On: Monomial orders and Ring of Multiplicative Invariants

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Abstract

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For a finite group $\mathcal{G}$ in $GL(n,\mathbb{Z}) \cong Aut(\mathbb{Z}^n)$, action of $\mathcal{G}$ on $\mathbb{Z}^n$ (via automorphism) can be uniquely extended to group algebra of $\mathbb{Z}^n$ i.e. to Laurent polynomial ring $\mathbb{K}[\mathbf{X}^{\pm 1}]$ over some base field $\mathbb{K}$. M. Lorenz in Lorenz (2001) showed that the invariant algebra $\mathbb{K}[\mathbf{X}^{\pm 1}]^{\mathcal{G}} = \mathbf{R}$ of this multiplicative action has a form of affine semigroup algebra $\mathbb{K}[M]$, provided the group acting is a reflection group. Further he conjured that, *if* $\mathbf{R} = \mathbb{K}[M]$ *an affine semigroup algebra, then must* $\mathcal{G}$ *act as a reflection group?* Lorenz (2005). Few partial answer was given, using different restriction and approach. In this dissertation we showed that the above conjuncture holds, provided $\mathcal{G}$ is taken from a class that satisfies certain linearization conditions. We used monomial ordering of $\mathbb{Z}^n$ intensively in relation to the initial algebra of invariant rings. Furthermore M. Tesemma and H. Wang in Tesemma and Wang (2011), described the initial algebra of invariant rings for arbitrary lattice (monomial) ordering can be represented using an archimedean order, giving same initial algebra hence SAGBI, provided the invariant algebra is induced by action of reflection group. Further confirmed that for the usual lex ordering all the non reflection groups (4 groups upto conjugacy) in $GL(2,\mathbb{Z})$ do not admit such representation of initial algebra. We further show that, such representation (via archimedean order) of initial algebra for multiplicative invariants of any monomial order is possible if and only if the invariant algebra are induced from action of reflection group.

Dedication

Dedicated to my children
daughter **Hewan Mulugeta** and son **Menata Mulugeta**,

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Abbreviations and Notation

The following are some of the most frequently used notations in this thesis:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, denote the usual sets of natural, integer, rational and real numbers respectively.
- **L** - a lattice a free $\mathbb{Z}$-module of finite rank (n).
- $\succcurlyeq$ to denote a multiplicative ordering on a lattice **L** (Example $\mathbb{Z}^n$) .
- Ω denotes the set of multiplicative orderings on $\mathbb{Z}^n$.
- $GL(n, \mathbb{Z})$ is used to denote the set of automorphisms of $\mathbb{Z}^n$.
- $\mathcal{G}$ denotes a finite group contained in $GL(n, \mathbb{Z})$.
- By δ_{ij} we denote the Kronecker-Delta with

$$\delta_{ij} = \begin{cases} 1, & i = j. \\ 0, & i \neq j. \end{cases}$$

- **R** multiplicative invariant rings,
- $\mathbf{R}_m$ monic subsemigroup of **R** with respect to ring multiplication.

Chapter 1

Introduction

Invariant theory is both classical and contemporary research area of mathematics. It played a central role in 19th century algebra and geometry, yet many of its techniques and algorithms were practically forgotten by the middle of 20th century. With the fields of combinatorics and computer science reviving old-fashioned algorithmic mathematics during the past twenty years or so, classical invariant theory has come to a renaissance. We quote from the expository article of Kung and Rota (1984):
"Like the Arabian phoenix rising out of its ashes, the theory of invariants, pronounced dead at the turn of the century, is once again at the forefront of mathematics. During its long eclipse, the language of modern algebra was developed, a sharp tool now at last being applied to the very purpose for which it was invented."
This quote refers to the fact that three of Hilbert fundamental contributions to modern algebra, namely, the *Nullstellensatz*, the *Basis Theorem* and the *Syzygy Theorem*, were first proved as lemmas in his invariant theory papers (Hilbert 1890, 1893) [Strumfels (2008)]. Some of the basic theorems and concepts of computational algebra can be found in 19th century papers on classical invariant theory. The roots of invariant theory can be traced back to Lagrange (1773-1775) and Gauss (1801) who were interested in the problem of representing integers by quadratic binary forms and used the discriminant to distinguish between non equivalent forms [Nausel (2007)].
Algebraic invariants, such as the discriminant show up also in algebraic geometry, when one asks for properties of geometric objects which are invariant under certain classes of transformations.
In this thesis our concern will be on multiplicative action and the associated invariant subalgebra (subrings), we study some properties of these rings, using ordering of the set (lattice) in which the group acts by automorphisms. For its natural course, and comparison of the multiplicative action to its older and comparatively well studied by invariant theory researcher, we consider some of the notion of linear actions, as both share similar mathematical terms, further more some of the question in multiplicative action are raced from the linear counterpart. But one should not consider multiplicative action as an extension of the linear, rather a variant, and it does stand on its own.

1.1 Linear Group Action

A linear representation of a group G, is a group homomorphism,

$$\rho : G \to GL(n, \mathbb{K}) \tag{1.1.1}$$

where $\mathbb{K}$ is a field and $GL(n,\mathbb{K})$ is the **G**eneral **L**inear group. The number n is called the *degree* of the representation ρ. For a finite dimensional vector space V of dimension n we have, $GL(n,\mathbb{K}) \cong GL(V)$. Where $GL(V)$ is a set containing all invertible linear transformation (automorphisms) of V. Moreover if ρ is injective (i.e. monomorphism) representation, then it is said to be *faithful*. Furthermore the linear representation of the group induces a group action on V of the vector space by automorphisms.

Definition 1.1.1. Let X be a non empty set and G be a group the map,

$$\circ : G \times X \to X \quad satisfying$$

$$e \circ x = x, \qquad (gg^{'}) \circ x = (g \circ (g^{'} \circ x))$$

where $x \in X$ and $g, g^{'} \in G$ is called a *group action* on the set X. The set X on which G acts is called a G-set.

Now consider the linear representation a group G (of degree n) as in equation:1.1.1 and define a map,

$$\circ\ : G \times V \to V \quad via\, \rho, \quad g \circ u = \rho(g)(u) \quad \forall g \in G, \forall u \in V.$$

with this definition $\circ$ is a group action on the vector space $V \cong \mathbb{K}^n$. The action of a group is called *faithful*, if for any $g \in G \quad g \circ x = x \quad \forall x \in X$, then $g = e$ where e is the identity element of G. If ρ is a faithful representation of G then, the induced group action on V is faithful. Conversely a group action of G on X by automorphisms also induces a homomorphism of G into $GL(X)$ (a representation). If the group action is faithful so is the induced representation. For $x \in X$ we define the *stabilizer* of x in G, denoted by $Stab_G(x) = \{g \in G : g \circ x = x\}$ and is a subgroup known as *isotropy subgroup* of G. The *orbit* denoted by $[x]$ is the subset of X given by $\{g \circ x : g \in G\}$ is an equivalence class containing x for the relation ($x, y \in X, \quad x \sim y$ if $\exists g \in G, g \circ y = x$), which is induced by the group action on X. Further more the index of $Stab_G(x)$ in G is equal to the cardinality of its orbit set (i.e. $|[x]|$).

Let $e_1, e_2, \cdots, e_n$ be a basis for V, a vector space over field $\mathbb{K}$ and $x_1, x_2, \cdots, x_n$ be the basis of its dual space V^* i.e. $(x_j(e_i) = \delta_{ij})$. The action of G on V can be extended to a group action on the dual space $V^*(= Hom_{\mathbb{K}}(V,\mathbb{K}))$,

$$\odot : G \times V^* \to V^* \quad \texttt{given by}$$

for each $j \in \{1, 2, \cdots, n\}$,

$$g \odot x_j(u) = x_j(g^{-1} \circ u) \quad (\forall u \in V).$$

This can be further extended to any $\tau \in V^*$ for any $g \in G$,

$$g \odot \tau(u) = \tau(g^{-1} \circ u) \quad \forall u \in V,$$

with these V^* become a G-set.

Remark 1.1.2. G acts on V^* faithfully if and only if G acts on V faithfully, observe that for $g \in G$

$$[g \circ u = u \leftrightarrow g^{-1} \circ u = u \quad \forall u \in V] \Leftrightarrow \quad g \odot \alpha = \alpha \quad \forall \alpha \in V^*.$$

Let $\mathbb{K}[x_1, x_2, \cdots, x_n] = \mathbb{K}[\mathbf{X}]$ be a polynomial ring with field $\mathbb{K}$, consider each linear form in V^* as a linear polynomial in the indeterminate (variables) $x_1, x_2, \cdots, x_n$, with this identification, $V^* \subset \mathbb{K}[\mathbf{X}]$ and we extend the action of G to the polynomial ring $\mathbb{K}[\mathbf{X}]$. This is done by defining the action of G on the monomials of $\mathbb{K}[\mathbf{X}]$ and extend it by linearity to each terms of polynomial $f = \sum_{i=1}^{n} k_i \mathbf{x}^{\mathbf{a}_i}$, where $\mathbf{a}_i = (a_{i1}, a_{i2}, \cdots, a_{in}) \in \mathbb{Z}^n_{\geq 0}$ and $k_i \in \mathbb{K}$ as follows. Let $\mathbf{x}^{\mathbf{a}}$ be a monomial where $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{Z}^n_{\geq 0}$ and $g \in G$,

$$g \odot \mathbf{x}^{\mathbf{a}} = g \odot (x_1^{a_1} \cdots x_n^{a_n}) = (g \odot x_1)^{a_1} \cdots (g \odot x_n)^{a_n} \tag{1.1.2}$$

The action of G over the monomials is defined and linearly extended to the polynomial that is,

$$g \odot f = g \odot \sum_{i=1}^{n} k_i \mathbf{x}^{\mathbf{a}_i} = \sum_{i=1}^{n} k_i (g \odot \mathbf{x}^{\mathbf{a}_i})$$

One can easily show that the polynomial ring $\mathbb{K}[\mathbf{X}]$ is a G-set.

Definition 1.1.3. A polynomial $f = \sum_{i=1}^{n} k_i \mathbf{x}^{\mathbf{a}_i} \in \mathbb{K}[\mathbf{X}]$ is said to be an invariant polynomial under the action of a group G if $\forall g \in G, \quad g \odot f = f$. The set of all invariant polynomials of $\mathbb{K}[\mathbf{X}]$ forms a subring and is denoted by $\mathbb{K}[\mathbf{X}]^G$ and hence,

$$\mathbb{K}[\mathbf{X}]^G = \{f \in \mathbb{K}[\mathbf{X}] : g \odot f = f \quad \forall g \in G\}$$

is called the *ring of polynomial invariants*.

An element $g \in G$ is called a pseudo-reflection on V if g acts trivially on a hyperplane in V or, equivalently, if the $n \times n$ matrix $g - I_V$ has rank one, where I_V is the identity automorphism of V. The group G is said to act as a pseudo-reflection group on V, if G is generated by elements that are pseudo-reflections on V. Note that pseudo-reflection g in lattice (free module) over rings contained in the real ($\mathbb{R}$) are reflections and $g^2 = I$, else are of finite order.
One of the most celebrated result in the invariant theory of the linear action is the **Shephard-Todd-Chevalley Theorem**(S-T-C): Nausel (2007) [Theorem 13.16]

Let $\rho : G \hookrightarrow GL(n, \mathbb{K})$ be a faithful representation of finite group G, Then the invariant algebra $\mathbb{K}[V]^G$ is a polynomial algebra if and only if ρ is pseudo-reflection representation.(i.e.$\rho(G)$ acts as a pseudo-reflection group on the vector space V (The action extended to polynomial ring).

1.2 Multiplicative Group Action

The linear representation of a group G induces an action of G on the polynomial ring $\mathbb{K}[\mathbf{X}]$, it sends each x_j to a $\mathbb{K}$ linear combination of the x_is'. Also the monomials,

$$M = \{\mathbf{x}^{\mathbf{a}} : \mathbf{a} \in \mathbb{Z}^n_{\geq 0}\}$$

forms a $\mathbb{K}$ basis for the polynomial ring and is a commutative monoid. In multiplicative action the monomials

$$\{\mathbf{x}^{\mathbf{a}} : \mathbf{a} \in \mathbb{Z}^n\},$$

will have free abelian group structure and the ring over some base field $\mathbb{K}$ forms a group ring structure (Laurent polynomial if $M \cong \mathbb{Z}^n$).

Definition 1.2.1. A *lattice* $\mathbf{L}$ is a free $\mathbb{Z}$-module (free abelian group) of finite rank, so $\mathbf{L} \cong \mathbb{Z}^n$ where $n = rank(\mathbf{L})$.

The group operation on $\mathbf{L}$ is traditionally denoted additively, if a group G acts on $\mathbf{L}$, via group homomorphism,

$$\gamma : G \longrightarrow GL(\mathbf{L}), \tag{1.2.1}$$

where $GL(\mathbf{L})$ is the set of invertible $\mathbb{Z}$-linear maps ($\mathbb{Z}$-module automorphisms), in such cases $\mathbf{L}$ is called *G-lattice* and with appropriate choice of basis for $\mathbf{L}$ we have the isomorphism of groups,

$$GL(\mathbf{L}) \cong GL(n, \mathbb{Z})$$

Further we put $\mathbf{L}^G = \{l \in \mathbf{L} : g(l) = l \quad \forall g \in G\}$ a G-invariant sub lattice of $\mathbf{L}$. A G-lattice is said to be *effective* if the G-invariant sub lattice is $\mathbf{L}^G = 1$ and *trivial* if $\mathbf{L}^G = \mathbf{L}$. The action of G on $\mathbf{L}$, extends uniquely to the group algebra of $\mathbf{L}$ over a commutative base ring with unity, written $\mathbb{K}[L]$. Further L is a subgroup of units of the group algebra, i.e. $U(\mathbb{K}[L])$. Working inside $\mathbb{K}[L]$, we pass from the additive notation of $\mathbf{L}$ to multiplication and denote $\mathbf{L}$ by L. To do this we write the basis element of $\mathbb{K}[L]$ (i.e. L) corresponding to each lattice point $m \in \mathbf{L}$ as $x^m \in L$ then we have,

$$\mathbf{x}^{\mathbf{0}} = 1, \quad \mathbf{x}^{m+l} = \mathbf{x}^m \mathbf{x}^l \quad and \quad \mathbf{x}^{-m} = (\mathbf{x}^m)^{-1},$$

hence,

$$f \in \mathbb{K}[L] \quad \texttt{is written} \quad f = \sum_{m \in \mathbf{L}} k_m \mathbf{x}^m$$

For each $f \in \mathbb{K}[L]$ the set $\{m : 0 \neq k_m \in \mathbb{K}\}$ is a finite subset of $\mathbf{L}$ is called the *support* of f and is denoted by $supp(f)$. A fixed choice of $\mathbb{Z}$-basis $\{m_i : i = 1, 2, \cdots, n\}$ of $\mathbf{L}$ or a fixed isomorphism $\mathbf{L} \cong \mathbb{Z}^n$ give rise to $\mathbb{K}$- algebra isomorphism $\mathbb{K}[L]$ to the Laurent polynomial ring $\mathbb{K}[x_1^{\pm 1}, x_2^{\pm 1}, \cdots, x_n^{\pm 1}] = \mathbb{K}[\mathbf{X}^{\pm 1}]$, via,

$$Via, \quad \mathbf{x}^{m_i} \mapsto x_i = \mathbf{x}^{e_i},$$

There for we may think of the representation $x^m \in \mathbb{K}[L]$ of a lattice element $m \in \mathbf{L}$ as a monomial in the free variable $x_1^{\pm 1}, x_2^{\pm 1}, \cdots, x_n^{\pm 1}$.

Definition 1.2.2. The action of G on $\mathbf{L}$ (via homomorphism 1.2.1) extends uniquely to the $\mathbb{K}$-algebra automorphisms of $\mathbb{K}[L]$ so,

$$\star : G \times \mathbb{K}[L] \to \mathbb{K}[L] \quad g \star f = \sum_{m \in supp(f)} k_m \mathbf{x}^{\gamma(g)(m)}$$

this type of group action is called *multiplicative (exponential or integral) action*, and the subalgebra of invariant polynomials,

$$\mathbb{K}[L]^G = \{f \in \mathbb{K}[L] : g \star f = f \quad \forall g \in G\}$$

is called *the multiplicative invariant* subring.

For each $g \in G$ and via the isomorphism $\mathbf{L} \to L$ given by $m \to \mathbf{x}^m$, for each $m \in \mathbf{L}$ one can identify $\gamma(g) \in GL(\mathbf{L})$ hence by a bit abuse of notation we assume $\gamma(g) : L \to L$ is automorphism of monomials. Here $L = \{\mathbf{x}^m : m \in \mathbf{L}\}$ form a $\mathbb{K}$ basis to the group algebra $\mathbb{K}[L]$ and hence,

$$\gamma(g) : \mathbb{K}[L] \to \mathbb{K}[L]$$

is a $\mathbb{K}$ algebra automorphism. Further more the multiplicative action of G via the representation depend on its image in $GL(\mathbf{L})$. Since a given group can have different representation, resulting in different action on L hence invariant subalgebra, we consider G to be a group contained in $GL(\mathbf{L})$, hence inclusion representation which induces faithful action on the group algebra $\mathbb{K}[L]$. Furthermore the action of arbitrary group G on a lattice $\mathbf{L}$ (via representation) can be reduced to an action of finite group, as long as the invariant algebra is concerned [Lorenz (2005), proposition 3.3.1]. Hence the multiplicative action is concerned with only finite group. The fact $GL(\mathbf{L}) \cong GL(n, \mathbb{Z})$ for some isomorphism further allow as to consider these groups to be in $GL(n, \mathbb{Z})$, in such cases these groups are denote by $\mathcal{G}$ and $\mathcal{G} \subseteq GL(n, \mathbb{Z})$. In fact for a given rank n the number of finite subgroup contained in $GL(n, \mathbb{Z})$ is finite (up to conjugacy).

1.3 Some Results on Invariant Subrings

In invariant theory two algebraic structures interact, a group G and a set X. Here in our case X has the algebraic structure a group ring (Laurent polynomial) in the multiplicative group action case, and polynomial ring in the case of linear action. Most of the studies are concerned with the algebraic properties of invariant subalgebra and, identify those inherited from the parent algebra. For example on the linear action we ask questions like 'is the invariant subalgebra a polynomial ring'? 'if not for which representation is the invariant algebra a polynomial ring? etc $\cdots$. Observe that these are very wide questions due to several parameter come to play.

Multiplicative action theory emerged relatively recently and has only been studied systematically during the past 30 years, beginning with the work of D.R Farkas [Farakas (1985)] in the 80s'. Prior to Farkas, only a few isolated results on multiplicative invariants, also known as *exponential invariants* or *monomial invariants*, were known, notably in the work of Bourbaki [Bourbaki (1968)].

1.3.1 The Semigroup Problem

The action of $\mathcal{G}$ stabilizes **L** and hence maps monomials to monomials. Despite some obvious formal similarities in the basic set up, multiplicative invariant theory and its linear counterpart exhibit many strikingly different features. For one, other than the theory of polynomial invariants, multiplicative invariant theory is only concerned with finite group actions, when studying the invariants of a multiplicative action under an arbitrary group presentation, one can gather all information by reducing to a suitable finite group Lorenz (2005) [proposition 3.3.1]. This tells us in particular that multiplicative invariant algebras $\mathbb{K}[L]^G$ are always affine K-algebras. Another notable feature of multiplicative actions is the fact that degree of Laurent polynomials is not preserved under the action (loss of homogeneity) unlike the linear see eq: 1.1.2. This is again in sharp contrast with the classical case of linear actions and causes a great deal of added difficulty when investigating multiplicative invariants. One of the problem in the theory of multiplicative action is existence of a variant of S-T-C theorem discussed in section [§1.1] of the linear action. Invariant subalgebra of a multiplicative action is a group ring (i.e. has same structure as $\mathbb{K}[L]$) if and only if the group acts trivially on **L** [Lorenz (2005), Corollary 3.4.2]. Hence Lorenz in [Lorenz (2001), Theorem 2.4], which is a refinement of Faraka's work [Farakas (1985), §3], proved that,

Theorem 1.3.1 (Lorenz (2001))**.** *Let $\mathcal{G}$ be a finite group acting as a reflection group on* **L***, then there is a submonoid M of $(\mathbb{Z}[L]^{\mathcal{G}}, \cdot)$ whose elements form a $\mathbb{Z}$-basis of the invariant algebra $\mathbb{Z}[L]^{\mathcal{G}}$. Consequently, for any commutative base ring $\mathbb{K}$, $\mathbb{K}[L]^{\mathcal{G}}$ is isomorphic to the semigroup algebra $\mathbb{K}[M]$.*

Lorenz uses tools from Lie algebra (crystallographic root system [§2.6]) to prove, and gives a way to construct the semigroup M using the weight lattice. He further posed (conjured) in his book a set of problems about invariant ring of this action among which, [Lorenz (2005), §10.2: Problem 5], and we restate,

Conjecture 1.3.2. *Let $\mathcal{G}$ be a finite group in $GL(\mathbf{L})$. If invariant ring $\mathbb{K}[L]^{\mathcal{G}} = \mathbb{K}[M]$ is affine semigroup algebra then $\mathcal{G}$ acts as a reflection group on* **L***.*

Further Tesemma [Tesemma (2004)], gives an alternative proof for 1.3.1 and a partial of the converse, in which he uses ordering of the lattice $\mathbb{Z}^n \cong \mathbf{L}$ and initial algebra, which is highly motivated by Reichstein's approach on SAGBI bases for ring of multiplicative invariant [Reichstein (2003)]. The acronym **SAGBI** stands for **S**ubalgebra **A**nalogue to

Gröbner **B**ases for **I**deals.

Before we state Tesemmas' result, some important definitions will be given as these definitions are needed for latter use in our work but also allow our reader to grasp the ideas contained in Z. Reichstein and M. Tesemmas' work.

Definition 1.3.1. A **monomial or multiplicative order on** a lattice $\mathbf{L} \cong \mathbb{Z}^n$, is a total order $\succcurlyeq$, which is compatible with the group operation of $\mathbb{Z}^n$, i.e $(\mathbb{Z}^n, \succcurlyeq)$ is a totally order set and $\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{Z}^n$ if $\mathbf{a} \succcurlyeq \mathbf{b}$, then $\mathbf{a} + \mathbf{c} \succcurlyeq \mathbf{b} + \mathbf{c}$.

There are uncountably infinite orders on $\mathbb{Z}^n$ for $n \geq 2$. We denote the set of monomial orders on $\mathbb{Z}^n$ by Ω and we write $\mathbf{a} \succ \mathbf{b}$ if $\mathbf{a} \succcurlyeq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$. The term **monomial** order is used as these order are used in ordering of the monomials $\{x^{\mathbf{a}} : \mathbf{a} \in \mathbb{Z}^n\}$ of the Laurent polynomial $\mathbb{K}[\mathbf{X}^{\pm 1}]$. Unlike the polynomial ring ordering which by definition is well ordered, the ordering on $\mathbb{K}[\mathbf{X}^{\pm 1}]$ via $\mathbb{Z}^n$ is not a well ordering, hence division of polynomials needs extra caution.

Example 1.3.2. The lexicographic order $\succcurlyeq_{lex}$ on $\mathbb{Z}^n$, is given by for $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$, $\mathbf{a} \succcurlyeq_{lex} \mathbf{b}$, if the first non zero component of $\mathbf{a} - \mathbf{b}$ is non negative. This gives a multiplicative order on $\mathbb{Z}^n$.

For $0 \neq f \in \mathbb{K}[\mathbf{X}^{\pm 1}]$, we define the *initial degree* of f with respect to a fixed $\succcurlyeq \in \Omega$, to be $max_{\succcurlyeq}\{\mathbf{a} : \mathbf{a} \in supp(f)\}$ and denote it $in_{\succcurlyeq}(f)$. Note since the $supp(f)$ is a finite subset of totally ordered $\mathbb{Z}^n$, the initial degree is a well defined function from $\mathbb{K}[\mathbf{X}^{\pm 1}] \setminus \{0\}$ to $\mathbb{Z}^n$. Further for any set $T \subseteq \mathbb{K}[\mathbf{X}^{\pm 1}]$,

$$in_{\succcurlyeq}(T) = \{in_{\succcurlyeq}(f) : 0 \neq f \in T\}.$$

It easy to see that $in_{\succcurlyeq}(1) = \mathbf{0}$, generally this holds for constant polynomials. Let R be a subalgebra of $\mathbb{K}[\mathbf{X}^{\pm 1}]$, and f and f' non zero polynomials in R,

$$in_{\succcurlyeq}(f.f') = in_{\succcurlyeq}(f) + in_{\succcurlyeq}(f')$$

Hence $in_{\succcurlyeq}$ is a monoid homomorphism ($in_{\succcurlyeq} : (R, \cdot) \to \mathbb{Z}^n$). In particular $in_{\succcurlyeq}(R)$ is a submonoid of $\mathbb{Z}^n$.

Definition 1.3.3. Let T be a subalgebra of $\mathbb{K}[\mathbf{X}^{\pm 1}]$ and let $\{f_\lambda\}_{\lambda \in \wedge}$ be a family of elements in T such that the monoid $in_{\succcurlyeq}(T)$ is generated by $\{in_{\succcurlyeq}(f_\lambda)\}_{\lambda \in \wedge}$. If $in_{\succcurlyeq}(T)$ is well ordered under $\succcurlyeq$ then T is generated by $\{f_\lambda\}_{\lambda \in \wedge}$ that is $T = K[f_\lambda : \lambda \in \wedge]$. Then the set $\{f_\lambda\}_{\lambda \in \wedge}$ is called a *SAGBI basis*.

The algorithm used to write an element of T in terms of the SAGBI basis is called *subduction algorithm*. Both the above terms, i.e. SAGBI and subduction algorithm are introduced by Robbiano and Sweedler [Robbiano and Sweedler (1988)]. For polynomial subalgebra and in the above definition of SAGBI basis the statement "if $in_{\succcurlyeq}(T)$ is

well ordered under $\succcurlyeq$" is the extra condition put for Laurent polynomial subalgebra to ensure that division of polynomials terminate. These would have been tautology for the initials subalgebra of a polynomial algebra. If $\wedge$ is a finite set we say T has a *finite SAGBI* basis.
Let $\mathcal{G}$ be a finite group in $GL(n, \mathbb{Z})$. We now state Z. Reichsteine main theorem [Reichstein (2003)]

Theorem 1.3.3. *(Reichsteine 2003). The following are equivalent statements,*

1. *$\mathcal{G}$ acts as reflection group on $\mathbb{Z}^n$.*
2. *The monoid $\mathfrak{A}^{\succcurlyeq}$ ($in_{\succcurlyeq}(\mathbb{K}[\mathbf{X}^{\pm 1}]^{\mathcal{G}})$) is finitely generated.*
3. *The invariant algebra $\mathbb{K}[\mathbf{X}^{\pm 1}]^{\mathcal{G}}$ has a finite SAGBI bases.*

Note in this theorem $\succcurlyeq \in \Omega$ is arbitrary fixed. Reichsteine used the concept of cones especially polyhedral cones in his proof, and further Tesemma [Tesemma (2004) and Tesemma (2007)], using these proved the following result.

Theorem 1.3.4. *(Tesemma 2004). Let $\succcurlyeq$ be a monomial order on $\mathbb{Z}^n$. Then $\mathcal{G}$ acts as a reflection group on $\mathbb{Z}^n$ if and only if there exist generators $f_1, \cdots, f_s$ in $\mathbb{K}[\mathbf{X}^{\pm 1}]^{\mathcal{G}}$ such that the restriction of the initial map to the monoid $M =< f_1, \cdots, f_s >_{mon}$ is injective.*

In his proof, Tesemma constructed the affine semigroup M using the semigroup of the initial algebra of the invariant ring, whenever $\mathcal{G}$ acts as reflection group on $\mathbb{Z}^n$. While showing the converse the injective initial map on M force $\mathbb{K}[\mathbf{X}^{\pm 1}]^{\mathcal{G}}$ to have a finite SAGBI and hence [Theorem 1.3.3], implies $\mathcal{G}$ must act as reflection group on $\mathbb{Z}^n$. Since the initial map depends on the head term and is a semigroup homomorphism, with source (domain) the whole $\mathbb{K}[\mathbf{X}^{\pm 1}]^{\mathcal{G}}$ the injectivity conditions seems strong while showing the converse. Hence in chapter 3, we construct a factor semigroup of $\mathbb{K}[\mathbf{X}^{\pm 1}]^{\mathcal{G}}$ in which the initial map injective condition is removed and a certain restriction on the finite group $\mathcal{G}$ is imposed.

1.3.2 Initial of Invariant Algebra Dependency on Ω.

The initial algebra of T, a subalgebra of $\mathbb{K}[\mathbf{X}^{\pm 1}]$ depends on the given multiplicative order $\succcurlyeq \in \Omega$. we know that there are orders which takes less time and memory (computer) to compute the initial algebra of a given subalgebra using algebra packages. It is also possible two distinct orders may give the same initial algebra (hence SAGBI basis) in such cases it would be more efficient to consider the one which consumes less time and floating memory of computer disregarding other packages short coming. We consider a class of multiplicative order satisfying certain extra conditions.

Definition 1.3.4. An order $\succcurlyeq \in \Omega$ is said to be *archimedean* if for $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$ and $\mathbf{a} \succcurlyeq \mathbf{b} \succ 0$ then there exist $t \in \mathbb{N}$ such that $t\mathbf{b} \succ \mathbf{a}$.

While computing the SAGBI basis of $\mathbb{K}[\mathbf{X}^{\pm 1}]^{\mathcal{G}}$ for reflection groups $\mathcal{G}$, Tesemma and Wang in the article [Tesemma and Wang (2011), Theorem 4.2] proved the following result.

Theorem 1.3.5. *Let $\mathcal{G}$ be a finite reflection group acting on $\mathbb{Z}^n$ and $\succcurlyeq \in \Omega$. Then there exists an archimedean order $\succcurlyeq_\omega \in \Omega$ such that*

$$in_{\succcurlyeq}(\mathbb{K}[\mathbf{X}^{\pm 1}]^{\mathcal{G}}) = in_{\succcurlyeq_\omega}(\mathbb{K}[\mathbf{X}^{\pm 1}]^{\mathcal{G}}).$$

further they showed for usual lexicographic order the finite non reflection subgroups of $GL(2, \mathbb{Z})$ do not admit an archimedean order $\succcurlyeq_\omega$ such that,

$$in_{\succcurlyeq}(\mathbb{K}[\mathbf{X}^{\pm 1}]^{\mathcal{G}}) = in_{\succcurlyeq_\omega}(\mathbb{K}[\mathbf{X}^{\pm 1}]^{\mathcal{G}}). \tag{1.3.1}$$

For a finite non reflection group in $GL(n, \mathbb{Z})$ for $n \geq 3$ for the usual lex ordering on $\mathbb{Z}^n$, can one have an archimedean odering such that equation: 1.3.1 holds? or such representation of initial algebra holds only for invariants of reflection group?. These problems were posed intrinsically on the same article [Tesemma and Wang (2011)], and we investigated, and latter showed that if equation: 1.3.1 holds for any multiplicative order then the invariant algebra must came from a reflection group action.

1.4 Our Contributions

(SEMIGROUP PROBLEM)

Both Lorenz in [Lorenz (2001)], and Tesemma in [Tesemma (2007)], showed that for $\mathcal{G}$ a reflection group the invariant algebra is affine semigroup algebra $\mathbb{K}[M]$, Conversely Tesemma with condition (letting the initial map $M \to \mathbb{Z}^n$ injective), showed the converse also holds. We will show that the forward proves of both are equivalent by creating a bridge between the two, and show the converse also holds given certain linearization condition on the finite group acting on $\mathbb{Z}^n$.
In chapter two, §2.6, we construct the relationship between weight lattice and the image (a semigroup in $\mathbb{Z}^n$) of the initial algebra of the invariant ring for a reflection group. These give the relationship on the construction of generators for the affine semigroup M. Most of the preparation is done in chapter three, which includes constructing monic semigroup $\mathbf{R}_m$ contained in $(\mathbb{K}[\mathbf{X}^{\pm 1}]^{\mathcal{G}}, \cdot)$. We define a monoid isomorphism over the factor semigroup of $\mathbf{R}_m$, to the image of initial algebra and try to avoid the effect of monomial ordering on the semigroup structure. To do that we impose a condition, and consider finitely generated subsemigroup (linearization) satisfying a number of conditions, which further create equivalence relation on Ω with finite class, two orders on the same class are identified as two distinct order giving the same semigroup structure over the given linearization.
Finally in chapter four, we showed the following equivalence.

