

**VARIATION IN MAGNETIC FIELD STRENGTH
AND DECAYING TIME AMONG
NEUTRON STARS**

**A Thesis Submitted To
The School Of Graduate Studies Of
Addis Ababa University**



**IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR
THE DEGREE OF MASTER OF SCIENCE IN PHYSICS**

**BY
KIFLE WOLDEMICHAEL**

ADDIS ABABA

ETHIOPIA

MARCH 2007

ADDIS ABABA UNIVERSITY

DEPARTMENT OF PHYSICS

The undersigned hereby certify that they have read and recommend to the School of Graduate Studies Faculty of Science for acceptance a thesis entitled the” **Variation in Magnetic Field Strength and decaying time among Neutron Stars**” by **kifle W/Michael** in partial fulfillment of the requirements for the degree of Master of Science.

A/Advisor:.....

Dr. Ramanand Jha

B/Examiners:1/.....

Dr. Legesse Wetro

2/.....

Prof. Ashok A. Chaubey

Date: March 2007

DEDICATED TO MY FAMILY

ACKNOWLEDGEMENTS

First of all I would like to thank the son of **God** who is the Springboard of Christian Religion and the father of everything. Next, I would like to express my heartfelt gratitude to my advisor **Dr. Ramanand Jha** for his unbounded assistance and encouragement, logical and technical guidance, and enthusiastic approach.

I am also indebted to **Ayalew Tassew, Ribka Kinde, Dereje Kinde, and Abera Shiferaw** for their invaluable assistance, comments and suggestion.

Last but not least, I am very much grateful to my son **Solomon Kifle** and my wife **Ejgayehu Kefelegn** for their kindheartedly entertainment.

ABSTRACT

Today astronomical observations confirm that both **radio Pulsars** and **Magnetars** tend to be found in the same locations of the **Milky Way**, in regions where stars have recently exploded as supernovae.

The question has been: if they are located in similar places and are born in similar ways, then why does a magnetar's magnetic field decay in just **10^4 years**, when a normal pulsar decays in **10^6 - 10^7 years** and why are **magnetars (10^{14} - 10^{15} Gauss)** formed with stronger magnetic fields than **radio pulsars (10^{12} Gauss)**.

In these work I will try to suggest possible answers for questions given above by investigating the cause of the variation of the magnetic field and decaying time among neutron stars. Moreover, I am also deriving an equation and stating an idea that describes the relation between magnetic field and decaying time.

Table of Contents

Approval	ii
Dedication	iii
Acknowledgments	iv
Abstract	v
Table of Contents	vi
List of Figures	vii
Introduction	1
1 Neutron Stars	7
1.1 Classification of Neutron Stars	7
1.2 Nature of Pulsars and Magnetars	8
1.2.1 Pulsars	8
1.2.2 Magnetars	8
1.3 History of Pulsars and Magnetars	9
1.3.1 Early Ideas	9
1.3.2 Discovery of Neutron Stars	10
1.4 Bulk Properties of Stars	12
1.4.1 Introduction	12
1.4.2 Luminosity of Stars	12
1.4.3 Surface Temperature of Stars	13
1.4.4 The Radius of a Star	15
1.5 Coordinate Conditions	16
1.5.1 Introduction	16
1.5.2 General Coordinate Conditions	17
1.5.3 Flat Minkowski Space in Cartesian Coordinates	18
1.5.4 Flat Minkowski Space in Cylindrical Coordinates	19
1.5.5 Flat Minkowski Space in Spherical Coordinates	20
1.5.6 Curved Minkowski Space in Spherical Coordinates	21
2 Generation of Magnetic Fields in Neutron Stars	23
2.1 Introduction	23
2.2 Magnetic Fields	24
2.2.1 Ultra-Strong Magnetic Fields	24
2.2.2 The Magnetic Fields due to Flux Conservation	25
2.3 Rotating Dipole Model	27
2.4 Magnetosphere	31
2.5 The Magnetohydrodynamics (MHD)	38
2.5.1 Introduction to Magnetohydrodynamics	38
2.5.2 The Magnetohydrodynamics Equations	38

2.5.3 The Induction Equation and the Alfve'n Theorem	44
2.6 Instability and Dynamo Mechanism in Proto Neutron Stars (PNSs)..	49
2.6.1 Introduction	49
2.6.2 Convection in Proto Neutron Stars	50
2.6.3 Dynamo Mechanism in Proto Neutron Stars	53
3 Magnetic Field Decay of Neutron Stars	58
3.1 Introduction	58
3.2 Magnetic Field Decay due to Magnetic Dipole Radiation	59
3.3 Magnetic Field Decay due to Gravitational Quadruple Radiation ...	67
3.4 The Decaying Time of Neutron's Star Magnetic Field	76
3.4.1 Age of Radiation Source Neutron Stars	76
3.4.2 The relation between Magnetic Field and Decaying Time	78
3.5 The Angular Velocity of Neutron Stars	89
3.5.1 The Initial Angular Velocity of Neutron Stars	89
3.5.2 The Spin Evolution of Neutron Stars	90
4 Summary and Conclusion	94
4.1 Summary	94
4.2 Conclusion	100
 Appendix	 101
 Bibliography	 103

List of Figures

1/ Fig. 1.1: Measured spectra of wavelengths emitted by a body.	15
2/ Fig. 2.1: Partitions of unstable region.	52
3/ Fig. 3.1: Rotation of frame of reference by angle $\theta=\Omega t$ in x-y plane.	69
4/ Fig. 4.1a: This graph illustrates the magnetic field versus decaying time diagram of a magnetar at the threshold surface magnetic field of 10^7 G and it is based on equations (3.72 and 3.73).	96
5/ Fig. 4.1b: This graph illustrates the magnetic field versus decaying time diagram of a magnetar at the threshold surface magnetic field of 10^7 G and it is based on equations (3.84 and 3.85).	97
6/ Fig. 4.1c: This graph illustrates the magnetic field versus decaying time diagram of a radio pulsar at the threshold surface magnetic field of 10^7 G and it is based on equations (3.72 and 3.73).	98
7/ Fig. 4.1d: This graph illustrates the magnetic field versus decaying time diagram of a radio pulsar at the threshold surface magnetic field of 10^7 G and it is based on equations (3.84 and 3.85).	99

INTRODUCTION

Our universe consists of elementary particles (electron, proton, neutron, positron, neutrino, and photon, etc.) and massive bodies (stars, galaxies, and so on). The components of interstellar medium usually consisting of molecular hydrogen and some dust are the “springboards” for the formation of stars. Dust is not the dust one finds around the house but it is an irregularly shaped grains of carbon or silicate measuring a fraction of a micron across which is found between the stars.

Basic factors in the formation of stellar are gravity, dust, gas pressure, rotation, magnetic fields, winds and radiation from nearby young stars, and radiative shock waves.

The night sky contains myriads of stars including those in the Milky Way, which is a “side view “ of our Galaxy looking along the plane of the disc. Our Galaxy includes about 10^{11} stars. Beyond our galaxy are billions of other Galaxies. The nearest star is about 4 light years away and the nearest large Galaxy is 2 million light year away. Our Galactic disc has a diameter of about 100,000 light years.

Stars are believed to begin life as collapsing masses of hydrogen gas (protostars). As they contract, they heat up (potential energy is transformed to kinetic energy). When the temperature reaches about 10 million degrees, nuclear fusion begins and forms heavier elements (nucleosynthesis), mainly helium at first [1]. The energy released during these

reactions heats the gas so its outward pressure balances the inward gravitational force, and the young star stabilizes as a main sequence star. The tremendous luminosity of stars comes from the energy released during these thermonuclear reactions. This thermonuclear fusion provides the energy for the “great furnace” in the phase of Eddington [2], which drives stars through various stage of combustion-hydrogen, helium, carbon, neon, oxygen, silicon, and magnesium. After billions of years, as helium is collected in the core and hydrogen is used up, the core contracts and heats further. The envelope expands and cools, and the star becomes a red giant (larger diameter, redder color). The next stage of stellar evolution depends on the mass of the star, which may have lost much of its original mass as its outer envelope escaped into space as a result of star’s explosion.

The explosion of the star is divided into two, namely, nova and supernova. The term “nova” refers to a class of exploding stars whose luminosity temporarily increases from several thousand to as much as 100,000 times its original level. Most novas are thought to involve special double-star systems (“close binaries”). The term “supernova” refers to a completely different phenomenon: a supernova is any of the class of violently exploding stars whose luminosity after eruption suddenly increases millions or billions of times its normal level, the super nova explosion, unlike a nova explosion, a cataclysmic event associated with the essential end of the active (energy-generating) life of the star.

During the life of the star, two opposing forces control the star’s equilibrium: the gravitational force, which drives the collapse of the star’s mass inward to the center of gravity, and the counteracting outward pressure derived from the nuclear fusion reac-

tions in the star's core. When the nuclear fuel burns out, the star begins its death and gravitational collapse occurs. If the terminal stages of the star death (during which large amounts of stellar material are blown away) leave a remnant star mass greater than three solar masses, the ultimate gravitational collapse will produce a black hole, a relativistic singularity. Present view of a black hole is that it is a mass that has collapsed to such a small volume that its gravity prevents the escape of all radiation. Space and time essentially have no meaning at the center of a black hole. The boundary of a black hole is called the "event horizon", because any event within the boundary is invisible outside, the invisibility resulting from the fact that no radiation can escape to be detected. The radius of the black hole depends upon how much matter has fallen into the region; it is called the "Schwarzschild radius", and it is usually a few kilometers [3].

If, after the blow-off at star death, the remnant mass is between 1.4 and 3 solar-masses, the star will collapse into a neutron star, a body with a radius of 10 to 15 kilometers, but with a core so dense that its components protons and electrons have merged into neutrons by inverse β -decay ($e^- + p = n + \nu$). The size of the star is no longer limited by the Pauli exclusion principle applied to electrons, but rather applied to neutrons (neutron degeneracy) [1].

The core of a neutron star contracts to the point at which all neutrons are as close together as they are in a nucleus. That is, the density of a neutron star is on the order of 10^{14} times greater than the normal solids and liquids on the Earth. A cupful of such dense matter would weigh billions of tons. Neutron stars apparently have an outer shell

of iron, but it is iron no Earth iron, an iron of 4 orders of magnitude greater density. Theory predicts that a neutron star should rotate very rapidly, be extremely hot, and have an intense magnetic field.

If, after the blow-off at star death, the remnant mass is less than 1.4 or approximately is equal to 1 solar- mass, the star will collapse into white dwarf a body with a radius of 5,000 kilometers and density about 10^9 kg/m^3 [4]. A white dwarf contracts to a point at which the electron clouds starts to overlap, but collapses no further because, as the Pauli exclusion principle claims, no two electrons can be in the same quantum state. Arriving at this point is called electron degeneracy.

In 1932, neutron was discovered by Chadwick [5]. After two years, in 1934, W. Baade and F. Zwicky [6] invented the notion of neutron stars in their investigation of supernovae. They were suggested that supernovae might be created by the collapse of a normal star to form a neutron star.

In 1940, Woltjer estimated [7] that the conservation of magnetic flux in a star that contracts from a red giant to a neutron star would produce fields of 10^{12} Gauss. After this, in 1966, the discovery of the first pulsar by Hewish and Bell as a highly magnetized neutron star was already available, but unknown to them, in the paper by Pacini published in the same year [8]. Following this discovery, and independently of Pacini, Gold's paper also provided a strong case for identifying the pulsar as a rotating neutron star which emits EM radiation.

The year 1979 was a fantastic time for the study of soft gamma repeaters (SGRs).

The first SGR burst ever detected, from a source in the constellation Sagittarius, occurred on January 7, 1979. Other bursts came two months later, on March 5, 1979 and March 14, 1979. Following this another two bursts came in 1992 and in 1998. In 1998, Chryssa Kouveliotou [9] showed that the brightness of this SGR burst (X-ray star) pulsates at a gradually-slowing rate, giving strong corroboration to the magnetar model.

The measured rate of spin-down allows the magnetic field to be estimated. For almost all young pulsars and magnetars the magnetic fields at magnetic poles are 10^{12} - 10^{13} Gauss and 10^{14} - 10^{15} Gauss, respectively. In addition to this, astronomers can estimate the age of a supernova remnant (SNR) that is, how long ago the super-nova occurred. This is done by measuring properties of SNR, like its size and rate of expansion. Therefore, the burster remains radio “loud” for approximately 10 million years in the case of pulsars [8] whereas the burster spit out bursts of high-energy X-rays or gamma-rays for 10,000 years in the case of magnetars [10].

The origin of strong magnetic fields in neutron stars is still a matter of controversy. The present understanding are: 1/ the standard theory simply related to the magnetic fields of the progenitor main sequence stars frozen during collapse (Woltjer, 1964); 2/ the magnetic fields with different strengths on different length-scales can be explained naturally if they are generated through a dynamo mechanism driven by turbulent motions [9], [11]; 3/ the generation of neutron star magnetic fields by thermomagnetic effects [12]; and 4/ the generation of neutron star magnetic fields by the spinning of separated charges [13].

This thesis analyses the possibility that the neutron star magnetic fields are generated due to fossil remnant plus large scale dynamo action in the case of magnetars and due to fossil remnant and small scale dynamo action in the case of radio pulsars. The thesis is organized as follows. Chapter one deals with classification, nature, history, size of neutron stars, and the coordinate conditions. Chapter two deals with the generation of magnetic fields in neutron stars. Chapter three deals with magnetic field decay, decaying time and angular velocity of neutron stars. In chapter four summary and conclusion are given.

Chapter 1

NEUTRON STARS

1.1 Classification of Neutron Stars

According to their magnetic fields, neutron stars are classified into two, namely, pulsars and magnetars with typical polar surface magnetic fields of 10^{12} - 10^{13} Gauss and 10^{14} - 10^{15} Gauss, respectively. Magnetars are subdivided into soft gamma repeaters (SGRs) and anomalous X-ray pulsars (AXPs) [9]. Pulsars are also subdivided into Crab pulsar, Vela pulsar, Millisecond pulsar, etc. [14].

1.2 Nature of Pulsars and Magnetars

1.2.1 Pulsars

Pulsars (pulsating radio stars) are any class of cosmic objects that populate the plane of our galaxy in the Milky Way and rapidly rotating by emitting extremely regular pulses of radio waves, with several such objects known to emit pulses of visible light, X-rays, and gamma-rays as well. Ordinary radio pulsars are neutron stars with magnetic fields 10^{12} Gauss and the spin periods between 16 milliseconds and 8.5 seconds. They are also radio “loud” for approximately 10 million years and are powered by slow loss of the rotational energy that the star is born with [10]. Radio pulsar is neutron star in which large-scale dynamos has essentially failed to operate, because it is not born fast enough (compared to the magnetar) as a result pulsar’s magnetic field is essentially stable; its main role is to passively facilitate the loss of rotational energy [9].

1.2.2 Magnetars

Magnetars (AXPs and SGRs) are a kind of neutron star—a city-sized ball of neutrons created from star’s core when the star explodes as a supernova at the end of its life. They are nearly 1 mile (1.6 km) thick and made of material so tightly packed that a single tablespoon of the stuff would weigh about 10 billion tons of material on Earth. Young magnetars have magnetic fields that range from approximately 10^{14} - 10^{15} Gauss and they are born spinning faster than pulsars. In magnetar, the combined effects of rotation and

convection can build up the magnetar's magnetic field through dynamo process. But the rotational energy quickly decreases and the magnetic field itself provides an energy source for emission. As a result of this a magnetar's magnetic energy is dissipated during the first 10,000 years [9].

1.3 History of Pulsars and Magnetars

1.3.1 Early Ideas

In 1932, neutron discovered by Chadwick. Fritz Zwicky (1938) had long ago proposed that the Crab Nebula might be found in supernova remnants, and the very term neutron star seems to have been coined by Baade and Zwicky (1934a). The modern quantitative theory of such objects began with Oppenheimer and Volkoff (1939). Just before the discovery of pulsars, it was theorized that a rotating magnetized neutron star could be the source of the nebular energy output (Pacini,1967). John Wheeler (1966) and Lev Landau (1932) had also predicted the existence of something like neutron star before the pulsar phenomenon came to apparently required such object. Piddington (1957) suggested the relative “modern” interpretation that the Crab Nebula's magnetic field was that of a wound-up magnetic field from a central star. The Piddington (1957) estimate is interesting. First he assumes a star like the Sun, so he takes a radius of $10^9 m$ and a magnetic field of 10^3 Gauss. He then images the magnetic field to be dragged out radially to the size of the Nebula, which only leaves 10^{-11} Gauss. To get instead a 10^{-3}

Gauss field then requires an “amplification factor” of 10^8 , which, if due to rotation, would require the star to rotate once every 5 minutes (his guess for the period) for 900 years, that is the earliest estimate of the Crab pulsar period. It was suggested in the discovery paper (Hewish et al.1968) that a neutron star might be responsible for the phenomenon, and this interpretation was pursued energetically by Gold (1968). Even observation in a sense preceded observation: hard X-ray data establishing the existence of a pulsar in the Crab Nebula were in hand even before the discovery of pulsars themselves as radio sources (Fish-man et al.1969 a,b). Only a few radio pulsars emit detectable levels of X-rays. The Crab Nebula itself has been an active center of attention and we should mention Dewhirst’s (1983) reproduction of Messier’s drawings from 1843, one of which fairly accurately reproduces the central continuum region [14].

1.3.2 Discovery of Neutron Stars

On 28 November 1967 the Mullard Radio Astronomy observatory array (2048 dipoles at 81.5 Mhz) observed a train of pulses of varying amplitudes but quite regular spacing near $19^h 19^m$ right ascension and 21 degrees of northern (plus) declination. It was not at first realized what exactly had been observed (given especially that an automobile ignition could produce such a regular pattern of interference), but Jocelyn Bell, a research student, eventually found the observation to be genuine and was assigned the task of identification. Ultimately a Nobel Prize went to the discovery director, Anthony Hewish. Like many Astrophysical objects pulsars are identified but their rough location in the sky, and this

first pulsar is now known as PSR 1919+21, where the right ascension and declination have been incorporated into designation. It is now generally accepted that pulsars are rapidly rotating neutron stars. In addition to being radio sources, pulsars radiate even more energy in the form of invisible “winds”. The Crab Nebula is a supernova remnant that is lit up by the pulsar in its center, as best as one can tell [14].

