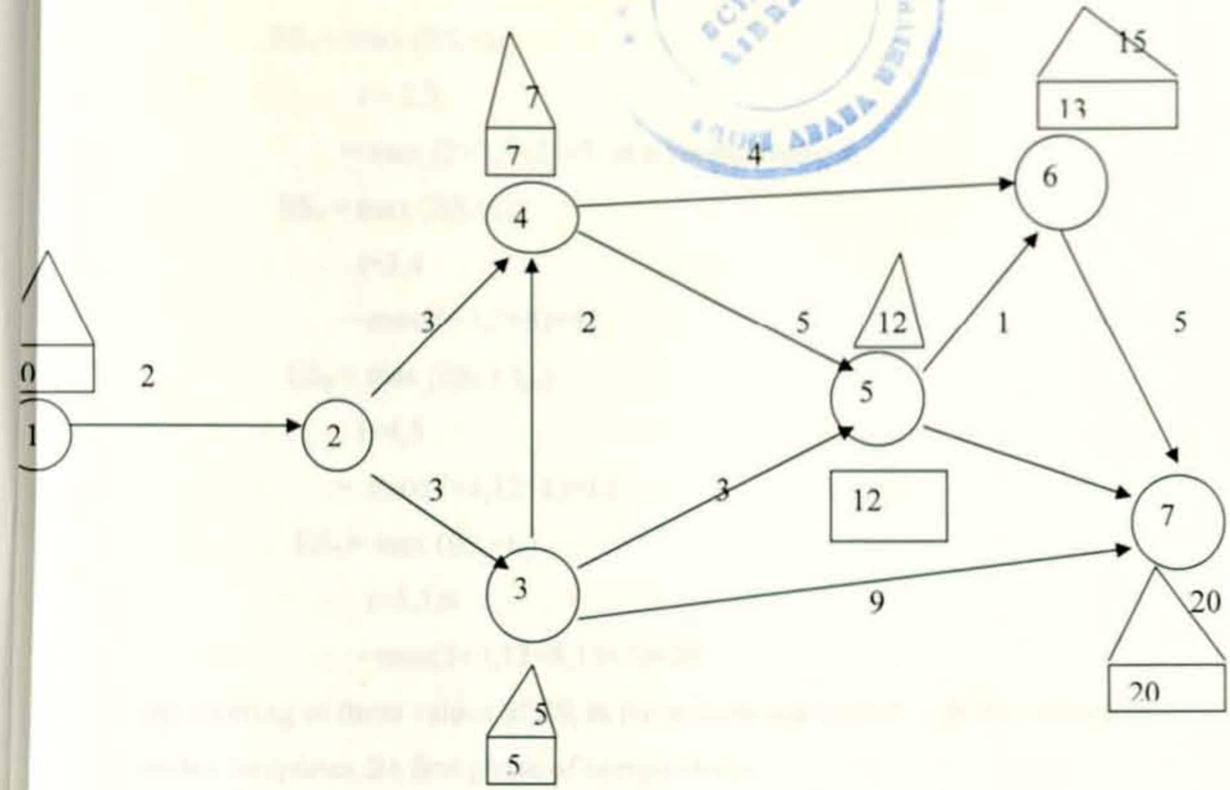



Fig. 2.2

this equation shows that ESj at any node j can be computed only if ESi corresponding to all incoming activity (i,j) are computed. The values of $ES_i$ are generally in rectangle , □ , ,in the network. For illustration consider the network shown in the figure 2.2. The duration of the activities (i,j) are shown above the arcs connecting the nodes I and j. the values of $ES_1$=0 is shown in the square above in the figure 2.2.

By moving to node 2, we obtain

$ES_2 = ES_1 + t_{12} = 0 + 2 = 2$; since there is only one incoming activity ,namely (1,2) for node. this value of 2 is shown in the square associated with node 2. We next consider node 3 to obtain $ES_3$.

$ES_3 = ES_2 + t_{23} = 2 + 3 = 5$. After entering this value in the square associated with node 3 we move to node 4.

since there are two incoming activities ,namely (2,4) and (3,4) at node 4, we have

$ES_4 = \max(ES_i+t_{i4})$

$i = 2,3$

$= \max(2+3,5+2)=7$. in a similar manner

$ES_5 = \max(ES_i+t_{i5})$

$i=3,4$

$= \max(5+3,7+5)=12$

$ES_6 = \max(ES_i + t_{i,6})$

$i=4,5$

$= \max(7+4,12+1)=13$

$ES_7 = \max(ES_i+t_{ij})$

$i=3,5,6$

$=\max(5+9,12+8,13+5)=20$

the entering of these values of $ES_i$ in the squares associated with the respective nodes completes the first phase of computations.

To carry out the second phase of computions;

Let $LC_i$ denote the latest completion time for all the activities coming into the node i .If n denotes the number of the end node , $LC_n= ES_n$ will be the starting point for these calculations. The general rule for computing LCi of any node is:

$LC_i=\min(LC_i- t_{ij})$…………..(2.2) for all out going activities (i,j).these values are entered in a triangle assocaited with the node i as shown in the figure 2.2.

