ADDIS ABEBA UNIVERSITY
COLLEGE OF SCIENCE
FACULTY OF MATHEMATICS AND COMPUTER SCIENCES
DEPARTMENT OF MATHEMATICS

GRADUATE PROJECT REPORT ON

FOURIER SERIES FUNCTIONS ON A HOMOGENEOUS BANACH SPACE

(IN PARTIAL FULFILLMENT OF THE M.SC DEGREE IN MATHEMATICS)

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Declaration

I, the undersigned, declare that this project work is my original work, has not been presented for degrees in any other university, and all sources of material used for the project work have been duly acknowledged.

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By

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INTRODUCTION

Just before 1800, the French mathematician/physicist/engineer Jean Baptist Joseph **Fourier** made a surprising discovery. As a result of his investigations into the partial differential equations, modeling vibration and heat propagation in bodies; Fourier was led to claim that “every” function could be represented by an infinite series of elementary trigonometric functions , sines and cosines.

Fourier analysis is an essential component of much of modern applied (and pure) mathematics. Applications in pure mathematics, physics and engineering are almost too numerous to catalogue — typing in “**Fourier**” in the subject index of a modern science library will dramatically demonstrate just how everywhere these methods are. Fourier analysis lies at the heart of signal processing, including audio, speech, images, videos, seismic data, radio transmissions, and so on. Many modern technological advances, including television, music CD’s and DVD’s, video movies, computer graphics, image processing, and fingerprint
Analysis and storage are, in one way or another, founded upon the many ramifications of Fourier’s discovery.

Furthermore, a remarkably large fraction of modern pure mathematics is the result of subsequent attempts to place Fourier series on a firm mathematical foundation.Thus,all of the student’s“favorite” analytical tools, including the definition of a function, the ε - δ definition of limit and continuity, **convergence** properties in function space, including uniform convergence, weak convergence, etc., the modern theory of integration and measure, generalized functions such as the delta function, and many others, all be obliged a profound debt to the prolonged struggle to establish a rigorous framework for Fourier analysis. Even more remarkably, modern set theory and as a result, mathematical logic and foundations can be traced directly back to Cantor's attempts to understand the sets upon which Fourier series converge.

The most far reaching of this was that of L.Fejer (1904) who invented a new procedure of summing the Fourier series; which made the series convergent at all points. The only conditions were that the function should be integrable,with out demanding continuity or differentiable.

Lebesgue and Haar have investigated the general conditions that at a particular point of continuity of the function, the Fourier series should fail to converge. Haar also investigated that the series converges at a point and the convergence should be non-uniform in any neighborhood of the point.

Fourier series is the main tool in the analysis of periodic function, which plays a major role in the solution of some fundamental problems in mathematical physics.

The overall **objective** of this project is to identify the convergence of Fourier series of functions on a Homogenous Banach space. That is, to address the natural question which now arises is" whether the Fourier series of f converges to f ",or, more generally, " *whether f is determined by its Fourier series "*.That is to say, if we know the Fourier coefficients of a function, can we find the function, and if so, how?

This project have two chapters. The first chapter focuses on the basic concepts on Fourier series and the second chapter is the convergence of Fourier series .

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CHAPTER ONE

FOURIER SERIES

1.1 INTRODUCTION

DEFINITION: 1.1.1 Let T be the unit circle in the complex plane, i.e., the set of all complex numbers of absolute value. If F is any function on T and if f is defined on ℝ by $f(t) = F(e^{it})$. (1)

Then f is a periodic function of period 2π, i.e. f(t+2π) = f(t) for all real t. Conversely, if f is a function on ℝ with period 2π, then there is a function F on T such that (1) holds. Thus we may identify functions on T with 2π periodic functions on ℝ and for simplicity of notation, we shall sometimes write f(t) rather than $f(e^{it})$, even if we think of f as being defined on T.

With these conventions in mind, we define $L^p(T)$, for $1 < p < \infty$, to be the class of all complex, Lebesgue measurable, 2π-periodic functions on ℝ for which the norm $\|f\|_p = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)|^p\,dt\right\}^{\frac{1}{p}}$ is finite.

DEFINITION: 1 .1.2 The space o f al l co mplex v alued Leb esgue integrable functions on T is denoted by $L^1(T)$ is a Banach space with the norm de fined by $\|f\|_{L^1} = \frac{1}{2\pi}\int_T |f(t)|\,dt$. Where a function f on T is Lebesgue integrable if the corresponding 2π-periodic functions which we denote a gain by f is l ebesgue i ntegrable on [0, 2π] and we set $\int_T f(t)\,dt = \int_0^{2\pi} f(x)\,dx$.

1.2 FOURIER COEFFICIENTS

DEFINITION: 1.2.1 A trigonometric polynomial on t is an expression of the form

$$P(t) = \sum_{-N}^{N} C_n e^{int}$$

Where the numbers n appearing in the expression are called the frequencies of P. the largest integers n such that $|C_n| + |C_{-n}| \neq 0$ is called the degree of P. Since each of the summand in the expression is a function on T and is finite. The expression represents a function, i.e. for each $t \in T$.

DEFINITION: 1.2.2 The conjugate series $\tilde{P}$ of trigonometric series P is the series

$$\tilde{P}(t) = \sum_{i=-\infty}^{\infty} -i \, \mathrm{Sign}(j) \, a_j e^{ijt}, \text{where } \mathrm{Sgn}(j) = \begin{cases} 0 & \text{if } j = 0 \\ \frac{n}{|n|} & \text{if } j \neq 0 \end{cases}$$

DEFINITION 1.2.3 Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π- periodic and integrable function in $[-\pi, \pi]$. Then the Fourier series of f is the series $\sum_{j=-\infty}^{+\infty} c_j e^{ijt}$. We write $f(t) \sim \sum_{j=-\infty}^{+\infty} c_j e^{ijt}$.

Where the Fourier coefficients $\mathbf{c_j}$ are defined by

$$c_j \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ijt} \, dt$$

These coefficients are denoted as $c_j = \hat{f}(j)$.

We also define the n^{th} partial sum of the Fourier series of f by

$$S_n(f)(x) \equiv \sum_{k=-n}^{n} c_k e^{ikt} .$$

It may be interesting to see where this formula came from. Suppose that

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

And we multiply both sides by e^{-imx} and take the integral , $\int_{-\pi}^{\pi}$, so that

$$\int_{-\pi}^{\pi} f(x)e^{-imx} dx = \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} c_k e^{ikx} e^{-imx} \ dx .$$

Now we switch the sum and the integral on the right side even though we have absolutely no reason to believe this makes any sense.

Then we get

$$\int_{-\pi}^{\pi} f(x)e^{-imx} dx = \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} c_k e^{ikx} e^{-imx} \ dx$$

$$= c_m \int_{-\pi}^{\pi} 1 \ dx = 2\pi \, c_m.$$

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-imx} \ dx .$$

In case, if f is real valued, we see that $\overline{c_k} = c_{-k}$ and so

$$S_n f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \, dy + \sum_{k=1}^{n} 2Re(c_k e^{ikx}).$$

Letting $c_k \equiv \alpha_k + i\beta_k$

$$S_n f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,dy + \sum_{k=1}^{n} 2(\alpha_k \cos kx - \beta_k \sin kx).$$

Where, $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, e^{-imy}\,dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)(\cos ky - i \sin ky)\,dy$.

This shows that

$$\alpha_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\cos(ky)dy, \quad \beta_k = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\sin(ky)dy.$$

Therefore, letting $a_k = 2\alpha_k$ and $b_k = -2\beta_k$, we see that

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos(ky)\,dy, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin(ky)\,dy.$$

And

$$S_n f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx.$$

where, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\,dy$.

DEFINITION: 1.2.4 Let $f \in L^1(T)$, then n^{th} Fourier coefficients of f denoted by $\hat{f}(n)$, is defined as

$$\hat{f}(n) = \frac{1}{2\pi}\int_T f(t)e^{-int}\,dt.$$

DEFINITION 1.2.5 Let $f \in L^1(T)$, the Fourier series of f is the trigonometric series $\sum_{-\infty}^{\infty} \hat{f}(n)\,e^{int}$.

THEOREM: 1.2.6 Let $f, g \in L^1(T)$ and $n \in \mathbb{Z}$. Then:

1) $\widehat{(f+g)}(n) = \hat{f}(n) + \hat{g}(n)$

2) For any complex α, $(\widehat{\alpha f})(n) = \alpha[\hat{f}(n)]$.

3) If $\bar{g}$ is the complex conjugate of f , i.e. $\bar{g}(x) = \overline{g(x)}$

for all x∈ T, then $\hat{\bar{g}}(n) = \overline{\hat{g}(-n)}$.

4) Denote $g_t(x) = g(x-t)$, for x ∈ T. Then $\hat{g}_t(n) = \hat{g}(n)\, e^{-int}$

5) $|\hat{g}(n)| \leq \frac{1}{2\pi}\int_T |g(t)|\, dt = \|g\|_L{}^1$

PROOF: The proof of (1) – (3) is a direct consequences of the definition of Fourier series.

4) $\hat{g}_t(n) = \frac{1}{2\pi}\int_T g_t(x)\, e^{-inx}\, dx = \frac{1}{2\pi}\int_T g(x-t)\, dx$

$= \frac{1}{2\pi}\int_T g(x-t)\, e^{-inx}\,(e^{inx}\, e^{-inx})\, dx = \frac{1}{2\pi}\int_T g(x-t)\, e^{-in(t-x)}\, e^{-inx}\, dx$

$= \frac{1}{2\pi}\int_T g(y)\, e^{-iny}\, e^{-int}\, dy = \hat{g}(n)e^{-int}$.

5) $|\hat{g}(n)| = \left|\frac{1}{2\pi}\int_T f(t)\, e^{-int}\, dt\right| \leq \frac{1}{2\pi}\int_T |f(t)e^{-int}|\, dt$

$= \frac{1}{2\pi}\int_T |f(t)|\, |e^{-int}|\, dt = \frac{1}{2\pi}\int_T |f|\, dt = \|f\|_L{}^1$.

COROLLARY: 1.2.7 Assume $f_k \in L^1(T)$, k = 0,1,2,3,…, and

$\|f_k - f_0\|_L{}^1 \to 0$ as $k\to\infty$, Then $\hat{f}_k(n) \to \hat{f}_0(n)$ as $k\to\infty$ uniformly .

PROOF: Using (5) in the above theorem,

$\Rightarrow |\widehat{(f_k - f_0)}(n)| \leq \frac{1}{2\pi}\int_T |f_k - f_0|\, dt = \|f_k - f_0\|_L{}^1$

$\Rightarrow |\widehat{(f_k - f_0)}(n)| \leq \|f_k - f_0\|_L{}^1$ but $\|f_k - f_0\|_L{}^1 \to 0$ as $k\to\infty$.

$\Rightarrow |\, \hat{f}_k(n) - \hat{f}_0(n) \,| \to 0$ as $k \to \infty$.

Hence, $\hat{f}_k(n) \to \hat{f}_0(n)$ as $k \to \infty$ uniformly.

THEOREM: 1.2.8 Let $f \in L^1(T)$. Assume that $\hat{f}(0) = 0$ and Define

$F(t) = \int_0^t f(x)\,dx$. Then F is continuous 2π-periodic functions and

$\hat{F}(n) = \frac{1}{in}\hat{f}(n), \quad n \neq 0$.

PROOF: since F is absolute continuous, the continuity of F is evident. The periodicity follows from

$$F(t + 2\pi) - F(t) = \int_0^{t+2\pi} f(x)dx - \int_0^t f(x)dx$$

$$= \int_0^{2\pi} f(x)\,dx = 2\pi\ \hat{f}(0) = 0.$$

We want to show, $\hat{F}(n) = \frac{1}{in}\hat{f}(n),\ n \neq 0$, use integration by parts.

$$\hat{F}(n) = \frac{1}{2\pi}\int_0^{2\pi} F(t)e^{-int}\,dt.$$

$$\text{Then, } \hat{F}(n) = \frac{1}{2\pi}\left(\left[\frac{F(t)}{in}\, e^{int}\right]_0^{2\pi} - \int_0^{2\pi}\frac{F'(t)}{in}\, e^{-int}\,dt\right)$$

$$= \frac{1}{in2\pi}\int_0^{2\pi}\int f(t)\, e^{-int}\,dt \quad = \frac{1}{in}\hat{f}(n)\ , n \neq 0.$$

DEFINITION: 1.2.9 Let f, g $\in L^1(T)$. The *convolution* of **f** and **g** on the $L^1(T)$ function is denoted as **f * g** and is defined by

$$(f * g)(x) = \frac{1}{2\pi}\int_T f(x - t)\, g(t)\, dt, \quad \text{for all t.}$$

THEOREM: 1.2.10 Let f, g $\in L^1(T)$. For almost all t, the function

f(x-t) g(t) is integrable (as a function of t on T), and if we write

$h(x) = \frac{1}{2\pi}\int f(x-t)g(t)\,dt$, then $h \in L^1(T)$, $\|h\|_{L^1} \leq \|f\|_{L^1}\|g\|_{L^1}$ and

$\hat{h}(n) = \hat{f}(n)\,\hat{g}(n)$, for all $n \in \mathbb{N}$.