The year 1979 was a remarkable one for the study of soft gamma repeaters (SGRs), although nobody realized it at the time. The first SGR burst ever detected, from a source in constellation Sagittarius, occurred on January 7, 1979. Then a truly powerful SGR outburst . . . indeed, by far the most intense blast of gamma-rays that had ever been detected from outside our solar system (until another SGR outburst broke the record in 1998). . . came just two months later, on March 5, 1979. This tremendous flare eventually allowed the SGR mystery to be unraveled. Only nine days after that a third SGR became active in a new part of the Galaxy, giving three bursts in a three day period. So during the first three months of 1979, three of the five known SGRs were discovered. Following this another two bursts came in 1992 and 1998. For years, astronomers didn’t distinguish between SGR bursts and the much more frequently observed “ordinary” or “classic” gamma-ray bursts (GRBs). (During the 1990’s, with modern detectors, about one GRB per day was detected from some place in the sky. Fewer SGR bursts were seen, perhaps 10 to 50 per year, although they came in groups.) It wasn’t until 1987 the SGRs were clearly recognized as a distinct set of objects. In 1998, Chryssa Kouveliotou [9] showed that the brightness of this SGR burst (X-ray star) pulsates at a gradually-slowng rate,

giving strong corroboration to the magnetar model.

1.4 Bulk Properties of Stars

1.4.1 Introduction

The interior of stars, in spite of the potentialities of neutrino astronomy, remain inaccessible to direct study. The properties and behavior of the material within stars must therefore be deduced from the laws of physics and conditions observed at the surface of the stars, together with a few global properties such as mass, size etc.

1.4.2 Luminosity of Stars

Vogt-Russell theorem [15] states that the mass, composition and age of an isolated star uniquely determine its luminosity and effective surface temperature. Over 90% of stars are formed essentially of hydrogen and helium in the ratio of about three to one by mass, the only variation occurring in the very small portion of the mass of the star which is in the form of the heavier elements. Furthermore, for a substantial part of the star's life its age has only a marginal effect upon its structure; whilst burning hydrogen to helium at its center, only the composition of these central regions changes, and that very slowly. Thus the only significant variable which remains to affect a star's luminosity is its mass.

We find therefore a relationship between the two, and it takes the form

$$L_s \propto M^n. \tag{1.1}$$

The value of the index n varies slightly with the mass, and in practice takes the values

$$n = 3.9 \text{ for } M_s < 7M_\odot, \tag{1.2}$$

$$n = 3.0 \text{ for } 7M_\odot < M_s < 25M_\odot, \tag{1.3}$$

$$n = 2.7 \text{ for } M_s > 25M_\odot, \tag{1.4}$$

where M_s is the mass of the star and M_\odot is the mass of the sun.

1.4.3 Surface Temperature of Stars

The rate at which objects radiate energy has been found to be proportional to the fourth power of the Kelvin temperature, T . At normal temperatures ($\sim 300\text{K}$), we are not aware of this electromagnetic radiation because of its low intensity. At higher temperatures, there is sufficient infrared radiation that we can feel that if we are close to the object. At still higher temperatures (in the order of 1000K), objects actually glow, such as a

red-hot electric stove burner or the element in a toaster. At temperatures above 2000K, objects glow with a yellow or whitish color, such as white-hot iron and the filament of a light bulb. The light emitted is a continuous range of wavelengths or frequencies, and the spectrum is a plot of intensity versus wavelength or frequency. As the temperature increases, the EM radiation emitted by objects not only increases in total intensity but also the peak of intensity shifts to higher frequencies. It is found experimentally that the wavelength at peak of the spectrum, λ_p , is related to the Kelvin temperature T by:

$$\lambda_p T = 2.90 \times 10^{-3} \text{ mK}. \quad (1.5)$$

This formula (1.5) is known as Wien's law [1], [16].

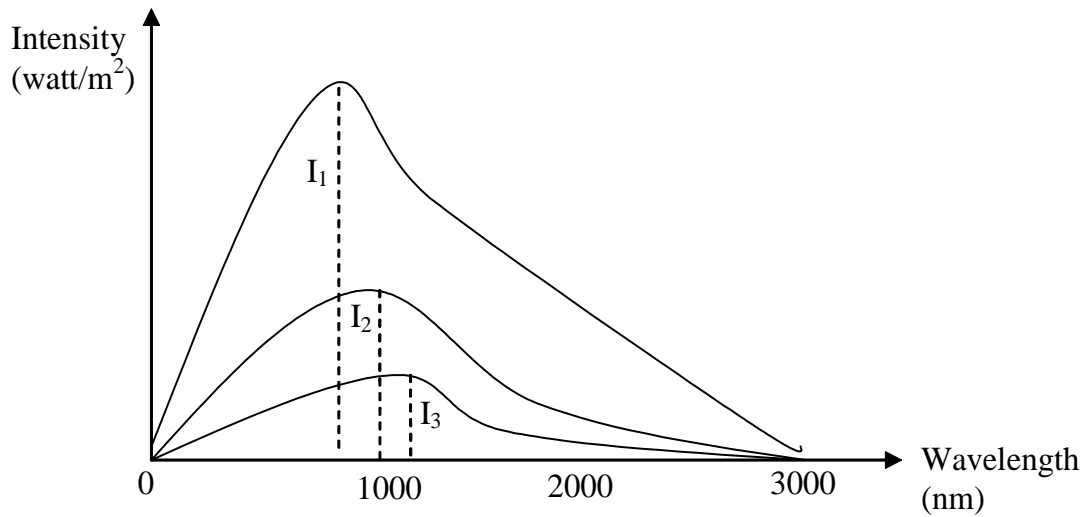


Fig. 1.1: Measured spectra of wavelengths emitted by a body. Where I_1 , I_2 , and I_3 are the peaks of intensity.

1.4.4 The Radius of a Star

All stars appear as point sources of light. However, from the luminosity and temperature of a star, we can calculate its actual radius using the Stefan-Boltzmann law. The Stefan-Boltzmann law says that radiancy (the energy emitted per second by a unit surface of a luminous body) is proportional to the fourth power the absolute temperature: $E = e\sigma T^4$ [17], where E is radiant energy; σ is the proportionality factor called the Stefan-

Boltzmann constant, which is equal to $5.67 \times 10^{-8} \frac{\text{watts}}{\text{k}^4}$; T is the absolute temperature; and e is called the emissivity between 0 and 1.

The surface of a star is spherical. Its surface is therefore, $4\pi r^2$, where r is the radius of the star. The energy emitted is given by [1]

$$L_s = 4\pi r^2 \sigma T^4, \tag{1.6}$$

where emissivity $e=1$.

And the radius is given by

$$r = \sqrt{\frac{L_s}{4\pi\sigma T^4}} = \frac{1}{2T^2} \sqrt{\frac{L_s}{\pi\sigma}}. \tag{1.7}$$

1.5 Coordinate Conditions

1.5.1 Introduction

Begin, now, with models for relativistic stars. As a major simplification, insist (initially) that all stars studied be static. Thereby exclude not only exploding and pulsating stars, but even quiescent ones with stationary rotational motion. From the static assumption, plus a demand that the star be made of “perfect fluid”, (no shear stress allowed), plus Einstein’s field equations, it probably follows that the star is spherically symmetric. However, nobody has yet given a proof. In the absence of a proof, assume the result: insist that all stars studied be spherical as well as static [4].

1.5.2 General Coordinate Conditions

The general form of the position vector (\vec{S}), the line element (\overrightarrow{dS}), and the squared length in Minkowski space (4D) are given by:

$$\vec{S} = \vec{S}(x^0, x^1, x^2, x^3) = x^0 \hat{e}_0 + x^1 \hat{e}_1 + x^2 \hat{e}_2 + x^3 \hat{e}_3, \quad (1.8)$$

$$\overrightarrow{dS} = \left| \frac{\partial \vec{S}}{\partial x^0} \right| dx^0 \hat{e}_0 + \left| \frac{\partial \vec{S}}{\partial x^1} \right| dx^1 \hat{e}_1 + \left| \frac{\partial \vec{S}}{\partial x^2} \right| dx^2 \hat{e}_2 + \left| \frac{\partial \vec{S}}{\partial x^3} \right| dx^3 \hat{e}_3, \quad (1.9)$$

$$\begin{aligned} \overrightarrow{dS}^2 &= dS^2 = g_{00} dx^0 dx^0 + g_{01} dx^0 dx^1 + g_{02} dx^0 dx^2 + g_{03} dx^0 dx^3 \\ &\quad + g_{10} dx^1 dx^0 + g_{11} dx^1 dx^1 + g_{12} dx^1 dx^2 + g_{13} dx^1 dx^3 \\ &\quad + g_{20} dx^2 dx^0 + g_{21} dx^2 dx^1 + g_{22} dx^2 dx^2 + g_{23} dx^2 dx^3 \\ &\quad + g_{30} dx^3 dx^0 + g_{31} dx^3 dx^1 + g_{32} dx^3 dx^2 + g_{33} dx^3 dx^3. \end{aligned} \quad (1.10)$$

The covariant metric tensor $g_{\mu\nu}$ is given by,

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}. \quad (1.11)$$

The contravariant metric tensor $g^{\mu\nu}$ is given by,

$$g^{\mu\nu} = \frac{g_{\mu\nu}^{cof}}{\det |g_{\mu\nu}|}, \quad (1.12)$$

where $g_{\mu\nu}^{cof}$ is the cofactor of $g_{\mu\nu}$ and $\det |g_{\mu\nu}|$ is the determinant of $g_{\mu\nu}$.

1.5.3 Flat Minkowski Space in Cartesian Coordinates (t,x,y,z):

The coordinate axes are given by

$$x^0 = t, x^1 = x, x^2 = y, \text{ and } x^3 = z. \quad (1.13)$$

The position vector, the line element, the squared length, and the metric tensor are given by:

$$\vec{S} = t\hat{e}_t + x\hat{e}_x + y\hat{e}_y + z\hat{e}_z, \quad (1.14)$$

$$\vec{dS} = dt\hat{e}_t + dx\hat{e}_x + dy\hat{e}_y + dz\hat{e}_z, \quad (1.15)$$

$$\vec{dS}^2 = dS^2 = -dt^2 + dx^2 + dy^2 + dz^2, \quad (1.16)$$

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (1.17)$$

where the minus sign in equation (1.16) is introduced because of “time like” has negative squared length whereas “space like” has positive squared length.

1.5.4 Flat Minkowski Space in Cylindrical Coordinates (t, ρ, ϕ, z) :

The coordinate axes are given by

$$x^0 = t, x^1 = \rho \cos \phi, x^2 = \rho \sin \phi, \text{ and } x^3 = z. \quad (1.18)$$

The position vector, the line element, the squared length, and the metric tensor are given by:

$$\vec{S} = t\hat{e}_t + \rho \cos \phi \hat{e}_x + \rho \sin \phi \hat{e}_y + z\hat{e}_z, \quad (1.19)$$

$$d\vec{S} = dt\hat{e}_t + d\rho\hat{e}_\rho + \rho d\phi\hat{e}_\phi + dz\hat{e}_z, \quad (1.20)$$

$$\overrightarrow{dS}^2 = dS^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2 + dz^2, \quad (1.21)$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (1.22)$$

1.5.5 Flat Minkowski Space in Spherical Coordinates (t, r, θ, ϕ):

The coordinate axes are given by

$$x^0 = t, x^1 = r \sin\theta \cos\phi, x^2 = r \sin\theta \sin\phi, \text{ and } x^3 = r \cos\theta. \quad (1.23)$$

The position vector, the line element, the squared length, and the metric tensor are given by:

$$\overrightarrow{S} = t\hat{e}_t + r \sin\theta \cos\phi \hat{e}_x + r \sin\theta \sin\phi \hat{e}_y + r \cos\theta \hat{e}_z, \quad (1.24)$$

$$\overrightarrow{dS} = dt\hat{e}_t + dr\hat{e}_r + r d\theta \hat{e}_\theta + r \sin\theta d\phi \hat{e}_\phi, \quad (1.25)$$

$$\overrightarrow{dS}^2 = dS^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \quad (1.26)$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}, \quad \text{and} \quad g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \end{pmatrix}. \quad (1.27)$$

1.5.6 Curved Minkowski Space in Spherical Coordinates

The general form of the squared length and the metric tensor for spherically symmetric space-time are [18],

$$\overrightarrow{dS}^2 = dS^2 = -g_{tt}dt^2 + 2g_{tr}drdt + g_{rr}dr^2 + r^2d\Omega^2, \quad (1.28)$$

$$g_{\mu\nu} = \begin{pmatrix} -g_{tt} & g_{tr} & 0 & 0 \\ g_{rt} & g_{rr} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}, \quad (1.29)$$

$$g^{\mu\nu} = \begin{pmatrix} \frac{-g_{rr}}{g_{rt}g_{tr}+g_{tt}g_{rr}} & \frac{g_{tr}}{g_{rt}g_{tr}+g_{tt}g_{rr}} & 0 & 0 \\ \frac{g_{rt}}{g_{rt}g_{tr}+g_{tt}g_{rr}} & \frac{g_{tt}}{g_{rt}g_{tr}+g_{tt}g_{rr}} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \end{pmatrix}, \quad (1.30)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

For static isotropic spherical coordinates (t, r, θ, ϕ) : the squared length, the squared proper time, and the metric tensor [8] are given by:

$$\overrightarrow{ds}^2 = ds^2 = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (1.31)$$

$$d\tau^2 = -dS^2 = e^{2\nu(r)} dt^2 - e^{2\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad (1.32)$$

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\nu(r)} & 0 & 0 & 0 \\ 0 & e^{2\lambda(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}, \quad (1.33)$$

$$g^{\mu\nu} = \begin{pmatrix} -e^{-2\nu(r)} & 0 & 0 & 0 \\ 0 & e^{-2\lambda(r)} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \end{pmatrix}, \quad (1.34)$$

where $d\tau^2$ is the squared proper time.

Chapter 2

Generation of Magnetic Fields in Neutron Stars

2.1 Introduction

As mentioned in the introduction of this thesis, the origin of strong magnetic fields in neutron stars is still a matter of controversy. The present understanding are:

1. the standard theory simply related to the magnetic fields of the progenitor main sequence stars frozen during collapse (Woltjer,1964);
2. the magnetic fields with different strengths on different length-scales can be explained naturally if they are generated through a dynamo mechanism driven by turbulent motions [9], [11];
3. the generation of neutron star magnetic fields by thermomagnetic effects [12]; and

4. the generation of neutron star magnetic fields by the spinning of separated charges [13].

In this chapter we are going to discuss the possibility that the neutron star magnetic fields are generated due to fossil remnant (flux conservation) plus small scale dynamo action in the case of radio pulsars and due to fossil ruminant plus large scale dynamo action in the case of magnetars.

2.2 Magnetic Fields

2.2.1 Ultra-Strong Magnetic Fields

Many fascinating physical effects occur in magnetic fields with strength exceeding the “quantum electrodynamics field strength” of $B_c = 4.4 \times 10^{13}$ Gauss. (This field-strength is given by a combination of fundamental constants: $B_c = m_e^2 c^3 / \hbar e$, where m_e is the mass of an electron, c is the speed of light, \hbar is Planck’s constant divided by 2π , and e is the charge of an electron.) This field strength determines the luminosity, the life time, and even the nature of the energy loss from isolated neutron stars. In fields stronger than B_c , electrons gyrate nearly the speed of light around magnetic field lines, even in their lowest quantum energy states. Consequently, the ultra-magnetized vacuum... which, according to quantum mechanics, seethes with virtual electron-positron pairs and other particles... becomes birefringent like a calcite crystal, capable of destroying and magnifying images ("magnetic lensing"). X-ray photons travelling through such strong fields

readily split into two, or merge together; and many other novel physical effects come into play. Because B_c lies between the field strengths observed in magnetars and in ordinary radio pulsars, this new physics is important only in magnetars, making the theory of magnetars especially rich and interesting.

Although magnetar fields are stronger by most measures, they are weak compared to the strongest possible field that could theoretically exist in nature, which is 10^{49} - 10^{53} Gauss [9]. A field stronger than this would literally break down the vacuum and decay, via the quantum mechanical process of magnetic monopole creation. However, there is no known way that such strong fields could be generated. Magnetars are the most strongly-magnetized objects yet known in the universe.

2.2.2 The Magnetic Fields due to Flux Conservation

Ordinary stars have magnetic fields (~ 100 Gauss) and rotate. When they collapse from a radius of 10^6 km to 10 km, both the rotational frequency and field are scaled up by the conservation laws of angular momentum and magnetic flux. The field is typically scaled to 10^{12} Gauss. The remnant of the Crab supernova, some nine hundred years after the explosion, is both accelerating and radiating in radio, optical, x-ray, and gamma-ray bands powered by an energy input from the pulsar of $\sim 4 \times 10^{31} \frac{J}{\text{sec}}$. By equating the input to the nebula to the power loss of the magnetic dipole radiation (eq.3.25), one can derive the magnetic field which is scaled up by the conservation of magnetic flux.

Consider the fastest rotating pulsar with a period $\sim \frac{1}{30}$ sec, PSR 1937+21 [8], now

we can find the magnetic field

$$\frac{B_p^2 R^6 \Omega^4 \sin^2 \alpha}{6c^3} = 4 \times 10^{31} \frac{J}{sec}, \quad (2.1)$$

where B_p is the dipolar magnetic field at the stellar pole, R is a canonical neutron star radius (~ 10 km), Ω is the angular velocity, and α is the angle of inclination of the magnetic axis to the rotational axis.

From the above relationship we can compute an expression for magnetic field,

$$B_p = \sqrt{\frac{4 \times 6 \times 10^{31} c^3}{R^6 \Omega^4 \sin^2 \alpha}}. \quad (2.2)$$

By assuming $\alpha = 90^\circ$ and replacing Ω by $\frac{2\pi}{p}$, we can obtain

$$B_p \approx 4 \times 10^{12} \text{ Gauss}, \quad (2.3)$$

where $p \sim \frac{1}{30}$ sec is the observed period and $c = 3 \times 10^8 \frac{m}{s}$ the speed of light.

