Asymptotic Zeros of Hypergeometric Bernoulli Polynomials

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Declaration

I, with student ID number $GSR/2973/05$, hereby declare that this thesis is my own work and that it has not been previously submitted for assessment or completion of any post graduate qualification to another university or for another qualification.

__________________________ Date ________________

Nasir Asfaw Kelifa

Certificate

I hereby certify that I have read this dissertation prepared by Nasir Asfaw under my supervision and recommended that, it should be accepted as fulfilling the dissertation requirement.

______________________________ Date __________________

Abdulkadir Hassen, Professor

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This is to certify that the thesis prepared by Nasir Asfaw Kelifa, entitled: *Asymptotic Zeros of Hypergeometric Bernoulli Polynomials* and submitted in fulfilment of the requirements for the Degree of Doctor of Philosophy (in Mathematics) compiles with the regulations of the University and meets the accepted standards with respect to originality and quality.

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Nasir Asfaw Kelifa
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Dedication

[1] I dedicated this thesis to:

My father Asfaw Kelifa

and

My mother Aysha Sayid

[1] "*I shall never flaunt the little learning that I have acquired through the care and help my father has given me. If I have learned anything, it is only because he took care to teach me. Had he not taken upon himself the trouble of instructing me, I would be as ignorant as many other children.*" Augustin-Louis Cauchy

Abstract

Bernoulli polynomials are named after the Swiss mathematician Jacob Bernoulli (1654 -1705). These are the class of polynomials $\{B_n(x)\}$ defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} \quad \text{for} \quad |z| < 2\pi.$$

With $B_n = B_n(x)$, the rational numbers B_n are called Bernoulli numbers.

In 1999, A. P. Veselov and J. P. Ward[28] established an asymptotic representation for $B_n(x)$ and described several properties of real zeros of $B_n(x)$ for large values of n. Later in 2008, *John Mangual*[24] considered another method and discussed the asymptotic real and complex zeros of $B_n(x)$. He precisely explained the asymptotic complex zeros of $B_n(nx)$ by introducing a curve to which the complex zeros are attracted as n goes to infinity.

In 2008, Abdulkadir Hassen and Hieu D. Nguyen [15] considered a generalization of $B_n(x)$ called *Hypergeometric Bernoulli polynomials* of order N, $B_n(N, x)$, defined by

$$\frac{z^N e^{xz}/N!}{e^z - T_{N-1}(z)} = \sum_{n=0}^{\infty} B_n(N, x)\frac{z^n}{n!},$$

where $T_N(z) = \sum_{k=0}^{N} \frac{z^k}{k!}$. When $N = 2$, we obtain the class of polynomials $\{B_n(2, x)\}$ first considered by F. T. Howard[21] (with another notation).

In this thesis, we introduce some properties of $B_n(N, x)$ which are analogous to that of Bernoulli polynomials. We establish an asymptotic formula for $B_n(2, x)$ and determine their asymptotic zeros. We briefly explain the behavior of the real and complex zeros of $B_n(2, x)$ for sufficiently large positive integers n. We prove that the complex zeros of $B_n(2, nz)$ asymptotically lie on a curve whose equation is given by

$$r_1 e|z| = \begin{cases} e^{y_1 \Im(z)} & : \ \Im(z) > 0 \\ e^{-y_1 \Im(z)} & : \ \Im(z) < 0 \end{cases},$$

where $z_1 = x_1 + iy_1$ and $\bar{z}_1 = x_1 - iy_1$ are roots of $\varphi(z) = e^z - 1 - z$ of the minimum modulus $r_1 = |z_1| = |\bar{z}_1|$.

Contents

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Introduction

This thesis report consists three chapters. In Chapter 1, we present several preliminary concepts and major known results about the Bernoulli polynomials $B_n(x)$. We briefly discuss basic properties of $B_n(x)$, their relation to the Hurwitz's zeta function $\zeta(s, x)$ and about asymptotic zeros of $B_n(n)$. Also we present some known results about the *hypergeometric Bernoulli polynomials of order N,* $B_n(N, x)$. We discuss some of the alternative definitions and basic properties of $B_n(N, x)$ which are analogous to that of the Bernoulli polynomials $B_n(x)$.

In Chapter 2, we introduce the first few results of our study. These results are about the properties of $B_n(N, x)$ which are analogous to that of the Bernoulli polynomials $B_n(x)$. These results will be discussed mainly under Theorem 2.1, Theorem 2.2 and Theorem 2.6. In the remaining part of Chapter 2, we discuss some important relations between hypergeometric Bernoulli polynomials $B_n(N, x)$ and hypergeometric Hurwitz zeta functions $\zeta_N(s, x)$. We present some known results about series representation of $\zeta_2(s, x)$ and relations between $B_n(2, x)$ and $\zeta_2(s, x)$. In fact, we made a little extension of the region on which these known results are valid.

Chapter 3 consists of the major results of our study. These are the asymptotic real and complex zeros of hypergeometric Bernoulli polynomials of order 2, $B_n(2, x)$. We briefly explain the results regarding asymptotic real zeros of $B_n(2, x)$ under Theorem 3.1, Corollary 3.3, Theorem 3.4, Theorem 3.6 and Theorem 3.7. For the complex zeros, we establish an asymptotic representation for the rescaled hypergeometric Bernoulli polynomials $B_n(2, nz)$ and we present these results under the several lemmas stated in Section 3.2. Then we prove Theorem 3.18 which provides a special curve (3.12) in the complex plane to which complex zeros of $B_n(2, nz)$ approach asymptotically as n goes to infinity. In Section 3.3, we discuss some interesting curves related to the H-shaped curve we obtained in (3.12), which are called Szegö curves. Then we state and prove our last result under Theorem 3.24 . Finally, we conclude our study by giving a brief summary of main results and conclusions of the study in Section 3.4.

Chapter 1

Review of Preliminary Concepts

In this chapter, we briefly discuss some basic concepts about Bernoulli polynomials $B_n(x)$, which we some times call classical Bernoulli polynomials. We consider alternative definitions and several interesting properties of $B_n(x)$. Then we explain some known results regarding the real and complex zeros of Bernoulli polynomials. Finally, we briefly discuss some known results about the hypergeometric Bernoulli polynomials of order N, $B_n(N, x)$. We describe several approaches for defining $B_n(N, x)$ which are analogous to that of $B_n(x)$.

1.1 Bernoulli Numbers and Polynomials

Bernoulli numbers and Bernoulli polynomials are named after the Swiss mathematician Jacob Bernoulli (1654-1705). He introduced these numbers and polynomials in his book *Ars Conjectandi*, published posthumously (Basel, 1713). The Bernoulli polynomials $B_n(x)$, as appeared in Ars Conjectandi, are:

Figure (1.1) shows a list of formulas as written on the book "Ars Conjectandi" of Jacob Bernoulli, in which Bernoulli numbers and Bernoulli polynomials first appeared in print (1713). The symbol "$\int$", an elongated S, is used for "summation" and the open "∞" symbol is used for "=".

In other words, the Bernoulli numbers B_n first appeared a sum over k of the power k^n. Indeed, Jacob Bernoulli himself calculated these sums up to $n = 10$, hence the Bernoulli numbers $B_1, B_2, \cdots, B_{10}$. The first three of the sums given in Figure (1.1), by using modern notations, are:

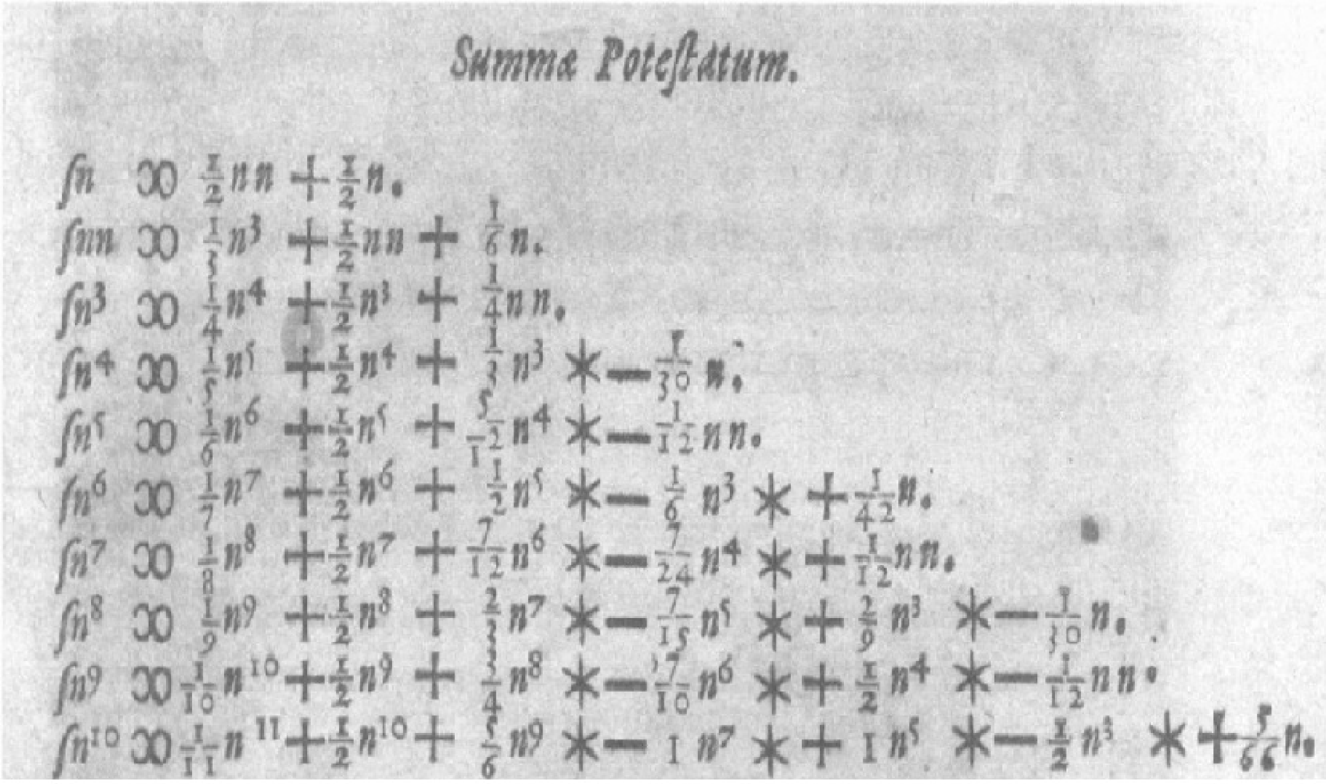
$$\sum_{k=1}^{n} k = \frac{1}{2}n^2 + \frac{1}{2} = \frac{1}{2}n(n+1),$$
$$\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{1}{6}n(n+1)(2n+1)\,,$$
$$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 = \frac{1}{4}n^2(n+1)^2.$$

In terms of Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$, we will see that each of these sums is expressed as:

$$\sum_{k=1}^{m-1} k^n = \frac{1}{n+1}\left(B_{n+1}(m) - B_{n+1}\right).$$

Moreover, we obtain the general formula,

$$\sum_{k=1}^{m-1} (x+k)^n = \frac{1}{n+1}\left(B_{n+1}(x+m) - B_{n+1}(x)\right).$$


Summæ Potestatum.

$\int n \infty \frac{1}{2}nn + \frac{1}{2}n.$

$\int nn \infty \frac{1}{3}n^3 + \frac{1}{2}nn + \frac{1}{6}n.$

$\int n^3 \infty \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}nn.$

$\int n^4 \infty \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 \ast - \frac{1}{30}n.$

$\int n^5 \infty \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 \ast - \frac{1}{12}nn.$

$\int n^6 \infty \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 \ast - \frac{1}{6}n^3 \ast + \frac{1}{42}n.$

$\int n^7 \infty \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 \ast - \frac{7}{24}n^4 \ast + \frac{1}{12}nn.$

$\int n^8 \infty \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 \ast - \frac{7}{15}n^5 \ast + \frac{2}{9}n^3 \ast - \frac{1}{30}n.$

$\int n^9 \infty \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 \ast - \frac{7}{10}n^6 \ast + \frac{1}{2}n^4 \ast - \frac{1}{12}nn.$

$\int n^{10} \infty \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 \ast - 1n^7 \ast + 1n^5 \ast - \frac{1}{2}n^3 \ast + \frac{5}{66}n.$

Figure 1.1: Bernoulli numbers and polynomials, as written on the book "Ars Conjectandi" (1713).

A Recurrence Formula

The Bernoulli numbers B_n are defined recursively as: $B_0 = 1$, and

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0\,, \quad for\ n = 2, 3, 4, \cdots . \tag{1.1}$$

The recursive formula (1.1) generates an interesting sequence of rational numbers $\{B_n\}$, the sequence of Bernoulli numbers. The sequence of Bernoulli numbers $\{B_n\}$ possess several interesting properties. By using the formula (1.1), we list the first few Bernoulli numbers as:

$$B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0,$$
$$B_6 = \frac{1}{42}, \quad B_7 = 0, \quad B_8 = -\frac{1}{30}, \quad B_9 = 0, \quad B_{10} = \frac{5}{66}, \cdots .$$

Observe that the B_n's with odd indices are vanishing terms. This is one of the properties of B_n which Bernoulli numbers possess and we discuss several other properties of Bernoulli numbers later. Jacob Bernoulli considered the sequence $\{B_n(x)\}$ of Bernoulli polynomials; they can be defined by a recursive formula, or by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \tag{1.2}$$

where the B_k's are the Bernoulli numbers given in the recurrence formula (1.1). Indeed, the Bernoulli numbers B_n are particular values of $B_n(x)$ with $x = 0$.

Combining the recurrence formulas (1.1) and (1.2), we obtain an explicit formula for Bernoulli polynomials. That is, the polynomials $B_n(x)$ are expressed explicitly as

$$B_n(x) = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x+j)^n.$$

Generating Functions

Bernoulli polynomials are also defined by a generating function as

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}. \tag{1.3}$$

If we put $x = 0$ in (1.3), we get the Bernoulli numbers $B_n = B_n(0)$. That is, the Bernoulli numbers B_n are defined by generating function as

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}. \tag{1.4}$$

Appell Sequence Definition

We have a third alternative approach towards the definition of Bernoulli polynomials and numbers. This is an Appell sequence definition of $B_n(x)$, given by

$$B_0(x) = 1 \tag{1.5}$$

$$B_n'(x) = nB_{n-1}(x) \tag{1.6}$$

$$\int_0^1 B_n(x)\, dx = \begin{cases} 1 & , \quad n = 0 \\ 0 & , \quad n > 0 \end{cases} \tag{1.7}$$

Each of the three definitions, that is, the recurrence formula (1.2), the generating function (1.3) and the Appell sequence (1.5) - (1.7), generate the same sequence of polynomials. This is the sequence $\{B_n(x)\}$ of Bernoulli polynomials whose first few terms are:

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6},$$
$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \quad B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30},$$
$$B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^2 - \frac{1}{6}x, \cdots.$$

The need of studying Bernoulli polynomials $B_n(x)$ is because it helps much for further understanding of the Bernoulli numbers B_n. Indeed, the Bernoulli numbers B_n are constant terms of the Bernoulli polynomials $B_n(x)$. That is, $B_n(0) = B_n$ for all $n \geq 1$.

For positive integers $n \geq 2$, the Bernoulli numbers B_n are also given by $B_n = B_n(1)$. To prove this, observe that

$$\begin{aligned} B_n(1) &= \sum_{k=0}^{n} \binom{n}{k} B_k \\ &= B_n + \sum_{k=0}^{n-1} \binom{n}{k} B_k. \end{aligned}$$

Then by (1.1), we conclude that $B_n(1) = B_n = B_n(0)$ for all $n \geq 2$.

The (classical) Bernoulli polynomials and Bernoulli numbers possess many interesting properties. Among such properties is that $B_{2k+1} = 0$ for all $k \geq 1$. To prove this, Tom M. Apostol [2] re-expressed (1.4) as

$$\frac{z}{e^z - 1} + \frac{z}{2} = 1 + \sum_{n=2}^{\infty} B_n \frac{z^n}{n!}.$$

Then the left hand expression represents an even function of z so that all the odd terms in the right hand series vanish.

Apostol also used generating functions in [2] and proved several properties of $B_n(x)$. Some of the well known properties of Bernoulli polynomials are:

- *Symmetry Property*

$$B_n(1-x) = (-1)^n B_n(x) \qquad for \quad n \geq 0. \tag{1.8}$$

- *Difference Equation*

$$B_n(x+1) - B_n(x) = nx^{n-1} \qquad for \ n \geq 1. \tag{1.9}$$

- *Addition Formula*

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k}. \tag{1.10}$$

- *Raabe's Multiplication Formula*

$$B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n \left(x + \frac{k}{m} \right), \tag{1.11}$$

  where m and n are integers with $n \geq 0$ and $m \geq 1$.

Apostol proved the above properties by using different identities. For instance, he proved (1.9) by using the generating function (1.3) in the identity

$$\frac{ze^{(x+1)z}}{e^z - 1} - \frac{ze^{xz}}{e^z - 1} = ze^{xz}.$$

The properties (1.8) - (1.11) of $B_n(x)$ are used in proving different properties of Bernoulli polynomials and Bernoulli numbers. Indeed, in each of these properties of $B_n(x)$, we put $x = 0$ and obtain the corresponding properties of Bernoulli numbers.

For any positive integer m, we replace x by $x+k$ in (1.9) and take sum over k from $k=1$ to $k=m$ and get

$$\sum_{k=1}^{m-1}(x+k)^n = \frac{1}{n+1}\left(B_{n+1}(m+x) - B_{n+1}(x)\right).$$

The particular case when $x = 0$ yields the sum of the powers k^n, the modern form of the sums appeared in "Ars Conjectandi" of Jacob Bernoulli,

$$\sum_{k=1}^{m-1} k^n = \frac{1}{n+1}\left(B_{n+1}(m) - B_{n+1}\right).$$

Moreover, with $x = \frac{a}{d}$, where a and d are integers, we get a formula for sum of n^{th} powers of m integers in an arithmetic progression, given by

$$\sum_{k=0}^{m-1}(a+dk)^n = \frac{d^n}{n+1}\left(B_{n+1}\left(m+\frac{a}{d}\right) - B_{n+1}\left(\frac{a}{d}\right)\right).$$

Each of the properties (1.8) - (1.11) will be used in the determination of asymptotic zeros of $B_n(x)$. Later in this chapter, we shall illustrate how people used these properties to obtain other interesting properties of $B_n(x)$, especially the asymptotic behavior of the real and complex zeros of $B_n(x)$.

1.2 Zeta Functions and Bernoulli Polynomials

The Riemann's Zeta Function $\zeta(s)$

The Riemann's zeta function, $\zeta(s)$, is defined by

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \quad \text{for } \Re(s) = \sigma > 1.$$

We have an interesting relation between the Riemann zeta function $\zeta(s)$ and the Bernoulli numbers B_n. We illustrate this relation as a consequence of Theorem 1.1 below. Indeed, we consider further description of the relation later as particular cases of the Hurwitz zeta function and Bernoulli polynomials.

Theorem 1.1. *For each positive integer m and $\Re(s) = \sigma > 1-m$, $s \neq 1$, the Riemann zeta function $\zeta(s)$ is given by*

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{k=2}^{m}(s)_{k-1}\frac{B_k}{k!} - \frac{(s)_m}{m!}\int_1^{\infty} B_m(x-[x])x^{-s-m}\,dx,$$

where $(s)_m = s(s-1)(s-2)\cdots(s-m+1)$.

In fact, Theorem 1.1 gives an analytic continuation of the Riemann zeta function $\zeta(s)$ to the half-plane $\Re(s) = \sigma > 1-m$ for each positive integer m, except for a simple pole at $s = 1$. It also enables us to find some special values of $\zeta(s)$. For instance, we easily obtain $\zeta(0) = -\frac{1}{2}$. More generally, we give relations between Bernoulli numbers and $\zeta(s)$ by taking the negative integer values $s = -n$. That is,

$$\zeta(-n) = \frac{-1}{n+1} + \frac{1}{2} - \sum_{k=2}^{n+1}\frac{B_k}{k!}n(n-1)\cdots(n-k+2).$$

Also a further simplification yields

$$(n+1)\zeta(-n) = -\sum_{k=0}^{n+1} \binom{n+1}{k} B_k.$$

Finally, we use the definition (1.1) of Bernoulli numbers and obtain

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}.$$

Since $B_{2k+1} = 0$ for all $k \geq 1$, we conclude that $\zeta(-2k) = 0$ for $k \geq 1$. This confirms the fact that all negative even integers are zeros of the Riemann zeta function.

The Hurwitz Zeta Function $\zeta(s,\, a)$

For $a \in \mathbb{R}$ such that $0 < a \leq 1$, the Hurwitz zeta function $\zeta(s,\, a)$ is defined by

$$\zeta(s,\, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}\,, \quad \text{for } \Re(s) = \sigma > 1. \tag{1.12}$$

The particular case when $a = 1$ reduces (1.12) to the Riemann zeta function

$$\zeta(s,\, 1) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Note that for $a = 1$ and $n \geq 2$, the Bernoulli polynomial $B_n(a)$ reduces to the Bernoulli number B_n, that is, $B_n(1) = B_n$. Therefore, we relate the Hurwitz zeta function $\zeta(s,\, a)$ to Bernoulli polynomials.

For any $\delta > 0$ and $\Re(s) = \sigma > 1$, we have

$$\sum_{n=1}^{\infty} \left|(n+a)^{-s}\right| = \sum_{n=1}^{\infty} (n+a)^{-\sigma} \leq \sum_{n=1}^{\infty} (n+a)^{-(1+\delta)}.$$

Therefore, the series (1.12) converges absolutely and uniformly in every closed half plane inside the region $\Re(s) = \sigma > 1$. Hence, the Hurwitz zeta function $\zeta(s,\, a)$ is an analytic function of s in the region $\Re(s) = \sigma > 1$. Moreover, for any $a \in \mathbb{R}$ such that $0 < a \leq 1$ and $\Re(s) = \sigma > 1$, we have an integral representation for $\zeta(s,\, a)$, given by

$$\zeta(s,\, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} e^{-ax}}{1 - e^{-x}}\, dx, \tag{1.13}$$

where $\Gamma(s)$ is the Γ-function, $\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x}\, dx$.
Indeed, (1.13) is obtained by multiplying $\Gamma(s)$ to (1.12) and using the fact that $\zeta(s,\, a)$ is analytic inside the half-plane $\Re(s) = \sigma > 1$.

Remark 1.2. *Many authors considered the definition of $\zeta(s,\, a)$ for a real number $a, 0 < a \leq 1$. However, we observed that the series (1.12) converges absolutely and uniformly in the region $\Re(s) = \sigma > 1$, for any real number $a > 0$. Therefore, given any real number $a > 0$, the function $\zeta(s,\, a)$ is analytic in the half plane $\Re(s) = \sigma > 1$.*

Analytic Continuation of $\zeta(s,\, a)$

For analytic continuation of the Hurwitz's zeta function $\zeta(s,\, a)$ to the left half-plane $\Re(s) = \sigma < 1$, we consider another representation of $\zeta(s,\, a)$. Consider the contour C, where C is the union of the curves C_1, C_2 and C_3, as given in Figure 1.2 below.

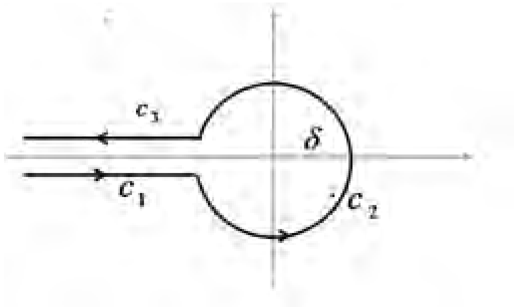


Figure 1.2: The key-hole contour C

Define $I(s,\, a)$ by a contour integral over C, as

$$I(s,\, a) = \frac{1}{2\pi i}\int_C \frac{z^{s-1}e^{az}}{1-e^z}\, dz. \tag{1.14}$$

Then $\zeta(s,\, a)$ is expressed in terms of $I(s,\, a)$ as given in Theorem 1.3 below.

Theorem 1.3. *If $a \in \mathbb{R}$, $a > 0$, then the function $I(s,\, a)$ defined in (1.14) is an entire function of s. Moreover, if $\Re(s) = \sigma > 1$, then we have*

$$\zeta(s,\, a) = \Gamma(1-s)I(s,\, a). \tag{1.15}$$

The proof of Theorem 1.3 appears in several books of analytic number theory. We may refer to the book by Tom M. Apostol [1].