PROOF: The function f(x-t) and g(t) are measurable function of the two variables (x, t), hence the product function F(x,t)=f(x-t)g(t) is also measurable. For almost all t, F(x,t) is just a constant multiple of f_t , hence it is integrable and

$$\frac{1}{2\pi}\int_0^{2t}\left(\frac{1}{2\pi}\int_0^{2\pi} |F(x,t)|\,dx\right)dt = \frac{1}{2\pi}\int_0^{2t}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(x-t)|\,|g(t)|dx\right)dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} |g(t)|\ \|f\|_{L^1}dt$$

$$= \|f\|_{L^1}\|g\|_{L^1}.$$

By Fubini's theorem, F(x,t) is integrable over (0,2π) as a function of t for almost all x.

Now, $\frac{1}{2\pi}\int_0^{2\pi} |h(x)|\,dx = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{2\pi}\int_0^{2\pi} F(x,t)dt\right|dx$

$$\leq \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} |F(x,t)|\,dx\,dt = \|f\|_{L^1}\ \|g\|_{L^1}.$$

Thus h is integrable.

For $n \in Z$, $\hat{h}(n) = \frac{1}{2\pi}\int_0^{2\pi} h(x)e^{-inx}\,dx$

$$= \frac{1}{2\pi.2\pi}\int_0^{2\pi}\int_0^{2\pi} f(x-t)g(t)\,e^{-in(x-t)}e^{-int}\,dx\,dt.$$

$$\hat{h}(n) = \hat{f}(n)\,\hat{g}(n)\,.$$

Therefore, $\hat{h}(n) = \hat{f}(n)\,\hat{g}(n)$ for all n.

REMARK: The c onvolution ope rator i n $L^1(T)$ is c ommutative, associative and distributive with respect to addition.

THEOREM: 1.2.11 Assume $f \in L^1(T)$ and $\psi(x) = e^{inx}$ for some integer n.

Then $(\psi * f)(x) = \hat{f}(n)\, e^{inx}$.

PROOF: $(\psi * f)(x) = \frac{1}{2\pi}\int_T f(t)\, e^{in(x-t)}dt = (\frac{1}{2\pi}\int_T f(t)\, e^{-int}\, dt)\, e^{inx}$

$$= \hat{f}(n)\, e^{inx} .$$

COROLLAR : 1 .2.12 If $f \in L^1(T)$ a nd $K(x) = \sum_{-N}^{N} a_n e^{inx}$, then $(K * f)(x) = \sum_{-N}^{N} \hat{f}(n) a_n e^{inx}$.

PROOF:

$$(K * f) = \frac{1}{2\pi}\int_T \sum_{-N}^{N} a_n f(x) e^{in(x-t)}\, dt$$

$$= \sum_{-N}^{N} a_n\, e^{inx} \frac{1}{2\pi}\int_T e^{-int} f(t) dt$$

$$= \sum_{-N}^{N} a_n \hat{f}\,(n) e^{inx}.$$

1.3 SUMMABILITY IN NORM AND HOMOGENEOUS BANACH SPACE ON T

THEOREM: 1.3.1 (Properties of the Banach space on T**)**

i.(Translation invariance)

If $f \in L^1(T)$ and $x \in T$, then $f_t(x) = f(x-t) \in L^1(T)$ and $\| f_t \|_{L^1} = \|f\|_{L^1}$.

ii.(Continuity of translation)

The $L^1(T)$ valued function $t \to f_t$ is continuous on T, i.e. for $f \in L^1(T)$ and $t_o \in T$, $\lim_{t\to t0} \|f_t - f_{t_0}\|_{L^1} = 0$.

PROOF: i) Let $f \in L^1(T)$ and $x \in T$. Then

$$\frac{1}{2\pi}\int_T |f_t(x)|\,dx = \frac{1}{2\pi}\int_T |f(x-t)|\,dx$$

$$= \frac{1}{2\pi}\int_T |f(x)|\,dx = \|f\|_{L^1} < \infty \text{. Then, } \|f_t\|_{L^1} < \infty.$$

$$\Longrightarrow f_t \in L^1(T) \text{ and } \|f\|_{L^1} = \|f_t\|_{L^1}.$$

ii) *Case 1*: Let f be a continuous function on T. Then given $\varepsilon > 0$ there exists $\delta>0$ such that $|x-t| < \delta \Longrightarrow |f(x) - f(t)| < \varepsilon$.

Now , $\|f_t - f_{t_0}\|_{L^1} = \frac{1}{2\pi}\int_T |f_t(x) - f_{t_0}(x)|\,dx$

$$= \frac{1}{2\pi}\int_T |f(x-t) - f(x-t_0)|\,dx$$

$$< \frac{1}{2\pi}\int_T \varepsilon\,dx = \varepsilon \text{, provided that } |t - t_0| < \delta.$$

Therefore, $\lim_{t\to t0} \|f_t - f_{t_0}\|_{L^1} = 0$.

Case 2: consider arbitrary $f \in L^1(T)$, since continuous function are dense in $L^1(T)$. For $\varepsilon > 0$, there exist a continuous function g on T, such that $\|g - f\|_{L^1} < \frac{\varepsilon}{4}$. Since g is continuous on T, there exist $\delta = \delta(\varepsilon) > 0$ such that $t, t_0 \in T$ and $|t - t_0| < \delta \Longrightarrow \|g_t - g_{t_0}\|_{L^1} < \frac{\varepsilon}{2}$.

Now, $\|f_t - f_{t_0}\|_{L^1} \le \|f_t - g_t\|_{L^1} + \|g_t - g_{t_0}\|_{L^1} + \|g_{t_0} - f_{t_0}\|_{L^1}$

$$= \|(f-g)_t\|_{L^1} + \|g_t - g_{t_0}\|_{L^1} + \|(g-f)_{t_0}\|_{L^1}$$

$= \|f - g\|_{L^1} + \|g_t - g_{t_0}\|_{L^1} + \|g - f\|_{L^1}$

$< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon$, Provided that $|t - t_0| < \delta$.

Thus, $\lim_{t\to t_0} \|f_t - f_{t_0}\|_{L^1} = 0$. Therefore continuity is established.

DEFINITION: 1 .3.2 A Summability Kernel i s a s equence $\{K_n\}$ of continuous 2π-periodic func tions sa tisfying:

S.1: $\frac{1}{2\pi}\int_T K_n(x)dx = 1.$

S.2: $\frac{1}{2\pi}\int_T |K_n(x)|dx \le c$, for some constant c

S.3: For all $0 < \delta < \pi$, $\lim_{n\to\infty} \int_\delta^{2\pi-\delta} |K_n(x)|\, dx = 0$.

A Positive Summability kernel is one such that $k_n(x) \ge 0$ for all x and n.

EXAMPLE ON SUMMABILITY KERNEL

1. One of the most useful summability kernels, and probably the best known is Fejér's kernel , denot e by $\{F_n\}$ and is defined by

$$F_n(x) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) e^{ijx}$$

2. The de la valle-poussin kernel defined by

$V_n(x) = 2F_{2n+1}(x) + F_n(x)$ is a summability kernel.

3. The poisson kernel: for $0 \le r \le 1$,

$$P(r, x) = 1 + 2\sum_{j=1}^{\infty} r^j \cos jx$$

$$= \frac{1 - r^2}{1 - 2r\cos t + r^2}$$ is a summability kernel.

LEMMA: 1.3.3 Let B be a Banach space, φ be a continuous B-valued function on T and $\{k_n\}$ is a summability kernel.

Then $\lim_{n\to\infty} \frac{1}{2\pi}\int_T K_n(\tau)\, \varphi(\tau)\, d\tau = \varphi(0)$.

PROOF: By (S-1), we have

$$\frac{1}{2\pi}\int_T K_n(\tau)\, \varphi(\tau)\, d\tau = \varphi(0)\, \frac{1}{2\pi}\int_T K_n(\tau) d\tau.$$

Continuity of φ at 0, then let ε > 0 be given, there exists δ = δ(ε) > 0 such that $|\tau| < \delta \Longrightarrow \| \varphi(\tau) - \varphi(0)\|_B < \varepsilon$.

By (S-3), for 0 < δ < π, there exists ℕ, such that

$$\frac{1}{2\pi}\int_\delta^{2\pi-\delta} |K_n(\tau)|\, d\tau < \varepsilon, \text{ for all } n > \mathbb{N}.$$

Now, $\frac{1}{2\pi}\int_T K_n(\tau)\varphi(\tau)d\tau - \frac{1}{2\pi}\int_T K_n(\tau)\varphi(0)d\tau = \frac{1}{2\pi}\int_T K_n(\tau)(\varphi(\tau) - \varphi(0))\, d\tau$

$$= \frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(\tau)(\varphi(\tau) - \varphi(0))d\tau + \frac{1}{2\pi}\int_\delta^{2\pi-\delta} K_n(\tau)(\varphi(\tau) - \varphi(0))\, d\tau,$$

$\|\frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(\tau)(\varphi(\tau) - \varphi(0))d\tau\|_B \le \max_{|\tau|\le\delta} \| \varphi(\tau) - \varphi(0)\|_B \|K_n\|_{L^1}$ and

$\|\frac{1}{2\pi}\int_\delta^{2\pi-\delta} K_n(\tau)(\varphi(\tau) - \varphi(0))d\tau\|_B \le \max_\tau \| \varphi(\tau) - \varphi(0)\|_B \frac{1}{2\pi}\int_\delta^{2\pi-\delta} |K_n(\tau)|\, d\tau$

$\|\frac{1}{2\pi}\int_T K_n(\tau)(\varphi(\tau) - \varphi(0))d\tau\|_B$

$\le \| \frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(\tau)(\varphi(\tau)-\varphi(0))d\tau\|_B + \| \frac{1}{2\pi}\int_\delta^{2\pi-\delta} K_n(\tau)(\varphi(\tau)-\varphi(0))d\tau\|_B$

$$< \varepsilon \|K_n\|_{L^1} + \varepsilon \max_\tau \| \varphi(\tau) - \varphi(0)\|_B$$

$$= \varepsilon (\|K_n\|_{L^1} + \max_\tau \| \varphi(\tau) - \varphi(0)\|_B), \text{ for } n \ge \mathbb{N}.$$

Hence $\lim_{n\to\infty} \frac{1}{2\pi}\int_T K_n(\tau)\varphi(\tau)d\tau = \varphi(0)$.

THEOREM: 1.3.4 Let $f \in L^1(T)$ and $\{K_n\}$ be a summability kernel.

Then, $f = \lim_{n\to\infty} \frac{1}{2\pi}\int_T K_n(\tau) f_\tau\, d\tau$.

PROOF: By the property of Banach space, the mapping $\tau \to f_\tau$ is continuous $L^1(T)$-valued function on T. By the above lemma, putting $\varphi(\tau) = f_\tau(t) = f(t-\tau)$, then we have $\varphi(0) = f(t)$.

Therefore, $\lim_{n\to\infty} \frac{1}{2\pi}\int_T K_n(\tau) f_\tau \, d\tau = f_0 = f$.

LEMMA 1.3.5 Let k be a continuous function on T and $f \in L^1(T)$, then

$$\frac{1}{2\pi}\int_T K(\tau)\, f_\tau \, d\tau = k*f.$$

PROOF: Case i) Assume first that f is continuous on T. We have

$$\frac{1}{2\pi}\int_T K(\tau) f_\tau d\tau = \lim_{N\to\infty} \sum_{j=0}^{N} (\tau_{j+1} - \tau_j) K(\tau_j) f_{\tau_j},$$

the limit being taken in the $L^1(T)$ norms the subdivision $\{\tau_j\}$ of $[0,2\pi]$ becomes finer and finer .

On the other hand

$$\lim_{N\to\infty} \sum_{j=0}^{N} (\tau_{j+1} - \tau_j) K(\tau_j) f_{\tau_j} = \frac{1}{2\pi}\int_T K(\tau) f(t-\tau) d\tau = k * f$$

Case ii) For arbitrary $f \in L^1(T)$ and $\varepsilon > 0$, there exists a continuous function g on T such that $\|f - g\|_B < \varepsilon$.

Now, since g is continuous, we have

$$\frac{1}{2\pi}\int_T K(\tau)\, f_\tau \, d\tau - k * f = \frac{1}{2\pi}\int_T K(\tau)\, f_\tau \, d\tau - \frac{1}{2\pi}\int_T K(\tau)\, g_\tau \, d\tau + k * g - k * f$$

$$= \frac{1}{2\pi}\int_T K(\tau)\, (f_\tau - g_\tau)\, d\tau + K * (f - g)$$

And consequently,

$$\left\|\frac{1}{2\pi}\int_T K(\tau) f_\tau \, d\tau - k*f\right\|_{L^1} \le \left\|\frac{1}{2\pi}\int_T K(\tau)(f_\tau - g_\tau)\, d\tau\right\|_{L^1} \| K*(f-g)\|_{L^1}.$$

$$\le \|k\|_{L^1} \|g_\tau - f_\tau\| + \|k\|_{L^1} \|g - f\|_{L^1}$$

$$= 2\|k\|_{L^1} \|g - f\|_{L^1} < 2\varepsilon\|k\|_{L^1} .$$

Since, ε is arbitrary, we h ave $\frac{1}{2\pi}\int_T K(\tau)f_\tau \, d\tau = k*f$.