From the conservation of magnetic flux, we can also obtain the magnetic field of a new born neutron star.

$$\Phi_{sM} = \Phi_{nM}$$

or

$$B_s A_s = B_n A_n, \quad (2.4)$$

where Φ_{sM} is the magnetic flux of an ordinary star, Φ_{nM} is the magnetic flux of a neutron star, $B_s=100$ G is the magnetic field of an ordinary star, B_n is the magnetic field of a neutron star, $A_s=4\pi R_s^2$ is the area of an ordinary star, $R_s=10^6$ km is the radius of an ordinary star, $A_n=4\pi R_n^2$ is the area of a neutron star, and $R_n=10$ km is the radius of a neutron star.

Now we can compute the magnetic field which is scaled up by the conservation of magnetic flux as

$$B_n = B_s \frac{A_s}{A_n} = B_s \frac{4\pi R_s^2}{4\pi R_n^2} = 100 \text{ G} \left(\frac{10^6 \text{ km}}{10 \text{ km}} \right)^2 = 1 \times 10^{12} \text{ G}. \quad (2.5)$$

2.3 Rotating Dipole Model

This model accounts for many of the observational properties of pulsars in which the neutron star is assumed to be a perfect conducting sphere of radius R rotating in vacuum at angular velocity $\vec{\Omega}$ and possesses a magnetic dipole moment \vec{m} inclined at an angle α to the rotational frequency [11], [19].

The magnetic moment can be decomposed into its components as

$$\vec{m} = m_x \hat{e}_x + m_y \hat{e}_y + m_z \hat{e}_z \quad (2.6)$$

or

$$\vec{m} = m_0(\sin\alpha\cos\Omega t\hat{e}_x + \sin\alpha\sin\Omega t\hat{e}_y + \cos\alpha\hat{e}_z), \quad (2.7)$$

where $m_x=m_0\sin\alpha\cos\Omega t$, $m_y =m_0\sin\alpha\sin\Omega t$, and $m_z=m_0\cos\alpha$, and m_0 is the magnitude of the magnetic moment.

With similar fashion the angular velocity can also be decomposed as

$$\vec{\Omega} = \Omega_x\hat{e}_x + \Omega_y\hat{e}_y + \Omega_z\hat{e}_z \quad (2.8)$$

or

$$\vec{\Omega} = \Omega(\sin\alpha\cos\Omega t\hat{e}_x + \sin\alpha\sin\Omega t\hat{e}_y + \cos\alpha\hat{e}_z), \quad (2.9)$$

where $\Omega_x= \Omega \sin\alpha\cos\Omega t$, $\Omega_y= \Omega \sin\alpha\sin\Omega t$, $\Omega_z=\Omega\cos\alpha$, and Ω is the magnitude of the angular velocity.

If the magnetic moment and angular velocity are acting along z-axis, then equations (2.7) and (2.9) reduce to

$$\vec{m} = m_0\hat{e}_z \quad (2.10)$$

and

$$\vec{\Omega} = \Omega\hat{e}_z. \quad (2.11)$$

To obtain the relationship between unit vectors in Cartesian and Spherical coordi-

nates, first note that

$$\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z = r \sin \theta \cos \phi \hat{e}_x + r \sin \theta \sin \phi \hat{e}_y + r \cos \theta \hat{e}_z, \quad (2.12)$$

$$\hat{e}_r = \frac{\frac{\partial \vec{r}}{\partial r}}{\left| \frac{\partial \vec{r}}{\partial r} \right|} = \sin \theta \cos \phi \hat{e}_x + \sin \theta \sin \phi \hat{e}_y + \cos \theta \hat{e}_z, \quad (2.13)$$

$$\hat{e}_\theta = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left| \frac{\partial \vec{r}}{\partial \theta} \right|} = \cos \theta \cos \phi \hat{e}_x + \cos \theta \sin \phi \hat{e}_y - \sin \theta \hat{e}_z, \quad (2.14)$$

and

$$\hat{e}_\phi = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left| \frac{\partial \vec{r}}{\partial \phi} \right|} = -\sin \phi \hat{e}_x + \cos \phi \hat{e}_y. \quad (2.15)$$

By multiplying equation (2.13) by $\cos \theta$ and equation (2.14) by $-\sin \theta$ and adding simultaneously, we will get

$$\hat{e}_z = \cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta. \quad (2.16)$$

Substituting equation (2.16) into equations (2.10) and (2.11), we will get

$$\vec{m} = m_0(\cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta) \quad (2.17)$$

and

$$\vec{\Omega} = \Omega(\cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta). \quad (2.18)$$

The magnetic field for magnetic dipole generally is given by [20]

$$\vec{B} = \mu_0 \vec{H} = \frac{\mu_0}{4\pi} \left\{ k^2 (\hat{n} \times \vec{m}) \times \hat{n} \frac{e^{ikr}}{r} + [3\hat{n}(\hat{n} \cdot \vec{m}) - \vec{m}] \left(\frac{1}{r^3} - \frac{ik}{r^2} \right) e^{ikr} \right\} \quad (\text{Gaussian})$$

or

$$\vec{B} = k^2 (\hat{n} \times \vec{m}) \times \hat{n} \frac{e^{ikr}}{r} + [3\hat{n}(\hat{n} \cdot \vec{m}) - \vec{m}] \left(\frac{1}{r^3} - \frac{ik}{r^2} \right) e^{ikr} \quad (S.I), \quad (2.19)$$

where $\hat{n} = \frac{\vec{r}}{r} = \hat{e}_r$ is a unit vector in the direction of the field point or \vec{r} and $k = \Omega/c$ is a wave number.

The zonal magnetic fields are,

$$\vec{B} = k^2 (\hat{n} \times \vec{m}) \times \hat{n} \frac{e^{ikr}}{r} \quad (\text{radiation - zone, } kr \gg 1) \quad (2.20)$$

and

$$\vec{B} = \frac{1}{r^3} [3\hat{n}(\hat{n} \cdot \vec{m}) - \vec{m}] \quad (\text{near - zone, } kr \ll 1). \quad (2.21)$$

Substituting equation (2.17) into equation (2.21) we will obtain,

$$\vec{B} = \frac{m_0}{r^3} \{ 3\hat{n}[\hat{n} \cdot (\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta)] - (\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta) \},$$

$$\begin{aligned}
&= \frac{m_0}{r^3} (3\cos\theta\hat{e}_r - \cos\theta\hat{e}_r + \sin\theta\hat{e}_\theta), \\
&= 2\frac{m_0}{r^3} \left(\cos\theta\hat{e}_r + \frac{1}{2}\sin\theta\hat{e}_\theta \right), \tag{2.22}
\end{aligned}$$

where $\hat{n}\cdot\hat{e}_r=1$ and $\hat{n}\cdot\hat{e}_\theta=0$.

The magnitude of the dipolar magnetic field at the pole and the magnetic dipole moment are given by:

$$B_p = \frac{2m_0}{R^3}, \tag{2.23}$$

$$m_0 = \frac{1}{2}B_p R^3, \tag{2.24}$$

where $r=R$ at the surface of the pole.

2.4 The Magnetosphere

The rotating dipole model provides explanations for many of the observed properties of pulsars, it is based on the hypothesis that the rotator is in vacuum. But this model was opposed by Goldreich and Julian (1969), they proposed that the electric field induced by the rotation of the magnetic moment can pull-out charges from the stellar surface and will fill the magnetosphere which corotates rigidly with the star.

Consider a perfectly conducting rotating sphere threaded by an external dipole magnetic field. By combining equations (2.22) and (2.24) you will get the external dipolar magnetic field,

$$\vec{B}_{ext} = \frac{B_p R^3}{r^3} \left(\cos\theta \hat{e}_r + \frac{1}{2} \sin\theta \hat{e}_\theta \right) \quad (r > R). \quad (2.25)$$

With similar fashion the magnetic field in the interior is,

$$\vec{B}_{int} = B_p \left(\cos\theta \hat{e}_r + \frac{1}{2} \sin\theta \hat{e}_\theta \right) \quad (r < R). \quad (2.26)$$

Due to perfect electric conductivity in stellar interior, the electric conductivity σ tends to be infinitely large and Ohm's law (2.68) can be reduced to

$$\frac{\vec{J}_{int}}{\sigma} = 0 = \vec{E}_{int} + \left(\frac{\vec{\Omega} \times \vec{r}}{c} \right) \times \vec{B}_{int}$$

or

$$\vec{E}_{int} = - \left(\frac{\vec{\Omega} \times \vec{r}}{c} \right) \times \vec{B}_{int} = \vec{B}_{int} \times \left(\frac{\vec{\Omega} \times \vec{r}}{c} \right). \quad (2.27)$$

Using the vector relation

$$\vec{U} \times (\vec{V} \times \vec{W}) = \vec{V}(\vec{U} \cdot \vec{W}) - \vec{W}(\vec{U} \cdot \vec{V}). \quad (2.28)$$

We get the electric field in the stellar interior to be

$$\vec{E}_{int} = \frac{1}{c} [\vec{\Omega}(\vec{B}_{int} \cdot \vec{r}) - \vec{r}(\vec{B}_{int} \cdot \vec{\Omega})]. \quad (2.29)$$

Substituting equations (2.18) and (2.25) into equation (2.29) we will get

$$\vec{E}_{int} = \frac{2B_p R \Omega}{c} \left(\frac{1}{2} \sin^2 \theta \hat{e}_r - \sin \theta \cos \theta \hat{e}_\theta \right). \quad (2.30)$$

According to the solution of Legendre's equation [21], the scalar potential is given by

$$\Phi(r, \theta) = D r^{-(n+1)} P_n(\theta) = \frac{D}{r^3} (3 \cos^2 \theta - 1), \quad (2.31)$$

where $n=2$ and $D = \frac{B_p R^5 \Omega}{6c}$ is a constant determined by requiring the continuity of the tangential electric field.

The electric field in the exterior is given by:

$$\vec{E}_{ext} = -\nabla \Phi = - \left(\frac{\partial \Phi}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \hat{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \varphi} \hat{e}_\varphi \right) \quad (2.32)$$

or

$$\vec{E}_{ext} = \frac{B_p R^5 \Omega}{c r^4} \left[\left(\frac{3 \cos^2 \theta - 1}{2} \right) \hat{e}_r + \sin \theta \cos \theta \hat{e}_\theta \right]. \quad (2.33)$$

Where the components of the electric field in the exterior can be computed as follows,

$$E_{ext}^r = -\frac{\partial \Phi}{\partial r} = \frac{B_p R^5 \Omega}{2c} \left(\frac{3 \cos^2 \theta - 1}{r^4} \right), \quad (2.34)$$

$$E_{ext}^\theta = -\frac{1}{r} \frac{\partial \Phi}{\partial \theta} = \frac{B_p R^5 \Omega}{c r^4} \sin \theta \cos \theta, \quad (2.35)$$

and

$$E_{ext}^\varphi = -\frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \varphi} = 0. \quad (2.36)$$

Now we can examine the orthogonality between the electric field and magnetic field by using equations (2.26) and (2.30) and applying dot product,

$$(\vec{E}_{int} \cdot \vec{B}_{int}) = \frac{2B_p^2 R \Omega}{c} \left(\frac{1}{2} \sin^2 \theta \cos \theta - \frac{1}{2} \sin^2 \theta \cos \theta \right) = 0. \quad (2.37)$$

Therefore, the electric and magnetic fields in the interior are mutually orthogonal.

Next, we can also examine the orthogonality between electric and magnetic fields in the exterior by using equations (2.25) and (2.33) and applying dot product,

$$\begin{aligned} (\vec{E}_{ext} \cdot \vec{B}_{ext}) &= \frac{B_p R^5 \Omega}{c r^4} \left[\left(\frac{3 \cos^2 \theta - 1}{2} \right) \hat{e}_r + \sin \theta \cos \theta \hat{e}_\theta \right] \cdot \left[\frac{B_p R^3}{r^3} \left(\cos \theta \hat{e}_r + \frac{1}{2} \sin \theta \hat{e}_\theta \right) \right], \\ &= \frac{B_p^2 R^8 \Omega}{c r^7} \cos^3 \theta, \\ &\approx \frac{B_p^2 R \Omega}{c} \left(\frac{R}{r} \right)^7. \end{aligned} \quad (2.38)$$

Therefore, the electromagnetic fields in the exterior are not orthogonal.

The magnitude of the external electric field around the surface is given by:

$$\left| \vec{E}_{ext} \right| = \frac{\vec{E}_{ext} \cdot \vec{B}_{ext}}{\left| \vec{B}_{ext} \right|} = \left[\frac{B_p^2 \Omega R}{2B_p (R/r)^3} \left(\frac{R}{r} \right)^7 \frac{\cos^3 \theta}{\sqrt{3 \cos^2 \theta + 1}} \right] \sim \frac{B_p \Omega R}{c}, \quad (2.39)$$

where $R=r$.

The ratio of the electric force to the gravitational force of attraction on charged particles around the surface will be

$$\left| \frac{\vec{F}_e}{\vec{F}_G} \right| = e \frac{\left| \vec{E}_{ext} \right|}{G \frac{M_s m_e}{R^2}} = \frac{e B_p \Omega \frac{R}{c}}{G \frac{M_s m_e}{R^2}} = \frac{2\pi e B_p R^3}{G c M_s m_e P}. \quad (2.40)$$

Given the observed period, $P = \frac{2\pi}{\Omega} \sim \frac{1}{30}$ sec, the dipolar magnetic field at the pole, $B_p \sim 4 \times 10^{12}$ Gauss, the canonical neutron star radius, $R \sim 10$ km, the charge of an electron, $e = -1.6 \times 10^{-19}$ C, the universal constant, $G = 6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$, the speed of light, $c = 3 \times 10^8 \frac{m}{s}$, the mass of the neutron star, $M_s \sim 2M_\odot = 4 \times 10^{30}$ kg, and the mass of an electron, $m_e = 1.67 \times 10^{-31}$ kg, then by substituting these into equation (2.40) we can obtain,

$$\left| \frac{\vec{F}_e}{\vec{F}_G} \right| \sim 10^9. \quad (2.41)$$

The ratio of the two forces in equation (2.41) assures that the electric force acting on charged particles at the stellar surface is much–much larger than the corresponding gravitational force. As a result, the charged particles will be attracted from the stellar surface. These charges will then corotate with their parent star and forming the so-called "magnetosphere". If no surface charge can be maintained, then all the charges must be in the magnetosphere.

However, equation (2.39) gives a force greatly exceeding gravity on charged particles near the neutron star surface, the electrostatic forces are expected to ultimately create a space charges such that $\frac{\vec{E}_{ext} \cdot \vec{B}_{ext}}{B_{ext}^2} \longrightarrow 0$ [14], which in turn suggests that the magnetic field lines become equipotential. If the magnetic field lines are equipotential, then it immediately follows that the plasma motion (which is just the $\frac{\vec{E} \times \vec{B}}{B^2}$ drift velocity, neglecting any motion along field lines) corresponds to rigid corotation of the plasma with the neutron star.

In the magnetosphere the magnetic fields are orthogonal and if the field lines are equipotential, the plasma corotates with

$$\vec{v} = \vec{\Omega} \times \vec{r}. \quad (2.42)$$

Thus we can write

$$\vec{E} = -\frac{\vec{v}}{c} \times \vec{B} = \left(\frac{\vec{r} \times \vec{\Omega}}{c} \right) \times \vec{B}, \quad (2.43)$$

$$\begin{aligned}
\nabla \cdot \vec{E} &= \nabla \cdot \left(\frac{\vec{r} \times \vec{\Omega}}{c} \right) \times \vec{B} = \frac{1}{c} \left\{ \vec{B} \cdot [\nabla \times (\vec{r} \times \vec{\Omega})] - (\vec{r} \times \vec{\Omega}) \cdot (\nabla \times \vec{B}) \right\}, \\
&= \frac{2}{c} \vec{B} \cdot \vec{\Omega},
\end{aligned} \tag{2.44}$$

where $\nabla \times \vec{v} = \nabla \times (\vec{\Omega} \times \vec{r}) = 2\vec{\Omega}$ and $\nabla \times \vec{B} = 0$, because the magnetic field lines are equipotential and \vec{v} is the velocity of the plasma in a rigid neutron star.

The charged distribution in the magnetosphere is given by:

$$\rho = \frac{1}{4\pi} \nabla \cdot \vec{E} = \frac{1}{2\pi c} \vec{\Omega} \cdot \vec{B}, \tag{2.45}$$

where $\nabla \cdot \vec{E} = 4\pi\rho$.

Because of corotation, the magnetic field lines of the magnetosphere will be closed out to "light cylinder", $R_L = \frac{c}{\Omega}$. Charged particles are accelerated along these lines ($\vec{E} \cdot \vec{B} \neq 0$) and stream out to infinity, producing the pulsed emission observed.

2.5 The Magnetohydrodynamics (MHD)

2.5.1 Introduction to Magnetohydrodynamics

It is now generally believed that collapsed stellar objects such as white dwarf stars and neutron stars (pulsars and magnetars) are very likely to possess strong magnetic fields of the order of 10^8 - 10^{15} Gauss [9], [22]. The discovery of pulsars and magnetars have stimulated considerable intrinsic interest in the effects brought about by ultra-strong magnetic fields. Generally, in a neutron star the gravitational binding energy per particle can be a tenth of the rest energy. Thus a fully consistent relativistic magnetohydrodynamics model or equations [11], [23] are necessary to study the evolution of a rotating neutron star's interior.