By using $LC_i=\min(LC_i-t_{ij})$, we obtain

$LC_7=ES_7=20$

$LC_6=LC_7-t_{67}=20-5=15$.

$LC_5=\min(LCj-t_{5j})$

$j=6,7$

$= \min(LC_7-t_{57},LC_6-t_{56})$

$=\min(20-8,15-1)=12$

$LC_4=\min(LCj-t_{4j})$

j=5,6

=min(15-4,12-5)=7

$LC_3$=min($LC_j$-$t_{3j}$)

j=4,5,7

=min(20-9,12-3,7-2)=5.

$LC_2$=min($LC_j$-$t_{3j}$)

=min(7-3,5-3)=2. and finally

$LC_1=LC_2-t_{12}=2-2=0$

this completes the second phase calculations.

the results of the two phases of calculations can now be used to identify the critical path.

the rule is that any activity ( i, j) satisfying the following the three conditions will be the critical path.

$$\left.\begin{aligned} &1.ES_i = LC_i \\ &2.ES_j = LC_j \\ &3.ES_j - ES_i = LC_j - LC_i \end{aligned}\right\} \quad (2.3)$$

These conditions actually denote that there is no slack(or float)time between the earliest stating time and the latest starting time of the activity. Hence this activity must be a critical activity. An application of equations (2.3) to the network of figure 3.2. gives (1,2),(2,3),(3,4),(4,5)and (5,7) as the critical activities. Hence the critical path 1-2-3-4-5-7.

## 2.3 Finding the critical path in PERT.

In PERT the time estimates for to ,tp and tm are given for each of the activities of the project. The expected times of completion of the various activities of the project are calculated and the critical path based on the expected time is determined by

$$t_{i,j} = \frac{t_0 + 4(t_m) + t_p}{6} \ldots\ldots\ldots\ldots\ldots(2.4).$$

where $t_o$ -optimistic time and the variance $\delta_t = \frac{(t_p - t_o)^2}{6^2} \ldots\ldots(2.5)$

$t_m$ –the most likely time

$t_p$ –pessimistic time

If T denotes the shortest time necessary for completing the whole project , the expected value and the variance of T are found from the relations (by assuming the critical activities to be statistically independent ):

T'=expected value of T =sum of the expected time of Completion of critical activities. and

$\delta^2$=variance of T=sum of the variances of completion times of critical activities.

The variance of T will be useful in finding the chances (probability) of completing the whole project within a specified period.

Example: 2.3.

As an example of how the critical path is found in PERT. Consider the network shown in the figure 2.3 below.

The estimates of $t_0$, $t_m$ and $t_p$ are shown in this order for each of the activities on the top of the arcs denoting the respective activities. Expected times of completion of various activities are found by using equation (2.4).

Thus if $t_{i,j}$ denotes the expected time of completion of the activity(i,j),we obtain

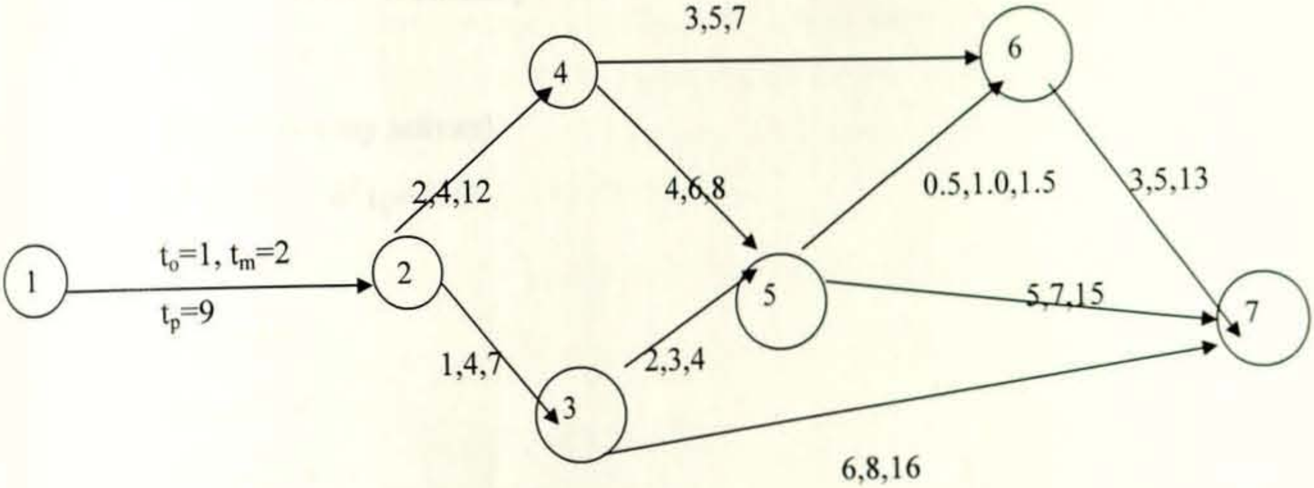



Fig. 2.3



$t_{1,2}$=(1+4(2)+9)/6=3

$t_{2,3}$=(1+4(4)+7)/6=4

$t_{2,4}$=(2+4(4)+12)6=5

$t_{3,4}$=0(dummy activity)