Theorem 1.4.1. *Let $\mathcal{G} \in GL^\star(n, \mathbb{Z})$ such that ($[\mathbb{Z}]^{\mathcal{G}} = \{0\}$) then the following statements are equivalent,*

1. *$\mathcal{G}$ acts as a reflection group.*

2. *$\mathbf{R} = \mathbb{Z}[T]$ where $T =< f_a : a \in \mathfrak{A}_{gl} >_{mon}$ a canonical semigroup representation for some $\mathcal{M}^i$ (hence for all).*

where $GL^\star(n,\mathbb{Z})$ contains all finite group in $GL(n,\mathbb{Z})$ satisfying linearization condition ($\mathfrak{A}_{gl}$, a generating set , and $\mathcal{M}^i$, factor semigroup associated with the linearization) see section [§:**??** & §:**??**] . This group (satisfying the linearization conditions) are characterized by being contained in a reflection group (Proposition 3.4.5).

REPRESENTATION OF INITIAL ALGEBRA BY ARCHIMEDEAN ORDER

The computational aspect of the initial algebra and representation of a given order by archimedean order without altering the initial algebra of invariant ring was done, by Tesemma and Wang, [Theorem: 1.3.5] for invariants of reflection group. In chapter 4 we show that the converse also holds,

Theorem 1.4.2. *Let $\mathcal{G}$ be a finite subgroup in $GL(n,\mathbb{Z})$ then the following statements are equivalent,*

1. *For each $\succcurlyeq \in \Omega$ there exists an archimedean order $\succcurlyeq_\omega$ such that,*

$$\mathfrak{A}^{\succcurlyeq} = \mathfrak{A}^{\succcurlyeq_\omega}.$$

2. *$\mathcal{G}$ is reflection group.*

The main tool we use to show the above equivalence includes, Robbiano result on characterizing of term ordering for polynomial ring [Robbiano (1985)], with extension so that it holds on ordering of monomials of Laurent polynomials i.e. on $\mathbb{Z}^n$. In addition rational dimension of a given vector or element of $\mathbb{R}^n$, which is used for classifying multiplicative order is characterized and utilized. These tools further simplify our work in representing the initial of invariant ring, Via its interaction of special generator (depending on the order) of a finite group $\mathcal{G}$ contained in $GL(n,\mathbb{Z})$.

Most of the preparation work is done in chapter two and three, both the problem share monomial ordering of $\mathbb{Z}^n$, and hence most of the construction or the preparation focus on Ω. We advise our reader that all of the proofs involved in term ordering are made self contained but that dose not mean these results are new. Typically Robbiano theorem and extension of term ordering to $\mathbb{Q}^n$ and $\mathbb{R}^n$ exist on different literature [Robbiano (1985), and Tesemma (2007)]. But the proofs provided in this thesis are different.

Chapter 2

Preliminaries

In this chapter, we describe some important characteristics of monomial ordering for $\mathbb{K}[\mathbf{X}^{\pm 1}]$, via the multiplicative order of $\mathbb{Z}^n$, and show its unique extension to $\mathbb{Q}^n$. Meanwhile after developing tools like, rational dimension of a vector (element) in $\mathbb{R}^n$, we revisit L. Robbiano theorem of term ordering on polynomial ring [Robbiano (1985)]. The rational dimension and Robbianos' theorem help us in characterizing multiplicative orders and also to show the existence of an extension to the real space $\mathbb{R}^n$. Further fundamental domains for the action of a finite group $\mathcal{G}$ on $\mathbb{Z}^n$ and $\mathbb{Q}^n$ is defined and discussed in relation with the image of initial algebra invariant subring. In addition some important theorems about semigroup algebra and semigroup rings is briefly discussed and some comparison between algebra and semigroup algebra generators is given.

In the last section, we considered finite reflection groups in $GL(n, \mathbb{Z})$ and show that for a given $\succcurlyeq \in \Omega$, image of the initial algebra of the associated invariant ring ($in_{\succcurlyeq}(\mathbf{R}) = \mathfrak{A}^{\succcurlyeq}$) is finitely generated semigroup. The semigroup algebra generators of the invariant ring constructed by Tesemma in [Theorem 1.3.4] and Lorenzs' in [Theorem 1.3.1], for finite reflection group are linked [Theorem 2.6.2]. In fact we show the arbitrary choice of base for the root system in Lorenz proof is fixed via the multiplicative order. We show that, a given multiplicative order in Ω uniquely determine a base for root system of the reflection group via the initial algebra.

2.1 Lattice ordering

In term ordering of the polynomial algebra $\mathbb{K}[\mathbf{X}]$ one uses a total order of the semigroup $\mathbb{Z}^n_{\geq 0}$, and are by definition totally ordered, compatible with the semigroup operation (addition) and is a well ordered i.e.($\mathbf{0}$ or ($\mathbf{x}^{\mathbf{0}} = 1$))is minimal (least) element. The last property ensure that the polynomial division stops after a finite steps. But the multiplicative order on $\mathbb{Z}^n$, which induces ordering on the monomials of Laurent polynomial, can't be at the same time well ordered and compatible with the group operation, the following note shows why.

Note 2.1.1. If one assumes $\succcurlyeq$ is a total order and compatible with the group operation of $\mathbb{Z}^n$, then for any nonzero $\mathbf{a} \in \mathbb{Z}^n$ compatibility property forces $\mathbf{0}$ to be between $\mathbf{a}$ and $-\mathbf{a}$. Let us assume $\mathbf{0} \succ \mathbf{a}$, then the strict decreasing sequence,

$$\mathbf{0} \succ \mathbf{a} \succ 2\mathbf{a} \succ 3\mathbf{a} \succ \cdots,$$

doesn't terminate, hence there is no least element. Conversely well ordering intern ruins compatibility with addition.

So well ordering is sacrificed to keep division (equivalently compatibility with the group operation) as division of polynomials comes a head of question of termination the division algorithm.
The following proposition shows extension of multiplicative order to $\mathbb{Q}^n$.

Proposition 2.1.1. *For each $\succcurlyeq \in \Omega$, there exists a unique order extension $\succcurlyeq_{\mathbb{Q}}$ to $\mathbb{Q}^n$, where $\succcurlyeq_{\mathbb{Q}}$ is a total order and compatible with addition of $\mathbb{Q}^n$.*

Proof. Consider $\mathbb{Q}^n$ as a module of fraction the free $\mathbb{Z}$ module $\mathbb{Z}^n$, i.e.

$$\mathbb{Q}^n = \{\frac{\mathbf{a}}{r} : \quad \mathbf{a} \in \mathbb{Z}^n, r \in \mathbb{Z}^*\}$$

where $\mathbb{Z}^* = \mathbb{Z} \setminus \{\mathbf{0}\}$ and, $\frac{\mathbf{a}}{r} = \{(\mathbf{b}, t) : \quad (\mathbf{b}, t) \sim (\mathbf{a}, r)\} \in \mathbb{Q}^n$, is the equivalent class of the relation $\sim$ on $\mathbb{Z}^n \times \mathbb{Z}^*$, given by,

$$(\mathbf{b}, t) \sim (\mathbf{a}, r) \quad \textit{if} \quad t\mathbf{a} = r\mathbf{b}$$

$\mathbb{Q}^n$ is a $\mathbb{Z}$ module and we have an embedding,

$$\mathbb{Z}^n \hookrightarrow \mathbb{Q}^n \quad \mathbf{a} \mapsto \frac{\mathbf{a}}{1}(\frac{m\mathbf{a}}{m})$$

for $m \in \mathbb{Z}^*$. Using the fact $(\mathbf{b}, t) \sim (-\mathbf{b}, -t)$, we can assume,

$$\mathbb{Q}^n = \{\frac{\mathbf{a}}{r} : \quad \mathbf{a} \in \mathbb{Z}^n, r \in \mathbb{N}\}.$$

Define $\succcurlyeq_{\mathbb{Q}}$ by,

$$\frac{\mathbf{a}}{r} \succcurlyeq_{\mathbb{Q}} \frac{\mathbf{b}}{t} \quad \textit{iff} \quad t\mathbf{a} \succcurlyeq r\mathbf{b} \tag{2.1.1}$$

clearly $\succcurlyeq_{\mathbb{Q}}$ is a total ordering and if $\frac{\mathbf{a}}{r} \neq \frac{\mathbf{b}}{t}$, then $t\mathbf{a} \neq r\mathbf{b}$ and vise versa. Since $\succcurlyeq$ is a total ordering in $\mathbb{Z}^n$ so is $\succcurlyeq_{\mathbb{Q}}$. Uniqueness and compatibility with the addition in $\mathbb{Q}^n$ is a direct forward. □

Note 2.1.2. For any given $\succcurlyeq \in \Omega$ its' extension to $\mathbb{Q}^n$ i.e. $\succcurlyeq_{\mathbb{Q}}$ is unique and hence we employ the same notation $\succcurlyeq$ for both.

Definition 2.1.3. A *Monoid* is a set M, with an associative binary operation $\star$ (*a semi-group*) which contain identity element. M is called a *commutative monoid* if $a \star b = b \star a$ for all $a, b \in M$.

A non empty subset S of a semigroup M is said to be *subsemigroup* if S is a semigroup with respect to the binary operation of M. Furthermore an element $m \in M$ is said to be invertible if there exist $m' \in M$ such that $m \star m' = m' \star m = e$ where e is the identity element of M. A subsemigroup T is called a *subgroup* of the monoid M if T contains the inverse of each of its element, i.e. T is a group. A monoid M is *reduced (positive)*, if M doesn't contain invertible elements or the maximal subgroup (with respect to inclusion) is the trivial (identity).

In this thesis, most of the semigroups will be subsemigroups of $\mathbb{Z}^n$, and these semigroups are *cancellative* i.e. ($\mathbf{a}+\mathbf{c}=\mathbf{b}+\mathbf{c} \implies \mathbf{a}=\mathbf{b}$) and *torison free* i.e. ($n\mathbf{a}=n\mathbf{b}$ for n positive integer $\mathbf{a}=\mathbf{b}$) for $\mathbf{a},\mathbf{b},\mathbf{c}\in S$. A subsemigroup S of $\mathbb{Z}^n$ is said to be *saturated* if $m\mathbf{a}\in S$ for $\mathbf{a}\in\mathbb{Z}^n$ and $m\in\mathbb{N}$, implies $\mathbf{a}\in S$. For detail on finitely generated commutative monoids of $\mathbb{Z}^n$, we refer the reader to [Rosales and García-Sánchez (1999)].

Let $\succcurlyeq\in\Omega$ define the positive set (associated with $\succcurlyeq$),

$$\mathbf{P}=\{\mathbf{a}\in\mathbb{Z}^n:\mathbf{a}\succcurlyeq\mathbf{0}\}\qquad \mathbf{P}(\mathbb{Q})=\{\mathbf{u}\in\mathbb{Q}^n:\mathbf{u}\succcurlyeq\mathbf{0}\}$$

The following proposition gives some properties of these sets over $\mathbb{Z}^n$ and $\mathbb{Q}^n$. Furthermore justify the use of term positive set as most mathematics literature dictates sum of two positive elements to be a positive element and the whole set is the union of positive, zero and inverses of the positive elements.

Proposition 2.1.2. *Let $\succcurlyeq\Omega$ be any ordering then,*

1. $\mathbb{Z}^n=\mathbf{P}\cup-\mathbf{P}$ *and* $\mathbf{P}\cap-\mathbf{P}=\{\mathbf{0}\}$, *(analogously this holds over $\mathbb{Q}^n$).*
2. $\mathbf{P}$ *is a saturated subsemigroup of $\mathbb{Z}^n$ with trivial maximal subgroup ($\{\mathbf{0}\}$).*
3. $\mathbb{Q}_+\mathbf{P}=\mathbf{P}(\mathbb{Q})$, *where* $\mathbb{Q}_+\mathbf{P}=\{\sum_{i=0}^{t}r_i\mathbf{a}_i:r_i\in\mathbb{Q}_+\quad \mathbf{a}_i\in\mathbf{P},t\in\mathbb{N}\}$.
4. *For a subspace Γ of $\mathbb{Q}^n$ then Γ is totally ordered and,*

$$\mathbf{P}_\Gamma=\{\mathbf{u}\in\Gamma:\ \mathbf{u}\succcurlyeq\mathbf{0}\}=\mathbf{P}(\mathbb{Q})\cap\Gamma,$$

and $\Gamma=\mathbf{P}_\Gamma\cup-\mathbf{P}_\Gamma$.

Proof. 1. For $\mathbf{0}\neq\mathbf{a}\in\mathbb{Z}^n$ then either $\mathbf{a}\succ\mathbf{0}$ or $\mathbf{0}\succ\mathbf{a}$ let assume $\mathbf{a}\succ\mathbf{0}$ since $\succcurlyeq$ is compatible with addition $\mathbf{0}=\mathbf{a}+(-\mathbf{a})\succ\mathbf{0}+(-\mathbf{a})=-\mathbf{a}$, and the fact that it is a total order yields

$$\mathbb{Z}^n=\{\mathbf{a}\in\mathbb{Z}^n:\mathbf{a}\succcurlyeq\mathbf{0}\}\cup\{\mathbf{a}\in\mathbb{Z}^n:\mathbf{0}\succcurlyeq\mathbf{a}\}.$$

clearly,

$$\{\mathbf{a}\in\mathbb{Z}^n:\mathbf{0}\succcurlyeq\mathbf{a}\}=\{\mathbf{a}\in\mathbb{Z}^n:-\mathbf{a}\succcurlyeq\mathbf{0}\}=\{\mathbf{a}\in\mathbb{Z}^n:-\mathbf{a}\in\mathbf{P}\}=-\mathbf{P}.$$

which further imply $\mathbf{P}\cap-\mathbf{P}=\{\mathbf{0}\}$. Similarly $\mathbf{P}(\mathbb{Q})\cup-\mathbf{P}(\mathbb{Q})=\mathbb{Q}^n$ and $\mathbf{P}(\mathbb{Q})\cap-\mathbf{P}(\mathbb{Q})=\{\mathbf{0}\}$ over $\mathbb{Q}^n$.

2. Clearly $\mathbf{P}$ is a subsemigroup of $\mathbb{Z}^n$, and to show $\mathbf{P}$ is saturated, Let $n\mathbf{a}\in\mathbf{P}$ for $\mathbf{a}\in\mathbb{Z}^n$ and $n\in\mathbb{N}$, suppose $\mathbf{a}\notin\mathbf{P}$, hence $\mathbf{0}\succ\mathbf{a}$ then by induction on $k=2,3,\cdots$

and using addition compatibility of $\succcurlyeq$,

$$\mathbf{0} \succ \mathbf{a} \succ \cdots \succ (k-1)\mathbf{a} \succ k\mathbf{a}.$$

for any $k \in \mathbb{N}$, which implies $\mathbf{0} \succ n\mathbf{a}$, a contradiction to the fact $n\mathbf{a} \in \mathbf{P}$ as $\mathbf{P} \cap -\mathbf{P} = \{\mathbf{0}\}$ by (1) above. Hence $\mathbf{a} \in \mathbf{P}$ and $\mathbf{P}$ is saturated semigroup. Further more $\mathbf{0} \neq \mathbf{a} \in \mathbb{Z}^n$, then either $\mathbf{a}$ or $-\mathbf{a}$ is in $\mathbf{P}$ (not both). Therefor the only subgroup of $\mathbf{P}$ is the trivial (i.e $\{\mathbf{0}\}$), and $\mathbf{P}$ is positive, saturated subsemigroup of $\mathbb{Z}^n$.

3. Setting $\mathbb{Q}^n = \{\frac{\mathbf{a}}{r} : \mathbf{a} \in \mathbb{Z}^n, r \in \mathbb{N}\}$, then from the definitions $\mathbf{P}, \mathbf{P}(\mathbb{Q})$ and extension eq:2.1.1,
$$\frac{\mathbf{a}}{r} \in \mathbf{P}(\mathbb{Q}) \quad \textit{iff} \quad \mathbf{a} \in \mathbf{P}.$$
Since $\frac{\mathbf{a}}{r} \succcurlyeq \mathbf{0}$ if and only if $\mathbf{a} \succcurlyeq \mathbf{0}$, and the fact finite sum, $\sum_{i=0}^{t} r_i \mathbf{a}_i \sim \frac{\mathbf{a}}{r}$, for some $\mathbf{a} \in \mathbb{Z}^n$ and $r \in \mathbb{N}$, implies that $\mathbb{Q}_+\mathbf{P} = \mathbf{P}(\mathbb{Q})$.
4. Clearly $\Gamma \subset \mathbb{Q}^n$ is totally ordered, and compatible with the addition of subspace Γ. By (1) $\Gamma = \mathbf{P}_\Gamma \cup -\mathbf{P}_\Gamma$, and further $\mathbf{u} \in \mathbf{P}_\Gamma$ then, $\mathbf{u} \in \mathbb{Q}^n$ and $\mathbf{u} \succcurlyeq \mathbf{0}$, hence $\mathbf{u} \in \Gamma \cap \mathbf{P}(\mathbb{Q})$. Conversely $\mathbf{P}(\mathbb{Q}) \cap \Gamma \subset \mathbf{P}_\Gamma$, therefore $\mathbf{P}(\mathbb{Q}) \cap \Gamma = \mathbf{P}_\Gamma$, for Γ a subspace of $\mathbb{Q}^n$.

□

In part (4) the above proposition one can infer that any subspace of $\mathbb{Q}^n$ is totally ordered and $\mathbf{P}(\mathbb{Q})$ contains the positive part of each such subspace with respect to the induced order of $\mathbb{Q}^n$. Observe that in the proposition $\mathbf{P}$ can be replaced by $-\mathbf{P}$ and the similar property holds. In fact $\mathbf{P}$ and $-\mathbf{P}$ are isomorphic semigroups.

2.2 Rational Dimension of Vectors in $\mathbb{R}^n$

The concept of rational dimension is used in classifying the term ordering and in representing a given order using some inequalities with finite vectors in $\mathbb{R}^n$ and the usual inner (dot) product.

Definition 2.2.1. Let $0 \neq \omega = (\gamma_1, \cdots, \gamma_n) \in \mathbb{R}^n$. The rational dimension of ω, denoted by $\mathfrak{rdim}(\omega)$ is the dimension of the vector space spanned by the set $\{\gamma_1, \cdots, \gamma_n\} \subset \mathbb{R}$ over the rationals $(\mathbb{Q})$.

From the definition one can directly infer that for $0 \neq \mathbf{u} \in \mathbb{Q}^n$, then the rational dimension is 1 ($\mathfrak{rdim}(\mathbf{u}) = 1$). Let $\mathbf{0} \neq \omega \in \mathbb{R}^n$ be any element, the subspace (in $\mathbb{Q}^n$),

$$\mathcal{H}_\omega = \{\mathbf{u} \in \mathbb{Q}^n : \langle \mathbf{u}, \omega \rangle = 0\} = \mathbb{Q}^n \cap \{\mathbf{v} \in \mathbb{R}^n : \langle \mathbf{v}, \omega \rangle = 0\},$$

is the maximal rational subspace contained in the hyperplane orthogonal to ω in $\mathbb{R}^n$. The following proposition give some properties of rational dimension of a vector in $\mathbb{R}^n$. It associates the rational dimension of a given vector to a dimension of the maximal subspace in $\mathbb{Q}^n$ contained in the hyperplane orthogonal to the vector in $\mathbb{R}^n$.

Proposition 2.2.1. *Let $\omega \in \mathbb{R}^n$, and $\langle \, , \rangle$, be the usual inner product of $\mathbb{R}^n$. Then,*

1. $\mathfrak{rdim}(\omega) = \min(\dim(\Gamma))$, *where Γ is a subspace in $\mathbb{Q}^n$, such that $\omega \in \mathbb{R}\Gamma$.*
2. $\mathfrak{rdim}(\omega) = n - \dim(\mathcal{H}_\omega)$.
3. *For $0 \neq r \in \mathbb{R}, \mathfrak{rdim}(r\omega) = \mathfrak{rdim}(\omega)$.*

Proof. 1. Let $\mathfrak{rdim}(\omega) = t$ since $\omega \in \mathbb{R}^n = \mathbb{R}\mathbb{Q}^n$, hence $t \leq n$. Let $\omega = (\gamma_1, \cdots, \gamma_n) \in \mathbb{R}^n$, then the vector space spanned by $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ has dimension t and there exists t linearly independent real numbers $\{r_1, r_2, \cdots, r_t\}$ over $\mathbb{Q}$ such that,

$$Span_{\mathbb{Q}}\{\gamma_1, \gamma_2, \cdots, \gamma_n\} = Span_{\mathbb{Q}}\{r_1, r_2, \cdots, r_t\}.$$

Without loss of generality let us assume each r_i to be a positive real numbers. Now for each $i = 1, 2, \cdots, n$ we have $\gamma_i = \sum_{j=1}^{t} u_{ij} r_j$ for some $u_{ij} \in \mathbb{Q}$. Hence

$$\omega = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1t} \\ u_{21} & u_{22} & \cdots & u_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nt} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_t \end{pmatrix} = \sum_{j=1}^{t} r_j \mathbf{u}^j,$$

where $\mathbf{u}^j \in \mathbb{Q}^n$ is the j^{th} column of the matrix $\mathbf{U} = (u_{ij})$. Furthermore $\{\mathbf{u}^j\}_{j=1}^t \subset \mathbb{Q}^n$ forms a linearly independent set over the rationals hence over the reals. If $\sum_{i=1}^{t} q_i \mathbf{u}^i = \mathbf{0}$, where $q_i \in \mathbb{Q}$ of which some are non zero. With out loss of generality assume $q_1 \neq 0$ then $\mathbf{u}^1 = \sum_{j=2}^{t} q_j^* \mathbf{u}^j$, where $q_j^* = \frac{q_j}{q_1}$ for $j = 2, 3, \cdots, t$. Hence each component of ω, i.e. $\gamma_i = \sum_{j=2}^{t} (r_1 q_j^* + r_j) u_{ij}$ and hence,

$$\{\gamma_1, \gamma_2, \cdots, \gamma_n\} \subset Span_{\mathbb{Q}}\{r_1 q_j^* + r_j\}_{j=2}^t$$

a contradiction to $Span_{\mathbb{Q}}\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ has dimension t and cannot be contained in dimension $t-1$ space. The vectors $\{\mathbf{u}^j\}_{j=1}^t$ are linearly independent. Setting $\Gamma = Span_{\mathbb{Q}}\{\mathbf{u}^j : j = 1, 2, \cdots, t\}$ one clearly observes Γ is of minimal dimension and $\omega \in \mathbb{R}\Gamma$.

2. Let assume the $\mathfrak{rdim}(\omega) = t$, from (1) $\omega = \sum_{j=1}^{t} r_j \mathbf{u}^j$, here we emphasis the inner product is stable over the rationals. i.e. $\forall\, \mathbf{u}, \mathbf{v} \in \mathbb{Q}^n \; < \mathbf{u}, \mathbf{v} > \in \mathbb{Q}$. It is enough to show that,

$$\{\mathbf{u}^1, \mathbf{u}^2, \cdots, \mathbf{u}^t\}^\perp = \{\mathbf{u} \in \mathbb{Q}^n : \langle \mathbf{u}, \mathbf{u}^j \rangle = 0 \quad \forall j = 1, 2, \cdots, t\} = \mathcal{H}_\omega.$$

Let $\mathbf{u} \in \mathcal{H}_\omega$, then,

$$0 = \langle \mathbf{u}, \omega \rangle = \langle \mathbf{u}, \sum_{j=1}^{t} r_j \mathbf{u}^j \rangle = \sum_{j=1}^{t} r_j \langle \mathbf{u}, \mathbf{u}^j \rangle,$$

since $\langle \mathbf{u}, \mathbf{u}^j \rangle \in \mathbb{Q}$ and $\{r_j\}_{j=1}^t$ are linearly independent over $\mathbb{Q}$ this implies that, $\forall j \quad \langle \mathbf{u}, \mathbf{u}^j \rangle = 0$. Therefor, $\mathcal{H}_\omega \subseteq \{\mathbf{u}^1, \mathbf{u}^2, \cdots, \mathbf{u}^t\}^\perp$.

Conversely $\{\mathbf{u}^1, \mathbf{u}^2, \cdots, \mathbf{u}^t\}^\perp \subseteq \mathcal{H}_\omega$, is clear. The result in (2) follows from the fact,

$$\mathbb{Q}^n = \{\mathbf{u}^1, \mathbf{u}^2, \cdots, \mathbf{u}^t\}^\perp \oplus Span_{\mathbb{Q}}\{\mathbf{u}^1, \mathbf{u}^2, \cdots, \mathbf{u}^t\}.$$

and $\{\mathbf{u}^1, \mathbf{u}^2, \cdots, \mathbf{u}^t\}$ is linearly independent from (1).

3. Follows from (1), since the linear dependency of the set $\{rr_1, rr_2, \cdots, rr_t\}$ implies linear dependency of the set $\{r_1, r_2, \cdots, r_t\}$ over $\mathbb{Q}$ for $0 \neq r \in \mathbb{R}$.

□

Remark 2.2.2. 1. In proposition 2.2.1, the definition of $\mathcal{H}_\omega$ depend on the inner product but the dimension remains same as long as the inner product $\langle\, , \rangle$ is stable on $\mathbb{Q}^n$.

2. $\forall \omega \in \mathbb{R}^n$, from proposition 2.2.1, part (2), $\mathfrak{rdim}(\omega) + dim(\mathcal{H}_\omega) = n$.

2.3 Characterizing Lattice Ordering

In this section, we give an alternative proof for Robbiano theorem of term ordering (polynomial ring), with slight change to multiplicative order (on Laurent polynomial ring). Further the construction of this proof also allow us to show the existence of extension of multiplicative order to $\mathbb{R}^n$.
For the above reasons and later use, some important concepts about cones (lattice cones) is briefly discussed below.

Definition 2.3.1. A subset **C** of $\mathbb{R}^n$ is called a **cone** (convex cone) if, $\sum_{i=1}^t r_i c_i \in \mathbf{C}$, where each $r_i \in \mathbb{R}_+$ and each $c_i \in \mathbf{C}$ and $t \in \mathbb{N}$.

From the definition one can easily infer that a cone is a semigroup in $\mathbb{R}^n$. The intersection of cones is again a cone. For $X \subset \mathbb{R}^n$ the set,

$$C = Pos(X) = \{\sum_{i=1}^t r_i x_i : x_i \in X, r_i \in \mathbb{R}_+, t \in \mathbb{N}\},$$

is the smallest cone in $\mathbb{R}^n$ containing X, (i.e. for any cone C' in $\mathbb{R}^n$ such that $X \subseteq C'$, then $C \subseteq C'$). Equivalently $Pos(X)$ is the intersection of all cones in $\mathbb{R}^n$ containing the set X, in this case C is called a cone generated by X (positive hull over $\mathbb{R}$). If $X \subset \mathbb{Z}^n$, then C is called an *integral cone*, and if the set X is finite, it is called *polyhedral*.
Let $(\mathbb{R}^n)^*$ be the dual space of $\mathbb{R}^n$ with the dual pairing $\langle\, , \rangle$. For any cone C in $\mathbb{R}^n$ the dual cone of C denoted by $C^\vee$ is defined by,

$$C^\vee = \{f \in \mathbb{R}^n : \langle c, f\rangle \geq 0 \quad \forall c \in C\}.$$

It follows from the definition that $C^\vee$ is a cone and if C polyhedral integral cone then so is $C^\vee$ [Günter (1996), Theorem 2.10]. For a given cone $C \subset \mathbb{R}^n$ define a *face* $\mathcal{F}$ of C is a subset of the form,

$$\mathcal{F} := C \cap f^{\perp} = \{c \in C : \langle c, e \rangle = 0 \quad for \quad some\, e \in f \subseteq C^{\vee}\}.$$

Each face of a cone is again a cone and the *dimension* of a cone C is the dimension of the subspace generated by C denoted by $dim(C)$. A face $\mathcal{F}$ of a cone C of dimension equal to $dim(C) - 1$ is called a *facets* and a face of dimension one is called an *edge* and of dimension zero is called a *vertex* respectively. For $\mathbf{0} \neq \alpha \in \mathbb{R}^n$ a subset,

$$\{\mathbf{u} \in \mathbb{R}^n : \langle \mathbf{u}, \alpha \rangle \geq 0\}$$

is a *half space*, and is denoted by $\mathbf{H}_{\alpha}^{+}$. Note for any positive real number r one can easily infer that, $\mathbf{H}_{\alpha}^{+} = \mathbf{H}_{r\alpha}^{+}$. Further a half space is cone in $\mathbb{R}^n$, and is a maximal proper ($\neq \mathbb{R}^n$) cone with respect to inclusion.

Remark 2.3.2. In these remark some important properties of cones in $\mathbb{R}^n$, are given. Note that these are nice results which can be found in books dealing about cones and algebraic geometry,

1. A polyhedral cone C is a closed subset of $\mathbb{R}^n$ with the usual euclidean norm topology. And clearly from the definition it is a submonoid of $\mathbb{R}^n$. Hence $C \cap \mathbb{Z}^n$ is a submonoid of $\mathbb{Z}^n$.
2. (Gordan's Lemma) Let C be an integral polyhedral cone, then $C \cap \mathbb{Z}^n$ is saturated and finitely generated monoid [Fulton (1993), Proposition 1].
3. If a cone C in $\mathbb{R}^n$ satisfies, $C \subset \cup_{i=1}^{n} W_i$, for subspaces $W_i \subset \mathbb{R}^n$, then $C \subset W_i$, for some i. In particular, the subspace generated by C is contained in W_i. Here we use the fact from linear algebra, that any finite dimensional vector space over infinite field can't be expressed as a finite union of proper subspaces.
4. Every proper cone C, i.e. ($C \neq \mathbb{R}^n$), is contained in a half space $\mathbf{H}_{\alpha}^{+}$, for some $\alpha \in \mathbb{R}^n$ (equivalently $C^{\vee}$ is non trivial).

For a semigroup $S \subseteq \mathbb{Z}^n$ we define the rank of S to be the rank of the smallest group in which S can be embedded. The linear span of S over $\mathbb{Q}$ denoted by $Span_{\mathbb{Q}}(S)$ is the smallest subspace of $\mathbb{Q}^n$ containing S, and the $dim(S)$ is the dimension of $Span_{\mathbb{Q}}(S)$. For further reading on cones (lattice) [Günter (1996)].

Example 2.3.3. For $\succcurlyeq \in \Omega$ the positive set $\mathbf{P}$ is not finitely generated semigroup for $(n \geq 2)$ in $\mathbb{Z}^n$. It has rank and dimension n. The positive hull over $\mathbb{R}$ i.e.

$$\mathbb{R}_{+}(\mathbf{P}) = \{\sum_{i=1}^{t} r_i \mathbf{a}_i : r_i \in \mathbb{R}_{+}, \quad \mathbf{a}_i \in \mathbf{P}\}$$

is a convex cone in $\mathbb{R}^n$ of dimension n.

The proof of following theorem is included as it is previously done in view of characterizing ordering on polynomial rings rather than Laurent polynomial rings. It will be used for forthcoming theorem extending the multiplicative order of $\mathbb{Z}^n$ to $\mathbb{R}^n$ and Proposition 2.4.2. One can also see the proof in original Robbiano article [Robbiano (1985)]. Here we give equivalent proof using cones (lattice).

Theorem 2.3.1. *(Robbiano 1985) Let $\succcurlyeq \in \Omega$ be any ordering. Then there exists mutually orthogonal vectors $\{\omega_1, \omega_2, ..., \omega_s\} \subset \mathbb{R}^n$ with $\sum_{i=1}^{s} \mathfrak{rdim}(\omega_i) = n$ and*

$$\omega_i \subseteq \mathbb{R}(\cap_{k=1}^{i-1} \mathcal{H}_{\omega_k}), \tag{2.3.1}$$

and for any $\mathbf{a}, \mathbf{b} \in \mathbb{Q}^n$,

$$\mathbf{a} \succcurlyeq \mathbf{b} \iff (\langle \mathbf{a}, \omega_1 \rangle, \cdots, \langle \mathbf{a}, \omega_s \rangle) \succcurlyeq_{lex} (\langle \mathbf{b}, \omega_1 \rangle, \cdots, \langle \mathbf{b}, \omega_s \rangle).$$

where $\langle\ ,\ \rangle$, is usual inner product and $\succcurlyeq_{lex}$, is the usual lex ordering in $\mathbb{R}^s$.