2.5.2 The Magnetohydrodynamics Equations

Let us consider a globally neutral plasma of particles of electrons (-e) and protons (+e) at the same temperature T_0 and with mean density N_0 [11], [23]. By using Boltzmann formula, the charge densities are given by:

$$\begin{aligned} N_e(\Phi) &= N_0 e^{\frac{e\Phi}{kT_0}} = N_0 \left[1 + \frac{e\Phi}{kT_0} + \frac{\left(\frac{e\Phi}{kT_0}\right)^2}{2!} + \dots \right], \\ &\approx N_0 \left(1 + \frac{e\Phi}{kT_0} \right), \end{aligned} \tag{2.46}$$

$$\begin{aligned}
N_p(\Phi) &= N_0 e^{\frac{-e\Phi}{kT_0}} = N_0 \left[1 - \frac{e\Phi}{kT_0} + \frac{\left(\frac{e\Phi}{kT_0}\right)^2}{2!} - \dots \right], \\
&\approx N_0 \left(1 - \frac{e\Phi}{kT_0} \right),
\end{aligned} \tag{2.47}$$

where Φ is electrostatic potential, k is the Boltzmann constant and the kinetic energy is much greater than the potential energy $\left(\left|\frac{e\Phi}{kT_0}\right| \ll 1\right)$.

In this a feebly coupled plasma, a test charge q_0 at the origin experiences the global electrostatic potential solution of the Poisson equation given by

$$\nabla^2 \Phi = -\frac{\rho_t}{\epsilon_0} = -\left(\frac{\rho_e + \rho_p + \rho_{q_0}}{\epsilon_0}\right) = \frac{e}{\epsilon_0} [N_e(\Phi) - N_p(\Phi)] - \frac{q_0}{\epsilon_0} \delta^{(3)}(x), \tag{2.48}$$

where $\rho_e = -eN_e(\Phi)$, $\rho_p = eN_p(\Phi)$, $\rho_{q_0} = \sum_{i=1}^n q_{0i} \delta(x - x_i) = q_0 \delta^{(3)}(x)$ and $\delta^{(3)}(x)$ is the three dimensional delta function.

By substituting equations (2.46) and (2.47) into equation (2.48), the solution of Poisson equation is given by

$$\nabla^2 \Phi = \frac{1}{\lambda_D^2} \Phi - \frac{q_0}{\epsilon_0} \delta^{(3)}(x), \tag{2.49}$$

where $\lambda_D = \sqrt{\frac{T_0 \epsilon_0 k}{2N_0 e^2}}$ is the Debye length and indicates the shortest distance at which

electrons are moving at random in the plasma, screen the coulomb field of the probe particles. Since the particles in the plasma exchange momentum through two-body system with a cross-section $\sigma \sim \frac{\alpha_{EM}^2}{T_0^2}$, where α_{EM} is the EM coupling constant, the mean-free-path will be $l_{mfp} \sim \frac{T_0^2}{\alpha_{EM}^2 N_0}$ and thus l_{mfp} is much greater than λ_D .

Using the equation of motion for globally neutral plasma of particles, one can obtain, the oscillation frequency commonly called the plasma frequency,

$$\omega_p = \sqrt{\frac{N_0 e^2}{m \epsilon_0}}. \quad (2.50)$$

By using the ratio of the thermal velocity ($v_{th} \approx \sqrt{\frac{kT_0}{m_e}}$) to the l_{mfp} , we can get the collision frequency ω_c ,

$$\omega_c \approx \frac{V_{th}}{l_{mfp}} = \frac{\alpha_{em}^2 N_0 \sqrt{k}}{\sqrt{m_e T_0^3}}. \quad (2.51)$$

By comparing equations (2.50) and (2.51) one can say that ω_c is much less than ω_p and the system undergoes many oscillations between two collisions. The number of particles interacting in the Debye sphere or, equivalently, the plasma parameter $n = \frac{1}{N_0 \lambda^3}$. Since $n \propto \frac{1}{N_0}$, the number of particles interacting in a low-density plasma is larger and vice versa. In plasma, charged particles will not be influenced fully by the EM fields of the other charged particles up until they enter in their Debye sphere. However, long-range EM fields are present and determine the collective motion.

In fluid dynamics the mechanical motion is described in terms of a single conducting

fluid with the usual hydrodynamic variables of density, velocity, pressure and gravitational force. The combined system of equations, of these usual hydrodynamic variables and EM field equations will describe the magnetohydrodynamics.

The dynamics of the plasma as a fluid can be described by the MHD equations as follows:

$$\textit{Gausslaw} : \nabla \cdot \vec{E} = \frac{\rho_e}{\epsilon_0}, \quad (2.52)$$

$$\textit{No mono pole} : \nabla \cdot \vec{B} = 0, \quad (2.53)$$

$$\textit{Ampere's law} : \nabla \times \vec{B} = \mu_0 \vec{J}, \quad (2.54)$$

$$\textit{Faraday's law} : \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad (2.55)$$

$$\textit{Ampere - Maxwell law} : \nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}, \quad (2.56)$$

where equations (2.52)—(2.56) generally are said to be Maxwell's equations.

$$\textit{Equation of continuity} : \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \quad (2.57)$$

$$\text{Equation of charge conservation: } \frac{\partial \rho_e}{\partial t} + \nabla \cdot \vec{J} = 0, \quad (2.58)$$

$$\text{Euler equation: } \rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v} \cdot \nabla) \vec{v} = -\rho \nabla \Phi - \nabla P + \vec{J} \times \vec{B}, \quad (2.59)$$

$$\text{Ohm's law: } \vec{J} = \sigma (\vec{E} + \vec{v} \times \vec{B}) - \frac{\sigma}{eN_0} (\vec{J} \times \vec{B} - \nabla P_e), \quad (2.60)$$

where σ is the electric conductivity, ρ is the fluid density, P is fluid pressure, Φ is gravitational potential, $\vec{J} = eN_0 \vec{v}$ is the charge-density current, P_e is the electron pressure, and $\vec{J} \times \vec{B}$ is the Lorentz force.

For path length much larger than the Debye length ($L \gg L_D$), the plasma is assumed to be neutral and charge density can be neglected ($\rho_e \approx 0$). So that equations (2.52) and (2.58) can be written as,

$$\nabla \cdot \vec{E} = \nabla \cdot \vec{J} = 0. \quad (2.61)$$

With similar fashion, on time scales much longer than that of the plasma oscillations, the displacement current in equation (2.56) can be dropped to obtain

$$\vec{J} = \frac{1}{\mu_0} \nabla \times \vec{B} \quad (\text{SI system})$$

or

$$\vec{J} = \frac{c}{4\pi} \nabla \times \vec{B} = \frac{1}{4\pi} \nabla \times \vec{B} \quad (c=1 \text{ in Gaussian system}). \quad (2.62)$$

The cross product of \vec{J} (2.62) and \vec{B} gives

$$\vec{J} \times \vec{B} = \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B} = -\frac{1}{\mu_0} \vec{B} \times (\nabla \times \vec{B}). \quad (2.63)$$

Using the identity

$$\nabla(\vec{A} \cdot \vec{B}) = (\vec{B} \cdot \nabla) \vec{A} + (\vec{A} \cdot \nabla) \vec{B} + \vec{B} \times \nabla \times \vec{A} + \vec{A} \times \nabla \times \vec{B}, \quad (2.64)$$

equation (2.63) becomes,

$$\vec{J} \times \vec{B} = -\frac{\nabla B^2}{2\mu_0} + \frac{1}{\mu_0} (\vec{B} \cdot \nabla) \vec{B} = -\nabla P_m + \frac{1}{\mu_0} (\vec{B} \cdot \nabla) \vec{B}, \quad (2.65)$$

where $P_m = \frac{B^2}{2\mu_0}$ may be regarded as the "magnetic pressure" which is perpendicular to the magnetic field lines, whereas the term $\frac{1}{\mu_0} (\vec{B} \cdot \nabla) \vec{B}$ represents the "magnetic tension" which is parallel to the magnetic field lines.

Next, substituting equation (2.65) into equation (2.59) we will get

$$\begin{aligned}
\rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v} \cdot \nabla) \vec{v} &= -\rho \nabla \Phi - \nabla P - \nabla P_m + \frac{1}{\mu_0} (\vec{B} \cdot \nabla) \vec{B}, \\
&= -\rho \nabla \Phi - \nabla (P + P_m) + \frac{1}{\mu_0} (\vec{B} \cdot \nabla) \vec{B}. \quad (2.66)
\end{aligned}$$

For force-free magnetic field $[\vec{J} \times \vec{B} = \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B} = 0]$ usually in astrophysical flows, the Euler equation (2.59) is reduced to

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v} \cdot \nabla) \vec{v} = -\rho \nabla \Phi - \nabla P. \quad (2.67)$$

2.5.3 The Induction Equation and the Alfve'n Theorem

Among the listed MHD equations Faraday's law (2.55) and Ampere-Maxwell law (2.56) best describe the time evolution of magnetic fields. Another most important expression can be obtained by neglecting the Lorentz force $(\vec{J} \times \vec{B})$ and the thermoelectric (∇P_e) terms from Ohm's law (2.60), so that Ohm's law is reduced to

$$\frac{\vec{J}}{\sigma} \approx \vec{E} + \vec{v} \times \vec{B} \quad (S.I - System)$$

or

$$\frac{\vec{J}}{\sigma} \approx \vec{E} + \frac{\vec{v} \times \vec{B}}{c} \quad (Gaussian - System). \quad (2.68)$$

Substituting equation (2.62) into equation (2.68) and finding its curl,

$$\begin{aligned}\nabla \times \left(\frac{\nabla \times \vec{B}}{4\pi\sigma} \right) &= \nabla \times \vec{E} + \nabla \times (\vec{v} \times \vec{B}), \\ &= -\frac{\partial \vec{B}}{\partial t} + \nabla \times (\vec{v} \times \vec{B}),\end{aligned}\tag{2.69}$$

where $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's law).

By rearranging equation (2.69), one can solve for time rate of change of \vec{B} ,

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) - \nabla \times (\eta \nabla \times \vec{B}),\tag{2.70}$$

where $\eta = \frac{1}{4\pi\sigma}$ is the magnetic diffusivity.

The other most important MHD equations are (2.66) and (2.70). They describe the dynamics of magnetized plasma even in the absence of heat loss and heat gain (i.e., adiabatic process).

Two possible cases of equation (2.70) are:

Case 1: Ideal MHD limit

Conductivity goes to infinity or diffusivity goes to zero, so that equation (2.70) reduces to:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}).\tag{2.71}$$

By using the total time derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v} \cdot \nabla)$ of the magnetic field one can get

$$\frac{d\vec{B}}{dt} = \frac{\partial\vec{B}}{\partial t} + (\vec{v} \cdot \nabla) \vec{B}. \quad (2.72)$$

Using the identity

$$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla) \vec{A} - \vec{B}(\nabla \cdot \vec{A}) - (\vec{A} \cdot \nabla) \vec{B} + \vec{A}(\nabla \cdot \vec{B}). \quad (2.73)$$

Equation (2.72) becomes

$$\begin{aligned} \frac{d\vec{B}}{dt} &= \frac{\partial\vec{B}}{\partial t} + \nabla \times (\vec{B} \times \vec{v}) + \vec{v}(\nabla \cdot \vec{B}) + (\vec{B} \cdot \nabla) \vec{v} - \vec{B}(\nabla \cdot \vec{v}), \\ &= \frac{\partial\vec{B}}{\partial t} + \nabla \times (\vec{B} \times \vec{v}) + (\vec{B} \cdot \nabla) \vec{v}, \end{aligned} \quad (2.74)$$

where $\nabla \cdot \vec{B} = 0$ in the absence of free magnetic pole and $\nabla \cdot \vec{v} = 0$ for an incompressible fluid.

Substituting equation (2.74) into integral of the Lagrangian time derivative of the magnetic flux over any surface made of fluid elements, so that

$$\frac{d}{dt} \int_s \vec{B} \cdot d\vec{s} = \int_s \frac{\partial\vec{B}}{\partial t} \cdot d\vec{s} + \int_s \nabla \times (\vec{B} \times \vec{v}) \cdot d\vec{s} + \int_s (\vec{B} \cdot \nabla) (\vec{v} \cdot d\vec{s}). \quad (2.75)$$

We consider an electric field $\vec{E} = \vec{B} \times \vec{v}$ (as seen from the rest frame) and neglecting the term $\int_s (\vec{B} \cdot \nabla) (\vec{v} \cdot d\vec{s})$, which gives the contribution of the sources of the vector field swept over by the moving fluids.

Then equation (2.75) reduces to

$$\begin{aligned} \frac{d}{dt} \int_s \vec{B} \cdot d\vec{s} &= \int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \int_s (\nabla \times \vec{E}) \cdot d\vec{s}, \\ &= \int_s \left(\frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} \right) \cdot d\vec{s} = 0, \end{aligned} \quad (2.76)$$

where $\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$ (Faraday's law).

Equation (2.76) describes a highly conducting plasma as a fluid in which magnetic field is " frozen-in" when $\sigma \rightarrow \infty$. This is called Alfvén's equation (theorem).

It can be shown that the stress due to a magnetic field is equivalent to an isotropic magnetic pressure $\frac{B^2}{2\mu_0}$ and a tension $\frac{B^2}{\mu_0}$ along the magnetic field lines tube [23]. Stretching the tube increases the tension. The situation is similar to that of a stretched elastic string. By plucking the tube, transverse waves (Alfvén waves) are propagating along the magnetic field lines.

In the case of strings, the wave velocity is given in terms of the tension T and the linear mass density ρ of the string as

$$v = \sqrt{\frac{T}{\rho}}. \quad (2.77)$$

Making the analogy to MHD plasma waves, we replace T by the magnetic tension $\frac{B^2}{\mu_0}$ and ρ becomes the plasma mass density.

Thus, for MHD waves we have for the phase velocity

$$v_A = \frac{B}{\sqrt{\mu_0 \rho}}, \quad (2.78)$$

where the velocity v_A is often called Alfvén velocity.

Another conserved quantity in the ideal MHD limit is the magnetic helicity. The magnetic helicity is conserved in a perfectly conducting plasma and it describes topological properties such as links, twists, and kinks of magnetic field lines. When the topology of the field lines changes, the conductivity should be finite, which implies non-conservation of the magnetic helicity [24].

In the covariant language, the 4-current of magnetic helicity is defined by

$$H^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} A_\nu F_{\alpha\beta}. \quad (2.79)$$

This is conserved in the ideal MHD,

$$H^\mu; \mu = 0. \quad (2.80)$$

Case 2: Resistive-MHD limit

If there exists a static fluid (i.e. $v=0$), then equation (2.70) reduces to

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times (\eta \nabla \times \vec{B}). \quad (2.81)$$

Using the relation

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}, \quad (2.82)$$

equation (2.81) becomes

$$\frac{\partial \vec{B}}{\partial t} = -\eta \nabla^2 \vec{B}, \quad (2.83)$$

where $\nabla \cdot \vec{B} = 0$ and assuming the diffusivity η doesn't depend on space.

2.6 Instability and Dynamo Mechanism in Proto Neutron Stars (PNSs)

2.6.1 Introduction

Nowadays, it is believed that proto neutron stars are obligated, for small time interval after they born, to hydrodynamic instabilities (convective and neutron-finger) and these can cease approximately 30-40 seconds. If the stars are born rotating fast enough, the combined effects of rotation and convection, which both drag field lines though the star, can build up the star's over all magnetic fields, though a complex process called "dynamo

action" [9], [11]. Dynamo action is driven by turbulent motions of fluid in the stellar.

2.6.2 Convection in Proto Neutron Stars

There are two types of hydrodynamic instabilities, namely, neutron-finger and convective. Neutron-finger instability is driven by lepton gradients (Epstein 1979) whereas convective instability is driven by negative entropy gradients which are commonly observed in simulations of supernova explosions (Bruenn and Mezzacappa 1994,1995).

Since the PNS is opaque to neutrinos, the approximated pressure length-scale and the effective flow velocity are given by

$$\begin{aligned} L &= \frac{P}{\left|\frac{dP}{dr}\right|}, \\ v &\approx \frac{L}{T_L}, \end{aligned} \tag{2.84}$$

where T_L is the growth-time of instability.

In the convectively unstable region, the lepton number gradient is dominated by the temperature gradient and the growth-time of instability can be approximated as

$$T_L \sim T_C \sim \sqrt{\frac{3T}{g\beta|\Delta\nabla T|}}, \tag{2.85}$$

where T_C is the growth-time of convection, β is the coefficient of thermal expansion,

$\Delta\nabla T$ is the difference between the actual and adiabatic temperature gradient, and g is the acceleration due to gravity.

The convective instability grows on a short dynamical time-scale. According to Miralles et al. (2000) cited by Luciano Rezzolla (2004), the estimated time $T_C \sim 0.1 - 1$ *ms* and velocity, $v_L \sim 10^6 - 10^7 \frac{m}{s}$.

In the case of the neutron-finger unstable region, the temperature gradient is dominated by the lepton number gradient and the growth-time is approximately given by

$$T_L \sim T_{nf} \sim \sqrt{\frac{3}{g\delta|\nabla Y|}}, \quad (2.86)$$

where g is the acceleration due to gravity, δ is the coefficient of chemical expansion, and $Y = \frac{(n_e + n_\nu)}{n}$ is the lepton fraction with n_e , n , and n_ν being the number density of electrons, baryons, and neutrons, respectively.

The neutron-finger instability grows on a larger dynamical time-scale compared to the convective one. According to Miralles et al.(2000) cited by Luciano Rezzolla (2004), the estimated time $T_{nf} \sim 30 - 100$ *ms* and the flow velocity $v_L \sim 1 \times 10^4 \frac{m}{s}$ up to $3 \times 10^4 \frac{m}{s}$. But according to Thompson and Duncan (1993) the whole PNS to be convectively unstable with turbulent velocity $v_L \sim 10^8 \frac{cm}{sec}$.

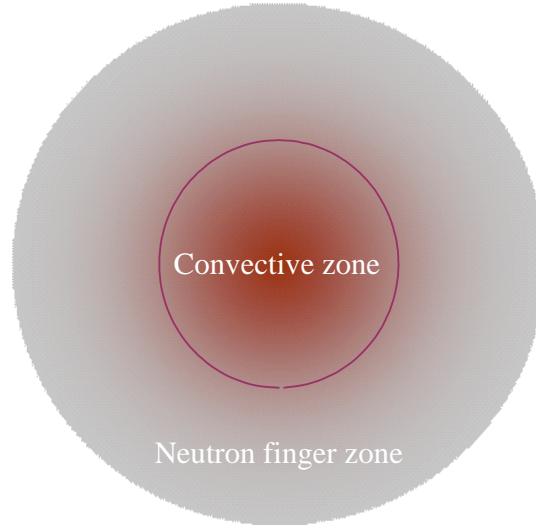


Fig. 2.1: Partitions of unstable region.