Remark 1.4. *The expression (1.15) extends the analyticity of $\zeta(s,\, a)$ to the left half plane $\Re(s) = \sigma < 1$. Clearly, $I(s,\, a)$ is entire and $\Gamma(1-s)$ is analytic in $\mathbb{C} \setminus \{1, 2, 3, \cdots\}$. Thus, (1.15) implies that $\zeta(s,\, a)$ is analytic in the left half plane $\Re(s) = \sigma < 1$. Thus, by combining (1.12) and (1.15), we conclude that $\zeta(s,\, a)$ is analytic in the whole complex plane except for a simple pole at $s = 1$ with residue* 1.

Fourier Series Representation of $\zeta(s,\, a)$

Now we discuss a Fourier series representation of the Hurwitz zeta function $\zeta(s,\, a)$. We state Hurwitz's Formula in which we extend the value of the parameter a to be any positive real number. Also we reproduce the proof here since we may use some concepts in the proof later for the case of Bernoulli polynomials.

Theorem 1.5 (Hurwitz's Formula)**.** *If $a \in \mathbb{R}$, $a > 0$ and $s \in \mathbb{C}$ such that $\Re(s) = \sigma > 1$, then we have*

$$\zeta(1-s\,,\, a) = \frac{\Gamma(s)}{(2\pi i)^s}\sum_{|k|\geq 1}\frac{e^{2\pi i k a}}{k^s}. \tag{1.16}$$

Proof. Let $I_N(s,\, a)$ be defined by a contour integral as

$$I_N(s,\, a) = \frac{1}{2\pi i}\int_{C(N)} \frac{z^{s-1}e^{az}}{1-e^z}\, dz,$$

where $C(N)$ is the annulus given in Figure 1.3 below.

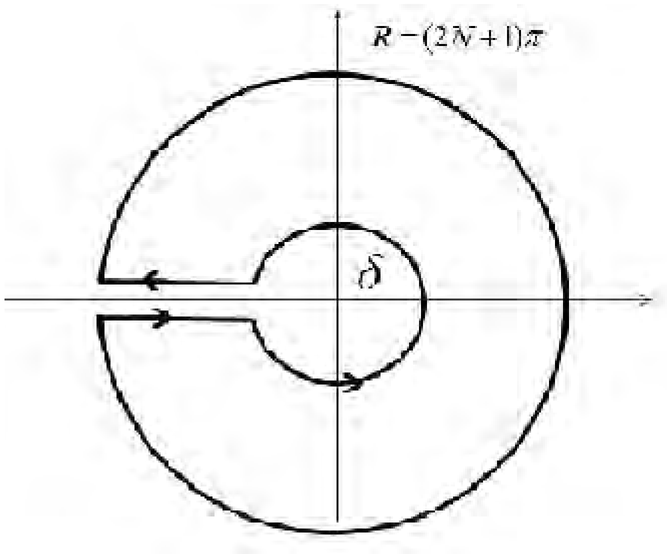


Figure 1.3: The contour (annulus) $C(N)$

Then we get $\lim_{N\to\infty} I_N(s,\, a) = I(s,\, a)$, where $I(s,\, a)$ is as given in (1.14). On the other hand, we evaluate $I_N(1-s,\, a)$ explicitly by using residue calculus. Clearly the poles of the integrand are $z = 2\pi i k$ for all $k = 0, \pm 1, \pm 2, \cdots$ among which only $k = \pm 1, \pm 2, \cdots, \pm N$ lie inside $C(N)$. Thus by the residue theorem we have

$$\begin{aligned} I_N(1-s,\, a) &= \frac{1}{2\pi i}\int_{C(N)} \frac{z^{-s}e^{az}}{1-e^z}\, dz \\ &= -\sum_{0<|k|\le N} Res\left(\frac{z^{-s}e^{az}}{1-e^z}\,;\, z = 2\pi i k\right). \end{aligned}$$

Then by calculating residues at $z = 2\pi i k$ and letting $N \to \infty$, we obtain

$$I(1-s,\, a) = \sum_{|k|\ge 1} \frac{e^{2\pi i k a}}{(2\pi i k)^s}.$$

Finally, we use this expression of $I(1-s,\, a)$ in (1.15) and complete the proof. □

Note that the Hurwitz formula gives a series representation of the function $\zeta(1-s,\, a)$ in the region $\Re(s) = \sigma > 1$. If we replace $1-s$ by s in (1.16), it becomes

$$\zeta(s\,,\, a) = \frac{\Gamma(1-s)}{(2\pi i)^{1-s}} \sum_{|k|\ge 1} \frac{e^{2\pi i k a}}{k^{1-s}} \quad \text{for } \Re(s) = \sigma < 0.$$

Moreover, by using an appropriate trigonometric identity, we obtain

$$\zeta(s,\, a) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}} \sum_{k=1}^{\infty} \frac{\sin\left(2\pi k a + \frac{\pi s}{2}\right)}{k^{1-s}}\,, \quad \text{for } \Re(s) = \sigma < 0. \tag{1.17}$$

Relations between $\zeta(s,\, a)$ and $B_n(x)$

To determine the relationship between the Hurwitz's zeta function $\zeta(s\,,\, a)$ and Bernoulli polynomials $B_n(x)$, we use negative integer values $s = -n$ in $\zeta(s\,,\, a)$. For a positive integer n, put $s = -n$ so that $\Gamma(1-s) = \Gamma(1+n) = n!$. Then (1.15) becomes

$$\zeta(-n,\, a) = n!\, I(-n,\, a).$$

Now applying the Residue Theorem to the contour integral (1.14) of $I(s,\, a)$, we get

$$I(-n,\, a) = Res(\frac{z^{-n-1}e^{az}}{1-e^{z}}\,;\, z=0).$$

To calculate this residue, we use the series in (1.3) for $B_n(a)$. Then we obtain an important relation between $\zeta(s,\, a)$ and $B_n(a)$, given by

$$\zeta(-n+1\,,\, a) = -\frac{B_n(a)}{n}\,, \quad \text{for } n > 1\,,\; a > 0. \tag{1.18}$$

Now combining (1.16) (with $s = n$, $n > 1$) and (1.18), we get a Fourier series representation for Bernoulli polynomials. Therefore, for any real number $a > 0$ and positive integers $n > 1$, the Bernoulli polynomials $B_n(a)$ are given by

$$B_n(a) = -\frac{n!}{(2\pi i)^n} \sum_{|k|\geq 1} \frac{e^{2\pi i k a}}{k^n}. \tag{1.19}$$

Moreover, by taking the term $(i)^{-n}$ in to the series, we express (1.19) as

$$B_n(a) = \frac{-2\, n!}{(2\pi)^n} \sum_{k=1}^{\infty} \frac{\cos\left(2\pi k a - \frac{\pi n}{2}\right)}{k^n}. \tag{1.20}$$

Further, a simple trigonometric identity yields

$$B_{2n}(a) = (-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos(2\pi k a)}{k^{2n}}$$

and

$$B_{2n+1}(a) = (-1)^{n+1} \frac{2(2n+1)!}{(2\pi)^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin(2\pi k a)}{k^{2n+1}}.$$

Since $B_n(1) = B_n$ for all $n \geq 2$, we put $a = 1$ and get a formula relating Bernoulli numbers B_n and Riemann's zeta $\zeta(s)$. That is,

$$B_{2n} = (-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n).$$

1.3 Asymptotic Zeros of Bernoulli Polynomials

By the word "asymptotic zeros", we mean zeros of $B_n(x)$ for sufficiently large positive integers n. The need to study asymptotic zeros is because zeros of $B_n(x)$ possess an interesting behavior when n tends to infinity. In 1999, A. P. Veselov and J. P. Ward[28] established an asymptotic representation for

$B_n(x)$ inside large intervals of the real line. They described several properties of real zeros of $B_n(x)$ for large values of n. Later in 2008, *John Mangual*[24] considered another method and discussed asymptotic real and complex zeros of Bernoulli polynomials. He precisely explained the asymptotic complex zeros of $B_n(nx)$ by introducing an H-shaped curve to which the complex zeros are attracted as n goes to infinity.

In this section, we discuss some known results about asymptotic zeros of Bernoulli polynomials. We carefully review the method used by Veselov and Ward[28] as well as that of *John Mangual*[24] because we apply it later to determine the asymptotic zeros of hypergeometric Bernoulli polynomials of order 2.

The Asymptotic Real Zeros of $B_n(x)$

Now we re-express the Fourier series (1.20) of $B_n(x)$ as

$$B_n(a) = \frac{-2\, n!}{(2\pi)^n}\left[\cos\left(2\pi ka - \frac{\pi n}{2}\right) + \sum_{k=2}^{\infty}\frac{\cos\left(2\pi ka - \frac{\pi n}{2}\right)}{k^n}\right].$$

If we let $Q_n = \frac{-2\, n!}{(2\pi)^n}$, then we obtain

$$\frac{B_n(a)}{Q_n} = \cos\left(2\pi a - \frac{\pi n}{2}\right) + R_n(a),$$

where $R_n(a) = \sum_{k=2}^{\infty}\frac{\cos\left(2\pi ka - \frac{\pi n}{2}\right)}{k^n}$. Clearly, $R_n(a) \to 0$ uniformly as $n \to \infty$. Therefore, for sufficiently large n, we have $R_n(a) \approx 0$ so that

$$\frac{B_n(a)}{Q_n} \approx \cos\left(2\pi a - \frac{\pi n}{2}\right).$$

By using the asymptotic notation 'little o', we write $R_n(a) = \circ(1)$ as $n \to \infty$ so that

$$\frac{B_n(a)}{Q_n} = \cos\left(2\pi a - \frac{\pi n}{2}\right) + \circ(1) \text{ as } n \to \infty. \tag{1.21}$$

The asymptotic formula (1.24) is very important for describing asymptotic zeros of Bernoulli polynomials. Similar results were obtained in [8]. We use these results and discuss asymptotic real zeros of Bernoulli polynomials.

Veselov and Ward [28] used (1.24) and described the asymptotic behavior of real zeros of $B_n(a)$. Indeed, the asymptotic formula (1.24) is derived for $0 < a \leq 1$. However, they extended it to large values of a by using the relation between $B_n(a)$ and $\zeta(s,\, a)$. In particular, for $0 < a \leq 1$ and any integer $m > 1$, they used the functional equation

$$\zeta(s,\, a) = \zeta(s,\, m + a) + \sum_{k=0}^{m-1}(k + a)^{-s}.$$

Their conclusion is that (1.24) holds for all a such that $0 < a < \frac{n}{2\pi e}$ which is obtained as the consequence of the following theorem.

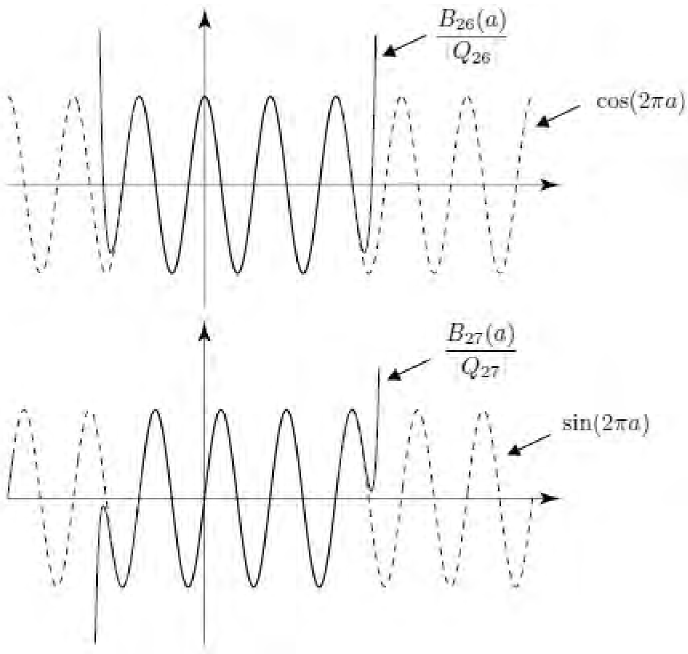


Figure 1.4: Graphs of $\cos a$ and $\sin a$ as compared to $B_n(a)$ ($n = 26$, $n = 27$)

Theorem 1.6. *Let n be any positive integer and $0 < a < \alpha n$ where $\alpha < \frac{1}{2\pi e}$. Then the Hurwitz zeta function satisfies the inequality*

$$\left|\frac{\zeta(-n,\, a)}{Q_n} - \sin(2\pi a - \frac{1}{2}\pi n)\right| < C_1 n^{-\frac{1}{2}} (2\pi e\alpha)^n + C_2 2^{-n},$$

where $Q_n = \frac{2\,\Gamma(1+n)}{(2\pi)^{1+n}}$, C_1 and C_2 are constants which are independent of n. In particular, for $0 < a < \frac{n}{2\pi e}$, we have the asymptotic behavior

$$\frac{\zeta(-n,\, a)}{Q(-n)} = \sin(2\pi a - \frac{1}{2}\pi n) + o(1) \quad \textit{when } n \to \infty.$$

Theorem 1.7. *For any constant $c > \frac{1}{4\pi e}$ there exists m_0 (depending on c) such that*

$$\zeta(-m,\, a) < 0,$$

for any $m > m_0$ and $a > \frac{m}{2\pi e} + c \log m$.

From Theorem 1.7, we conclude that $\zeta(-n,\, a) \neq 0$ for all n larger than some fixed positive integer m_0 and $a > \frac{n}{2\pi e} + c \log n$. Then by using the relation between $\zeta(-n,\, a)$ and $B_n(a)$ given in (1.18), we determine a certain large interval of the real line outside of which $B_n(a)$ do not vanish. This in turn helps to determine the largest possible root $A(n)$ of $B_n(a)$. In general, the following theorem summarizes the major results obtained in [28]. For $x \in \mathbb{R}$, we use the notation $\lfloor x \rfloor$ to represent the largest integer less than or equal to x.

Theorem 1.8. *Let $N(n)$ be the number of real roots of $B_n(a)$and let $A(n)$ be the largest root. Let α be any constant such that $\alpha > 1$. For n sufficiently large,*

$$\frac{n}{2\pi e} - \frac{1}{2} < A(n) < \frac{n}{2\pi e} + \frac{\alpha}{4\pi e} \log n. \tag{1.22}$$

and

$$1 + 4\lfloor\frac{n-2}{4\pi e}\rfloor < N(n) < \frac{2n}{\pi e} + \alpha \log n. \tag{1.23}$$

The roots of $B_n(a)$ in the interval $-\lfloor\frac{n-2}{2\pi e}\rfloor < a < 1 + \lfloor\frac{n-2}{2\pi e}\rfloor$ are simple and close to the half-integer lattices

$$a = 0,\ \pm\frac{1}{2},\ \pm 1,\ \pm\frac{3}{2},\ \pm 2,\ \pm\frac{5}{2},\ \cdots \quad \textit{if } n \textit{ is odd}$$

and

$$a = \pm\frac{1}{4},\ \pm\frac{3}{4},\ \pm\frac{5}{4},\ \pm\frac{7}{4},\ \cdots \quad \textit{if } n \textit{ is even.}$$

In Theorem 1.8, the authors [28] described several properties of asymptotic real zeros of $B_n(a)$. For a Bernoulli polynomial $B_n(a)$ of large degree n, they approximated the maximum number of its real zeros $N(n)$, and the largest possible real zero $A(n)$. For sufficiently large positive integers n, the real zeros of $B_n(a)$ are close enough to roots of either

$$\cos\left(2\pi a - \frac{\pi n}{2}\right) = 0$$

or

$$\sin\left(2\pi a - \frac{\pi n}{2}\right) = 0.$$

It is by this asymptotic relation to trigonometric functions that Veselov and Ward [28] illustrated many asymptotic behaviors of the real zeros of Bernoulli polynomials. Clearly, $\cos(2\pi a - \frac{1}{2}\pi n) = 0$ if and only if $2\pi a - \frac{1}{2}\pi n = \frac{(2k+1)\pi}{2}$. Therefore, the roots of $B_n(a)$ are asymptotically approximated to be

$$a = \frac{m+1}{4} + \frac{k}{2},$$

where m and k are integers. If n is odd, then $a = \frac{m}{2} + \frac{k}{2} = \frac{m+k}{2}$. Hence the asymptotic real zeros of $B_n(a)$ are near the half-integers $a = \frac{m+k}{2}$. Similarly, if n is even, then $a = \frac{q}{4} + \frac{k}{2} = \frac{q+2k}{4}$ where q is an odd integer.

Alternative Approach to Asymptotic Real Zeros of $B_n(a)$

John Mangual [24] considered asymptotic zeros of the re-scaled Bernoulli polynomials $B_n(na)$. He also used the Fourier series representation (1.19) to investigate asymptotic real zeros of $B_n(a)$ but the method is quite different. It is another method for how to extend the asymptotic formula (1.24) to larger values of a. Now we discuss this method and how the asymptotic real zeros are determined.

Any real number $a > 0$ can be expressed as $a = \lfloor a\rfloor + \{a\}$, where $\lfloor a\rfloor$ is an integer and $0 \le \{a\} < 1$ is the fractional part. Moreover, $B_n(na)$ is expressed as

$$B_n(na) = B_n(\{na\}) + B_n(na) - B_n(\{na\}).$$

Then to get asymptotic representation for $B_n(na)$, it suffices to establish asymptotic representations for $B_n(\{na\})$ and $B_n(na) - B_n(\{na\})$ separately. Clearly, an asymptotic representation for $B_n(\{na\})$ is easily obtained from (1.19). That is, for sufficiently large n, we obtain

$$\frac{B_n(\{na\})}{Q_n} \approx e^{2\pi ina} \pm e^{-2\pi ina},$$

where $Q_n = \frac{-2\,n!}{(2\pi)^n}$. Mangual also used Stirling's formula for $n!$ and expressed Q_n as

$$Q_n = -2\left(\frac{n}{2\pi e}\right)^n \sqrt{2\pi n}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right).$$

Lemma 1.9. *Let a be a positive real number such that $0 \leq a < \frac{1}{2\pi e}$. As n approaches infinity,*

$$B_n(ma) - B_n(\{ma\}) \quad \textit{approaches} \quad nm^{n-1}\int_0^a mx^{n-1}\,dx.$$

This convergence is uniform in the interval $[0\,,\,\epsilon]$, for $0 < \epsilon < \frac{1}{2\pi e}$.

We refer to the paper by Mangual [24] for a proof of Lemma 1.9. By replacing m by n, we obtain the important conclusion

$$B_n(na) - B_n(\{na\}) \quad \longrightarrow \quad n^n a^n \quad \text{as } n \to \infty.$$

Moreover, we express this in an appropriate way as

$$\frac{B_n(na) - B_n(\{na\})}{Q_n} \quad \longrightarrow \quad \frac{(2\pi e a)^n}{\sqrt{2\pi n}} \quad \text{as } n \to \infty.$$

Combining asymptotic representations of $B_n(\{na\})$ and $B_n(na) - B_n(\{na\})$ yields

$$\frac{B_n\,(na)}{Q_n} \quad \longrightarrow \quad e^{2\pi i n a} \pm e^{-2\pi i n a} + \frac{(2\pi e a)^n}{\sqrt{2\pi n}} \quad \text{as } n \to \infty.$$

Since $0 \leq a < \frac{1}{2\pi e}$, it follows that $\frac{(2\pi e a)^n}{\sqrt{2\pi n}} \to 0$ as $n \to \infty$. Thus, for sufficiently large values of n, we have

$$\frac{B_n\,(na)}{Q_n} \approx e^{2\pi i n a} \pm e^{-2\pi i n a}.$$

Therefore, the asymptotic real zeros of $B_n(na)$ are approximated by the roots of

$$e^{2\pi i n a} \pm e^{-2\pi i n a} = 0.$$

Indeed, the asymptotic real zeros of $B_n(na)$ are the roots of $e^{2\pi i n a} + e^{-2\pi i n a} = 0$ if n is even and the roots of $e^{2\pi i n a} - e^{-2\pi i n a} = 0$ if n is odd.

Remark 1.10. *If we replace a by na in the result obtained by the previous method, then the real zeros of $B_n(na)$ are asymptotically given by the roots of $\cos\left(2\pi n a - \frac{\pi n}{2}\right) = 0$ or $\sin\left(2\pi n a - \frac{\pi n}{2}\right) = 0$. This is equivalently expressed as*

$$e^{2\pi i n a - i\frac{\pi n}{2}} \pm e^{-2\pi i n a + i\frac{\pi n}{2}} = 0.$$

Moreover, considering cases when n is even or odd, we obtain the asymptotic real zeros of $B_n(na)$ are approximately the roots of $e^{2\pi i n a} \pm e^{-2\pi i n a} = 0$. Therefore, the method used by Veselov and Ward in [28] and that John Mangual used in [24] yield the same results regarding the asymptotic real zeros of $B_n(na)$.

Asymptotic Complex Zeros of $B_n(z)$

For complex zeros of $B_n(z)$, Mangual [24] also considered properly re-scaled polynomials $B_n(nz)$ when n is sufficiently large. He investigated the location of asymptotic complex zeros of $B_n(nz)$ in the complex plane by a curve to which these zeros approach when the values of n goes to infinity.

$$\frac{B_n(a)}{Q_n} = \cos\left(2\pi a - \frac{\pi n}{2}\right) + \circ(1) \text{ as } n \to \infty. \quad (1.24)$$

So we have some relation between the zeros of $B_n(a)$ and that of $\cos\left(2\pi a - \frac{\pi n}{2}\right)$ when n is sufficiently large. Let $c_n(z)$ and $s_n(z)$ represent the n^{th} Taylor polynomials of the cosine an sine functions, respectively. That is,

$$c_n(z) = \sum_{k=0}^{n}(-1)^k \frac{z^{2k}}{(2k)!} \quad \text{and} \quad s_n(z) = \sum_{k=0}^{n}(-1)^k \frac{z^{2k+1}}{(2k+1)!}.$$

For large n, the complex zeros of $B_n(nz)$ are related to the zeros of either of the polynomials of $c_n(nz)$ or $s_n(nz)$. In Figure 1.5, zeros of $c_{60}(60z)$ are indicated by '$\times$'. The solid curve indicated in the figure is $|\phi(wz)| = 1$, where $\phi(wz) = zwe^{1-wz}$ for $w = \pm 2\pi i$. We will explain about this curve later in Chapter 3, Section 3.3.

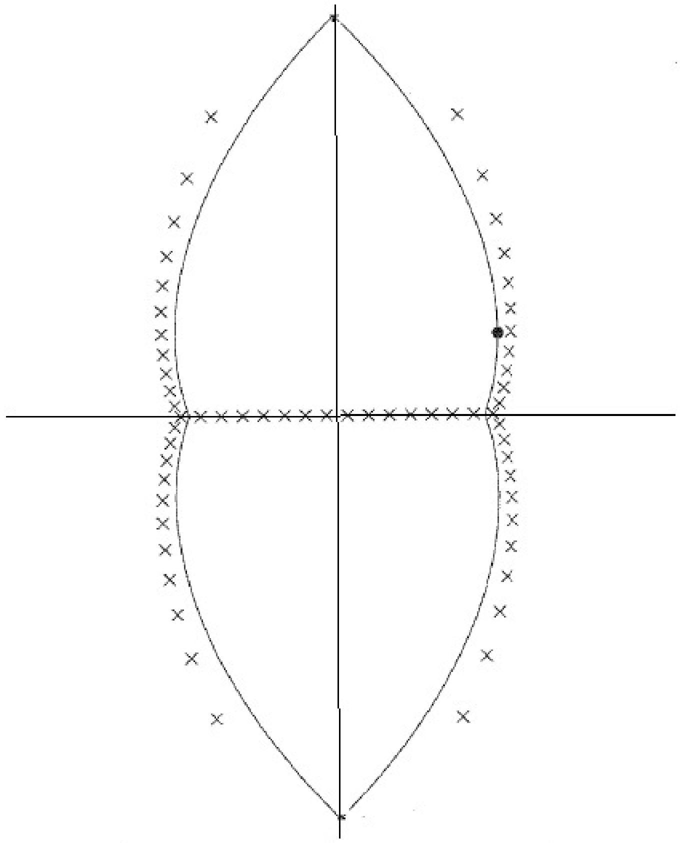


Figure 1.5: Zeros of the n^{th} Taylor polynomial of the cosine function, for $n = 60$

For a complex number $z = a + ib$, we use addition formula (1.10) and express $B_n(z)$ as

$$B_n(z) = B_n(a+ib) = \sum_{k=0}^{n} \binom{n}{k} B_k(a)(ib)^{n-k}.$$

This formula will be used for extending the method used for the real case above to the case of complex variables. The following lemma is the complex version of Lemma 1.9 and its proof is given in [24].