Proof. We show by induction on the dimension of $\mathbb{Q}^n$. For $n = 1$, take $0 \neq \gamma \in \mathbb{R}_+$, then $\{\gamma\}$ has rational dimension 1, and satisfies the above conditions vacuously. Suppose the theorem holds for dimension $k < n$.

Let $\succcurlyeq$ be a multiplicative order on $\mathbb{Z}^n$ consider the unique extension to $\mathbb{Q}^n$ and let $\mathbf{C}$ be the positive hull of $\mathbf{P}$ in $\mathbb{R}$ i.e.

$$\mathbf{C} = \mathbb{R}_+\mathbf{P}(\mathbb{Q}) = \mathbb{R}_+\mathbf{P} = \{\sum_{i=1}^{t} r_i \mathbf{a}_i : \mathbf{a}_i \in \mathbf{P}, r_i \in \mathbb{R}_+\} \subseteq \mathbb{R}^n$$

then C is a proper cone in $\mathbb{R}^n$ (since $\mathbf{C} \cap \mathbb{Q}^n = \mathbf{P}(\mathbb{Q})$) and $\mathbf{P}$ is a positive semigroup.

$$\mathbf{C}^\vee = \{\varphi \in \mathbb{R}^n : \langle \mathbf{v}, \varphi \rangle \geq 0 \quad \forall \quad \mathbf{v} \in \mathbf{C}\}$$

then $\mathbf{C}^\vee$ non trivial and let $\mathbf{0} \neq \alpha \in \mathbf{C}^\vee$,

Claim $\mathbf{C}^\vee = \mathbb{R}_+\alpha$

Suppose there exists $\mathbf{0} \neq \beta \in \mathbb{R}^n$ such that $\beta \in \mathbf{C}^\vee$. It is enough to show that

$$\mathbf{H}_\beta^+ = \mathbf{H}_\alpha^+.$$

We show this by contradiction, assume $\mathbf{H}_\beta^+ \neq \mathbf{H}_\alpha^+$ then the half spaces $\mathbf{H}_\alpha^+$ &$\mathbf{H}_\beta^+$ each containing $\mathbf{C}$ and divide $\mathbb{R}^n$ in to four open (chamber) compartments A, B, C and D see: Fig.1.

$$\begin{aligned} A =& \{\mathbf{v} \in \mathbb{R}^n : \langle \mathbf{v}, \beta \rangle > 0 \quad \textit{and} \quad \langle \mathbf{v}, \alpha \rangle < 0\}, \\ B =& \{\mathbf{v} \in \mathbb{R}^n : \langle \mathbf{v}, \beta \rangle < 0 \quad \textit{and} \quad \langle \mathbf{v}, \alpha \rangle > 0\}, \\ C =& \{\mathbf{v} \in \mathbb{R}^n : \langle \mathbf{v}, \beta \rangle < 0 \quad \textit{and} \quad \langle \mathbf{v}, \alpha \rangle < 0\}, \\ D =& \{\mathbf{v} \in \mathbb{R}^n : \langle \mathbf{v}, \beta \rangle > 0 \quad \textit{and} \quad \langle \mathbf{v}, \alpha \rangle > 0\}. \end{aligned}$$

Each chamber has dimension n and for any rational point $\mathbf{u} \in B$ then $-\mathbf{u} \in A$ and vice versa, which implies $\pm\mathbf{u} \notin \mathbf{P}(\mathbb{Q}) \subseteq \mathbf{C}$, a contradiction to the fact $\mathbf{P}(\mathbb{Q})$ contain one of $\pm\mathbf{u}$ for any $\mathbf{u} \in \mathbb{Q}^n$ [by proposition 2.1.2, (1)]. Hence the chamber A (equivalently B) is

empty implies both α, β to be orthogonal to the same hyperplane and $\mathbf{H}_\beta^+ = \mathbf{H}_\alpha^+$, there fore $\beta = r\alpha$ for $r \in \mathbb{R}_+$.
Set $\alpha = \omega_1$ and [by Proposition 2.2.1 (2)] there exists linearly independent set $\{\mathbf{u}_{1j} : j = 1, 2, \cdots, d_1\} \subseteq \mathbb{Q}^n$, such that $\omega_1 = \sum_{i=1}^{d_1} r_j \mathbf{u}_{1j}$ where, $d_1 = \mathfrak{rdim}(\omega_1)$. Furthermore

$$\mathcal{H}_{\omega_1} = \mathbf{H}_{\omega_1} \cap \mathbb{Q}^n = \{\mathbf{u}_{1j} : j = 1, 2, \cdots, d_1\}^\perp =: \Gamma$$

is of dimension $n - \mathfrak{rdim}(\omega_1) = n - d_1$. Since Γ is a subspace of $\mathbb{Q}^n$, and [by proposition 2.1.2,(4)] $\succcurlyeq$ is a total order compatible with addition of Γ. By induction there exists mutually orthogonal vectors $\{\omega_2, \omega_3, \cdots, \omega_s\}$, in $\mathbb{R}(\Gamma)$ such that, $\sum_{i=2}^{s} \mathfrak{rdim}(\omega_i) = n - d_1$ and $\omega_i \subseteq \mathbb{R}(\cap_{k=1}^{i-1} \mathcal{H}_{\omega_k})$, for $i = 3, 4, \cdots, s$. Further more $\mathbf{P}_\Gamma = \Gamma \cap \mathbf{P}(\mathbb{Q})$ for $\mathbf{a}, \mathbf{b} \in \Gamma$ and $\mathbf{a} \succcurlyeq \mathbf{b} \iff \mathbf{a} - \mathbf{b} \succcurlyeq \mathbf{0} \iff \mathbf{a} - \mathbf{b} \in \mathbf{P}_\Gamma$. There fore

$$\mathbf{a} - \mathbf{b} \succcurlyeq \mathbf{0} \iff (\langle \mathbf{a} - \mathbf{b}, \omega_2 \rangle, \cdots, \langle \mathbf{a} - \mathbf{b}, \omega_s \rangle) \succcurlyeq_{lex} \mathbf{0}.$$

now for any $\mathbf{a}, \mathbf{b} \in \mathbb{Q}^n$ such that $\mathbf{a} \succcurlyeq \mathbf{b}$ if and only if $\mathbf{a} - \mathbf{b} \in \mathbf{P}(\mathbb{Q})$ then,

$$\langle \mathbf{a} - \mathbf{b}, \omega_1 \rangle > 0 \quad \texttt{else} \quad \langle \mathbf{a} - \mathbf{b}, \omega_1 \rangle = 0 \quad \& \quad \mathbf{a} - \mathbf{b} \in \mathbf{P}_\Gamma.$$

whence merging the two and $\mathbb{R} \times \mathbb{R}^{s-1} = \mathbb{R}^s$ one have,

$$\begin{aligned} \mathbf{a} - \mathbf{b} \succcurlyeq \mathbf{0} &\iff (\langle \mathbf{a} - \mathbf{b}, \omega_1 \rangle, \cdots, \langle \mathbf{a} - \mathbf{b}, \omega_s \rangle) \succcurlyeq_{lex} \mathbf{0} \\ &\iff (\langle \mathbf{a}, \omega_1 \rangle, \cdots, \langle \mathbf{a}, \omega_s \rangle) \succcurlyeq_{lex} (\langle \mathbf{b}, \omega_1 \rangle, \cdots, \langle \mathbf{b}, \omega_s \rangle). \end{aligned}$$

□

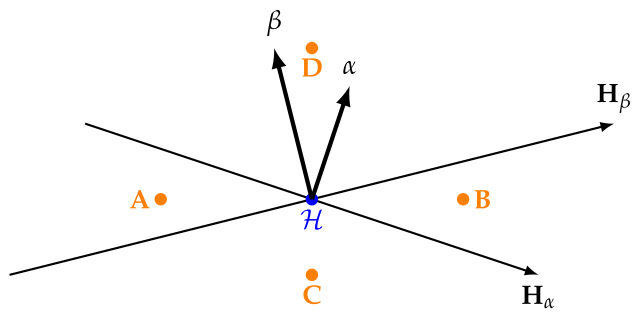


Fig.1

Definition 2.3.4. In the above proof the positive integer s, gives the **type** of $\succcurlyeq$, which is the minimum number of vectors in $\mathbb{R}^n$, used for the lex representation of $\succcurlyeq$.

Example 2.3.5. The usual lex ordering on $\mathbb{Z}^n$ is type n, and is not archimedean for $n > 1$. Since $e_1 \succ e_2$ and for any $t \in \mathbb{N} \quad e_1 \succ te_2$.

Remark 2.3.6. 1. For a fixed inner product (where the rational vectors are stable as in the proof of proposition 2.2.1,2), the vectors are unique up to positive scalar multiple and adding or subtracting any linear combination of $\{\omega_1, \omega_2, \cdots, \omega_{k-1}\}$ on ω_k for $k = 2, 3, \cdots, s$.

2. $\succcurlyeq \in \Omega$ is *archimedean* if and only if $\succcurlyeq$ is *type* 1. See (Tesemma and Wang, 2011, Proposition 2.5), In such case there exist $\omega \in \mathbb{R}^n$ of rational dimension n, such that $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$ and,

$$\mathbf{a} \succcurlyeq \mathbf{b} \quad \textit{iff} \quad \langle \mathbf{a}, \omega \rangle \geq \langle \mathbf{b}, \omega \rangle$$

We denote archimedean order by $\succcurlyeq_\omega$ where ω is defined as above. Note that for non zero real number r, $\succcurlyeq_\omega$ is same ordering as $\succcurlyeq_{r\omega}$.

3. In view of proposition 2.2.1, and theorem 2.3.1, if we set $\omega_i = \sum_{j=1}^{d_i} r_{ij}\mathbf{u}_{ij}$, for $i = 1, 2, \cdots, s$, where $d_i = \mathfrak{rdim}(\omega_i)$, then we have s collection of sets,

$$\{\{\mathbf{u}_{ij}\}_{j=1}^{d_i} : i = 1, 2, \cdots, s\},$$

which are mutually orthogonal (i.e $\langle \mathbf{u}_{ij}, \mathbf{u}_{kl} \rangle = \delta_{ik}\langle \mathbf{u}_{ij}, \mathbf{u}_{il} \rangle$), and also the n vectors $\{\mathbf{u}_{ij} : 1 \leq j \leq d_i \quad i = 1, 2, \cdots, s\} \subset \mathbb{Q}^n$ span $\mathbb{R}^n$. Further $(d_1, d_2, \cdots, d_s)$ gives partition type of the order. see [Robbiano (1985)].

4. Let $\succcurlyeq \in \Omega$ and $\mathbf{P}$ is the positive set for $\succcurlyeq$, then $\mathbf{a} - \mathbf{b} \in \mathbf{P}$ if and only if $\mathbf{a} \succcurlyeq \mathbf{b}$, hence the positive set $\mathbf{P}$ uniquely determine $\succcurlyeq$ on $\mathbb{Z}^n$ similarly for $\mathbb{Q}^n$.

Example 2.3.7. Let $\succcurlyeq_{glex}$ be the graded lexicographic order on $\mathbb{Z}^n$ i.e. $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ and $\mathbf{b} = (b_1, b_2, \cdots, b_n) \in \mathbb{Z}^n$,

$$\mathbf{a} \succcurlyeq_{glex} \mathbf{b} \quad \textit{if} \sum_{i=1}^{n} a_i > \sum_{i=1}^{n} b_i,$$

or if

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i \quad \textit{then} \quad \mathbf{a} \succcurlyeq_{lex} \mathbf{b}$$

using the usual inner product $\mathbf{a} \in \mathbf{P}$ if $\langle \mathbf{a}, (1, 1, \cdots, 1) \rangle = \sum_{i=1}^{n} a_i > 0$, or if $\sum_{i=1}^{n} a_i = 0$, then first non zero $a_i > 0$, we take $\langle \mathbf{a}, e_1 \rangle = a_1 > 0$, while $\langle \mathbf{a}, (1, 1, \cdots, 1) \rangle = 0$ take ω_2 orthogonal projection of e_1 to $\omega_1 = (1, 1, \cdots, 1)$. In a similar manner $\langle \mathbf{a}, \omega_1 \rangle = \langle \mathbf{a}, \omega_2 \rangle = 0$ i.e. $\sum_{i=1}^{n} a_i = 0$ and $a_1 = 0$, consider $a_2 > 0$ and in this case $\sum_{i=2}^{n} a_i = 0$ and $a_2 > 0$, take the orthogonal projection of e_2 to $\omega_1, \omega_2 = (0, 1, 1, \cdots, 1)$ to get ω_3, continuing this way one can evaluate n orthogonal vectors (up to positive multiple),

$$\begin{aligned}
\omega_1 &= (1, 1, 1, \cdots, 1, 1) \\
\omega_2 &= (n-1, -1, -1, \cdots, -1, -1) \\
\omega_3 &= (0, n-2, -1, \cdots, -1, -1) \\
\omega_4 &= (0, 0, n-3, \cdots, -1, -1) \\
&\vdots \\
\omega_n &= (0, 0, 0, \cdots, 1, -1)
\end{aligned}$$

which shows $\succcurlyeq_{glex}$ is of type n.

Corollary 2.3.2. *$\succcurlyeq \in \Omega$ can be extended to a total order on $\mathbb{R}^n$ compatible with addition. Such extensions are not necessarily unique except $\succcurlyeq$ is type n.*

Proof. By theorem 2.3.1 for any $\succcurlyeq \in \Omega$ there exist vectors $\{\omega_1, \omega_2, \cdots, \omega_s\} \in \mathbb{R}^n$ where $s \leq n$, and satisfy the lex representation therewith, consider the subspace of $\mathbb{R}^n$,

$$\mathbf{L} = \{v \in \mathbb{R}^n : \quad \langle v, \omega_i \rangle = 0 \quad \forall i = 1, 2, \cdots, s\} = \{\omega_1, \omega_2, \cdots, \omega_s\}^{\perp}$$

$\mathbf{L}$ is non trivial if $s < n$, now the $dim(\mathbf{L}) = n - s$ and consider the vectors $\{\omega_{s+1}, \cdots, \omega_n\}$ generating $\mathbf{L}$, then the lex ordering on $(\mathbb{R}^n)$ by $\{\omega_1, \omega_2, \cdots, \omega_n\}$ form a total ordering compatible with addition of $\mathbb{R}^n$, further both the associated lex ordering of the vectors $\{\omega_1, \omega_2, \cdots, \omega_s\}$ and $\{\omega_1, \omega_2, \cdots, \omega_n\}$ give the same ordering on $\mathbb{Q}^n$. Observe that, $\mathbf{u} \in \mathbb{Q}^n \quad \langle \mathbf{u}, \omega_i \rangle = 0 \quad i = 1, 2, \cdots, s$ if and only if $\langle \mathbf{u}, \mathbf{u}_{ij} \rangle = 0 \quad \forall i = 1, 2, \cdots, s$ and $1 \leq j \leq d_i$, if and only if $\mathbf{u} = \mathbf{0}$. There fore both order give the same ordering on $\mathbb{Q}^n$, hence on $\mathbb{Z}^n$.
If $\succcurlyeq \in \Omega$ is type n, i.e. $s = n$ then the extension is unique, since each ω_i has rational dimension 1 and mutually orthogonal. □

Note 2.3.8. For a given $\succcurlyeq \in \Omega$ we denote its unique extension in $\mathbb{Q}^n$ and any one of its extension to $\mathbb{R}^n$ by the same symbol. i.e. $\succcurlyeq$. Further we use the notation for $\succcurlyeq \in \Omega$ by

$$\{\langle\, , \omega_1 \rangle, \langle\, , \omega_2 \rangle, \cdots \langle\, , \omega_s \rangle\}$$

where $\{\omega_1, \omega_2, \cdots, \omega_s\}$ is as in the proof of 2.3.1.

Further for $n > 1$ a total ordering on $\mathbb{R}^n$ compatible with addition are forced to be type n and hence there is no archimedean order on $\mathbb{R}^n$ for $n > 1$.

2.4 Initial Algebra of Invariant rings and Fundamental Domains

In this section we show the relationship between initial algebra of the invariant ring and image of initial map, a subsemigroup of $\mathbb{Z}^n$. First let consider examples of initial algebra for a subalgebra of Laurent polynomial,

Example 2.4.1. Let $\succcurlyeq_{lex}$ be the usual lexicographic ordering of $\mathbb{Z}^2$ with $e_1 \succcurlyeq_{lex} e_2$ consider the Laurent polynomial, $\mathbb{K}[x^{\pm 1}, y^{\pm 1}]$.

1. Let $R = \mathbb{K}[f_1, f_2, f_3]$ be a subalgbra of $\mathbb{K}[x^{\pm 1}, y^{\pm 1}]$, where,

$$\begin{aligned} f_1 =& x + y + x^{-1} + y^{-1} + xy + x^{-1}y^{-1} \\ f_2 =& xy^{-1} + x^{-2}y^{-1} + xy^2 \\ f_3 =& x^2y + x^{-1}y + x^{-1}y^{-2} \end{aligned}$$

The initials of this algebra generators are, $\mathbf{x}^{in_{\succcurlyeq}(f_1)} = xy = \mathbf{x}^{(1,1)}, \quad \mathbf{x}^{in_{\succcurlyeq}(f_2)} = xy^2 = \mathbf{x}^{(1,2)}$ and $\mathbf{x}^{in_{\succcurlyeq}(f_3)} = x^2y = \mathbf{x}^{(2,1)}$ and the initial algebra for R is

$$in_{\succcurlyeq}(R) = \mathbb{K}[xy, xy^2, x^2y]$$

So f_1, f_2 and f_3 form a SAGBI basis for R. One can also observe the distribution of the support of f_i for $i = 1,2,3$, in $\mathbb{Z}^2$, and changing the order may give different initial algebra but f_1, f_2 and f_3 remain SAGBI basis.

2. Let $R = \mathbb{K}[f_1, f_2, f_3]$, where

$$\begin{aligned} f_1 =& x + x^{-1} \\ f_2 =& y^{-1} + y \\ f_3 =& xy + x^{-1}y^{-1} \end{aligned}$$

If we claim $in_{\succcurlyeq}(R) = \mathbb{K}[x,y]$, using the initial of the algebra generators, Then $g = xy^{-1} + yx^{-1} = f_1f_2 - f_3 \in R$, but $\mathbf{x}^{in_{\succcurlyeq}(g)} = xy^{-1} \notin \mathbb{K}[x,y]$, In fact the initial algebra is not finitely generated and,

$$in_{\succcurlyeq}(R) = \oplus_{\mathbf{a}\in P}\mathbb{K}\mathbf{x}^{\mathbf{a}}$$

where $p = \{\mathbf{a} \in \mathbb{Z}^2 : \mathbf{a} \succcurlyeq_{lex} \mathbf{0}\}$, a semigroup of $\mathbb{Z}^2$, which is not finitely generated.

The initial algebra of a subalgebra may not be finitely generated algebra, even if the sub-algebra is finitely generated, (see in the case polynomial algebra [Robbiano and Sweedler (1988)]) and also depend on the order.

Definition 2.4.2. (Fundamental Domain for a Group Action) Suppose a group $\mathcal{G}$ acts on a set X, we call a subset $F \subset X$ a *fundamental domain* for the action if each $\mathcal{G}$-orbit in X intersect F in exactly one point. Equivalently $\forall x \in X \quad \exists g \in \mathcal{G} \quad g \circ x \in F$ and if some $x_1, x_2 \in F$ and $g \in \mathcal{G}$ satisfies $x_1 = g \circ x_2$ then $x_1 = x_2$.

Note in the equivalent definition the first statement declares that $X = \cup_{g\in\mathcal{G}} g(F)$. Hence for a finite group acting by automorphism on a finite dimensional vector space over an infinite field, (In view of the fact a finite dimensional vector space can't be covered by union of finitely many proper subspaces.) these fundamental set have equals dimension to the vector space in which the finite group acts.

For the action of a finite group $\mathcal{G}$ in $GL(n,\mathbb{Z})$, let denote the invariant ring $\mathbb{K}[\mathbf{X}^{\pm 1}]^{\mathcal{G}}$ by $\mathbf{R}$. i.e.

$$\mathbf{R} = \{f \in \mathbb{K}[\mathbf{X}^{\pm 1}] : g \star f = f \quad \forall g \in \mathcal{G}\}$$

Let $\succcurlyeq \in \Omega$ and $I \neq g \in \mathcal{G}$ define the following sets,

$$A_g = \{\mathbf{a} \in \mathbb{Z}^n : \mathbf{a} \succcurlyeq g(\mathbf{a})\} \qquad A_g(\mathbb{Q}) = \{\mathbf{u} \in \mathbb{Q}^n : \mathbf{u} \succcurlyeq g(\mathbf{u})\}$$

and the lead element of each orbits over $\mathbb{Z}^n$,

$$\mathfrak{A}^{\succcurlyeq} = \{\mathbf{a} \in \mathbb{Z}^n : \mathbf{a} \succcurlyeq g(\mathbf{a}) \ \forall g \in \mathcal{G}\} = \cap_{g\in\mathcal{G}} A_g = \{max_{\succcurlyeq}[\mathbf{b}] : \mathbf{b} \in \mathbb{Z}^n/\mathcal{G}\}$$

where $\mathbb{Z}^n/\mathcal{G}$ is the transverse element for the orbits, further $max_{\succcurlyeq}[\mathbf{b}]$ is well defined since $[\mathbf{b}]$ is a totally ordered finite subset of $\mathbb{Z}^n$. In a similarly manner we define analogously for $\mathbb{Q}^n$,

$$\mathfrak{A}^{\succcurlyeq}(\mathbf{Q}) = \{\mathbf{u} \in \mathbb{Q}^n : \mathbf{u} \succcurlyeq g(\mathbf{u})\, \forall g \in \mathcal{G}\} = \cap_{g\in\mathcal{G}} A_g(\mathbb{Q}).$$

To each orbit $[\mathbf{b}] = \{g(\mathbf{b}) : g \in \mathcal{G}\}$ define the **orbit sum** of $\mathbf{b}$ (denoted by $\mathfrak{o}(\mathbf{b})$) to be the Laurent polynomial,

$$\mathfrak{o}(\mathbf{a}) = \sum_{\mathbf{b}\in[\mathbf{a}]} \mathbf{x}^{\mathbf{b}} \tag{2.4.1}$$

the orbits partitioned $\mathbb{Z}^n$ any two orbit sum has either identical support (hence equal) or disjoint. Further more $\mathfrak{o}(\mathbf{b}) \in \mathbf{R}$ (i.e. invariant polynomial). In fact the collection $\{\mathfrak{o}(\mathbf{b}) : \mathbf{b} \in \mathbb{Z}^n/\mathcal{G}\}$ forms a $\mathbb{K}$ basis for the invariant algebra $\mathbf{R}$ hence,

$$\mathbf{R} = \oplus_{\mathbf{a}\in\mathbb{Z}^n|\mathcal{G}} \mathbb{K}\mathfrak{o}(\mathbf{a}) \tag{2.4.2}$$

For any given $\succcurlyeq \in \Omega$ the semigroup $\mathfrak{A}^{\succcurlyeq}$ forms a fundamental domain for the $\mathcal{G}$ action on $\mathbb{Z}^n$. This follows from the equivalent definition, i.e. $\mathfrak{A}^{\succcurlyeq}$ containing a head term of each orbit sum. Hence,

$$in_{\succcurlyeq}(\mathbf{R}) = \mathbb{K}[\mathbf{x}^{\mathbf{a}} : \mathbf{a} \in \mathfrak{A}^{\succcurlyeq}]$$

for detail of these relationship see [Tesemma (2007), lemma 2.2.]. In the next proposition we derive a simple presentation of the set A_g in relation to a given $\succcurlyeq \in \Omega$. Here we need the inner product $\langle\, ,\, \rangle$ to be $\mathcal{G}$-invariant ($\langle hx, hy\rangle = \langle x, y\rangle$ for $x, y \in \mathbb{R}^n$ and for any $h \in \mathcal{G}$). We consider the usual inner product $(\, ,\,)$ averaged over $\mathcal{G}$ i.e.

$$\langle x, y\rangle = \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} (gx, gy) = x^T M y, \tag{2.4.3}$$

where $|\mathcal{G}|$ is the order of $\mathcal{G}$ and $M = \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} g^T g$. Note M is a symmetric and positive definite matrix with entries in $\mathbb{Q}$ and hence $\langle \mathbf{u}, \mathbf{v}\rangle \in \mathbb{Q}$ for any $\mathbf{u}, \mathbf{v} \in \mathbb{Q}^n$. In the next lemma we give some properties of the $\mathcal{G}$-invariant inner product to elements of $\mathcal{G}$ and vectors in $\mathbb{R}^n$.

Lemma 2.4.1. *Let $\langle\, ,\, \rangle$ be $\mathcal{G}$ invariant inner product and $\omega \in \mathbb{R}^n$ then,*

a) $\forall h \in \mathcal{G} \quad h^T M h = M \quad (h^T M = M h^{-1})$.
b) *For any* $\mathbf{u} \in \mathbb{Q}^n$, $\quad \langle \mathbf{u} - g\mathbf{u}, \omega\rangle = \langle \mathbf{u}, \omega - g^{-1}\omega\rangle$.
c) *For any* $\mathbf{u} \in \mathbb{Q}^n$, $\quad \langle \mathbf{u}, g\mathbf{u} - g^{-1}\mathbf{u}\rangle = 0$.
d) *For any* $\mathbf{u} \in \mathbb{Q}^n$, $\quad \langle \mathbf{u} - g\mathbf{u}, \mathbf{u}\rangle = 0 \iff g\mathbf{u} = \mathbf{u}$.

Proof. (a) Holds since $\langle h\mathbf{u}, h\mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle$ for any $\mathbf{u}, \mathbf{v} \in \mathbb{Q}^n$, implies $h^T M h = M$ and M is symmetric, positive definite and non-singular gives $(Mh)^T = h^T M = M h^{-1}$ (b) and (c) follows from (a) and further $\langle \mathbf{u} - g\mathbf{u}, \mathbf{u} - g\mathbf{u}\rangle = 2\langle \mathbf{u} - g\mathbf{u}, \mathbf{u}\rangle$ and $\langle\, ,\, \rangle$ is positive definite implies (d). ☐

Proposition 2.4.2. *Let $\mathcal{G}$ be a finite group, and $\succcurlyeq \in \Omega$ with the above notations,*

1. *For each $g \in \mathcal{G}$, A_g is a saturated subsemigroup of $\mathbb{Z}^n$ with maximal subgroup $[\mathbb{Z}^n]^g = \{\mathbf{a} \in \mathbb{Z}^n : \mathbf{a} = g(\mathbf{a})\}$.*
2. *$\forall \mathbf{a} \in \mathbb{Z}^n$ and $g \in \mathcal{G}$,*
   (a) $\mathbf{a}$ and $-\mathbf{a} \in A_g$ if and only if $\mathbf{a}$ equivalently $-\mathbf{a}$ is in $[\mathbb{Z}^n]^g$,
   (b) $\mathbf{a} \notin [\mathbb{Z}^n]^g$ either $\mathbf{a}$ or $-\mathbf{a}$ is in A_g (not both).
3. *$\mathfrak{A}^{\succcurlyeq}$ is saturated subsemigroup of $\mathbb{Z}^n$, and $\mathfrak{A}^{\succcurlyeq}(\mathbb{Q}) = \mathbb{Q}_+(\mathfrak{A}^{\succcurlyeq})$.*
4. *$C = \mathbb{R}_+ A_g = \mathbb{R}_+ A_g(\mathbb{Q})$, is a cone in $\mathbb{R}^n$ and there exist a unique vector (up to positive scalar multiple) $\alpha_g^1 \in C \subsetneqq \mathbb{R}^n$ such that,*

$$C \subset \mathbf{H}^+_{\alpha_g^1} = \{v \in \mathbb{R}^n : \quad \langle v, \alpha_g^1 \rangle \geq 0\}.$$

   Further $\mathfrak{rdim}(\alpha_g^1) \leq n - \dim([\mathbb{Q}^n]^g)$.
5. *For $g \in \mathcal{G}$ $\quad v \in \mathbb{R}^n$, if $gv = v$ then $\mathfrak{rdim}(v) \leq \dim([\mathbb{Q}^n]^g)$.*

Proof. 1. Clearly A_g is a semigroup, and if $n\mathbf{a} \in A_g$ for some $n \in \mathbb{N}$, and $\mathbf{a} \in \mathbb{Z}^n$, then, $n\mathbf{a} - g(n\mathbf{a}) = n(\mathbf{a} - g(\mathbf{a})) \succcurlyeq \mathbf{0}$, by lemma 2.1.2, $\mathbf{a} - g(\mathbf{a}) \succcurlyeq \mathbf{0}$, hence $\mathbf{a} \in A_g$. That $[\mathbb{Z}^n]^g$ is maximal subgroup will follow from 2.

2. (a) If $\mathbf{a}$ and $-\mathbf{a}$ are in A_g then $\mathbf{a} - g(\mathbf{a}) \succcurlyeq \mathbf{0}$ and $-\mathbf{a} - g(-\mathbf{a}) = -(\mathbf{a} - g(\mathbf{a})) \succcurlyeq \mathbf{0}$, but $\succcurlyeq$ is a total order implies $\mathbf{a} - g(\mathbf{a}) = \mathbf{0}$ and $\pm\mathbf{a} \in [\mathbb{Z}^n]^g$.
   (b) If $\mathbf{a} \notin [\mathbb{Z}^n]^g$ then $\mathbf{a} - g(\mathbf{a}) \neq \mathbf{0}$ gives $\mathbf{a} - g(\mathbf{a}) \succ \mathbf{0} \iff \mathbf{0} \succ -(\mathbf{a} - g(\mathbf{a}) = -\mathbf{a} - g(-\mathbf{a}))$ implies $\mathbf{a}$ or $-\mathbf{a}$ is in A_g.
3. $\mathfrak{A}^{\succcurlyeq}$ is saturated follows from (1) (i.e. The finite intersection of saturated subsemigroup is saturated). With $\mathbb{Q}^n = \{\frac{\mathbf{a}}{r} : \mathbf{a} \in \mathbb{Z}^n, r \in \mathbb{N}\}$ then,

$$\begin{aligned} \frac{\mathbf{a}}{r} \in \mathfrak{A}^{\succcurlyeq}(\mathbb{Q}) &\iff \frac{\mathbf{a}}{r} - g(\frac{\mathbf{a}}{r}) = \frac{\mathbf{a} - g(\mathbf{a})}{r} \succcurlyeq \mathbf{0} \quad \forall g \in \mathcal{G} \\ &\iff \mathbf{a} - g(\mathbf{a}) \succcurlyeq \mathbf{0} \quad \forall g \in \mathcal{G} \iff \mathbf{a} \in \mathfrak{A}^{\succcurlyeq} \end{aligned}$$

   where the middle equivalence is from [2.1.2 (3)] therefore $\mathfrak{A}^{\succcurlyeq}(\mathbb{Q}) = \mathbb{Q}_+(\mathfrak{A}^{\succcurlyeq})$.
4. Let $\succcurlyeq \in \Omega$ then by theorem, 2.3.1 there exists, $\{\omega_1, \omega_2, \cdots, \omega_s\} \subset \mathbb{R}^n$ such that,

$$\mathbf{a} \succcurlyeq \mathbf{0} \quad \textit{if} \quad (\langle \mathbf{a}, \omega_1 \rangle, \langle \mathbf{a}, \omega_2 \rangle, \cdots, \langle \mathbf{a}, \omega_s \rangle) \succcurlyeq_{lex} \mathbf{0}$$

   $\mathbf{u} \in A_g(\mathbb{Q})$, if $\mathbf{u} - g(\mathbf{u}) \succcurlyeq \mathbf{0}$, hence,

$$(\langle \mathbf{u} - g(\mathbf{u}), \omega_1 \rangle, \langle \mathbf{u} - g(\mathbf{u}), \omega_2 \rangle, \cdots, \langle \mathbf{u} - g(\mathbf{u}), \omega_s \rangle) \succcurlyeq_{lex} \mathbf{0}$$

   by lemma [2.4.1,(2)],

$$(\langle \mathbf{u}, \omega_1 - g^{-1}(\omega_1) \rangle, \langle \mathbf{u}, \omega_2 - g^{-1}(\omega_2) \rangle, \cdots, \langle \mathbf{u}, \omega_s - g^{-1}(\omega_s) \rangle) \succcurlyeq_{lex} \mathbf{0}$$

   Let $k_1 = min_i\{i : \omega_i - g^{-1}(\omega_i) \neq \mathbf{0}\}$ and set $\alpha_g^1 = \omega_{k_1} - g^{-1}(\omega_{k_1})$ then we claim that (a) $C = \mathbb{R}_+ A_g \subseteq \mathbf{H}^+_{\alpha_g^1}$ and (b) $\alpha_g^1 \in C$ holds.