According to Narayan (1987) cited by Luciano Rezzolla (2004), PNS is rotating at a period $P \sim 100 \text{ ms}$. As a result the Rossby number R_0 is given by

$$R_0 = \frac{P}{T_L} \sim \left\{ \begin{array}{l} 1 \text{ neutron-finger zone} \\ 100 \text{ convective zone} \end{array} \right\}. \quad (2.87)$$

The existence of two unstable regions with different mean velocities is a considerable difference because the efficiency of the mean-field dynamo will be different in the two

regions, i.e. low in convective zone and high in neutron-finger zone. As a result, the influence of rotation on the turbulence is strong in neutron-finger zone whereas weak in the case of convective zone.

Generally, the evolution of these instabilities are summarized as follows [11]:

1. The instability first is developed in the outer layer and gradually transmitted to the inner layer;
2. after few seconds the two unstable zones move towards the inner portion of the stellar;
3. after ten seconds 90% of the stellar mass is in turbulence motion;
4. after 20 seconds the temperature and the lepton gradients are rapidly reduced and the two unstable regions begin to shrink, living the outer regions of the star;
5. after 30 seconds most of the PNS becomes stable; and
6. after 40 seconds the PNS becomes fully stable.

2.6.3 Dynamo Mechanism in Proto Neutron Stars

Since the characteristic cooling time-scale for the PNS $t \sim 40$ sec exceeds both $t_c = 0.1 - 1$ ms and $t_{nf} = 30 - 100$ ms (Miralles et al.2000), then the dynamo process (turbulence) can be treated adiabatically (without heat lost or gained). By modifying equation (2.70), the mean-field induction equation for turbulent, magnetized and conducting plasma can be given by

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B} + \alpha \vec{B}) - \nabla \times (\tilde{\eta} \nabla \times \vec{B}), \quad (2.88)$$

where α is a pseudo-scalar measuring the efficiency of the "alpha-dynamo" and $\tilde{\eta}$ is the turbulent magnetic diffusivity.

The velocity of the turbulent motion \vec{v} can be given by:

$$\vec{v} = \vec{\Omega} \times \vec{r}, \quad (2.89)$$

where $\vec{\Omega} = \vec{\Omega}(r) = \Omega_0 + r^2 \Omega_1$ is the differential rotation.

In equation (2.88), the coupling between the induction and Euler equations is neglected as a result the magnetic field evolving will grow through a kinetic dynamo process. A continuous velocity field \vec{v} is responsible for a dynamo when there exists an initial magnetic field configuration of a finite magnetic diffusivity.

The evolution of the magnetic field by making use of equation (2.70) yields the total magnetic energy of the form

$$\begin{aligned} E_M &= \frac{1}{2} \int_V (\vec{B} \cdot \vec{H}) d^3x, \\ &= \frac{1}{2\mu_0} \int_V B^2 d^3x \quad (\text{SI system}), \end{aligned} \quad (2.90)$$

or

$$E_M = \frac{1}{8\pi} \int_V B^2 d^3x \quad (\text{Gaussian system}), \quad (2.91)$$

where $\vec{H} = \frac{\vec{B}}{\mu}$ and $\mu = \mu_0 + 4\pi x_m \approx \mu_0$ since x_m is in the order of 10^{-5} up to 10^{-8} .

According to Rüdiger and Kitchatinov (1993), the α -parameter for rotating turbulence is,

$$\alpha \approx -\Omega l^2 \nabla \ln(\rho v_L^2), \quad (2.92)$$

where l is the length-scale, ρ is the density, Ω is the angular velocity and v_L is the turbulence velocity.

Since the density scale-height is proportional to the pressure-scale L , then we can express the isotropic turbulence in the neutron-finger zone as

$$\alpha_{nf} \approx \Omega_L = (\Omega_0 + r^2 \Omega_1) L. \quad (2.93)$$

For a given amount of differential rotation, the original magnetic field will increase if $\alpha_{nf} \succ \alpha_0$ and decrease if $\alpha_{nf} \prec \alpha_0$.

Two possible cases of dynamo mechanisms, namely, α^2 -dynamo and $\alpha\Omega$ -dynamo. The α^2 -dynamo process is corresponding to a small differential rotation and stationary magnetic field whereas the $\alpha\Omega$ -dynamo process is strongly attached to large range differential rotation and oscillating magnetic field.

The magnetic field generation through a mean-field dynamo action is possible if and

only if the stellar spin period is smaller than the critical spin period. The critical value of the spin period is given by

$$P_0 = \frac{2\pi L}{\alpha_0}, \quad (2.94)$$

where α_0 is the critical value of α -parameter.

The mean-field induction equation for a turbulent, magnetized and conducting plasma given in (2.88) has been solved in axisymmetry by Bonanno, Rezzolla and Uprin (2003) by changing steadily the degree of differential rotation $q=R^2 \frac{\Omega_1}{\Omega(r)}$ at $r=R$ to determine the critical value α_0 corresponding to the marginal stability of the dynamo.

Major cases of the degree of differential rotation are:

- a/ If $q > 0$, then the stellar surface rotates slower than the center.
- a/ If $q < 0$, then the stellar surface rotates faster than the center.
- c/ If $|q| < 1$, then the α^2 -dynamo dominates the process of generating the magnetic field. This leads to $\frac{B_{t0}}{B_{p0}} \sim 10$.
- d/ If $q < -1$, then it corresponds to anticlockwise rotation but not important.
- e/ If $|q| > 1$, then an oscillating $\alpha\Omega$ -dynamo dominates the magnetic field generation process. This leads to $\frac{B_{t0}}{B_{p0}} \sim 100 - 200$, B_{t0} is the magnetic field around the polar region (toroidal magnetic field) and B_{p0} is the magnetic field along the latitude (poloidal magnetic field).

Both results in cases (c) and (e) assure that the internal magnetic fields in neutron stars could be considerably larger than the surface value.

In general, we can say that as the star cools, convection and dynamo action cease. This happens after only about 10–20 seconds [9] or 30–40 sec (Miralles et al. 2000, 2002) in a neutron star, but 10 seconds is enough time for a very strong field to build up. After that, the field can remain trapped by the heavy, stratified liquid of neutrons and protons inside the neutron star. This led us to conclude that the radio pulsars were neutron stars in which large-scale dynamos had essentially failed to operate, probably because they were not born rotating fast enough whereas large-scale dynamo is operated in the case of magnetars due to their fast spinning when they born.

Chapter 3

Magnetic Field Decay of Neutron Stars

3.1 Introduction

Careful measurements and mathematical manipulations have shown that both pulsars and magnetars periods increase over time at different rates, implying that the neutron stars are gradually spinning down. This is due to the loss of energy by the emission of the EM radiation (predominantly dipolar) and gravitational radiation (usually quadrupolar) [4]. In this chapter we are going to derive the power dissipated due to gravitational and EM radiations, the decaying time of neutron's star magnetic field, the relation between decaying time and magnetic field, and the angular velocity of neutron stars [11], [20].

3.2 Magnetic Field Decay due to Magnetic Dipole Radiation

The crust of the neutron star has odd properties. It is very difficult to compress crust material very much, or to move elements of crust up and down, because strong gravity and pressure forces maintain a firm balance. But it is simpler to move parts of the crust horizontally, in which by applying only shear strains to it.

In the magnetar, the magnetic field is strong enough to push horizontally and twist the crust. The magnetic field lines outside the star also get twisted because they are anchored to the crust. The rotational energy of the magnetar quickly decreases after its birth. This crust's twist strengthens the current outside the magnetar and generates X-rays. As a result the magnetic field itself can provide an energy source for potentially observable emissions. This leads the magnetar to dissipate a significant amount of magnetic energy during the first ten thousand years. But in the radio pulsar, the magnetic field is not strong enough (compared to the magnetar) to push and twist the crust. As a result the field doesn't play a significant role to strengthen the current outside the radio pulsar and the magnetic energy is not dissipated very quickly. Pulsar's magnetic field is essentially stable; its main role is passively facilitate the loss of rotational energy [9].

As shown in chapter two, the magnetic moment is given by

$$\vec{m} = m_o(\sin \alpha \cos \Omega t \hat{e}_x + \sin \alpha \sin \Omega t \hat{e}_y + \cos \alpha \hat{e}_z). \quad (3.1)$$

If the magnetic dipole moment acts along z-axis, then only its perpendicular components generate radiation and expressed as

$$\begin{aligned} \vec{m} &= m_o(\sin \alpha \cos \Omega t \hat{e}_x + \sin \alpha \sin \Omega t \hat{e}_y), \\ &= R m_o \sin \alpha (\hat{e}_x + i \hat{e}_y) e^{-i\Omega t}, \end{aligned} \quad (3.2)$$

where R is real.

Let we consider only the magnetization term, we have the vector potential,

$$\vec{A}(r) = \frac{ik}{c} (\hat{n} \times \vec{m}) \left(1 - \frac{1}{ikr}\right) \frac{e^{ikr}}{r}, \quad (3.3)$$

where \hat{n} is a unit vector in the direction of the field point given by $\hat{n} = \frac{\vec{r}}{|\vec{r}|} = \hat{e}_r$.

Generally, the magnetic field can be computed as

$$\vec{B} = k^2 (\hat{n} \times \vec{m}) \times \hat{n} \frac{e^{ikr}}{r} + [3\hat{n}(\hat{n} \cdot \vec{m}) - \vec{m}] \left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{ikr} \quad (S.I)$$

or

$$\vec{B} = \nabla \times \vec{A} = \frac{\mu_0}{4\pi} \left\{ k^2 (\hat{n} \times \vec{m}) \times \hat{n} \frac{e^{ikr}}{r} + [3\hat{n}(\hat{n} \cdot \vec{m}) - \vec{m}] \left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{ikr} \right\} \quad (Gaussian). \quad (3.4)$$

The zonal magnetic fields are

$$\vec{B} = k^2(\hat{n} \times \vec{m}) \times \hat{n} \frac{e^{ikr}}{r} \quad (\text{radiation - zone, } kr \gg 1)$$

and

$$\vec{B} = \frac{1}{r^3}[3\hat{n}(\hat{n} \cdot \vec{m}) - \vec{m}] \quad (\text{near - zone, } kr \ll 1). \quad (3.5)$$

Since an electric field is a time dependent harmonic field, then one can use the Maxwell's equation [20]

$$\begin{aligned} \nabla \times \vec{E} &= i\Omega \vec{B}, \\ &= i\Omega(\nabla \times \vec{A}), \\ &= \nabla \times i\Omega \vec{A}. \end{aligned} \quad (3.6)$$

By using equations (3.3) and (3.6) we can solve the general expression for an electric field

$$\begin{aligned}
\vec{E} &= i\Omega\vec{A}, \\
&= i\Omega\left(\frac{ik}{c}\right)(\hat{n} \times \vec{m})\left(1 - \frac{1}{ik}\right)\frac{e^{ikr}}{r}, \\
&= -k^2(\hat{n} \times \vec{m})\left(1 - \frac{1}{ik}\right)\frac{e^{ikr}}{r},
\end{aligned} \tag{3.7}$$

where $k = \frac{\Omega}{c}$.

The zonal electric fields are

$$\vec{E} = -k^2(\hat{n} \times \vec{m})\frac{e^{ikr}}{r} \quad (\text{radiation - zone, } kr \gg 1) \tag{3.8}$$

and

$$\vec{E} = -ik(\hat{n} \times \vec{m})\frac{e^{ikr}}{r} \quad (\text{near - zone, } kr \ll 1). \tag{3.9}$$

Using the identity

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}). \tag{3.10}$$

By substituting equation (3.2) into radiation zone of electric and magnetic fields, we can obtain

$$\vec{E} = -k^2 m_o R \left[\sin \alpha \hat{e}_r \times (\hat{e}_x + i\hat{e}_y) \frac{e^{i(kr-\Omega t)}}{r} \right], \quad (3.11)$$

$$\vec{B} = \frac{k^2}{r} m_o R \{ \sin \alpha [\hat{e}_r \times (\hat{e}_x + i\hat{e}_y)] \times \hat{e}_r e^{i(kr-\Omega t)} \},$$

or

$$(3.12)$$

$$\vec{B} = \frac{k^2}{r} m_o R \sin \alpha \{ (\hat{e}_x + i\hat{e}_y) - \hat{e}_r [\hat{e}_r \cdot (\hat{e}_x + i\hat{e}_y)] \} e^{i(kr-\Omega t)}.$$

Using the relation

$$\hat{e}_x = \sin \theta \cos \phi \hat{e}_r + \cos \theta \cos \phi \hat{e}_\theta - \sin \phi \hat{e}_\phi$$

and

$$\hat{e}_y = \sin \theta \sin \phi \hat{e}_r + \sin \phi \cos \theta \hat{e}_\theta + \cos \phi \hat{e}_\phi. \quad (3.13)$$

Substituting equation (3.13) into equations (3.11) and (3.12), one can obtain the

electric and magnetic fields,

$$\begin{aligned}
\vec{E} &= -\frac{k^2}{r} m_o \sin \alpha R \left\{ \begin{array}{l} \hat{e}_r \times [(\sin \theta \cos \phi \hat{e}_r + \cos \theta \cos \phi \hat{e}_\theta - \sin \phi \hat{e}_\phi) + \\ i(\sin \theta \sin \phi \hat{e}_r + \sin \phi \cos \theta \hat{e}_\theta + \cos \phi \hat{e}_\phi)] e^{i(kr - \Omega t)} \end{array} \right\}, \\
&= -\frac{k^2}{r} m_o \sin \alpha R [(\sin \phi - i \cos \phi) \hat{e}_\theta + (\cos \theta \cos \phi + i \sin \phi \cos \theta) \hat{e}_\phi] e^{i(kr - \Omega t)}, \\
&= -\frac{k^2}{r} m_o \sin \alpha \left\{ \begin{array}{l} \cos(kr - \Omega t) [\sin \phi \hat{e}_\theta + \cos \theta \cos \phi \hat{e}_\phi] + \\ \sin(kr - \Omega t) [\cos \phi \hat{e}_\theta - \sin \phi \cos \theta \hat{e}_\phi] \end{array} \right\}, \tag{3.14}
\end{aligned}$$

$$\begin{aligned}
\vec{B} &= \frac{k^2}{r} m_o \sin \alpha R \left\{ \begin{array}{l} [(\sin \theta \cos \phi \hat{e}_r + \cos \theta \cos \phi \hat{e}_\theta - \sin \phi \hat{e}_\phi) + \\ i(\sin \theta \sin \phi \hat{e}_r + \sin \phi \cos \theta \hat{e}_\theta + \cos \phi \hat{e}_\phi)] (\hat{e}_r \cdot \hat{e}_r) - \\ \hat{e}_r [\hat{e}_r \cdot (\sin \theta \cos \phi \hat{e}_r + \cos \theta \cos \phi \hat{e}_\theta - \sin \phi \hat{e}_\phi) + \\ i(\sin \theta \sin \phi \hat{e}_r + \sin \phi \cos \theta \hat{e}_\theta + \cos \phi \hat{e}_\phi)] \end{array} \right\} e^{i(kr - \Omega t)}, \\
&= \frac{k^2}{r} m_o \sin \alpha R \left\{ \begin{array}{l} [(\cos \theta \cos \phi + i \sin \phi \cos \theta) \hat{e}_\theta + (i \cos \phi - \sin \phi) \hat{e}_\phi] \times \\ [\cos(kr - \Omega t) + i \sin(kr - \Omega t)] \end{array} \right\}, \\
&= \frac{k^2}{r} m_o \sin \alpha \left[\begin{array}{l} \cos(kr - \Omega t) (\cos \theta \cos \phi \hat{e}_\theta - \sin \phi \hat{e}_\phi) - \\ \sin(kr - \Omega t) (\sin \phi \cos \theta \hat{e}_\theta + \cos \phi \hat{e}_\phi) \end{array} \right]. \tag{3.15}
\end{aligned}$$

Since the emitted EM radiation is elliptically polarized, then we can choose $\phi = 0$ in the case of electric fields,

$$\vec{E} = -\frac{k^2}{r} m_o \sin \alpha [\sin(kr - \Omega t) \hat{e}_\theta + \cos(kr - \Omega t) \cos \theta \hat{e}_\phi]. \tag{3.16}$$

The vector \vec{S} , representing the power radiated per unit area, is called the poynting vector. It is given by

$$\vec{S} = \frac{c}{4\pi}(\vec{E} \times \vec{B}). \quad (3.17)$$

Substituting equations (3.15) and (3.16) into equation (3.17) we get

$$\begin{aligned} \vec{S} = \frac{c}{4\pi} \left(-\frac{k^4}{r^2} m_o^2 \sin^2 \alpha \right) & [\sin(kr - \Omega t) \hat{e}_\theta + \cos(kr - \Omega t) \hat{e}_\phi] \times \\ & [\cos(kr - \Omega t) (\cos \theta \cos \phi \hat{e}_\theta - \sin \phi \hat{e}_\phi) - \sin(kr - \Omega t) (\sin \phi \cos \theta \hat{e}_\theta + \cos \phi \hat{e}_\phi)]. \end{aligned} \quad (3.18)$$

Since the EM fields behave as radially outgoing waves for which $\left| \frac{Br}{B_\theta} \right| \sim \left| \frac{Br}{B_\phi} \right| \sim \frac{1}{\Omega} \ll 1$ [11], then equation (3.18) becomes

$$\vec{S} = \frac{ck^4}{4\pi r^2} m_o^2 \sin^2 \alpha [1 - \sin^2 \theta \cos^2(kr - \Omega t - \phi)] \hat{e}_r$$

or

$$\vec{S} = \frac{\Omega^4}{4\pi c^3 r^2} m_o^2 \sin^2 \alpha [1 - \sin^2 \theta \cos^2(kr - \Omega t - \phi)] \hat{e}_r. \quad (3.19)$$

The magnitude of the poynting vector along the poles is constant. This is called circular polarization at $\theta=0^\circ$. But the magnitude of the poynting vector varies along the equator between zero and some maximum value with linear polarization. Based on these facts, the poynting vector is given by:

$$\vec{S}_{\min} = \frac{\Omega^4}{4\pi c^3 r^2} m_o^2 \sin^2 \alpha \hat{e}_r \quad (\text{at } \theta = 0^\circ) \quad (3.20)$$

and

$$\vec{S}_{\max} = \frac{\Omega^4}{4\pi c^3 r^2} m_o^2 \sin^2 \alpha \cos^2 \theta \hat{e}_r \quad [\text{at } (kr - \Omega t - \phi) = 0]. \quad (3.21)$$

Generally, the power radiated from neutron stars through EM waves is,

$$\begin{aligned} P_{EM} &= \frac{dE}{dt} = \oint \oint \vec{S} \cdot d\vec{A} = \oint \oint \vec{S} \cdot \hat{n} dA, \\ &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} S r^2 \sin \theta d\theta d\phi, \end{aligned} \quad (3.22)$$

where $dA = r^2 \sin \theta d\theta d\phi$.