Lemma 1.11. *As n approaches infinity,*

$$B_n(na+inb) - B_n(\{na\}+inb) \quad \textit{approaches} \quad n^n \int_0^a n(x+ib)^{n-1}\,dx.$$

From Lemma 1.11, we conclude that for sufficiently large positive integers n, we obtain the approximation

$$B_n(na+inb) - B_n(\{na\}+inb) \;\approx\; n^n\left((a+ib)^n - (ib)^n\right).$$

Therefore, the asymptotic zeros of $B_n(na+inb) - B_n(\{na\}+inb)$ are approximated by the zeros of $(a+ib)^n - (ib)^n$.

Lemma 1.12. *As n approaches infinity,*

$$B_n(\{na\}+inb) \quad \textit{approaches} \quad \frac{-\left(n\mathcal{O}\left\{n^{\frac{1}{2n}}(1+\frac{1}{n})\right\}\right)^n}{(2\pi i e)^n}\left(T_n(2\pi nb) + T_n(-2\pi nb)\right),$$

where $T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$.

Proof. We use (1.10) and (1.19) for $B_n(\{nx\})$ as:

$$\begin{aligned} B_n(\{nx\}+iny) &= \sum_{m=0}^{n}\binom{n}{k} B_m(\{nx\})(iny)^{n-m} \\ &= \sum_{m=0}^{n} \frac{n!(iny)^{n-m}}{m!(n-m)!}.\frac{(-m!)}{(2\pi i)^m}\sum_{|k|\geq 1}^{\infty}\frac{e^{2\pi iknx}}{k^m}. \end{aligned}$$

Then multiplying both numerator and denominator in the first summation by $(2\pi i)^{n-m}$ and writing the second sum for $k=-1$ and $k=1$ explicitly, we have

$$\begin{aligned} B_n(\{nx\}+iny) &= \frac{-n!}{(2\pi i)^n}\sum_{m=0}^{n}\frac{(-2\pi ny)^{n-m}}{(n-m)!}\left(e^{2\pi inx} + (-1)^m e^{-2\pi inx} + \mathcal{O}\left(\frac{1}{2^m}\right)\right) \\ &= \frac{-n!}{(2\pi i)^n}\left\{e^{2\pi inx}\sum_{m=0}^{n}\frac{(-2\pi ny)^{n-m}}{(n-m)!} + (-1)^n e^{-2\pi inx}\sum_{m=0}^{n}\frac{(2\pi ny)^{n-m}}{(n-m)!} + \sum_{m=0}^{n}\frac{(-2\pi ny)^{n-m}}{(n-m)!}\mathcal{O}\left(\frac{1}{2^m}\right)\right\} \\ &= \frac{-n!}{(2\pi i)^n}\left\{T_n(2\pi ny) + T_n(-2\pi ny) + \frac{1}{2^n}\mathcal{O}\left(T_n(\pm 4\pi ny)\right)\right\}. \end{aligned}$$

Observe that $\left|e^{2\pi inx}\right| = 1$ and

$$\frac{1}{2^n}T_n(4\pi n|y|) = \frac{1}{2^n}\sum_{k=0}^{n}\frac{(4\pi n|y|)^k}{k!} < \sum_{k=0}^{n}\frac{(2\pi n|y|)^k}{k!} = T_n(2\pi n|y|).$$

Since $\frac{1}{2^n}\mathcal{O}\left(T_n(\pm 4\pi ny)\right) \to 0$, we apply Stirling's formula. □

Theorem 1.13. *As n approaches infinity, the complex zeros of $B_n(nz)$ approach the H-shaped curve whose equation is given by*

$$2\pi e|z| = \begin{cases} e^{2\pi\Im(z)} & : \ \Im(z) > 0 \\ e^{-2\pi\Im(z)} & : \ \Im(z) < 0 \end{cases}. \tag{1.25}$$

Proof. For sufficiently large n, we need to have

$$T_n(2\pi inz) \longrightarrow e^{2\pi inz} \text{ uniformly in the disk } \frac{1}{2\pi}D,$$

where $D = \{z \in \mathbb{C} : |z| < 1\}$. For this we use *Dieudonné's Estimate*

$$n!(e^{nz} - T_n(nz)) = \frac{z^n}{1-z}(1 + \lambda_n(z)),$$

where $\lambda_n(z) \to 0$ as $n \to \infty$, uniformly in $\frac{1}{2\pi}D$. This enables us to conclude that

$$T_n(\pm 2\pi nb) \to e^{\pm 2\pi nb} \text{ as } n \to \infty.$$

Therefore, for sufficiently large positive integers n, Lemma 1.12 implies

$$B_n(\{na\} + inb) \approx -\left[n\mathcal{O}\left(n^{\frac{1}{2n}}\left(1 + \frac{1}{n}\right)\right)\right]^n \left(\frac{e^{2\pi nb} + e^{-2\pi nb}}{(2\pi ie)^n}\right).$$

On the other hand, for sufficiently large n, Lemma 1.11 implies

$$\begin{aligned} B_n(na + inb) - B_n(\{na\} + inb) &\approx n^n\left((a+ib)^n - (ib)^n\right) \\ &= n^n(a+ib)^n\left(1 - \left(\frac{ib}{a+ib}\right)^n\right). \end{aligned}$$

Thus, noting that $B_n(na+inb) = B_n(\{na\}+inb) + B_n(na+inb) - B_n(\{na\}+inb)$, we use $b = \Im(z)$ and obtain the approximation

$$B_n(nz) \approx -\left[n\mathcal{O}\left(n^{\frac{1}{2n}}\left(1 + \frac{1}{n}\right)\right)\right]^n \left(\frac{e^{2\pi n\Im(z)} + e^{-2\pi n\Im(z)}}{(2\pi ie)^n}\right) + n^n z^n\left(1 - \left(\frac{i\Im(z)}{z}\right)^n\right).$$

Hence the complex zeros of $B_n(nz)$ are asymptotically the same as the roots of

$$\frac{e^{2\pi n\Im(z)} + e^{-2\pi n\Im(z)}}{(2\pi ie)^n} + z^n = 0.$$

Therefore, the asymptotic complex zeros of $B_n(nz)$ are approximately the roots of

$$e^{2\pi n\Im(z)} + e^{-2\pi n\Im(z)} + (2\pi iez)^n = 0.$$

If $\Im(z) = b > 0$, then $e^{-2\pi nb} \to 0$ as $n \to \infty$. Hence, $(2\pi iez)^n = -e^{2\pi nb}$ so that we get $(2\pi e|z|)^n = e^{2\pi nb}$. Thus, $2\pi e|z| = e^{2\pi b}$. Similarly, if $\Im(z) = b < 0$, we get $2\pi e|z| = e^{-2\pi b}$. Thus, the zeros of $B_n(nz)$ are asymptotically the roots of the equation

$$2\pi e|z| = \begin{cases} e^{2\pi b} & : \ b > 0 \\ e^{-2\pi b} & : \ b < 0 \end{cases}.$$

□

Remark 1.14. *The curve in the complex plane whose equation is given by (1.25) is the H-shaped curve given in Figure 1.6 below. That is, for sufficiently large positive integers n, the complex zeros of $B_n(nz)$ approximately lie on the curve*

$$2\pi e|z| = \begin{cases} e^{2\pi\Im(z)} & : \ \Im(z) > 0 \\ e^{-2\pi\Im(z)} & : \ \Im(z) < 0 \end{cases}.$$

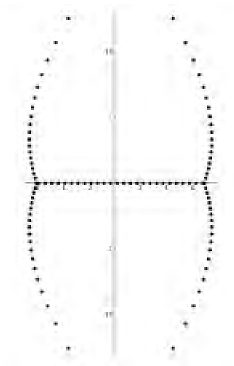


Figure 1.6: The roots of $B_{100}(z) = 0$ via Mathematica

1.4 Hypergeometric Bernoulli Polynomials of Order N

From the generating function definition of Bernoulli polynomials, several generalizations were made by different authors. Among these are the polynomials called Generalized Bernoulli Polynomials of order μ, $B_n^{\mu}(x)$, given by

$$\frac{z^{\mu}e^{xz}}{(e^z-1)^{\mu}} = \sum_{n=0}^{\infty} B_n^{\mu}(x)\frac{z^n}{n!}.$$

Another generalization of Bernoulli polynomials is the polynomials $\{A_n(x)\}$ introduced by F. T. Howard [19], given by

$$\frac{z^2e^{xz}/2}{e^z-1-z} = \sum_{n=0}^{\infty} A_n(x)\frac{z^n}{n!}.$$

Several authors considered similar generalizations of Bernoulli polynomials. We may refer to the papers [20], [9], [15] and [10] for some other related concepts to generalization of Bernoulli polynomials.

In this thesis, we focus on the generalization made by Abdulkadir Hassen and Hieu D. Nguyen in [15], the class of Bernoulli polynomials of higher order called *Hypergeometric Bernoulli polynomials of order N.*

Definition 1.15. *For any integer $N \geq 1$, hypergeometric Bernoulli polynomials of order N, $B_n(N,x)$ are defined as:*

$$\frac{z^N e^{xz}/N!}{e^z - T_{N-1}(z)} = \sum_{n=0}^{\infty} B_n(N,x)\frac{z^n}{n!}, \tag{1.26}$$

where $T_m(z) = 1 + z + \frac{z^2}{2!} + ... + \frac{z^m}{m!} = \sum_{k=0}^{m} \frac{z^k}{k!}$.

In particular if we put $N = 1$ in (1.26), it reduces to the classical Bernoulli polynomials $B_n(x)$ and when $N = 2$ it represents the polynomials $A_n(x)$ considered by Howard.

If we put $x = 0$ in (1.26), we get a sequence of rational numbers $\{B_n(N)\}$, where $B_n(N,0) = B_n(N)$. The numbers $B_n(N)$ are called *hypergeometric Bernoulli numbers of order N* and these are generated by

$$\frac{z^N}{N!(e^z - T_{N-1}(z))} = \sum_{n=0}^{\infty} B_n(N)\frac{z^n}{n!}. \tag{1.27}$$

Analogous to that of classical Bernoulli polynomials, we have several approaches for defining hypergeometric Bernoulli polynomials. Accordingly, the hypergeometric Bernoulli polynomials $B_n(N, x)$ are equivalently defined by a recurrence formula as

$$B_n(N,\, x) = \sum_{k=0}^{n} \binom{n}{k} B_k(N) x^{n-k}. \tag{1.28}$$

The hypergeometric Bernoulli polynomials $B_n(N,\, x)$ are also defined in terms of an Appell sequence with zero moments as

$$B_0(N,\, x) = 1 \tag{1.29}$$

$$B_n'(N,\, x) = nB_{n-1}(N,\, x) \tag{1.30}$$

$$\int_0^1 (1-x)^{N-1} B_n(N,\, x)\, dx = \begin{cases} 1/N & : \quad n = 0 \\ 0 & : \quad n > 0 \end{cases}. \tag{1.31}$$

Hassen and Nguyen proved the equivalence of these different definitions of hypergeometric Bernoulli polynomials. They proved this equivalence in [15] and we reproduce their proof here with a little rearrangement of steps.

Lemma 1.16. *If the sequence of polynomials $\{B_n(N,\, x)\}$ is given by the generating function (1.26), then the $B_n(N,\, x)$'s satisfy the recurrence formula (1.28).*

Proof. We use (1.26) and (1.27) together with the power series of e^z as

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{B_n(N,\, x)}{n!} z^n &= \left(\sum_{n=0}^{\infty} B_n(N) \frac{z^n}{n!} \right) e^{xz} \\ &= \left(\sum_{n=0}^{\infty} B_n(N) \frac{z^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} z^n \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{B_k(N)}{k!(n-k)!} x^{n-k} \right) z^n. \end{aligned}$$

Then comparing the coefficients of z^n, we obtain (1.28). □

Lemma 1.17. *Suppose $\{B_n(N,\, x)\}$ satisfy the Appell sequence given in (1.29) - (1.31). Then the sequence $\{B_n(N,\, x)\}$ is generated by $G(x,\, z)$, where*

$$G(x,\, z) = \frac{z^N e^{xz}}{N!(e^z - T_{N-1}(z))}.$$

Proof. Suppose $B_n(N,\, x)$ satisfy (1.29), (1.30) and (1.31). Let $G(x,\, z)$ be given by

$$G(x,\, z) = \sum_{n=0}^{\infty} B_n(N,\, x) \frac{z^n}{n!}.$$

Differentiating with respect to x and by (1.30), we have

$$\begin{aligned} \frac{\partial}{\partial x} G(x,\, z) &= \sum_{n=1}^{\infty} B_n'(N,\, x) \frac{z^n}{n!} \\ &= \sum_{n=1}^{\infty} nB_{n-1}(N,\, x) \frac{z^n}{n(n-1)!} \end{aligned}$$

$$= zG(x,\, z).$$

Then we obtain $G(x,z) = e^{xz}g(z)$ for some function g of z. If we multiply both sides by $(1-x)^{N-1}$, then we use (1.31) so that

$$\begin{aligned} g(z)\int_0^1 (1-x)^{N-1}e^{xz}\ dx &= \int_0^1 (1-x)^{N-1}\sum_{n=0}^{\infty} B_n(N,\, x)\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{z^n}{n!}\int_0^1 (1-x)^{N-1}B_n(N,x)\ dx \\ &= \frac{1}{N}. \end{aligned}$$

Using integration by parts $N-1$ times, we get

$$\begin{aligned} \int_0^1 (1-x)^{N-1}e^{xz}\ dx &= -\frac{1}{z} - \frac{N-1}{z^2} - \cdots - \frac{(N-1)!}{z^{N-1}} - \frac{(N-1)!}{z^N} + \frac{(N-1)!(e^z-1)}{z^N} \\ &= \frac{(N-1)!(e^z - T_{N-1}(z))}{z^N}. \end{aligned}$$

Hence $g(z) = \frac{z^N}{N!(e^z - T_{N-1}(z))}$ so that we get

$$G(x,\, z) = \frac{z^N e^{xz}}{N!(e^z - T_{N-1}(z))}.$$

□

Theorem 1.18. *The generating function (1.26), the recurrence formula (1.28) and the Appell sequence definition (1.29) - (1.31) of hypergeometric Bernoulli polynomials are equivalent.*

Proof. Since two of the equivalences are proved in the preceding lemmas, we need to prove only the third equivalence. Suppose $B_n(N,x)$ are generated by (1.26). Then we show that the $B_n(N,\, x)$'s satisfy each of the equations (1.29), (1.30) and (1.31).

Now differentiating (1.26) with respect to x, we get

$$z\frac{z^N e^{xz}}{N!(e^z - T_{N-1}(z))} = \sum_{n=1}^{\infty}\frac{B_n'(N,\, x)}{n!}\, z^n.$$

Again we use (1.26) for the left hand expression and then after re-indexing we obtain

$$\sum_{n=1}^{\infty}\frac{B_{n-1}(N,\, x)}{(n-1)!}\, z^n = \sum_{n=1}^{\infty}\frac{B_n'(N,\, x)}{n!}\, z^n.$$

Then comparing coefficients of z^n for $n = 1, 2, 3, \cdots$, we conclude that (1.30) holds.

To prove (1.29), observe that e^{xz} in (1.26) is expressed as

$$\begin{aligned} e^{xz} &= \frac{N!(e^z - T_{N-1}(z))}{z^N}\sum_{n=0}^{\infty} B_n(N,\, x)\frac{z^n}{n!} \\ &= \left(\frac{N!}{z^N}\sum_{n=N}^{\infty}\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} B_n(N,\, x)\frac{z^n}{n!}\right) \\ &= \left(\sum_{n=0}^{\infty}\frac{z^n}{(N+1)_n}\right)\left(\sum_{n=0}^{\infty} B_n(N,\, x)\frac{z^n}{n!}\right), \end{aligned}$$

where the Pochhammer symbol $(\alpha)_m$ is given by $(\alpha)_m = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+m-1)$.

Then using the power series of e^{xz}, we have

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} z^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{B_k(N,\, x)}{k!(N+1)_{n-k}} \right) z^n.$$

Then comparing coefficients of the series, we get

$$\frac{x^n}{n!} = \sum_{k=0}^{n} \frac{B_k(N,\, x)}{k!(N+1)_{n-k}}.$$

Now assuming $x \neq 0$, we take $n = 0$ and get $B_0(N,\, x) = 1$ which proves (1.29).

Finally, to prove (1.31), we multiply (1.28) by $(1-x)^{N-1}$ and integrate as

$$\int_0^1 (1-x)^{N-1} B_n(N,\, x)\; dx = \sum_{k=0}^{n} \binom{n}{k} B_k(N) \int_0^1 (1-x)^{N-1} x^{n-k}\; dx.$$

The integral on the right hand is similar to that of the Beta function

$$B(p,\, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\; dx.$$

On the other hand, we use the $\Gamma-$ function and express $B(p,\, q)$ as

$$B(p,\, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \frac{(p-1)!(q-1)!}{(p+q-1)!}.$$

Thus we use this with $p = n-k+1$ and $q = N$, and obtain

$$\int_0^1 (1-x)^{N-1} B_n(N, x)\; dx = (N-1)! \sum_{k=0}^{n} \binom{n}{k} \frac{(n-k)!}{(N+n-k)!} B_k(N).$$

But by (2.6), the sum on the right hand side vanishes for $n > 0$. When $n = 0$, we note that $B_0(N) = 1$ and hence the right hand side reduces to $\frac{1}{N}$ and this proves (1.31). □

Remark 1.19. *We obtained an interesting analogy between hypergeometric Bernoulli polynomials and classical Bernoulli polynomials. The three alternative definitions; (1.26), (1.28) and (1.29) - (1.31) of $B_n(N,\, x)$ are analogous to the three definitions of $B_n(x)$ given in (1.3), (1.2) and (1.5) - (1.7), respectively. Indeed, it is this interesting relation between $B_n(N,\, x)$ and $B_n(x)$ that motivated us to study more analogous properties of these new class of polynomials.*

Chapter 2

Hypergeometric Bernoulli Polynomials of Order 2

Hypergeometric Bernoulli polynomials of order 2, $B_n(2, x)$, are the particular case of hypergeometric Bernoulli polynomials $B_n(N, x)$ defined in (1.26) with $N = 2$. That is, we define $B_n(2, x)$ by a generating function as

$$\frac{z^2 e^{xz}/2}{e^z - 1 - z} = \sum_{n=0}^{\infty} B_n(2, x)\frac{z^n}{n!}. \tag{2.1}$$

The corresponding numbers $B_n(2) = B_k(2, 0)$ are called hypergeometric Bernoulli numbers of order 2. These are the sequence $\{B_n(2)\}$ of rational numbers generated by

$$\frac{z^2/2}{e^z - 1 - z} = \sum_{n=0}^{\infty} B_n(2)\frac{z^n}{n!}. \tag{2.2}$$

2.1 Some Properties of Hypergeometric Bernoulli Polynomials of Order 2

In this section, we discuss our results consisting of some properties of hypergeometric Bernoulli polynomials. We state and prove two important properties of $B_n(2, x)$ which are analogous to those of the classical Bernoulli polynomials given in (1.9) and (1.10).

Now we state and prove one property which holds for the general case $B_n(N, x)$. This is an addition formula for $B_n(N, x)$ which is analogous to the property (1.10) of the classical Bernoulli polynomials $B_n(x)$.

Theorem 2.1 (Addition Formula). *For hypergeometric Bernoulli polynomials of order N, we have an addition formula*

$$B_n(N, x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(N, x) y^{n-k}. \tag{2.3}$$

Proof. Replacing x by $x + y$ in the generating function (1.26), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{B_n(N, x+y)}{n!} z^n &= \frac{z^N e^{(x+y)z}/N!}{e^z - T_{N-1}(z)} = \frac{z^N e^{xz}/N!}{e^z - T_{N-1}(z)} e^{yz} \\
&= \left(\sum_{n=0}^{\infty} \frac{B_n(N, x)}{n!} z^n\right)\left(\sum_{n=0}^{\infty} \frac{y^n}{n!} z^n\right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{B_k(N, x)}{k!} \frac{y^{n-k}}{(n-k)!}\right) z^n.
\end{aligned}$$

Then equating the coefficients of z^n of both sides, we get

$$\frac{B_n(N,\, x+y)}{n!} = \sum_{k=0}^{n} \frac{B_k(N,\, x)}{k!}\frac{y^{n-k}}{(n-k)!}.$$

Hence (2.3) automatically follows from this last equation. □

Analogous to (1.9) of the classical Bernoulli polynomials, we establish a difference equation for hypergeometric Bernoulli polynomials of order 2.

Theorem 2.2 (Difference Equation)**.** *For each $n = 2, 3, 4, \ldots$, hypergeometric Bernoulli polynomials of order 2, $B_n(2,\, x)$, satisfy the equation*

$$B_n(2,\, x+1) - B_n(2,\, x) = nB_{n-1}(2,\, x) + \binom{n}{2} x^{n-2}. \tag{2.4}$$

Proof. We use the identity

$$\frac{e^{(x+1)z}}{e^z - 1 - z} - \frac{(1+z)e^{xz}}{e^z - 1 - z} = e^{xz}.$$

By multiplying $\frac{z^2}{2}$ to both sides of this identity and using (2.1), we get

$$\sum_{n=0}^{\infty} \frac{B_n(2,\, x+1) - B_n(2,\, x)}{n!} z^n - \sum_{n=0}^{\infty} \frac{B_n(2,\, x)}{n!} z^{n+1} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{x^n}{n!} z^{n+2}.$$

Re-indexing sums and since $B_n(2,\, x+1) - B_n(2,\, x) - nB_{n-1}(2,\, x) = 0$ for $n = 0$ and $n = 1$, we obtain

$$\sum_{n=2}^{\infty} \frac{B_n(2,\, x+1) - B_n(2,\, x) - nB_{n-1}(2,\, x)}{n!} z^n = \frac{1}{2}\sum_{n=2}^{\infty} \frac{x^{n-2}}{(n-2)!} z^n.$$

Finally, we equate coefficients of z^n and get

$$\frac{B_n(2,\, x+1) - B_n(2,\, x) - nB_{n-1}(2,\, x)}{n!} = \frac{1}{2}\frac{x^{n-2}}{(n-2)!} \quad \text{for all} \quad n \geq 2.$$

Thus, we obtain

$$B_n(2,\, x+1) - B_n(2,\, x) - nB_{n-1}(2,\, x) = \frac{n(n-1)}{2} x^{n-2} \quad \text{for all} \quad n \geq 2.$$

□

Remark 2.3. *The property (2.4) can be extended to the general case for $B_n(N,\, x)$. That is, for all $n \geq N$, we have*

$$B_n(N,\, x+1) - \sum_{k=0}^{N-1} \binom{n}{k} B_{n-k}(N,\, x) = \binom{n}{N} x^{n-N}. \tag{2.5}$$

In this case, the identity to be used is

$$\frac{e^{(x+1)z}}{e^z - T_{N-1}(z)} - \frac{T_{N-1}(z)e^{xz}}{e^z - T_{N-1}(z)} = e^{xz}.$$

Then we multiply both sides by $\frac{x^N}{N!}$ and then use (1.26).

Recall that Hassen and Nguyen [15] established a recurrence formula for hypergeometric Bernoulli numbers $B_n(N)$. For the particular case when $N = 2$, we use the above results and provide an alternative expression.

Corollary 2.4. *The Hypergeometric Bernoulli numbers of order* 2*,* $B_n(2)$*, are given by a recurrence formula as:* $B_0(2) = 1$ *and*

$$\sum_{k=0}^{n-2} \binom{n}{k} B_k(2) = 0 \ \text{ for } n \geq 3. \tag{2.6}$$

Proof. Since $n \geq 3$, we put $x = 0$ in (2.4) and get

$$B_n(2,\, 1) = B_n(2) + nB_{n-1}(2).$$

On the other hand, we put $x = 1$ in formula (1.28) (with $N = 2$) and get

$$\begin{aligned} B_n(2,\, 1) &= \sum_{k=0}^{n} \binom{n}{k} B_k(2) \\ &= B_n(2) + nB_{n-1}(2) + \sum_{k=0}^{n-2} \binom{n}{k} B_k(2). \end{aligned}$$

Then combining these two expressions of $B_n(2,\, 1)$, we obtain the desired result. □

Remark 2.5. *For the classical Bernoulli numbers, we know that* $B_{2k+1} = 0$ *for each* $k = 1, 2, \ldots$ *. But this doesn't hold in the case of hypergeometric Bernoulli numbers. Observe that each of* $B_3(2), B_5(2)$ *and* $B_7(2)$ *is nonzero. Also we know that* $B_n(1) = B_n(0) = B_n$*. However,* $B_n(2,\, 1) \neq B_n(2,\, 0) = B_n(2)$*, which is different from the classical case.*

Next, we give one of our results regarding properties of $B_n(2,\, x)$. The following is a more general form of the difference equation (2.4).