$t_{3,5}$=(2+4(3)+4)/6=3

$t_{3,7}$=(6+4(8)+16)/6=9

$t_{4,5}$=(4+4(6)+8)/6=6

$t_{4,6}$=(3+4(5)+7)/6=5

,$t_{56}$=(0.5+4(1)+1.5)/6=1

$t_{5,7}$=(5+4(7)+15)/6=8

$t_{6,7}$=(3+4(5)+13)/6=6

Similarly the variances of the various activities can be obtained as:

$\delta^2 t_{1,2}=(9-1)^2/6^2=16/9$

$\delta^2 t_{2,3}=(7-1)^2/6^2=1$. Similarly

$\delta^2 t_{2,4}=25/9$

$\delta^2 t_{3,4}=0$(dummy activity)

$\delta^2 t_{3,5}=1/9$ , $\delta^2 t_{3,7}=25/9$

$\delta^2 t_{4,5}=4/9$

$\delta^2 t_{4,6}=4/9$

$\delta^2 t_{5,6}=1/36$

$\delta^2 t_{5,7}=25/9$

$\delta^2 t_{6,7}=25/9$.

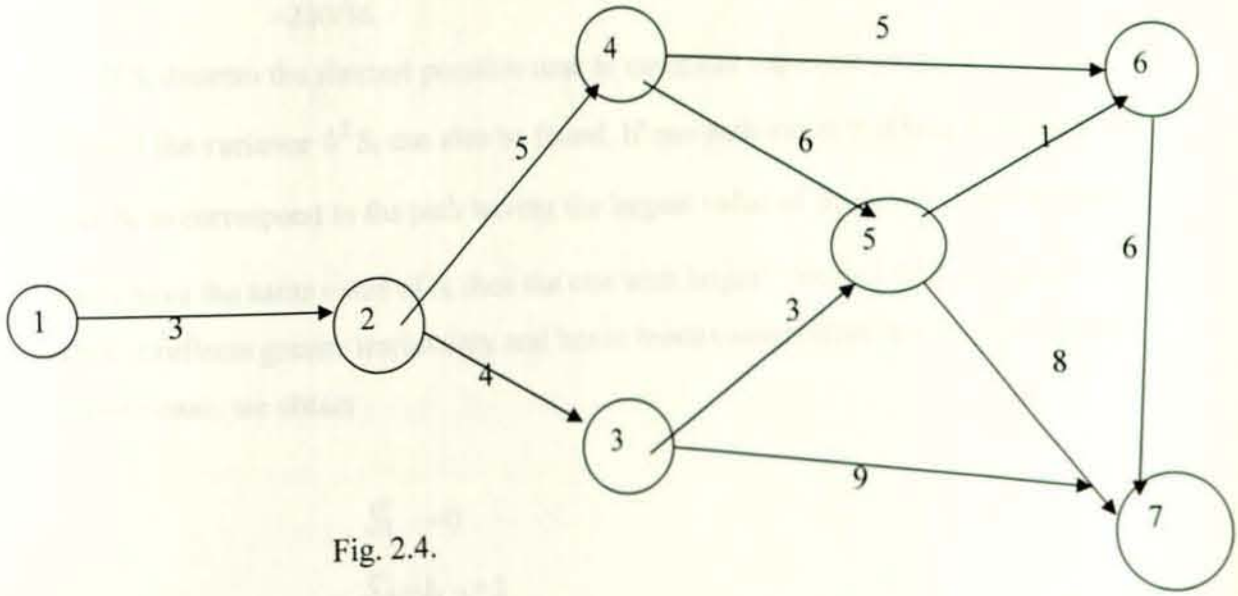



Fig. 2.4.

Figure 2.4 shows the same network with the expected time ti,j indicated on the top of the arc indicating the activity (i,j)

The critical path (which has the largest expected total time for completion of the project)can be identified to be

1-2-4-5-7.

The expected time of completion of the whole project can be obtained as:

$T'=t_{1,2}+t_{2,4}+t_{4,5}+t_{5,7}=3+5+6+8=22.$

If there is more than one path giving the same value of the largest expected total time of completion, the path having the largest value of variance has to be taken as the critical path.