(a) To show $C = \mathbb{R}_+ A_g \subseteq \mathbf{H}^+_{\alpha^1_g}$.
Let $\mathbf{u} \in A_g(\mathbb{Q})$ then $\mathbf{u} - g\mathbf{u} \succcurlyeq \mathbf{0}$ hence $\langle \mathbf{u}, \alpha^1_g \rangle \geq 0$ and for

$$i < k_1, \quad \langle \mathbf{u} - g(\mathbf{u}), \omega_i \rangle = \langle \mathbf{u}, \omega_i - g^{-1}\omega_i \rangle = \langle \mathbf{u}, \mathbf{0} \rangle = 0$$

now consider $\mathbf{v} \in \mathbb{R}_+ A_g = C$ then, $\mathbf{v} = \sum_{i=1}^t r_i \mathbf{u}_i$ where $r_i \in \mathbb{R}_+$ and $\mathbf{u}_i \in A_g(\mathbb{Q})$ and

$$\langle \mathbf{v}, \alpha^1_g \rangle = \sum_{i=1}^t r_i \langle \mathbf{u}_i, \alpha^1_g \rangle \geq 0$$

implies $\mathbf{v} \in \mathbf{H}^+_{\alpha^1_g}$ and the claim holds.

(b) Since $\alpha^1_g = \omega_k - g^{-1}(\omega_k) \neq \mathbf{0}$ implies $\langle \alpha^1_g, \alpha^1_g \rangle > 0$ therefore $\alpha^1_g \in C$. Finally the rational dimension is in general less than or equal to $n - \dim([\mathbb{Q}^n]^g)$ since $\mathbf{u} \in [\mathbb{Q}^n]^g$ implies $\langle \mathbf{u}, \alpha^1_g \rangle = 0$.

5. Assume $\mathfrak{rdim}(\mathbf{v}) = t$ then by [proposition 2.2.1 (1)] $\mathbf{v} = \sum_{j=1}^t r_j \mathbf{u}^j$ where $\mathbf{u}^j \in \mathbb{Q}^n$ the fact g fixes v implies

$$\mathbf{0} = \mathbf{v} - g(\mathbf{v}) = \sum_{j=1}^t r_j (\mathbf{u}^j - g(\mathbf{u}^j))$$

and r_j's linearly independence over $\mathbb{Q}$ implies $\forall j \ \mathbf{u}^j - g(\mathbf{u}^j) = \mathbf{0}$, therefore $\mathfrak{rdim}(\mathbf{v}) \leq \dim([\mathbb{Q}^n]^g)$.

□

Remark 2.4.3. If $\succcurlyeq$ is of type n (as in the usual lexicography order) then each ω_i can be chosen in $\mathbb{Q}^n$. Hence $\alpha^1_g \in \mathbb{Q}^n$ and $\mathbf{H}_{\alpha^1_g}$ is rational hyperplane. Furthermore $\mathcal{H}_{\alpha^1_g} \cap A_g(\mathbb{Q})$ contains a subsemigroup of A_g of dimension $n-1$ containing the maximal subgroup $[\mathbb{Z}^n]^g$ of A_g.

2.5 Semigroup Algebras

Definition 2.5.1. Let $\mathbb{K}$ be a field a $\mathbb{K}$-algebra $\mathbb{A}$ is called a semigroup algebra if there exist a submonoid $M \subset (\mathbb{A}, \cdot)$ such that the elements of M forms a $\mathbb{K}$ basis of $\mathbb{A}$.

In general for a given semigroup $(S, *)$, and a ring R the semigroup ring

$$R[X:S] = \{f : S \to R | f, a function \quad supp(f) \quad finite\}$$

where support of f is $\{s \in S : f(s) \neq 0\}$, for $f, g \in R[X:S]$ addition $f+g$ is defined point wise i.e. for each $s \in S \quad (f+g)(s) = f(s) + g(s)$, multiplication $f.g$, for each $s \in S \quad f.g(s) = \sum_{r*t=s} f(r)g(t)$. These semigroup ring $R[X:S]$, will have quite a number algebraic properties inherited from the semigroup S, and vise versa, for detail on commutative semigroup ring and algebraic properties relating $R[X:S]$ to S see, [Gilmer (1984)].

One can easily observe for R a domain S commutative cancellative and torsion free monoid, the semigroup ring $R[X : S]$, the set (multiplicative monoid) $\{\mathbf{x}^s : s \in S\}$, forms an R basis i.e. $R[X : S] = \oplus_{s \in S} R\mathbf{x}^s$. The Laurent polynomial ring over a field $\mathbb{K}$ is semigroup algebra generated by

$$M =< x_1{}^{\pm 1}, \cdots, x_n{}^{\pm 1} >_{mon} \cong (\mathbb{Z}_n, +)$$

For affine semigroup algebras associated with finite generated semigroups of $\mathbb{Z}^n$, see [Ene and Herzog (2012), §5]. Some of the main result associated with semigroup algebra and their multiplicative monoid is listed below for future references.

Remark 2.5.2. 1. If M is finitely generated commutative monoid. Then M is isomorphic to a submonoid of the additive group $\mathbb{Z}^r$ for some $r \in \mathbb{N}$ if and only if M is both cancellative and torsion free [Rosales and García-Sánchez (1999), Corollary 3.4].

2. Let $K[M]$ be a semi group algebra then;
   - $K[M]$ is finitely generated (affine) K-algebra if and only if M is a finitely generated monoid.
   - $K[M]$ a domain if and only if M is cancellative and torsion free. [Gilmer (1984), Theorem 8.1]

For a given sub algebra R of $\mathbb{K}[\mathbf{X}^{\pm 1}]$, it is not apparent from the algebra generators to decide whether R is affine semigroup algebra. If there exists an affine semigroup $M \subseteq (R, \cdot)$, such that $\mathbb{K}[M] = R$, then the semigroup generators are automatically algebra generator but the algebraic relation among the generators is in such a way two power product of these generators ('monomials') are equal. i.e. if $M =< g_1, g_2, \cdots, g_t >_{mon}$ then,

$$M = \{\prod_{i=1}^{t} g_i^{a_i} : a_i \in \mathbb{Z}_{\geq 0}\}.$$

and any algebraic relation would have the form (if any),

$$\prod_{i=1}^{t} g_i^{a_i} = \prod_{i=1}^{t} g_i^{b_i}.$$

Hence with some ordering on the generators can be identified as a semigroup in $\mathbb{Z}^t_{\geq 0}$, since M is conciliative and torsion free.

Example 2.5.3. Consider example 2.4.1, 1, R is an affine semigroup algebra, with semigroup generators

$$\begin{aligned} g_1 =& f_1 + 3f_3 + 6 \\ g_2 =& f_2 + 3f_3 + 6 \\ g_3 =& f_3 + 3 \end{aligned}$$

Observe that $g_1 g_2 = g_3^3$ is the algebraic relation. Clearly f_1, f_2 and f_3 dose not have such simple algebraic relation, rather

$$f_1f_2 - f_3^3 + 3f_1f_3 + 3f_2f_3 + 6f_1 + 6f_2 + 9f_3 + 9 = 0$$
$$g_1g_2 = g_3^3 \iff (f_1 + 3f_3 + 6)(f_2 + 3f_3 + 6) = (f_3 + 3)^3$$

hence their power product (monomials formed by the products $\{f_1, f_2, f_3\}$) forms linear dependence. In this case $M =< g_1, g_2, g_3 >_{mon} \cong < (2,1), (1,2), (1,1) >_{mon}$ semigroup generated by support of the initial algebra generators for the usual lex ordering. Latter in chapter four we show that R is an invariant algebra of a reflection group and describe how these semigroup generators are constructed.

2.6 Reflection Groups and Root System

Let $\mathbf{E} = \mathbb{R}^n$ be the Euclidean space, and $\langle \, , \, \rangle$, be usual inner product define a reflection either with respect to a hyperplane or a non zero vector. For a given hyperplane $\mathbf{H} \subseteq \mathbf{E}$, Let L be a line through the origin that is orthogonal to $\mathbf{H}$ ($\mathbf{E} = \mathbf{H} \oplus L$), then define $s_{\mathbf{H}}$ by,

$$s_{\mathbf{H}}(x) = \begin{cases} x, & \text{if } x \in \mathbf{H}, \\ -x, & \text{if } x \in L, \end{cases}$$

equivalently give $0 \neq \alpha \in \mathbb{R}^n$ and $\mathbf{H}_\alpha = \{x \in \mathbb{R}^n : \langle x, \alpha \rangle = 0\}$ then,

$$s_\alpha(x) = \begin{cases} x, & \text{if } x \in \mathbf{H}_\alpha, \\ -\alpha, & \text{if } x = \alpha, \end{cases}$$

One can easily observe that $0 \neq k \in \mathbb{R}$, $\mathbf{H}_{k\alpha} = \mathbf{H}_\alpha$, and $s_{k\alpha} = s_\alpha$. The automorphism have a form,

$$s_\alpha(x) = x - \frac{2\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha \quad \forall x \in \mathbb{R}^n.$$

Furthermore $\langle s_\alpha(x), s_\alpha(y) \rangle = \langle x, y \rangle$, i.e. s_α is *orthogonal*, $s_\alpha^2 = I$ i.e. involution. A group $G \subseteq GL(n, \mathbb{R})$ is said to be *an euclidean reflection group* if G admits a generator set $R_{\mathcal{G}}$, consists of reflections only.
Here our concern is subgroups of automorphism of a lattice $\mathbf{L} \cong \mathbb{Z}^n$ i.e $\mathcal{G} \subseteq Aut(\mathbf{L}) = GL(n, \mathbb{Z})$, in case of reflection group we consider the finite euclidean reflection group which admits $\mathcal{G}$-stable (invariant) lattice $\mathbf{L}$ ($\mathcal{G} \subseteq GL(n, \mathbb{Z})$), such reflection group are usually referred as *Weyl groups* in the literature.

Definition 2.6.1. A subset Φ of an euclidean space $\mathbb{R}^n$, is called a *root system* in $\mathbb{R}^n$ if,

1. Φ is finite, spans $\mathbb{R}^n$ and doesn't contain $\mathbf{0}$.
2. If $\alpha \in \Phi$, then the only multiple of α in Φ is $\pm\alpha$.
3. If $\alpha \in \Phi$, the reflection s_α leave Φ invariant. i.e. for $\beta \in \Phi$, $\quad s_\alpha(\beta) \in \Phi$.
4. (*Crystallographic property*) If $\alpha, \beta \in \Phi$, then $\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.

Let $g \in \mathcal{G}$ be a reflection and $\mathbf{L}$ be a $\mathcal{G}$ stable lattice then the submodule $\{\mathbf{a} \in \mathbf{L} : g(\mathbf{a}) =$

$-\mathbf{a}\} = \mathbb{Z}\alpha_g$ is one dimensional and is a direct summand of $\mathbf{L}$ and hence $\mathbf{L}/\mathbb{Z}\alpha_g$ is torsion free. Note α_g is atomic element of $\mathbf{L}$. $l \in \mathbf{L}$ is said to be *atomic* element if the one dimensional semigroup generated by l is saturated in $\mathbf{L}$.

Theorem 2.6.1. *Let $\mathcal{G}$ be a finite euclidean reflection group and $\mathbf{L}$ be $\mathcal{G}$ stable lattice in $\mathbb{R}^n$, and $\{\mathbf{0}\} = [\mathbb{R}^n]^{\mathcal{G}} = \{v \in \mathbb{R}^n : v = g(v) \quad \forall g \in \mathcal{G}\}$ (i.e. $\mathcal{G}$ acts effectively) then the set*

$$\Phi_{\mathcal{G}} = \{\pm\alpha_g : s_{\alpha_g} \in R_{\mathcal{G}}\} \subseteq \mathbf{L}.$$

where $R_{\mathcal{G}} = \{g \in \mathcal{G} : g \quad \text{reflection}\}$ and each α_g is an atomic element in $\mathbf{L}$ forms a root system.

Proof. (1) Clearly $\Phi_{\mathcal{G}}$ is finite and doesn't contain $\mathbf{0}$. since $s_{\mathbf{0}} = I \notin R_{\mathcal{G}}$ further $[\mathbb{R}^n]^{\mathcal{G}} = \{\mathbf{0}\}$ implies that $\Phi_{\mathcal{G}}^{\perp} = \{v \in \mathbb{R}^n : \langle v, \alpha\rangle = 0 \,\forall \alpha \in \Phi_{\mathcal{G}}\} = [\mathbb{R}^n]^{\mathcal{G}} = \{\mathbf{0}\}$ therefor $\Phi_{\mathcal{G}}$ spans $\mathbb{R}^n$.
(2) follows directly from the definition of $\Phi_{\mathcal{G}}$.
(3) For $\alpha, \beta \in \Phi_{\mathcal{G}}$ then $s_\alpha, s_\beta \in R_{\mathcal{G}}$ and $s_\alpha s_\beta s_\alpha = s_{s_\alpha(\beta)}$ fixes the hyperplane $s_\alpha(\mathbf{H}_\beta)$ and sends $s_\alpha(\beta)$ to $-s_\alpha(\beta)$ hence $s_{s_\alpha(\beta)} \in R_{\mathcal{G}}$ is a reflection therefore, $s_\alpha(\beta) \in \Phi_{\mathcal{G}}$. Further more $s_\alpha(\beta)$ is atomic follows from the fact s_α is lattice automorphism and sends atomic elements to atomic.
(4) For $\alpha, \beta \in \Phi_{\mathcal{G}} \subseteq \mathbf{L}$ are chosen to be atomic and

$$s_\alpha(\beta) = \beta - \frac{2\langle\beta, \alpha\rangle}{\langle\alpha, \alpha\rangle}\alpha$$

whence,

$$\beta - s_\alpha(\beta) = \frac{2\langle\beta, \alpha\rangle}{\langle\alpha, \alpha\rangle}\alpha \in \mathbf{L}$$

and α being atomic in $\mathbf{L}$ implies $\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$. therefor $\Phi_{\mathcal{G}}$ forms a root system in $\mathbb{R}^n$. $\square$

The study of root system in axiomatic structure came from the study of reflection groups, using the orthogonal vector α of each reflection s_α in the group. Note that not all euclidean reflection group admit crystallographic root system, which is a consequence of $\mathcal{G}$ being contained as a subgroup of $Aut(\mathbf{L})$.

Definition 2.6.2. A subset $\triangle$ of a root system Φ is called a *base or fundamental system* if,

1. $\triangle$ is a basis of $\mathbb{R}^n$.
2. Each root $\beta \in \Phi$, can be written as

$$\beta = \sum_{\alpha \in \triangle} k_\alpha \alpha,$$

   with integral coefficient k_α's all non negative or all non positive.

Each $\alpha \in \triangle$ is called a *simple roots* and the associated reflection s_α is called *simple reflections*. From the definition we have that the cardinality of $\triangle$ is n ($\triangle$ form a basis for

$\mathbb{R}^n$), and the expression for β is unique. We consider the set

$$\Phi^+ = \{\beta \in \Phi : k_\alpha \geq 0 \quad \forall \alpha \in \triangle\}.$$

to be the *positive roots*, and the *negative roots* are given by,

$$\Phi^- = \{\beta \in \Phi : k_\alpha \leq 0 \quad \forall \alpha \in \triangle\} = -\Phi^+$$

For a given root system a base exists, [Humphreys (1980), Theorem 10.1], and each $\alpha, \beta \in \triangle$, in such base satisfy $\langle \beta, \alpha \rangle \leq 0$. [Humphreys (1980), lemma 10.1.1].

Let $\gamma \in \mathbb{R}^n$ be called a *regular element* if $\gamma \in \mathbb{R}^n \setminus \cup_{\alpha \in \Phi} \mathbf{H}_\alpha$, equivalently the $\mathcal{G}$ orbit length of γ equals $|\mathcal{G}|$, other wise it is called *singular*. For regular γ,

$$\Phi^+(\gamma) = \{\alpha \in \Phi : \langle \gamma, \alpha \rangle > 0\},$$

and $\Phi = \Phi^+(\gamma) \cup -\Phi^+(\gamma)$. Further $\alpha \in \Phi^+(\gamma)$ is called *decomposable* if $\alpha = \beta_1 + \beta_2$, for some $\beta_1, \beta_2 \in \Phi^+(\gamma)$, else α is called *indecomposable*. The set of $\triangle(\gamma)$ of indecomposable elements of $\Phi^+(\gamma)$ forms a base of Φ and every base of Φ is obtained in this manner [Humphreys (1980), Theorem' 10.10.1].
The hyperplanes $\mathbf{H}_\alpha$ for $\alpha \in \Phi^+$ partitions the regular elements of $\mathbb{R}^n$ in to finitely many n dimensional regions by the equivalence relation that $\gamma_1 \sim \gamma_2$ if both γ_1 and γ_2 are on the same side of each hyperplane $\mathbf{H}_\alpha$ for $\alpha \in \Phi$. Equivalently γ_1 and γ_2 form the same positive root ($\Phi^+(\gamma_1) = \Phi^+(\gamma_2)$) for the root system Φ. The connected component (equivalent class) of $\mathbb{R}^n \setminus \cup_{\alpha \in \Phi} \mathbf{H}_\alpha$ are called (open) *Weyl chambers*, from the above argument one can see a $1 - to - 1$ correspondence between a base (equivalently a positive root) to a Weyl chamber. For a given regular element γ the equivalent class,

$$\begin{aligned}
&=\{v \in \mathbb{R}^n : v \sim \gamma\} \\
&=\{v \in \mathbb{R}^n : \langle v, \alpha \rangle > 0 \quad \forall \alpha \in \Phi^+(\gamma)\} \\
&=\{v \in \mathbb{R}^n : \langle v, \alpha \rangle > 0 \quad \forall \alpha \in \triangle(\gamma)\}
\end{aligned}$$

is a Weyl chamber. Furthermore $\mathcal{G}$ is generated by the simple reflection i.e. $\mathcal{G} =< s_\alpha : \alpha \in \triangle >$. For any given base $\triangle$ of a root system Φ the lattice,

$$\mathbf{Q} = \oplus_{\alpha \in \triangle} \mathbb{Z}\alpha$$

is called the *root lattice*. The crystallographic property of Φ implies that $\mathbf{Q}$ is $\mathcal{G}$ stable and $\mathbf{Q} \subseteq \mathbf{L}$. Further more the lattice,

$$\mathbf{P} = \{u \in \mathbb{R}^n : u - g(u) \in \mathbf{Q} \, \forall g \in \mathcal{G}\}$$

is called the *weight lattice*. For a given base $\triangle$ of the root system Φ, let $\triangle_{\mathcal{G}} = \{g_\alpha : \alpha \in$

$\triangle\}$ be the simple reflections and set,

$$\begin{aligned}\mathbf{P}^{\star} =&\{u \in \mathbb{R}^n : u - g_\alpha(u) \in \mathbf{Q} \quad g_\alpha \in \triangle_{\mathcal{G}}\} \\ =&\{u \in \mathbb{R}^n : \frac{2\langle u, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha \in \mathbf{Q} \quad \forall \alpha \in \triangle\} \\ =&\{u \in \mathbb{R}^n : \frac{2\langle u, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z} \quad \forall \alpha \in \triangle\}\end{aligned}$$

where the third equality above is due to the fact each $\alpha \in \Phi$ are atomic element of $\mathbf{L}$ which contains the root lattice $\mathbf{Q}$. Clearly from the definitions $\mathbf{P} \subseteq \mathbf{P}^{\star}$, conversely using the fact that $\mathcal{G} =< \triangle_{\mathcal{G}} >$, any $g \in \mathcal{G}$ is finite product of the simple reflection, $g = g_1 g_2 \cdots g_t$. For $u \in \mathbf{p}^{\star}$ setting $h = g_1 g_2 \cdots g_{t-1}$, then,

$$u - g(u) = u - h(u) + \frac{2\langle u, \alpha_t\rangle}{\langle \alpha_t, \alpha_t\rangle} h(\alpha_t)$$

where $g_t = g_{\alpha_t} \in \triangle_{\mathcal{G}}$ and $h(\alpha_t) \in \mathbf{Q}$ and its coefficient $\frac{2\langle u,\alpha_t\rangle}{\langle \alpha_t,\alpha_t\rangle} \in \mathbb{Z}$, (since $u - g_t(u) = \frac{2\langle u,\alpha_t\rangle}{\langle \alpha_t,\alpha_t\rangle}\alpha_t$ and α_t is atomic), hence by induction one have that $u \in \mathbf{P}$ and $\mathbf{P} = \mathbf{P}^{\star}$.
For a given base $\triangle = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of a root system Φ the weight lattice $\mathbf{P}$ is given by

$$\mathbf{P} = \oplus_{i=1}^{n} \mathbb{Z} u_i$$

where $\{u_1, u_2, \cdots, u_n\}$ are called the *fundamental weight* and satisfies,

$$\frac{2\langle u_j, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle} = \delta_{ij} \qquad \Longleftrightarrow \qquad u_j - g_i(u_j) = \delta_{ij}\alpha_i \quad 1 \le i, j \le n.$$

One notes that the fundamental weights are always singular element of $\mathbb{R}^n$. Further $\mathbf{L} \subseteq \mathbf{P}$ since for $\mathbf{a} \in \mathbf{L}$ and each simple reflection $g_\alpha \in \triangle_{\mathcal{G}}$ we have $\mathbf{a} - g_\alpha(\mathbf{a}) = \frac{2\langle \mathbf{a},\alpha\rangle}{\langle \alpha,\alpha\rangle}\alpha \in \mathbf{L}$ and each $\alpha \in \triangle$ is atomic implies $\frac{2\langle \mathbf{a},\alpha\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$ therefore $\mathbf{a} \in \mathbf{P}$. Hence every $\mathcal{G}$ invariant lattice is sandwiched between the root and weight lattice [Kane (2001), Proposition 9.5].

In the above few paragraphs we describe how the weight lattice of a given root system can be constructed using the simple roots (equivalently simple reflection). In the following theorem we show that the image of initial algebra for the invariant ring i.e. $\mathfrak{A}^{\succcurlyeq}$ the fundamental set will have a strong relation with the weight lattice. In doing so we create a link between the profs of 1.3.1 and 1.3.4 intrinsically.

Theorem 2.6.2. *Let $\mathcal{G} \subseteq GL(n, \mathbb{Z})$ be a finite reflection group acting effectively (i.e. $[\mathbb{Z}^n]^{\mathcal{G}} = \{\mathbf{0}\}$). Let $\succcurlyeq \in \Omega$ be any and $\langle\, ,\, \rangle$ be $\mathcal{G}$ invariant inner product then,*

1. *$\Phi_{\succcurlyeq} = \{\alpha \in \Phi : \alpha \succcurlyeq \mathbf{0}\}$ forms a set of positive roots for the root system Φ of $\mathcal{G}$.*

2. *Let $\triangle^{\succcurlyeq}$ be the base (simple roots) for the positive root $\Phi_{\succcurlyeq}$ (from 1) then,*

$$\begin{aligned}\mathfrak{A}^{\succcurlyeq} &= \{\mathbf{a} \in \mathbb{Z}^n : \mathbf{a} \succcurlyeq g(\mathbf{a}) \quad \forall g \in \mathcal{G}\} \\ &= \{\mathbf{a} \in \mathbb{Z}^n : \mathbf{a} \succcurlyeq g(\mathbf{a}) \quad \forall g \in \triangle_{\mathcal{G}}^{\succcurlyeq}\} \\ &\subseteq < u_1, u_2, \cdots, u_n >_{mon} \subseteq \mathbb{Q}_+ \mathfrak{A}^{\succcurlyeq},\end{aligned}$$

where $\triangle_{\mathcal{G}}^{\succcurlyeq}$ is the simple reflections and $\{u_1, u_2, \cdots, u_n\}$ is the fundamental weights with respect to $\triangle^{\succcurlyeq}$.

3. *Let λ_i be the smallest positive integer, such that $\mathbf{a}_i = \lambda_i u_i \in \mathbb{Z}^n$ then $\mathfrak{A}^{\succcurlyeq}$ is saturated subsemigroup with critical generators $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$. i.e. for any $\mathbf{b} \in \mathfrak{A}^{\succcurlyeq}$ there exist some positive integer n_a such that $n_a \mathbf{b} \in < \mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n >_{mon}$.*

Proof. (1) Let S_Φ be the semigroup generated by $\Phi_{\succcurlyeq}$ then $S_\Phi \subset \mathbf{P}$ positive set of $\succcurlyeq$ on $\mathbb{Z}^n$, hence S_Φ is positive and n dimensional semigroup ($span_Q(\Phi) = n = span_Q(\Phi_{\succcurlyeq})$). Now set $S_\Phi = < \alpha : \alpha \in \triangle >_{mon}$ for $\triangle \subseteq \Phi_{\succcurlyeq}$ is the minimal generators for S_Φ. For any $\beta \in \Phi$ then either $\beta \in \Phi_{\succcurlyeq} \subset S_\Phi$ hence,

$$\beta = \sum_{\alpha \in \triangle} \lambda_\alpha \alpha \quad \lambda_\alpha \in \mathbb{N}_{\geq 0}$$

else if $\beta \notin \Phi_{\succcurlyeq}$ then $-\beta \in \Phi_{\succcurlyeq}$ so in this case β have an expression with non positive integer coefficient with respect to $\triangle$. Therefore

$$\Phi = \Phi_{\succcurlyeq} \cup -\Phi_{\succcurlyeq},$$

which shows $\Phi_{\succcurlyeq}$ form a positive roots and in this case $\triangle$ form a base (indecomposable elements). We denoted the simple roots associated with $\succcurlyeq$ by $\triangle^{\succcurlyeq}$.
(2) Let $\triangle_{\mathcal{G}}^{\succcurlyeq} = \{g_1, g_2, \cdots, g_n\}$ be the simple reflection associated with the base $\triangle^{\succcurlyeq} = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ and set,

$$\mathbf{A}^\star = \{\mathbf{a} \in \mathbb{Z}^n : \mathbf{a} \succcurlyeq g_i(\mathbf{a}) \quad (\forall i) 1 \leq i \leq n\}.$$

for any $i, 1 \leq i \leq n$ and $\mathbf{a} \in \mathbb{Z}^n$,

$$\begin{aligned}\mathbf{a} \succcurlyeq g_i(\mathbf{a}) &\iff \mathbf{a} - g_i(\mathbf{a}) \succcurlyeq \mathbf{0} \\ &\iff \frac{2\langle \mathbf{a}, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i \succcurlyeq \mathbf{0} \\ &\iff \langle \mathbf{a}, \alpha_i \rangle \geq 0\end{aligned}$$

where the last equivalence is due to the fact $\alpha_i \succ \mathbf{0}$ and for $k \in \mathbb{Z}, k\alpha_i \succcurlyeq \mathbf{0} \iff k \in \mathbb{N}_{\geq 0}$ then,

$$\mathbb{A}^\star = \{\mathbf{a} \in \mathbb{Z}^n : \langle \mathbf{a}, \alpha_i \rangle \geq 0 \quad 1 \leq i \leq n\}.$$

clearly $\mathfrak{A}^{\succcurlyeq} \subseteq \mathbb{A}^\star$ by the definition of $\mathfrak{A}^{\succcurlyeq}$ and conversely let $\mathbf{b} \in \mathbb{A}^\star$ we need to show that for each $g \in \mathcal{G}\, \mathbf{b} - g(\mathbf{b}) \succcurlyeq \mathbf{0}$ since $\mathcal{G} = < g : g \in \triangle_{\mathcal{G}}^{\succcurlyeq} >$ (i.e. generated by the simple reflections). Let $g = \prod_{i=1}^{r} g_{a_i}$ where $a_i \in \triangle^{\succcurlyeq}$, be the minimal (number of simple reflection used to represent g) length representation of g with respect to the simple reflections (non consecutive a_i and a_j can be the same simple root). Now,

$$\begin{aligned} g(\mathbf{b}) = \mathbf{b} - &\frac{2\langle \mathbf{b}, a_1\rangle}{\langle a_1, a_1\rangle} a_1 - \frac{2\langle \mathbf{b}, a_2\rangle}{\langle a_2, a_2\rangle} g_{a_1}(a_2) \\ &- \frac{2\langle \mathbf{b}, a_3\rangle}{\langle a_3, a_3\rangle} g_{a_1} g_{a_2}(a_3) \cdots - \frac{2\langle \mathbf{b}, a_r\rangle}{\langle a_r, a_r\rangle} g_{a_1} g_{a_2} \cdots g_{a_{r-1}}(a_r). \end{aligned}$$

hence,

$$\begin{aligned} \mathbf{b} - g(\mathbf{b}) = &\frac{2\langle \mathbf{b}, a_1\rangle}{\langle a_1, a_1\rangle} a_1 + \frac{2\langle \mathbf{b}, a_2\rangle}{\langle a_2, a_2\rangle} g_{a_1}(a_2) \\ &+ \frac{2\langle \mathbf{b}, a_3\rangle}{\langle a_3, a_3\rangle} g_{a_1} g_{a_2}(a_3) \cdots + \frac{2\langle \mathbf{b}, a_r\rangle}{\langle a_r, a_r\rangle} g_{a_1} g_{a_2} \cdots g_{a_{r-1}}(a_r). \end{aligned}$$

since, $\mathbf{b} \in \mathbb{A}^{\star}$,

$$\frac{2\langle \mathbf{b}, a_i\rangle}{\langle a_i, a_i\rangle} \geq 0 \quad (\forall a_i \in \triangle^{\succcurlyeq}).$$

There for it is enough to show for each $i, 1 \leq i \leq r-1$,

$$g_{a_1} g_{a_2} \cdots g_{a_i}(a_{i+1}) \succ 0 \quad \Longleftrightarrow g_{a_1} g_{a_2} \cdots g_{a_i}(a_{i+1}) \in \Phi_{\succcurlyeq}$$

Observe that $\mathcal{G}$ permutes and stabilize Φ by [Humphreys (1980), Lemma 10.2.B] (for $\alpha \in \triangle$ the simple reflection σ_α permute the positive roots other than α) in particular for $\beta \in \triangle$ such that $\alpha \neq \beta$ then $\sigma_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha$ and $\langle \beta, \alpha\rangle \leq 0$ shows that $\sigma_\alpha(\beta)$ is a positive root. Moreover if $\sigma_\alpha(\gamma)$ is negative root for α simple and γ a positive root implies $\alpha = \gamma$.