NOTATION: Let $f \in L^1(T)$ and F_n be fejer's kernel,then $F_n*f = \frac{1}{2\pi}\int_T F_n(t)\, f_t \, dt$ denoted by $\sigma_n(f)$ and

$$\sigma_n(f,x) = (F_n * f)(x) = \sum_{j=-n}^{n}\left(1 - \frac{|j|}{n+1}\right)\hat{f}(j)e^{ijx}.$$

NOTE: The fact that $\sigma_n(f) \to f$ in the $L^1(T)$ norm for every $f \in L^1(T)$, which i s th e s pecial c ase $\lim_{n\to\infty} k_n*f$, w here K_n any summability kernel, a nd from t he fact t hat $\sigma_n(f)$ i s a t rigonometric polynomial imply that trigonometric polynomial are dense in $L^1(T)$.

THEOREM: 1.3.6 (UNIQUENESS THEOREM)

Let $f \in L^1(T)$. If $\hat{f}(n) = 0$ for all $n \in Z$, then $f = 0$.

PROOF: By the above note, $\sigma_n(f) = 0$, for all n. since $\sigma_n(f) \to f$ in $L^1(T)$.

We have $f = \lim_{n\to\infty} \sigma_n(f) = 0$. Therefore, $f = 0$.

COROLLARY: 1.3.7 Let $f, g \in L^1(T)$. If $\hat{f}(n) = \hat{g}(n)$ for all n, then $f = g$.

PROOF: $\hat{f}(n) = \hat{g}(n) \implies (\widehat{f+g})(n) = 0$. By uni queness t heorem, $\sigma_n(f-g) = 0$, for all n as $\sigma_n(f-g) \to f - g$ in $L^1(T)$.

Therefore, $f - g = 0$. Hence $f = g$.

THEOREM: 1.3.8 (*Riemann-Lebesgue Lemma*)

If $f \in L^1(T)$,then $\lim_{|n|\to\infty} \hat{f}(n) = 0$.

PROOF: Let $\varepsilon > 0$ and let p be a trigonometric polynomial on T of degree d such that $\|f - p\|_{L^1} < \varepsilon$ .Then $\hat{p}(n) = 0$ for all $|n| > d$.

consequently, $|\hat{f}(n)| = |\hat{f}(n)-\hat{p}(n)| \leq \|f - p\|_{L^1} < \varepsilon$ for all $|n| > d$.

Therefore, $\lim_{|n|\to\infty} \hat{f}(n) = 0$.

THEOREM:1.3.9 If K is a compact set in $L^1(T)$ and $\varepsilon > 0$, there exist a finite number of trigonometric polynomials $P_1, \ldots, P_N$ such that for every $f \in K$ there exists a j, $1 \leq j \leq N$ such that $\|f - P_j\|_{L^1} < \varepsilon$.

If $|n| > \max_{1\leq j\leq N} N_j$, then $|\hat{f}(n)| < \varepsilon$ for all $f \in K$. Where N_j is the degree of p_j.

PROOF: Since trigonometric polynomial are dense in $L^1(T)$,there exists a collection β of ε-spheres about trigonometric polynomial that cover $L^1(T)$, then β also covers K. By the compactness of β, there exists ε-sphere $S_1, S_2, \ldots, S_N$ such that

$$K \in \bigcup_{j=1}^{N} S_j$$

By the definition of β ,we get trigonometric polynomials $p_1, p_2, \ldots, p_N$ such that $S_j = (p_j, \varepsilon)$, $j = 1,2,\ldots N$. Let $f \in K$, there exists j, $1 \leq j \leq N$, such that $f \in S_j$, i.e $\|f - P_j\|_{L^1} < \varepsilon$. If $|n| > \max_{1\leq j\leq N} N_j$, then $|\hat{P}_j(n)| = 0$ for all $j=1,2,\ldots N$ and $|\hat{f}(n)| \leq \|f - P_j\|_{L^1} < \varepsilon$ for all $j, 1 \leq j \leq N$.

REMARK:

1. The Riemann-Lebesgue lemma holds uniformly on a compact subset of $L^1(T)$.
2.

i) $\sigma_n(f) = \frac{1}{n+1}\sum_{j=0}^{n} \hat{S}_j(f)$

ii) If the sequence $\sigma_1, \sigma_2, \ldots$, converges to σ, we call σ the cesaro sum of the series.

THEOREM: 1.3.10 If S[f] converges in $L^1(T)$,then the limit is necessary f.

PROOF: Assume $S_n(f) \to g$ as $n \to \infty$ in $L^1(T)$-norm .

Then $\hat{g}(j) = \lim_{n\to\infty} (\hat{S}_n(f))(j) = \lim_{n\to\infty} \hat{f}(n) = \hat{f}(n)$, for all $j \in Z$.

By the uniqueness theorem, f = g.

THEOREM: 1.3.11 $S_n(f) = D_n * f$, where D_n is the Dirichlet kernel

defined by

$$D_n(t) = \sum_{j=-n}^{n} e^{ijt} = \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}$$

PROOF: $(D_n * f)(x) = \frac{1}{2\pi}\int_T D_n(t)\, f(x-t)\, dt.$

$$= \frac{1}{2\pi}\int_T \sum_{j=-n}^{n} e^{ijt} f(x-t)\, dt. = \sum_{j=-n}^{n} \frac{1}{2\pi}\int_T f(x-t)\, e^{ijt}\, dt.$$

$$= \sum_{j=-n}^{n} \hat{f}(j)\, e^{ijx} = S_n(f,x) = S_n f(x).$$

$$(D_n * f)(x) = S_n f(x).$$

DEFINITION: 1.3.12 A homogeneous Banach Space on T is a linear subspace B of $L^1(T)$ having $\|.\|_B \geq \|.\|_{L^1}$ under which it is a Banach space and having the following properties:

H-1 (Translation invariance)

If $f \in B$ and $t \in T$, then $f_t \in B$ and $\|f\|_B = \|f_t\|_B$.

H-2 (continuity of translations)

For all $f \in B$ and $t, t_0 \in T$, $\lim_{t \to t_0} \| f_t - f_{t_0} \|_B = 0$.

REMARK: Let f be a homogeneous Banach space on T.

If $f \in B$ and $t, t_0 \in T$. Then, $\left\| f_t - f_{t_0} \right\|_B = \left\| f_{t-t_0} - f \right\|_B$.

By H-1, we get $\left\| f_t - f_{t_0} \right\|_B = \left\| (f_t - f_{t_0})_{t_0} \right\|_B$

$$= \left\| f_{t-t_0} - f_{0_0} \right\|_B = \left\| f_{t-t_0} - f \right\|_B .$$

EXAMPLES OF HOMOGENEOUS BANACH SPACE

1. $L^1(T)$ is a homogeneous Banach space .
2. C(T) is the space of all continuous 2π-periodic functions with the norm $\|f\|_\infty = \max_t |f(t)|$. C(T) with the defined norm is a Banach space satisfying $\|.\|_B \geq \|.\|_{L^1}$. Let $f \in C(T)$ and $t \in T$, since f is uniformly continuous, f_t is continuous 2π-periodic function and $f_t \in C(T)$. By the uniform continuity of f and for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ and $x_1, x_2 \in T$ such that $|x_1 - x_2| < \delta$ implies $|f(x_1) - f(x_2)| < \varepsilon$. Now, $\|f_t - f_{t_0}\|_\infty = \max_x |f_t(x) - f_{t_0}(x)|$ $= \max_t |f(x-t) - f(x-t_0)| < \varepsilon$, provided that $|t-t_0| < \delta$. Therefore, C(T) is a homogeneous Banach space on T.
3. $C^{(n)}(T)$ be the subspace of C (T) of all n -times continuously differentiable function (n being a rational integer) with the norm

$$\|f\|_{C^n} = \sum_{j=0}^{n} \frac{1}{j!} \max_t |f^{(j)}(t)| = \sum_{j=0}^{n} \frac{1}{j!} \|f^{(j)}\|_\infty .$$

With the defined norm $C^{(n)}(T)$ is a Banach space satisfying

$\|f\|_{C^n} \geq \|f\|_\infty \geq \|f\|_{L^1}$.

For $0 \leq j \leq n$, we have, $f_t^{(j)}(x) = f^{(j)}(x-t)$, for all $x \in T$.

Therefore, $f_t \in C^{(n)}(T)$ and

$$\|f_t\|_{C^n} = \sum_{j=0}^{n} \frac{1}{j!}\|f_t^{(j)}\|_\infty = \sum_{j=0}^{n} \frac{1}{j!}\|f^{(j)}\|_\infty = \|f\|_{C^n}.$$

Let $f \in C^{(n)}(T)$ and $t, t_0 \in T$. Then

$$\lim_{t\to t_0}\|f_t - f_{t_0}\|_{C^n} = \sum_{j=0}^{n} \frac{1}{j!}\lim_{t\to t_0}\left\|f_t^{(j)} - f_{t_0}^{(j)}\right\|_\infty = 0.$$

Since $f^{(j)} \in C(T)$.

Therefore, $C^{(n)}(T)$ is a homogeneous Banach space.

4. $L^p(T)$, $1 \leq p < \infty$ - the subspace of $L^1(T)$ consisting of all the functions f for which $\int_T |f(t)|^p \, dt < \infty$, with the norm

$$\|f\|_{L^p} = \left[\frac{1}{2\pi}\int_T |f(t)|^p\right]^{1/p} dt.$$

Clearly, $L^p(T)$, $1 \leq p < \infty$, with the defined norm is a Banach space satisfying $\|f\|_{L^p} \geq \|f\|_{L^1}$. By the translation invariance of the Lebesgue measure, we have:

$$\int_T |f_t(x)|^p dx = \int_T |f(x-t)|^p dx = \int_T |f(x)|^p dx < \infty.$$

Hence, $f_t \in L^p(T)$.

Since continuous functions are dense in $L^p(T)$, Given $\varepsilon > 0$, there exists a continuous function $g \in L^p(T)$ such that $\| f - g \|_{L^1} < \frac{\varepsilon}{4}$, for $f \in L^p(T)$. Since g is continuous on T, there exists $\delta(\varepsilon) = \delta > 0$ such that $t, t_0 \in T$ and $|t - t_0| < \delta \Rightarrow \|g_t - g_{t_0}\|_{L^p} < \frac{\varepsilon}{2}$. Now, $\|f_t - f_{t_0}\|_{L^p} \leq \| f_t - g_t \|_{L^1} +$

$\|g_t - g_{t_0}\|_{L^p} + \|g_t - f_{t_0}\|_{L^1}$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$, provided that $|t - t_0| < \delta$.

Therefore, $\lim_{t \to t_0} \|f_t - f_{t_0}\|_{L^p} = 0$.

Hence $L^p(T)$ is a homogeneous Banach space .

LEMMA: 1.2.13 Let $B \subseteq L^1(T)$ be a Banach space satisfying

H-1. Denote by B_c the set of all $f \in B$ such that $t \to f_t$ is continuous B-valued function. Then B_c is closed subspace of B.

PROOF: Let g be a limit point of B_c in B. then given $\varepsilon > 0$ there exist $f \in B_c$ such that $\|g - f\|_B < \frac{\varepsilon}{4}$. Then there exist also $\delta = \delta(\varepsilon)$ such that $|t - t_0| < \delta \Rightarrow \|f_{t-t_0} - f\|_B < \frac{\varepsilon}{2}$.

Now, $\|g_t - g_{t_0}\|_B \leq \|g_t - f_t\|_B + \|f_t - f_{t_0}\|_B + \|f_{t_0} - g_{t_0}\|_B$

$$\|g_t - g_{t_0}\|_B \leq \|g - f\|_B + \|f_t - f_{t_0}\|_B < \varepsilon, \text{ provided that } |t - t_0| < \delta.$$

Thus $t \to f_t$ is continuous B-valued function on T . Hence $g_t \in B_c$. Therefore, B_c is a closed subspace of B.

THEREOM: 1.3.14 Let B be a homogeneous Banach space on T, $f \in B$ and let $\{K_n\}$ be a summability kernel. Then , $\|K_n * f - f\|_B = 0$.

PROOF: Since $\|.\|_B \geq \|.\|_{L^1}$, then B-valued integral $\frac{1}{2\pi}\int_T K_n(\tau)\, f_\tau\, d\tau$ is the same as the $L^1(T)$-valued integral which by lemma 1.3.5 is equal to $K_n * f$ and by lemma 1. 3.4 $\lim_{n\to\infty} K_n * f = f_0 = f$ in B norm.

THEOREM: 1.3.15 Let B be a homogeneous Banach space on T, then the trigonometric polynomial in B are everywhere dense.

PROOF: For every f ϵB and by the above theorem $\sigma_n(f) \rightarrow f$ in B-norm.

COROLLARY: 1.3.16 (*Weierstrass Approximation Theorem*)

Every continuous 2π-periodic function can be approximated uniformly by trigonometric polynomials.

PROOF: Let f $\in$C(T). Then by the above theorem, there exists a sequence $\{P_n\}$ of trigonometric polynomial in C(T) such that $\lim_{n\to\infty} \| P_n - f \|_\infty = 0$. Since each P_n is uniformly continuous and f is also uniformly continuous on T, P_n converges to f uniformly.

1.4 POINTWISE CONVERGENCE OF $\sigma_n(f)$

If f $\in$ L^1(T), then $\sigma_n(f)$ converges to f in the topology of any homogeneous Banach space that contains f. In particular, if f $\in$C(T) then $\sigma_n(f)$ converges to f uniformly. However, if f is not continuous, we cannot usually deduce pointwise convergence of $\sigma_n(f)$ from its convergence in norm, nor can we relate the limit of $\sigma_n(f, t_0)$. In case it exists, to $f(t_0)$. We have to re-examine the integrals defining $\sigma_n(f)$ for pointwise convergence.