Theorem 2.6. *For any positive integer* m*, we have*

$$B_n(2,\, x+m) = \sum_{k=0}^{m} \binom{m}{k} B_n^{(k)}(2,\, x) + \frac{n!}{2} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1-k} \binom{m-1-k}{j} \frac{(x+k)^{n-2-j}}{(n-2-j)!}, \tag{2.7}$$

where $B_n^{(k)}(2,\, x) = n(n-1)\ldots(n-k+1)B_{n-k}(2,\, x)$ *is the* k^{th} *derivative of* $B_n(2,\, x)$*.*

Proof. We prove this by using induction on m. If $m = 1$, then $m - 1 = 0$ so that $k = j = 0$ in the double sum of the right side. Therefore the corollary holds for $m = 1$ because (2.7) reduces to (2.4). Now assume that (2.7) is true for m. To prove that it also holds for $m + 1$, we replace x by $x + m$ in (2.4) and use our induction hypothesis as follows.

$$\begin{aligned} B_n(2,\, x+m+1) &= B_n(2,\, x+m) + B_n'(2,\, x+m) + \frac{n!}{2} \frac{(x+m)^{n-2}}{(n-2)!} \\ &= \sum_{k=0}^{m} \binom{m}{k} B_n^{(k)}(2,\, x) + \frac{n!}{2} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1-k} \binom{m-1-k}{j} \frac{(x+k)^{n-2-j}}{(n-2-j)!} \\ &\quad + \sum_{k=1}^{m+1} \binom{m}{k-1} B_n^{(k)}(2,\, x) + \frac{n!}{2} \sum_{k=0}^{m-1} \sum_{j=1}^{m-k} \binom{m-1-k}{j-1} \frac{(x+k)^{n-2-j}}{(n-2-j)!} \end{aligned}$$

$$+\frac{n!}{2}\frac{(x+m)^{n-2}}{(n-2)!}.$$

Observe that the last term is the value of the double sum when $k = m$ and $j = 0$. Also for any two positive integers q and r, we have the identity

$$\binom{q}{r} + \binom{q}{r-1} = \binom{q+1}{r}.$$

Now by using this identity and appropriate re-indexing of the above sums, we get

$$B_n(2,\, x+m+1) \;=\; \sum_{k=0}^{m+1}\binom{m+1}{k}B_n^{(k)}(2,\, x) + \frac{n!}{2}\sum_{k=0}^{m}\sum_{j=0}^{m-k}\binom{m-k}{j}\frac{(x+k)^{n-2-j}}{(n-2-j)!}.$$

Note that this last expression is identical to (2.7) with m replaced by $m+1$. Therefore, (2.7) holds for $m+1$ whenever it holds for m. □

2.2 Hypergeometric Hurwitz Zeta Functions of Order 2

Abdulkadir Hassen and Hiew D. Nguyen also considered a generalization of Hurwitz zeta function called *Hypergeometric Hurwitz zeta functions of order N, $\zeta_N(s,\, a)$*. They defined $\zeta_N(s,\, a)$ in [16] and discussed the basic properties including its relation to the hypergeometric Bernoulli polynomials $B_n(N,\, x)$. Now we give the definition of $\zeta_N(s,\, a)$ for the particular case when $N = 2$.

Definition 2.7. *For a real number a such that $0 < a \le 1$, the* hypergeometric Hurwitz zeta function of order 2*, $\zeta_2(s,\, a)$, is defined by*

$$\zeta_2(s,\, a) = \frac{1}{\Gamma(s+1)}\int_0^\infty \frac{x^s e^{(1-a)x}}{e^x - 1 - x}\, dx. \tag{2.8}$$

For any real number a such that $0 < a \le 1$, Hassen and Nguyen showed in [16] for the general case that the integral in (2.8) converges absolutely and uniformly in the right half-plane $\Re(s) = \sigma > 1$. In other words, $\zeta_2(s,\, a)$ is analytic in the region $\Re(s) = \sigma > 1$.

Note that many of the proofs in [16] are done for $a \in \mathbb{R}$ such that $0 < a \le 1$. In this section, we consider different properties of $\zeta_2(s,\, a)$ by making a little change to the parameter a. We reproduce some of the proofs given in [16] by noting that it holds for any positive real number, $a > 0$.

Lemma 2.8. *For any real number $a > 0$, the integral in (2.8) converges absolutely and uniformly in every closed subset of the right half-plane $\Re(s) = \sigma > 1$.*

Proof. Let $a > 0$ and α be such that $0 < \alpha < 1 < a + \alpha$. Then there exist $R > 0$ such that $e^x \ge e^{\alpha x} + 1 + x$ for all $x \ge R$. Also $e^x - 1 - x > \frac{x^2}{2}$ for $x > 0$. Then for $\Re(s) = \sigma > 1$, we have

$$\begin{aligned} |\zeta_2(s,\, a)| &\le \frac{1}{|\Gamma(s+1)|}\left\{\int_0^R \left|\frac{e^{(1-z)x}x^s}{e^x - 1 - x}\right| dx + \int_R^\infty \left|\frac{e^{(1-z)x}x^s}{e^x - 1 - x}\right| dx\right\} \\ &\le \frac{1}{|\Gamma(s+1)|}\left\{\int_0^R \frac{e^{(1-a)x}x^\sigma}{x^2/2}\, dx + \int_R^\infty e^{(1-a-\alpha)x}x^\sigma\, dx\right\}. \end{aligned}$$

Note that in the last expression, the first integral is finite and the second integral converges because $1 - a - \alpha < 0$. □

Next we establish a series representation for $\zeta_2(s,\, a)$ and its relation to hypergeometric Bernoulli polynomials of order 2. The proof of Lemma 2.9 is given in [16]. However, we put the proof here by noting that the proof holds for any real number $a > 0$. Also the particular case $N = 2$ results in a simpler expression.

Lemma 2.9. *For $a > 0$ and $\sigma = \Re(s) > 1$, we have*

$$\zeta_2(s,\, a) = \sum_{m=0}^{\infty} \frac{\mu_m(s,a)}{(m+a)^{s+1}},$$

where

$$\mu_m(s,\, a) = \sum_{k=0}^{m} \binom{m}{k} \frac{(s+1)_k}{(m+a)^k}.$$

Proof. Since $(1+x)e^{-x} < 1$ for all $x > 0$, the integrand in (2.8) can be expressed as

$$\begin{aligned}
\frac{x^s e^{(1-a)x}}{e^x - 1 - x} &= \frac{x^s e^{-ax}}{1-(1+x)e^{-x}} \\
&= x^s e^{-ax} \sum_{m=0}^{\infty} \left[(1+x)e^{-x}\right]^m \\
&= x^s \sum_{m=0}^{\infty} (1+x)^m e^{-(m+a)x}.
\end{aligned}$$

Using this in (2.8) and by Lebesgue's Convergence Theorem, we get

$$\zeta_2(s,\, a) = \sum_{m=0}^{\infty} \frac{1}{\Gamma(s+1)} \int_0^{\infty} x^s (1+x)^m e^{-(m+a)x} \; dx.$$

Now let

$$f_m(s,\, a) := \frac{1}{\Gamma(s+1)} \int_0^{\infty} x^s (1+x)^m e^{-(m+a)x} \; dx.$$

Then we express the binomial term $(1+x)^m$ as

$$(1+x)^m = \sum_{k=0}^{m} \binom{m}{k} x^k$$

and rewrite $\zeta_2(s,\, a)$ as $\zeta_2(s,\, a) = \sum_{m=0}^{\infty} f_m(s,\, a)$, where

$$f_m(s,\, a) = \frac{1}{\Gamma(s+1)} \sum_{k=0}^{m} \binom{m}{k} \int_0^{\infty} x^{s+k} e^{-(m+a)x} \; dx.$$

Furthermore, we use the substitution $u = (m+a)x$ and the definition of Γ-function to conclude

$$\begin{aligned}
f_m(s,\, a) &= \frac{1}{\Gamma(s+1)} \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma(s+1+k)}{(m+a)^{s+1+k}} \\
&= \frac{1}{(m+a)^{s+1}} \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma(s+1+k)}{(m+a)^k \Gamma(s+1)} \\
&= \frac{1}{(m+a)^{s+1}} \sum_{k=0}^{m} \binom{m}{k} \frac{(s+1)_k}{(m+a)^k} \\
&= \frac{\mu_m(s,a)}{(m+a)^{s+1}}.
\end{aligned}$$

□

Analytic Continuation of $\zeta_2(s,\, a)$

Similar to the classical case, we consider analytic continuation of $\zeta_2(s,\, a)$ to the left half-plane $\sigma = \Re(s) < 1$. For this, we define a function $I_2(s,\, a)$ by contour integral as

$$I_2(s,\, a) = \frac{1}{2\pi i}\int_C \frac{(-w)^{s+1}e^{(1-a)w}}{e^w - 1 - w}\frac{dw}{w}, \tag{2.9}$$

where C is the contour given in Figure 2.1 below.

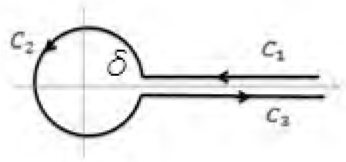


Figure 2.1: A key-hole contour C (Alternative type)

We take the circle in the figure to be of small radius $0 < \delta < r_1$ so that all zeros of $\varphi(w) = e^w - 1 - w$ lie outside of the contour C. Hence the integral in (2.9) converges for any complex number s. Therefore, for any real number $a > 0$, $I_2(s,\, a)$ represents an entire function of s. In terms of the curves C_1, C_2 and C_3, we express $I_2(s, a)$ as

$$\begin{aligned} I_2(s,\, a) &= I_{21}(s,\, a) + I_{22}(s,\, a) + I_{23}(s,\, a), \quad \text{where} \\ I_{21}(s,\, a) &= \frac{1}{2\pi i}\int_\infty^\delta \frac{e^{(1-a)x}e^{(s+1)[\log x - \pi i]}}{e^x - 1 - x}\frac{dx}{x}, \\ I_{22}(s,\, a) &= \frac{1}{2\pi i}\int_{|w|=\delta} \frac{(-w)^{s+1}e^{(1-a)w}}{e^w - 1 - w}\frac{dw}{w} \\ \text{and} \qquad I_{23}(s,\, a) &= \frac{1}{2\pi i}\int_\delta^\infty \frac{e^{(1-a)x}e^{(s+1)[\log x + \pi i]}}{e^x - 1 - x}\frac{dx}{x}. \end{aligned}$$

For $a > 0$ and $\Re(s) = \sigma > 1$, it follows that $I_{22}(s,\, a) \to 0$ as $\delta \to 0$. Thus, letting $\delta \to 0$ yields $I_2(s,\, a) = I_{21}(s,\, a) + I_{23}(s,\, a)$. Therefore,

$$I_2(s,\, a) = \left(\frac{e^{(s+1)\pi i} - e^{-(s+1)\pi i}}{2\pi i}\right)\int_0^\infty \frac{x^s e^{(1-a)x}}{e^x - 1 - x}\, dx.$$

Moreover, by using (2.8) for the integral in this expression, we get

$$I_2(s,\, a) = \frac{\sin \pi(s+1)}{\pi}\Gamma(s+1)\zeta_2(s,\, a).$$

Finally, by using the functional equation $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$, we obtain (2.10) which extends $\zeta_2(s,\, a)$ to the left half-plane.

Theorem 2.10. *For any real number $a > 0$ and $\sigma = \Re(s) > 1$, the function $I_2(s,\, a)$ defined by 2.9 is entire and we have*

$$\zeta_2(s,\, a) = \Gamma(-s)I_2(s,\, a). \tag{2.10}$$

Relations between $\zeta_2(s,\, a)$ and $B_n(2,\, a)$

For integer values $s = n$, we have $I_{21}(n,\, a) = -I_{23}(n,\, a)$ so that $I_2(n,\, a) = I_{22}(n,\, a)$. Then, (2.9) becomes

$$I_2(n,\, a) = \frac{1}{2\pi i}\int_{|w|=\delta} \frac{w(-w)^{n-1}e^{(1-a)w}}{e^w - 1 - w}\, dw.$$

Here observe that the integrand has removable singularity at $w = 0$ when $n > 1$. Then Cauchy's Theorem implies that $I_2(n,\, a) = 0$ for all integers $n > 1$. When $n \leq 1$, we use the Residue Theorem and the generating function of $B_n(2,\, a)$ and get

$$\begin{aligned} I_2(n,\, a) &= 2(-1)^{n+1}\frac{1}{2\pi i}\int_{|w|=\delta} \frac{w^2e^{(1-a)w}/2}{e^w - 1 - w}\,\frac{dw}{w^{2-n}} \\ &= 2(-1)^{n+1}\frac{1}{2\pi i}\int_{|w|=\delta} \left(\sum_{k=0}^{\infty}\frac{B_k(2,\, 1-a)}{k!}w^k\right)\frac{dw}{w^{2-n}} \\ &= 2(-1)^{n+1}\frac{B_{1-n}(2,\, 1-a)}{(1-n)!}. \end{aligned}$$

Therefore, by using integer values of s in (2.9), we obtain

$$I_2(n,\, a) = 2(-1)^{n+1}\frac{B_{1-n}(2,\, 1-a)}{(1-n)!}.$$

Replacing n by $1-n$ and a by $1-a$, we use this expression of $I_2(s,a)$ in (2.10). Therefore, for any real number $a < 1$ and $n = 2, 3, \cdots$, we have

$$\zeta_2(1-n,\, 1-a) = \frac{2(-1)^n}{n(n-1)}B_n(2,\, a). \tag{2.11}$$

Note that (2.10) extends $\zeta_2(s,\, a)$ analytically to the left half-plane $\sigma = \Re(s) < 1$. Next, we show that the function $\zeta_2(s,\, a)$ is analytic in the whole complex plane except for two simple poles, at $s = 0$ and $s = 1$.

Theorem 2.11. *The function $\zeta_2(s,\, a)$ is analytic on the entire complex plane except for two simple poles at $s = 1$ and $s = 0$ whose residues are*

$$\begin{aligned} &Res\,(\zeta_2(s,\, a)\,;\; s = 0) = -2B_1(2,\, 1-a)\,, \quad \textit{and} \\ &Res\,(\zeta_2(s,\, a)\,;\; s = 1) = -2B_0(2,\, 1-a). \end{aligned}$$

Proof. Clearly, the integers $m \leq 0$ are simple poles of $\Gamma(s)$ with residue

$$Res\,(\Gamma(s)\,;\; s = m) = \frac{(-1)^m}{|m|!}.$$

Then we evaluate the residue at $m = 0$ and $m = 1$ as

$$\begin{aligned} Res\,(\zeta_2(s,\, a)\,;\; s = m) &= \lim_{s\to m}(s-m)\zeta_2(s,\, a) = \lim_{s\to m}(s-m)\Gamma(-s)I_2(s,\, a) \\ &= \frac{(-1)^{-m}}{m!}I_2(m,\, a) = \frac{(-1)^{-m}}{m!}2(-1)^{m+1}\frac{B_{1-m}(2,\, 1-a)}{(1-m)!} \\ &= -2B_{1-m}(2,\, 1-a)\,, \quad \text{for } m \in \{0, 1\}. \end{aligned}$$

□

2.3 Series Representation of Hypergeometric Bernoulli Polynomials of Order 2

The function $\varphi(z) = e^z - 1 - z$ appears in the generating function of $B_n(2,\, a)$ given in (2.1) as well as in the definition (2.8) of $\zeta_2(s,\, a)$. The roots of this function are basic quantities for the series representation of $\zeta_2(s, a)$ and $B_n(2,\, a)$. Hassen and Nguyen discussed in [14] several concepts related to the roots of $\varphi(z) = e^z - 1 - z$. Now we briefly discuss some known results about the roots of $\varphi(z) = e^z - 1 - z$ and how to use these roots to establish a series representation for $B_n(2,\, a)$.

Now let $z_k = x_k + iy_k = r_k e^{\theta_k}$, for $k = 1, 2, 3, \cdots$, be the zeros of $\varphi(z) = e^z - 1 - z$ which lie in the upper half of the complex plane and such that $0 < r_1 < r_2 < r_3 < \cdots$. As proved in [**?**], we have

$$2\pi k + \frac{\pi}{4} < y_k < 2\pi k + \frac{\pi}{2},$$

for each root $z_k = x_k + iy_k$ of $\varphi(z)$. We reproduce this result by making a little change to the lower bound of the interval as shown in Lemma 2.12 below.

Lemma 2.12. *Let $z_k = x_k + iy_k = r_k e^{\theta_k}$ be zeros of $\varphi(z) = e^z - 1 - z$ which are located in the upper half-plane.*

i) For each $k = 1, 2, 3, \cdots$, the imaginary parts y_k satisfy the inequality

$$2\pi k + \frac{\pi}{3} < y_k < 2\pi k + \frac{\pi}{2}.$$

Moreover, there is exactly one zero z_k with imaginary part in the stated interval, and there are no other zeros elsewhere in the complex plane.

ii) Both $\{x_k\}$ and $\{y_k\}$ are increasing sequences. As $k \to \infty$, the y_k's grow faster, nearly exponentially while the growth of x_k's is logarithmic. Moreover, the y_k's increase in such a way that

$$\left| y_{k+1} - \left(2\pi(k+1) + \frac{\pi}{2}\right) \right| < \left| y_k - \left(2\pi k + \frac{\pi}{2}\right) \right|.$$

Proof. Suppose $z = x + iy$ is a zero of $\varphi(z) = e^z - 1 - z$ with $y > 0$. Then the equation $e^z - 1 - z = 0$ is equivalent to the system of two equations

$$e^x \cos y = 1 + x$$
$$e^x \sin y = y.$$

We have $y > 0$ and $\frac{y}{\sin y} = e^x > 0$ which implies $\sin y > 0$ so that y must lie in the interval $2k\pi < y < (2k+1)\pi$ for some $k = 1, 2, 3, \cdots$. Also $\cos y > 0$ for otherwise, we get $x < -1$ so that $y = e^x \sin y < e^{-1}$, which is a contradiction because $y > 2\pi > e^{-1}$. Thus, $2k\pi < y < (2k + 1/2)\pi$. Further, from the above system of equations, we get $x = -1 + y\cot y$ and $x = \log\left(\frac{y}{\sin y}\right)$ so that

$$-1 + y\cot y - \log\left(\frac{y}{\sin y}\right) = 0.$$

Let $f(y) = -1 + y\cot y - \log\left(\frac{y}{\sin y}\right)$. Then we consider the roots of $f(y)$ which are the y_k's of our $z_k = x_k + iy_k$. From elementary calculus, we see that $f(y)$ is strictly decreasing on the interval

$2\pi k < y < 2\pi k + \frac{\pi}{2}$. Then $f(y) = 0$ for at most one y in this interval. Moreover, we have

$$f\left(2\pi k + \frac{\pi}{3}\right) > 0 \quad \text{and} \quad f\left(2\pi k + \frac{\pi}{2}\right) < 0.$$

Therefore, $f(y)$ has exactly one root in $2\pi k + \frac{\pi}{3} < y < 2\pi k + \frac{\pi}{2}$. □

From Lemma 2.12, we get

$$(6k+1)\frac{\pi}{3} < \frac{y_k}{\sin y_k} < \frac{(2k+1/2)\pi}{\sqrt{3}/2}.$$

Then we use the relation $x = \log\left(\frac{y}{\sin y}\right)$ and the fact that the logarithm is increasing in its domain to conclude that $x_k > 0$ for each $k = 1, 2, 3, \cdots$. This is true for all the roots in the upper half plane and the other roots are exactly the $\bar{z}_k$'s, the complex conjugates. Indeed, $\bar{z}_k$'s lie in the right half plane whenever z_k's do.

Also observe that $\theta_k = \tan^{-1}\left(\frac{y_k}{x_k}\right)$. But the growth of x_k's is logarithmic when compared to the growth of the y_k's so that $\frac{y_1}{x_1} < \frac{y_2}{x_2} < \frac{y_3}{x_3} < \cdots$. Moreover, the function $h(t) = \tan^{-1}(t)$ is strictly increasing to $\frac{\pi}{2}$. Note that we have $\frac{\sqrt{3}}{2} < \sin(y_k) < 1$ and

$$y_k = \sin(y_k)e^{x_k}.$$

But from the above Lemma, we have $y_k \to 2\pi k + \frac{\pi}{2}$ as $k \to \infty$ so that $\sin(y_k) \to 1$ as $k \to \infty$. Hence, $y_k \to e^{x_k}$ as $k \to \infty$. In general, we put the following remark which summarizes approximate locations of zeros of $\varphi(z) = e^z - 1 - z$.

Remark 2.13. *If $z_k = x_k + iy_k$ is a root of $\varphi(z) = e^z - 1 - z$, then its complex conjugate $\bar{z}_k = x_k - iy_k$ is also a root of $\varphi(z) = e^z - 1 - z$. We usually list all the roots in pairs as $\{z_k, \bar{z}_k\}$. Moreover,*

- *All the zeros $z_k = x_k + iy_k$ and $\bar{z}_k = x_k - iy_k$ of $\varphi(z) = e^z - 1 - z$ lie inside the right half-plane.*
- *The arguments θ_k of z_k are such that $\theta_1 < \theta_2 < \theta_3 < \cdots < \frac{\pi}{2}$ and eventually grows to the value $\theta = \frac{\pi}{2}$.*
- *As k increases, all the quantities x_k, y_k and θ_k increase. In addition, we have $y_k < e^{x_k}$ and eventually, $y_k \to e^{x_k}$ as $k \to \infty$.*

Note that we have two roots of $\varphi(z) = e^z - 1 - z$ with minimum modulus. These are $z_1 = x_1 + iy_1 = r_1 e^{\theta_1}$ and $\bar{z}_1 = x_1 - iy_1 = r_1 e^{-\theta_1}$ and the approximate values of x_1, y_1, r_1 and θ_1 (as calculated by Mathematica) are given by

$$x_1 = 2.0888, \quad y_1 = 7.4615, \quad r_1 = 7.7484, \quad \theta_1 = 1.2978. \tag{2.12}$$

That is, $z_1 \approx 2.0888 + i\,7.4615$ and $\bar{z}_1 \approx 2.0888 - i\,7.4615$. Moreover, we list the first few approximate values of $z_k = x_k + iy_k = r_k e^{\theta_k}$ as follows.

k	x_k	y_k	r_k	θ_k
1	2.0888	7.4615	7.7484	1.2978
2	2.6641	13.879	14.132	1.3812
3	3.0263	20.224	20.449	1.4223
4	3.2917	26.543	26.747	1.4474
5	3.5013	32.851	33.037	1.4646
6	3.6745	39.151	39.323	1.4772
7	3.8222	45.447	45.608	1.4869

Also we plot the first few z_k's as indicated in Figure 2.2 below.

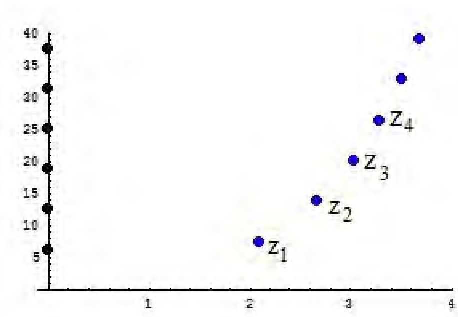


Figure 2.2: Plots of some roots of $\varphi(z) = e^z - 1 - z$

Series Representation of $\zeta_2(s, a)$ and $B_n(2, a)$

Now we illustrate how we use the roots of $\varphi(z) = e^z - 1 - z$ to construct a series representation for $\zeta_2(s, a)$. Consider a semicircular contour C_R given in Figure 2.3.

Define $I_R(s, a)$ by the contour integral as

$$I_R(s, a) = \frac{1}{2\pi i} \int_{C_R} \frac{(-z)^{s+1} e^{(1-a)z}}{e^z - 1 - z} \frac{dz}{z}. \tag{2.13}$$

Let C_{R_1} be the half circle of radius R, C_{R_2} be the inner key-hole contour and C_{R_3} be the segment of the imaginary axis from $-iR$ to iR, shifted to the left in order to avoid the keyhole. Then the contour C_R is the union of C_{R_1}, C_{R_2} and C_{R_3}. We choose $R = (2m + 1)\pi$ for some positive integer m so that none of the zeros of $\varphi(z) = e^z - 1 - z$ lie on the contour C_R. Also we take a small $\delta > 0$ so that all zeros of $\varphi(z)$ lie outside the keyhole. For any zero z_k of $\varphi(z) = e^z - 1 - z$, there exist R such that z_k lies inside the contour C_R. Note that for any $R > 0$, only finitely many z_k's lie inside the closed contour C_R.