The variance of T can be computed as:

$$\delta^2 T=\sum\delta^2{}_{t_{i,j}}=\delta^2{}_{t\,1,2}+\delta^2{}_{t\,2,4}+\delta^2{}_{t\,4,5}+\delta^2 t\,5,7$$

$$=(9-1)^2/6^2+(12-2)^2/6^2+(8-4)^2/6^2$$

$$+(15-5)^2/6^2$$

$$=1/36(64+100+16+100)$$

$$=280/36$$

If $S_i$ denotes the shortest possible time to complete the event (node) i, the mean $\overline{S}_i$ and the variance $\delta^2 S_i$ can also be found. If one path exists that lead to event i, we take $S_i$ to correspond to the path having the largest value of $S_i$.Again if two or more paths have the same value of $S_i$ then the one with largest value of $\delta^2 S_i$ is selected since it reflects greater uncertainty and hence more conservative results. Thus in the present case, we obtain

$$\overline{S}_1=0$$

$$\overline{S}_2=\bar{t}_{1,2}=3$$

$$\overline{S}_3=\bar{t}_{1,2}+\bar{t}_{2,3}=7$$

$$\bar{S}_4 = \bar{t}_{1,2} + \bar{t}_{2,4} = 8$$

$$\bar{S}_5 = \bar{t}_{1,2} + \bar{t}_{2,4} + \bar{t}_{4,5} = 14$$

$$\bar{S}_6 = \bar{t}_{1,2} + \bar{t}_{2,4} + \bar{t}_{4,5} + \bar{t}_{5,6} = 15$$

$$\bar{S}_7 == \bar{t}_{1,2} + \bar{t}_{2,4} + \bar{t}_{4,5} + \bar{t}_{5,7} = 22$$

$$\sigma_1^2 == 0$$

$$\sigma_2^2 == 16/9$$

$$\sigma_3^2 = \delta^2_{t\,1,2} + \delta^2 t_{2,3} = 25/9$$

$$\sigma_4^2 = \delta^2_{t\,1,2} + \delta^2 t_{2,4} = 41/9$$

$$\sigma_5^2 = \delta^2_{t\,1,2} + \delta^2 t_{2,4} + \delta^2_{t\,4,5} = 5$$

$$\sigma_6^2 = \delta^2_{t\,1,2} + \delta^2 t_{2,4} + \delta^2_{t\,4,5} + \delta^2 t_{5,6} = 181/136$$

$$\sigma_7^2 == \delta^2_{t\,1,2} + \delta^2 t_{2,4} + \delta^2_{t\,4,5} + \delta^2 t\ 5,7 = 70/9$$

The variance for each activity and for the entire gives the manager additional information about variations in job and project times that he can anticipate. He will be therefore able to plan for and cope with such variation.

# chapter III. DETERMINATION OF NUMBER OF SERVICES TO BE OPERATED ONTO ROUTES.

**3.1 .Introduction**: In view of developments the varied demand patterns for various routes, a time comes when it becomes necessary to study the system a fresh. The task is to make best utilization the existing fleet of size.

In designing the working of a bus transport system we have to find solutions to problems namely ;fleet size , allocation of trips of routes and scheduling buses.

The first article on bus schedules is by Foulkes Prager and Warner (1954) in which they tried to schedule buses on the minimum waiting criterion. Since then several researchers have been conducted concerning routing and scheduling of school buses , public transport facilities like buses , aeroplanes ,railway trains; heavy vehicles like trucks ,coal carriers ,cargo carrying ships, military tankers . An overview of bus scheduling problems and their overall relationship can be obtained in the view by Wren (1983).Yang and Wilkinson (1970) are first to devise a computer program for routing buses. Other promising works are by Saha (1970)Shaw(1970) Lampkin and Saalmans (1967) Patel(1970) Ishmael(1972) ,Gavish (1978).

For the sake of completeness we explain here the terminologies we often use:

For city road network G =(N,A), 'N' is a set of nodes on city road layout and 'A' is the set of road links connecting these nodes.

'Demand Matrix' means the matrix showing demand of passengers between all pairs of nodes in the network.

'Routing a bus' means specifying a chain of nodes which a bus has to cover.

'scheduling a bus' means specifying a list of trips a bus operates. A trip can be designated by starting place, ending place, starting time and ending time.' A trip on a route' signifies that a bus starts at one end of the route , travels to the other and then travels back to the original place.

## 3.2. Basic Assumptions And Commuters Expectations For Developing Model And Solution methodology.

### 3.2.1. Basic assumptions for developing model and solution methodology.

1. Every entry in demand matrix is a static representation of the passenger flow between the corresponding origin and destination node pair (actually the demand matrix gives the passenger flow between all node pairs, in a matrix form ).With this assumption we are separating the dynamic nature of the demand in the transportation network. The same can be achieved by making a less general assumption like, the demand in uniformly distributed over a period of time example .a day, a month etc.
2. For ordinary routes, unlike circuit routes, the operation of a trip comprises to and from journey of the bus on the route that is ,terminus to terminal and back to terminus. This ensures availability of bus at terminus for further allocations. On the so- called circuit route, the buses travel in directionally(either clockwise or anticlockwise) to travel a trip.
3. All the indirect demand travels up to the terminus lying on their respective routes and from there catches another routes (or other routes ) leading to their destination .We, therefore, do not assume any changeover of place other than the terminal .