Remark: The proof of the following theorem will be based on the fact that $\{K_n(t)\}$ is a positive summability kernel which has the following properties:

For $0 < \delta < \pi$, $\lim_{n\to\infty} (\text{Sup}_{\delta < t < 2\pi-\delta} K_n(t)) = 0$.

And $K_n(t) = K_n(-t)$.

THEOREM: 1.4.1 (Fejer's) Let f $\in$$L^1$(T).

a) Assume that $\lim_{h\to 0}$ $(f(t_0+h)+f(t_0-h))$ exists (we allow the values $-\infty$ and $+\infty$); then $\sigma_n(f_t, t) \to \lim_{h\to 0}$ $(f(t_0+h)+f(t_0-h))$.

In particular, if t_0 is a point of continuity of f, then $\sigma_n(f, t_0) \to f(t_0)$.

b) If every point of a closed interval I is a point of continuity for f, $\sigma_n(f, t)$ converges to f(t) uniformly on I.

c) If for a.e. t, $m \leq f(t)$, then $m \leq \sigma_n(f,t)$; if for a.e. t, $f(t) \leq M$, then $\sigma_n(f,t) \leq M$.

PROOF: a) The above Remark is valid if we replace $\sigma_n(f)$ by $k_n(f)$, where $\{k_n\}$ is a positive Summability kernel.

For example: the Poisson kernel satisfies all these requirements and the statement of the theorem remains valid if we replace $\sigma_n(f)$ by the Abel means of the Fourier series of f.

We assume for simplicity that $\check{f}(t_0) = \lim_{h\to 0}\frac{1}{2}(f(t_0+h)+f(t_0-h))$ is finite; the modifications needed for the cases $\check{f}(t_0) = +\infty$ or $\check{f}(t_0) = -\infty$ being obvious.

Now, $\sigma_n(f,t_0) - \check{f}(t_0) = \frac{1}{2\pi}\int_T K_n(\tau)(f(t_0-\tau) - \check{f}(t_0))\,d\tau$

$$= \frac{1}{2\pi}\left(\int_{-\delta}^{\delta} + \int_{\delta}^{2\pi-\delta}\right) K_n(\tau)(f(t_0-\tau) - \check{f}(t_0))\,d\tau$$

$$= \frac{1}{2\pi}\left(\int_0^{\delta} + \int_{\delta}^{\pi}\right) K_n(\tau)\left(\frac{f(t_0-\tau)+f(t_0+\tau)}{2} - \check{f}(t_0)\right) d\tau.$$

Given $\varepsilon > 0$, we choose $\delta = \delta(\varepsilon) > 0$, $N = N(\varepsilon)$ such that

$|\tau| < \delta \Rightarrow \left|\frac{f(t_0+\tau)+f(t_0-\tau)}{2} - \check{f}(t_0)\right| < \varepsilon$ and then n_0 so large that $n > n_0$ implies $\sup_{\delta<\tau<2\pi-\delta} K_n(\tau) < \varepsilon$.

We have, $|\sigma_n(f,t_0) - \check{f}(t+0)| < \varepsilon + \varepsilon \| f - \check{f}(t_0)\|_{L^1}$ for all $n \geq N$.

Thus $\lim_{n\to\infty} \sigma_n(f,t_0) = \check{f}(t_0)$.

b) Since the closed interval I is compact and f is uniform continuous on I. Thus given $\varepsilon > 0$, there exists $\delta > 0$ such that $t, \tau \in T$ and $|t - \tau| < \delta \Rightarrow |f(t) - f(\tau)| < \varepsilon$.

There exists n_0 such that $\sup_{\delta<\tau<2\pi-\tau} F_n(\tau) < \varepsilon$, for $n > n_0$.

Now, $|\sigma_n(f, t) - f(t)| \le \frac{1}{\pi}\int_{-\delta}^{\delta} F_n(\tau)\,|f(t-\tau) - f(t)|$

$+ \frac{1}{\pi}\int_{\delta}^{2\pi-\delta} F_n(\tau)\,|f(t-\tau) - f(t)|\,d\tau < \varepsilon + 2\varepsilon\,\|f\|_{L^1}$, for all $n > n_0$ and $t \in T$.

C) Depends only on the fact that $K_n(t) \ge 0$ and $\frac{1}{2\pi} K_n(t)\,dt = 1$;

if $m \le f$, then $\sigma_n(f, t_0) - m = \frac{1}{2\pi}\int_0^{2\pi} K_n(\tau)\,(f(t - \tau) - m)\,d\tau \ge 0$.

The integrand being non-negative.

If $f \le M$, then $M - \sigma_n(f,t) = \frac{1}{2\pi}\int_0^{2\pi} K_n(\tau)\,(M - f(t - \tau))\,d\tau \ge 0$.

Therefore, $m \le \sigma_n(f, t) \le M$.

COROLLARY: 1.4.2 If t_0 is a point of continuity of f and if the Fourier series of f converges at $t_{0,}$ then its sum is $f(t_0)$.

PROOF: $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int}$, the series is convergent, the arithmetic means of the partial sums $S_n(f, t_0)$ are convergent. Let's assume that $S_n(f,t_0)$ convergence to $g(t_0)$. Then $\frac{1}{n+1}\sum_{j=0}^{n} S_j(f,t_0) \to g(t_0)$.

By Fejer's theorem we have $f(t_0) = g(t_0)$.

THEOREM: 1.4.3 (*Lebesgue Theorem*)

Let $f \in L^1(T)$ and if $\lim_{h\to 0}\frac{1}{h}\int_0^h \left| \frac{f(t_0+\tau)+f(t_0-\tau)}{2} - \check{f}(t_0) \right| d\tau = 0$. ------ (*)

Then $\sigma_n(f,t_0) \to \check{f}(t_0)$ as $n\to\infty$.

In particular, $\sigma_n(f,t) \to \check{f}(t)$ as $n\to\infty$, almost everywhere.

PROOF: $\sigma_n(f,t) - \check{f}(t) = \frac{1}{\pi}\int_0^{\delta} F_n(\tau)\left[\frac{f(t0+\tau)+f(t0-\tau)}{2} - \check{f}(t_0)\right] d\tau$

$$+ \frac{1}{\pi}\int_{\delta}^{\pi} F_n(\tau)\ [\frac{f(t0+\tau)+f(t0-\tau)}{2} - \breve{f}(t_0)]\ d\tau\text{------------}(**)$$

Since $F_n(t) = \frac{1}{n+1}\left(\frac{\sin(\frac{n+1}{2})t}{\sin\frac{t}{2}}\right)^2$ and $\sin\frac{t}{2} > \frac{t}{\pi}$, $0 < t < \pi$.

We obtain $F_n(t) \leq \min\ (n+1\ \frac{\pi^2}{(n+1)\tau^2})$.

Therefore, $\frac{1}{\pi}\int_0^{\delta} F_n(\tau)\left|\frac{f(t_0+\tau)+f(t_0-\tau)}{2} - \breve{f}(t_0)\right| d\tau \leq \frac{\pi}{(n+1)\delta^2}\ \|f - \breve{f}(t_0)\|_{L}^{1} \to$ 0 as $n\to\infty$. Provided that $(n+1)\delta^2$ tends to ∞.

Now we turn to evaluate the first integral of (**).

We pick $\delta = n^{1/4}$ and denote $\varphi(h) = \frac{1}{\pi}\int_0^{h}\left|\frac{f(t_0+\tau)+f(t_0-\tau)}{2} - \breve{f}(t_0)\right| d\tau$.

Then, $\left|\frac{1}{\pi}\int_0^{\delta} F_n(\tau)\ [\frac{f(t_0+\tau)+f(t_0-\tau)}{2} - \breve{f}(t_0)]\ d\tau\right|$

$$\leq \frac{1}{\pi}\left|\int_0^{1/n} F_n(\tau)[\frac{f(t_0+\tau)+f(t_0-\tau)}{2} - \breve{f}(t_0)]d\tau\right| + \frac{1}{\pi}\left|\int_{1/n}^{\delta} F_n(\tau)[\frac{f(t_0+\tau)+f(t_0-\tau)}{2} - \breve{f}(t_0)]\ d\tau\right| \leq$$

$$\frac{n+1}{\pi}\varphi\left(\frac{1}{n}\right) + \frac{\pi}{\pi+1}\int_{1/n}^{\delta}\left|\frac{f(t_0+\tau)+f(t_0-\tau)}{2} - \breve{f}(t_0)\right|\frac{d\tau}{\tau^2}$$

Since $\lim_{h\to 0}\frac{1}{h}\int_0^{h}\left|\frac{f(t_0+\tau)+f(t_0-\tau)}{2} - \breve{f}(t_0)\right| d\tau = 0$.

We get $\lim_{n\to\infty}\frac{n+1}{\pi}\varphi(\frac{1}{n}) = 0$ and integrating by parts ,we obtain

$$\frac{\pi}{n+1}\int_{1/n}^{\delta}\left|\frac{f(t0+\tau)+f(t0-\tau)}{2} - \breve{f}(t_0)\right|\frac{d\tau}{\tau 2} = \frac{\pi}{n+1}\left[\frac{\varphi(\tau)}{\tau 2}\right]_{1/n}^{\delta} + \frac{2\pi}{n+1}\int_{1/n}^{\delta}\frac{\varphi(\tau)}{\tau 3}d\tau$$

$$= \frac{\pi}{n+1}\left[\frac{\varphi(n1/4)}{n1/2}\right]_{1/n}^{\delta} - \frac{\pi}{n+1}\left[\frac{\varphi(\frac{1}{n})}{n-2}\right]\int_{1/n}^{\delta}\frac{\varphi(\tau)}{\tau 3}d\tau + \frac{2\pi}{n+1}\int_{1/n}^{\delta}\frac{\varphi(\tau)}{\tau 3}d\tau\ \text{-----}\ (***)$$

As n tends to ∞, (***) tends to $\frac{2\pi}{n+1}\int_{1/n}^{\delta}\frac{\varphi(\tau)}{\tau^3}d\tau$.

Now again $\varepsilon > 0$, there exists n_0 such that $n > n_0$ and $0 < \tau < \delta$,

we have $\varphi(t) < \frac{\varepsilon\tau}{3\pi}$. Hence, $\frac{2\pi}{n+1}\int_{1/n}^{\delta}\frac{\varphi(\tau)}{\tau^3}d\tau \leq \frac{2\pi}{(n+1)3\pi}\int_{1/n}^{\delta}\frac{1}{\tau^2}d\tau < \varepsilon$.

Therefore, $\lim_{n\to\infty}\frac{2\pi}{n+1}\int_{1/n}^{\delta}\frac{\varphi(\tau)}{\tau^3}d\tau = 0$ ------- (#)

Hence, (**) , (***) and (#) , we get $\lim_{n\to\infty} \sigma_n(f,t_0) = \check{f}(t_0)$.

COLLORAY: 1.4.4 If the Fourier series of $f \in L^1(T)$ converges on a set E of positive measure, its sum coincides with f almost everywhere on E. In particular, if a Fourier series converges to zero almost everywhere, all its coefficients must vanish.

PROOF: By Fejer's Theorem $\sigma_n(f,t) \to \check{f}(t)$ as $n\to\infty$ for all $t\in E$ and since $\check{f}(t) = f(t)$ almost everywhere on E, we obtain $\sigma_n(f,t) \to f(t)$ almost everywhere. If the Fourier series of f converges to zero almost everywhere then f is zero almost everywhere. Thus $\check{f}(t) = 0$, $n\in Z$.

THEOREM: 1.4.5(*Fatou's Theorem*)

If $\psi(h) = \int_0^h \left(\frac{f(t_0+\tau)+f(t_0-\tau)}{2} - \check{f}(t_0)\right) d\tau = 0(h)$, then

$$\lim_{r\to 1} \sum_{-\infty}^{\infty} \hat{f}(j) r^{|j|} e^{ijt_0} = \hat{f}(t_0).$$

1.5 THE ORDER OF MAGNITUDE OF FOURIER COEFFICIENTS

A complete characterization, that is, a necessary and sufficient condition expressed in terms of order of magnitude, for a sequence $\{a_n\}$ to be the Fourier coefficients of a function in the space are $L^2(T)$ and its "derivatives", Such as the space of absolutely continuous functions with derivatives in $L^2(T)$.

THEOREM:1.5.1 Let $\{a_n\}_{-\infty}^{\infty}$ be an even sequence of nonnegative numbers tending to zero at infinity. Assume $n > 0$, $a_{n-1} + a_{n+1} - 2a_n \geq 0$. Then there exists a nonnegative function $f \in L^1(T)$ such that $\hat{f}(n) = a_n$.

PROOF: It is given that $(a_n - a_{n+1})$ is monotonically decreasing with n.

Hence $\lim_{n\to\infty}$ (an − a_{n+1}) = 0, a nd c onsequently,

$$\sum_{n=1}^{N}(a_{n-1}+a_{n+1}-2a_n) = a_0 - a_N - N(a_N - a_{N+1})$$

converges to a_0 as N→∞.

Put

$$f(t) = \sum_{n=1}^{\infty} n(a_{n-1}+a_{n+1}-2a_n)K_{n-1}(t),$$

where K_n denotes, as usual, the Fejér kernel. Since $\|K_n\|_{L^1}$ =1, the series converges in $L^1(T)$ and, all its terms being nonnegative, its limit f is non negative.