Lemma 2.14. *Let $I_2(s, a)$ and $I_R(s, a)$ be as defined in (2.9) and (2.13), respectively. For any real*

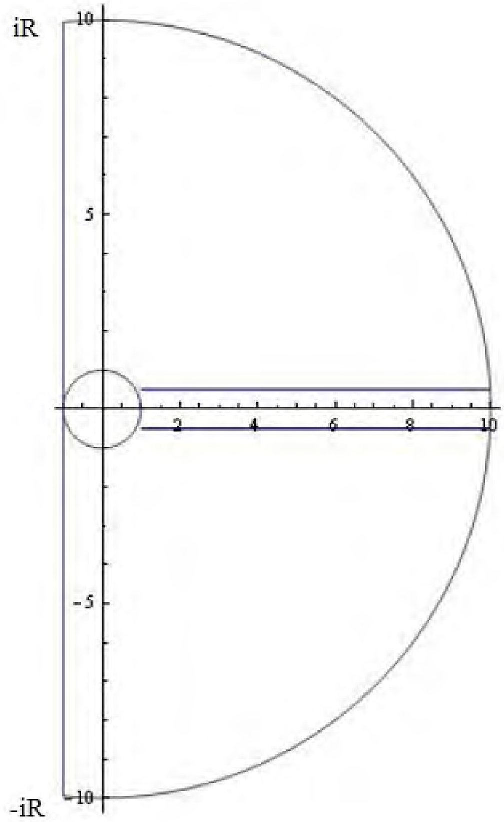


Figure 2.3: Semicircular key-hole contour $\mathbf{C_R}$

number $a > 0$ *and* $\Re(s) = \sigma < 0$, *we have*

$$\lim_{R\to\infty} I_R(s,\, a) = I_2(s,\, a).$$

Proof. We express $I_R(s,\, a)$ as $I_R(s,\, a) = I_{R_1}(s,\, a) + I_{R_2}(s,\, a) + I_{R_3}(s,\, a)$, where

$$I_{R_1}(s,\, a) = \frac{1}{2\pi i}\int_{C_{R_1}} \frac{-(-z)^s e^{(1-a)z}}{e^z - 1 - z}\, dz,$$
$$I_{R_2}(s,\, a) = \frac{1}{2\pi i}\int_{C_{R_2}} \frac{-(-z)^s e^{(1-a)z}}{e^z - 1 - z}\, dz \quad \text{and}$$
$$I_{R_3}(s,\, a) = \frac{1}{2\pi i}\int_{C_{R_3}} \frac{-(-z)^s e^{(1-a)z}}{e^z - 1 - z}\, dz.$$

Then for $\Re(s) = \sigma < 0$, we claim that $I_{R_1}(s,\, a) \to 0$ and $I_{R_3}(s,\, a) \to 0$ as $R \to \infty$. If z is on the curve C_{R_1}, then $z = x + iy = Re^{i\theta}$ for some θ such that $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$. Then assuming R to be large, we have

$$\frac{R-1}{R}|z| \le |1+z| \le \frac{R+1}{R}|z|.$$

Then we have $|e^z - 1 - z| > e^x - (R-1)$ and hence

$$\left|\frac{z^s e^{(1-a)z}}{e^z - 1 - z}\right| \le \frac{R^\sigma e^{(1-a)x}}{e^x - (R-1)} = \frac{R^\sigma e^{-aR\cos\theta}}{1 - (R-1)e^{-R\cos\theta}}.$$

Thus, for $\Re(s) = \sigma < 0$, we obtain

$$|I_{R_1}(s,\, a)| = \frac{1}{2\pi}\left|\int_{C_{R_1}} \frac{-(-z)^s e^{(1-a)z}}{e^z - 1 - z}\, dz\right| \le \frac{1}{2\pi}\frac{R^{\sigma+1}e^{-aR\cos\theta}}{1-(R-1)e^{-R\cos\theta}}(\pi).$$

Then noting that $a > 0$, $\cos\theta > 0$ and $\sigma + 1 < 1$, we conclude $I_{R_1}(s,\, a) \to 0$ as $R \to \infty$.
Now let $z = iy$ be on C_{R_3}. Then, we have

$$\left|\frac{z^s e^{(1-a)z}}{e^z - 1 - z}\right| \le \frac{R^\sigma}{|1-(R-1)|} = R^{\sigma-1}.$$

Thus, since $\Re(s) = \sigma < 0$, we get

$$|I_{R_3}(s,\, a)| = \frac{1}{2\pi}\left|\int_{C_{R_3}} \frac{-(-z)^s e^{(1-a)z}}{e^z - 1 - z}\, dz\right| \le \frac{1}{2\pi}R^{\sigma-1}(2R) = \frac{1}{\pi}R^\sigma.$$

Thus, $|I_{R_3}(s,\, a)| \to 0$ as $R \to \infty$ because $\sigma = \Re(s) < 0$.Therefore, since both $I_{R_1}(s,\, a)$ and $I_{R_3}(s,\, a)$ converge to zero, we conclude that $I_R(s,\, a) \to I_{R_2}(s,\, a)$ as $R \to \infty$. On the other hand, when $R \to \infty$, the key-hole C_{R_2} becomes identical to the key-hole contour C of $I_2(s,\, a)$ given in *Figure 4*. Therefore,

$$\lim_{R\to\infty} I_R(s,\, a) = I_2(s,\, a)\,, \quad \text{for } \Re(s) = \sigma < 0.$$

□

Theorem 2.15. *Let z_k and $\bar{z}_k$ be the roots of $\varphi(z) = e^z - 1 - z$ for $k = 1, 2, 3, \ldots$. Then for any real number $a > 0$ and $\Re(s) = \sigma < 0$, we have*

$$\zeta_2(s,\, a) = (-1)^s\Gamma(-s)\sum_{k=1}^{\infty}\left(\frac{e^{(1-a)z_k}}{z_k^{1-s}} + \frac{e^{(1-a)\bar{z}_k}}{\bar{z}_k^{1-s}}\right). \tag{2.14}$$

Proof. For the contour C_R above, only finitely many z_k's lie inside C_R, say $z_1, z_2, \ldots, z_M$ and their respective complex conjugates. Then with $F(z) = \frac{-(-z)^s e^{(1-a)z}}{e^z-1-z}$ and by the Residue Theorem, we have

$$I_R(s,\, a) = -\sum_{k=1}^{M} Res\big(F(z)\,;\; z = z_k\big) + Res\big(F(z)\,;\; z = \bar{z}_k\big).$$

The residue of $F(z)$ at each z_k is calculated as

$$\begin{aligned} Res\,(F(z)\,;\; z = z_k) &= -(-z_k)^s e^{(1-a)z_k}\lim_{z\to z_k}\frac{z - z_k}{e^z - 1 - z} \\ &= -(-z_k)^s e^{(1-a)z_k}\frac{1}{e^{z_k} - 1} \\ &= -(-z_k)^s e^{(1-a)z_k}\frac{1}{e^{z_k} - 1 - z_k + z_k} \\ &= (-z_k)^{s-1}e^{(1-a)z_k}. \end{aligned}$$

Thus, the function $I_R(s,\, a)$ is expressed as

$$I_R(s,\, a) = (-1)^s\sum_{k=1}^{M}\left(\frac{e^{(1-a)z_k}}{z_k^{1-s}} + \frac{e^{(1-a)\bar{z}_k}}{\bar{z}_k^{1-s}}\right).$$

Moreover, $M \to \infty$ when $R \to \infty$. Therefore,

$$I_2(s,\, a) = (-1)^s\sum_{k=1}^{\infty}\left[\frac{e^{(1-a)z_k}}{z_k^{1-s}} + \frac{e^{(1-a)\bar{z}_k}}{\bar{z}_k^{1-s}}\right].$$

Finally, using this expression of $I_2(s, a)$ in (2.10), we obtain the desired result. □

Corollary 2.16. *Let $z_k = x_k + iy_k = r_k e^{i\theta_k}$ be the zeros of $\varphi(z) = e^z - 1 - z$ that lie in the upper half-plane. For each positive integer n and any real number $a < 1$, the hypergeometric Bernoulli polynomials $B_n(2,\, a)$ are given by*

$$B_n(2,\, a) = -\frac{n!}{2} \sum_{k=1}^{\infty} \left[\frac{e^{z_k a}}{z_k^n} + \frac{e^{\bar{z}_k a}}{\bar{z}_k^n} \right]. \tag{2.15}$$

Further, the $B_n(2,\, a)$'s are equivalently expressed as

$$B_n(2,\, a) = -n! \sum_{k=1}^{\infty} \frac{e^{x_k a} \cos(y_k a - n\theta_k)}{r_k^n}. \tag{2.16}$$

Proof. Since (2.14) holds for all s such that $\Re(s) = \sigma < 0$ and $a > 0$, we use $s = 1 - n$ for positive integers $n > 1$. Then by replacing a by $1 - a$ in (2.14) and using the relation between $\zeta_2(s,\, a)$ and $B_n(2,\, a)$ given in (2.11), we conclude (2.15). Moreover, we use the fact that

$$\frac{e^{z_k a}}{z_k^n} + \frac{e^{\bar{z}_k a}}{\bar{z}_k^n} = 2\,\Re\left(\frac{e^{z_k a}}{z_k^n}\right).$$

Therefore, (2.16) follows automatically from (2.15). □

Some Inequalities Involving $B_n(2,\, a)$

Next we establish some inequalities related to the hypergeometric Bernoulli polynomials $B_n(2, a)$. If we put $a = 0$ in (2.16), we get a series expression for hypergeometric Bernoulli numbers $B_n(2)$,

$$B_n(2) = -n! \sum_{k=1}^{\infty} \frac{\cos(n\theta_k)}{r_k^n}. \tag{2.17}$$

Hassen and Nguyen established in [14] an inequality for $B_n(2)$, given by

$$|B_n(2)| < \frac{n!}{r_1^n}. \tag{2.18}$$

They mentioned that (2.18) proves Howard's conjecture [21] noting that $7 < r_1$. From the series (2.16), observe that $e^{x_1 a} < 1$ for $a < 0$. Hence we conclude that

$$|B_n(2, a)| \le |B_n(2)|\ , \text{ for any real number } \ a \le 0.. \tag{2.19}$$

Also combining (2.18) and (2.19), we obtain

$$|B_n(2, a)| < \frac{n!}{r_1^n}\ , \text{ for any } \ a < 0.. \tag{2.20}$$

Observe that since (2.16) is derived only for $a < 1$, we can't use it for positive real numbers $a > 1$. However, we use the general recurrence definition (1.28) for $B_n(2, a)$, $a \in \mathbb{R}$.

Theorem 2.17. *For every real number a, the polynomials $B_n(2,\, a)$ satisfy*

$$|B_n(2,\, a)| < e^{r_1|a|} \frac{n!}{r_1^n}. \tag{2.21}$$

Proof. Using the definition (1.28) (with $N = 2$) and (2.18), we have

$$\begin{aligned}
|B_n(2,\, a)| &\leq \sum_{k=0}^{n} \binom{n}{k} \left| B_k(2) a^{n-k} \right| \\
&< \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{r_1^k} |a|^{n-k} = \frac{n!}{r_1^n} \sum_{k=0}^{n} \frac{(r_1|a|)^{n-k}}{(n-k)!} \\
&\leq \frac{n!}{r_1^n} \sum_{k=0}^{\infty} \frac{(r_1|a|)^k}{k!} \\
&= e^{r_1|a|} \frac{n!}{r_1^n}.
\end{aligned}$$

□

Chapter 3

Zeros of Hypergeometric Bernoulli Polynomials of Order 2

In this chapter, we present the major results of our study. First, we briefly discuss our results regarding asymptotic real zeros of the hypergeometric Bernoulli polynomials of order 2, $B_n(2,\, x)$. Also we use another (alternative) method and describe asymptotic real zeros of the properly re-scaled polynomials $B_n(2,\, nx)$. Finally, we extend the second method to complex variables and briefly discuss the asymptotic complex zeros of $B_n(2,\, nz)$. In both cases, we use the method applied in determining asymptotic zeros of the classical Bernoulli polynomials by making suitable modifications.

3.1 Asymptotic Real Zeros of Hypergeometric Bernoulli Polynomials of Order 2

In this section, we shall make use of the series (2.16) to derive an asymptotic formula for the hypergeometric Bernoulli polynomials of order 2, $B_n(2,\, a)$. Moreover, we use the asymptotic representation and describe several concepts regarding the asymptotic real zeros of $B_n(2,\, a)$.

Theorem 3.1. *Let $z_1 = x_1 + iy_1 = r_1 e^{i\theta_1}$ be the zero of $\varphi(z) = e^z - 1 - z$ of minimum modulus and let $Q_n = -\frac{n!}{r_1^n}$. For any real number $a < 1$, $B_n(2,\, a)$ is expressed asymptotically as*

$$\frac{B_n(2,\, a)}{Q_n} = e^{x_1 a}\cos\left(y_1 a - n\theta_1\right) + \mathcal{O}\left(2^{-n}\right). \tag{3.1}$$

Proof. For each $k = 1, 2, 3, \ldots$, let $z_k = x_k + iy_k = r_k e^{i\theta_k}$ be the roots of $\varphi(z) = e^z - 1 - z$. Then from Lemma 2.12, we have $2\pi k < r_k < 6\pi k$ so that there exist $\lambda_k > 1$ such that

$$r_k = 2\pi k \lambda_k.$$

Also $r_1 < r_2 < r_3 < \cdots$ so that $1 < \lambda_1 < \lambda_2 < \cdots < 3$. Then we express (2.16) as

$$\begin{aligned} B_n(2,\, a) &= -n!\sum_{k=1}^{\infty}\frac{e^{x_k a}\cos\left(y_k a - n\theta_k\right)}{(2\pi k\lambda_k)^n} \\ &= \frac{-n!}{(2\pi\lambda_1)^n}\sum_{k=1}^{\infty}\frac{e^{x_k a}\cos\left(y_k a - n\theta_k\right)}{\left(k\lambda_k/\lambda_1\right)^n} \\ &= \frac{-n!}{r_1^n}\left[e^{x_1 a}\cos\left(y_1 a - n\theta_1\right) + \rho_n(a)\right], \end{aligned}$$

where

$$\rho_n(a) = \sum_{k=2}^{\infty}\frac{e^{x_k a}\cos\left(y_k a - n\theta_k\right)}{\left(k\lambda_k/\lambda_1\right)^n}.$$

From Lemma 2.12, we conclude that $y_k < e^{x_k}$ and $|e^{x_k} - y_k| \to 0$ as $k \to \infty$. Since $a < 1$, $e^{x_k a} < e^{x_k}$ so that $e^{x_k a} < y_k$, for sufficiently large values of k. Also since $\frac{\lambda_k}{\lambda_1} > 1$ and $y_k < 2\pi k + \frac{\pi}{2} < 3\pi k$, there exist a positive integer M such that

$$|\rho_n(a)| \leq \sum_{k=2}^{M} \frac{e^{x_k a}}{(k\lambda_k/\lambda_1)^n} + \sum_{k=M+1}^{\infty} \frac{1}{k^{n-1}}.$$

Moreover, we have

$$\begin{aligned} \sum_{k=M+1}^{\infty} \frac{1}{k^{n-1}} &\leq \sum_{k=2}^{\infty} \frac{1}{k^{n-1}} = \frac{1}{2^{n-1}} \sum_{k=2}^{\infty} \frac{1}{(k/2)^{n-1}} \\ &< \frac{\zeta(n-1)}{2^{n-1}} < \frac{2\zeta(2)}{2^n}, \quad \text{for all } \; n > 3. \end{aligned}$$

Therefore, $\rho_n(a) = \mathcal{O}\left(2^{-n}\right)$ and this completes the proof. □

Remark 3.2. *The asymptotic formula* (3.1) *enables us to approximate the asymptotic negative real zeros of* $B_n(2,\, a)$ *by the zeros of the function* $f(a) = \cos\left(y_1 a - n\theta_1\right)$*. That is, when* n *is sufficiently large, the real zeros of* $B_n(2,\, a)$ *are also (approximately) zeros of* $\cos\left(y_1 a - n\theta_1\right)$*.*

Clearly, $\cos\left(y_1 a - n\theta_1\right)$ *has infinitely many real zeros while* $B_n(2,\, a)$*, being a polynomial, has always a finite number of zeros (what ever large integer* n *is). Therefore, given a certain large (fixed) positive integer* n*, we can not say every zero of* $\cos\left(y_1 a - n\theta_1\right)$ *is approximately a zero of* $B_n(2,\, a)$*. However, given any zero* α *of* $\cos\left(y_1 a - n\theta_1\right)$*, we assume* n *to be large enough (we can assume* n *as large as we wish) so that* α *is also (approximately) a zero of* $B_n(2,\, a)$*.*

Before presenting more results of our study, we would like to explain the meanings of some ambiguous mathematical terms or statements we may use in this section or through out this chapter. We often use terms or phrases such as “sufficiently large n”, “asymptotically equal (the same)”, “approximately equal” , or we may use the approximation symbol “ $\approx$ ” in some expressions. These mathematical terms or phrases will be used in the following sense.

> The functions $B_n(2,\, a)$ and $\cos\left(y_1 a - n\theta_1\right)$ are asymptotically the same (or, approximately equal for sufficiently large n) means: for any $\epsilon > 0$, there exist a real number $R > 0$ such that $|B_n(2,\, a) - \cos\left(y_1 a - n\theta_1\right)| < \epsilon$, for all $n \geq R$.

When we say “*the real (or, complex) zeros of* $B_n(2,\, a)$ *and that of* $\cos\left(y_1 a - n\theta_1\right)$ *are approximately the same*”, it means: if α and β are zeros of $B_n(2,\, a)$ and $\cos\left(y_1 a - n\theta_1\right)$, respectively, then $|\alpha - \beta| < \epsilon$. In particular, we may write $B_n(2,\, \alpha) \approx 0$ when α is approximately a zero of $B_n(2,\, a)$, where the word “approximately” is in the sense introduced above.

Corollary 3.3. *Let* n *be a sufficiently large positive integer.*

(i) *For each integer* $m \leq 0$ *(not less than all zeros of* $B_n(2,\, a)$*), there are at least two real zeros of* $B_n(2,\, a)$ *in the open interval* $I_m = (m,\; m+1)$*.*

(ii) *If* $\alpha\mathbb{R}$, $0 < \alpha < 1$ *is a zero of* $B_n(2,\, a)$*, then* $\alpha_k = \alpha - \frac{2\pi k}{y_1}$ *is also (approximately) a zero of* $B_n(2,\, a)$ *for all* $k = 1, 2, 3, \cdots$ *(provided that* $-k$ *is not less than all real zeros of* $B_n(2,\, a)$*).*

(iii) $B_n(2,\, m) \neq 0$*, for all integers* $m \leq 0$*.*

Proof. **i**). Let $b, c \in \mathbb{R}$ be such that $b < c < 1$ and K be the interval $K = [b\,,\,c]$. As a consequence of Theorem 3.1, we have

$$\left|\frac{B_n(2,\,a)}{Q_n} - e^{x_1 a}\cos\left(y_1 a - n\theta_1\right)\right| \to 0 \text{ uniformly for all } \ a \in K, \text{ as } n \to \infty.$$

Therefore, for sufficiently large positive integers n, the real zeros of $B_n(2,\,a)$ are approximated by the roots of

$$\cos\left(y_1 a - n\theta_1\right) = 0.$$

But $\cos\left(y_1 a - n\theta_1\right) = 0$ if and only if $y_1 a - n\theta_1 = k\pi + \frac{\pi}{2}$, for some $k \in \mathbb{Z}$.

Thus, for large values of n, the real zeros of $B_n(2,\,a)$ are approximately given by

$$a = \frac{n\theta_1}{y_1} + \frac{k\pi}{y_1} + \frac{\pi}{2y_1}, \quad k \in \mathbb{Z},$$

where $y_1 \approx 7.461$ and $\theta_1 \approx 1.298$, as given in (2.12).

Let $m \leq 0$ be any non-positive integer. Then we fix any (sufficiently large) n and determine the number of integers k such that

$$m < a = \frac{n\theta_1}{y_1} + \frac{k\pi}{y_1} + \frac{\pi}{2y_1} < m + 1.$$

Solving this inequality for k, we get

$$b_n(m) < k < b_n(m) + \frac{y_1}{\pi},$$

where $b_n(m) = \frac{y_1 m - n\theta_1}{\pi} - \frac{1}{2}$. But $2 < \frac{y_1}{\pi} < 3$ so that we have at least two integers k_1 and k_2 between the real numbers $b_n(m)$ and $b_n(m) + \frac{y_1}{\pi}$. Therefore, we have at least two real roots of $B_n(2,\,a)$ between m and $m+1$, namely

$$a_1 = \frac{n\theta_1}{y_1} + \frac{k_1\pi}{y_1} + \frac{\pi}{2y_1}$$

and

$$a_2 = \frac{n\theta_1}{y_1} + \frac{k_2\pi}{y_1} + \frac{\pi}{2y_1}.$$

ii). By (3.1), the negative real zeros of $B_n(2,\,a)$ are asymptotically the same as zeros of $f(a) = \cos\left(y_1 a - n\theta_1\right)$. For $\cos\left(y_1 a - n\theta_1\right)$, if β is a zero, then so is $\beta - \frac{2\pi k}{y_1}$ because $f(a) = \cos\left(y_1 t - n\theta_1\right)$ is a periodic function of period $p = \frac{2\pi}{y_1}$. For $\epsilon > 0$, since α is a zero of $B_n(2,\,a)$ and n is arbitrarily large, we have $\left|\alpha - \frac{2\pi k}{y_1} - (\beta - \frac{2\pi k}{y_1})\right| = |\alpha - \beta| < \epsilon$. Therefore, for each $k = 1, 2, \ldots,$ $\alpha - \frac{2\pi k}{y_1}$ is (approximately) a zero of $B_n(2,\,a)$.

iii). Assume $B_n(2,\,m) = 0$ for some integer $m \leq 0$. Then by (*ii*), $a_1 = m - \frac{4\pi}{y_1}$, $a_2 = m - \frac{2\pi}{y_1}$ and $a_3 = m$ are consecutive zeros of $B_n(2,\,a)$. But $m - \frac{4\pi}{y_1} < m - 1 < m - \frac{2\pi}{y_1} < m$. Thus, $a_2 = m - \frac{2\pi}{y_1}$ is the only zero of $B_n(2,\,a)$ between $m-1$ and m. This is a contradiction to (*i*). Therefore, no integer $m \leq 0$ can be a zero of $B_n(2,\,a)$. □

Theorem 3.4. *For each integer $m \geq 0$, let $B_n^{(m)}(2,\,a)$ be the m^{th} derivative of $B_n(2,\,a)$.*

i) For every real number $a < 0$, the m^{th} derivative of $B_n(2,\,a)$, $B_n^{(m)}(2,\,a)$, is expressed asymptotically as

$$\frac{B_n^{(m)}(2,\,a)}{Q_n} = e^{x_1 a} r_1^m \cos\left(y_1 a - (n-m)\theta_1\right) + \mathcal{O}\left(2^{-n}\right). \tag{3.2}$$

ii) For sufficiently large n, if α is a real zero of $B_n^{(m)}(2,\, a)$, then

$$\alpha = \alpha_0 - \frac{m\theta_1}{y_1} \quad \textit{for some real zero } \alpha_0 \textit{ of } B_n(2,\, a).$$

Proof. For each positive integer m, (1.30) implies

$$B_n^{(m)}(2,\, a) = n(n-1)\cdots(n-m+1)\, B_{n-m}(2,\, a).$$

On the other hand, for any real number $a < 1$, we use (3.1) and obtain

$$\frac{B_{n-m}(2,\, a)}{Q_{n-m}} = e^{x_1 a}\cos\left(y_1 a - (n-m)\theta_1\right) + \mathcal{O}\left(2^{-(n-m)}\right),$$

where $Q_{n-m} = \frac{-(n-m)!}{r_1^{n-m}}$. Note that $(2r_1)^m \mathcal{O}\left(2^{-n}\right) = \mathcal{O}\left(2^{-n}\right)$ because m is fixed. Now eliminating $B_{n-m}(2,\, a)$ between the two equations, we get

$$\frac{B_n^{(m)}(2,\, a)}{r_1^m Q_n} = e^{x_1 a}\cos\left(y_1 a - (n-m)\theta_1\right) + 2^m \mathcal{O}\left(2^{-n}\right).$$

To prove (ii), let n be large enough. By (i), the real zeros of $B_n^{(m)}(2,\, a)$ are approximated by roots of

$$\cos\left(y_1 a - (n-m)\theta_1\right) = 0.$$

That is, a real number $\alpha < 1$ is a zero of $B_n^{(m)}(2,\, a)$ if

$$\alpha = \frac{(n-m)\theta_1}{y_1} + \frac{\pi k}{y_1} + \frac{\pi}{2y_1}.$$

But from Corollary 3.3, $\alpha_0 = \frac{n\theta_1}{y_1} + \frac{\pi k}{y_1} + \frac{\pi}{2y_1}$ is a real zero of $B_n(2,\, a)$. Therefore,

$$\begin{aligned} \alpha &= \frac{n\theta_1}{y_1} + \frac{\pi k}{y_1} + \frac{\pi}{2y_1} - \frac{m\theta_1}{y_1} \\ &= \alpha_0 - \frac{m\theta_1}{y_1}. \end{aligned}$$

□

Remark 3.5. *Suppose n is sufficiently large and $a < 1$. By using the derivative formula (1.30) and Theorem 3.4, we determine a relation between asymptotic real zeros of $B_n(2,\, a)$ and that of $B_{n-1}(2,\, a)$. That is, if $B_{n-1}(2,\, \alpha) = 0$, then $B_n\left(2,\, \alpha + \frac{\theta_1}{y_1}\right) \approx 0$. In general, for every positive integer $m > 0$, we have*

$$B_{n-m}\left(2,\, \alpha\right) = 0 \;\;\Rightarrow\;\; B_n(2,\, \alpha + \frac{m\theta_1}{y_1}) \approx 0.$$

Next, we illustrate how to determine the positive real zeros of hypergeometric Bernoulli polynomials of order 2. Note that since the series (2.16) is derived for real numbers $a < 1$, every consequence of this series is valid only for real numbers $a < 1$.