Now $a_1 \succ 0$ simple root hence positive and $g_{a_1}(a_2)$ is a negative root implies $a_1 = a_2$ the same simple root and $g = \prod_{i=1}^{r} g_{a_i} = \prod_{i=3}^{r} g_{a_i}$ contradict the minimality of representation hence $g_{a_1}(a_2)$ is positive root. Suppose for some t, $3 \leq t \leq r$

$$g_{a_1} g_{a_2} \cdots g_{a_{t-1}}(a_t) \prec 0$$

then by [Humphreys (1980), Lemma 10.2.C] there exists some index s and $1 \leq s < t$ such that,

$$g_{a_1} g_{a_2} \cdots g_{a_{t-1}} g_{a_t} = g_{a_1} g_{a_2} \cdots g_{a_{s-1}} g_{a_{s+1}} \cdots g_{a_{t-1}}$$

again this contradict the minimality of the representation for g therefor for each $1 \leq i \leq r-1$,

$$g_{a_1} g_{a_2} \cdots g_{a_i}(a_{i+1}) \in \Phi_{\succcurlyeq}$$

hence $\mathbf{b} \in \mathfrak{A}^{\succcurlyeq}$ and $\mathbb{A}^{\star} = \mathfrak{A}^{\succcurlyeq}$. Observe that

$$\mathfrak{A}^{\succcurlyeq} = \mathbb{A}^{\star} = \{\mathbf{a} \in \mathbb{Z}^n : \frac{2\langle \mathbf{a}, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle} \in \mathbb{N}_{\geq 0} \quad (\forall i) \quad 1 \leq i \leq n\}$$

Let $\{u_1, u_2, \cdots, u_n\} \subset \mathbb{Q}^n$, such that

$$\frac{2\langle u_j, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle} = \delta_{ij}$$

clearly $\{u_1, u_2, \cdots, u_n\}$ are the fundamental weight associated with the positive roots $\Phi_{\succcurlyeq}$ which shows that,

$$\mathfrak{A}^{\succcurlyeq} \subseteq < u_1, u_2, \cdots, u_n >_{mon} \subseteq \mathbb{Q}_+ \mathfrak{A}^{\succcurlyeq}.$$

(3) follows from (2) Note $< u_1, u_2, \cdots, u_n >_{mon} = \mathbf{P} \cap \{\mathbf{u} \in \mathbb{Q}^n : \langle \mathbf{u}, \alpha_i\rangle \geq 0 \forall 1 \leq i \leq n\}$ where $\mathbf{P}$ is the weight lattice and $\mathbb{Q}_+(\mathfrak{A}^{\succcurlyeq}) = \mathbb{Q}_+ < u_1, u_2, \cdots, u_n >_{mon}$. □

Chapter 3

Linearization of Invariant Rings via Ω

3.1 The Setup

Let $\mathcal{G}$ be a finite group in $GL(n,\mathbb{Z}) \subseteq GL(n,\mathbb{R})$, in the previous chapter we have shown that the action of $\mathcal{G}$ on $\mathbb{Z}^n$ can be uniquely extended to a multiplicative action on the group ring $\mathbb{K}[\mathbf{X}^{\pm 1}]$ (Laurent polynomial algebra). The multiplicative invariant ring $\mathbb{K}[\mathbf{X}^{\pm 1}]^{\mathcal{G}} =: \mathbf{R}$ is always an affine algebra. In section [§2.4], we define for a given $\succcurlyeq \in \Omega$, a semigroup (image of the initials of invariant ring)

$$\begin{aligned}\mathfrak{A}^{\succcurlyeq} &= \{\mathbf{a} \in \mathbb{Z}^n : \quad \mathbf{a} \succcurlyeq g(\mathbf{a}) \quad \forall g \in \mathcal{G}\} \\ &= \{max_{\succcurlyeq}\{\mathbf{b} \in [\mathbf{a}]\} \in \mathbb{Z}^n : \mathbf{a} \in \mathbb{Z}^n|\mathcal{G}\},\end{aligned}$$

which also forms a fundamental domain for the $\mathcal{G}$ action on $\mathbb{Z}^n$. Let $\mathfrak{A}$ denote the image of initial algebra of $\mathbf{R}$ with respect to usual lex ordering ($\succcurlyeq_{lex}$) i.e. $\mathfrak{A} = \mathfrak{A}^{\succcurlyeq_{lex}}$. Observe that $\mathfrak{A}^{\succcurlyeq}$ for any $\succcurlyeq \in \Omega$ contain a single element from each orbits (head term of the associated orbit sum with respect to $\succcurlyeq$). We use elements of $\mathfrak{A}$ instead of arbitrary transverse set to denote the orbits and orbit sum. Hence the invariant ring in equation:2.4.2 will have the form,

$$\mathbf{R} = \oplus_{\mathbf{a} \in \mathfrak{A}} \mathbb{K}\mathfrak{o}(\mathbf{a})$$

$\mathfrak{A}$ is both a subsemigroup of $\mathbb{Z}^n$ and also a transverse set (fundamental domain for the $\mathcal{G}$ action on $\mathbb{Z}^n$). Moreover the action of $\mathcal{G}$ on $\mathbb{Z}^n$ is assumed to be effective, therefore the semigroup $\mathfrak{A}^{\succcurlyeq}$ is positive, n dimensional and saturated subsemigroup of $\mathbb{Z}^n$ for any $\succcurlyeq \in \Omega$. Further $\{\mathfrak{o}(\mathbf{a}) : \mathbf{a} \in \mathfrak{A}\}$ is totally ordered by any $\succcurlyeq \in \Omega$ via the corresponding head term of the orbit sums in $\mathfrak{A}^{\succcurlyeq}$ which give an ordering on $\mathbf{R}$. In the remaining sections of this chapter we exploit this property to define a monic subsemigroup of $\mathbf{R}$ and linearization of $\mathcal{G}$ via a finite generated subsemigroups of $\mathfrak{A}^{\succcurlyeq}$.

3.2 Monic Subsemigroup of R

In this section we will construct a multiplicative subsemigroup of $\mathbf{R}$ containing only monic members of $\mathbf{R}$ for any given ordering $\succcurlyeq \in \Omega$. Let us associate a function to each $\succcurlyeq \in \Omega$ via $\mathfrak{A}$,

Definition 3.2.1. Let $\succcurlyeq \in \Omega$ and define,

$$\phi(\succcurlyeq) : \mathfrak{A} \to \mathfrak{A}^{\succcurlyeq} \quad \mathbf{a} \to \phi_(\succcurlyeq)(\mathbf{a}) = [\mathbf{a}] \cap \mathfrak{A}^{\succcurlyeq} \quad \forall \mathbf{a} \in \mathfrak{A}, \tag{3.2.1}$$

the map $\phi(\succcurlyeq)$ is called $\succcurlyeq$-associated function (where $\mathfrak{A} = \mathfrak{A}^{\succcurlyeq_{lex}}$).

Since for each orbit there exist exactly one maximal member with respect to $\succcurlyeq$ the above map is well defined. We use ϕ in place of $\phi(\succcurlyeq)$ if the particular order is arbitrary. Further some of the immediate properties of $\phi(\succcurlyeq)$ is given in the lemma below,

Lemma 3.2.1. *Let $\succcurlyeq \in \Omega$ and ϕ is the associated function equation:3.2.1 then the following holds,*

1. $\phi(\mathbf{0}) = \mathbf{0}$ *[In fact for any $\mathbf{a} \in [\mathbb{Z}^n]^{\mathcal{G}}$ $\quad \phi(\mathbf{a}) = \mathbf{a}$].*
2. $\forall \mathbf{a} \in \mathfrak{A}$ *and* $n \in \mathbb{N}$, $\phi(n\mathbf{a}) = n\phi(\mathbf{a})$.
3. *ϕ is bijective map (hence ϕ^{-1} exists).*
4. $\forall \mathbf{a} \in \mathfrak{A} \quad \mathfrak{o}(\mathbf{a}) = \mathfrak{o}(\phi(\mathbf{a}))$.

Proof.
1. Follows from the fact that $[\mathbf{0}] = \{\mathbf{0}\}$ and for any $\mathbf{a} \in [\mathbb{Z}^n]^{\mathcal{G}}$, the orbit $[\mathbf{a}] = \{\mathbf{a}\}$ contain a single element.
2. By definition $a \sim \phi(a)$ both are in the same orbit then so is $n\mathbf{a} \sim n\phi(\mathbf{a})$, the property once follow from the fact $\mathfrak{A}$ and $\mathfrak{A}^{\succcurlyeq}$ are saturated subsemigroup of $\mathbb{Z}^n$ and are transverse set (fundamental domain) for the $\mathcal{G}$ action on $\mathbb{Z}^n$.
3. Holds since both set contain a single element from the equivalent class i.e. orbits.
4. Follows from the definition of orbit sum in equation:2.4.1.

□

Remark 3.2.2.
1. In general ϕ is not a semigroup homomorphism but the above lemma 3.2.1,(2) shows that, ϕ restricted to any ray (subsemigroup of $\mathfrak{A}$ generated by a single element) is a semigroup homomorphism. Further $\phi(\mathbf{a}+\mathbf{b}) \neq \phi(\mathbf{a}) + \phi(\mathbf{b})$ implies that the product $\mathfrak{o}(\mathbf{a})\mathfrak{o}(\mathbf{b})$ is associated with at least two distinct orbit sums in $\mathbf{R}$ contributing head term i.e. with $\mathfrak{o}(\mathbf{a}+\mathbf{b}) = \mathfrak{o}(\phi(\mathbf{a}+\mathbf{b}))$ and $\mathfrak{o}(\phi(\mathbf{a}) + \phi(\mathbf{b}))$. Note all properties of lemma holds for ϕ^{-1}. But since the product $\mathfrak{o}(\mathbf{a})\mathfrak{o}(\mathbf{b})$ has only finitely many such orbits sums appearing as a head term, which we try to characterize in order to study the factor subsemigroup of $\mathbf{R}$ in later sections.
2. The use of $\mathfrak{A}$ as a source (domain) for the $\succcurlyeq$ associated function could be any transverse set $\mathbb{Z}^n|\mathcal{G}$. Since each orbit has a unique representation in the transverse set $\mathfrak{A}^{\succcurlyeq}$ for any $\succcurlyeq \in \Omega$, one can take $\phi(\succcurlyeq) = f_{\succcurlyeq} f_{lex}^{-1}$ where $f_{\succcurlyeq} : \mathbb{Z}^n|\mathcal{G} \to \mathfrak{A}^{\succcurlyeq}$ and $f_{lex} : \mathbb{Z}^n|\mathcal{G} \to \mathfrak{A} = \mathfrak{A}^{\succcurlyeq_{lex}}$ and hence the source (domain) use of $\mathfrak{A}$ is for its convenience as it ($\mathfrak{A}$) is both a transverse set and a semigroup. Note that for any $\mathbf{a} \in \mathbb{Z}^n/\mathcal{G}$ we have $[[\mathbf{a}] \cap \mathfrak{A}] \cap \mathfrak{A}^{\succcurlyeq} = [\mathbf{a}] \cap \mathfrak{A}^{\succcurlyeq} = \phi(\mathbf{a})$.

Definition 3.2.3.
1. Let $f \in \mathbf{R}$ then f has a unique representation,

$$f = \sum_{a \in A_f} k_a \mathfrak{o}(\mathbf{a})$$

where $k_a \in \mathbb{K}$ and $A_f = \{\mathbf{a} \in \mathfrak{A} : \ k_a \neq 0\}$ is a finite subset of $\mathfrak{A}$ is known as the *support* of f in $\mathbf{R}$.

2. Let $\succcurlyeq \in \Omega$ then the support of f is *totally ordered* via the image of $\succcurlyeq$-associated function. i.e. $A_f^{\succcurlyeq} = \{\phi(\mathbf{a}) : \mathbf{a} \in A_f\} \subseteq \mathfrak{A}^{\succcurlyeq}$ and,
   (a) $\mathfrak{o}(\mathbf{a})$ is said to be the *lead orbit sum* of f with respect to $\succcurlyeq$ if

   $$\phi(\mathbf{a}) = max_{\succcurlyeq}\{\phi(\mathbf{b}) : \mathbf{b} \in A_f\}$$

   we denote the lead orbit sum by $(a_f, \succcurlyeq) \in A_f$ or simply by a_f if the $\succcurlyeq$ is implicit.
   (b) f is said to be *monic* with respect to $\succcurlyeq$ if $K_{(a_f,\succcurlyeq)} = 1$ i.e. the lead orbit sum has a unit coefficient.

Here the definition for support of $f \in \mathbf{R}$, an invariant polynomial is based on the orbit sums rather than the usual terms of the Laurent polynomial ring. As one can observe that if $\mathbf{x^a}$ is a term in $f \in \mathbf{R}$ then if $\mathbf{b} \in [\mathbf{a}]$ the term $\mathbf{x^b}$ exists in f with exactly the same coefficient. This implies that the orbit set $[\mathbf{a}]$ is in the support with common coefficient. Observe that in this definition of support, we have a contracted (one element for each orbit sum) notation for support of polynomials in the invariant ring.

Theorem 3.2.2. *For $\succcurlyeq \in \Omega$ define the set $\mathbf{R}_{\succcurlyeq}$ to be the set of monic members of $\mathbf{R}$ with respect to $\succcurlyeq$ then, with the ring multiplication, $\mathbf{R}_{\succcurlyeq}$ form a subsemigroup of $(\mathbf{R}, \cdot)$.*

Proof. Clearly $1 \in \mathbf{R}_{\succcurlyeq}$ and let $f, g \in \mathbf{R}_{\succcurlyeq}$ then, $f = \mathfrak{o}(a_f) + f_t$ and $g = \mathfrak{o}(a_g) + g_t$ where $f_t, g_t \in \mathbf{R}$ are the tail of f and g with respect to $\succcurlyeq$. Further $\phi(a_f) \succcurlyeq \phi(\mathbf{a})\ \forall \mathbf{a} \in A_f$ and $\phi(a_g) \succcurlyeq \phi(\mathbf{b})\ \forall \mathbf{b} \in A_g$ which implies,

$$\phi(a_f) + \phi(a_g) \succcurlyeq \phi(\mathbf{a}) + \phi(\mathbf{b})\ \forall \mathbf{a} \in A_f, \forall \mathbf{b} \in A_g.$$

hence it is enough to show that the product of two monic orbit sum is a monic. Let $\phi(a_f) = \mathbf{c} \quad \phi(a_g) = \mathbf{d} \in \mathfrak{A}^{\succcurlyeq}$ then,

$$\mathfrak{o}(\mathbf{c}) \cdot \mathfrak{o}(\mathbf{d}) = \sum_{x \in [\mathbf{c}], y \in [\mathbf{d}] \ni x+y \in \mathfrak{A}^{\succcurlyeq}} \mathfrak{o}(x + y),$$

since $x + y = \mathbf{c} + \mathbf{d}$ if and only if $x = \mathbf{c}$ and $y = \mathbf{d}$ further $\mathfrak{A}^{\succcurlyeq}$ is totally ordered together imply that $\mathfrak{o}(\mathbf{c} + \mathbf{d})$ is the lead orbit with coefficient unity. Further,

$$fg = \mathfrak{o}(\phi(a_f) + \phi(a_g)) + (fg)_t \in \mathbf{R}_{\succcurlyeq}.$$

thus $\mathbf{R}_{\succcurlyeq}$ is a subsemigroup of $\mathbf{R}$. □

For any order $\succcurlyeq$ the orbit sums $\{\mathfrak{o}(\mathbf{a}) : \mathbf{a} \in \mathfrak{A}\} \subseteq \mathbf{R}_{\succcurlyeq}$, are contained in the monic semigroup $\mathbf{R}_{\succcurlyeq}$ and is totally ordered. $f \in \mathbf{R}$ can be a monic for some multiplicative order and may fail for others so to remove these type of polynomials we define the following,

Definition 3.2.4. Let $\{\mathbf{R}_{\succcurlyeq}\}_{\succcurlyeq \in \Omega}$ be the family of monic subsemigroups of $\mathbf{R}$, define

$$\mathbf{R}_m = \cap_{\succcurlyeq \in \Omega} \quad \mathbf{R}_{\succcurlyeq},$$

a subsemigroup $\mathbf{R}$ to be a *monic subsemigroup*. For $f \in \mathbf{R}_m$ define,

$$A_f(d) = \{\mathbf{a} \in A_f : \mathfrak{o}(\mathbf{a}) \textit{ lead orbit sum of } f \quad \textit{for some } \succcurlyeq \in \Omega\}$$

the *dominant* orbit sums of f.

$\mathbf{R}_m$ is a monic subsemigroup of $\mathbf{R}$ without prescribing to any particular multiplicative orders. This does not mean $f \in \mathbf{R}_m$ will have a unique lead orbit sum rather f will be monic for any given order with possible different lead orbit sum. Hence definition of dominant orbit sum $A_f(d)$ may contain more than one elements. i.e. f is monic in finitely many ways. In general invariant polynomials with all of the coefficient 1 will appear in $\mathbf{R}_m$. More over we have the inclusion of semigroup, for any $\succcurlyeq \in \Omega$,

$$\mathbf{R}_m \subseteq \mathbf{R}_{\succcurlyeq} \subseteq \mathbf{R}$$

$\{\mathfrak{o}(\mathbf{a}) : \mathbf{a} \in \mathfrak{A}\} \subseteq \mathbf{R}_m$. In the coming sections we use the multiplicative order on the monic semigroup $\mathbf{R}_m$, since it contains monic polynomials for any given $\succcurlyeq \in \Omega$.
Let consider $\succcurlyeq \in \Omega$, define a congruence (an equivalence relation compatible with the semigroup operation of $\mathbf{R}_m$) $\precsim$ on $\mathbf{R}_m$ by,

$$f, g \in \mathbf{R}_m \quad f \precsim g \quad \textit{if} \quad (a_f, \succcurlyeq) = (a_g, \succcurlyeq).$$

i.e. two elements $f, g \in \mathbf{R}_m$ are equivalent if they have the same lead orbit sum with respect to the given $\succcurlyeq$. This congruence will be used in the following theorem.

Theorem 3.2.3. *For $\succcurlyeq \in \Omega$ then,*

$$\mathbf{R}_m / \precsim \;\; \cong \;\; \mathfrak{A}^{\succcurlyeq}$$

Proof. Let

$$\varphi : \mathbf{R}_m \longmapsto \mathfrak{A}^{\succcurlyeq} \quad f \longmapsto \varphi(f) = \phi(a_f)$$

Here we use a_f for $(a_f, \succcurlyeq)$ to simplify notation and for any $f, g \in \mathbf{R}_m$ we have

$$fg = \mathfrak{o}(\phi(a_f) + \phi(a_g)) + (fg)_t \in \mathbf{R}_m$$

where a_f and a_g are the elements in A_f and A_g respectively such that,

$$\phi(a_f) = \max_{\succcurlyeq}\{\phi(\mathbf{a}) : \mathbf{a} \in A_f\}$$

$$\phi(a_g) = \max_{\succcurlyeq}\{\phi(\mathbf{a}) : \mathbf{a} \in A_g\}$$

this implies $f = \mathfrak{o}(a_f) + f_t$ and $g = \mathfrak{o}(a_g) + g_t$ by lemma 3.2.1,4, $\mathfrak{o}(a_f) = \mathfrak{o}(\phi(a_f))$ similarly for g and,

$$\begin{aligned}\varphi(fg) &= \varphi(\mathfrak{o}(\phi(a_f) + \phi(a_g)) + (fg)_t) \\ &= \phi(a_f) + \phi(a_g) \\ &= \varphi(f) + \varphi(g).\end{aligned}$$

Since $\{\mathfrak{o}(\mathbf{a}) : \mathbf{a} \in \mathfrak{A}\} \subseteq \mathbf{R}_m$ implies φ is epimorphism and the isomorphism at once follows from the fact that the kernel congruence of φ is the same as $\precsim$ i.e. $\varphi(f) = \varphi(g)$ if and only if $f \precsim g$ for $f, g \in \mathbf{R}_m$. We denote the isomorphism by ψ and $\mathbf{R}_m / \precsim$ by $\mathcal{M}^{\succcurlyeq}$ for simplicity. Hence

$$\psi : \mathcal{M}^{\succcurlyeq} \to \mathfrak{A}^{\succcurlyeq},$$

and $\psi([f]) = \varphi(f)$ for $[f] \in \mathcal{M}^{\succcurlyeq}$. □

In the above $\mathfrak{A}$ is used for notational purpose does not play any role in the isomorphism. The semigroup epimorphism φ is the initial map (restricted to $\mathbf{R}_m$) associated with $\succcurlyeq \in \Omega$, and the congruence collect those elements of $\mathbf{R}_m$ which shares the same lead orbit sum.

Note 3.2.5. Observe for $\succcurlyeq \in \Omega$, transverse elements $T \subset \mathbf{R}_m$ of $\mathcal{M}^{\succcurlyeq}$ form a $\mathbb{K}$ basis for $\mathbf{R}$. The semigroup structure of $\mathcal{M}^{\succcurlyeq}$ depend on $\mathfrak{A}^{\succcurlyeq}$ via the isomorphism hence on $\succcurlyeq \in \Omega$. To characterize semigroup structure of $\mathbf{R}$, one have to look invariance on $\mathfrak{A}^{\succcurlyeq}$ or subsemigroup of $\mathfrak{A}^{\succcurlyeq}$ over $\succcurlyeq \in \Omega$.

3.3 The Linear Semigroup

The isomorphism in theorem 3.2.3 and note 3.2.5, confirms that for $\succcurlyeq \in \Omega$ a transverse set $T \subseteq \mathbf{R}_m$ of the factor subsemigroup $\mathcal{M}^{\succcurlyeq}$ exists and,

$$\mathbf{R} = \oplus_{s \in T} \mathbb{K} s$$

the element of T that form a $\mathbb{K}$ basis for $\mathbf{R}$. One can considers the set,

$$\{\mathfrak{o}(\mathbf{a}) : \mathbf{a} \in \mathfrak{A}\} \tag{3.3.1}$$

which forms a set of transverse element for $\mathcal{M}^{\succcurlyeq}$ any given order and is contained in $\mathbf{R}_m$. But this set is often not closed under ring multiplication. Further more for two distinct order the product,

$$\mathfrak{o}(\mathbf{a})\mathfrak{o}(\mathbf{b}),$$

for $\mathbf{a} \neq \mathbf{b} \in \mathfrak{A}$ can have distinct lead orbit sum. In this section we define a finitely generated subsemigroup S of $\mathfrak{A}^{\succcurlyeq}$ via $\mathfrak{A}$ and the function $\phi : \mathfrak{A} \to \mathfrak{A}^{\succcurlyeq}$, in such a way that the product of orbit sums corresponding to the generators of S gives every possible elements in the set equation:3.3.1, as a lead orbit sum for different multiplicative order. Those ordering in Ω giving the same lead orbit for the products will latter be classified as equivalent orders. Before defining this semigroup lets consider the following terminologies.

A finitely generated, and positive subsemigroup S of $\mathbb{Z}^n$ is said to be *simplical* if there exist r linearly independent elements $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_r\} \subseteq \mathbb{Q}^n$ such that,

$$S \subseteq \langle \mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_r \rangle_{mon} \subseteq \mathbb{Q}_+ S. \tag{3.3.2}$$

where r is the dimension $span_{\mathbb{Q}}(S)$. For some minimal $\lambda_i \in \mathbb{N}$ one have $\lambda_i \mathbf{u}_i = \mathbf{a}_i$ is in the generator set of S for $i = 1, 2, \cdots, r$ and the generators $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_r\}$ are called the *critical* generators of S.

Let $\mathcal{G}$ be finite group in $GL(n, \mathbb{Z})$ and $\mathfrak{A}^{\succcurlyeq}$ and $\mathfrak{A}$ are the semigroup of head term the orbits associated with $\succcurlyeq$ and the usual lex respectively. Let $\langle\, ,\, \rangle$, be the $\mathcal{G}$-invariant inner product (averaged usual dot product over $\mathcal{G}$). Then,

Definition 3.3.1. Let $\mathfrak{A}_l = \langle \mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t \rangle_{mon} \subseteq \mathfrak{A}$ be a minimal, simplical and saturated subsemigroup of $\mathfrak{A}$ satisfying,

1. $\mathbf{a}, \mathbf{b} \in \mathfrak{A}_l \quad \langle \mathbf{a}, \mathbf{b} \rangle \geq 0$ and non of the critical generators of $\mathfrak{A}_l$ can be written as a sum, $\mathbf{a} + \mathbf{b}$ and $\langle \mathbf{a}, \mathbf{b} \rangle \geq 0$ in $\mathfrak{A}$.
2. For any $\mathbf{a} \in \mathfrak{A}$ there exists $\succcurlyeq \in \Omega$ such that,

$$\phi(\mathbf{a}) \in \langle \phi(\mathbf{a}_1), \phi(\mathbf{a}_2), \cdots, \phi(\mathbf{a}_t) \rangle_{mon} =: \mathfrak{A}_l^{\succcurlyeq}$$

where ϕ is $\succcurlyeq$ associated function.

The minimal (with respect to inclusion) is taken among semigroup satisfying the other conditions of the definition. One observes that if this semigroup $\mathfrak{A}_l$ exists then every, $\mathbf{a} \in \mathfrak{A}$ if not contained in $\mathfrak{A}_l$ then must appear as element of $\mathfrak{A}_l^{\succcurlyeq}$ for some $\succcurlyeq \in \Omega$. Note the generating set of the semigroup $\mathfrak{A}_l^{\succcurlyeq}$ are strictly the image of generators of $\mathfrak{A}_l$ via ϕ. Now we give some properties of such semigroups enforced by the criteria in definition 3.3.1.

Proposition 3.3.1. *Let $\mathcal{G}$ admits such a semigroup $\mathfrak{A}_l$ satisfying definition 3.3.1 then,*

1. *The semigroups $\mathfrak{A}_l \cong \mathfrak{A}_l^{\succcurlyeq}$ for any $\succcurlyeq \in \Omega$.*
2. *The semigroup $\mathfrak{A}_l^{\succcurlyeq}$ is n dimensional.*

Proof. 1. From the definition of $\mathfrak{A}_l$ and $\mathfrak{A}_l^{\succcurlyeq}$, we have generating sets,

$$\{\mathbf{a}_i : i = 1, 2, \cdots, t\} =: \mathfrak{A}_{gl} \quad \{\phi(\mathbf{a}_i) : i = 1, 2, \cdots, t\} =: \mathfrak{A}_{gl}^{\succcurlyeq}$$

and since ϕ preserve orbits both the above set associate to the same orbit sums, i.e.

$$\{\mathfrak{o}(\mathbf{a}_i) = \mathfrak{o}(\phi(\mathbf{a}_i)) : i = 1, 2, \cdots, t\}.$$

Hence the semigroup isomorphism follows from the fact $\phi_l(\succcurlyeq) : \mathfrak{A}_l \to \mathfrak{A}_l^{\succcurlyeq}$ relates the head terms of the product the orbit sums in $\{\mathfrak{o}(\mathbf{a}_i) : i = 1, 2, \cdots, t\}$.

$$\prod_{i=1}^{t} \mathfrak{o}(\mathbf{a}_i)^{n_i} = \prod_{i=1}^{t} \mathfrak{o}(\phi(\mathbf{a}_i))^{n_i}$$
$$\sum_{i=1}^{t} n_i \mathbf{a}_i \xrightarrow{\phi_l(\succcurlyeq)} \sum_{i=1}^{t} n_i \phi(\mathbf{a}_i).$$

Since both $\mathfrak{A}_l$ and $\mathfrak{A}_l^{\succcurlyeq}$ are totally ordered (in $\mathfrak{A}$ and $\mathfrak{A}^{\succcurlyeq}$ respectively) and the lead orbit sum of products of orbit sums generated by $\{\mathfrak{o}(\mathbf{a}_i) = \mathfrak{o}(\phi(\mathbf{a}_i)) : i = 1, 2, \cdots, t\}$ are uniquely identified depending the multiplicative orders, we have $\phi_l(\succcurlyeq) : \mathfrak{A}_l \to \mathfrak{A}_l^{\succcurlyeq}$ one to one correspondence.

2. Let denote the set containing all possible generator of the semigroup $\mathfrak{A}_l^{\succcurlyeq}$ for some $\succcurlyeq \in \Omega$, by Σ_l hence,

$$\Sigma_l = \{\mathbf{a} \in \mathbb{Z}^n : \mathbf{a} = \phi(\succcurlyeq)(\mathbf{a}_i), \quad \mathbf{a}_i \in \mathfrak{A}_{gl}, \quad \succcurlyeq \in \Omega\} = \cup_{i=1}^{t} [\mathbf{a}_i].$$

is a finite subset of $\mathbb{Z}^n$ and hence there are finitely many semigroups of the form,

$$\langle \phi(\mathbf{a}_1), \phi(\mathbf{a}_2), \cdots, \phi(\mathbf{a}_t) \rangle_{mon} =: \mathfrak{A}_l^{\succcurlyeq}, \quad \exists \succcurlyeq \in \Omega. \tag{3.3.3}$$

But since $\mathfrak{A}$ is n dimensional definition 3.3.1,2, $\mathfrak{A}$ is contained in union of finitely many isomorphic semigroups. Each such semigroup must be of dimension n. Here we use the fact $\mathbb{Q}^n = span_{\mathbb{Q}}(\mathfrak{A})$ cannot be contained in the union of finitely many proper subspaces, at least one of $span_{\mathbb{Q}}(\mathfrak{A}_l^{\succcurlyeq})$ must have dimension n and hence all have dimension n follows from isomorphism in (1).

□

Let ℓ_Σ denotes semigroups in $\mathbb{Z}^n$ which are of the form eq:3.3.3. The fact that $\mathfrak{A}$ is a fundamental domain for $\mathcal{G}$ action on $\mathbb{Z}^n$ implies,

$$\mathbb{Z}^n = \cup_{g \in \mathcal{G}} g(\mathfrak{A}).$$

Further $\mathfrak{A}$ is contained in finite union of elements of ℓ_Σ together with each $g \in \mathcal{G}$ sends semigroups of the form Eq:3.3.3 to some semigroup in ℓ_Σ. Therefore,

$$\mathbb{Z}^n = \cup_{S \in \ell_\Sigma} S. \tag{3.3.4}$$

In the following lemma and the coming section we characterize, which type of elements (orbits) are a must to appear in the generating set of $\mathfrak{A}_l^{\succcurlyeq}$ for any $\succcurlyeq \in \Omega$. Let us denote the set containing the generators $\mathfrak{A}^{\succcurlyeq}$ by Σ_l, and

$$\Sigma_l = \{\mathbf{a} \in \mathbb{Z}^n : \mathbf{a} = \phi(\succcurlyeq)(\mathbf{a}_i), \quad \mathbf{a}_i \in \mathfrak{A}_{gl}, \quad \succcurlyeq \in \Omega\} = \cup_{i=1}^{t} [\mathbf{a}_i]. \tag{3.3.5}$$

The irreducibility of elements in $\mathfrak{A}^{\succcurlyeq}$ often depends on the multiplicative order $\succcurlyeq \in \Omega$.

But there are irreducible elements irrespective of the order. In lemma 3.3.2 we consider irreducibility of elements in $\mathfrak{A}^{\succcurlyeq}$ for every $\succcurlyeq\in\Omega$ forces these elements to be in Σ_l. Further in section 3.4 we show singular elements of $\mathfrak{A}^{\succcurlyeq}$ will be always contained in $\mathfrak{A}_l^{\succcurlyeq}$ for any $\succcurlyeq\in\Omega$.

Lemma 3.3.2. *Let* $\mathbf{a}\in\mathbb{Z}^n$ *such that for any* $\succcurlyeq\in\Omega \quad [\mathbf{a}]\cap\mathfrak{A}^{\succcurlyeq}=\phi(\mathbf{a})$ *is irreducible in* $\mathfrak{A}^{\succcurlyeq}$ *then* $[\mathbf{a}]\subseteq\Sigma_l$.