Now,
$$\hat{f}(t) = \sum_{n=1}^{\infty} n(a_{n-1}+a_{n+1}-2a_n)\widehat{K}_{n-1}(t),$$
$$= \sum_{n=1}^{\infty} n(a_{n-1}+a_{n+1}-2a_n)\left(1-\frac{|j|}{n}\right)$$
$$= a_{|j|}.$$

THEOREM: 1.5.2 Let f ∈ $L^1(T)$ and assume that

$\hat{f}(|n|) = -\hat{f}(-|n|) \geq 0$. Then, $\sum_{n>0}\frac{1}{n}\hat{f}(n) < \infty$.

PROOF: Without loss of generality we may assume that $\hat{f}(0) = 0$.

Write $F(t) = \int_0^t f(\tau)\, d\tau$; then F∈ C(T) and, $\widehat{F}(n) = \frac{1}{in}\hat{f}(n)$, n≠0.

Since F is continuous, we can apply Fejér's theorem for $t_0 = 0$ and obtain,

$$\lim_{N\to\infty} 2\sum_{n=1}^{N}\left(1-\frac{n}{N+1}\right)\frac{\hat{f}(n)}{n} = i\left(F(0)-\widehat{F}(0)\right) \text{ and since } \frac{\hat{f}(n)}{n} \geq 0.$$

Therefore, $\sum_{n>0}\frac{1}{n}\hat{f}(n) < \infty$.

COROLLARY: 1 .5.3 If $a_n > 0$, $\sum \frac{a_n}{n} = \infty$, then $\sum a_n \sin nt$ is not a Fourier series. Hence there exist trigonometric series with coefficients tending to zero which are not Fourier series. The series,

$$\sum_{n=2}^{\infty} \frac{\cos nt}{\log n} = \sum_{n\geq 2} \frac{e^{int}}{2\log|n|}$$

is a Fourier series while, its conjugate series

$$\sum_{n=2}^{\infty} \frac{\sin nt}{\log n} = -\pi \sum_{n\geq 2} \frac{\operatorname{sgn}(n)}{2\log|n|} e^{int} \quad \text{is not.}$$

We turn now to some simple results about the order of magnitude of Fourier coefficients of functions satisfying various smoothness conditions.

DEFINITION:1.5.4 Given functions two f(t) and g(t) for $f(t) > 0$, $g(t) > 0$, $t > t_0$. We say that,

1. $f(t) = 0(g(t)) \Leftrightarrow \frac{f(t)}{g(t)} \longrightarrow 0$ as $t \longrightarrow 0$.
2. $f(t) = 0(g(t)) \Leftrightarrow \frac{f(t)}{g(t)}$ is bounded for all t sufficiently large.
3. f and g are of the same order in the neighborhood of t_0 if and only if

There exists two constants $p > 0$ and $q > 0$ such $p \leq \frac{f(t)}{g(t)} \leq q$, for all t sufficiently near to t_0 denoted as $f(t) \sim g(t)$.

THEOREM: 1.5.5 If $f \in L^1(T)$ is absolutely continuous, then

$$\hat{f}(n) = o\left(\frac{1}{n}\right), \text{ for all } n\neq 0.$$

PROOF: since f is absolutely continuous, then $f(t) = \int_0^t f'(t)\, dt$. Then we have, $\hat{f}(n) = \frac{1}{in} \hat{f'}(n)$ for all $n\neq 0$. By the Riemann- Lebesgue lemma, $\hat{f}(n)$ tends to zero. Then, $\hat{f}(n) = \frac{1}{in} \hat{f'}(n)$ tends to zero as n tends to infinite.

Therefore, $n\hat{f}(n) \to 0$ as $n \to \infty$. Hence, $\hat{f}(n) = o\left(\frac{1}{n}\right)$.

REMARK: if f is k-times differentiable and $f^{(k-1)}$ is absolutely continuous (So that $f(k) \in L^1(T)$ and $f^{(k-1)}$ is its primitive), Then, $\hat{f}(n) = \mathbf{o}(n^{-k})$ as $|n| \to \infty$. We have to do is notice that if $0 \le j \le k$, then , $\hat{f}(n) = (in)^{-j}\,\hat{f}^{(j)}(n)$ and hence, $|\hat{f}(n)| \le |n|^{-j}\,\|f^{(j)}\|_{L^1}$.

NOTE: 1. If f is k-times differentiable, and $f^{(k-1)}$ is absolutely continuous,

Then, $|\hat{f}(n)| \le \min_{0\le\tau\le k} \frac{\|f^{(\tau)}\|_{L^1}}{|n|^{\tau}}$.

2. If f is infinitely differentiable, $|\hat{f}(n)| \le \min_{0\le\tau} \frac{\|f^{(j)}\|_{L^1}}{|n|^{j}}$.

THEOREM: 1.5.6 If f is of bounded variation on T, then $|\hat{f}(n)| \le \frac{Var(f)}{2\pi|n|}$.

PROOF: We integrate by parts using Stieltjes integrals

$$|\hat{f}(n)| = \left|\frac{1}{2\pi}\int_T e^{-int} f(t)\,dt\right| = \left|\frac{1}{2\pi in}\int_T e^{-int}\,df(t)\right| \le \frac{Var(f)}{2\pi|n|}.$$

Hence, $|\hat{f}(n)| \le \frac{Var(f)}{2\pi|n|}$.

DEFINITION: 1.5.7

1) The modulus of continuity of a function $f \in L^1(T)$ denoted by $\omega(f, g)$, defined as , $\omega(f, g) = \max_{|y|\le g} |f(y+t) - f(t)|$.

2) The integral modulus of continuity of a function $f \in L^1(T)$ denote by $\Omega(f, g)$, defined as , $\Omega(f, g) = \|f(y+t) - f(t)\|_{L^1}$.

Then, $\Omega(f, g) \le \omega(f, g)$.

THEOREM: 1.5.8 For $n \ne 0$, $|\hat{f}(n)| \le \frac{1}{2}\Omega\left(f, \frac{\pi}{|n|}\right)$.

PROOF: $\hat{f}(n) = \frac{1}{2\pi}\int_T f(t)e^{-int}\,dt = -\frac{1}{2\pi}\int_T f(t)e^{-in(t+\pi/n)}\,dt$.

By change of variables, $\hat{f}(n) = -\frac{1}{4\pi}\int_T [f(t+\frac{\pi}{n}) - f(t)]e^{-int}\,dt$

Hence, $|\hat{f}(n)| \le \frac{1}{2}\Omega\left(f, \frac{\pi}{|n|}\right)$.

1.6 FOURIER SERIES OF SQUARE SUMMABLE FUNCTIONS

If f, g $\in L^2(T)$, then the inner product in $L^2(T)$ denoted by $< f, g >$ is defined as $< f, g > = \frac{1}{2\pi}\int_T f(t)\overline{g(t)}\,dt$.

$L^2(T)$ with its inner product as defined is a Hilbert space .

DEFINITION: 1.6.1 Let H be a complex Hilbert space. Let f, g $\in$H. We say that f is orthogonal to g if $< f, g > = 0$. This relation is clearly symmetric. If E is a subset of H, we say that f $\in$ H is orthogonal to E if f is orthogonal to every element of E. A set E $\subseteq$ H is *orthogonal* if any two vectors in E are orthogonal to each other. A set E$\subseteq$ H is *an orthonormal system* if it is orthogonal and the norm of each vector in E is one, that is if, whenever f, g $\in$ E, $< f, g > = 0$, if $f \neq g$ and $< f, f > =1$.

LEMMA: 1.6.2 Let $\{\varphi_n\}_{n=1}^{N}$be a finite orthonormal system.

Let $a_1 . . . a_N$ be complex numbers. Then $\left\|\sum_1^N a_n\varphi_n\right\|^2 = \sum_1^N |a_n|^2$.

PROOF:

$$\left\|\sum_1^N a_n\varphi_n\right\|^2 = < \sum_1^N a_n\varphi_n , \sum_1^N a_n\varphi_n >$$

$$= \sum_1^N a_n < \varphi_n , \sum_1^N a_n\varphi_n > = \sum_1^N a_n\bar{a}_n = \sum_1^N |a_n|^2.$$

COROLLARY:1.6.3 Let $\{\varphi_n\}_1^\infty$ be an orthonormal system in H and let $\{a_n\}_1^\infty$ be a sequence of complex numbers such that $\sum |a_n|^2 < \infty$.

Then, $\sum_1^N a_n\varphi_n$ converges in H.

PROOF: Since H is complete, we have to show is that the partial sums

$$S_N = \sum_{1}^{N} a_n \varphi_n \quad \text{form a Cauchy sequence in H.}$$

Now, for N > M,

$$\|S_N - S_M\|^2 = \left\|\sum_{M+1}^{N} a_n \varphi_n\right\|^2 = \sum_{M+1}^{N} |a_n|^2 \to 0 \text{ as } M \to \infty.$$

LEMMA: 1 .6.4 Let H be a Hilbert space. Let $\{\varphi_n\}$ be a finite orthonormal system in H. For f ∈H write $a_n = < f, \varphi_n >$. Then

$$0 \le \left\|f - \sum_{1}^{N} a_n \varphi_n\right\|^2 = \|f\|^2 - \sum_{M+1}^{N} |a_n|^2.$$

PROOF:

$$0 \le \left\|f - \sum_{1}^{N} a_n \varphi_n\right\|^2 = < f - \sum_{1}^{N} a_n \varphi_n , f - \sum_{1}^{N} a_n \varphi_n >$$

$$= \|f\|^2 - \sum_{1}^{N} \bar{a}_n < f, \varphi_n > - \sum_{1}^{N} a_n < f, \varphi_n > + \sum_{1}^{N} |a_n|^2$$

$$= \|f\|^2 + \sum_{1}^{N} |a_n|^2.$$

REMARK: **(Bessel's inequality)**

Let H be a Hilbert space and $\{\varphi_\alpha\}$ be an orthonormal system in H. For f ∈H write aα = $<f, \varphi_\alpha>$. Then $\sum |a_\alpha|^2 \le \|f\|^2$.

The family $\{\varphi_\alpha\}$ in the statement of Bessel's inequality need not be finite nor even countable. The inequality is equivalent to saying that for every finite subset of $\{\varphi_\alpha\}$. In particular $a_\alpha = 0$, except for countably many values of α and the series $\sum |a_\alpha|^2$ converges.

If $H = L^2(T)$ all orthonormal systems in H are finite or countable and we write them as sequences $\{\varphi_n\}$.

DEFINITION: 1.6.5 A complete orthonormal syst em in H is a n orthonormal system having the additional property that the only vector in H orthogonal to it is the zero vector.

THEOREM: 1.6.6 Let $f \in L^2(T)$, then

$$\text{i.} \quad \sum_{-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{2\pi}\int_T |f(t)|^2\, dt.$$

$$\text{ii.} \quad f = \lim_{N \to \infty} \sum_{-N}^{N} \hat{f}(n) e^{int} \text{ in the } L^2(T) \text{norm.}$$

PROOF: i) Since $\{e^{int}\}$ is a complete orthonormal system in $L^2(T)$ and for every $f \in L^2(T)$.

We h ave , $\frac{1}{2\pi}\int_T |f(t)|^2 \, dt = (\|f\|_{L^2})^2$

$$= \sum_{-\infty}^{\infty} |< f, e^{int} >|^2 = \sum_{-\infty}^{\infty} |\hat{f}(n)|^2$$

ii) Given $\varepsilon > 0$, there exists N such that. $\sum_{|n|=N+1}^{\infty} |\hat{f}(n)|^2 < \varepsilon$.

Now, $\left\| \sum_{-N}^{N} \hat{f}(n) e^{int} - f \right\|_{L^2}^2 = < \sum_{-N}^{N} \hat{f}(n) e^{int} - f , \sum_{-N}^{N} \hat{f}(n) e^{int} - f >$

$$=< \sum_{-N}^{N} \hat{f}(n) e^{int} , \sum_{-N}^{N} \hat{f}(n) e^{int} - \sum_{-N}^{N} \hat{f}(n) < e^{int}, f > - \sum_{-N}^{N} \overline{\hat{f}(n)} < f, e^{int} > + < f, f >.$$

$$= \sum_{-N}^{N} |\hat{f}(n)|^2 - 2\sum_{-N}^{N} |\hat{f}(n)|^2 + (\|f\|_{L^2})^2 = -\sum_{-N}^{N} |\hat{f}(n)|^2 + (\|f\|_{L^2})^2$$

By i) we have ,

$$\sum_{-N}^{N} |\hat{f}(n)|^2 + (\|f\|_{L^2})^2 = \sum_{-\infty}^{\infty} |\hat{f}(n)|^2 < \varepsilon .$$

Therefore,

$$f = \lim_{N \to \infty} \sum_{-N}^{N} \hat{f}(n) e^{int} \text{ in the } L^2(T) \text{ norm.}$$

THEOREM: 1.6.7 For any square summable sequence $\{a_n\}$ for $n \in \mathbb{Z}$ of complex numbers such that $\sum |a_n|^2 < \infty$, then there exists a unique $f \in L^2(T)$ such that $a_n = \hat{f}(n)$.

PROOF: Let $f(t) = a_n e^{int}$.

Then $\hat{f}(n) = \frac{1}{2\pi}\int_T a_n e^{int} e^{-int} dt = a_n$.