To determine the real zeros of $B_n(2,\, a)$ in the interval $1 \le a < \infty$, we fix any positive integer m and consider real zeros of $B_n(2,\, a + m)$ for $0 < a < 1$. For this, we first establish an asymptotic formula for $B_n(2,\, a + m)$, where $0 < a < 1$ and m is any positive integer.

Theorem 3.6. *Let $a \in \mathbb{R}$ be such that $0 < a < 1$. For any positive integer m, we have*

$$\frac{B_n(2,\, a+m)}{Q_n} = e^{x_1 a} \left|1+z_1\right|^m \cos\left(y_1 a - n\theta_1 + m\theta_1'\right) + \mathcal{O}\left(2^{-n}\right), \tag{3.3}$$

where $\theta_1' = \arg(1+z_1)$ and $Q_n = -\frac{n!}{r_1^n}$.

Proof. For each positive integer m, we use (2.7) of Theorem 2.6 and get

$$B_n(2,\, a+m) = \sum_{k=0}^{m} \binom{m}{k} B_n^{(k)}(2,\, a) + \binom{n}{2} \sum_{k=0}^{m-1} (a+k)^{n-2} + P(m,a),$$

where $P(m,a)$ is a polynomial of degree $n-3$, given by

$$P(m,a) = \frac{n!}{2} \sum_{k=0}^{m-1} \sum_{j=1}^{m-1-k} \binom{m-1-k}{j} \frac{(x+k)^{n-2-j}}{(n-2-j)!}.$$

By using a trigonometric identity for $\cos\left(y_1 a - (n-m)\theta_1\right)$, (3.2) can be expressed as

$$\frac{B_n^{(k)}(2,\, a)}{Q_n} = e^{x_1 a} \left[\cos\left(y_1 a - n\theta_1\right) r_1^k \cos(k\theta_1) - \sin\left(y_1 a - n\theta_1\right) r_1^k \sin(k\theta_1)\right] + \mathcal{O}\left(2^{-n}\right).$$

Now we take the sum over k, from $k=0$ to $k=m$ to get

$$\begin{aligned}
\frac{B_n(2,\, a+m)}{Q_n} &= \sum_{k=0}^{m} \binom{m}{k} \frac{B_n^{(k)}(2,a)}{Q_n} + \frac{r_1^2 r_1^{n-2}}{(n-2)!} \mathcal{O}\left(a^{n-2}\right) + \mathcal{O}\left(2^{-n}\right) \\
&= e^{x_1 a} \cos\left(y_1 a - n\theta_1\right) \sum_{k=0}^{m} \binom{m}{k} r_1^k \cos(k\theta_1) \\
&\quad - e^{x_1 a} \sin\left(y_1 a - n\theta_1\right) \sum_{k=0}^{m} \binom{m}{k} r_1^k \sin(k\theta_1) + \mathcal{O}\left(2^{-n}\right).
\end{aligned}$$

Noting that $z_1 = r_1 e^{i\theta_1}$, we have

$$r_1^k \cos(k\theta_1) = \frac{z_1^k + \bar{z}_1^k}{2} \quad \text{and} \quad r_1^k \sin(k\theta_1) = \frac{z_1^k - \bar{z}_1^k}{2i}.$$

Moreover, from the binomial sum formula we have

$$\sum_{k=0}^{m} \binom{m}{k} z_1^k = (1+z_1)^m \quad \text{and} \quad \sum_{k=0}^{m} \binom{m}{k} \bar{z}_1^k = (1+\bar{z}_1)^m.$$

Then using these relations in the above expression, we get

$$\begin{aligned}
\frac{B_n(2,\, a+m)}{Q_n} &= e^{x_1 a} \cos\left(y_1 a - n\theta_1\right) \left(\frac{(1+z_1)^m + (1+\bar{z}_1)^m}{2}\right) \\
&\quad - e^{x_1 a} \sin\left(y_1 a - n\theta_1\right) \left(\frac{(1+z_1)^m - (1+\bar{z}_1)^m}{2i}\right) + \mathcal{O}\left(2^{-n}\right) \\
&= e^{x_1 a} \left\{\Re\left(e^{i(y_1 a - n\theta_1)}\right) \Re\left((1+z_1)^m\right) - \Im\left(e^{i(y_1 a - n\theta_1)}\right) \Im\left((1+z_1)^m\right)\right\} + \mathcal{O}\left(2^{-n}\right) \\
&= \Re\left(e^{x_1 a} (1+z_1)^m e^{i(y_1 a - n\theta_1)}\right) + \mathcal{O}\left(2^{-n}\right).
\end{aligned}$$

Finally, by letting $\theta_1' = \arg(1+z_1)$ we obtain

$$\frac{B_n(2,\, a+m)}{Q_n} = e^{x_1 a} \left|1+z_1\right|^m \cos\left(y_1 a - n\theta_1 + m\theta_1'\right) + \mathcal{O}\left(2^{-n}\right).$$

□

Theorem 3.7. *Let m be any positive integer. For a fixed large n, the real zeros of $B_n(2,x)$ in the interval $I_m = (m\,,\ m+1)$ are given by*

$$a \approx \left(1 - \frac{\theta_1'}{y_1}\right) m + \frac{2n\theta_1 + \pi}{2y_1} + \frac{\pi k}{y_1},$$

for appropriate integers k such that $m < a < m+1$. Moreover, there are exactly two such values of k.

Proof. Let a be any real number such that $m < a < m+1$. Then there exists an a_m such that $0 < a_m < 1$ and $a = a_m + m$. Then (3.3) implies

$$\frac{B_n(2,\, a)}{Q_n} = e^{x_1 a_m} \left|1 + z_1\right|^m \cos\left(y_1 a_m - n\theta_1 + m\theta_1'\right) + \mathcal{O}\left(2^{-n}\right).$$

Thus, for sufficiently large n, real zeros of $B_n(2,\, a)$ are approximated by the roots of

$$\cos\left(y_1 a_m - n\theta_1 + m\theta_1'\right) = 0.$$

That is, $B_n(2,\, a) = B_n(2,\ a_m + m) = 0$ if $y_1 a_m - n\theta_1 + m\theta_1' = \pi k + \frac{\pi}{2}$, where k is an integer. Solving for a_m, the zeros $a = a_m + m$ are given by

$$a = \left(1 - \frac{\theta_1'}{y_1}\right) m + \frac{2n\theta_1 + \pi}{2y_1} + \frac{\pi k}{y_1},$$

for appropriate integers k such that $m < a < m+1$.
Now we need $k \in \mathbb{Z}$ such that

$$m < \left(1 - \frac{\theta_1'}{y_1}\right) m + \frac{2n\theta_1 + \pi}{2y_1} + \frac{\pi k}{y_1} < m + 1.$$

For this we solve for k to get

$$\frac{m\theta_1' - n\theta_1}{\pi} - \frac{1}{2} < k < \frac{m\theta_1' - n\theta_1}{\pi} - \frac{1}{2} + \frac{y_1}{\pi}.$$

Observe that k lies in an interval of length $\frac{y_1}{\pi}$. Since $2 < \frac{y_1}{\pi} < 3$, we conclude that there are two integers k_1 and k_2, hence two real zeros, a_1 and a_2, of $B_n(2,\, a)$ in the interval $I_m = (m\,,\ m+1)$. Indeed, these zeros are given by

$$a_1 \approx \left(1 - \frac{\theta_1'}{y_1}\right) m + \frac{2n\theta_1 + \pi}{2y_1} + \frac{\pi k_1}{y_1}$$

and

$$a_2 \approx \left(1 - \frac{\theta_1'}{y_1}\right) m + \frac{2n\theta_1 + \pi}{2y_1} + \frac{\pi k_1 2}{y_1}.$$

□

3.2 Asymptotic Complex Zeros of Hypergeometric Bernoulli Polynomials of Order 2

In this section, we establish asymptotic formula for $B_n(2,\, z)$ through an alternative method different from the one we used in the previous section. Then we briefly describe the asymptotic real and complex zeros of the properly re-scaled hypergeometric Bernoulli polynomials of order 2, $B_n(2,\, nz)$.

In Chapter 2, we have seen that several properties of $\zeta_2(s,\, a)$ which are proved for $0 < a \leq 1$ in [16] also hold for any real number $a > 0$. We obtain similar results for the case when a is a complex number. Now we begin by a series representation of $B_n(2,\, z)$.

Series Representation of $B_n(2,\, z)$

Let $z = a + ib$ be a complex number such that $\Re(z) = a > 0$. The *hypergeometric Hurwitz zeta function of order 2*, $\zeta_2(s, z)$, is simply defined by replacing a by z in (2.8). The following statement is an extension of Lemma 2.8 from real to a complex variable and the proof is the same.

Lemma 3.8. *Given $z = a + ib \in \mathbb{C}$ with $\Re(z) = a > 0$, the hypergeometric Hurwitz zeta function $\zeta_2(s,\, z)$ converges absolutely in the half-plane $\Re(s) = \sigma > 1$. Moreover, the convergence is uniform in the closed region $\Re(s) = \sigma \geq 1 + \delta$, for any $\delta > 0$.*

Now let z be any complex number such that $\Re(z) > 0$. As a consequence of Lemma 3.8, we conclude that $\zeta_2(s, z)$ represents an analytic function of s in the region $\Re(s) = \sigma > 1$. Moreover, we extend the analytic continuation process discussed in Lemma 2.10 and Lemma 2.11 and obtain an extension of formula (2.11) to the complex values of a. That is, for $z = a + ib \in \mathbb{C}$ with $\Re(z) = a > 0$, we have

$$\zeta_2(1-n\,,\, z) = \frac{2(-1)^n}{n(n-1)} B_n(2\,,\, 1-z) \quad \text{for integers } n > 1. \tag{3.4}$$

Similarly, the series representation established in Theorem 2.15 holds for complex values of a. That is, for $\Re(s) = \sigma < 0$ and z such that $\Re(z) = a > 0$, we have

$$\zeta_2(s,\, z) = (-1)^s \Gamma(-s) \sum_{k=1}^{\infty} \left(\frac{e^{(1-z)z_k}}{z_k^{1-s}} + \frac{e^{(1-z)\bar{z}_k}}{\bar{z}_k^{1-s}} \right), \tag{3.5}$$

where z_k and $\bar{z}_k$ are zeros of $\varphi(z) = e^z - 1 - z$. For any $z \in \mathbb{C}$ with $\Re(z) < 1$, we combine (3.4) and (3.5) and obtain

$$B_n(2,\, z) = -\frac{n!}{2} \sum_{k=1}^{\infty} \left(\frac{e^{z_k z}}{z_k^n} + \frac{e^{\bar{z}_k z}}{\bar{z}_k^n} \right). \tag{3.6}$$

If $z = a + ib$ with $\Re(z) = a < 0$, then $\Re(nz) = na < 0$ for any positive integer n. Thus,

$$B_n(2,\, nz) = -\frac{n!}{2} \sum_{k=1}^{\infty} \left(\frac{e^{z_k n z}}{z_k^n} + \frac{e^{\bar{z}_k n z}}{\bar{z}_k^n} \right) \quad \text{for } z \in \mathbb{C} \text{ with } \Re(z) < 0.$$

Lemma 3.9. *Let $z_k = x_k + iy_k = r_k e^{i\theta_k}$, $k = 1, 2, \cdots$ be the zeros of $\varphi(z) = e^z - 1 - z$. For any $z \in \mathbb{C}$ such that $\Re(z) < 1$ and $-1 < \Im(z) < 1$, we have*

$$B_n(2,\, z) = -n! \sum_{k=1}^{\infty} \frac{e^{x_k z} \cos\left(y_k z - n\theta_k\right)}{r_k^n}.$$

Proof. Let $z \in \mathbb{C}$ with $\Re(z) < 1$ and $-1 < \Im(z) < 1$. Then with $z_k = x_k + iy_k = r_k e^{i\theta_k}$ and $\bar{z}_k = x_k - iy_k = r_k e^{-i\theta_k}$, we use (3.6) as follows.

$$\begin{aligned} -\frac{n!}{2} \sum_{k=1}^{\infty} \left(\frac{e^{z_k z}}{z_k^n} + \frac{e^{\bar{z}_k z}}{\bar{z}_k^n} \right) &= -\frac{n!}{2} \sum_{k=1}^{\infty} e^{x_k z} \left(\frac{e^{iy_k z}}{z_k^n} + \frac{e^{-iy_k z}}{\bar{z}_k{}^n} \right) \\ &= -\frac{n!}{2} \sum_{k=1}^{\infty} \frac{e^{x_k z}}{r_k^n} \left(e^{i(y_k z - n\theta_k)} + e^{-i(y_k z - n\theta_k)} \right) \end{aligned}$$

$$= -n! \sum_{k=1}^{\infty} \frac{e^{x_k z}}{r_k^n} \cos(y_k z - n\theta_k).$$

□

The Asymptotic Real Zeros of $B_n(2,\, na)$

We illustrate how the method used by John Mangual in [24] is applicable to the case of the hypergeometric Bernoulli polynomials of order 2, $B_n(2,\, na)$. This is an alternative method to the result about real zeros that we obtained in Section 3.1. Indeed, we extend this method to the asymptotic complex zeros of $B_n(2,\, nx)$.

Lemma 3.10. *For any real number $a < 0$ and a sufficiently large n, $B_n(2,\, na)$ is expressed asymptotically as*

$$B_n(2,\, na) = -\left[n\mathcal{O}\left(n^{1/2n}\left(1+\frac{1}{n}\right)\right)\right]^n \left[\frac{e^{in(y_1 a-\theta_1)} + e^{-in(y_1 a-\theta_1)}}{\left(r_1 e^{1-x_1 a}\right)^n} + \mathcal{O}\left(\frac{1}{r_1^n}\right)\right]. \tag{3.7}$$

Proof. Clearly, if $a < 0$, then $na < 0$ for any positive integer n. Then by using (3.6) and Theorem 3.1, we get

$$\begin{aligned} B_n(2,\, na) &= -\frac{n!}{2}\sum_{k=1}^{\infty}\left[\frac{e^{z_k na}}{z_k^n} + \frac{e^{\bar{z}_k na}}{\bar{z}_k^n}\right] \\ &= -\frac{n!}{2}\left[\frac{e^{z_1 na}}{z_1^n} + \frac{e^{\bar{z}_1 na}}{\bar{z}_1^n} + \sum_{k=2}^{\infty}\frac{e^{z_k na}}{z_k^n} + \frac{e^{\bar{z}_k na}}{\bar{z}_k^n}\right] \\ &= -\frac{n!}{2}\left[\frac{e^{in(y_1 a-\theta_1)} + e^{-in(y_1 a-\theta_1)}}{\left(r_1 e^{-x_1 a}\right)^n} + \mathcal{O}\left(\frac{1}{r_1^n}\right)\right]. \end{aligned}$$

Then by using Stirling's estimate for $n!$, we obtain

$$\begin{aligned} B_n(2,\, na) &= -\frac{\left[n\mathcal{O}\left(n^{1/2n}\left(1+\frac{1}{n}\right)\right)\right]^n}{e^n}\left[\frac{e^{in(y_1 a-\theta_1)} + e^{-in(y_1 a-\theta_1)}}{\left(r_1 e^{-x_1 a}\right)^n} + \mathcal{O}\left(\frac{1}{r_1^n}\right)\right] \\ &= -\left[n\mathcal{O}\left(n^{1/2n}\left(1+\frac{1}{n}\right)\right)\right]^n\left[\frac{e^{in(y_1 a-\theta_1)} + e^{-in(y_1 a-\theta_1)}}{\left(r_1 e^{1-x_1 a}\right)^n} + \mathcal{O}\left(\frac{1}{r_1^n}\right)\right]. \end{aligned}$$

□

Corollary 3.11. *For sufficiently large positive integers n, the negative real zeros of $B_n(2,\, na)$ are approximated by the roots of the equation*

$$\cos(ny_1 a - n\theta_1) = 0.$$

That is, the negative real zeros α_k of $B_n(2,\, na)$ are given by

$$\alpha_k = \frac{\theta_1 - (2k+1)\frac{\pi}{2}}{y_1}, \text{ for } k = 1, 2, 3, \cdots.$$

Proof. For large values of n and $a < 0$, we have $na < 0$ so that (3.7) implies that the negative real zeros of $B_n(2,\, na)$ are approximated by the roots of

$$\frac{e^{in(y_1 a-\theta_1)} + e^{-in(y_1 a-\theta_1)}}{\left(r_1 e^{1-x_1 a}\right)^n} = 0.$$

Therefore, the negative real zeros of $B_n(2, na)$ are approximately given by the roots of

$$e^{i(y_1 a-\theta_1)} + e^{-i(y_1 a-\theta_1)} = 0.$$

Hence $\cos(ny_1 a - n\theta_1) = 0$. Alternatively, note that when $a < 0$, we have $na < 0$ for any positive integer n. Then replacing "a" by "na", the asymptotic formula (3.1) becomes

$$\frac{B_n(2,\, na)}{Q_n} = e^{nx_1 a}\cos(ny_1 a - n\theta_1) + \mathcal{O}\left(2^{-n}\right).$$

Therefore, the negative real zeros of $B_n(2, na)$ are asymptotically given by the roots of of the equation $\cos(ny_1 a - n\theta_1) = 0$. Hence, the negative real zeros α_k of $B_n(2, na)$ are given by $\alpha_k \approx \frac{\theta_1-(2k+1)\frac{\pi}{2}}{y_1}$. □

Remark 3.12. *Note that Lemma 3.10 holds only for $a < 0$ so that Corollary 3.11 gives only the negative real zeros of $B_n(2, na)$. To determine the asymptotic positive real zeros of $B_n(2, na)$, let a be any real number such that $0 < a < 1$. Then $na > 0$ for any positive real number n and we express it as*

$$na = \{na\} + \lfloor na \rfloor,$$

where $\lfloor na \rfloor$ is the greatest integer such that $\lfloor na \rfloor \le na$ and $0 \le \{na\} < 1$ is the fractional part. Moreover, we express $B_n(2, na)$ as

$$B_n(2,\, na) = B_n(2,\, \{na\}) + B_n(2,\, na) - B_n(2,\, \{na\}).$$

Then, to get asymptotic representation for $B_n(2, na)$, we establish asymptotic representation for both $B_n(2, \{na\})$ and $B_n(2, na) - B_n(2, \{na\})$.

Clearly, we have $0 \le \{na\} < 1$ so that the series (2.16) holds for $B_n(2, \{na\})$. Therefore, it remains to establish an asymptotic formula for $B_n(2, na) - B_n(2, \{na\})$.

Lemma 3.13. *Let a be any real number such that $0 < a < 1$. For sufficiently large positive integer m, we have*

$$B_n(2,\, a+m) - B_n(2,\, a) \approx \sum_{k=0}^{m-1} n(a+k)^{n-1}. \tag{3.8}$$

Proof. We use the difference equation (2.4) repeatedly as follows.

$$\begin{aligned}
&B_n(2,\, a+1) - B_n(2,\, a) = nB_{n-1}(2,\, a) + \frac{n(n-1)}{2}a^{n-2}\\
&B_n(2,\, a+2) - B_n(2,\, a+1) = nB_{n-1}(2,\, a+1) + \frac{n(n-1)}{2}(a+1)^{n-2}\\
&\vdots\\
&B_n(2,\, a+m) - B_n(2,\, a+m-1) = nB_{n-1}(2,\, a+m-1) + \frac{n(n-1)}{2}(a+m-1)^{n-2}.
\end{aligned}$$

Then adding both sides we get

$$B_n(2,\, a+m) - B_n(2,\, a) = \sum_{k=0}^{m-1} nB_{n-1}(2,\, a+k) + \frac{n(n-1)}{2}(a+k)^{n-2}.$$

The right-hand sum is a polynomial, say $P(x)$. By using definition of $B_{n-1}(2, a+k)$, we get $B_n(2, a+m) - B_n(2, a) = P(x)$, where

$$P(x) \;=\; \sum_{k=0}^{m-1} nB_{n-1}(2,\, a+k) + \frac{n(n-1)}{2}(a+k)^{n-2}$$

$$= \sum_{k=0}^{m-1} n \sum_{j=0}^{n-1} \binom{n-1}{j} B_j(2)(a+k)^{n-1-j} + \frac{n(n-1)}{2} \sum_{k=0}^{m-1} (a+k)^{n-2}.$$

Clearly, the leading term of $P(x)$ is the sum consisting of $n(a+k)^{n-1}$, and it is when $j = 0$. We express the term for $j = 0$ separately and get

$$P(x) = \sum_{k=0}^{m-1} n(a+k)^{n-1} + Q(x),$$

where

$$Q(x) = \binom{n}{2} \sum_{k=0}^{m-1} (a+k)^{n-2} + n \sum_{k=0}^{m-1} \sum_{j=1}^{n-1} \binom{n-1}{j} B_j(2)(a+k)^{n-1-j}.$$

Observe that the polynomial $Q(x)$ is of degree $n-2$. Also we can express $P(x)$ as

$$P(x) = n(a+m-1)^{n-1} + \sum_{k=0}^{m-2} n(a+k)^{n-1} + Q(x).$$

For large values of $|x|$, the polynomial $P(x)$ is dominated by its leading term. That is, for any $\epsilon > 0$, there exist $R > 0$ such that $\left|P(x) - x^{n-1}\right| < \epsilon$, for all $|x| > R$. Thus, for large values of m such that $|x| = |a+m-1| > R$, we conclude

$$P(x) \approx \sum_{k=0}^{m-1} n(a+k)^{n-1}.$$

Therefore, since m is assumed to be sufficiently large, we conclude

$$B_n(2,\, a+m) - B_n(2,\, a) \approx \sum_{k=0}^{m-1} n(a+k)^{n-1}.$$

□

Lemma 3.14. *Let a be any real number such that $0 < a < 1$. For sufficiently large positive integer n, we have*

$$B_n(2,\, na) - B_n(2,\, \{na\}) \;\approx\; n^n \int_0^a nx^{n-1}\, dx. \tag{3.9}$$

Moreover, for $0 < \delta < 1$, the convergence is uniform for all a in the interval $[0\ ,\ 1-\delta]$.