Proof. Let assume $\mathbf{a}\in\mathfrak{A}$.
Claim: $\mathbf{a}$ in the generating set of $\mathfrak{A}_l$.
Suppose not, then $\mathbf{a}\notin\mathfrak{A}_l$ since $\mathbf{a}$ is irreducible in $\mathfrak{A}$ implies it is irreducible in $\mathfrak{A}_l$. From definition 3.3.1, (2) there exist $\succcurlyeq\in\Omega$ such that $\phi(\mathbf{a})\in\mathfrak{A}_l^{\succcurlyeq}$. But since $\phi(\mathbf{a})$ is irreducible in $\mathfrak{A}^{\succcurlyeq}$ implies $\phi(\mathbf{a})\in\mathfrak{A}_{gl}^{\succcurlyeq}$ the generating set for $\mathfrak{A}_l^{\succcurlyeq}$ a contradiction since $\mathfrak{A}_{gl}^{\succcurlyeq}=\{\phi(\mathbf{b}):\mathbf{b}\in\mathfrak{A}_{gl}\}$. Therefor $[\mathbf{a}]\cap\mathfrak{A}^{\succcurlyeq}=\phi(\mathbf{a})\in\mathfrak{A}_{gl}^{\succcurlyeq}$ for any $\succcurlyeq\in\Omega$ and hence $[\mathbf{a}]\subseteq\Sigma_l$. □

The proof of the following theorem is given at the end of section 3.4 which needs some preparation,

Theorem 3.3.3. *Let* $\mathbf{a}\in\mathbb{Z}^n$ *be a singular element i.e.* $\mathcal{G}_{\mathbf{a}}\neq\{I\}$ *then for each* $\succcurlyeq\in\Omega$,

$$[\mathbf{a}]\cap\mathfrak{A}^{\succcurlyeq}=\phi(\mathbf{a})\in\mathfrak{A}_l^{\succcurlyeq}.$$

Proof. The proof depend on the properties of multiplicative order associated map and stabilizer subgroups and is given by corollary 3.4.4. □

3.4 Decomposition of $\mathfrak{A}^{\succcurlyeq}$ via Isotropy Subgroups

Let $\succcurlyeq\in\Omega$ define the following relations on $\mathfrak{A}^{\succcurlyeq}$ for $\mathbf{a},\mathbf{b}\in\mathfrak{A}^{\succcurlyeq}$,

$$\begin{aligned}\mathbf{a}\lesssim_s\mathbf{b} &\quad \textit{if} \quad \mathcal{G}_{\mathbf{a}}\subseteq\mathcal{G}_{\mathbf{b}},\\ \mathbf{a}\sim_s\mathbf{b} &\quad \textit{if} \quad \mathcal{G}_{\mathbf{a}}=\mathcal{G}_{\mathbf{b}}, \quad [\iff]\mathbf{a}\lesssim_s\mathbf{b}\wedge\mathbf{b}\lesssim_s\mathbf{a}.\end{aligned}$$

Clearly $\lesssim_s$ is reflexive and transitive (a pre-order). It is neither symmetric nor antisymmetric on $\mathfrak{A}^{\succcurlyeq}$, while the relation $\sim_s$ is an equivalence on $\mathfrak{A}^{\succcurlyeq}$ further let denote $\mathfrak{A}^{\succcurlyeq}/\sim_s$ the set of equivalent classes for $\sim_s$ then, the following lemma gives some important properties of the above relations,

Lemma 3.4.1.
1. $|\mathfrak{A}^{\succcurlyeq}/\sim_s|$ *is finite (at least 2) and* $\lesssim$ *is a partial order on the equivalence classes (i.e.* $\mathfrak{A}^{\succcurlyeq}/\sim_s$*).*
2. *For* $\mathbf{a},\mathbf{b}\in\mathfrak{A}^{\succcurlyeq}$, $\quad\mathcal{G}_{\mathbf{a}+\mathbf{b}}=\mathcal{G}_{\mathbf{a}}\cap\mathcal{G}_{\mathbf{b}}$.
3. *For* $\mathbf{a}\in\mathfrak{A}^{\succcurlyeq}$ *the equivalent class* $[\mathbf{a}]_{\sim_s}$ *is a semigroup.*
4. *For* $\mathbf{a},\mathbf{b}\in\mathfrak{A}^{\succcurlyeq}$, $\quad\mathbf{a}+\mathbf{b}\lesssim_s\mathbf{a}$ *and* $\mathbf{a}+\mathbf{b}\lesssim_s\mathbf{b}$*, in addition if* $\mathbf{a}\lesssim_s\mathbf{b}$ *then* $\mathbf{a}+\mathbf{b}\sim_s\mathbf{a}$.

Proof. 1. Since $\mathcal{G}$ is a finite group it has finitely many subgroups hence stabilizer (isotropy) subgroups, of which $\mathcal{G}$ and $\{I\}$ exist therefore $\mathfrak{A}^{\succcurlyeq}/\sim_s$ has finitely many classes (at least two). These two classes are $(\mathfrak{A}^{\succcurlyeq})^{\circ} = [\mathbf{0}]_{\sim_s} = [\mathbb{Z}^n]^{\mathcal{G}}$ the trivial class and $(\mathfrak{A}^{\succcurlyeq})_{\circ} = \{\mathbf{a} \in \mathfrak{A}^{\succcurlyeq} : \mathcal{G}_{\mathbf{a}} = \{I\}\}$ the regular class for any non trivial $\mathcal{G}$. The other classes are determined by singular elements. Clearly $\lesssim_s$ satisfies the antisymmetric property on the equivalence classes (see the equivalence definition of $\sim_s$). The set $(\mathfrak{A}^{\succcurlyeq}/\sim_s, \lesssim_s)$ is partially ordered with unique maximal $(\mathfrak{A}^{\succcurlyeq})^{\circ}$ and minimal $(\mathfrak{A}^{\succcurlyeq})_{\circ}$.

2. Clearly $\mathcal{G}_{\mathbf{a}} \cap \mathcal{G}_{\mathbf{b}} \subseteq \mathcal{G}_{\mathbf{a}+\mathbf{b}}$ conversely let $h \in \mathcal{G}$ and $h \notin \mathcal{G}_{\mathbf{a}} \cap \mathcal{G}_{\mathbf{b}}$ and consider two cases,
   - h fixes one without loss of generality let $h \in \mathcal{G}_{\mathbf{a}}$ and $h \notin \mathcal{G}_{\mathbf{b}}$ then,

     $$h(\mathbf{b}) \neq \mathbf{b} \Leftrightarrow h(\mathbf{a}+\mathbf{b}) = \mathbf{a} + h(\mathbf{b}) \neq \mathbf{a}+\mathbf{b},$$

     hence $h \notin \mathcal{G}_{\mathbf{a}+\mathbf{b}}$ (note that $\mathfrak{A}^{\succcurlyeq}$ is cancellative).
   - h fixes none $\mathbf{a} \neq h(\mathbf{a}) = \mathbf{a}\prime$ and $\mathbf{b} \neq h(\mathbf{b}) = \mathbf{b}\prime$. Since $\mathbf{a} \succ \mathbf{a}\prime \Rightarrow \mathbf{0} \neq \mathbf{a} - \mathbf{a}\prime \succ \mathbf{0}$ and similarly $\mathbf{0} \neq \mathbf{b} - \mathbf{b}\prime \succ \mathbf{0}$, implies $(\mathbf{a}-\mathbf{a}\prime) + (\mathbf{b}-\mathbf{b}\prime) \neq \mathbf{0}$ and $\mathbf{a}' + \mathbf{b}' = h(\mathbf{a}+\mathbf{b}) \neq \mathbf{a}+\mathbf{b}$ hence $h \notin \mathcal{G}_{\mathbf{a}+\mathbf{b}}$.

   therefore $\mathcal{G}_{\mathbf{a}+\mathbf{b}} \subseteq \mathcal{G}_{\mathbf{a}} \cap \mathcal{G}_{\mathbf{b}}$, which implies $\mathcal{G}_{\mathbf{a}+\mathbf{b}} = \mathcal{G}_{\mathbf{a}} \cap \mathcal{G}_{\mathbf{b}}$.
3. Let $\mathbf{a} \in \mathfrak{A}^{\succcurlyeq}$ then $[\mathbf{a}]_{\sim_s} = \{\mathbf{b} : \mathbf{b} \sim_s \mathbf{a}\} = \{\mathbf{b} : \mathcal{G}_{\mathbf{b}} = \mathcal{G}_{\mathbf{a}}\}$ hence $\mathcal{G}_a = \mathcal{G}_b$ and (2) implies $\mathcal{G}_{\mathbf{a}+\mathbf{b}} = \mathcal{G}_{\mathbf{a}} \cap \mathcal{G}_{\mathbf{b}} = \mathcal{G}_{\mathbf{a}}$ there fore for $\mathbf{a}, \mathbf{b} \in [\mathbf{a}]_{\sim_s}$ then, $\mathbf{a}+\mathbf{b} \sim_s \mathbf{a}$ and $\mathbf{a}+\mathbf{b} \in [\mathbf{a}]_{\sim_s}$ hence a semigroup.
4. Follows from the fact $\mathcal{G}_{\mathbf{a}} \cap \mathcal{G}_{\mathbf{b}} \subseteq \mathcal{G}_{\mathbf{a}} | \mathcal{G}_{\mathbf{b}}$, and $\mathcal{G}_{\mathbf{a}} \cap \mathcal{G}_{\mathbf{b}} = \mathcal{G}_{\mathbf{a}}$ if $\mathcal{G}_{\mathbf{a}} \subseteq \mathcal{G}_{\mathbf{b}}$ respectively.

□

For the equivalence class (semigroups) in $\mathfrak{A}^{\succcurlyeq} | \sim_s$ in view of (3) and (4) of above lemma we define the semigroup $[\mathbf{b}]_{\sim_s}$ is said to be a *face (or boundary) semigroup* of $[\mathbf{a}]_{\sim_s}$ if $\mathcal{G}_{\mathbf{a}} \subsetneq \mathcal{G}_{\mathbf{b}}$. Further the *closure semigroup* of $[\mathbf{a}]_{\sim_s}$ denoted by $\overline{[a]}_{\sim_s}$ to be,

$$\{\mathbf{b} \in \mathfrak{A}^{\succcurlyeq} | \, g(\mathbf{b}) = \mathbf{b} \quad \forall g \in \mathcal{G}_{\mathbf{a}}\}.$$

Theorem 3.4.2. *Let* $\mathbf{a} \in \mathfrak{A}^{\succcurlyeq}$,

1. $\overline{[\mathbf{a}]}_{\sim_s} = \cap_{g \in \mathcal{G}_{\mathbf{a}}} g(\mathfrak{A}^{\succcurlyeq}) = [\mathbb{Z}^n]^{\mathcal{G}_{\mathbf{a}}} \cap \mathfrak{A}^{\succcurlyeq} = \sum_{[\mathbf{a}]_{\sim_s} \lesssim_s [\mathbf{b}]_{\sim_s}} [\mathbf{b}]_{\sim_s}$.
2. *if* $\mathbf{u} \in \overline{[\mathbf{a}]}_{\sim_s}$ *is irreducible, then it is irreducible in* $\mathfrak{A}^{\succcurlyeq}$.

Proof. 1. Follows from the fact $\mathfrak{A}^{\succcurlyeq}$ is a fundamental set i.e. $\mathbf{u}, g(\mathbf{u}) \in \mathfrak{A}^{\succcurlyeq}$ then $\mathbf{u} = g(\mathbf{u})$. Therefore $\mathbf{u} \in \cap_{g \in \mathcal{G}_{\mathbf{a}}} g(\mathfrak{A}^{\succcurlyeq})$ then $\mathbf{u} \in \mathfrak{A}^{\succcurlyeq}$ and $\mathbf{u} \in g(\mathfrak{A}^{\succcurlyeq}) \, \forall g \in \mathcal{G}_{\mathbf{a}}$ hence $\mathbf{u} = g(\mathbf{u})$ for each $g \in \mathcal{G}_{\mathbf{a}}$. The other equalities are directly forward from the definition of closure semigroup. i.e. it contains $[\mathbf{a}]_{\sim_s}$ and all of its face semigroup.

2. Let $\mathbf{u} \in \overline{[\mathbf{a}]}_{\sim_s}$, be irreducible and there exist $\mathbf{b}, \mathbf{c} \in \mathfrak{A}^{\succcurlyeq}$, such that $\mathbf{u} = \mathbf{b} + \mathbf{c}$ then $\mathcal{G}_{\mathbf{a}} \subseteq \mathcal{G}_{\mathbf{u}} = \mathcal{G}_{\mathbf{c}+\mathbf{b}}$. Hence $\mathcal{G}_{\mathbf{a}} \subseteq \mathcal{G}_{\mathbf{b}}$ and $\mathcal{G}_{\mathbf{a}} \subseteq \mathcal{G}_{\mathbf{c}}$. This forces both $\mathbf{b}, \mathbf{c} \in \overline{[\mathbf{a}]}_{\sim_s}$ a contradiction. There for $\mathbf{u}$ is irreducible in $\mathfrak{A}^{\succcurlyeq}$.

□

Remark 3.4.1. 1. $(\mathfrak{A}^{\succcurlyeq})^{\circ}$ is independent of $\succcurlyeq \in \Omega$. Observe that ϕ is identity on $(\mathfrak{A}^{\succcurlyeq})^{\circ}$. [lemma 3.2.1].

2. $|\mathfrak{A}^{\succcurlyeq}/\sim_S|$ doesn't depend on the particular order.

Theorem 3.4.3. *Let $\mathbf{a} \in \mathfrak{A}$ be singular and $\succcurlyeq \in \Omega$ then,*

1. $[\mathbf{a}]_{\sim_s}(\mathfrak{A}) \cong [\phi(\mathbf{a})]_{\sim_s}(\mathfrak{A}^{\succcurlyeq})$.
2. *The restriction $\phi : \overline{[\mathbf{a}]}_{\sim_s}(\mathfrak{A}) \to \overline{[\phi(\mathbf{a})]}_{\sim_s}(\mathfrak{A}^{\succcurlyeq})$ is semigroup isomorphism and is order preserving ($\mathfrak{A}| \sim_S \to \mathfrak{A}^{\succcurlyeq}| \sim_S$).*

Proof. Each $g \in \mathcal{G}$ be considered us a semigroup homomorphism on $[\mathbf{a}]_{\sim_s}$ i.e.

$$g : [\mathbf{a}]_{\sim_s} \to \mathbb{Z}^n \quad ([\mathbf{a}]_{\sim_s} \in \mathfrak{A}/\sim_s)$$

is a semigroup automorphism. Define a relation on $\mathcal{G}$ by $g, h \in \mathcal{G}$ for any $\mathbf{u} \in [\mathbf{a}]_{\sim_s}$

$$g \sim h \quad if \quad g(\mathbf{u}) = h(\mathbf{u}) \quad [\Leftrightarrow g = h(on\, [\mathbf{a}]_{\sim_s})]$$

Note for $\mathbf{u} \in [\mathbf{a}]_{\sim_s}$ and $g(\mathbf{u}) = h(\mathbf{u})$ then $g^{-1}h \in G_{\mathbf{u}} = \mathcal{G}_{\mathbf{a}} = \mathcal{G}_t\, \forall t \in [\mathbf{a}]_{\sim_s}$, hence $h(t) = g(t)$ which show the equivalence of both definition. Clearly $\sim$ is an equivalence on $\mathcal{G}$. For each $g \in \mathcal{G}$ $\; g([\mathbf{a}]_{\sim_s}) \cong [\mathbf{a}]_{\sim_s}$ and also if $g([\mathbf{a}]_{\sim_s}) \cap h([\mathbf{a}]_{\sim_s}) \neq \emptyset$ then $g \sim h$ (i.e. if $v = h(\mathbf{u}) = g(\mathbf{u}')$ for $\mathbf{u}, \mathbf{u}' \in [\mathbf{a}]_{\sim_s}$ then $\mathbf{u} = \mathbf{u}'$ since $[\mathbf{a}]_{\sim_s} \subset \mathfrak{A}$ is in the fundamental domain). Hence there are $|[\mathbf{a}]| = [\mathcal{G} : \mathcal{G}_{\mathbf{a}}]$ disjoint semigroup in $\mathbb{Z}^n$, isomorphic to $[\mathbf{a}]_{\sim_s}$ and $\mathcal{G}$ permutes these semigroups.
The first part of the theorem once follows from the fact that for any $\succcurlyeq \in \Omega$ then $\mathfrak{A}^{\succcurlyeq}$ contains one of these $|[\mathbf{a}]|$ semigroup. Furthermore (ϕ) preserves orbit implies $[\phi(\mathbf{a})]_{\sim_s}$ the corresponding semigroup and for any $h \in \{g \in \mathcal{G} : g(\mathbf{a}) = \phi(\mathbf{a})\}$ one has $\mathbf{u} \in [\mathbf{a}]_{\sim_s}, h(\mathbf{u}) = \phi(\mathbf{u})$ there for we have,

$$\phi = h : [\mathbf{a}]_{\sim_s} \cong [\phi(\mathbf{a})]_{\sim_s}.$$

The second part follows from the above isomorphism and $\mathcal{G}_{\mathbf{a}} \subseteq \mathcal{G}_{\mathbf{b}}$ imply $h\mathcal{G}_{\mathbf{a}} \subseteq h\mathcal{G}_{\mathbf{b}}$ and hence the same h mapping $[\mathbf{a}]_{\sim_s} \to [\phi(\mathbf{a})]_{\sim_s}$ maps the corresponding boundaries (face) semigroup to both,

$$\phi = h : \overline{[\mathbf{a}]}_{\sim_s} \cong \overline{[\phi(\mathbf{a})]}_{\sim_s}.$$

Furthermore $h\mathcal{G}_{\mathbf{a}}h^{-1} = \mathcal{G}_{h\mathbf{a}} = \mathcal{G}_{\phi(\mathbf{a})} \subseteq h\mathcal{G}_{\mathbf{b}}h^{-1} = \mathcal{G}_{h\mathbf{b}} = \mathcal{G}_{\phi(\mathbf{b})}$ hence ϕ preserves the partial order $\lesssim_s$ i.e.

$$[\mathbf{a}]_{\sim_s} \lesssim_s (\mathfrak{A})[\mathbf{b}]_{\sim_s} \iff [\phi(\mathbf{a})]_{\sim_s} \lesssim_s (\mathfrak{A}^{\succcurlyeq})[\phi(\mathbf{b})]_{\sim_s}.$$

□

From the above theorem the $\succcurlyeq$ associated function ϕ is a monomorphism on each class determined by a singular elements. In addition,

Corollary 3.4.4. *Let $\mathbf{a}, \mathbf{b}$ be singular elements in $\mathfrak{A}$ and $\succcurlyeq \in \Omega$ then the associated function restricted to $\overline{[\mathbf{a}]}_{\sim_s} + \overline{[\mathbf{b}]}_{\sim_s}$ equals h for some h in $\mathcal{G}$. Furthermore $\overline{[\mathbf{a}]}_{\sim_s} \subseteq \mathfrak{A}_l$. (Hence each singular elements of $\mathfrak{A}^{\succcurlyeq}$ is contained in $\mathfrak{A}_l^{\succcurlyeq}$ for any $\succcurlyeq \in \Omega$)*

Proof. By the above theorem 3.4.3 ϕ restricted to $\overline{[\mathbf{a}]}_{\sim_s} \underset{\rightarrow}{\cong} \overline{[\phi(\mathbf{a})]}_{\sim_s}$ and $\overline{[\mathbf{b}]}_{\sim_s} \underset{\rightarrow}{\cong} \overline{[\phi(\mathbf{b})]}_{\sim_s}$ semigroup homomorphism i.e.

$$\begin{aligned} \phi = h \quad on \quad & [\mathbf{a}]_{\sim_s}, (\forall h)\, h \in \{t \in \mathcal{G} : t(\mathbf{a}) = \phi(\mathbf{a})\} =: T_{\mathbf{a}} \\ \phi = g \quad on \quad & [\mathbf{b}]_{\sim_s}, (\forall g)\, g \in \{t \in \mathcal{G} : t(\mathbf{b}) = \phi(\mathbf{b})\} =: T_{\mathbf{b}}, \end{aligned}$$

Now $T_{\mathbf{a}} \cap T_{\mathbf{b}} \neq \emptyset$ in fact $|T_{\mathbf{a}} \cap T_{\mathbf{b}}| = |\mathcal{G}_{\mathbf{a}+\mathbf{b}}| \geq 1$ with $|T_{\mathbf{a}} \cap T_{\mathbf{b}}| = |\mathcal{G}_{\mathbf{a}+\mathbf{b}}| = 1$ if $\mathbf{a}+\mathbf{b}$ become a regular element. Hence there exist $h \in \mathcal{G}$ such that $h \in T_{\mathbf{a}} \cap T_{\mathbf{b}}$ and therefore,

$$\phi = h : \overline{[\mathbf{a}]}_{\sim_s} + \overline{[\mathbf{a}]}_{\sim_s} \to \overline{[\phi(\mathbf{a})]}_{\sim_s} + \overline{[\phi(\mathbf{b})]}_{\sim_s}.$$

Note h depend on the given order $\succcurlyeq \in \Omega$. Since both $T_{\mathbf{a}}$ and $T_{\mathbf{b}}$ depend on ϕ hence $\succcurlyeq$. For the second statement of the corollary, let $\mathbf{u} \in \overline{[\mathbf{a}]}_{\sim_s}$ and be irreducible then $\mathbf{u}$ is irreducible in $\mathfrak{A}$ by theorem 3.4.2 further $\phi(\mathbf{u})$ is irreducible in $\overline{[\phi(\mathbf{a})]}_{\sim_s}$ since $\overline{[\mathbf{a}]}_{\sim_s} \cong \overline{[\phi(\mathbf{a})]}_{\sim_s}$ hence therefore in $\mathfrak{A}^{\succcurlyeq}$ for any $\succcurlyeq \in \Omega$ by lemma 3.3.2, $[\mathbf{u}] \subseteq \Sigma_l$. That is each irreducible element $\mathbf{u}$ of $\overline{[\mathbf{a}]}_{\sim_s}$ is contained in $\mathfrak{A}_{gl}$ the generator for $\mathfrak{A}_l$, hence $\mathfrak{A}_l$ contain $\overline{[\mathbf{a}]}_{\sim_s}$. Therefor every singular points of $\mathfrak{A}^{\succcurlyeq}$ is contained in $\mathfrak{A}_l^{\succcurlyeq}$. □

In $\mathfrak{A}| \sim_s$ the sum $[\mathbf{a}]_{\sim_s} + [\mathbf{b}]_{\sim_s} \subseteq [\mathbf{a}+\mathbf{b}]_{\sim_s}$ and this inclusion can be strict. If $\mathbf{a}+\mathbf{b}$ is singular element then the critical generators of $\overline{[\mathbf{a}+\mathbf{b}]}_{\sim_s}$ are contained in $\overline{[\mathbf{a}]}_{\sim_s}$ and $\overline{[\mathbf{b}]}_{\sim_s}$.

Example 3.4.2. Let G be the 2-reflection (fix $n-2$ dimensional subspace instead of $n-1$ of usual reflection) group in $GL(3, \mathbb{Z})$ given by,

$$\left\{ s_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, s_y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, s_z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I_3 \right\}$$

For the usual lex ordering

$$\mathfrak{A} = \{(x,y,z) \in \mathbb{Z}^3 : x \geq 0, y \geq 0\} \setminus F_1 \cup F_2.$$

where $F_1 = \{(x,0,z) \in \mathbb{Z}^3 : x \geq 0, z < 0\}$ and $F_2 = \{(0,y,z) \in \mathbb{Z}^3 : y > 0, z < 0\}$ are the two dimensional semigroups see fig.3 of example 4.2.1. The decomposition of $\mathfrak{A}$ via isotropy subgroups,

$$\mathfrak{A}| \sim_s = \{[e_1]_{\sim_s}, [e_2]_{\sim_s}, [e_3]_{\sim_s}, \{\mathbf{0}\} = [\mathbf{0}]_{\sim_s}, \mathfrak{A}_\circ\}$$

One can easily verify for $1 \leq i \neq j \leq 3, \quad e_i + e_j$ is regular element and hence $[e_i]_{\sim_s} + [e_j]_{\sim_s} = \langle e_i, e_j \rangle_{mon} \subsetneq [e_i + e_j]_{\sim_s} = \mathfrak{A}_\circ$. and

$$\mathfrak{A}_l = \langle e_1, e_2, e_3 \rangle_{mon}$$

Further more the semigroups $[e_i]_{\sim_s}$ is permuted by G to $[-e_i = g_j(e_i)]_{\sim_s}$ for $i \neq j$. For any $\succcurlyeq \in \Omega$ the semigroup $\mathfrak{A}^{\succcurlyeq}$ contain exactly one of these two semigroup, and

$$\Sigma_l = \{\pm e_1, \pm e_2, \pm e_3\}.$$

3.4.1 Existence of The Linear Semigroup for $\mathfrak{A}^{\succcurlyeq}$

Definition 3.4.3. For $\mathcal{G}$ a finite subgroup of $GL(n,\mathbb{Z})$ the semigroup $\mathfrak{A}_l^{\succcurlyeq}$ defined in 3.3.1 is called the linear semigroup associated with $\mathfrak{A}^{\succcurlyeq}$.

Note from the definition 3.3.1 if the generators of $\mathfrak{A}_l^{\succcurlyeq}$ for some $\succcurlyeq \in \Omega$ is known then for any other order the generators are uniquely identified via ϕ. Before we proceed to associate linear semigroup to the semigroup of the invariant ring, we provide for which finite subgroup $\mathcal{G}$ in $GL(n,\mathbb{Z})$ the linear semigroup exist. In the previous two sections we have shown that semigroup $\mathfrak{A}_l^{\succcurlyeq}$ defined in 3.3.1 has the following property,

- It has dimension n (Proposition 3.3.1).
- Contains all the singular points of $\mathfrak{A}^{\succcurlyeq}$ (Corollary 3.4.4).

From section 3.4 we have that each class determined by singular elements are boundary (face) semigroup for the n dimensional class of regular elements $(\mathfrak{A}^{\succcurlyeq})_\circ$. Hence the irreducible singular element (if exist) contribute often the critical generators for the linear semigroup.
Clearly the existence of linear semigroup defined in 3.3.1 for arbitrary finite group is non trivial matter. specially when there is no definitive common characteristics among these groups. Hence for our work we employ a certain additional condition on the linear semigroup and this give a common characteristics for those group admitting the extra condition.
Let $\mathfrak{A}_l$ satisfy definition 3.3.1 and further for each $\succcurlyeq \in \Omega$ the associated map ϕ preserves 'angle' (with respect $\mathcal{G}$ invariant inner product) between each critical generators of $\mathfrak{A}_l^{\succcurlyeq}$, i.e.

$$\langle \mathbf{a}_i, \mathbf{a}_j \rangle = \langle \phi(\mathbf{a}_i), \phi(\mathbf{a}_j) \rangle \quad 1 \le i, j \le n. \tag{3.4.1}$$

where $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$ are the critical generators of $\mathfrak{A}_l$.

Definition 3.4.4. Let $\mathcal{G}$ be a finite group in $GL(n,\mathbb{Z})$ we say $\mathcal{G}$ *admits linearization*, if there exist a linear semigroup $\mathfrak{A}_l$ (as in definition 3.4.3) and the critical generators of $\mathfrak{A}_l$ satisfies equation:3.4.1 for any $\succcurlyeq \in \Omega$.

If $\mathcal{G}$ admits linearization, then we say $\mathfrak{A}_l$ is the lex associated linearization of $\mathcal{G}$. We denote all finite groups in $GL(n,\mathbb{Z})$ which admits linearization by $GL^\star(n,\mathbb{Z})$. Note from proposition 3.3.1 for each $\succcurlyeq \in \Omega$ we have isomorphisms $\phi_l(\succcurlyeq) : \mathfrak{A}_l \to \mathfrak{A}_l^{\succcurlyeq}$ hence for a given finite group admit linearization then $\mathfrak{A}_l^{\succcurlyeq}$ an isomorphic copy of $\mathfrak{A}_l$ and its generators are uniquely identify by the generators of $\mathfrak{A}_l$ (i.e. $\mathfrak{A}_{gl}$).
From this section onward unless otherwise expressed all the finite group $\mathcal{G}$ considered admit linearization. Note that case of finite groups in $GL(2,\mathbb{Z})$ all 13 of them (up to conjugacy) admit linearization (see annex-1). And we have a characterization of those finite groups of $GL(n,\mathbb{Z})$ which admits linearization,

Proposition 3.4.5. *Let $\mathcal{G}$ be a group in $GL(n,\mathbb{Z})$ then the following are equivalent,*

1. *$\mathcal{G}$ admits linearization, ($\mathcal{G} \in GL^{\star}(n,\mathbb{Z})$)*
2. *There exist a reflection group $\mathcal{G}^{\star}$ in $GL(n,\mathbb{Z})$ such that $\mathcal{G}$ is a subgroup.*

Proof. ($1 \implies 2$) Assume $\mathcal{G}$ admits linearization, consider the generators $\mathfrak{A}_{gl} = \{\mathbf{a}_i : 1 \leq i \leq t\}$ and let assume the critical generators to be $\{\mathbf{a}_i : 1 \leq i \leq n\} = \mathfrak{A}^{\star}_{gl}$. Now for each i there exist a semigroup $T_i \in \ell_{\Sigma}$ such that,

$$\mathfrak{A}^{\star}_{gl} \cap T_i \quad contains \quad \{\mathbf{a}_j : j \neq i\}$$

i.e. the semigroups $\mathfrak{A}_l$ and T_i share $(n-1)$ dimensional semigroup. One should note such T_i is unique as both are isomorphic and simplical. There exist $\mathbf{a}^{\star}_i \in T_i$, ($\mathbf{a}^{\star}_i = \phi(\mathbf{a}_i)$ for some $\succcurlyeq \in \Omega$). Then by 3.4.1,

$$\langle \mathbf{a}_i, \mathbf{a}_j \rangle = \langle \mathbf{a}^{\star}_i, \mathbf{a}_j \rangle \quad \forall j \neq i$$

from which,

$$\mathbf{a}_i - \mathbf{a}^{\star}_i \in \{\mathbf{a}_j : j \neq i\}^{\perp}$$

the orthogonal complement is one dimensional and setting $\mathbf{a}_i - \mathbf{a}^{\star}_i = \lambda_i \mathbf{w}_i$ where $\mathbf{w}_i \in \mathbb{Z}^n$ is atomic element and hence λ_i is minimal positive integer. Let $u_i \in \mathbb{Q}^n$ such that $\lambda_i u_i = \mathbf{a}_i$ since i is arbitrary we can construct the set $\{u_1, u_2, \cdots, u_n\}$ and we have,

$$\mathfrak{A}_l \subseteq < u_1, u_2, \cdots, u_n >_{mon}$$

define a reflection,

$$g_i(u_j) = \begin{cases} u_j, & \text{if } j \neq i \\ u_i - \mathbf{w}_i, & \text{if } j = i \end{cases}$$

Observe that $g_i(\mathbf{a}_i) = g_i(\lambda_i u_i) = \lambda_i(u_i - \mathbf{w}_i) = \mathbf{a}^{\star}_i$ and for $j \neq i \quad g_i(\mathbf{a}_j) = \mathbf{a}_j$, further $g_i(\mathbf{w}_i) = -\mathbf{w}_i$. Consider the reflection group generated by $\{g_i : i = 1, 2, \cdots, n\}$ denote it by $\mathcal{G}^{\star}$. Both $\mathcal{G}$ and $\mathcal{G}^{\star}$ permutes transitively the fundamental weyl chamber (for $\mathcal{G}^{\star}$) and $\mathfrak{A}_l$ for the lex associated linearization of $\mathcal{G}$ each element of $\mathcal{G}$ is identified as an elements of $\mathcal{G}^{\star}$ as two automorphism which agree on n dimensional simplical semigroup are identical.

($2 \implies 1$) Let $\mathcal{G}^{\star}$ be a reflection group containing $\mathcal{G}$ take the minimal order $\mathcal{G}^{\star}$ which contain all Σ_l of $\mathcal{G}$ as singular members and act effectively on $\mathbb{Z}^n$ which is possible knowing the fact there are finitely many reflection group in $GL(n,\mathbb{Z})$ by assumption $\mathcal{G}$ is contained in reflection group. Then for $\succcurlyeq_{lex} \in \Omega$ we have that,

$$\mathfrak{A}_l = \cap_{g \in \mathcal{G}^{\star}} A_g \quad \subseteq \quad \cap_{g \in \mathcal{G}} A_g = \mathfrak{A}$$

Since $\mathfrak{A}_l$ is finitely generated and simplical further more $\langle \mathbf{a}_i, \mathbf{a}_j \rangle \geq 0$, these is due to the fact the critical generators of $\mathfrak{A}_l$ is contained in the dual (weight lattice) of simple root associated with $\succcurlyeq_{lex}$ (i.e. $\triangle_{\succcurlyeq_{lex}}$) of $\mathcal{G}^\star$. The simple root satisfy $\langle \mathbf{w}_i, \mathbf{w}_j \rangle \leq 0$ [see section:§ 2.6]. Clearly $\mathfrak{A}_l$ satisfies the definition 3.3.1 where the minimality is imposed by the way $\mathcal{G}^\star$ is selected (minimal order). Further more since $\mathcal{G}^\star$ permute transitively $\mathfrak{A}_l$ and each order $\succcurlyeq \in \Omega$ gives $\mathfrak{A}_l^{\succcurlyeq}$ which is uniquely identified by $\mathfrak{A}_l$. This further implies there exist some $g \in \mathcal{G}^\star$ such that $g(\mathfrak{A}_l) = \mathfrak{A}_l^{\succcurlyeq}$ hence the angle condition equation:3.4.1 holds as each $g \in \mathcal{G}^\star$ preserves the invariant inner product, therefore $\mathcal{G}$ admits linearization.