THEOREM: 1.6.8 Let $f, g \in L^2(T)$, then

$$\frac{1}{2\pi}\int_T f(t)\overline{g(t)}\,dt = \sum_{-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}.$$

PROOF: *Let* $f = \sum_{-\infty}^{\infty} \hat{f}(n)e^{int}$ and $g = \sum_{-\infty}^{\infty} \hat{g}(n)e^{int}$

Then, $< f, g > = < \sum_{-\infty}^{\infty} \hat{f}(n)e^{int}, \sum_{-\infty}^{\infty} \hat{g}(n)e^{int} >$

$$= \sum_{-\infty}^{\infty} \hat{f}(n) \sum_{-\infty}^{\infty} \overline{\hat{g}(n)} < e^{int}, e^{int} >$$

$$= \sum_{-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}.$$

CHAPTER TWO

CONVERGENCE OF FOURIER SERIES

2.1 CONVERGENCE IN NORM

DEFINITION: 2.1.1 Let B be a homogeneous Banach space on T.

Let $f \in B$ and $S_n f(t) = S_n(f,t) = \sum_{j=-n}^{n} \hat{f}(j)e^{ijt}$, We say that B admits convergence in norm if $\lim_{n\to\infty} \| S_n(f) - f \|_B = 0$.

The operator $S_n : f \to S_n(f)$ is well defined in every homogeneous Banach space B. We denote its norm as an operator on B by $\| S_n \|^B$.

THEOREM:2.1.2 A homogeneous Banach space B admits convergence in norm if and only if $\|S_n\|^B$ is bounded (as $n\to\infty$), that is if there exists a constant k, such that $\|S_n(f)\|_B \le k\|f\|_B$ for all $f \in B$ and $n > 0$.

PROOF: Assume B admits convergence in norm. Then for each $f \in B$, there exists $t > 0$ such that $\| S_n(f) \|_B \le t$. Thus the family $S_n(f)$ is bounded. By the uniform boundedness theorem, $\| S_n \|^B$ is bounded.

Conversely, Let k be constant, such that $\| S_n(f) \|_B \le k\|f\|_B$ for all $f \in B$ and $n \ge 0$. Let $f \in B$, $\varepsilon \ge 0$ be given, then there exists a trigonometric polynomial P of degree P such that $\| f - P \| \le \frac{\varepsilon}{2k}$. For $n > \mathbb{N}$, $S_n(P) = P$.

Now $\| S_n(f) - f \|_B = \| S_n(f) - S_n(P) + p - f \|_B$

$$\le \| S_n(f - P) \|_B + \| P - f \|_B \le k\frac{\varepsilon}{2k} + \frac{\varepsilon}{2k} \le \varepsilon.$$

Therefore, $\lim_{n\to\infty} \| S_n(f) - f \|_B = 0$, for all $f \in B$.

Thus B admits convergence in norm.

NOTE:

The fact that $S_n(f) = D_n * f$, where D_n is the Dirichlet kernel,

$$D_n(t) = \sum_{j=-n}^{n} e^{ijt},$$

$$D_n(t) = \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}$$ yields a simple bound for $\|S_n\|^B$.

Since $\| D_n * f \|_B \le \|D_n\|_{L^1} \|f\|_B$. It follows that $\|S_n\|^B \le \|D_n\|_{L^1}$.

The numbers $L_n = \|D_n\|_{L^1}$ are called the *Lebesgue constants.*

DEFINITION: 2.1.3 If $f \in L^1(T)$ and if the series conjugate $\sum \hat{f}(n)\, e^{int}$ to the series $-i \sum Sgn(n)\, c_n\, e^{int}$ is the Fourier series of some function $g \in L^1(T)$, then we call ***g*** the conjugate function of ***f*** and denoted by $\tilde{f}$.

DEFINITION: 2.1.4 A space of functions $B \in L^1(T)$ admits conjugation if for every $f \in B$, $\tilde{f}$ is defined and belongs to B.

THEOREM: 2.1.5 The sequence $\{L_n\}$ of the Lebesgue constants is not bounded .That is, $L_n = \frac{4}{\pi^2} \log^n + \mathbf{0}(1)$.

PROOF: Since $t \geq 2 \sin\frac{t}{2}$, we have $\frac{1}{t} \leq \frac{1}{2\sin\frac{t}{2}}$, then we get

$$L_n = \frac{1}{2\pi}\int_0^{2\pi} \left| \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right| dt \geq \frac{2}{\pi}\int_0^{\pi} \frac{\left|\sin\left(n+\frac{1}{2}\right)t\right|}{t}\, dt$$

By change of variables, we get

$$L_n \geq \frac{2}{\pi}\int_0^{\pi} \frac{|\sin u|}{u}\, du > \frac{2}{\pi}\sum_{k=1}^{n} \frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi} |\sin u|\, dt = \frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k}$$

$$> \frac{4}{\pi^2}\log n.$$

Therefore, $\lim_{n\to\infty} L_n = \infty$.

THEOREM: 2.1.6 $L^1(T)$ does not admit convergence in norm.

PROOF: It is true that for a homogeneous Banach space B, $\|S_n\|^B \leq L_n$. In particular, if $B = L^1(T)$ then $\|S_n\|^{L1(T)} \leq L_n$, n = 0,1,2,3,…, we know that $S_n(F_N) = \sigma_N(D_n)$, where F_N is the fejer kernel and we have $\|\sigma_N(D_n)\|_{L^1} = \|S_n(F_N)\|_{L^1} \leq \|S_n\|^{L^1(T)}$. And since $\sigma_N(D_n) \to D_n$ as $N\to\infty$ in $L^1(T)$ norm, we have $\|\sigma_N(D_n)\|_{L^1} \geq \lim_{n\to\infty}\|\sigma_N(D_n)\|_{L^1} = \|D_n\|_{L^1} = L_n$.

Hence , $\|S_n\|^{L^1(T)} = \|D_n\|_{L^1}$ as L_n is not bounded .

Therefore, by the above theorems, it follows that $L^1(T)$ does not admit convergence in norm.

THEOREM: 2.1.7 C(T) does not admit convergence in norm.

PROOF: Let φ be a continuous function satisfying

$\|\varphi_n\|_\infty = \sup_t |\varphi_n(t)| \leq 1$ such that $\varphi_n(t) = \text{Sgn}(D_n(t))$, except in small intervals around the points of discontinuity of $\text{sgn}(D_n(t))$. If the sum of the length of these interval is smaller than $\frac{\varepsilon}{2n}$, then we have

$$\|S_n\|^{C(T)} \geq |S_n(\varphi_n,0)| = \frac{1}{2\pi}\int D_n(t)\,\varphi_n(t)\,dt > L_n - \varepsilon.$$

Since ε is arbitrary, $\|S_n\|^{C(T)} \geq L_n$.

Therefore, by the above theorems, C(T) does not admit convergence in norm.

THEOREM: 2.1.8 If B is a homogeneous Banach space which admits conjugation, then the mapping $f \to \tilde{f}$ is bounded linear operator on B.

PROOF: Let G: B→B is a mapping defined by $G(f) = \tilde{f}$. Let f, g ∈B and α is a complex number and if G(f + g) = h, then for each integer n,

$$\hat{h}(n) = -i\,\text{sgn}(n)[\hat{f}(n) + \hat{g}(n)] = \hat{\tilde{f}}(n) + \hat{\tilde{g}}(n) = \left(\widehat{\tilde{f} + \tilde{g}}\right)(n)$$

By uniqueness theorem, $h = \tilde{f} + \tilde{g}$.

Therefore, $G(f + g) = \tilde{f} + \tilde{g} = G(f) + G(g)$.

Let G(αf) = k, then $\hat{k}(n) = -i\,\text{sgn}(n)(\widehat{\alpha f})(n) = -i\,\text{sgn}(n)\alpha(\hat{f}(n))$

$$= \alpha(\hat{\tilde{f}}(n)) = (\widehat{\alpha\tilde{f}})(n) \text{ for all } n \in Z.$$

By the uniqueness theorem $k = \alpha\tilde{f}$, i.e G(αf) = αG(f).

Hence, G is homogeneous. Therefore G is linear.

Let $\{f_n\}$ be a sequence of function in B such that

$\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} G(f_n) = g$ in B norm .

Now for each integer j, $\hat{g}(j) = \lim_{n\to\infty} \hat{\tilde{f}}_n(j) = \lim_{n\to\infty} [- i \, \mathrm{Sgn}(j) \, \hat{f}_n(j)]$

$$= - i \, \mathrm{Sgn}(j) \lim_{n\to\infty} \hat{f}_n(j) = - i \, \mathrm{Sgn}(j) \, \hat{f}(j) = \hat{\tilde{f}}(j).$$

By uniqueness theorem, $g = \tilde{f}$. Thus the operator G is closed. Then by the closed graph theorem, G is continuous and hence bounded.

THEOREM 2.1.9 Let B be a homogeneous Banach space on T, which admits c onjugation, for e ach f ∈B, d efine $f^{\flat} = \frac{1}{2}\hat{f}(0) + \frac{1}{2}(f + i\tilde{f})$

$$f^{\flat} \sim \sum_{j=0}^{\infty} \hat{f}(j)e^{ijt}.$$

Then the mapping $f \to f^{\flat}$ is well defined, bounded linear operator on B.

PROOF: Let f, g ∈B and α be a complex number. Then

$$(f + g)^{\flat} = \frac{1}{2}\left(\widehat{f+g}\right)(0) + \frac{1}{2}[(f + g) + i(\tilde{f} + \tilde{g})] = f^{\flat} + g^{\flat}, \text{ and}$$

$(\alpha f)^{\flat} = \frac{1}{2}(\alpha\hat{f})(0) + \frac{1}{2}[(\alpha f) + i(\alpha\tilde{f})] = \alpha f^{\flat}$. Therefore, the mapping is linear.

Let $\{f_n\}$ be a sequence of function in B such that $f_n \to f$ as $n\to\infty$ and $f_n^{\flat} \to g$ in B norm. For each integer j,

$$\hat{g}(j) = \lim_{n\to\infty} \hat{f}_n^{\flat}(j) = \begin{cases} \lim_{n\to\infty} \hat{f}_n(j), & \text{if } j \geq 0 \\ 0, & \text{if } j < 0 \end{cases} = \begin{cases} \hat{f}(j), & \text{if } j \geq 0 \\ 0,, & \text{if } j < 0 \end{cases} = \hat{f}^{\flat}(j).$$

By the uniqueness theorem, $g = f^{\flat}$. Hence the operator is closed .By the closed graph theorem, the mapping is continuous and hence bounded.

THEOREM: 2.1.10 Let B be a homogeneous Banach space on T. If the mapping $f \to f^{\flat}$ is well defined in B, then B admits conjugation.

PROOF: From the hypothesis and if f ∈B, then

$\tilde{f} = -i\,[2f^{\flat} - f - \hat{f}(0)] \in B$, from $f^{\flat} = \frac{1}{2}\hat{f}(0) + \frac{1}{2}(f + i\tilde{f})$.

Therefore, B admits conjugation.

THEOREM: 2.1.11 Let B be a homogeneous Banach space on T and assume that for $f \in B$ and for every integer n, $e^{int} f \in B$ and $\|e^{int} f\|_B = \|f\|_B$, then B admits conjugation if and only if B admits convergence in norm.

PROOF: Assume B admits conjugation ,then the mapping $f \to f^{\flat}$ is well defined and bounded linear operator on B .

$$\text{Defined } S_n^{\flat}: B \to B \text{ by } S_n^{\flat}(f) = \sum_{j=0}^{2n} \hat{f}(j)e^{ijt} = e^{int} S_n\left(e^{-int} f\right).$$

$$\text{Then} \quad S_n^{\flat}(f) = f^{\flat} - e^{i(2n+1)t}\,(e^{-i(2n+1)t} f)^{\flat}.$$

Let us assume a constant k such that $\|f^{\flat}\|_B \leq k\,\|f\|_B$ for all $f \in B$.

For each $f \in B$, $\|S_n^{\flat}(f)\|_B \leq \|f^{\flat}\|_B + \|e^{i(2n+1)t}(e^{-i(2n+1)t} f)^{\flat}\|_B$.

$$= \|f^{\flat}\|_B + \|(e^{-i(2n+1)t} f)^{\flat}\|_B.$$

$$\leq k\|f\|_B + k\,\|(e^{-i(2n+1)t} f)\|_B = 2k\|f\|_B.$$

Therefore, the sequence $\{\|S_n^{\flat}(f)\|_B\}$ is bounded.

$\|S_n^{\flat}(f)\|_B = \|e^{int} S_n(e^{-int} f)\|_B \leq \|S_n\|^B \|f\|_B$, for all $f \in B$.

Therefore, $\|S_n^{\flat}\|^B \leq \|S_n\|^B$.

Now, since $S_n(e^{-int} f) = e^{-int} S_n^{\flat}(f)$,

$\Longrightarrow \|S_n(e^{-int} f)\|_B = \|e^{-int} S_n^{\flat}(f)\|_B = \|S_n^{\flat}(f)\|_B \leq \|S_n^{\flat}\|^B \|f\|_B$, for all $f \in B$.

Thus, $\|S_n\|^B \leq \|S_n^{\flat}\|^B$. Therefore, $\|S_n\|^B = \|S_n^{\flat}\|^B$. Hence $\|S_n\|^B$ is bounded, by theorem 2.1.2, B admits convergence in norm. Conversely, assume that B admits convergence in norm. By Theorem 2.1.2, there exists a constant $\alpha > 0$ such that $\|S_n\|^B \leq \alpha$, $n = 0,1,2,\ldots$, . For the operators defined above, we have,

$$\|S_n^{\flat}(f)\|_B = \|e^{int} S_n(e^{-int} f)\|_B \leq \|S_n\|^B \|f\|_B \leq \alpha\|f\|_B \text{ for all } f \in B.$$

Thus, we get $\| S_n^{\flat} \|^B \leq \alpha$ for n = 0,1,2, . . . , .