### 3.2.2. Commuters' Expectations for developing model And Solution methodology.

a). A passenger should get a bus within a reasonable period that is the waiting time at the stop should be low.

b). A passenger should be able reach his destination by a direct bus that is without having to changeover.

c). A passenger should be able to travel to his destination by the short possible route so that he has to travel less ,has to pay less and reaches faster.

It is obvious that it is not possible to meet all these objectives squarely as they are conflicting each other.

.If every passenger wants to get a bus as soon as he comes to a bus stop, numerous trips will have to be operated . Since the fleet is limited in size, it will have to be distributed over fewer routes .Idea of fewer routes is to make the routes zig-zag to connect as many places as possible to avoid changeover of passengers from one route to another. This means the passengers may have to travel longer and may not be able to travel along the shortest route. We see that though the waiting time is reduced (expectation(a) fulfilled) by operating high frequency bus service on one route ,this in reduction in the total number of routes, which can be operated .Though changeovers are reduced (expectation(b)fulfilled) the passengers have to travel more(expectation(c)unsatisfied).

If routes should take the passengers directly to their destination along shortest distance (expectation(c) fulfilled) ,then large number of routes will have to be operated requiring higher number of passengers changing over routes. .Moreover, with limited fleet, less number of trips can be run on a large number of routes and the waiting time at stops will increase (expectation(a) violated .

From the above discussion we see that most important features of service offered to public are:

1.To satisfy maximum possible demand.

2.To minimize waiting time.

3.To minimize the travel time or travel distance.

### 3.3. Concept of 'Utility' of a Route .

In general, utility of a route 'R' is a function of both the demand satisfied on route 'R' and the distance , which the demand travels on the route 'R'.

Mathematically,

$$UT(R) = \psi \{D(R), S(R)\} \quad \ldots\ldots\ldots\ldots(3.1.)$$

where UT(R)=Utility of route'R'

D(R) =Demand satisfied on route 'R'.

S(R) =Distance, the demand D(R) has to travel on route 'R'.

A service level measure should improve as more demand is satisfied and decrease as the passengers are put to the inconvenience of traveling more. Since the demand satisfied is between each pairs of nodes lying on the route, the equation(1) should be rewritten in the following generalized form,

$$UT(R) = \sum_{I=1}^{NNR_R-1} \sum_{J=I+1}^{NNR_R} \frac{D(NSET(R,T), NSET(R,J))}{\sum_{L=1}^{J-1} S(NSET(R,L), NSET(R,L+1))} \quad \ldots\ldots\ldots\ldots(3.2)$$

Where NSET (R, J) is the symbol number representing

The node in the $J^{th}$ position.

$NNR_R$ = Number of nodes covered by the route 'R'.

Route 'R' is a sequence of nodes.

D(I,J) = Demand between nodes (I) to node (J).

$S_R(I, J)$ = Distance between nodes (I) and (J) along route 'R'.

Here $S_R(I, J) = \sum_{L=1}^{J-1} S(L, L+1)$ and S (L, L+1) = distance between adjacent nodes (L ) and (L+1). For illustration we consider a sequence of nodes $\{I_R\}$ forming a route 'R' as in the following figure .The total number of nodes in the route ,NNR=N.

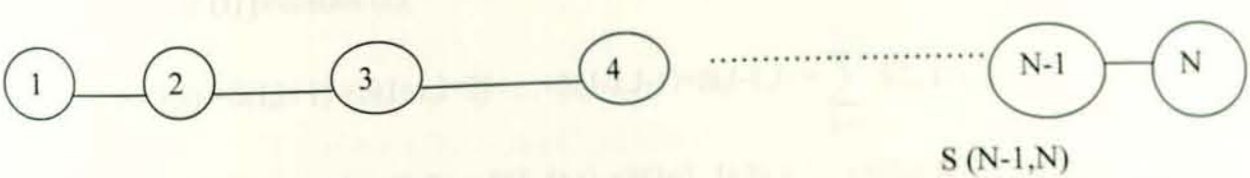



Fig 3.1.

A trip on route 'R' mentioned above implies that the bus starts from (1) goes to (2),then to (3) and so on, and reaches finally to (N) and then this route is traversed backward by the bus.

At (1) some passengers enter the bus, with the inversion of getting down at certain other nodes on the route 'R', example (2) ,(3) .....(N).When the bus comes to node (2),passengers with origin (1)and destination (2) that is D(1,2) passengers get down, while passengers D(2,3),D(3,4) ......D(2,N) enter the bus with the intension of getting down at nodes (3),(4), ....,(N) respectively (Note: All these passengers will have to travel along the route 'R' .The passengers' liking for this route decrease if this route takes longer time and distance to carry passengers from their respective origins to their respective destinations).