□

Example 3.4.5. Let $\mathcal{G} = \{\pm I\} \subseteq GL(n, \mathbb{Z})$ then $\mathcal{G} \in GL^\star(n, \mathbb{Z})$
Here for the usual lex ordering ($e_1 \succ e_2 \succ \cdots \succ e_n$) we have that

$$\mathfrak{A} = \{\mathbf{a} \in \mathbb{Z}^n : \mathbf{a} \succcurlyeq (-I(\mathbf{a}))\} = \{\mathbf{a} \in \mathbb{Z}^n : \mathbf{a} \succcurlyeq \mathbf{0}\}$$

which is the positive set of the lex ordering $\mathbf{P}$. The usual inner product is $\mathcal{G}$ invariant and one can easily find,

$$\mathfrak{A}_l =< e_1, e_2, \cdots, e_n >_{mon}$$

Note for any $\mathbf{a} \in \mathfrak{A}$ given by $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{Z}^n$ let,

$$r_i = \begin{cases} 1, & \text{if } a_i \geq 0 \\ -1, & \text{if } a_i < 0 \end{cases}$$

then the lex ordering with $r_1e_1 \succ r_2e_2 \succ \cdots \succ r_ne_n$ gives,

$$\mathbf{a} \in \mathfrak{A}_l^{\succcurlyeq} =< r_1e_1, r_2e_2, \cdots, r_ne_n >_{mon} .$$

Both semigroups are isomorphic and $\mathfrak{A}_l$ satisfies all the conditions of definition 3.3.1 and 3.4.1, there for $\mathfrak{A}_l$ is the lex associated linearization of $\mathcal{G}$.

3.5 Branching and Linearization of R

In this section we give a relation between the subsemigroup of $\mathbb{Z}^n$ and invariant ring $\mathbf{R}$ for a finite group in $GL^\star(n, \mathbb{Z})$, which we developed in the previous sections of this chapter. We employ two important semigroup isomorphism in the profs of theorems 3.2.3 and proposition 3.3.1. First let us define linear (unit degree) orbit sums,

Definition 3.5.1. Let $\mathcal{G}$ be a group in $GL^\star(n, \mathbb{Z})$ and $\mathfrak{A}_l =< \mathbf{a} : \mathfrak{A}_{gl} >_{mon}$ be the lex associated linearization of $\mathcal{G}$. Define $\mathfrak{o}(\mathbf{a})$ for each $\mathbf{a} \in \mathfrak{A}_{gl}$ to be a *unit degree (linear)* orbit sum in $\mathbf{R}$.

The linear orbit sum definition though given via the lex linearization does not depend on the orders rather each order give different head term of the linear orbits (specifically $\mathfrak{A}_{gl}^{\succcurlyeq}$). The isomorphism given in proposition 3.3.1 i.e. $\phi_l(\succcurlyeq) : \mathfrak{A}_l \to \mathfrak{A}_l^{\succcurlyeq}$ which relates the *head term* of the *lead orbit sum* of *products of linear orbits* is called for. Note that,

$$\prod_{i=1}^{t} \mathfrak{o}(\mathbf{a}_i)^{n_i}$$

can have different lead orbit sum depending on the order so ϕ_l plays a key role in recording these properties.
For $\succcurlyeq \in \Omega$ the factor semigroup $\mathcal{M}^{\succcurlyeq}$ of $\mathbf{R}_m$ restricted to $\mathfrak{A}_l^{\succcurlyeq}$ (see theorem 3.2.3) gives a subsemigroup of $\mathcal{M}^{\succcurlyeq}$ which is isomorphic to $\mathfrak{A}_l^{\succcurlyeq}$ and generated by the equivalent class of $\{\mathfrak{o}(\mathbf{a}), \mathbf{a} \in \mathfrak{A}_{gl}^{\succcurlyeq}\}$ i.e. the linear orbit sums.

$$< \mathbf{a} : \mathfrak{A}_{gl}^{\succcurlyeq} >_{mon} \rightarrow < [\mathfrak{o}(\mathbf{a})]_{\precsim} : \mathbf{a} \in \mathfrak{A}_{gl}^{\succcurlyeq} >_{mon} := \mathcal{M}_l^{\succcurlyeq} \subseteq \mathcal{M}^{\succcurlyeq} \tag{3.5.1}$$

$\mathcal{M}_l^{\succcurlyeq}$ is a finitely generated semigroup. The lead orbit sum of product of linear orbit sums in relation to singular generators has the following properties.

Proposition 3.5.1. *Let $S \subset \mathfrak{A}_{gl}$ such that there exist $I \neq g \in \mathcal{G}$ and $g(\mathbf{a}) = \mathbf{a} \,\forall \mathbf{a} \in S$ and $\mathbf{c}$ be any singular element in $\mathfrak{A}_{gl}$, then the product of linear orbit sums $\mathfrak{o}(\mathbf{c})^{n_c} \prod_{\mathbf{a} \in S} \mathfrak{o}(\mathbf{a})^{n_a}$ for n_a, n_c non negative integer has a unique lead orbit sum for any $\succcurlyeq \in \Omega$.*

Proof. This follows from the fact that ϕ is semigroup isomorphism on $[\mathbf{a}]_{\sim_s}$ for all $\mathbf{a} \in S$ and $[\mathbf{c}]_{\sim_s}$ to any $\mathfrak{A}^{\succcurlyeq}$ ($\succcurlyeq \in \Omega$) by corollary 3.4.4. Note that $\mathbf{b} = \sum_{\mathbf{a} \in S} \mathbf{a}$ is singular element and for any $\succcurlyeq \in \Omega$ we have,

$$\phi : [\mathbf{b}]_{\sim_s} + [\mathbf{c}]_{\sim_s} \rightarrow [\phi(\mathbf{b})]_{\sim_s} + [\phi(\mathbf{c})]_{\sim_s} \subseteq \mathfrak{A}_l^{\succcurlyeq} \subseteq \mathfrak{A}^{\succcurlyeq}$$

is a semigroup homomorphism preserving orbits. i.e.

$$[\sum_{\mathbf{a} \in S} n_a \mathbf{a} + n_c \mathbf{c}] = [\phi(\sum_{\mathbf{a} \in S} n_a \mathbf{a} + n_c \mathbf{c})] = [n_a \sum_{\mathbf{a} \in S} \phi(\mathbf{a}) + n_c \phi(\mathbf{c})].$$

There for the lead orbit sum of the product $\mathfrak{o}(\mathbf{c})^{n_c} \prod_{\mathbf{a} \in S} \mathfrak{o}(\mathbf{a})^{n_a}$ is unique. ☐

In the above proof one should know that the lead orbit sum is unique but the head term of the lead orbit for a given order varies. In view of the above proposition collecting $\succcurlyeq \in \Omega$ which gives the same lead orbits for the product of linear orbit sum in one class seems viable, as our work is to use the ordering to study the semigroup structure of $\mathbf{R}$. Hence we define,

Definition 3.5.2. Let $\succcurlyeq_1$ and $\succcurlyeq_2$ be in Ω, define $\succcurlyeq_1$ and $\succcurlyeq_2$ give $\mathcal{G}$-equivalent semigroup structure on the linear orbit sums if each product $\prod_{\mathbf{a} \in S} \mathfrak{o}(\mathbf{a})^{n_a}$, where $S \subseteq \mathfrak{A}_{gl}$ and n_a non negative integers has the same lead orbit sum for both orders.

The following theorem gives a criteria when the order associated function ϕ restricted to $\mathfrak{A}_l$ become semigroup homomorphism and its relation to $\phi_l(\succcurlyeq)$. Note ϕ_l as in theorem 3.3.1, need not preserve orbits i.e $[\mathbf{u}]$ and $[\phi_l(\succcurlyeq)(\mathbf{u})]$ need not be the same orbit but when it does one has,

Theorem 3.5.2. *Let $\succcurlyeq \in \Omega$ then the following are equivalent,*

1. $\phi_l(\succcurlyeq) : \mathfrak{A}_l \to \mathfrak{A}_l^{\succcurlyeq}$, *preserves orbit. i.e.* $[\mathbf{u}] = [\phi_l(\succcurlyeq)(\mathbf{u})]$, $\forall \mathbf{u} \in \mathfrak{A}_l$.
2. $\phi_l(\succcurlyeq) = \phi$ *(on* $\mathfrak{A}_l$*).*
3. $\exists g \in \mathcal{G}$ *such that* $g = \phi_l(\succcurlyeq)$ *on* $\mathfrak{A}_l$.

Proof.

1. ($1 \implies 2$) Let denote $\phi_l(\succcurlyeq)$ by ϕ_l since $\mathfrak{A}^{\succcurlyeq}$ is a fundamental set and for any $\mathbf{u} \in \mathfrak{A}_l$ $[\mathbf{u}] = [\phi(\mathbf{u})] = [\phi_l(\mathbf{u})]$ and the fact $\phi(\mathbf{u}), \phi_l(\mathbf{u})$ both represent same orbit in $\mathfrak{A}^{\succcurlyeq}$ implies $\phi_l(\mathbf{u}) = \phi(\mathbf{u}) \, \forall \mathbf{u} \in \mathfrak{A}_l$. Hence ϕ is a semigroup homomorphism (on $\mathfrak{A}_l$).
2. ($2 \implies 3$) Let first show $\phi_l(\mathfrak{A}_l) \cap g(\mathfrak{A}) = \phi_l(\mathfrak{A}_l) \cap g(\mathfrak{A}_l)$. Clearly $\phi_l(\mathfrak{A}_l) \cap g(\mathfrak{A}) \supseteq \phi_l(\mathfrak{A}_l) \cap g(\mathfrak{A}_l)$ and the reverse inclusion follows from the fact if $\phi_l(t) = g(l)$ and $\mathfrak{A}$ is a fundamental set force $t = l \in \mathfrak{A}_l$.
   Furthermore since $\mathbb{Z}^n = \cup_{g \in \mathcal{G}} g(\mathfrak{A})$,

   $$\begin{aligned} \phi_l(\mathfrak{A}_l) &= \phi_l(\mathfrak{A}_l) \cap \mathbb{Z}^n \\ &= \phi_l(\mathfrak{A}_l) \cap (\cup_{g \in \mathcal{G}} g(\mathfrak{A})) \\ &= \cup_{g \in \mathcal{G}} (\phi_l(\mathfrak{A}_l) \cap g(\mathfrak{A})) \\ &= \cup_{g \in \mathcal{G}} (\phi_l(\mathfrak{A}_l) \cap g(\mathfrak{A}_l)) \end{aligned}$$

   $\phi_l(\mathfrak{A}_l) \cong g(\mathfrak{A}_l)$ for each $g \in \mathcal{G}$, and for $g \neq h \in \mathcal{G} \quad g(\mathfrak{A}_l) \cap h(\mathfrak{A}_l) \subseteq g(\mathfrak{A}) \cap h(\mathfrak{A})$, has dimension at most $n - 1$. Since the dimension of $\phi_l(\mathfrak{A}_l)$ is n, there exist g such that $\phi_l(\mathfrak{A}_l) \cap g(\mathfrak{A}_l)$ has dimension n. Since $\phi_l (= \phi|_{\mathfrak{A}_l})$ preserve orbits and two isomorphisms agreeing on n dimension are the same, hence for any $\mathbf{a} \in \mathfrak{A}_l$ one have $\phi_l(\mathbf{a}) = g(\mathbf{a})$. There fore $g = \phi_l$ on $\mathfrak{A}_l$.
3. ($3 \implies 1$) Clear since each g in $\mathcal{G}$ preserves orbit.

□

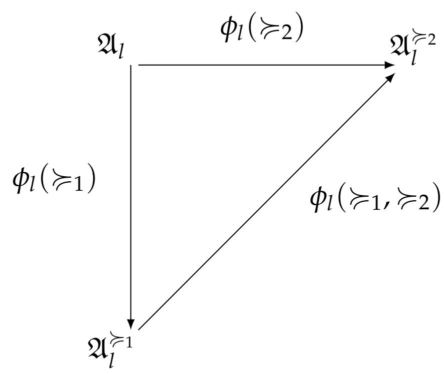
Observe that for two orders $\succcurlyeq_1, \succcurlyeq_2 \in \Omega$ if $\phi_l(\succcurlyeq_1, \succcurlyeq_2) : \mathfrak{A}_l^{\succcurlyeq_1} \to \mathfrak{A}_l^{\succcurlyeq_2}$ given by

$$\phi_1(\mathbf{a}) \quad \xrightarrow{\phi_l(\succcurlyeq_1, \succcurlyeq_2)} \quad \phi_2(\mathbf{a}) \quad (\forall) \, \mathbf{a} \in \mathfrak{A}_{gl}$$

which is linearly extended i.e.

$$\sum_{\mathbf{a} \in S} n_a \phi_1(\mathbf{a}) \quad \xrightarrow{\phi_l(\succcurlyeq_1, \succcurlyeq_2)} \quad \sum_{\mathbf{a} \in S} n_a \phi_2(\mathbf{a}) \quad S \subseteq \mathfrak{A}_{gl}$$

then we have isomorphisms of semigroup homomorphism,



If any two preserve orbits then so is the other. The following corollary of theorem 3.5.2 gives a relation between two $\mathcal{G}$-equivalent orders,

Corollary 3.5.3. $\succcurlyeq_1, \succcurlyeq_2 \in \Omega$, *give* $\mathcal{G}$*-equivalent semigroup structure on the linear orbit sums if and only if there exists* g *in* $\mathcal{G}$ *such that* $g(\mathfrak{A}_l^{\succcurlyeq_1}) = \mathfrak{A}_l^{\succcurlyeq_2}$.

Proof. Follows from theorem 3.5.2, note $\prod_{\mathbf{a}\in S} \mathfrak{o}(\mathbf{a})^{n_a} \quad S \subseteq \mathfrak{A}_{gl}$ has the same lead orbit sum for both order implies

$$\phi_l(\succcurlyeq_1, \succcurlyeq_2) : \mathfrak{A}_l^{\succcurlyeq_1} \to \mathfrak{A}_l^{\succcurlyeq_2} \quad \phi_1(\mathbf{a}) \mapsto \phi_2(\mathbf{a}) \quad \mathbf{a} \in \mathfrak{A}_{gl}.$$

preserves orbit and conversely. □

The fact ℓ_Σ has finitely many semigroups of the form $\mathfrak{A}_l^{\succcurlyeq}$ further imply we have finitely many $\mathcal{G}$-equivalent ordering on the linear orbit sums. Let define equivalence relation on Ω by,

$$\succcurlyeq_1, \succcurlyeq_2 \in \Omega \quad \succcurlyeq_1 \sim_\phi \succcurlyeq_2 \quad \textit{iff} \quad \exists g \in \mathcal{G} \quad g(\mathfrak{A}_l^{\succcurlyeq_1}) = \mathfrak{A}_l^{\succcurlyeq_2}$$

two ordering are equivalent if and only if they form the same $\mathcal{G}$-equivalent semigroup structure on the linear orbit sums. Let Ω_ϕ denote the collection equivalent classes of $\sim_\phi$.

Definition 3.5.3. The natural number $|\Omega_\phi|$ (the number of equivalent classes) gives the number of branching of $\mathbf{R}_m$ in to distinct isomorphic factor semigroup generated by the class of linear orbit sums.

In view of these facts

$$\mathbb{Z}^n \subseteq \cup_{\succcurlyeq \in \Omega} \mathfrak{A}_l^{\succcurlyeq}$$

we have

$$\mathbb{Z}^n \subseteq \cup_{g\in\mathcal{G}} \cup_{i=1}^{|\Omega_\phi|} g(\mathfrak{A}_l^{[\succcurlyeq_i]}) \tag{3.5.2}$$

where $\mathfrak{A}_l^{[\succcurlyeq_i]} \quad i = 1, \cdots, |\Omega_\phi|$ is a semigroup for some order $\succcurlyeq \in [\succcurlyeq_i]_{\sim_\phi}$ where

$$\Omega_\phi = \{[\succcurlyeq_1]_{\sim_\phi}, [\succcurlyeq_2]_{\sim_\phi}, \cdots, [\succcurlyeq_{|\Omega_\phi|}]_{\sim_\phi}\}$$

Let $\succcurlyeq_1, \succcurlyeq_2 \in [\succcurlyeq_i]_{\sim_\phi}$ then we know these two order form $\mathcal{G}$-equivalent semigroup structure on the linear orbit sum, but the relative ordering of the linear orbit sum can be

different in $\mathcal{M}_l^{\succcurlyeq}$ but the product of the linear orbits (much of our concern) always gives the same lead orbit sum for both. Hence same semigroup structure, i.e. $\succcurlyeq_1, \succcurlyeq_2$ are two distinct ordering over the same semigroup (determined by linear orbit sums).
Further more following proposition 3.4.5 gives the number of branching of $\mathbf{R}_m$.

Corollary 3.5.4. *Let $\mathcal{G}$ be a finite group in $GL^{\star}(n,\mathbb{Z})$ and $\mathcal{G}^{\star}$ be the minimal reflection group containing $\mathcal{G}$ then,*

$$|\Omega_{\phi}| = [\mathcal{G}^{\star} : \mathcal{G}]$$

.

Proof. Let $\mathfrak{A}_l$ be lex associated linearization of $\mathcal{G}$ and $\mathcal{G}^{\star}$ be the reflection group constructed in the proof of [Proposition 3.4.5]. Since each chambers are associated to a unique element of $g \in \mathcal{G}^{\star}$ where $\mathfrak{A}_l$ is contained the fundamental chamber associated to I. Let $\{g(\mathfrak{A}_l) : g \in \mathcal{G}\}$ is given collection of chambers permuted (transitively) by $\mathcal{G}$ and if there are no more chambers then $\mathcal{G} = \mathcal{G}^{\star}$. Hence we have $1 = |\Omega_{\phi}| = [\mathcal{G}\star : \mathcal{G}]$, else if there exist any T a chamber such that $T \notin \{g(\mathfrak{A}_l) : g \in \mathcal{G}\}$ then for any $g \in \mathcal{G} \quad g(T) \notin \{h(\mathfrak{A}_l) : h \in \mathcal{G}\}$. Observe that, if for some $h \in \mathcal{G} \quad h(T) = g(\mathfrak{A}_l)$ then there exist a unique $h' \in \mathcal{G}^{\star}$ such that $h'(\mathfrak{A}_l) = T$ and hence $hh' = g$, there for $h' = h^{-1}g \in \mathcal{G}$, which forces $T \in \{g(\mathfrak{A}_l) : g \in \mathcal{G}\}$ a contradiction. Therefore there are $[\mathcal{G}^{\star} : \mathcal{G}]$ distinct semigroup in which $\mathcal{G}$ permutes each transitively hence the corollary. □

Since we employ strict convexity on $\mathfrak{A}_l^{\succcurlyeq}$, (i.e. $\langle \mathbf{a}, \mathbf{b} \rangle \geq 0$ for $\mathbf{a}, \mathbf{b} \in \mathfrak{A}_l^{\succcurlyeq}$) for the effective action of $\mathcal{G}$, $|\mathcal{G}||\Omega_{\phi}| \geq 2^n$.

Example 3.5.4. Let $\mathcal{G} \subseteq GL(2,\mathbb{Z})$ be the group of order 3 generated by

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

For the usual *lex* with $e_1 \succcurlyeq e_2$,

$$\mathfrak{A} = \{(x,y) \in \mathbb{Z}^2 | \quad 2x \geq y \wedge x > -y\}$$

The irreducible elements of $\mathfrak{A}$ are $\{(1,2),(1,1),(1,0)\} \cup \{(a+1,-a) | a \in \mathbb{N}\}$. Let consider order $\succcurlyeq$ to be *lex* with $e_2 \succcurlyeq e_1$,

$$\mathfrak{A}^{\succcurlyeq} = \{(x,y) \in \mathbb{Z}^2 | \quad 2y \geq x \wedge y > -x\}$$

with irreducible elements of $\mathfrak{A}^{\succcurlyeq}$ are $\{(2,1),(1,1),(0,1)\} \cup \{(-a,a+1) | a \in \mathbb{N}\}$. If ϕ is the associated map for $\succcurlyeq$, $\phi((1,2)) = (1,2) = (1,1) + (0,1)$ $\phi((a+1,-a)) = (a, 2a+1) = a(1,1) + (a+1)(0,1)$, showing the possible linear orbit sum to be $\mathfrak{o}(e_1+e_2) = x^{e_1+e_2} + x^{-e_1} + x^{-e_2}$ and $\mathfrak{o}(e_1) = x^{e_1} + x^{e_2} + x^{-e_1-e_2}$, indeed for any order $\phi((1,1))$ and

$\phi((1,0))$ are irreducible in $\mathfrak{A}^{\succcurlyeq}$, hence

$$\Sigma_l = \{\pm e_1, \pm e_2, \pm(e_1+e_2)\}$$

this gives six possible semigroups for $\mathfrak{A}_l^{\succcurlyeq}$ for any $\succcurlyeq\in\Omega$ of which $\mathcal{G}$ permutes each three giving two branching.

Example 3.5.5. Continued from 3.4.5 for the usual lex ordering

$$\mathfrak{A}_l = < e_1, e_2, \cdots, e_n >_{mon}$$

The semigroups associated to non $\mathcal{G}$-equivalent ordering are

$$< e_1, \pm e_2 \cdots, \pm e_n >_{mon}$$

up to $\mathcal{G}$ permutation hence $\mathbf{R}_m$ has 2^{n-1} branching.and hence,

$$\Sigma_l = \{\pm e_i | i = 1, 2, \cdots, n\}.$$

The linear orbits sums are $\mathfrak{o}(e_i) = \mathbf{x}^{e_i} + \mathbf{x}^{-e_i}$ for $1 \leq i \leq n$. These are algebraically independent and the semigroup with the ring multiplication given by

$$< \mathfrak{o}(e_1), \mathfrak{o}(e_2), \cdots, \mathfrak{o}(e_n) >_{mon} = M_{-I}$$

is isomorphic to any one of the above sub semigroups of $\mathbb{Z}^n$. One can easily deduce that each of $\phi_l(\succcurlyeq)$ are a reflection (up to $\mathcal{G}$ translation). If we define g_i to be a reflection fixing the hyperplane generated by $\{e_j : j \neq i\}$ for any $\succcurlyeq\in\Omega$ then $\phi_l(\succcurlyeq) : \mathfrak{A}_l \to \mathfrak{A}_l^{\succcurlyeq}$ equals $g \in \mathcal{G}^\star$ where $\mathcal{G}^\star$ is the reflection group generated by $\{g_i : i \in \{1, 2, \cdots, n\}\}$. The invariant algebra over some field $\mathbb{K}$ properly contain $\mathbf{R} \supsetneq \mathbb{K}[M_{-I}]$ for $n \geq 2$.
For $n = 3$, the semigroup associated to non $\mathcal{G}$-equivalent order are $|\Omega_\phi| = 4$ and $gg_1(\mathfrak{A}_l), g_2(\mathfrak{A}_l), g_3(\mathfrak{A}_l), \mathfrak{A}_l$ more over the fundamental domain (for lex) can be given by

$$\mathfrak{A} = \mathfrak{A}_l \sqcup ((e_1 - e_3) + gg_1(\mathfrak{A}_l)) \sqcup ((e_1 - e_2) + g_2(\mathfrak{A}_l)) \sqcup ((e_2 - e_3) + g_3(\mathfrak{A}_l))$$

As a $\mathbb{K}$ algebra the invariant ring is generated by the linear orbit sums and any one of from each pair $\{\mathfrak{o}(e_i + e_j), \mathfrak{o}(e_i - e_j) : 1 \leq i < j \leq 3\}$. Observe that these are the two possible lead orbit sums of $\mathfrak{o}(e_i)\mathfrak{o}(e_j)$. The same lattice points are used for the decomposition of $\mathfrak{A}$, Further $\mathbf{R}$ has a direct sum module representation with $T = \mathbb{K}[M_I]$, Then,

$$\mathbf{R} = T \oplus (\oplus_{1 \leq i < i \leq 3} \mathfrak{o}(e_i - e_j)T)$$

each module in the above has trivial intersection.

Let $\{a_1, a_2, \cdots, a_n\}$ be the critical generators then the linear orbit sum associated are algebraically independent, observe that for any $(l_1, l_2, \cdots, l_n) \in \mathbb{N}^n_{\geq 0}$ the linear product associate it self to a unique (up to the branch) lead orbit sum and any finitely many of these are $\mathbb{K}$ independent.

Chapter 4

The Main Results

4.1 The Semigroup Problem

The semigroup problem stated in the conjuncture (1.3.2) was first raised and partially done by M. Lorenz in his article [Lorenz (2001)]. He used the concept of Lie algebra to show, and later on his book [Lorenz (2005),§: 10.2] gave a detailed layout of this problem including many suggestions and possible alternative approach. Further M. Tesemma, (a Phd student of Lorenz) in his PhD thesis, [Tesemma (2004)], work on the same problem dropping the technical lie algebra which is more or less associated to the reflection groups and is not that much of a tool for the converse. Hence he used monomial ordering and SAGBI basis which was studied by Z. Reichstein [Reichstein (2003)] a bit earlier. He later compiled with some additional condition in article [Tesemma (2007)], in which theorem 1.3.4 is proved. In the first section of this chapter we give a proof of semigroup problem for multiplicative action with some restriction on $\mathcal{G}$ i.e.($\mathcal{G}$ in $GL^{\star}(n,\mathbb{Z})$.

4.1.1 Linearization Associated with Reflection Group

In chapter three Theorem 3.2.3, we have constructed a monic semigroup $\mathbf{R}_m$ and showed that for any given order $\succcurlyeq \in \Omega$,

$$\mathbf{R}_m| \precsim := \mathcal{M}^{\succcurlyeq} \cong \mathfrak{A}^{\succcurlyeq},$$

where the map;

$$\varphi : \mathbf{R}_m \to \mathfrak{A}^{\succcurlyeq}$$

given for $0 \neq f \in \mathbf{R}_m$ by,

$$\varphi(f) = max_{\succcurlyeq}\{\phi(\mathbf{a}) : \mathbf{a} \in supp(f)\}$$

Note that the map φ is similar to the initial map of Tessema's work in [Tesemma (2004) and Tesemma (2007)], except we have restricted the source (domain) to the monic semigroup $\mathbf{R}_m$. The congruence ($\precsim$) collect all monic polynomials of $\mathbf{R}_m$ which share the same lead orbit sum for $\succcurlyeq$. In section [§2.6 theorem 2.6.2] we have associated a given order to the positive roots of the root system of a given reflection groups. Further we have,

Theorem 4.1.1. *Let $\mathcal{G}$ in $GL^{\star}(n,\mathbb{Z})$ be a finite group, then the following are equivalent,*

1. *$\mathcal{G}$ is reflection group.*

2. *For each $\succcurlyeq \in \Omega$, $\phi(\succcurlyeq)$ is semigroup homomorphism.*
3. $|\Omega_\phi| = 1$.
4. $\mathfrak{A} = \mathfrak{A}_l$ *(Hence $\mathfrak{A}$ is finitely generated).*

Proof. $(1 \Rightarrow 2)$Let $\mathcal{G}$ be a reflection group, then for $\succcurlyeq_{lex} \in \Omega$, by theorem 2.6.2

$$\mathfrak{A} = \mathfrak{A}^{\succcurlyeq_{lex}} = \{\mathbf{a} \in \mathbb{Z}^n : \mathbf{a} \succcurlyeq_{lex} g_i(\mathbf{a}) \quad 1 \le i \le n\}$$

where $\{g_i : i = 1, 2, \cdots, n\} = \triangle_{\mathcal{G}}^{\succcurlyeq_{lex}}$, are the simple reflections associated with the simple roots (base) $\triangle^{\succcurlyeq_{lex}}$ of the positive roots associated with $\succcurlyeq_{lex}$ i.e.

$$\Phi_{\succcurlyeq_{lex}} = \{\alpha \in \Phi : \alpha \succcurlyeq_{lex} \mathbf{0}\}.$$

Let $\{u_1, u_2, \cdots, u_n\}$, be the fundamental weight associated with base $\triangle^{\succcurlyeq_{lex}}$. Further,

$$\mathfrak{A} =< \mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n, \cdots, \mathbf{a}_t >_{mon} \subseteq < u_1, u_2, \cdots, u_n >_{mon}$$

where $\mathbf{a}_i = \lambda_i u_i$ for $1 \le i \le n$ are the critical generators of $\mathfrak{A}$. Now consider the decomposition $\mathfrak{A}/ \sim_s$ of $\mathfrak{A}$, the critical generators are $\{\mathbf{a}_i : i = 1, 2, \cdots n\}$ and for any i the simple reflection g_i fixes $\mathbf{a}_j$ for all $j \neq i$. Since for any $\succcurlyeq \in \Omega$ the associated function ϕ is semigroup homomorphism on the equivalent class (generated by singular elements) (semigroups). Let

$$h_i = \sum_{j \neq i}^{n} \mathbf{a}_j$$

Following corollary 3.4.4 then ϕ is a semigroup homomorphism on the n dimensional semigroup, $(h_i) + (\mathbf{a}_i)$ for any $i = 1, 2, \cdots, n$. Further more since $\mathcal{G}_{h_i} = \{g_i, I\} \cap \mathcal{G}_{\mathbf{a}_i} = \{I\}$ hence there exist a unique $h \in \mathcal{G}$ such that $h = \phi$. Therefor ϕ is semigroup isomorphism for any $\succcurlyeq \in \Omega$.
$(2 \Rightarrow 3)$ Let for each $\succcurlyeq \in \Omega$ the map $\phi : \mathfrak{A} \to \mathfrak{A}^{\succcurlyeq}$ is a semigroup homomorphism, hence the restriction $\phi : \mathfrak{A}_l \to \mathfrak{A}_l^{\succcurlyeq}$ is a semigroup homomorphism and by theorem 3.5.2 there exist $g \in \mathcal{G}$ such that $g = \phi$ on $\mathfrak{A}_l$. Since $\succcurlyeq$ is arbitrary and $lex \sim_\phi \succcurlyeq$ imply $|\Omega_\phi| = 1$.
$(3 \Rightarrow 4)$ By definition $\mathfrak{A}_l \subseteq \mathfrak{A}$ and $|\Omega_\phi| = 1$, decomposition of $\mathbb{Z}^n$, eq:3.5.2 i.e.

$$\mathbb{Z}^n \subseteq \cup_{g \in \mathcal{G}} \cup_1^{|\Omega_\phi|} g(\mathfrak{A}_l^{[\succcurlyeq_i]}).$$

We have $\mathbb{Z}^n \subseteq \cup_{g \in \mathcal{G}} g(\mathfrak{A}_l)$ hence $\mathfrak{A}_l$ is a fundamental set for the $\mathcal{G}$ action on $\mathbb{Z}^n$. Since there is no proper inclusion between two fundamental sets we must have $\mathfrak{A}_l = \mathfrak{A}$. Hence $\mathfrak{A}$, finitely generated.
$(4 \Rightarrow 1)$ This once follows from forward proof of proposition 3.4.5 i.e. constructing the reflection group from $\mathfrak{A}_l$.

□

4.1.2 Canonical Semigroup Representation

Fix order class $[\succcurlyeq_i] \in \Omega_\phi$ for any two $\succcurlyeq_1, \succcurlyeq_2 \in [\succcurlyeq_i]$, then $\mathfrak{A}_l^{\succcurlyeq_1} = g(\mathfrak{A}_l^{\succcurlyeq_2})$ for some $g \in \mathcal{G}$. Furthermore $\mathcal{M}_l^{\succcurlyeq_1}$ and $\mathcal{M}_l^{\succcurlyeq_2}$ have the same lead orbit sum for each product of the linear orbit sums. The set $\{\mathfrak{o}(\mathbf{a}) : \mathbf{a} \in \mathfrak{A}\}$ contain identical class representative for both the factor semigroup. Let define,

$$\mathcal{M}^i = \{[\mathfrak{o}(\mathbf{a})]_i : \quad [\mathfrak{o}(\mathbf{a})]_i = \cap_{\succcurlyeq \in [\succcurlyeq_i]}[\mathfrak{o}(\mathbf{a})]_{\precsim}, \quad \phi(\mathbf{a}) \in \mathfrak{A}_l^{\succcurlyeq} \; \exists \succcurlyeq \in [\succcurlyeq_i]\}$$

to be the factor semigroup associated with the class order $[\succcurlyeq_i] \in \Omega_\phi$. Here the filtration removes f in $\mathbf{R}_m$ with dominant lead orbits $A_f(d)$ more than one. In particular if $\succcurlyeq \in [\succcurlyeq_i]$ and $\{\phi(\mathbf{a}_j) : 1 \leq j \leq n\}$ are the critical generators of $\mathfrak{A}_l^{\succcurlyeq}$. If $f \in [\mathfrak{o}(\mathbf{a}_j)]_i$, then f does not contain any of the other orbit sum associated with the critical generators, $\mathbf{a}_i \neq \mathbf{a}_j$ for $1 \leq i \leq n$. Further more f have a form

$$f = \mathfrak{o}(\mathbf{a}_j) + \sum_{k=n+1}^{t} n_k \mathfrak{o}(\mathbf{a}_k)$$

where $\mathbf{a}_k$ for $n+1 \leq k \leq t$ are the non critical generators of $\mathfrak{A}_l^{\succcurlyeq}$ dominated by $\mathfrak{o}(\mathbf{a}_j)$. One can construct multiplicative order (of type n) given by $\{\langle\,, \mathbf{a}_k\rangle = \omega_k : 1 \leq k \leq n\}$ in any order give the same linearization semigroup with different ordering on the critical generators. Any class representation of $\mathcal{M}^i$ is a representation for $\mathcal{M}_l^{\succcurlyeq}$ for $\succcurlyeq \in [\succcurlyeq_i]$.