By substituting equation (3.20) into equation (3.22) we can compute the power radiated from neutron stars due to minimum poynting vector,

$$\begin{aligned} P_{1EM} &= \left(\frac{dE}{dt} \right)_1 = \frac{m_o^2 \Omega^4}{4\pi c^3} \sin^2 \alpha \int_0^{2\pi} \int_0^{\pi} \sin \theta d\theta d\phi, \\ &= \frac{m_o^2 \Omega^4}{c^3} \sin^2 \alpha, \end{aligned} \quad (3.23)$$

where $\int_0^{2\pi} \int_0^{\pi} \sin \theta d\theta d\phi = \int_0^{2\pi} [-\cos \theta]_0^{\pi} d\phi = [2\phi]_0^{2\pi} = 4\pi$.

By substituting equation (3.21) into equation (3.22), one can obtain, the power

radiated from neutron stars due to maximum poynting vector,

$$\begin{aligned}
P_{2EM} &= \left(\frac{dE}{dt}\right)_2 = \frac{m_o^2 \Omega^4}{4\pi c^3} \sin^2 \alpha \int_0^{2\pi} \int_0^\pi \cos^2 \theta \sin \theta d\theta d\phi, \\
&= \frac{m_o^2 \Omega^4}{3c^3} \sin^2 \alpha,
\end{aligned} \tag{3.24}$$

where $\int_0^{2\pi} \int_0^\pi \cos^2 \theta \sin \theta d\theta d\phi = \frac{-1}{3} \int_0^{2\pi} [\cos^3 \theta]_0^\pi d\phi = \frac{2}{3}(2\pi) = \frac{4}{3}\pi$.

Using equations (3.23) and (3.24), one can compute the total power (average power)

radiated from neutron stars,

$$\begin{aligned}
P_{EM} &= \frac{P_{1EM} + P_{2EM}}{2} = \frac{m_o^2 \Omega^4}{c^3} \sin^2 \alpha \left(\frac{1 + \frac{1}{3}}{2}\right), \\
&= \frac{-B_p^2 R^6 \Omega^4}{6c^3} \sin^2 \alpha,
\end{aligned} \tag{3.25}$$

where $m_o = \frac{1}{2} B_p R^3$ and the minus sign is introduced to indicate the power wastage.

3.3 Magnetic Field Decay due to Gravitational Quadrupole Radiation

Any EM wave can be decomposed into its constituent plane waves. With similar fashion, any gravitational wave can also be decomposed into a superposition of plane waves. In linearized theory one can consider a localized source of gravitational waves in steady

oscillation, radiating a periodic wave. But the exact theory insists that the energy of the sources decreases secularly, to counter balance the energy carried off by the radiation which has quadrupole nature. This quadrupole system is used to approximate the gravitational radiation in the non-relativistic case [4].

The gravitational waves cannot be dipolar because:

The gravitational analog of the electric dipole moment is the mass dipole moment,

$$\vec{d} = \sum_i m_i \vec{r}_i. \quad (3.26)$$

The first time derivative of the mass dipole moment is the total momentum of the system,

$$\dot{\vec{d}} = \sum_i m_i \dot{\vec{r}}_i = \vec{p}. \quad (3.27)$$

The second time derivative of the mass dipole moment is zero because of the conservation of linear momentum,

$$\ddot{\vec{d}} = \dot{\vec{p}} = 0. \quad (3.28)$$

The second type of moment is the magnetic dipole moment. The gravitational analog of the magnetic dipole moment is the angular momentum,

$$\vec{\mu} = \sum_i \vec{r}_i \times \vec{p}_i = \vec{J}. \quad (3.29)$$

The first time derivative of $\vec{\mu}$ is zero because of the law of conservation of angular momentum,

$$\dot{\vec{\mu}} = \dot{\vec{J}} = 0. \quad (3.30)$$

Therefore there can be no gravitational dipole radiation of any kind.

The next possible moment which is applicable in the case of gravitational radiation is electric-quadrupole moment.

If the neutron star is rotating rigidly about z-axis with angular velocity $\vec{\Omega}$, then one can transform the coordinate from S frame to S' frame or vice versa.

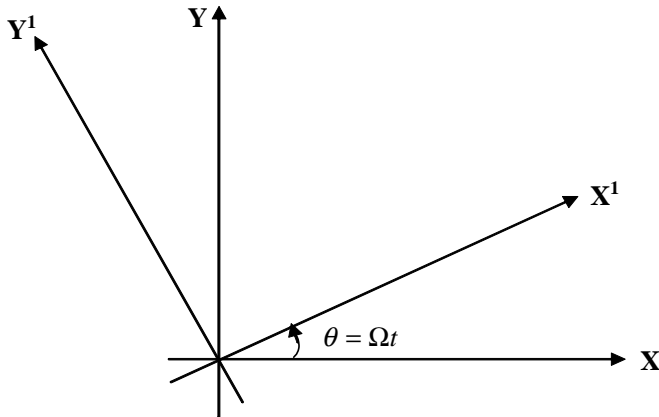


Fig. 3.1: Rotation of frame of reference by angle $\theta = \Omega t$ in x-y plane.

Position in S and S'-frames

$$x_1 = x_1' \cos \theta - x_2' \sin \theta,$$

$$x_1' = x_1 \cos \theta + x_2 \sin \theta,$$

$$x_2 = x_1' \sin \theta + x_2' \cos \theta,$$

$$x_2' = -x_1 \sin \theta + x_2 \cos \theta, \tag{3.31}$$

$$x_3 = x_3', \quad \text{and} \quad x_3' = x_3.$$

The moment of inertia in fixed coordinate system (S-frame) or the second moment of the mass distribution is,

$$I_{ij} = \int x_i' x_j' \rho(x') d^3 x', \tag{3.32}$$

where $\rho(x') = T^{oo}(x, t)$ is the rest mass density in S' frame.

A case of special importance in the study of star's rotation is one where there is rotational symmetry about an axis, which we call the x_3' -axis. Rotational symmetry here implies that one can not distinguish one orientation of the distribution about the x_3' -axis from any other. In such case $I_{12}=I_{13}=I_{23}=0$ or in short all the "off-diagonal" can be made to equal to zero [21].

The moment of inertia of the rotating star or the components of the star's reduced quadrupole moment is,

$$I_{ij}^l(t) = \int x_i x_j T^{00}(x, t) d^3 x. \quad (3.33)$$

Using equations (3.31) and (3.33) one can derive the components of $I_{ij}^l(t)$ as,

$$\begin{aligned} I_{ij}^l(t) &= \int x_i x_j \rho(x') d^3 x', \\ I_{11}^l(t) &= \int (x_1^2 \cos^2 \Omega t - 2x_1 x_2 \sin \Omega t \cos \Omega t + x_2^2 \sin^2 \Omega t) \rho(x') d^3 x', \\ &= I_{11} \cos^2 \Omega t - 2I_{12} \sin \Omega t \cos \Omega t + I_{22} \sin^2 \Omega t, \\ &= I_{11} \left(\frac{1 + \cos 2\Omega t}{2} \right) + I_{22} \left(\frac{1 - \cos 2\Omega t}{2} \right), \\ &= \frac{1}{2}(I_{11} + I_{22}) + \frac{1}{2}(I_{11} - I_{22}) \cos 2\Omega t, \end{aligned} \quad (3.34)$$

$$\begin{aligned} I_{12}^l(t) &= \int (x_1 \cos \Omega t - x_2 \sin \Omega t)(x_1 \sin \Omega t + x_2 \cos \Omega t) \rho(x') d^3 x', \\ &= \int (x_1^2 \sin \Omega t \cos \Omega t - x_1 x_2 \sin^2 \Omega t + x_1 x_2 \cos^2 \Omega t - x_2^2 \sin \Omega t \cos \Omega t) \rho(x') d^3 x', \\ &= I_{11} \sin \Omega t \cos \Omega t - I_{12} \sin^2 \Omega t + I_{12} \cos^2 \Omega t - I_{22} \sin \Omega t \cos \Omega t, \\ &= \frac{1}{2}(I_{11} - I_{22}) \sin 2\Omega t, \end{aligned} \quad (3.35)$$

where $I_{12} = 0$, $\cos^2 \Omega t = \frac{1 + \cos 2\Omega t}{2}$, $\sin^2 \Omega t = \frac{1 - \cos 2\Omega t}{2}$, and $\sin 2\Omega t = 2 \sin \Omega t \cos \Omega t$.

With the same fashion one can also derive the other components,

$$\begin{aligned}
I_{13}^l(t) &= I_{31}^l(t) = I_{23}^l(t) = I_{32}^l(t) = 0, \\
I_{21}^l(t) &= I_{12}^l(t) = \frac{1}{2}(I_{11} - I_{22}) \sin 2\Omega t, \\
I_{22}^l(t) &= \frac{1}{2}(I_{11} + I_{22}) + \frac{1}{2}(I_{22} - I_{11}) \cos 2\Omega t,
\end{aligned} \tag{3.36}$$

and

$$I_{33}^l(t) = I_{33}.$$

Using Fourier transformation one can write the quadrupole moment as,

$$I_{ij}^l(t) = \sum e^{-i\omega t} I_{ij}^l(\omega). \tag{3.37}$$

Due to the equilateral oblateness of the star, the quadrupole radiation undergoes two oscillations per rotation of the star ($\omega = 2\Omega$) [11].

The Fourier component for the particular frequency is given by,

$$I_{ij}^l(2\Omega) = I_{ij}^l(t) e^{2i\Omega t}. \tag{3.38}$$

Using equations (3.34), (3.35), and (3.38) one can derive the components $I_{11}^l(2\Omega)$ and $I_{12}^l(2\Omega)$:

$$\begin{aligned}
I'_{11}(2\Omega) &= I'_{11}(t)e^{2i\Omega t}, \\
&= \frac{1}{2}(I_{11} + I_{22})e^{2i\Omega t} + \frac{1}{2}(I_{11} - I_{22})\cos 2\Omega t e^{2i\Omega t}, \\
&= \frac{1}{2}(I_{11} + I_{22})e^{2i\Omega t} + \frac{1}{2}i(I_{11} - I_{22})\sin 2\Omega t \cos 2\Omega t + \frac{1}{2}(I_{11} - I_{22})\cos^2 2\Omega t,
\end{aligned} \tag{3.39}$$

$$\begin{aligned}
I'_{12}(2\Omega) &= I'_{21}(2\Omega) = I'_{12}(t)e^{2i\Omega t} = I'_{21}(t)e^{2i\Omega t}, \\
&= \frac{1}{2}(I_{11} - I_{22})\sin 2\Omega t e^{2i\Omega t}, \\
&= \frac{1}{2}(I_{11} - I_{22})(\sin 2\Omega t \cos 2\Omega t + i \sin^2 2\Omega t).
\end{aligned} \tag{3.40}$$

In a similar way one can also derive the other components,

$$\begin{aligned}
I'_{13}(2\Omega) &= I'_{31}(2\Omega) = I'_{23}(2\Omega) = I'_{32}(2\Omega) = 0, \\
I'_{22}(2\Omega) &= \frac{1}{2}(I_{11} + I_{22})e^{2i\Omega t} + \frac{1}{2}(I_{22} - I_{11})(\cos^2 2\Omega t + i \sin 2\Omega t \cos 2\Omega t), \\
&\text{and} \\
I'_{33}(2\Omega) &= I'_{33}e^{2i\Omega t}.
\end{aligned} \tag{3.41}$$

Now, over a complete period oscillation, the time average functions are given as [23],

$$\begin{aligned}
\langle \cos^2 2\Omega t \rangle &= \langle \sin^2 2\Omega t \rangle = \frac{1}{2}, \\
\langle \sin 2\Omega t \cos 2\Omega t \rangle &= \langle e^{2i\Omega t} \rangle = 0.
\end{aligned}
\tag{3.42}$$

Using the conditions in equation (3.42), the time-average of the quadrupole components are given by,

$$\begin{aligned}
\langle I_{11}'(2\Omega) \rangle &= \frac{1}{4} (I_{11} - I_{22}), \\
\langle I_{12}'(2\Omega) \rangle &= \langle I_{21}'(2\Omega) \rangle = \frac{i}{4} (I_{11} - I_{22}), \\
\langle I_{22}'(2\Omega) \rangle &= \frac{1}{4} (I_{22} - I_{11}),
\end{aligned}
\tag{3.43}$$

and

$$\langle I_{13}'(2\Omega) \rangle = \langle I_{31}'(2\Omega) \rangle = \langle I_{23}'(2\Omega) \rangle = \langle I_{32}'(2\Omega) \rangle = \langle I_{33}'(2\Omega) \rangle = 0.$$

The total gravitational power radiated by a rotating neutron star can be given by [25],

$$\begin{aligned}
P_{GW} &= \frac{2G\omega^6}{5c^5} \left[I'_{ij}(\omega)^* I'_{ij}(\omega) - \frac{1}{3} |I'_{ij}(\omega)|^2 \right], \\
&= \frac{128G\Omega^6}{5c^5} \left[I'_{ij}(2\Omega)^* I'_{ij}(2\Omega) - \frac{1}{3} |I'_{ij}(2\Omega)|^2 \right],
\end{aligned} \tag{3.44}$$

where $\omega = 2\Omega$.

By substituting equation (3.43) into equation (3.44), we can derive the gravitational power radiated by a quadrupole field,

$$\begin{aligned}
P_{GW} &= \frac{128G\Omega^6}{5c^5} \left[\begin{aligned} &\frac{1}{16} (I_{11} - I_{22})^2 - \frac{1}{48} (I_{11} - I_{22})^2 + \frac{1}{16} (I_{11} - I_{22})^2 \\ &+ \frac{1}{48} (I_{11} - I_{22})^2 + \frac{1}{16} (I_{11} - I_{22})^2 + \frac{1}{48} (I_{11} - I_{22})^2 \\ &+ \frac{1}{16} (I_{11} - I_{22})^2 - \frac{1}{48} (I_{11} - I_{22})^2 \end{aligned} \right], \\
&= \frac{128G\Omega^6}{5c^5} \left[\frac{1}{4} (I_{11} - I_{22})^2 \right], \\
&= \frac{-32G\Omega^6 I^2 \epsilon^2}{5c^5},
\end{aligned} \tag{3.45}$$

where $\epsilon = \frac{I_{11} - I_{22}}{I}$ is the stellar equatorial gravitational oblateness and the negative sign is introduced to indicate the power dissipation.

3.4 The Decaying Time of Neutron's Star Magnetic Field

3.4.1 Age of Radiation Source Neutron Stars

Neutron stars are born in supernova explosions at the end point of the evolution of massive stars usually $M_s \geq 8 M_\odot$ [8]. Initially the angular velocity of the star is determined by the conservation of the angular momentum of its core and the conditions accompanying its birth. Careful measurements have shown that a rotating neutron star periods increase over time, implying that the star is gradually spinning down due to the lost in its energy through radiation.

Assuming the spin rate follows a generic power-law dependence as [14]

$$\dot{\Omega} = -k \Omega^n. \quad (3.46)$$

Substituting $\Omega = \frac{2\pi}{p}$ in equation (3.46), one can obtain

$$\dot{P} = k \frac{(2\pi)^{n-1}}{p^{n-2}}, \quad (3.47)$$

where $n=3$ for magnetic dipole electromagnetic radiation and $n=5$ for quadrupole gravitational radiation.

Next integrating equations (3.46) and (3.47) by taking $t_o = 0$ at birth, Ω_o as the initial

spin, and P_o as the initial period, then the time, t in terms of angular velocity and period are given by:

$$\begin{aligned}
t &= \frac{1}{(n-1)k} \left(\frac{1}{\Omega^{n-1}} - \frac{1}{\Omega_o^{n-1}} \right), \\
&= \frac{-1}{(n-1)} \frac{\Omega^n}{\dot{\Omega}} \left(\frac{1}{\Omega^{n-1}} - \frac{1}{\Omega_o^{n-1}} \right), \\
&= \frac{-1}{(n-1)} \frac{\Omega}{\dot{\Omega}} \left[1 - \left(\frac{\Omega}{\Omega_o} \right)^{n-1} \right], \tag{3.48}
\end{aligned}$$

$$\begin{aligned}
t &= \frac{1}{k(2\pi)^{n-1}} \left(\frac{p^{n-1}}{n-1} - \frac{p_o^{n-1}}{n-1} \right), \\
&= \frac{1}{\dot{p} p^{n-2}} \left(\frac{p^{n-1}}{n-1} - \frac{p_o^{n-1}}{n-1} \right), \\
&= \frac{1}{(n-1)} \frac{p}{\dot{p}} \left[1 - \left(\frac{p_o}{p} \right)^{n-1} \right]. \tag{3.49}
\end{aligned}$$

After a long time, such that $\Omega \ll \Omega_o$ or $P \gg P_o$, equations (3.48) and (3.49) are reduced to give an estimate of the present age of the rotating neutron star,

$$T_{age} = T = \frac{1}{(1-n)} \frac{\Omega}{\dot{\Omega}} = \frac{1}{(n-1)} \frac{p}{\dot{p}}. \tag{3.50}$$

The ‘‘dipole’’ and ‘‘quadrupole’’ ages are given, respectively as

$$T_{dipole} = \frac{-1}{2} \frac{\Omega}{\dot{\Omega}} = \frac{1}{2} \frac{p}{\dot{p}} \quad (n=3) \tag{3.51}$$

and

$$T_{quad} = \frac{-1}{4} \frac{\dot{\Omega}}{\Omega} = \frac{1}{4} \frac{\dot{p}}{p} \quad (n = 5). \quad (3.52)$$

Using equation (3.49) one can write the general expression for the age of the radiation source neutron stars as,

$$P_{(t)} = [p_0^{n-1} + (2\pi)^{n-1}(n-1)kt]^{1/(n-1)}, \quad (3.53)$$

where the initial period $p_o = p(1 - \frac{t}{T})^{1/(n-1)} = p[1 - (n-1)\frac{\dot{p}}{p}t]^{1/(n-1)}$ can be derived by rearranging equation (3.49).