Therefore at each node (I) two events occur:

1 . Passengers having destination (I), with origins as nodes preceding (I) in route 'R', get down. Mathematically this can be represented as,

$$D(1,I) + D(2,I) + \ldots + D(I-1,I) = \sum_{J=1}^{I-1} D(J,I) = \sum_{J=1}^{I} D(J,I) \text{ as } D(I,I) = 0 \ldots\ldots\ldots\ldots(3.3)$$

2. Passengers having origin (I) and destination as any of the nodes following (1) in route 'R', get in. Mathematically this can be represented as,

$$D(I,I+1) + D(I,I+2) + \ldots + D(I,N) = \sum_{J=I+1}^{N} D(I,J) = \sum_{J=1}^{N} D(I,J) \text{ as } D(I,I) = 0 \ldots\ldots\ldots\ldots(3.4)$$

In any route a single road connects the adjacent nodes. This link of road has a fixed distance. The distance between nodes (I) and (J), on the route 'R',can be calculated as below:

If (I) precedes (J);

$$S_R(I,J) = S(I,I+1) + S(I+1,I+2) + \ldots + S(J-2,J-1) + S(J-1,J) = \sum_{L=I}^{J-1} S(L,L+1) \ldots\ldots\ldots\ldots(3.5)$$

If (J) precedes (I); $S_R = S(J,J+1) + S(J+1,J+2) + \ldots + S(I-2,I-1) + S(I-1,1)$

$$= \sum_{L=J}^{I-1} S(L,L+1) \ldots\ldots\ldots\ldots(3.6)$$

As (1) and (N) are the first and the last nodes respectively in the route 'R', the length of the route 'R' in $S_R$ (1,N).

### 3.4. Direct and Indirect demand for a Route.

Definition: Direct demand of route 'R' comprises the number of passengers (i.e. demand )whose origins and destinations both lie on the route 'R'.

Many time some passengers can not get a direct route from their origin to their destination and therefore ,have to change routes to reach their destinations. It is assumed that all passengers ,whose destinations do not be on route 'R' under consideration, travel up to the terminus lying on the route. Similarly some passengers whose origins do not be on the route 'R' travel by other routes and reach the terminus lying on the route under consideration. From this terminus they travel along the route to reach their destinations which lie on the route 'R' we therefore define ;indirect demand of route 'R' from their origins to their destinations.

A route having high indirect demand suggests its inability to promote direct travel. On the other hand the route is giving some service to certain 'indirect' passengers by offering them transportation to and from the terminus. The measure for this service level for route 'R' is called 'Deficient' utility of a route 'R'.

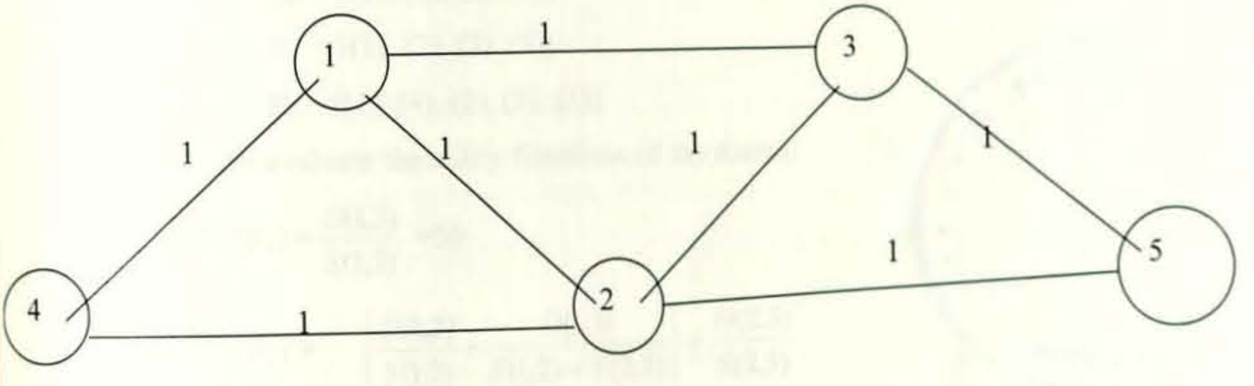
Analogues to utility of route 'R' (eqn. ), Deficient utility of a route 'R' is defined as the cumulative number of indirect passengers traveling per unit distance of the route. Mathematically,

$$\text{DUT(R)} = \sum^{NNR} \frac{D'(NSET(R,I), NSET(R,J))}{\sum_{L=1}^{J-1} S(NSET(R,L), NSET(R,L+1))} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.7)$$

Where NSET(R, J) is the symbol number representing the node in the $J^{th}$ position of the indirect demand and D'(1,J) is indirect demand from terminus (1) to node (J) lying on the route 'R'.

**Example on the utility functions (number of services to be operated on to the routes) of the routes.**

Consider the network shown in the figure ( 3.2) below, which is some what artificial.




.fig 3.2.