Let f $\in$B and $\varepsilon > 0$, there exists a trigonometric polynomial p of degree n such that $\|f - p\|_B \leq \frac{\varepsilon}{2\alpha}$, then $\|S_n^{\flat}(f) - S_n^{\flat}(p)\|_B \leq \|S_n^{\flat}(f - p)\|_B$

$$\leq \| S_n^{\flat} \|^B \| f - p \|_B \leq \alpha \frac{\varepsilon}{2\alpha} = \frac{\varepsilon}{2}, \text{ for all } n \geq N.$$

If n, m $\geq$ N, then $S_n^{\flat}(p) = S_m^{\flat}(p)$ and it follows that

$$\| S_n^{\flat}(f) - S_m^{\flat}(f)\|_B \leq \| S_n^{\flat}(f) - S_n^{\flat}(p) + S_m^{\flat}(p) - S_m^{\flat}(f) \|_B$$

$$\leq \| S_n^{\flat}(f) - S_n^{\flat}(p)\| + \| S_m^{\flat}(p) - S_m^{\flat}(f) \|_B < \varepsilon.$$

Therefore, the sequence $\{S_n^{\flat}(f)\}$ is a Cauchy sequence in B.

Then $\{S_n^{\flat}(f)\}$ converges to g in B norm for g $\in$B.

By applying the uniqueness theorem, $g = f^{\flat}$. Thus $f^{\flat} \in$B.

Therefore, the mapping $f \to f^{\flat}$ is well defined in B. By theorem 2.1.10, B

THEOREM: 2.1.12 For $1 < p < 1$, the Fourier series of every $f \in L^p(T)$ converges to f in the $L^p(T)$ norm.

Example:

1. Show that the Dirichlets kernel does not converge in $L^1(T)$.

$$\text{where}, D_n(t) = \sum_{j=-n}^{n} e^{ijt} = \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}.$$

Proof:- $\|D_n\|_{L^1} = \frac{1}{2\pi}\int_T |D_n(t)|\, dt$, since $D_n \in L^1(T)$.

$$\|D_n\|_{L^1} = \frac{1}{2\pi}\int_T |D_n(t)|\, dt$$

$$= \frac{1}{2\pi}\int_T \left|\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\right| dt \quad , \text{since } \frac{1}{|\sin\frac{t}{2}|} \geq \frac{\pi}{t}, \text{for } 0<t<\pi.$$

$$\geq \frac{1}{2}\int_T \frac{\left|\sin\left(n+\frac{1}{2}\right)t\right|}{t}\, dt = \frac{n+\frac{1}{2}}{2}\int_T \frac{|\sin u|}{u}\, du, \text{ Put } u = \left(n+\frac{1}{2}\right)t.$$

$$\frac{n+\frac{1}{2}}{2}\int_T \frac{|\sin u|}{u}\, du \to \infty \text{ as } n \to \infty.$$

Therefore, $\|D_n\|_{L^1} = \frac{1}{2\pi}\int_T |D_n(t)|\, dt \to \infty.$

Hence the Dirichlets kernel does not converge in $L^1(T)$.

2. show t hat i f t he se quence { N_j} te nds to m inus in finity, then the Fourier series of the function, $f(t) = \sum_{1}^{\infty} 2^{-j} K_{N_j}(t)$ does not Convergent in $L^1(T)$.

Proof: – For $f(t) = \sum_{1}^{\infty} 2^{-j} K_{N_j}(t)$, then $S_{N_j}(f,t) = \sum_{j=1}^{N_j} 2^{-j} K_{N_j}(t)$.

$$S_{N_j}(f,t) - f(t) = \sum_{j=N_j+1}^{\infty} 2^{-j} K_{N_j}(t).$$

Now, $\left\|S_{N_j}(f,t) - f(t)\right\|_{L^1} = \left\|\frac{1}{2\pi}\int_T \left(\sum_{j=N_j+1}^{\infty} 2^{-j} K_{N_j}(t)\right) dt\right\|$

$\leq \frac{1}{2\pi}\int_T \sum_{j=N_j+1}^{\infty} |2^{-j}| \left\|K_{N_j}(t)\right\|_{L^1} dt$. Since $\left\|K_{N_j}(t)\right\|_{L^1} = 1$.

Therefore, $\left\|S_{N_j}(f,t) - f(t)\right\|_{L^1} \leq \sum_{j=N_j+1}^{\infty} |2^{-j}|$.

As $\{N_j\}$tends to minus infinity, $\sum_{j=N_j+1}^{\infty} |2^{-j}| \to \infty$.

Hence, $\left\|S_{N_j}(f,t) - f(t)\right\|_{L^1} \to \infty$.

Therefore, the given Fourier series does not convergence in $L^1(T)$.

2.2 CONVERGENCE AND DIVERGENCE AT A POINT

We have seen in the previous section that the Fourier series of a continuous functions need not converge uniformly. In this section, we show that it may even fail to converge pointwise and then give two criteria for the convergence of Fourier series at a point.

THEOREM 2.2.1 There exists a continuous function whose Fourier series diverges at a point. We give two proofs which are in fact one; the first is "abstract" based on the Uniform Boundedness Principle and is very short. The second is a construction of a concrete example in essentially the way one proves the Uniform Boundedness Principle.

PROOF A: The mapping $f \to S_n(f, 0)$ is a continuous linear functional on C(T). We saw in the previous section that these functionals are not uniformly bounded and consequently, by the Uniform Boundedness theorem, there exists an $f \in C(T)$ such that $\{S_n(f,0)\}$ is not bounded.

In other words, the Fourier series of **f** diverges unboundedly at t = 0.

PROOF B : As we have seen in section A, there exists a sequence of functions $\psi_n \in C(T)$ satisfying $\|\psi_n\|_\infty \leq 1$ and

$$|S_n(\psi_n, 0)| > \frac{1}{2} \| D_n \|_{L^1} > \frac{1}{10} \log n.$$

We put $\varphi_n(t) = \sigma_n^2(\psi_n, t)$ and notice that φ_n is a trigonometric polynomial of degree n^2 satisfying $\|\varphi_n\|_\infty \leq 1$ and

$$\left| S_{n(\varphi_n,t)} - S_{n(\psi_n,t)} \right| < 2. \text{ Hence } |S_{n(\varphi_n,0)}| > \frac{1}{10} \log(n-2).$$

To show this, $\|\varphi_n\|_\infty = \| \sigma_{n_2}(\psi_n)\|_\infty \leq \| F_{n_2}\|_{L^1} \|\psi_n\|_\infty = \|\psi_n\|_\infty \leq 1$.

$$\text{And } |S_n(\varphi_n, t) - S_n(\psi_n, t)| = \left| \sum_{j=-n}^{n} \hat{\varphi}_n(j) e^{ijt} - \sum_{j=-n}^{n} \hat{\psi}_n(j) e^{ijt} \right|$$

$$= \left| \sum_{j=-n}^{n} (\widehat{\varphi}_n(j) - \widehat{\psi}_n(j))\, e^{ijt} \right| = \left| - \sum_{j=-n}^{n} \frac{|j|}{n^2+1}\, \widehat{\psi}_n(j))\, e^{ijt} \right|$$

$$= \left| \frac{1}{n^2} \sum_{j=-n}^{n} |j| \right| = \frac{n(n+1)}{n^2+1} = 1 + \frac{n+1}{n^2+1} < 2\,.$$

In particular at t = 0, $|S_n(\varphi_n,0) - S_n(\psi_n,0)| < 2$ and

Hence $|S_n(\psi_{n,}0| - |S_n(\varphi_n,0)| \le |S_n(\varphi_n,0) - S_n(\psi_n,0)| < 2$.

$\Rightarrow |S_n(\varphi_n\ ,\ 0)| > |S_n(\psi_n\ ,0| - 2 > \frac{1}{10}\log\ n-2,$ with $\lambda_n 2^{3^n}$.

We define $f(t) = \sum_{n=1}^{\infty} \frac{1}{n^2}\, \varphi_{\lambda_n}(\lambda_n t)$.

Claim: f is a continuous function whose Fourier series diverges at t = 0. Since t he se quence { f_N},

where $f_N(t) = \sum_{n=1}^{\infty} \frac{1}{n^2}\, \varphi_{\lambda_n}(\lambda_n t)$ of continuous function

Converging point w ise to **f**, $f_N \to f$ as $N \to \infty$ uniformly on T. Therefore, **f** is c ontinuous on T. To show t he di vergence of t he Four ier series of **f** at zer o, we notice that

$$\varphi_{\lambda_j}(\lambda_j t) = \sigma_{\lambda^2 j}\left(\psi_{\lambda_j}, \lambda_j t\right) = \sum_{m=-\lambda^2 j}^{\lambda^2 j} \left(1 - \frac{|j|}{(\lambda^2 j + 1)}\right) \widehat{\psi}_{\lambda_j}(m)\, e^{i\lambda_j mt}$$

$$= \sum_{m=-\lambda^2 j}^{\lambda^2 j} \widehat{\psi}_{\lambda_j}(m)\, e^{i\lambda_j mt}$$

Since $\widehat{\varphi}_{\lambda j}(m) = \hat{F}_{\lambda^2 j}(m)\, \widehat{\psi}_{\lambda j}(m) = \begin{cases} \left(1 - \frac{|m|}{\lambda 2j}\right) \widehat{\psi}\lambda j(m) & \text{if } |m| \le \lambda^2 j. \\ 0 & \text{if } |m| > \lambda^2 j \end{cases}$

Therefore, $\varphi_{\lambda_j}(\lambda_j t) = \sum_{m=-\lambda^2 j}^{\lambda^2 j} \widehat{\psi}_{\lambda_j}(m)\, e^{i\lambda_j mt}$.

$$\text{Hence, } \left|S_{\lambda_j^2}(f,0)\right| = \left|S_{\lambda_j^2}\left(\sum_{j=1}^{n}\frac{1}{j^2}\varphi_{\lambda_j}(\lambda_j t),0\right)+\sum_{n+1}^{\infty}\frac{1}{j^2}\widehat{\varphi}_{\lambda_j}(0)\right|$$

$$= \left|S_{\lambda_j^2}\left(\sum_{j=1}^{n-1}\frac{1}{j^2}\varphi_{\lambda_j}(0)\right)+\frac{1}{n^2}S_{\lambda_N}\left(\varphi_{\lambda_j},0\right)+\sum_{n+1}^{\infty}\frac{1}{j^2}\widehat{\varphi}_{\lambda_j}(0)\right|$$

$$\geq \frac{k}{n^2}\log\lambda_n - 3 \text{, which tends to } \infty.$$

Therefore, $\{S_n(f, 0)\}$ is unbounded .

Hence the Fourier series diverges unbounded at t = 0.

LEMMA: 2.2.2 Let $f \in L^1(T)$ and $\hat{f}(n) = O(\frac{1}{n})$ as $|n| \to \infty$.

Then for every $\varepsilon > 0$, *there exists* $\lambda > 1$ *such that* $\limsup_{n\to\infty} \sum_{n<|j|\leq\lambda n} |\hat{f}(j)| < \varepsilon$.

PROOF: Since $\hat{f}(n) = 0\left(\frac{1}{n}\right)$, as there exists M such that $|n\,\hat{f}(n)| \leq M$, i.e $|\hat{f}(n)| \leq \frac{M}{|n|}$ as $n \to \infty$. If n is sufficiently large and $\lambda = 1+\frac{\varepsilon}{2M}$, then

$$\sum_{n<|j|\leq\lambda n} |\hat{f}(j)| \leq M \sum_{n<|j|\leq\lambda n} \frac{1}{|j|} < M\,2(\lambda n - n)\frac{1}{n} = 2M(\lambda-1) = \varepsilon\,.$$

Our first convergence criterion is really a simple **Tauberian theorem due to Hardy.**

THEOREM: 2.2.3 Let $f \in L^1(T)$ and assume $\hat{f}(n) = 0\left(\frac{1}{n}\right)$ as $|n| \to \infty$.

Then $S_n(f,t)$ and $\sigma_n(f,t)$ converges for the same values of t and to the same limits. Also, if $\sigma_n(f,t)$ converges uniformly on some set, so does $S_n(f,t)$.

PROOF: By the above lemma, for every $\varepsilon > 0$, there exists

$\lambda > 1 \quad such\ that \ \lim_{n\to\infty} \text{Sup} \sum_{n<|j|\leq\lambda n} |\hat{f}(j)| < \varepsilon.$

$$S_n(f,t) = \frac{[\lambda n]+1}{[\lambda n]-1}\sigma_{[\lambda n]}(f,t) - \frac{[\lambda n]+1}{[\lambda n]-1}\sum_{n<|j|\leq\lambda n}\left(1-\frac{|j|}{[\lambda n]+1}\right)\hat{f}(j)e^{ijt},$$

Where $[\lambda n]$ denotes the integral part of λn.