3.4.2 The relation between Magnetic Field and Decaying Time among Neutron Stars

As described in the introduction part, the measured magnetic fields at the poles and the decaying time are $10^{12} - 10^{13}$ Gauss and 10 million years, respectively in the case of radio pulsars whereas $10^{14} - 10^{15}$ Gauss and 10,000 years, respectively in the case of magnetars. From this astronomical data one can observe that, the pulsar remains radio “loud” for a longer time than magnetars and the stronger the original magnetic field of the neutron star is the shorter will be the decaying time or vice versa. Therefore, we can conclude that the magnetic field is a nonlinear time dependent variable. This approach leads to formulate (derive) the fundamental relationship between magnetic field and decaying time.

Since the magnetic field is a nonlinear time dependent variable then one can apply the generic power-law dependence for magnetic field rate as,

$$\frac{dB(t)}{dt} = \dot{B}(t) = -kB(t)^n, \quad (3.54)$$

where n is the braking index.

The rate of the magnetic field decay due to EM radiation is given by:

$$\frac{dB(t)}{dt} = \dot{B}(t) = -kB(t)^3, \quad (3.55)$$

where n=3 for magnetic dipole.

Equation (3.55) may be written of the form

$$\frac{dB(t)}{B(t)^3} = -kdt. \quad (3.56)$$

On integrating equation (3.56), we can obtain

$$-\frac{1}{2}B(t)^{-2} = -kt + C, \quad (3.57)$$

where t is decaying time.

We now solve the value of the constant of integration by use of the initial condition, $B(t=0)=B_0$. We obtain $-\frac{1}{2}B(0)^{-2} = -\frac{1}{2}B_0^{-2} = C$; hence equation (3.57) becomes

$$-\frac{1}{2}B(t)^{-2} = -kt - \frac{1}{2}B_0^{-2}$$

or

$$\frac{1}{2}B(t)^{-2} = kt + \frac{1}{2}B_0^{-2}. \quad (3.58)$$

Solving for k from equation (3.55) and substituting into equation (3.58),

$$\frac{1}{2}B(t)^{-2} = -B(t)\dot{B}(t)B(t)^{-3}t + \frac{1}{2}B_0^{-2}. \quad (3.59)$$

On multiplying equation (3.59) by $\frac{B(t)^3}{t}$, we can obtain

$$\frac{1}{2t}B(t) = -B(t)\dot{B}(t) + \frac{1}{2B_0^2t}B(t)^3$$

or

$$B(t)\dot{B}(t) + \frac{1}{2t}B(t) = \frac{1}{2B_0^2t}B(t)^3. \quad (3.60)$$

Equation (3.60) is a nonlinear equation since it contains a nonlinear term in the dependent variable, $B(t)^3$. This equation can be reduced to the linear form. To reduce equation (3.60) to the linear form, we first write in general form and make a change of variable.

The standard form of a nonlinear differential equation is given by

$$\dot{B}(t) + P(t)B(t) = Q(t)B(t)^n \quad (n \neq 1). \quad (3.61)$$

Let

$$U(t) = B(t)^{1-n}, \quad (3.62)$$

so that

$$\dot{U}(t) = (1-n)B(t)^{-n}\dot{B}(t). \quad (3.63)$$

Multiplying equation (3.61) by $(1-n)B(t)^{-n}$, one can obtain

$$(1-n)B(t)^{-n}\dot{B}(t) + (1-n)B(t)^{1-n}P(t) = (1-n)Q(t). \quad (3.64)$$

In terms of the variable $U(t)$, we can write

$$\dot{U}(t) + (1-n)P(t)U(t) = (1-n)Q(t). \quad (3.65)$$

Let we multiply equation (3.65) by $e^{(1-n)\int P(t)dt}$ (integrating factor) so that

$$\left[\dot{U}(t) + (1-n)P(t)U(t) \right] e^{(1-n)\int P(t)dt} = (1-n)Q(t)e^{(1-n)\int P(t)dt}. \quad (3.66)$$

Since

$$\frac{d}{dt} \left[U(t) e^{(1-n) \int P(t) dt} \right] = \left[\dot{U}(t) + (1-n) P(t) U(t) \right] e^{(1-n) \int P(t) dt}, \quad (3.67)$$

equation (3.66) may be written as

$$\frac{d}{dt} \left[U(t) e^{(1-n) \int P(t) dt} \right] = (1-n) Q(t) e^{(1-n) \int P(t) dt}. \quad (3.68)$$

By integrating equation (3.68), we can obtain

$$U(t) e^{(1-n) \int P(t) dt} = \int (1-n) Q(t) e^{(1-n) \int P(t) dt} dt + C$$

or

$$U(t) = e^{-(1-n) \int P(t) dt} \int (1-n) Q(t) e^{(1-n) \int P(t) dt} dt + C e^{-(1-n) \int P(t) dt}. \quad (3.69)$$

Equation (3.60) has the standard form of

$$P(t) \longrightarrow \frac{1}{2t}; \quad Q(t) \longrightarrow \frac{1}{2B_0^2 t}; \quad n \longrightarrow 3.$$

On applying equation (3.69), we get

$$U(t) = e^{2 \int \frac{1}{2t} dt} \int (-2) \frac{1}{2B_0^2 t} e^{-2 \int \frac{1}{2t} dt} dt + C e^{2 \int \frac{1}{2t} dt},$$

$$\begin{aligned}
&= \left(-\frac{1}{B_0^2}\right) e^{\ln t} \int \frac{1}{t} e^{-\ln t} dt + C e^{\ln t}, \\
&= -\frac{1}{B_0^2} t \int \frac{1}{t^2} dt + Ct, \\
&= -\frac{t}{B_0^2} \left(\frac{-1}{t}\right) + Ct, \tag{3.70} \\
&= \frac{1}{B_0^2} + Ct.
\end{aligned}$$

Since $U(t) = B(t)^{(1-n)}$, we have

$$U(t) = B(t)^{(1-3)} = B(t)^{-2} = \frac{1}{B_0^2} + Ct. \tag{3.71}$$

By rearranging equation (3.71), we can obtain the magnetic field at any time t ,

$$B(t) = \frac{B_0}{\sqrt{1 + CB_0^2 t}}$$

or

$$B(t) \propto \frac{1}{\sqrt{t}}. \tag{3.72}$$

Based on equation (3.72), one can deduce that for radiation emitting neutron stars, the magnetic field $B(t)$ is inversely proportional to the square root of the decaying time

t.

By using equation (3.71), we can also derive an expression for integrating constant C,

$$C = \frac{B_0^2 - B(t)^2}{B_0^2 B(t)^2 t}. \quad (3.73)$$

Given the observed original magnetic field, $B_{0M}=10^{15}$ G, the decaying time, $t_{DM}=10^4$ yrs, and the threshold surface magnetic field, $B_{TM}=10^7$ G [13] or $B_{TM}=10^8$ G [9] of the magnetar, then by substituting these into equation (3.73) we can obtain the integrating constant for the magnetar C_M ,

$$C_M = \frac{B_{0M}^2 - B_{TM}^2}{B_{0M}^2 B_{TM}^2 t_{DM}} \approx 1 \times 10^{-18} / G^2 yr \quad (B_{TM} = 10^7 G)$$

or

$$C_M \approx 1 \times 10^{-20} / G^2 yr \quad (B_{TM} = 10^8 G). \quad (3.74)$$

With the same fashion, given the observed original magnetic field, $B_{0P}=10^{12}$ G, the decaying time, $t_{DP}=10^7$ yrs, and the threshold surface magnetic field, $B_{TP}=10^7$ G [13] or $B_{TP}=10^8$ G [9] of the radio pulsar, then by substituting these into equation (3.73) we can also obtain the integrating constant for the radio pulsar C_P ,

$$C_P = \frac{B_{0P}^2 - B_{TP}^2}{B_{0P}^2 B_{TP}^2 t_{DP}} \approx 1 \times 10^{-21} / G^2 yr \quad (B_{TP} = 10^7 G)$$

or

$$C_P \approx 1 \times 10^{-23} / G^2 yr \quad (B_{TP} = 10^8 G). \quad (3.75)$$

With similar fashion the rate of magnetic field decay due to gravitational quadrupole radiation can be given by:

$$\frac{dB(t)}{dt} = \dot{B}(t) = -kB(t)^5, \quad (3.76)$$

where n=5 for gravitational quadrupole radiation.

Equation (3.76) may be written of the form

$$\frac{dB(t)}{B(t)^5} = -kdt. \quad (3.77)$$

On integrating equation (3.77), we can obtain

$$-\frac{1}{4}B(t)^{-4} = -kt + C, \quad (3.78)$$

where t is decaying time.

We now solve the value of the constant of integration by use of the initial condition, B(t=0)=B₀. We obtain $-\frac{1}{4}B(0)^{-4} = -\frac{1}{4}B_0^{-4} = C$; hence equation (3.78) becomes

$$-\frac{1}{4}B(t)^{-4} = -kt - \frac{1}{4}B_0^{-4}$$

or

$$\frac{1}{4}B(t)^{-4} = kt + \frac{1}{4}B_0^{-4}. \quad (3.79)$$

Solving for k from equation (3.76) and substituting into equation (3.79),

$$\frac{1}{4}B(t)^{-4} = -B(t)\dot{B}(t)B(t)^{-5}t + \frac{1}{4}B_0^{-4}. \quad (3.80)$$

On multiplying equation (3.80) by $\frac{B(t)^5}{t}$, we can obtain

$$\frac{1}{4t}B(t) = -B(t)\dot{B}(t) + \frac{1}{4B_0^4t}B(t)^5$$

or

$$B(t)\dot{B}(t) + \frac{1}{4t}B(t) = \frac{1}{4B_0^4t}B(t)^5. \quad (3.81)$$

Equation (3.81) is a nonlinear equation since it contains a nonlinear term in the dependent variable, $B(t)^5$. This equation can be reduced to the linear form. To reduce equation (3.81) to the linear form, we first write in the standard form of

$$P(t) \longrightarrow \frac{1}{4t}; \quad Q(t) \longrightarrow \frac{1}{4B_0^4t}; \quad n \longrightarrow 5.$$

On applying equation (3.69), we get

$$\begin{aligned}
U(t) &= e^{-(1-5) \int \frac{1}{4t} dt} \int (1-5) \frac{1}{4B_0^4 t} e^{(1-5) \int \frac{1}{4t} dt} dt + C e^{-(1-5) \int \frac{1}{4t} dt}, \\
&= \left(-\frac{1}{B_0^4} \right) e^{\ln t} \int \frac{1}{t} e^{-\ln t} dt + C e^{\ln t}, \\
&= -\frac{1}{B_0^4} t \int \frac{1}{t^2} dt + Ct, \\
&= -\frac{t}{B_0^4} \left(\frac{-1}{t} \right) + Ct, \tag{3.82} \\
&= \frac{1}{B_0^4} + Ct.
\end{aligned}$$

Since $U(t) = B(t)^{(1-n)}$, we have

$$U(t) = B(t)^{(1-5)} = B(t)^{-4} = \frac{1}{B_0^4} + Ct. \tag{3.83}$$

By rearranging equation (3.83), we can obtain the magnetic field at any time t ,

$$B(t) = \frac{B_0}{(1 + CB_0^4 t)^{\frac{1}{4}}}$$

or

$$B(t) \propto \frac{1}{t^{\frac{1}{4}}}. \quad (3.84)$$

Based on equation (3.84), one can conclude that for gravitational quadrupole radiation emitting neutron stars, the magnetic field $B(t)$ is inversely proportional to the fourth root of the decaying time, t .

By using equation (3.83), we can also derive an expression for integrating constant C in the case of gravitational quadrupole radiation,

$$C = \frac{B_0^4 - B(t)^4}{B_0^4 B(t)^4 t}. \quad (3.85)$$

Given the observed original magnetic field, $B_{0M}=10^{15}$ G, the decaying time, $t_{DM}=10^4$ yrs, and the threshold surface magnetic field, $B_{TM}=10^7$ G [13] or $B_{TM}=10^8$ G [9] of the magnetar, then by substituting these into equation (3.85) one can obtain the integrating constant for the magnetar C_M ,

$$C_M = \frac{B_{0M}^4 - B_{TM}^4}{B_{0M}^4 B_{TM}^4 t_{DM}} \approx 1 \times 10^{-32}/G^2 yr \quad (B_{TM} = 10^7 G)$$

or

$$C_M \approx 1 \times 10^{-36}/G^2 yr \quad (B_{TM} = 10^8 G). \quad (3.86)$$

With the same fashion, given the observed original magnetic field, $B_{0P}=10^{12}$ G, the

decaying time, $t_{DP}=10^7$ yrs, and the threshold surface magnetic field, $B_{TP}=10^7$ G [13] or $B_{TP}=10^8$ G [9] of the radio pulsar, then by substituting these into equation (3.85) we can also obtain the integrating constant for the radio pulsar C_P ,

$$C_P = \frac{B_{0P}^4 - B_{TP}^4}{B_{0P}^4 B_{TP}^4 t_{DP}} \approx 1 \times 10^{-35} / G^2 yr \quad (B_{TP} = 10^7 G)$$

or

$$C_P \approx 1 \times 10^{-39} / G^2 yr \quad (B_{TP} = 10^8 G). \quad (3.87)$$

3.5 The Angular Velocity of Neutron Stars

3.5.1 The Initial Angular Velocity of Neutron Stars

All star rotate, some slowly, some fast, some very fast. They have a specific angular momentum. When proto neutron star is transformed into neutron star, the angular velocity increases for the same reason that a diver rotates faster when arms and legs are tucked in than when they are outstretched. Angular momentum is conserved.

If a neutron star is a homogeneous sphere of radius R rotating at angular velocity Ω , then its angular momentum is given by

$$L = \frac{2}{5} M_s \Omega R^2. \quad (3.88)$$

Using equation (3.88) and the law of conservation of angular momentum, one can compute the original velocity of a new born rotating neutron star Ω_o ,

$$\Omega_o = \Omega_p \left(\frac{R_p}{R}\right)^2 \quad (R \gg R_p), \quad (3.89)$$

where R_p is the radius of the proto neutron star, R is the radius of the neutron star and Ω_p is the angular velocity of the proto neutron star.

3.5.2 The Spin Evolution of Neutron Stars

The rotational energy dissipated per unit time for a neutron star is

$$\frac{dE}{dt} = \frac{d}{dt} \left(\frac{1}{2} I \Omega^2 \right) = I \Omega \frac{d\Omega}{dt} = I \Omega \dot{\Omega}, \quad (3.90)$$

where I is the moment of inertia assumed to be constant.

In the previous topic equation (3.25), we have derived the power dissipated by magnetic dipole radiation as,

$$P_{EM} = \frac{-B_p^2 R^6 \Omega^4}{6c^3} \sin^2 \alpha. \quad (3.91)$$

By assuming that the EM losses are tapping energy from the rotational kinetic energy, we can equate equations (3.90) and (3.91) and integrating yields,

$$\begin{aligned}
\frac{I\Omega d\Omega}{dt} &= \frac{-B_p^2 R^6 \Omega^4}{6c^3} \sin^2 \alpha, \\
\int_{\Omega_i}^{\Omega_f} \frac{d\Omega}{\Omega^3} &= -\int_0^t \left(\frac{B_p^2 R^6 \sin^2 \alpha}{6c^3 I} \right) dt, \\
\frac{-1}{2} (\Omega_{(t)}^{-2} - \Omega_i^{-2}) &= -\left(\frac{B_p^2 R^6 \sin^2 \alpha}{6c^3 I} \right) t, \\
\Omega_{(t)}^{-2} &= \Omega_i^{-2} + 2 \left(\frac{B_p^2 R^6 \sin^2 \alpha}{6c^3 I} \right) t.
\end{aligned} \tag{3.92}$$

Again equate equations (3.90) and (3.91) at present time and solve for moment of inertia I,

$$\begin{aligned}
I\Omega_0 \dot{\Omega}_o &= \frac{B_p^2 R^6 \Omega_o^4}{6c^3} \sin^2 \alpha, \\
I &= \frac{B_p^2 R^6 \Omega_o^3}{6c^3 \dot{\Omega}_o} \sin^2 \alpha.
\end{aligned} \tag{3.93}$$

By substituting equation (3.93) into equation (3.92), one can obtain

$$\begin{aligned}
\Omega_{(t)}^{-2} &= \Omega_i^{-2} + 2 \frac{\dot{\Omega}_o}{\Omega_o^3} t, \\
\Omega_{(t)} &= \Omega_i \left(1 + 2 \frac{\Omega_i^2}{\Omega_o^2} \frac{t}{T_{EM}} \right)^{-\frac{1}{2}},
\end{aligned} \tag{3.94}$$

where $\Omega(t)$ is future spin, Ω_0 is present spin, Ω_i is initial spin, and $T_{EM} = \frac{\Omega_0}{\dot{\Omega}_o}$ is the characteristic age at present time, and $t \approx \frac{T_{EM}}{2}$ is the present age of a neutron star.