Proof. Clearly, we write $na = \{na\} + \lfloor na \rfloor$ and $0 \le \{na\} < 1$. For a sufficiently large positive integer n, we replace a by $\{na\}$ and m by $\lfloor na \rfloor$ in (3.8) and get

$$\begin{aligned} B_n(2,\, na) - B_n(2,\, \{na\}) \;&\approx\; \sum_{k=0}^{\lfloor na \rfloor - 1} n\big(\{na\} + k\big)^{n-1} \\ &=\; n^n \times n \sum_{k=0}^{\lfloor na \rfloor - 1} \left(\frac{\{na\}+k}{n}\right)^{n-1} \frac{1}{n}. \end{aligned}$$

Now we consider the right hand side expression. Let a be any real number such that $0 < a < 1$ and consider the function $f(x) = x^{n-1}$ on the interval $\left[\frac{\{na\}}{n}\ ,\ a\right]$. The sum on the right-hand side is the lower sum of f, $L(f, \mathbb{P})$, corresponding to the partition

$$\mathbb{P} = \left\{ x_k = \frac{\{na\}+k}{n} \;:\; k = 0, 2, \ldots, \lfloor na \rfloor \right\}.$$

Here $\Delta x_k = \frac{1}{n}$ and the upper sum $U(f, \mathbb{P})$ is the sum from $x_1 = \frac{\{na\}+1}{n}$ to $x_{\lfloor na \rfloor} = a$.
Now noting that $f(x)$ is increasing on the interval $\left[\frac{\{na\}}{n}\ ,\ a\right]$, we get

$$U(f, \mathbb{P}) - L(f, \mathbb{P}) = f(x_{\lfloor na \rfloor}) - f(x_0) = a^{n-1} - \left(\frac{\{na\}}{n}\right)^{n-1}.$$

Since $0 < a < 1$ and n is large, $U(f, \mathbb{P}) - L(f, \mathbb{P}) \to 0$ as $n \to \infty$. Therefore, the above Riemann (lower) sum converges and hence it converges to the integral of f from $x_0 = \frac{\{na\}}{n}$ to $x_{\lfloor na \rfloor} = a$. Also since n is large, we use $x_0 \approx 0$ and conclude

$$\sum_{k=0}^{\lfloor na \rfloor - 1} \left(\frac{\{na\}+k}{n}\right)^{n-1} \frac{1}{n} \longrightarrow \int_0^a f(x)\ dx \ \text{ as } n \to \infty.$$

Therefore, for sufficiently large n (or, when $n \to \infty$), we conclude that

$$B_n(2,\, na) - B_n(2,\, \{na\}) \ \longrightarrow\ n^n \times \int_0^a nx^{n-1}\ dx.$$

Moreover, if $0 < \delta < 1$, then for every a in $[0\ ,\ 1-\delta]$, we have

$$a^{n-1} - \left(\frac{\{na\}}{n}\right)^{n-1} < a^{n-1} < (1-\delta)^{n-1}.$$

Therefore, the convergence is uniform in the interval $[0\ ,\ 1-\delta]$. □

Corollary 3.15. *Let $a \in \mathbb{R}$ be such that $0 < a < 1$ and n be sufficiently large positive integer. Then the re-scaled hypergeometric Bernoulli polynomial $B_n(2\,,\, na)$ is given by*

$$B_n(2\,,\, na) \approx B_n(2,\, \{na\}) + n^n a^n.$$

Moreover, the asymptotic positive real zeros of $B_n(2,\, na)$ are approximately given by the roots of the equation

$$\cos\left(y_1 na - n\theta_1\right) = 0.$$

Proof. Clearly, $B_n(2,\, na) = B_n(2,\, \{na\}) + B_n(2,\, na) - B_n(2,\, \{na\})$ and we use (3.9) for $B_n(2,\, na) - B_n(2,\, \{na\})$. To prove the second assertion, first we use (2.15) and get

$$B_n(2,\, \{na\}) = -\frac{n!}{2}\left[\frac{e^{z_1\{na\}}}{z_1^n} + \frac{e^{\bar{z}_1\{na\}}}{\bar{z}_1^n} + \mathcal{O}\left(\frac{1}{r_1^n}\right)\right].$$

For a sufficiently large positive integer n, Stirling's estimate and Lemma 3.13 yields

$$\begin{aligned} B_n(2,\, na) &= B_n(2,\, \{na\}) + B_n(2, na) - B_n(2,\, \{na\}) \\ &= -\frac{\left[n\mathcal{O}\left(n^{1/2n}\left(1+\frac{1}{n}\right)\right)\right]^n}{2e^n}\left[\frac{e^{z_1\{na\}}}{z_1^n} + \frac{e^{\bar{z}_1\{na\}}}{\bar{z}_1^n} + \mathcal{O}\left(\frac{1}{r_1^n}\right)\right] + n^n a^n \\ &\approx -\left[n\mathcal{O}\left(n^{1/2n}\left(1+\frac{1}{n}\right)\right)\right]^n\left[\frac{e^{-z_1\lfloor na \rfloor}}{2\left(z_1 e^{1-z_1 a}\right)^n} + \frac{e^{-\bar{z}_1\lfloor na \rfloor}}{2\left(\bar{z}_1 e^{1-\bar{z}_1 a}\right)^n} + a^n\right]. \end{aligned}$$

Since $\left|e^{-z_1\lfloor na \rfloor}\right| = \left|e^{-\bar{z}_1\lfloor na \rfloor}\right| = e^{-x_1\lfloor na \rfloor} \to 0$ as $n \to \infty$, we have

$$B_n(2,\, na) \approx -\left[n\mathcal{O}\left(n^{1/2n}\left(1+\frac{1}{n}\right)\right)\right]^n\left[\frac{1}{2\left(z_1 e^{1-z_1 a}\right)^n} + \frac{1}{2\left(\bar{z}_1 e^{1-\bar{z}_1 a}\right)^n} + a^n\right].$$

Thus, the real zeros of $B_n(2, na)$ are approximated by the roots of

$$\frac{-1}{2\left(z_1 e^{1-z_1 a}\right)^n} + \frac{-1}{2\left(\bar{z}_1 e^{1-\bar{z}_1 a}\right)^n} + a^n = 0.$$

Then multiplying $\left|z_1 e^{1-z_1 a}\right|^n$ to both sides, we get

$$-2\Re\left(z_1^n e^{n-z_1 na}\right) + 2\left|a z_1 e^{1-z_1 a}\right|^n = 0.$$

Further simplification of this expression yields

$$r_1^n e^{n-x_1 na}\left[a^n - \cos\left(y_1 na - n\theta_1\right)\right] = 0.$$

Thus, we get $a^n = \cos\left(y_1 na - n\theta_1\right)$. But since $0 \leq a < 1$, we conclude that $a^n \to 0$ as $n \to \infty$. Therefore, for sufficiently large n, we obtain

$$\cos\left(y_1 na - n\theta_1\right) \approx 0.$$

□

The Asymptotic Complex Zeros of $B_n(2, nz)$

For any $z = a + ib \in \mathbb{C}$, we also express $B_n(2\,,\,nz) = B_n(2\,,\,na + inb)$ as

$$B_n(2\,,\,nz) = B_n(2\,,\,\{na\} + inb) + B_n(2\,,\,na + inb) - B_n(2\,,\,\{na\} + inb).$$

Similar to the previous case, we establish asymptotic formulas for $B_n(2\,,\,\{na\} + inb)$ and $B_n(2\,,\,na + inb) - B_n(2\,,\,\{na\} + inb)$ separately and obtain an asymptotic formula for $B_n(2\,,\,na + inb)$. First we establish an asymptotic formula for $B_n(2\,,\,\{na\} + inb)$.

Recall that the Taylor polynomials of the exponential function e^x, $T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$, appeared in the asymptotic formulas of the classical Bernoulli polynomials. Now we prove a statement analogous to Lemma 1.12 of the classical case.

Lemma 3.16. *Let $z_1 = x_1 + iy_1$ and $\bar{z}_1 = x_1 - iy_1$ be the two roots of $\varphi(z) = e^z - 1 - z$ with the minimum modulus $r_1 = |z_1| = |\bar{z}_1|$. If $z = a + ib$ is any complex number such that $0 < \Re(z) = a < 1$, then for sufficiently large positive integers n, we have*

$$B_n\left(2,\,\{na\} + inb\right) \approx -\left[n\mathcal{O}\left(n^{1/2n}\left(1 + \frac{1}{n}\right)\right)\right]^n \left(\frac{T_n\left(iz_1 nb\right) + T_n\left(i\bar{z}_1 nb\right)}{\left(r_1 e\right)^n}\right). \tag{3.10}$$

Proof. Since $\{na\} \in \mathbb{R}$ and $0 < \{na\} < 1$, we use (2.15) for $B_m\left(2\,,\,\{na\}\right)$ and the addition formula (2.3) as follows.

$$\begin{aligned}
B_n(2\,,\,\{na\} + inb) &= \sum_{m=0}^{n} \binom{n}{m} B_m\left(2\,,\,\{na\}\right)(inb)^{n-m} \\
&= -\frac{n!}{2}\sum_{m=0}^{n} \frac{(inb)^{n-m}}{(n-m)!}\left[\frac{e^{z_1\{na\}}}{z_1^m} + \frac{e^{\bar{z}_1\{na\}}}{\bar{z}_1^m} + \mathcal{O}\left(\frac{1}{r_1^m}\right)\right] \\
&= -\frac{n!}{2}\left[\frac{e^{z_1\{na\}}}{z_1^n}\sum_{m=0}^{n}\frac{(iz_1 nb)^{n-m}}{(n-m)!} + \frac{e^{\bar{z}_1\{na\}}}{\bar{z}_1^n}\sum_{m=0}^{n}\frac{(i\bar{z}_1 nb)^{n-m}}{(n-m)!} + \sum_{m=0}^{n}\frac{(inb)^{n-m}}{(n-m)!}\mathcal{O}\left(\frac{1}{r_1^m}\right)\right]
\end{aligned}$$

$$= -\frac{n!}{2}\left[\frac{e^{z_1\{na\}}}{z_1^n}T_n(iz_1nb) + \frac{e^{\bar{z}_1\{na\}}}{\bar{z}_1^n}T_n(i\bar{z}_1nb) + \frac{1}{r^n}\sum_{m=0}^{n}\frac{(ir_1nb)^{n-m}}{(n-m)!}\mathcal{O}\left(r^{n-m}\right)\right]$$
$$= -\frac{n!e^{x_1\{na\}}}{2r_1^n}\left[e^{i(y_1\{na\}-n\theta_1)}T_n(iz_1nb) + e^{-i(y_1\{na\}-n\theta_1)}T_n(i\bar{z}_1nb) + \frac{1}{r_1^n}\mathcal{O}\left(T_n(r_1nb)\right)\right].$$

Observe that $\left|e^{i(y_1\{na\}-n\theta_1)}\right| = 1$ and $e^{x_1\{na\}} < e^{x_1}$ for any n. Also we have

$$\frac{1}{r_1^n}T_n(r_1nb) < T_n(r_1nb) = T_n(|z_1|\,nb).$$

Thus, when n is sufficiently large, we use Stirling's estimate and conclude that

$$B_n(2, \{na\}) \approx -\left[n\mathcal{O}\left(n^{1/2n}\left(1+\frac{1}{n}\right)\right)\right]^n\left(\frac{T_n(iz_1nb)+T_n(i\bar{z}_1nb)}{(r_1e)^n}\right).$$

□

Next, we extend the method we used for the real case and establish an asymptotic formula for $B_n(2\,, na+inb) - B_n(2\,, \{na\}+inb)$.

Lemma 3.17. *Let $z = a+ib$ be any complex number such that $0 < a < 1$. For sufficiently large positive integers n, we have*

$$B_n(2\,, na+inb) - B_n(2\,, \{na\}+inb) \;\approx\; n^n\int_0^a n(x+ib)^{n-1}\,dx. \tag{3.11}$$

Moreover, $B_n(2\,, na+inb) - B_n(2\,, \{na\}+inb)$ is asymptotically expressed as

$$B_n(2\,, na+inb) - B_n(2\,, \{na\}+inb) \approx n^n(a+ib)^n.$$

Proof. By addition formula (2.3) (with $N = 2$), we get

$$B_n(2\,, na+inb) - B_n(2\,, \{na\}+inb) = \sum_{m=0}^{n}\binom{n}{m}\Big(B_m(2\,, na) - B_m(2\,, \{na\})\Big)(inb)^{n-m}.$$

As shown in the proof of Lemma 3.13, we see that

$$B_m(2, na) - B_m(2, \{na\}) = mn^{m-1}\sum_{k=0}^{\lfloor na\rfloor-1}\left(\frac{\{na\}+k}{n}\right)^{m-1}.$$

Then noting that $B_m(2, na) - B_m(2, \{na\}) = 0$ when $m = 0$, we get

$$\begin{aligned}
B_n(2\,, na+inb) - B_n(2\,, \{na\}+inb) &= \sum_{k=0}^{\lfloor na\rfloor-1}\sum_{m=1}^{n}\binom{n}{m}mn^{m-1}\left(\frac{\{na\}+k}{n}\right)^{m-1}(inb)^{n-m}\\
&= n^{n-1}\sum_{k=0}^{\lfloor na\rfloor-1}\sum_{m=1}^{n}\binom{n}{m}m\left(\frac{\{na\}+k}{n}\right)^{m-1}(ib)^{n-m}\\
&= n^n\sum_{k=0}^{\lfloor na\rfloor-1}\sum_{m=0}^{n-1}\frac{(n-1)!}{m!(n-1-m)!}\left(\frac{\{na\}+k}{n}\right)^{m}(ib)^{n-1-m}\\
&= n^n\sum_{k=0}^{\lfloor na\rfloor-1}\sum_{m=0}^{n-1}\binom{n-1}{m}\left(\frac{\{na\}+k}{n}\right)^{m}(ib)^{n-1-m}.
\end{aligned}$$

In this last expression, the inner sum is a binomial sum so that

$$\sum_{m=0}^{n-1} \binom{n-1}{m} \left(\frac{\{na\}+k}{n}\right)^m (ib)^{n-1-m} = \left(\frac{\{na\}+k}{n} + ib\right)^{n-1}.$$

Thus, we obtain

$$\begin{aligned} B_n(2\,,\,na+inb) - B_n(2\,,\,\{na\}+inb) &= n^n \sum_{k=0}^{\lfloor na\rfloor -1} \left(\frac{\{na\}+k}{n} + ib\right)^{n-1} \\ &= n^n \times n \sum_{k=0}^{\lfloor na\rfloor -1} \left(\frac{\{na\}+k}{n} + ib\right)^{n-1} \frac{1}{n}. \end{aligned}$$

The right-hand side corresponds to a Riemann (lower) sum of $f(x) = (x+ib)^{n-1}$, on the interval $\left[\frac{\{na\}}{n}\,,\ a\right]$ with respect to the partition

$$\mathbb{P} = \left\{ x_k = \frac{\{na\}+k}{n} \ :\ k = 0, 2, \cdots, \lfloor na \rfloor \right\}.$$

(Note that f is complex-valued function of the real variable x. The lower sum of f over a partition $\mathbb{P}$ of the real interval is in the sense that $|f(x_k)|$ treated instead of $f(x_k)$).
Therefore, for sufficiently large n,

$$B_n(2\,,\,na+inb) - B_n(2\,,\,\{na\}+inb) \quad \approx \quad n^n \int_0^a n(x+ib)^{n-1}\, dx.$$

Finally, a simple integration yields

$$B_n(2\,,\,na+inb) - B_n(2\,,\,\{na\}+inb) \quad \approx \quad n^n(a+ib)^n.$$

□

Theorem 3.18. *Let $z_1 = x_1 + iy_1$ and $\bar{z}_1 = x_1 - iy_1$ be the two zero of $\varphi(z) = e^z - 1 - z$ with the minimum modulus $r_1 = |z_1| = |\bar{z}_1|$. For sufficiently large values of n, the complex zeros $z = a+ib$ of $B_n(2,\,nz)$ asymptotically lie on the curve*

$$r_1 e|z| = \begin{cases} e^{y_1 b} & : \ \Im(z) = b > 0 \\ e^{-y_1 b} & : \ \Im(z) = b < 0 \end{cases}. \tag{3.12}$$

Moreover, (3.12) describes the asymptotic complex zeros of $B_n(2,\,nz)$ and it represents the H-like shaped curve given in Figure 3.1 below.

Proof. Let $z = a+ib$ be such that $0 < a < 1$ and suppose n is sufficiently. Clearly, $T_n(z) \to e^z$ as $n \to \infty$. By using Dieudonné's estimate (see [24]), we conclude that

$$T_n\left(iz_1 nz\right) \longrightarrow e^{iz_1 nz} \text{ uniformly for } z \text{ in } \frac{1}{r_1}D,$$

where $D = \{z \in \mathbb{C} : |z| < 1\}$. Then combining (3.11) and (3.10), we obtain

$$B_n(2,\,na+inb) \quad \approx \quad -\left[n\mathcal{O}\left(n^{1/2n}\left(1+\frac{1}{n}\right)\right)\right]^n \left[\frac{e^{iz_1 nb} + e^{i\bar{z}_1 nb}}{(r_1 e)^n} + z^n\right].$$

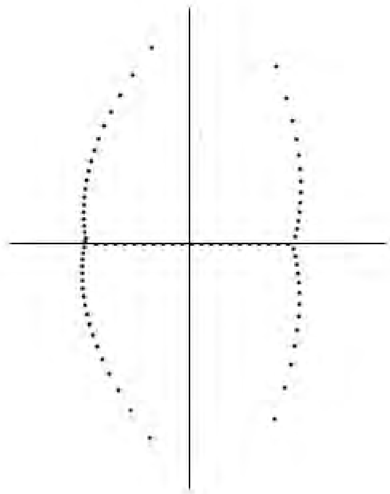


Figure 3.1: The complex zeros of $B_n(2, nz)$ (for $n = 200$)

Thus, the complex zeros of $B_n(2, nz)$ are approximately the roots of

$$e^{iz_1nb} + e^{i\bar{z}_1nb} + (zr_1e)^n = 0. \tag{3.13}$$

Case 1. If $\Im(z) = b > 0$, then $e^{iz_1nb} \to 0$ as $n \to \infty$. Hence (3.13) becomes $e^{i\bar{z}_1nb} + (zr_1e)^n = 0$. Therefore, the zeros z of $B_n(2, nz)$ satisfy the equation

$$r_1e\,|z| = \left|e^{i\bar{z}_1b}\right| = e^{y_1b}\,,\ \text{if } \Im(z) = b > 0.$$

Case 2. Suppose $\Im(z) = b < 0$. Then $e^{i\bar{z}_1nb} \to 0$ as $n \to \infty$ so that (3.13) reduces to $e^{iz_1nb} + (zr_1e)^n = 0$. Thus, the zeros of $B_n(2, nz)$ satisfy

$$r_1e\,|z| = \left|e^{iz_1b}\right| = e^{-y_1b}\,,\ \text{if } \Im(z) = b < 0.$$

Finally, we combine *Case 1* and *Case 2* and obtain (3.12). □

Observe that (3.12) describes a curve in the complex plane. Indeed, it is analogous to (1.25) of the classical Bernoulli polynomials. From the two cases considered in the proof of Theorem 3.18, we express (3.12) alternatively as

$$\left|zz_1e^{1-iz_1b}\right| = 1\,,\ \text{if } \Im(z) = b < 0$$

and

$$\left|z\bar{z}_1e^{1-i\bar{z}_1b}\right| = 1\,,\ \text{if } \Im(z) = b > 0.$$

3.3 Geometry of Curves Related to z_1 and $\bar{z}_1$

The Taylor polynomials of the function e^z, $T_n(z)$, are given by

$$T_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}.$$

The zeros of $T_n(z)$ have an interesting asymptotic behavior. As $n \to \infty$, the zeros of $T_n(z)$ go to ∞ since e^z has no complex zeros. However, if we rescale by a factor of n, the zeros of $T_n(nz)$ approach the curve $\left|ze^{1-z}\right| = 1$.

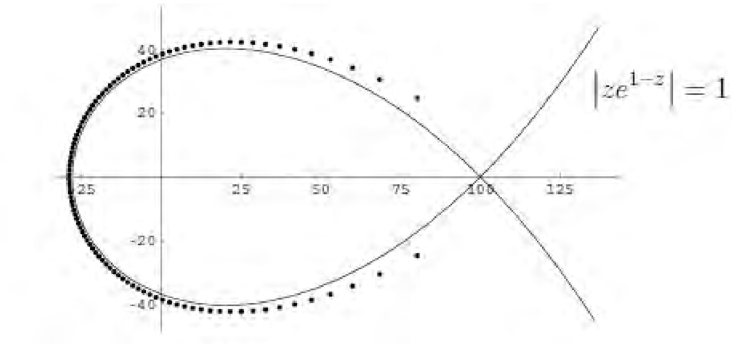


Figure 3.2: The complex zeros of $T_{100}(100z)$ along with the curve $\left|ze^{1-z}\right| = 1$

In Figure 3.2, the complex zeros of $T_{100}(100z)$ are indicated by 'dots' whereas the solid curve is the graph of $\left|ze^{1-z}\right| = 1$. In general, for sufficiently large n, the complex zeros of $T_n(nz)$ are uniformly distributed around the the curve $\left|ze^{1-z}\right| = 1$ with $|z| \leq 1$. The curve $\left|ze^{1-z}\right| = 1$ was first introduced by Gabor Szegö in 1924.

Definition 3.19 (Standard Szegö Curve)**.** *Let $\phi(z) = ze^{1-z}$. The curve $\mathbb{S}$ in the complex plane defined by*

$$\mathbb{S} = \left\{z \in \mathbb{C} : \left|ze^{1-z}\right| = 1 \text{ and } |z| \leq 1\right\}$$

is called Standard Szegö Curve.

In other words, the standard Szegö curve $\mathbb{S}$ is the set of all points $z \in \mathbb{C}$ such that

$$\left|ze^{1-z}\right| = 1 \text{ and } |z| \leq 1. \tag{3.14}$$

Remark 3.20. *By using generating functions, we define the polynomials $T_n(z)$ as*

$$\frac{e^{zw}}{1-w} = \sum_{n=0}^{\infty} T_n(z)w^n.$$

Observe that the denominator of the generating function is $\psi(w) = 1 - w$ and the only root of $\psi(w)$ is $w = 1$. Moreover, the function $\phi(z) = ze^{1-z}$ is a particular case of $\phi(wz) = zwe^{1-wz}$ with $w = 1$. On the other hand, we have $\varphi(w) = e^w - 1 - w$ in the denominator of the generating function (2.1) of $B_n(2, z)$ and the z_k's are the roots of $\varphi(z)$. Also we have $\phi(z_k z) = zz_k e^{1-z_k z}$ as a particular case of $\phi(wz) = zwe^{1-wz}$ with $w = z_k$. Therefore, being motivated by such interesting relations, we consider different curves related to $\phi(z_k z) = zz_k e^{1-z_k z}$, where the z_k's are the roots of $\varphi(w) = e^w - 1 - w$ and investigate how such curves are related to the asymptotic complex zeros of $B_n(2, nz)$.

Recall the roots $z_k = x_k + iy_k = r_k e^{i\theta_k}$ and $\bar{z}_k = x_k - iy_k = r_k e^{-i\theta_k}$ of $\varphi(z) = e^z - 1 - z$ discussed in Chapter 2. Although there are infinitely many roots, we used only the roots with minimum modulus, z_1 and $\bar{z}_1$, for describing the asymptotic zeros of $B_n(2, z)$. These two roots of $\varphi(z)$ helped us to determine the curve described by (3.12). Moreover, the location of the points z_1 and $\bar{z}_1$ in the complex plane are related to the upper and lower openings of the H-shaped curve shown in Figure 3.1. Therefore, we are interested in studying more about curves related to z_1 and $\bar{z}_1$.

Definition 3.21 (Szegö Curves)**.** *Let $z_k = x_k + iy_k$ be the zeros of $\varphi(z) = e^z - 1 - z$ with modulus $|z_k| = r_k$. The curves $\frac{1}{z_k}\mathbb{S}$ in the complex plane defined by the equation*

$$\frac{1}{z_k}\mathbb{S} = \left\{ z \in \mathbb{C} \,:\, \left|zz_k e^{1-z_k z}\right| = 1 \text{ and } |z| \leq \frac{1}{r_k} \right\}$$

are called Szegö Curves.

In particular, we consider the Szegö curves $\frac{1}{z_1}\mathbb{S}$ and $\frac{1}{\bar{z}_1}\mathbb{S}$. Note that $\frac{1}{z_1}\mathbb{S}$ is described as the set of points $z \in \mathbb{C}$ such that

$$|\phi\,(z_1 z)| = 1 \text{ and } |z| \leq \frac{1}{r_1}. \tag{3.15}$$

Similarly, $\frac{1}{\bar{z}_1}\mathbb{S}$ is given by the equation

$$|\phi\,(\bar{z}_1 z)| = 1 \text{ and } |z| \leq \frac{1}{r_1}. \tag{3.16}$$

The Szegö curves $\frac{1}{z_1}\mathbb{S}$ and $\frac{1}{\bar{z}_1}\mathbb{S}$ are roughly sketched as shown in Figure 3.3. We define regions $\mathbb{G}_{z_1}$ and $\mathbb{G}_{\bar{z}_1}$ to be the interior points of the curves $\frac{1}{z_1}\mathbb{S}$ and $\frac{1}{\bar{z}_1}\mathbb{S}$, respectively. Since $\phi(z) = ze^{1-z}$ is a conformal map in the unit disk $B(0,\,1)$, we see that both $\phi\,(z_1 z)$ and $\phi\,(\bar{z}_1 z)$ are conformal in the disk $B(0,\,\frac{1}{r_1})$.