Consider the demand matrix as:

| Dest.J → | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Orig.I ↓ | | | | | |
| 1 | 0 | 10 | 50 | 1 | 10 |
| 2 | 0 | 0 | 20 | 0 | 10 |
| 3 | 0 | 0 | 0 | 0 | 0 |
| 4 | 1 | 1 | 1 | 0 | 1 |
| 5 | 0 | 0 | 50 | 0 | 0 |

Different possible routes between nodes (1) and (3) can be

$R_1 = \{(1),(3)\}$

$R_2 = \{(1),(2),(3)\}$

$R_3 = \{(1), (4), (2), (3)\}$

$R_4 = \{(1), (2), (3), (5)\}$

$R_5 = \{(1), (4), (2), (5), (3)\}$

We evaluate the utility functions of the routes:

$$UT(R_1) = \frac{D(1,3)}{S(1,3)} = 50$$

$$UT(R_2) = \left\{\frac{D(1,2)}{S(1,2)} + \frac{D(1,3)}{S(1,2)+S(2,3)}\right\} + \frac{D(2,3)}{S(2,3)}$$

$$= \left\{\frac{10}{1} + \frac{50}{2}\right\} + \frac{20}{1} = 55.00$$

$UT(R_3)$

$$= \left\{\frac{D(1,4)}{S(1,4)} + \frac{D(1,2)}{S(1,4)+S(4,2)} + \frac{D(1,3)}{S(1,4)+S(4,2)+S(2,3)}\right\} + \left\{\frac{D(4,2)}{S(4,2)} + \frac{D(4,3)}{S(4,2)+S(2,3)}\right\}$$

$$+ \frac{D(2,3)}{S(2,3)}$$

$$= \left\{\frac{1}{1} + \frac{10}{2} + \frac{50}{3}\right\} + \left\{\frac{1}{1} + \frac{1}{2}\right\} + \frac{20}{1} = 44.2$$

$$UT\ (R_4) = \left\{ \frac{D(1,2)}{S(1,2)} + \frac{D(1,5)}{S(1,2)+S(2,5)} + \frac{D(1,3)}{S(1,2)+S(2,5)+S(5,3)} \right\}$$

$$+ \left\{ \frac{D(2,5)}{S(2,5)} + \frac{D(2,3)}{S(2,5)+S(5,3)} \right\} + \frac{D(5,3)}{S(5,3)}$$

$$= \left\{ \frac{10}{1} + \frac{10}{2} + \frac{50}{3} \right\} + \left\{ \frac{10}{1} + \frac{20}{2} \right\} + \frac{50}{1} = 101.7$$

UT ($R_5$)

=

$$\left\{ \frac{D(1,4)}{S(1,4)} + \frac{D(1,2)}{S(1,4)+S(4,2)} + \frac{D(1,5)}{S(1,4)+S(4,2)+S(2,5)} + \frac{D(1,3)}{S(1,4)+S(4,2)+S(2,5)+S(5,3)} \right\}$$

$$+ \left\{ \frac{D(4,2)}{S(4,2)} + \frac{D(4,5)}{S(4,2)+S(2,5)} + \frac{D(4,3)}{S(4,2)+S(2,5)+S(5,3)} \right\}$$

$$+ \left\{ \frac{D(2,5)}{S(2,5)} + \frac{D(2,3)}{S(2,5)+S(5,3)} \right\} + \frac{D(5,3)}{S(5,3)}$$

$$= \left\{ \frac{1}{1} + \frac{10}{1} + \frac{50}{3} + \frac{50}{4} \right\} + \left\{ \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \right\} + \left\{ \frac{10}{1} + \frac{20}{2} \right\} + \frac{50}{1} = 93.7$$

The above calculations show that route $R_4$ has the highest utility. It may be noticed that more demand is satisfied by $R_5$ but the distance traveled is comparatively more.

It is not easy to evaluate the sensitivity of the utility function with respect to routes. Since sensitivity of the utility with respect to demand alone does not have any significant importance, to maintain the meaning of the concept of utility as a whole, it is necessary to consider variation of demand and variation of distance between nodes for various routes.

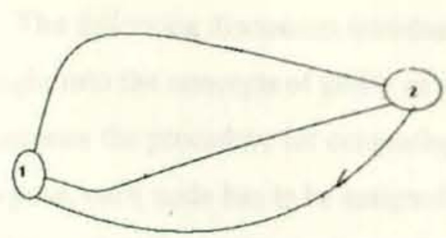


Fig. 3.3

Consider fig.( 3.3 ). The nodes (1) and (2) are connected by three routes $R_1$, $R_2$ and $R_3$ of lengths $DI(R_1)$, $DI(R_2)$, $DI(R_3)$ respectively. If the objective is to find a route joining the two nodes alone and there is no other node or nodes to be covered by the route joining nodes (1) and (2), then this is a trivial problem and the route $R^*$ with minimum length ($DIR^*$) should be chosen. Since the demand D (1,2). Will always be same irrespective of the routes, there is no gain by choosing a route other than the shortest one.

On the other hand consider fig. (3.4 ).