With similar fashion what done for a neutron star being spun-down by dipolar EM radiation and after defining the characteristic age at present time as $T_{GW} = \frac{\Omega_o}{\dot{\Omega}_o}$, we can integrate equation (3.45) for the gravitational energy-loss rate to get the spin evolution of a neutron star.

$$P_{GW} = \frac{-32}{5c^5} G \Omega^6 I^2 \epsilon^2 = I \Omega \frac{d\Omega}{dt}. \quad (3.95)$$

Integrating this

$$\begin{aligned} \int_{\Omega_i}^{\Omega_f} \frac{d\Omega}{\Omega^5} &= \frac{-32}{5} G I \frac{\epsilon^2}{c^5} \int_0^t dt, \\ \Omega^{-4}(t) &= \Omega_i^{-4} + 4 \left(\frac{32}{5} \frac{G I \epsilon^2}{c^5} t \right), \\ &= \Omega_i^{-4} \left[1 + 4 \left(\frac{32}{5} \right) \frac{G I \epsilon^2}{c^5} \Omega_i^4 t \right]. \end{aligned} \quad (3.96)$$

Taking equation (3.95) at present time and rearranging the expression,

$$I \Omega_o \dot{\Omega}_o = \frac{32}{5} \frac{G \Omega_o^6}{c^5} I^2 \epsilon^2, \quad (3.97)$$

$$\frac{\dot{\Omega}}{\Omega_o^5} = \frac{32G}{5c^5} I \epsilon^2. \quad (3.98)$$

Substituting equation (3.98) into equation (3.96), one can get

$$\Omega^{-4} = \Omega_i^{-4} \left(1 + 4 \frac{\dot{\Omega}_o \Omega_i^4}{\Omega_o^5} t \right), \quad (3.99)$$

$$\Omega_{(t)} = \Omega_i \left(1 + 4 \frac{\Omega_i^4}{\Omega_o^4} \frac{t}{T_{GW}} \right)^{-\frac{1}{4}}, \quad (3.100)$$

where $T_{GW} = \frac{\Omega_o}{\dot{\Omega}_o}$.

Chapter 4

Summary and Conclusion

4.1 Summary

For many years the magnetic fields in neutron stars have been simply related to the fossil remnant frozen during collapse (Woltjer,1964) and amplified by flux conservation. This approach is now challenged by the shortage of having sufficiently strong seed fields in the fossil remnant frozen during collapse and many views are merging out such as dynamo process, spinning effect and thermomagnetic effect, etc. The difficulty of having sufficiently strong seed fields in the progenitors can be solved by analyzing the possibility that neutron star magnetic fields are generated due to fossil remnant plus small-scale dynamo action in the case of radio pulsars and due to fossil remnant plus large scale dynamo action in the case of magnetars. According to Gaensler and his team [26], they suggest that heavy stars will form neutron stars (magnetars) spinning at up to 500-1000

times per second. Such rapid rotation should power a large-scale dynamo and generate a super-strong magnetic fields. Normal neutron stars (radio pulsars) are born spinning at only 50-100 times per second, preventing large-scale dynamo from working and leaving them with a magnetic field 1000 times weaker.

As described in chapter three, the crust of neutron star has odd properties. It is very difficult to compress the crust material very much, or to move elements of crust up down, because strong gravity and pressure forces maintain a firm balance. But it is simpler to move parts of the crust horizontally, in which by applying only shear strains to it.

In magnetar, the magnetic field is strong enough to push horizontally and twist the crust. The magnetic field lines outside the star also get twisted because they are anchored to the crust. This crust twist strengthen the current outside the magnetar and generate X-rays. The rotational energy quickly decreases and the magnetic field only itself can provide an energy source for potentially observable emissions. As a result a significant amount of magnetic energy is dissipated during the first 10,000 years. But in a radio pulsar, the magnetic field is not strong enough (compared to the magnetar) to push and twist the crust because the magnetic field is essentially stable; its main function is to passively facilitate the loss of rotational energy. As a result the field doesn't play a significant role to strengthen the current outside a radio pulsar and the magnetic energy is not dissipated very quickly.

Next, let we consider the competitive nature of magnetic field strength and decaying time.

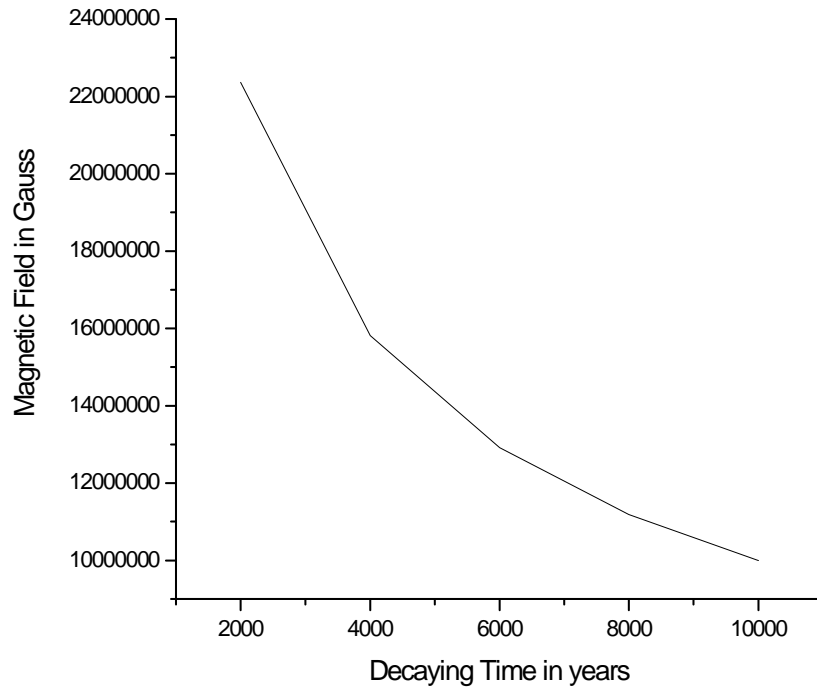


Fig. 4.1a: This graph illustrates the magnetic field versus decaying time diagram of a magnetar at the threshold surface magnetic field of 10^7 G and it is based on equations (3.72 and 3.73).

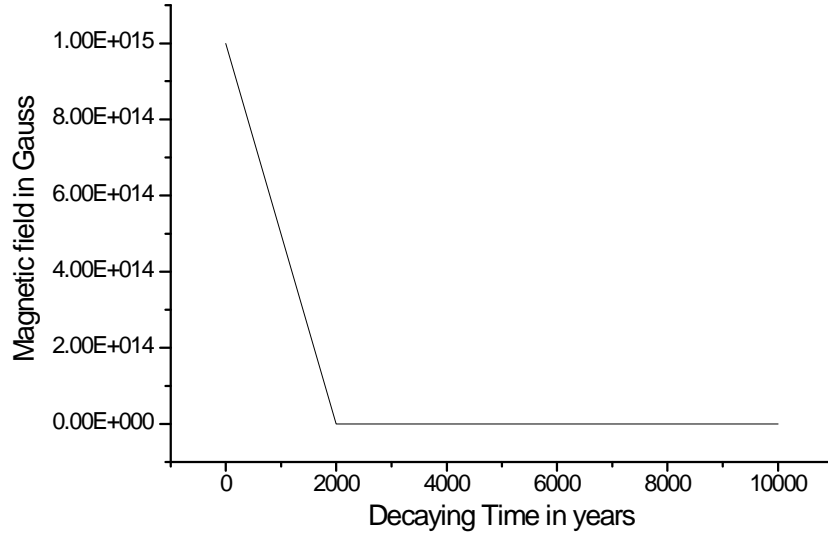


Fig. 4.1b: This graph illustrates the magnetic field versus decaying time diagram of a magnetar at the threshold surface magnetic field of 10^7 G and it is based on equations (3.84 and 3.85).

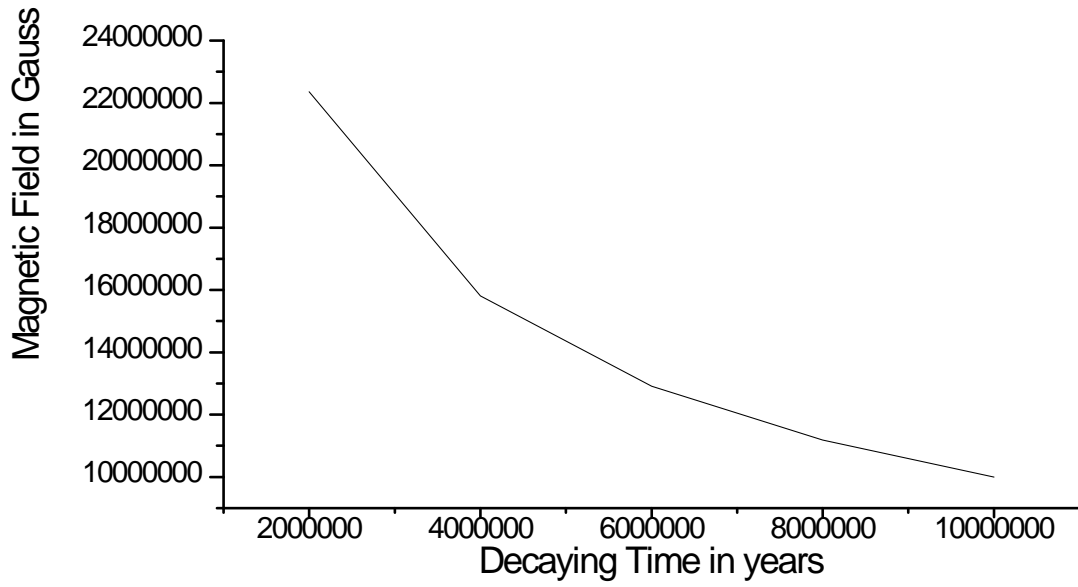


Fig. 4.1c: This graph illustrates the magnetic field versus decaying time diagram of a radio pulsar at the threshold surface magnetic field of 10^7 G and it is based on equations (3.72 and 3.73).

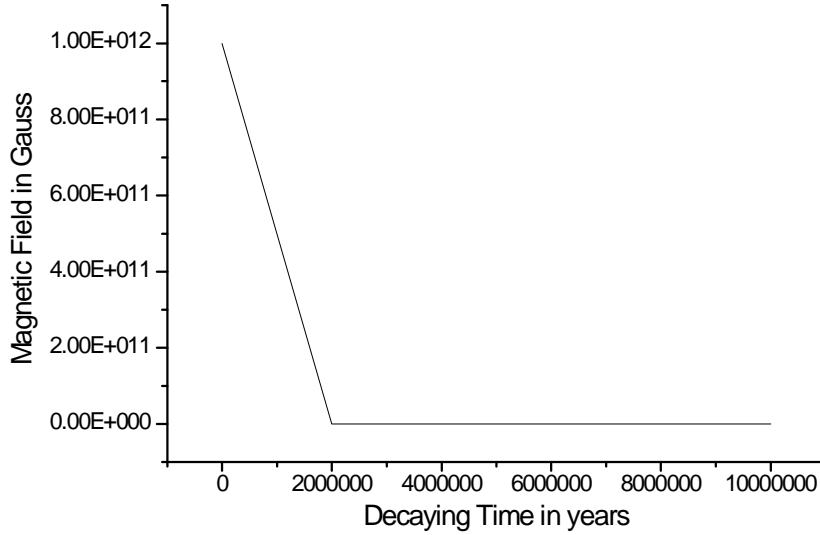


Fig. 4.1d: This graph illustrates the magnetic field versus decaying time diagram of a radio pulsar at the threshold surface magnetic field of 10^7 G and it is based on equations (3.84 and 3.85).

As mentioned in the previous chapter, the measured magnetic fields at the poles and decaying time are 10^{12} - 10^{13} Gauss and 10 million years, respectively in the case of radio pulsars whereas 10^{14} - 10^{15} Gauss and 10,000 years, respectively in the case of magnetars. From these astronomical data and equations (3.72 and 3.84), one can suggest that, a pulsar remains radio “loud” for a longer time than magnetars and the magnetic field in a neutron star varies nonlinearly with time. This suggestion leads to state that for radiation emitting neutron stars, the magnetic field, $B(t)$ is inversely proportional to the square root of the decaying time, t in the case of EM Radiation and $B(t)$ is inversely proportional to the fourth root of the decaying time, t in the case of Gravitational

Radiation (see Fig.4.1a, 4.1b, 4.1c, and 4.1d).

4.2 Conclusion

As we have discussed in the summary part, heavy stars will form neutron stars (magnetars) spinning at up to 500-1000 times per second. Such rapid rotation should power a large-scale dynamo and generate a super strong magnetic field. This strong magnetic field pushes horizontally and twist the crust as a result much of its magnetic energy is quickly dissipated. Normal neutron stars (radio pulsars) are born spinning at only 50-100 times per second, preventing large-scale dynamo from working and leaving the magnetic field 1000 times weaker. This field is not strong enough (compared to magnetar) to push horizontally and twist the crust as a result much of its magnetic energy is not quickly dissipated. In addition to this in both neutron stars the magnetic field varies nonlinearly with time and the stronger the original magnetic field of a neutron star is the shorter will be the decaying time or vice versa.

Therefore, we can conclude that the cause of the variation of magnetic field and decaying time among neutron stars is the variation of their original angular velocities. To the best of our knowledge no previous research has addressed this worrisome issue.

APPENDIX

The next table gives the symbolic relationships of SI variables to Gaussian variables. It is useful for quick conversion from SI to Gaussian or Gaussian to SI by replacing the relevant one on both sides of the equation. The SI system is a convenient unit that makes no numerical factors of 4π appear in Maxwell's equations, and for this reason such a system is called a rationalized system of units.

<u>Quantity</u>	<u>SI System of variables</u>	<u>Gaussian System of variables</u>
Charge	q	$\sqrt{4\pi\epsilon_0}q$
Charge density	$\rho(\sigma, \lambda)$	$\sqrt{4\pi\epsilon_0}\rho(\sigma, \lambda)$
Conductivity	σ_c	$4\pi\epsilon_0\sigma_c$
Current	I	$\sqrt{4\pi\epsilon_0}I$
Current density	\vec{J}	$\sqrt{4\pi\epsilon_0}\vec{J}$
Dielectric constant	ϵ	$\epsilon_0\epsilon$
Magnetic permeability	μ	$\mu_0\mu$
Susceptibility	$\chi(\chi_m)$	$4\pi\chi(\chi_m)$
Electric dipole moment	\vec{P}	$\sqrt{4\pi\epsilon_0}\vec{P}$
Magnetic dipole moment	\vec{m}	$\sqrt{\frac{4\pi}{\mu_0}}\vec{m}$
Displacement vector	\vec{D}	$\sqrt{\frac{\epsilon_0}{4\pi}}\vec{D}$
Magnetization	\vec{M}	$\sqrt{\frac{4\pi}{\mu_0}}\vec{M}$
Electric field	\vec{E}	$\frac{1}{\sqrt{4\pi\epsilon_0}}\vec{E}$
Magnetic field	\vec{B}	$\sqrt{\frac{\mu_0}{4\pi}}\vec{B}$
Electric flux	Φ_E	$\frac{1}{\sqrt{4\pi\epsilon_0}}\Phi_E$
Magnetic flux	Φ_B	$\sqrt{\frac{\mu_0}{4\pi}}\Phi_B$
Scalar potential	Φ	$\frac{1}{\sqrt{4\pi\epsilon_0}}\Phi$
Vector potential	\vec{A}	$\sqrt{\frac{\mu_0}{4\pi}}\vec{A}$
Magnetic intensity	\vec{H}	$\frac{1}{\sqrt{4\pi\mu_0}}\vec{H}$
Velocity of light	$\frac{1}{\sqrt{\epsilon_0\mu_0}}$	c

Bibliography

- [1] Giancoli, Douglas C., 2005, Physics principles with applications, 6th ed. (Pearson).
- [2] A.S.Eddington, Observatory 30 (1920) 353.
- [3] Michael A.Seeds: Horizons: Exploring the Universe, Wadsworth 1995, p. 310.
- [4] Gravitation, Misner, Thorne, Wheeler (1997).
- [5] Chadwick, J., 1932, "Possible existence of a neutron," Nature 129, 312.
- [6] Baade, W., and F.Zwicky, 1934c, "Supernovae and Cosmic rays," Phys. Rev. 45, 138.
- [7] L. Woltjer, Astrophys, J. 140 (1964) 1309.
- [8] Glendenning, Norman K., 2ed ed. (2000).
- [9] Robert, C. Duncan, May, 1998, Updated March, 2003.
- [10] ScienceWeek, 2003, Vol. 7, Number 22 A.
- [11] Rezzolla, L., 2004. SISSA. It., 1-93.

- [12] Urpin, V.A., Levshakov, S. A. and Yakovlev, D. G., 1986. *Mon. Not. astr. Soc.*, 219, 703-717.
- [13] Mohammed K., M. Sc. thesis, AAU, 2005.
- [14] Michael, F. C., 1991, *Theory of Neutron Star, Magnetospheres* (Chicago Press).
- [15] Kitchin, C.R., 1987, *Stars, Nebulae and the Interstellar Medium* (Adam Hilger).
- [16] Mc Gillivray, D., 1987, *Physics and Astronomy* (Macmillan).
- [17] Emiliani, C., 1988, *The Scientific Companion* (Wiley).
- [18] Padmanabhan, T., 1993, *Structure formation in the Universe* (Cambridge).
- [19] Pacini, F., *Nature* (1968).
- [20] Jackson, J. D., 1999, *Classical Electrodynamics*, 3rd ed. (Wiley).
- [21] Nayfeh, M. H. and Brussel, M. K, 1985, *Electricity and Magnetism* (John Wiley).
- [22] Singh, R. S. and Sinha, B. K., 2005, *Gen. Relativ. Gravit.*, 37(7), 1163–1175.
- [23] Laud, B. B., 1987, *Electromagnetics*, 2nd ed. (Wiley).
- [24] Ioka, K. and Sasaki, M., 2004, *The Astrophysical Journal*, 600:296–316.
- [25] *Gravitation and Cosmology, 1972, Principles and Applications of the General Theory of Relativity*, Steven Weinberg.
- [26] ScienceDaily, 2005, CSIRO Australia.