By the above lemma , there exists an n_0 such that $n > n_0$ implies that

$$\frac{[\lambda n]+1}{[\lambda n]-1}\sum_{n<|j|\leq\lambda n}\left(1-\frac{|j|}{[\lambda n]+1}\right)\hat{f}(j)\, e^{ijt} < \frac{\varepsilon}{2}.$$

If $\sigma_n(f,t_0)$ converges to a limit $\sigma(f, t_0)$, then there exists n_1 sufficiently large such that $n > n_1$ implies

$$|S_n(f,t_0) - \sigma(f,t_0)| \leq \frac{[\lambda n]+1}{[\lambda n]-1}|\sigma_{[\lambda_n]}(f,t_0) - \sigma(f,t_0)| + \frac{n+1}{[\lambda n]-n}|\sigma(f,t_0) - \sigma_n(f,t_0)|$$

$$+\frac{[\lambda n]+1}{[\lambda n]-1}\sum_{n<|j|\leq\lambda n}\left(1-\frac{|j|}{[\lambda n]+1}\right)\hat{f}(j)\, e^{ijt} < \frac{\varepsilon}{2}+\frac{\varepsilon}{4}+\frac{\varepsilon}{2} = \varepsilon.$$

In other words, $\lim_{n\to\infty} S_n(f,t_0) = \sigma(f,t_0)$.

Therefore, If $\sigma_n(f,t_0)$ converges uniformly on the same set, then $S_n(f,t_0)$ also converges on the same set.

COROLLARY:2.2.4 Let f be of bounded variation on T, then the partial sums $S_n(f,t)$ converge to $\frac{1}{2}(f(t+0)+f(t-0))$ and in particular to $f(t)$ at every point of continuity. The convergence is uniform on closed intervals of continuity of f.

PROOF: By Fejer's theorem, $\sigma_n(f, t) \to \frac{1}{2}(f(t+0)+f(t-0))$ as $n \to \infty$.

Since a function **f** is of bounded variation on T , we have $\hat{f}(n) = \mathbf{0}\left(\frac{1}{n}\right)$ as $|n| \to \infty$. Therefore by theorem 2.2.3, $S_n(f, t) \to \frac{1}{2}(f(t+0)+f(t-0))$.

Hence by fe jer's t heorem, the c onvergence i s uniform on t he c losed interval.

LEMMA: 2.2.5 Let $f \in L^1(T)$ and assume $\int_{-1}^{1} \left|\frac{f(t)}{t}\right| dt < \infty$.

Then $\lim_{n\to\infty} S_n(f,0) = 0$.

PROOF: $S_n(f,0) = \frac{1}{2\pi}\int_T f(t)\,\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dt$

$$= \frac{1}{2\pi}\int_T f(t)\cos nt\,dt + \frac{1}{2\pi}\int_T f(t)\,\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\sin nt\,dt.$$

By assumption $\frac{f(t)\cos\frac{t}{2}}{\sin\frac{t}{2}} \in L^1(T)$. Hence by Riemann- Lebesgue lemma,

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_T f(t)\cos nt\,dt = 0 \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{2\pi}\int_T f(t)\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\sin nt\,dt = 0.$$

Therefore, $\lim_{n\to\infty} S_n(f,0) = 0$.

THEOREM: 2.2.6 (*Principle of localization)*. Let $f \in L^1(T)$ and assume that **f** vanish in an open interval I. Then $S_n(f,t)$ converge to zero for $t \in I$, and the convergence is uniform on a closed subsets of I .

PROOF: T he c onvergence t o z ero a t e very t $\in$ I i s an i mmediate consequence of Lemma 2.2.5. Let $0 < \delta < \pi$, then

$$S_n(f,t_0) = \frac{1}{2\pi}\int_{-\delta}^{\delta} f(t+t_0)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt + \frac{1}{2\pi}\int_{\delta}^{\pi} f(t+t_0)\,\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

$$+\frac{1}{2\pi}\int_{-\pi}^{-\delta} f(t)\,\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dt.$$

On the closed interval $[\delta, \pi]$ and $[-\pi, -\delta]$, the function $\frac{1}{\sin\frac{t}{2}}$ is continuous.

Therefore, the function $\frac{f(t+t_0)}{\sin\frac{t}{2}}$ is absolutely integrable.

By Riemann-Lebesgue lemma :

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{\delta}^{\pi} f(t)\cos nt\,dt = 0, \quad \lim_{n\to\infty} \frac{1}{2\pi}\int_{\delta}^{\pi} f(t)\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} \sin nt\,dt = 0,$$

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{\delta}^{\pi} f(t+t_0)\cos nt\,dt = 0 \text{ and } \lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{-\delta} f(t)\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} \sin nt\,dt = 0.$$

Therefore, $\lim_{n\to\infty} S_n(f,t_0) = 0$. Since t_0 is arbitrary element of I, for all t in I, then $S_n(f,t)$ converges uniformly.

REMARK: The principle of localization is often stated as follows. Let f, g $\in L^1(T)$ and assume that $f(t) = g(t)$ in some neighborhood of a point t_0. Then the Fourier series of f and g at t_0 are either both convergent and to the same limit or both divergent and in the same manner.

THEOREM: 2.2.7 (*Dini's* test)

Let $f \in L^1(T)$. If $\int_{-1}^{1}\left|\frac{f(t+t_0)-f(t_0)}{t}\right| dt < \infty$, then $S_n(f,t_0) \to f(t_0)$.

PROOF: $S_n(f,t_0) - f(t_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(t+t_0) - f(t_0)]\, D_n(t)\, dt$.

$$= \frac{1}{2\pi}\int_{0}^{\pi} [f(t+t_0) + f(t_0-t) - 2f(t_0)]\, D_n(t)\, dt.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(t+t_0) - f(t_0)] \cos nt\, dt + \frac{1}{2\pi}\int_{0}^{\pi} [f(t+t_0) - f(t_0)] \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} \sin nt\, dt.$$

Since $\tan\frac{t}{2} \cong \frac{t}{2}$ as $t \to 0$, from the assumption $\frac{f(t+t_0)-f(t_0)}{\tan\frac{t}{2}}$ is integrable.

By Riemann-Lebesgue lemma,

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(t+t_0) - f(t_0)] \cos(nt)\, dt = 0 \text{ and}$$

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{0}^{\pi} [f(t+t_0) - f(t_0)] \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} \sin(nt)\, dt = 0.$$

Therefore, $\lim_{n\to\infty} [S_n(f, t_0) - f(t_0)] = 0$.

Hence, $\lim_{n\to\infty} S_n(f, t_0) = f(t_0)$.

2.3 SETS OF DIVERGENCE

DEFINITION: 3.2.1 Let B be a homogeneous Banach space on T. A set E ⊂ T is a set of divergence for B if there exists an $f \in B$ whose Fourier series diverges at every point of E.

NOTATION: For $f \in L^1(T)$, we put

a. $S_n^*(f,t) = \sup_{m \le n} |S_m(f,t)|$

b. $S^*(f,t) = \sup_{n} |S_n(f,t)|$

***LEMMA*: 2.3.2** Let $g \in B$. There exist an element $f \in B$, and a positive even sequence $\{\Omega_j\}$ such that $\lim_{j\to\infty} \Omega_j = \infty$ monotonically, and such that

$\hat{f}(j) = \Omega_j\, \hat{g}(j)$, for all $j \in Z$.

PROOF: Let $\lambda(n)$ be such that $\|\sigma_{\lambda(n)}(g) - g\|_B < 2^{-n}$. We write

$$f = g + \sum_{n=1}^{\infty} \left(g - \sigma_{\lambda(n)}(g)\right).$$

The series defining f converges in norm.

Hence, $f \in B$ and also $\hat{f}(j) = \Omega_j\, \hat{g}(j)$,

where $\Omega_j \sum_{n=1}^{\infty} \min\left(1, \frac{|j|}{\lambda_n + 1}\right)$

THEOREM: 2.3.3 Let **B** be a homogeneous Banach space on T and $E \in T$. E is a set of divergence for B if and only if, there exists an element $f \in B$ such that $S^*(f, t) = \infty$, for $t \in E$.

PROOF: The condition, $S^*(f, t) = \infty$ for $t \in E$ is sufficient for the divergence of $\sum \hat{f}(j)\, e^{ijt}$, for all $t \in E$. On the other hand, assume for some

$g \in \sum \hat{f}(j)\, e^{ijt}$ diverges at every point of E. Let f $\in$B and $\{\Omega_j\}$ be the function and the sequence corresponding to g by lemma 2.3.2.

Claim: $S^*(f, t) = \infty$.

For $n > m$, $S_n(g,t) - S_m(g,t) = \sum_{m+1}^{n} \left(S_j(f,t) - S_{j-1}(f,t)\right)\Omega_j^{-1}$

$$\frac{S_n(f,t)}{\Omega_n} - \frac{S_m(f,t)}{\Omega_{m+1}} + \sum_{m+1}^{n-1} \left(\Omega_j^{-1} - \Omega_{j+1}^{-1}\right) S_j(f,t).$$

$$\text{Hence, } \left| S_n(g,t) - S_m(g,t) \right| \le \frac{2S_*(f,t)}{\Omega_{m+1}}.$$

It follows that if $S^*(f, t) < \infty$, then the Fourier series of g converges and $t \notin E$. Hence this contradicts to the divergence of E. Therefore, $S^*(f, t) = \infty$.

REMARK:

Let ω_n, $n \ge 1$ be the sequence of positive numbers such that $\omega_j = \mathbf{0}(\Omega_j)$,

$$\sum_{1}^{\infty} \left(\frac{1}{\Omega_j} - \frac{1}{\Omega_{j+1}}\right)\omega_j < \infty, \text{then for all } t \in E, S_j(f,t) \neq 0(\omega_j).$$

LEMMA: 2.3.4 Let B be a homogeneous Banach space on T such that if f $\in$B and n $\in$Z, then $e^{int} f \in B$ and $\|e^{int} f\|_B = \|f\|_B$. Then E is a set of divergence for B if and only if there exists a sequence of trigonometric polynomial $P_j \in B$

Such that $\sum \|P_j\|_B < \infty$ *and sup* $S^*(P_j, t) = \infty$ on E.

PROOF: assume the existence of a sequence of trigonometric polynomial $\{P_j\}$ satisfying $\sum \|P_j\|_B < \infty$ and $\sup_j S^*(P_j, t) = \infty$ on E. Denote by m_j the degree of P_j and let V_j be integer satisfying $V_j > V_{j-1} + m_{j-1} + m_j$.

Put $f(t) = \sum e^{ijt} P_j(t)$. For $n \le m$, we have, $S_{v_{j+n}}(f,t) - S_{v_{j-n-1}}(f,t) = e^{iv_j t} S_n(P_j, t)$ and hence $\sum \hat{f}(j)e^{ijt}$ diverges on

E. Conversely, assume that e is a set of divergence for B. then there exists a monotonic sequence $\omega_n \to \infty$ and a function $f \in B$ such that $|S_n(f, t)| > \omega_n$ infinitely often for every t in E. We now pick a sequence of integers $\{\lambda_j\}$ such that $\|f - \sigma_{\lambda j}(f)\|_B < 2^{-j}$ and integers μ_j such that

$\omega_{\mu j} > 2\mathrm{Sup}_t\, S^*(\sigma_{\lambda_j}(f), t)$ and write $P_j = \nu_{\mu_{j+1}} * (f - \sigma_{\lambda_j}(f))$, where ν_μ denotes de la valle poussin's kernel, then

$\|P_j\|_B \leq \|\nu_{\mu_{j+1}}\|_{L^1} \|f - \sigma_{\lambda_j}(f)\|_B \leq \|\nu_{\mu_{j+1}}\|_{L^1} \sum 2^{-j} < \infty.$

If $t \in E$ and n integer such that $|S_n(f, t)| > \omega_n$, $\mu_j < n < \mu_{j+1}$ and $S_n(P_j, t) = S_n(f - \sigma_{\lambda_j}(f), t) = S_n(f, t) - S_n(\sigma_{\lambda_j}(f, t))$.

Hence, $|S(P_j, t)| > \frac{1}{2}\omega_n$. Therefore, $S^*(f, t) = \infty$ for every $t \in E$.

CONCLUSION

Carleson's theorem shows that for a given continuous function, the Fourier s eries c onverges a lmost a nywhere. If f is 2π periodic a nd absolutely c ontinuous o n $[0,2\pi]$, t hen t he Four ier s eries of f converges uniformly, but not necessarily absolutely, to f.

However, the Fourier series of a continuous function need not converge pointwise. Perhaps the easiest proof uses the non-boundedness of Dirichlet's kernel in $L^1(\mathbf{T})$ and the Banach–Steinhaus uniform bounded ness principle. It shows that the family of continuous functions whose Fourier series converges at a given point x is of first Baire category, in the Banach space of continuous functions on the circle. So in some sense point wise convergence is a typical, and for most continuous functions the Fourier series does not converge at a given point.

If f is square integrable, then $S_N f$ converges to f in the norm of L^2. If 2 in the exponents above is replaced with some p, the question becomes much harder. It turns out that the convergence still holds if $1 < p < \infty$. In other words, for f in Lp, $S_N(f)$ converges to f in the L^p norm. For $p = 1$ and infinity, the result is not true.

In general for, $B = L^1(T)$, it was shown by kolmogorov that there exists a function $f \in B$ whose Fourier series diverges everywhere. The case $B = L^2(T)$ was settled only recently by L.Carleson (1966) who proved that Fourier series of function in $L^2(T)$ converges almost everywhere.

There is a continuous function whose Fourier series diverges at all points of a given set of reals if and only if the set has measure 0.

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