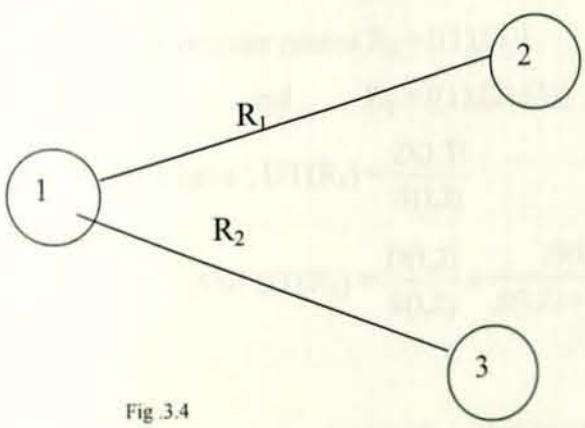



Fig 3.4

If DI ($R_1$)=DI ($R_2$), D (1,2)<D (1,3); as there is no road connecting nodes (2) and (3), this is again a trivial problem and $R_2$ is always a better choice.

### 3.5. Application Of Service Level Measure:

**The** following discussion introduces some additional symbols and gives better insight into the concepts of utility of a route 'R' namely UT(R). This discussion also illustrates the procedure for comparing routes through an example. For computational purpose, each node has to be assigned a certain number (section 3.3). This number will be referred to as the 'node symbol number' henceforth. A route, therefore, will be represented by a sequence of number corresponding to the sequence of nodes lying on the route.

Let NSET (R,I) denote the node symbol number of the node in the $I^{th}$ position on the route 'R'.

Consider the network shown in the figure 3.2.

Data available; Demand matrix showing demand between each pair of nodes,

Distance Matrix (Distances of links are shown in the figure)

Terminus-node (1),

Terminal-node (3).

Objective: To analyze sensitivity of utility function with respect to various routes

and varied demand

To find best route between terminus and terminal for a fixed demand pattern.

Consider routes $R_1 = \{(1),(3)\}$

and $R_2 = \{(1),(2),(3)\}$

We have , $UT(R_1) = \dfrac{D(1,3)}{S(1,3)}$

And $UT(R_2) = \dfrac{D(1,2)}{S(1,2)} + \dfrac{D(1,3)}{S(1,2)+S(2,3)} + \dfrac{D(2,3)}{S(2,3)}$.

Case I: If D(1,2)=D(1,3) = D(2,3) =D (say) then UT $(R_1) = \dfrac{D}{1}$ =D

and $UT(R_2) = \dfrac{D}{1} + \dfrac{D}{2} + \dfrac{D}{1} = \dfrac{5}{2} D$.

Implies $\frac{UT(R_1)}{UT(R_2)} = 2.5$

$\frac{UT(R_1)}{UT(R_2)} > 1$, we say that route $R_2$ is better than $R_1$.

The ratio $\frac{UT(R_2)}{UT(R_1)}$ in case I (viz., 2.5) is more than that in case II (viz.,1.5). Under such circumstances we say that the choice of route $R_2$ in case I is at a better advantage than that in case II.

**Case III:** If $D(1,2) = \frac{D(1,3)}{4} = D''$ (say)

Then $UT(R_1) = \frac{4D''}{1} = 4D''$

and $UT(R_1) = \frac{D'}{1} + \frac{4D''}{2} + \frac{D''}{1} = 4D''$

$\Rightarrow \frac{UT(R_2)}{UT(R_1)} = 1$

Here we say that selecting route $R_2$ as compared with route $R_1$, is as good a decision as selecting route $R_1$ as compared with route $R_2$.

**Discussion:** Selecting of $R_2$ implies D(1,3) passengers have to travel more by a distance of 1 unit (Ref. Fig. (3.2)).But this guarantees direct route facility for the passengers between nodes (2) and (3).We say that we are offering services D(1,2)and D(2,3) passengers at the cost of best possible service to D(1,3) passengers.

In case III we are indirectly assuming that the resulting discomfort to D (1,3) by selecting $R_2$ is exactly balanced by the added service to the passengers D(1,2) and D(2,3).

**Case IV .If** $D(1,2) = D(2,3) = \frac{D(1,3)}{5} = D'''$(say)

Then $UT(R_1) = 5D'''$.

and $UT(R_2) = D''' + \frac{5}{2} D''' + D''' = \frac{9}{2} D'''$.

Implies $\dfrac{UT(R_1)}{UT(R_2)} = \dfrac{9}{10} < 1.$

Here we say that route $R_2$ is a bad choice as compared to route $R_1$.

As in case III, if $\dfrac{UT(R_1)}{UT(R_2)}$=1, then even slight increase in values of both UT ($R_1$) and UT($R_2$). Since, now the amount of discomfort to which the passengers between node (1) and node (3) are subjected on route $R_2$ is more as compared with the service offered to passengers between nodes (1) and (2) and (3). This is so, as no demand is added to D(1,2) and D(2,3).Hence $R_1$ is a better choice as compared with $R_2$. The same will result with a drop in D(1,2) +D(2,3).

On the other hand, a slight increase in D(1,2)+D(2,3) will render route $R_2$ as a better choice. Now, we are, indirectly, giving more weight to the service offered to the additional passengers on route $R_2$ as compared to the weight assignment to the discomfort caused to D(1,3) passengers. The same will result with a drop is the demand D(1,3).

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