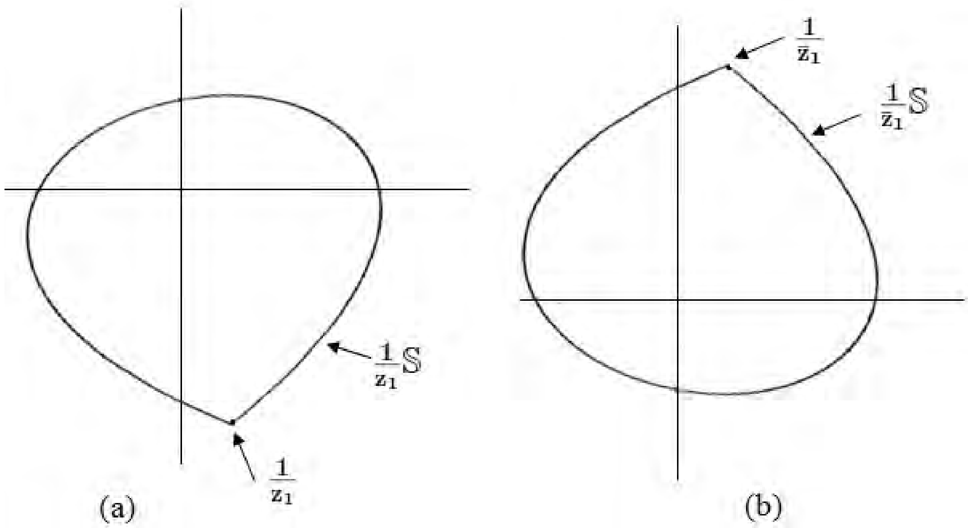


Figure 3.3: The Szegö curves $\frac{1}{z_1}\mathbb{S}$ and $\frac{1}{\bar{z}_1}\mathbb{S}$

Note that (3.15) is obtained when z is replaced by $z_1 z$ in (3.14). Similarly, if z is replaced by $\bar{z}_1 z$ in (3.14), we get (3.16). Therefore, the curves $\frac{1}{z_1}\mathbb{S}$ and $\frac{1}{\bar{z}_1}\mathbb{S}$ are obtained from the standard Szegö curve $\mathbb{S}$ by a stretch (dilation) with $\frac{1}{r_1}$ and rotation by $\pm\theta_1$.

Observe that each of the equations (3.14) - (3.16) consists of two conditions. For instance, (3.15) consists of the equation $|\phi\,(z_1 z)| = 1$ and the restriction $|z| \leq \frac{1}{r_1}$. Clearly, the equation $|\phi\,(z_1 z)| = 1$ by itself represents an unbounded curve in the complex plane. Let Γ_{z_1} be the unbounded curve determined by $|\phi(z_1 z)| = 1$. That is,

$$\Gamma_{z_1} = \{z \in \mathbb{C} : |\phi(z_1 z)| = 1\}. \tag{3.17}$$

Then $\frac{1}{z_1}\mathbb{S} \subseteq \Gamma_{z_1}$. Indeed, $\frac{1}{z_1}\mathbb{S}$ is the portion of Γ_{z_1} that lies in the closed disk $\bar{B}\left(0, \frac{1}{r_1}\right)$. Equivalently, $\frac{1}{z_1}\mathbb{S}$ can be described by the equation

$$|\phi(z_1 z)| = 1 \text{ and } \Re(z_1 z) \le 1. \tag{3.18}$$

That is, $\frac{1}{z_1}\mathbb{S}$ is the portion of Γ_{z_1} that lies in the closed half-plane $\Re(z_1 z) \le 1$. In other words, (3.15) and (3.18) define the same curve, the Szegö curve $\frac{1}{z_1}\mathbb{S}$.

Note that the curve Γ_{z_1} divides the complex plane into three different open regions. We denote these regions by $\mathbb{G}_{z_1}$, $\mathbb{G}^{+}_{z_1}$ and $\mathbb{G}^{-}_{z_1}$, where

1. $\mathbb{G}_{z_1}$ is the interior of the Szegö curve $\frac{1}{z_1}\mathbb{S}$, given by

$$\mathbb{G}_{z_1} = \left\{z \in \mathbb{C} : |\phi(z_1 z)| < 1 \text{ and } |z| \le \frac{1}{r_1}\right\}. \tag{3.19}$$

2. $\mathbb{G}^{+}_{z_1}$ is the unbounded region given by

$$\mathbb{G}^{+}_{z_1} = \left\{z \in \mathbb{C} : |\phi(z_1 z)| < 1 \text{ and } |z| > \frac{1}{r_1}\right\}. \tag{3.20}$$

3. $\mathbb{G}^{-}_{z_1}$ is the unbounded region given by

$$\mathbb{G}^{-}_{z_1} = \{z \in \mathbb{C} : |\phi(z_1 z)| > 1\}. \tag{3.21}$$

For the function $\phi(\bar{z}_1 z)$, we replace z_1 by $\bar{z}_1$ in equations (3.17), (3.19), (3.20) and (3.21) to define $\Gamma_{\bar{z}_1}$, $\mathbb{G}_{\bar{z}_1}$, $\mathbb{G}^{+}_{\bar{z}_1}$ and $\mathbb{G}^{-}_{\bar{z}_1}$, respectively. Clearly, $z_1 = x_1 + iy_1$ lies in the upper half-plane $\mathcal{H}^+$ while $\bar{z}_1 = x_1 - iy_1$ is in the lower half-plane $\mathcal{H}^-$. Hence $\frac{1}{z_1} \in \mathcal{H}^-$ and $\frac{1}{\bar{z}_1} \in \mathcal{H}^+$. The regions $\mathbb{G}_{\bar{z}_1}$, $\mathbb{G}^{+}_{\bar{z}_1}$ and $\mathbb{G}^{-}_{\bar{z}_1}$ are shown in Figure 3.4 below.

Next, we describe some relations between the Szegö curves $\frac{1}{z_1}\mathbb{S}$ and $\frac{1}{\bar{z}_1}\mathbb{S}$.

Lemma 3.22. *The curves $\frac{1}{z_1}\mathbb{S}$ and $\frac{1}{\bar{z}_1}\mathbb{S}$ intersect each other at exactly two points. Moreover, their point of intersections are on the real axis.*

Proof. The two curves intersect at point z if $|\phi(z_1 z)| = |\phi(\bar{z}_1 z)|$. But $z = x + iy$ satisfies this equation if $\left|e^{1-(x_1+iy_1)(x+iy)}\right| = \left|e^{1-(x_1-iy_1)(x+iy)}\right|$. From this, we get $e^{y_1 y} = e^{-y_1 y}$ which implies $y = 0$. $\square$

Observe that the line joining the two common points of $\frac{1}{z_1}\mathbb{S}$ and $\frac{1}{\bar{z}_1}\mathbb{S}$, $L_{z_1,\bar{z}_1}$, is the real axis. Hence $L_{z_1,\bar{z}_1}$ divides the complex plane into the upper half-plane $\mathcal{H}^+$ and lower half-plane $\mathcal{H}^-$. Indeed, we express these half-planes as:

$$\mathcal{H}^+ = \{z : |\phi(z_1 z)| > |\phi(\bar{z}_1 z)|\}$$

and

$$\mathcal{H}^- = \{z : |\phi(z_1 z)| < |\phi(\bar{z}_1 z)|\}.$$

Now we define several regions related to $\frac{1}{z_1}\mathbb{S}$, $\frac{1}{\bar{z}_1}\mathbb{S}$ and $L_{z_1,\bar{z}_1}$.

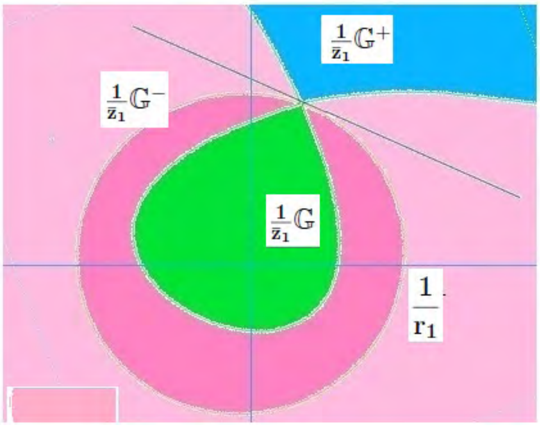


Figure 3.4: The Szegö regions $\mathbb{G}_{\bar{z}_1}$, $\mathbb{G}^+_{\bar{z}_1}$ and $\mathbb{G}^-_{\bar{z}_1}$

Definition 3.23. *Consider the Szegö curves $\frac{1}{z_1}\mathbb{S}$ and $\frac{1}{\bar{z}_1}\mathbb{S}$. We define Szegö regions $\mathbb{D}_{z_1}$ and $\mathbb{D}_{\bar{z}_1}$ as*

$$\mathbb{D}_{z_1} = \mathbb{G}_{z_1} \cap \mathcal{H}^- \quad \text{and} \quad \mathbb{D}_{\bar{z}_1} = \mathbb{G}_{\bar{z}_1} \cap \mathcal{H}^+,$$

where $\mathbb{G}_{z_1}$ and $\mathbb{G}_{\bar{z}_1}$ are the sets of points inside the curves $\frac{1}{z_1}\mathbb{S}$ and $\frac{1}{\bar{z}_1}\mathbb{S}$, respectively.
We define a region $\mathbb{D}_1$ to be the union of $\mathbb{D}_{z_1}$ and $\mathbb{D}_{\bar{z}_1}$. That is,

$$\mathbb{D}_1 = \mathbb{D}_{z_1} \cup \mathbb{D}_{\bar{z}_1}.$$

The Szegö curves $\frac{1}{z_1}\mathbb{S}$ and $\frac{1}{\bar{z}_1}\mathbb{S}$ intersect each other at two points, say P and Q, which are on the real axis. The Szegö regions $\mathbb{D}_{z_1}$ and $\mathbb{D}_{\bar{z}_1}$ and the boundary of their union, $\partial\mathbb{D}_1 = \partial\mathbb{D}_{z_1} \cup \partial\mathbb{D}_{\bar{z}_1}$, are shown in Figure 3.5 (a) and (b), respectively.

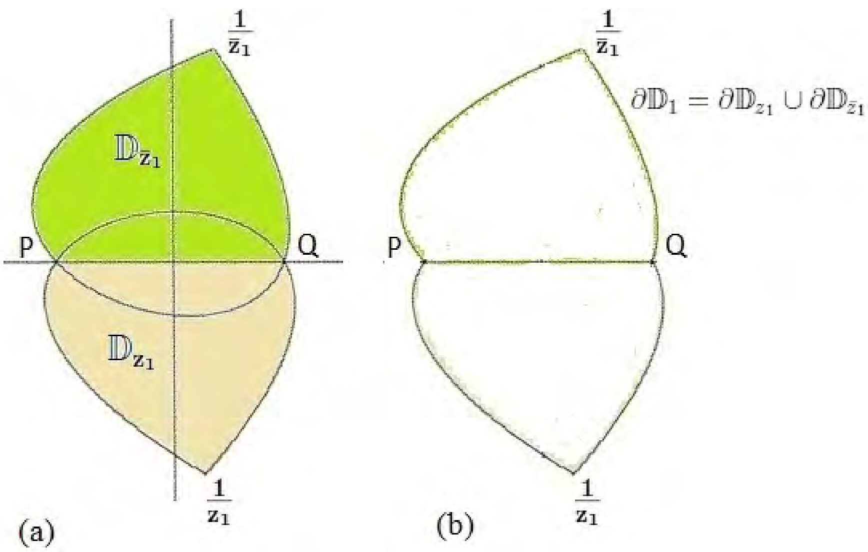


Figure 3.5: The Szegö regions $\mathbb{D}_{z_1}$ and $\mathbb{D}_{\bar{z}_1}$ and their boundaries

Clearly, both $\mathbb{D}_{z_1}$ and $\mathbb{D}_{\bar{z}_1}$ are open sets, each exclude the boundary points. For $\mathbb{D}_{z_1}$ and $\mathbb{D}_{\bar{z}_1}$, we have $\mathbb{D}_{z_1} \cap \mathbb{D}_{\bar{z}_1} = \emptyset$ and they have a common boundary, namely, the line segment $\overline{PQ}$ on the real axis. The closure of $\mathbb{D}_1$, $\bar{\mathbb{D}}_1 = \bar{\mathbb{D}}_{z_1} \cup \bar{\mathbb{D}}_{\bar{z}_1}$, represents all points of the colored regions in Figure 3.5 (a) together with the boundary points. Indeed, the boundary set $\partial\mathbb{D}_1$ includes the outer boundary points and points of the segment $\overline{PQ}$. The boundary of the region $\mathbb{D}_1$, $\partial\mathbb{D}_1$, is the the curve given in Figure 3.5 (b). Observe that $\mathbb{G}_{z_1} \cap B\left(\frac{1}{z_1}, \delta\right) \neq \emptyset$ for any $\delta > 0$. Therefore, the set $\mathbb{D}_{z_1}$ is an open non-empty subset of the region $\mathbb{G}_{z_1}$. Moreover, the regions $\mathbb{D}_{z_1}$ and $\mathbb{D}_{\bar{z}_1}$ can be expressed as

$$\mathbb{D}_{z_1} = \{z \in \mathbb{G}_{z_1} \,:\, |\phi(z_1 z)| < |\phi(\bar{z}_1 z)|\}$$

and

$$\mathbb{D}_{\bar{z}_1} = \{z \in \mathbb{G}_{\bar{z}_1} \,:\, |\phi(\bar{z}_1 z)| < |\phi(z_1 z)|\}\,.$$

Note that $\mathbb{D}_1 \subseteq \mathbb{G}_1 = \mathbb{G}_{z_1} \cup \mathbb{G}_{\bar{z}_1}$ but $\mathbb{D}_1 \neq \mathbb{G}_1$ because $\mathbb{G}_1$ contains the line segment $\overline{PQ}$. However, their closures are equal, that is, $\bar{\mathbb{D}}_1 = \bar{\mathbb{G}}_1$.

Next, we establish a special property of z_1 and $\bar{z}_1$ relative to all the other zeros of $\varphi(z) = e^z - 1 - z$. In fact, this is one of our results which we think is new in our study.

Theorem 3.24. *If $z_k = x_k + iy_k = r_k e^{i\theta_k}$ is any zero of $\varphi(z) = e^z - 1 - z$ with $y_k > 0$ and $|z_k| = r_k \geq r_2$, then $\frac{1}{z_k}$ lies inside the region $\mathbb{D}_{z_1}$. More generally, for any z_k such that $|z_k| > r_1$, we have $\frac{1}{z_k} \in \mathbb{D}_1 = \mathbb{D}_{z_1} \cup \mathbb{D}_{\bar{z}_1}$.*

Proof. Let $z_k = x_k + iy_k = r_k e^{i\theta_k}$ be as given in the hypothesis. For each $k \geq 2$, we have $\left|\frac{1}{z_k}\right| = \frac{1}{r_k} < \frac{1}{r_1}$ showing that $\frac{1}{z_k} \in B(0, \frac{1}{r_1})$. This shows that $\frac{1}{z_k} \notin \bar{\mathbb{G}}^+_{z_1}$. Now consider the function $\phi(z_1 z) = z z_1 e^{1 - z_1 z}$. If we evaluate $|\phi(z_1 z)|$ at $z = \frac{1}{z_k}$, then noting that $x_k \geq x_2$, $y_k \geq y_2$ and $r_k \geq r_2$ for all $k \geq 2$, we get

$$\left|\phi\left(z_1 \frac{1}{z_k}\right)\right| = \frac{r_1}{r_k} e^{1 - \frac{(x_1 x_k + y_1 y_k)}{r_k^2}} \leq \frac{r_1}{r_2} e^{1 - \frac{(x_1 x_2 + y_1 y_2)}{r_2^2}} \qquad \text{for all } k \geq 2.$$

Then using the values of x_1, y_1, r_1, x_2, y_2 and r_2 given in (2.12), we get

$$\frac{r_1}{r_2} e^{1 - \frac{(x_1 x_2 + y_1 y_2)}{r_2^2}} < 1.$$

Thus, $\left|\phi\left(z_1 \frac{1}{z_k}\right)\right| < 1$ and $\left|\frac{1}{z_k}\right| < \frac{1}{r_1}$ so that $\frac{1}{z_k} \in \mathbb{G}_{z_1}$. Moreover, since $y_k > 0$, we have $\frac{1}{z_k} \in \mathcal{H}^-$. Hence, $\frac{1}{z_k} \in \mathbb{D}_{z_1}$. On the other hand, if we assume $y_k < 0$, then we evaluate $|\phi(\bar{z}_1 z)|$ at $z = \frac{1}{z_k}$ and get $\frac{1}{z_k} \in \mathbb{D}_{\bar{z}_1}$. Therefore, $\frac{1}{z_k} \in \mathbb{D}_1 = \mathbb{D}_{z_1} \cup \mathbb{D}_{\bar{z}_1}$ for each $k \geq 2$. □

3.4 Summary of Results and Conclusion of the Study

Analogous to the classical Bernoulli polynomials, the hypergeometric Bernoulli polynomials of order 2, $B_n(2, a)$, possess interesting asymptotic behavior. For any real number $a < 0$, $B_n(2, a)$ and the respective m^{th} derivative $B_n^{(m)}(2, a)$ have asymptotic representations given by (3.1) and (3.2), respectively. Consequently, for sufficiently large positive integers n, the negative real zeros of $B_n(2, a)$ are approximately the same as the roots of $\cos(y_1 a - n\theta_1) = 0$, where $z_1 = x_1 + iy_1 = r_1 e^{i\theta_1}$ is the zero of $\varphi(z) = e^z - 1 - z$ with minimum modulus.

For positive real numbers $a > 0$, we established an asymptotic representation in (3.3) which is given for $B_n(2, a + m)$ for any integer m. Then we described the asymptotic positive real zeros of $B_n(2, a)$. These are given in Theorem 3.6 and Theorem 3.7 in terms of the root z_1 of $\varphi(z) = e^z - 1 - z$.

In order to determine the asymptotic complex zeros of $B_n(2, a)$, we established asymptotic formulas for the re-scaled polynomials $B_n(2, na)$. We obtained asymptotic representations for the case when na is real as well as for the complex variable case. These are given an (3.8), (3.9), (3.10) and under respective Lemmas and Corollaries in Section 3.2. For sufficiently large positive integers n, the complex zeros of $B_n(2, nz)$ are illustrated by using a curve in the complex plane. These asymptotic complex zeros are precisely described by an H-shaped curve whose equation is given by (3.12) as proved in Theorem 3.18.

The polynomials $B_n(2, x)$ and the classical Bernoulli polynomials $B_n(x)$ have many analogous properties. In particular, their asymptotic real and complex zeros are given by slightly similar H-shaped curves. We can make several conclusions about analogous results obtained regarding the asymptotic zeros of $B_n(x)$ and those of $B_n(2, x)$.

Remark 3.25. *For the polynomials $B_n(x)$ and $B_n(2, x)$, we have:*

- *In the case of $B_n(x)$, we have $\varphi_1(z) = e^z - 1$ in the generating function (1.3) of $B_n(x)$ and the two roots of $e^z - 1 = 0$ with minimum modulus are $w_1 = 2\pi i$ and $-w_1 = -2\pi i$. John Mangual proved that points of the H-shaped curve described by (1.25) are accumulation points of the complex zeros of $B_n(nz)$ as $n \to \infty$.*
- *Similarly, we have $\varphi(z) = e^z - 1 - z$ in the generating function (2.1) of $B_n(2, x)$ and $z_1 = x_1 + iy_1$ and $\bar{z}_1 = x_1 - iy_1$ are the roots of $e^z - 1 - z = 0$ with minimum modulus.*

Note that (1.25) and (3.12) are similar except that the former involves the roots $w_1 = 2\pi i$ and $-w_1 = -2\pi i$ of $\varphi_1(z) = e^z - 1$ while the latter consists of the roots $z_1 = x_1 + iy_1$ and $\bar{z}_1 = x_1 - iy_1$ of $\varphi(z) = e^z - 1 - z$.

Now consider the roots $z_1 = x_1 + iy_1 = r_1 e^{i\theta_1}$ and $\bar{z}_1 = x_1 - iy_1 = r_1 e^{-i\theta_1}$ of $\varphi(z) = e^z - 1 - z$. In several asymptotic formulas of $B_n(2, a)$ given in Section 3.1 and Section 3.2, we used z_1 and $\bar{z}_1$, especially the imaginary part y_1 and the argument θ_1 were the determining quantities in each asymptotic formula of $B_n(2, a)$. In Section 3.3, we considered the two curves related to the standard Szegö $\phi(z) = ze^{1-z}, |z| \leq 1$. These are the Szegö curves $\frac{1}{z_1}\mathbb{S}$ and $\frac{1}{\bar{z}_1}\mathbb{S}$ given by

$$\phi(z_1 z) = z z_1 e^{1 - z_1 z} \quad \text{and} \quad \phi(\bar{z}_1 z) = z z_1 e^{1 - \bar{z}_1 z}, \ \text{ for } |z| \leq \frac{1}{r_1}.$$

We discussed some interesting properties of these Szegö curves and some related regions of the complex plane. In Theorem 3.24, we obtained an important result regarding the special properties that z_1 and $\bar{z}_1$ have as compared to all the other roots z_k of $\varphi(z) = e^z - 1 - z$.

Remark 3.26. *The two Szegö curves $\frac{1}{z_1}\mathbb{S}$ and $\frac{1}{\bar{z}_1}\mathbb{S}$ (defined by 3.15 and 3.16) have an interesting relation to the asymptotic zeros of $B_n(2, nz)$. In our future study, we focus on these two Szegö curves and try to illustrate that the boundary of the Szegö region $\mathbb{D}_1 = \mathbb{D}_{z_1} \cup \mathbb{D}_{\bar{z}_1}$ (Figure 3.5 (b)) is a zero attractor for $B_n(2, nz)$ as $n \to \infty$. That is, the boundary points of $\mathbb{D}_1$ are accumulation points for the zeros of $B_n(2, nz)$ as $n \to \infty$.*

This method of studying asymptotic zeros of polynomials is possibly extendable to the hypergeometric Bernoulli polynomials of arbitrary order, $B_n(N, x)$,

$$\frac{z^N e^{xz}/N!}{e^z - T_{N-1}(z)} = \sum_{n=0}^{\infty} B_n(N, x) \frac{z^n}{n!},$$

where $T_N(z) = \sum_{k=0}^{N} \frac{z^k}{k!}$.
If $\{z_k, \bar{z}_k\}$ *are the roots of* $\varphi_N(z) = e^z - T_{N-1}(z)$, *then we expect some similar properties for* z_1 *and* $\bar{z}_1$ *as obtained in Theorem 3.24 (or, some more* z_k*'s with such a dominating property). We also have corresponding Szegö curves* $\frac{1}{z_k}\mathbb{S}$ *and* $\frac{1}{\bar{z}_k}\mathbb{S}$ *as well as related regions in the complex plane analogous to those discussed in Section 3.3 for the particular case when* $N = 2$.

Some Problems for Future Works

In our future works, we would like to consider more concepts related to the asymptotic representations and asymptotic real and complex zeros of hypergeometric Bernoulli polynomials of order 2, $B_n(2, x)$ as well as for the general case, $B_n(N, x)$.

- We consider hypergeometric Bernoulli polynomials of order $N = 3$ and establish similar asymptotic formulas for $B_n(3, x)$. We try to apply the methods discussed in Section 3.1 and Section 3.2 to determine the asymptotic real and complex zeros of $B_n(3, x)$. If possible, we extend these methods to the hypergeometric Bernoulli polynomials of arbitrary order, $B_n(N, x)$.
- We consider and go through details of the Szegö curve analysis (the concepts discussed in Section 3.3). We try to use the dominating properties of z_1 and $\bar{z}_1$ (the properties given in Theorem 3.24) to determine the asymptotic real and complex zeros of $B_n(2, x)$. Then we extend the method to the general case for $B_n(N, x)$; to determine the asymptotic real and complex zeros of $B_n(N, nz)$ in terms of the dominating zeros of $\varphi_N(z)$.
- We are interested in studying other concepts related to hypergeometric Bernoulli polynomials; such as **integral representations** and **integral asymptotics** of $B_n(2, x)$ as well as for the general case $B_n(N, x)$.

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