Example 4.1.1. In example 3.5.4 we have $|\Omega_\phi| = 2$. Fig 2. illustrates the two semigroups $\mathcal{G}$ permutes s and s'. Hence $\Omega_\phi = \{[\succcurlyeq_s], [\succcurlyeq_{s'}]\}$ for $\succcurlyeq \in [\succcurlyeq_s]$ (taking usual lex), The product of linear orbits,

$$\mathfrak{o}(e_1 + e_2)\mathfrak{o}(e_1) \sim_s \mathfrak{o}(2e_1 + e_2)$$

while for any $\succcurlyeq \in [\succcurlyeq_{s'}]$

$$\mathfrak{o}(e_1 + e_2)\mathfrak{o}(e_1) \sim_{s'} \mathfrak{o}(e_1 + 2e_2)$$

note that

$$\mathfrak{o}(e_1 + e_2)\mathfrak{o}(e_1) = \mathfrak{o}(2e_1 + e_2) + \mathfrak{o}(e_1 + 2e_2) + 3$$

$\mathbf{R}$ is not generated by the linear orbits as a $\mathbb{K}$ algebra.

Corollary 4.1.2. *If $\mathcal{G}$ acts as a reflection group the lead orbit sum of the product of linear orbits is unique, and conversely for $\mathcal{G}$ in $GL^\star(n, \mathbb{Z})$ the lead orbit sum of the product of linear orbit sums is unique then $\mathcal{G}$ reflection group.*

Proof. Since $\mathcal{G}$ is reflection, it has a single branching, and the semigroup isomorphism $\phi_l(\succcurlyeq) : \mathfrak{A}_l \to \mathfrak{A}_l^{\succcurlyeq}$, between the linearization of $\mathcal{G}$ preserves orbits for any $\succcurlyeq \in \Omega$. which implies the lead orbit sum of products of linear orbits is unique. Conversely if the product of the linear orbits give a unique lead orbit sum, then we have $\phi_l : \mathfrak{A}_l \to \mathfrak{A}_l^{\succcurlyeq}$, which relates the lead orbit of the product of linear orbit sums, preserve orbit for any $\succcurlyeq \in \Omega$. Hence $\succcurlyeq_{lex} \sim_\phi \succcurlyeq$, there for Ω_ϕ, contain a single class and hence $\mathcal{G}$ is reflection

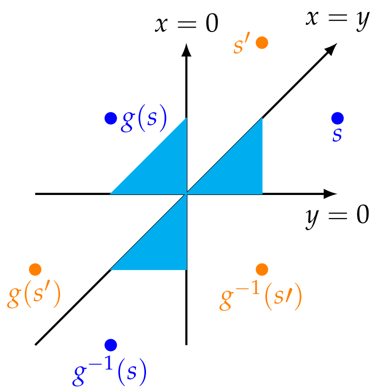


Fig.2

group. □

Definition 4.1.2. Let S be a semigroup and T a subsemigroup of the factor semigroup of S ($\bar{S} = S/\sim$) for some congruence $\sim$. a semigroup $S_T \subseteq S$ is a canonical semigroup representation of T, if $S_T \cong T$ and

$$T = \{[s]_\sim : s \in S_T\}$$

Lemma 4.1.3. *If $T \subseteq \mathbf{R}_m$ is a canonical semigroup representation of $\mathcal{M}^i$ (the factor semigroup associated with the order class $[\succcurlyeq_i]$), then T is also a canonical representation of $\mathcal{M}^j$ $\forall j$ where j runs over each order class in Ω_ϕ.*

Proof. Let T be the canonical semigroup representation of $\mathcal{M}^i$, Since $\mathcal{M}^i$ is finitely generated by the class of linear orbit sums, i.e.

$$\langle [\mathfrak{o}(\mathbf{a})]_i | \mathbf{a} \in \mathfrak{A}_{gl} \rangle_{mon} = \mathcal{M}^i$$

Then T is finitely generated and have the form,

$$T = \langle f_\mathbf{a} : \mathbf{a} \in \mathfrak{A}_{gl} \rangle_{mon} \subseteq \mathbf{R}_m$$

Here for each $\mathbf{a} \in \mathfrak{A}_{gl}$, we have $f_\mathbf{a} \in [\mathfrak{o}(\mathbf{a})]_i$, and has a unique lead orbit sum $\mathfrak{o}(\mathbf{a})$. Since each factor semigroup $\mathcal{M}^j$ is generated by the class of linear orbit sums, and are isomorphic to each other. Difference appear due to lead orbit sum for product of linear orbit sums i.e. (the branchings). Hence we have $f_\mathbf{a} \in [\mathfrak{o}(\mathbf{a})]_j \in \mathcal{M}^j$. Since all share the same linear orbit sum, and the fact each semigroup $\mathfrak{A}_l^{\succcurlyeq}$ are isomorphic, we have for any $S \subseteq \mathfrak{A}_{gl}$. The products

$$\prod_{\mathbf{a} \in S} f_\mathbf{a} \quad \& \prod_{\mathbf{a} \in S} \mathfrak{o}(\mathbf{a})$$

gives the same lead orbit sums for any given $\succcurlyeq \in \Omega$. i.e. under a given order $f_\mathbf{a}$ and $\mathfrak{o}(\mathbf{a})$ behaves identically in the factor semigroup associated to the order. There fore T is also a canonical representation of $\mathcal{M}^j$, for all $[\succcurlyeq_j]$ associated factor semigroup. □

Note in the above lemma linear orbit sums $\{\mathfrak{o}(\mathbf{a}) : \mathbf{a} \in \mathfrak{A}_{gl}\} = \{\mathfrak{o}(\phi(\mathbf{a})) : \mathbf{a} \in \mathfrak{A}_{gl}\}$ are the same also,

$$\prod_{i=1}^{t} \mathfrak{o}(\mathbf{a}_i)^{n_i} = \prod_{i=1}^{t} \mathfrak{o}(\phi(\mathbf{a}_i))^{n_i}$$
$$\sum_{i=1}^{t} n_i \mathbf{a}_i \xrightarrow{\phi_l(\succcurlyeq)} \sum_{i=1}^{t} n_i \phi(\mathbf{a}_i).$$

But the orbits $[\sum_{i=1}^{t} n_i \mathbf{a}_i]$ and $[\sum_{i=1}^{t} n_i \phi(\mathbf{a}_i)]$ need not be same, unless ϕ_l preserve it. which further imply the orders are $\mathcal{G}$ equivalent on the linear orbit. Note the semigroup generated by the linear orbit sums, though may not be canonical still contain a transverse element for any factor semigroup $\mathcal{M}^i$.

Example 4.1.3. Continued, 3.5.4 Let $T =< \mathfrak{o}(e_1 + e_2), \mathfrak{o}(e_1) >_{mon}$ is a canonical semigroup representation for both $\mathcal{M}^s$ and $\mathcal{M}^{s'}$, note $\mathfrak{o}(e_1 + e_2), \mathfrak{o}(e_1)$ are algebraically independent and the semigroup $s\,, s'$ are freely generated by two elements, but $\mathbb{Z}[T] \subsetneq \mathbf{R}$.

Lemma 4.1.4. *Let $M =< \mathfrak{o}(\mathbf{a}) : \mathbf{a} \in \mathfrak{A}_{gl} >_{mon}$ and $\mathcal{G}$ a finite group in $GL^\star(n, \mathbb{Z})$ then the following statement are equivalent,*

1. $|\Omega_\phi| = 1$.
2. *M forms a $\mathbb{K}$-span of $\mathbf{R}$.*
3. $\mathbb{K}[M] = \mathbb{K}[\mathfrak{o}(\mathbf{a}) : \mathbf{a} \in \mathfrak{A}_{gl}]$.

Proof. ($1 \implies 2$) $|\Omega_\phi| = 1$, then for any $\succcurlyeq \in \Omega$, and theorem 3.2.3,

$$\mathbf{R}_m / \precsim \;= \mathcal{M}^{\succcurlyeq} \quad \xrightarrow{\Psi_{\succcurlyeq}} \quad \mathfrak{A}^{\succcurlyeq} \quad \xleftarrow{\phi_{\succcurlyeq}} \quad \mathfrak{A}$$

M is a subsemigroup of $\mathbf{R}_m$, generated by the linear orbit sums, now consider the isomorphism $\xi : \mathfrak{A} \to \mathcal{M}^{\succcurlyeq}$ and define the set

$$M^\star = \{\xi(\mathbf{a}) \cap M : \mathbf{a} \in \mathfrak{A}\}$$

It is enough to show, for each $\mathbf{a} \in \mathfrak{A}$, $\xi(\mathbf{a}) \cap M$, is non empty, the result follows from 3.2.5, (i.e. $M^\star$ form a transverse element for $\mathcal{M}^{\succcurlyeq}$). $\mathfrak{A}$ is generated by $\{\mathbf{a}_i : \mathbf{a}_i \in \mathfrak{A}_{gl}\} = \{\mathbf{a}_i : i = 1, 2, \cdots, t\}$, for any $\mathbf{b} \in \mathfrak{A}$ and its representation $\mathbf{b} = \sum_{1=1}^{t} n_i \mathbf{a}_i$, then,

$$\prod_{i=1}^{t} \mathfrak{o}(\mathbf{a}_i)^{n_i} \in \xi(\mathbf{b}) \cap M$$

Hence M forms $\mathbb{K}$ span for $\mathbf{R}$.
($2 \iff 3$) Clear, since $M \subseteq \mathbb{K}[\mathfrak{o}(\mathbf{a}) : \mathbf{a} \in \mathfrak{A}_{gl}]$ and $\mathbf{R} = \mathbb{K}[M] \subseteq \mathbb{K}[\mathfrak{o}(\mathbf{a}) : \mathbf{a} \in \mathfrak{A}_{gl}] \subset \mathbf{R}$.
($3 \implies 1$) $M \subseteq \mathbf{R}_m$ and for any $\succcurlyeq \in \Omega$ restricting φ in the proof of 3.2.3 to $M \subseteq \mathbf{R}_m$, we have the isomorphism,

$$M / \sim \;\; \cong \mathfrak{A}_l^{\succcurlyeq}$$

where $\sim$,

$$m_1 = \prod_{i=1}^{t} \mathfrak{o}(\mathbf{a}_i)^{n_i} \sim m_2 = \prod_{i=1}^{t} \mathfrak{o}(\mathbf{a}_i)^{k_i} \quad \textit{iff} \quad \sum_{i=1}^{t} n_i \phi(\mathbf{a}_i) = \sum_{i=1}^{t} k_i \phi(\mathbf{a}_i)$$

which is the kernel congruence of the presentation $\mathfrak{A}_l^{\succcurlyeq}$ (i.e. $\mathbb{N}^t \rightarrow \mathfrak{A}_l^{\succcurlyeq}$ map each $(n_1, n_2, \cdots, n_t) \rightarrow \sum_{i=1}^{t} n_i \phi(\mathbf{a}_i)$). Further that $M/\sim$, is finitely generated and well ordered (via $\mathfrak{A}_l^{\succcurlyeq}$) and the kernel congruence finitely generated. Let,

$$ker_{\sim} = \{((n_1, n_2, \cdots, n_t)_i, (k_1, k_2, \cdots, k_t)_i) : i = 1, 2, 3, \cdots, k\} \tag{4.1.1}$$

be the minimal generating set then these corresponds to the set

$$\{(m_i, m_i^{\star}) : i = 1, 2, \cdots, k.\}$$

where $m_i = \prod_{i=1}^{t} \mathfrak{o}(\mathbf{a}_i)^{n_i}$ and $m_i^{\star} = \prod_{i=1}^{t} \mathfrak{o}(\mathbf{a}_i)^{k_i}$ and $m_i - m_i^{\star} \in \mathbb{R} = \mathbb{K}[M]$ since both m_i and $m_i^{\star}$ are monic with the same lead orbit sum, $m_i - m_i^{\star}$ has a unique monomial expression as a product of $\mathfrak{o}(\mathbf{a}) : \mathbf{a} \in \mathfrak{A}_{gl}$, by the minimality of kernel generators 4.1.1. There for the set $\{\mathfrak{o}(\mathbf{a}) : \mathbf{a} \in \mathfrak{A}_{gl}\}$ forms a SAGBI bases for $\mathbf{R}$ by 1.3.3 $\mathcal{G}$ acts as a reflection and $|\Omega_{\phi}| = 1$. □

Note in here though we did not explicitly stated that on the semigroup M, φ acts as initial map ($in_{\succcurlyeq}$) and is a semigroup epimorphism, with finitely generated kernel congruence. The map is injective if $\mathfrak{A}_l^{\succcurlyeq}$ is free semigroup (generated by only critical generator) in general. Other wise not injective on M, as two distinct product of the linear orbit sums giving identical lead orbit sum implies two distinct way representation of the same element of $\mathfrak{A}_l^{\succcurlyeq}$, while the two product are often different on the next theorem these is ratified by special construction of canonical semigroup.

Theorem 4.1.5. *The following statements are equivalent for $\mathcal{G} \in GL^{\star}(n, \mathbb{Z})$ such that ($[\mathbb{Z}]^{\mathcal{G}} = \{0\}$)*

1. *$\mathcal{G}$ acts as a reflection group.*
2. *$\mathbf{R} = \mathbb{Z}[T]$ where $T = < f_a : a \in \mathfrak{A}_{gl} >_{mon}$ a canonical semigroup representation for some $\mathcal{M}^i$ (hence for all).*

Proof. (1 $\implies$ 2) Suppose $\mathcal{G}$ acts as a reflection group for a given order $\succcurlyeq \in \Omega$, we have that,

$$\mathfrak{A}^{\succcurlyeq} = < m_1, m_2, \cdots, m_n, \cdots, m_t >_{mon},$$

each $i = 1, 2, 3, \cdots$ $m_i = \lambda_i u_i$, where λ_i are the smallest positive integer so that $\lambda_i u_i \in \mathbb{Z}^n$, and the set $\{u_1, u_2, \cdots, u_n\}$ is the fundamental weight associated with the simple roots $\{w_1, w_2, \cdots, w_n\} = \triangle^{\succcurlyeq}$ (simple root associated with $\succcurlyeq$) and,

$$\mathfrak{A}^{\succcurlyeq} = < u_1, u_2, \cdots, u_n >_{mon} \cap \mathbb{Z}^n,$$

(see for the construction of $\mathfrak{A}^{\succcurlyeq}$ in [§2.6, Theorem 2.6.2]). Setting $m_j = \sum_{i=1}^{n} \lambda_{ji} u_i$ where λ_{ji} are non negative integers for $j = n+1, n+2, \cdots, t$ non critical generators of $\mathfrak{A}^{\succcurlyeq}$, then the semigroup generators of $\mathbf{R}$ are given by

$$f_{m_j} = \prod_{i=1}^{n} \mathfrak{o}(u_i)^{\lambda_{ji}} \precsim \mathfrak{o}(\sum_{i=1}^{n} \lambda_{ji} u_i) = \mathfrak{o}(m_j)$$

Here the fact that f_{m_j} is indeed in $\mathbf{R}_m$ is not a trivial matter, $\mathcal{G}$ acts trivially on $P/\mathbb{Z}^n$ where P is the weight lattice takes a crucial fact, see reduction lemma [Lorenz (2001),§2.3] or [Tesemma (2007), Corollary 3.6]. Note $\{\mathfrak{o}(m_j) : j = 1, 2, \cdots, t\}$ are the linear orbit sum. For the non critical generators (if exist) m_j, for $n+1 \le i \le t$, there exist a positive integer n_j such that,

$$n_j m_j = \sum_{i=1}^{n} n_j \lambda_{ji} u_i = \sum_{i=1}^{n} t_i m_i$$

where each t_i is a non negative integers this gives the algebraic relation of the generators,

$$f_{m_j}^{n_j} = \prod_{i=1}^{n} f_{m_i}^{t_i}$$

Hence,

$$T = < f_{m_1}, f_{m_2}, \cdots, f_{m_t} >_{mon} \simeq \mathfrak{A}^{\succcurlyeq} = < m_1, m_2, \cdots, m_t >_{mon}$$

Therefore T is canonical semigroup representation of $\mathbf{R}_m / \precsim$ for any $\succcurlyeq \in \Omega$.

($2 \implies 1$) By lemma 4.1.3 T is a canonical semigroup representation for $\mathcal{M}^i$ implies that T is a canonical semigroup representation for all factor semigroup $\mathcal{M}^j$ associated with the order class $[\succcurlyeq_j] \in \Omega_\phi$. Further $\mathbf{R} = \mathbb{Z}[T]$, it is enough to show $|\Omega_\phi| = 1$. Assume $|\Omega_\phi| > 1$, there exist factor semigroup $\mathcal{M}^i$ and $\mathcal{M}^j$ isomorphic but distinct. Let $\succcurlyeq_{lex}$ be in the order class determining $\mathcal{M}^i$. Hence there exist $\mathfrak{o}(\mathbf{c}) \in \{\mathfrak{o}(\mathbf{a}) | \, \mathbf{a} \in \mathfrak{A}\}$ such that $\mathfrak{o}(\mathbf{c})$ is not in any class of $\mathcal{M}^i$. Take the branching orbit sum (the minimal product of linear orbit sums, which doesn't appear as a lead orbit in $\mathcal{M}^i \cong \mathfrak{A}_l \quad \mathbf{c} \in \mathfrak{A} \setminus \mathfrak{A}_l$), then for any finite set $\{f_i\} \cup \{\mathfrak{o}(\mathbf{c})\}$ where f_i's are from distinct classes of $\mathcal{M}^i$ are $\mathbb{K}$ independent. Hence,

$$\mathfrak{o}(\mathbf{c}) \notin \mathbb{K}\, span \, \{f_i\}$$

For any representation (not necessarily semigroup) S of $\mathcal{M}^i$ by elements of $\mathbf{R}_m$. we have then,

$$\mathfrak{o}(\mathbf{c}) \notin \mathbb{K}\, span \, S$$

But since T is a canonical semigroup representation of $\mathcal{M}^i$, and $\mathfrak{o}(\mathbf{c}) \notin \mathbb{K}[T]$, is a contradiction, to fact $\mathbb{Z}[T] = \mathbf{R}$. Therefor $|\Omega_\phi| = 1$. □

From the above lemma one can conclude that $\mathbf{R}$ is as a $\mathbb{K}$ algebra generated by the linear orbit if and only if $\mathcal{G}$ acts as a reflection group (equivalently $|\Omega_\phi| = 1$), Further in the construction of semigroup,

$$< f_{m_1}, f_{m_2}, \cdots, f_{m_t} >_{mon}$$

The fact that lead orbit sum for each f_{m_i} is the linear orbit sum $\mathfrak{o}(m_i)$, there for $supp(f_{m_i}) \subseteq \{m_j : j = 1, 2, \cdots, t\}$. Hence each semigroup generator are $\mathbb{K}$ linear combination of the linear orbit sums. There exist an affine transformation,

$$\begin{pmatrix} f_{m_1} \\ f_{m_2} \\ \vdots \\ f_{m_t} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1t} \\ u_{21} & u_{22} & \cdots & u_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ u_{t1} & u_{t2} & \cdots & u_{tt} \end{pmatrix} \begin{pmatrix} \mathfrak{o}(m_1) \\ \mathfrak{o}(m_2) \\ \vdots \\ \mathfrak{o}(m_t) \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_t \end{pmatrix},$$

sending linear orbit sum to the semigroup generators and vise versa.

Example 4.1.4. Let $\mathcal{G}$ be a reflection group generated by $\{s = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, t = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\}$, Here the reflections are $\{t, ts, st\}$ with the root system

$$\Phi = \{\pm e_1, \pm e_2, \pm(e_1 + e_2)\}$$

Usual lex ordering give the positive roots, $\Phi^+ = \{e_1, e_2, e_1 + e_2\}$, giving a base, $\{e_1, e_2\}$, the $\mathcal{G}$ invariant inner product (averaged the usual dot product over $\mathcal{G}$) is associated with the matrix $M = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ i.e. $< a, b >= a^t M b$ for $a, b \in \mathbb{Z}^2$ since the norm of each root is 2, we have the dual (weight) lattice generated by $\{u_1 = (\frac{2}{3}, \frac{1}{3}), u_2 = (\frac{1}{3}, \frac{2}{3})\}$ Hence

$$\mathfrak{A} =< u_1, u_2 >_{mon} \cap \mathbb{Z}^2 =< m_1 = (2,1), m_2 = (1,2), m_3 = (1,1) >_{mon},$$

here $m_1 = 3u_1, m_2 = 3u_2$ and $m_3 = u_1 + u_2$ and we have semigroup generators,

$$\begin{aligned} f_1 =& \mathfrak{o}(u_1)^3 = \mathfrak{o}(m_1) + 3\mathfrak{o}(m_3) + 6 \\ f_2 =& \mathfrak{o}(u_2)^3 = \mathfrak{o}(m_2) + 3\mathfrak{o}(m_3) + 6 \\ f_3 =& \mathfrak{o}(u_1)\mathfrak{o}(u_2) = \mathfrak{o}(m_3) + 3 \end{aligned}$$

$\mathbf{R} = \mathbb{Z}[T]$, where $T =< f_1, f_2, f_3 >_{mon} \simeq \mathfrak{A}$ and the linear orbits also generate $\mathbf{R}$ as a $\mathbb{Z}$ algebra.

4.2 Archimedean Ordering and Initial Algebra $\mathbf{R}_m$

The initial algebra of a given subalgebra depend on the monomial orders. There are monomial orders which demands less time and smaller spaces (data) compared to other orders. Nowadays the computation of Gróbener basis for Ideals, and SAGBI basis of an algebra is done via softwares (Algebra package) like SAGE, SINGULAR,$\cdots$. The efficiency of these package among the other technical parameters, is measured based on the time consumption and the required floating space (memory) for a given task.
While computing a SAGBI, or Gróbener basis one uses a monomial orders. In certain cases two different order may give the same basis but may take or consume different time and/or space to arrive the same output. Some algebra may allow a certain class of orders to represent initial algebra acquired by any monomial order. In the case of multiplicative action the invariant ring $\mathbf{R}$ for $\mathcal{G}$ reflection group is one such algebra allowing archimedean order representation for any $\succcurlyeq \in \Omega$.
One way to categorize an order in $\mathbb{Z}^n$ is using its type (i.e. the minimal number of vectors in $\mathbb{R}^n$ used to represent the order), or within fixed type, the partition can be used See [§2.3, remarks]. Here type of order especially type one, (archimedean) order is considered. One can easily see the computational efficiency to the other extreme type *typen*, for a given two integer points, while a single check is enough for the archimedean the latter (type n) may need up to n check ups in a loop.

4.2.1 Archemedian Order and Automorphisms of $\mathbb{Z}^n$

In the article [Tesemma and Wang (2011)] M. Tesemma and H. Wang, gave a detail characterization of archimedean orders, in addition here we use properties developed in [§2.2] on rational dimension of a given vector in $\mathbb{R}^n$. We continue the same notation within. Further more proposition 2.4.2, which relates a positive set associated with each $g \in \mathcal{G}$ i.e. (A_g) is used intensively.
Computing the initial algebra of a given invariant ring soon or latter end up with intersection A_g, of some automorphisms in $\mathcal{G}$. Hence we analyze the relationship of a given $g \in \mathcal{G}$ to a given order and vise versa. In the coming lemma the archimedean order for any $g \in \mathcal{G}$ and A_g of a reflection g to any $\succcurlyeq \in \Omega$, is characterized.

Lemma 4.2.1. *Let $I \neq g \in \mathcal{G}$, Then,*

1. *If $\succcurlyeq \in \Omega$ is type 1 (archimedean) then, $\mathbf{H}_{\alpha_g^1} \cap \mathbb{Q}^n = \mathcal{H}_{\alpha_g^1} = [\mathbb{Q}^n]^g$. Further $\mathfrak{rdim}(\alpha_g^1) = n - \dim([\mathbb{Q}^n]^g)$, where $\mathbf{H}_{\alpha_g^1} = \{v \in \mathbb{R}^n : \langle v, \alpha_g^1 \rangle = 0\}$.*
2. *If g acts as reflection in $\mathcal{G}$ with $g(\pm\mathbf{w}) = \mp\mathbf{w}$, where $\mathbf{w}$ is the normal vector, then $\alpha_g^1 = r\mathbf{w}$ for $r \in \mathbb{R}_+$ and $\mathbf{w} \succcurlyeq \mathbf{0}$. In general $\alpha_g^1 \in \mathbb{R}\mathbf{w}$, for any $\succcurlyeq \in \Omega$.*

Proof. 1. Let $\succcurlyeq$ be type 1, then there exist $\omega \in \mathbb{R}^n$ of rational dimension n and $\succcurlyeq =: \succcurlyeq_\omega$, where,

$$\mathbf{u} \succcurlyeq \mathbf{v} \quad \textit{iff} \quad \langle \mathbf{u}, \omega \rangle \geq \langle \mathbf{v}, \omega \rangle.$$

For any $\mathbf{u} \in \mathbb{Q}^n \quad \langle \mathbf{u}, \omega \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$. Hence $\alpha_g^1 =: \alpha_g^\omega = \omega - g^{-1}(\omega)$, where α_g^1 is as in proposition 2.4.2, then

$$A_g(\mathbb{Q}) = \{\mathbf{u} \in \mathbb{Q}^n : \langle \mathbf{u}, \alpha_g^\omega \rangle \geq 0\}.$$

For $\mathbf{u} \in \mathbb{Q}^n$,

$$\langle \mathbf{u} - g(\mathbf{u}), \omega \rangle = \langle \mathbf{u}, \alpha_g^\omega \rangle = 0 \iff \mathbf{u} - g(\mathbf{u}) = 0 \iff \mathbf{u} \in [\mathbb{Q}^n]^g.$$

Further if $\mathbf{v} \in \mathcal{H}_{\alpha_g^\omega}$, then $\langle \mathbf{v}, \alpha_g^\omega \rangle = 0$, imply $\mathbf{v} \in [\mathbb{Q}^n]^g$. There for by proposition 2.2.1,2, $\mathfrak{rdim}(\alpha_g^\omega) = n - \dim([\mathbb{Q}^n]^g)$.

2. Let g acts as a reflection then for any $\mathbf{u} \in \mathbb{Q}^n$

$$g(\mathbf{u}) = \mathbf{u} - 2\frac{\langle \mathbf{u}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle}\mathbf{w}$$

hence $\mathbf{u} - g(\mathbf{u}) = 2\frac{\langle \mathbf{u}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle}\mathbf{w}$ and for any $\succcurlyeq \in \Omega$, one of the two normal vectors is in the positive set $\mathbf{P}$ of $\succcurlyeq$. Let $\mathbf{w} \succ \mathbf{0}$ then,

$$\mathbf{u} \in A_g(\mathbb{Q}) \iff \mathbf{u} - g(\mathbf{u}) \succcurlyeq \mathbf{0} \iff 2\frac{\langle \mathbf{u}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle}\mathbf{w} \succcurlyeq \mathbf{0} \iff \langle \mathbf{u}, \mathbf{w} \rangle \geq 0$$

Further if $\{\omega_1, \cdots, \omega_s\}$ is the lex associated representation of $\succcurlyeq$ then $\alpha_g^1 = \omega_k - g^{-1}(\omega_k) = \omega_k - g(\omega_k) = 2\frac{\langle \omega_k, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle}\mathbf{w}$. Hence setting $r = 2\frac{\langle \omega_k, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle} > 0$, and $\alpha_g^1 = r\mathbf{w}$ for any $\succcurlyeq$ (provided $\mathbf{w} \succ \mathbf{0}$). Therefor, for g reflection α_g^1 admits only two vector i.e. $\alpha_g^1 \in \mathbb{R}\mathbf{w}$.

□

Let $\succcurlyeq \in \Omega$, and $\mathcal{G}$ be a finite subgroup of $GL(n, \mathbb{Z})$, and let $\{\omega_1, \omega_2, \cdots, \omega_s\}$ be the lex associated vectors to $\succcurlyeq$ such that,

$$\mathbf{u} \in \mathbf{P}(\mathbb{Q}) \iff (\langle \mathbf{u}, \omega_1 \rangle, \langle \mathbf{u}, \omega_2 \rangle, \cdots, \langle \mathbf{u}, \omega_s \rangle) \succcurlyeq_{lex} \mathbf{0} \tag{4.2.1}$$

where $\langle\, , \rangle$ is $\mathcal{G}$ invariant inner product as in 2.4.3 and $\mathbf{P}(\mathbb{Q}) = \{\mathbf{u} \in \mathbb{Q}^n : \mathbf{u} \succcurlyeq \mathbf{0}\}$ is positive set of $\succcurlyeq$ with respect to $\mathbb{Q}^n$. For each $g \in \mathcal{G}$,

$$\begin{aligned} A_g(\mathbb{Q}) =& \{\mathbf{u} \in \mathbb{Q}^n : \mathbf{u} \succcurlyeq g(\mathbf{u})\} \\ =& \{\mathbf{u} \in \mathbb{Q}^n : \mathbf{u} - g(\mathbf{u}) \succcurlyeq \mathbf{0}\} = \{\mathbf{u} \in \mathbb{Q}^n : \mathbf{u} - g(\mathbf{u}) \in \mathbf{P}(\mathbb{Q})\} \\ =& \{\mathbf{u} \in \mathbb{Q}^n : (\langle \mathbf{u} - g(\mathbf{u}), \omega_1 \rangle, \cdots, \langle \mathbf{u} - g(\mathbf{u}), \omega_s \rangle) \succcurlyeq_{lex} \mathbf{0}\} \\ =& \{\mathbf{u} \in \mathbb{Q}^n : (\langle \mathbf{u}, \omega_1 - g^{-1}(\omega_1) \rangle, \cdots, \langle \mathbf{u}, \omega_s - g^{-1}(\omega_s) \rangle) \succcurlyeq_{lex} \mathbf{0}\} \end{aligned}$$

where the last equality is by lemma 2.4.1,2. Further more there exist α_g^1 as in proposition 2.4.2,4 such that, $\mathbb{R}_+(A_g) \subseteq \mathbf{H}^+_{\alpha_g^1}$. For each $g \in \mathcal{G}$ we have that,

$$\{\mathbf{u} \in \mathbb{Q}^n : \langle \mathbf{u}, \alpha_g^1 \rangle > 0\} \subseteq A_g(\mathbb{Q}) \subseteq \{\mathbf{u} \in \mathbb{Q}^n : \langle \mathbf{u}, \alpha_g^1 \rangle \geq 0\}$$

Now from the definition of $\mathfrak{A}^{\succcurlyeq}(\mathbb{Q})$ and above inclusion,

$$\bigcap_{g\in\mathcal{G}}\{\mathbf{u}\in\mathbb{Q}^n : \langle \mathbf{u},\alpha_g^1\rangle > 0\} \subseteq \bigcap_{g\in\mathcal{G}} A_g(\mathbb{Q})$$
$$= \mathfrak{A}^{\succcurlyeq}(\mathbb{Q}) \subseteq \bigcap_{g\in\mathcal{G}}\{\mathbf{u}\in\mathbb{Q}^n : \langle \mathbf{u},\alpha_g^1\rangle \geq 0\}.$$

From Proposition 2.4.2 4,b for each $g\in\mathcal{G}$ $\alpha_g^1\in\mathbb{R}_+(\mathbf{P})$, hence

$$\mathbb{R}_+\langle \alpha_g^1 : g\in\mathcal{G}\rangle_{mon} \subseteq \mathbf{H}_{\omega_1}$$

Further let $\triangle_{\succcurlyeq} \subseteq \{\alpha_g^1 : g\in\mathcal{G}\} =: \Phi_{\succcurlyeq}^+$, be the minimal set such that

$$\bigcap_{\alpha\in\Phi_{\succcurlyeq}^+}\{\mathbf{u}\in\mathbb{Q}^n : \langle \mathbf{u},\alpha\rangle \geq 0\} = \bigcap_{\alpha\in\triangle_{\succcurlyeq}}\{\mathbf{u}\in\mathbb{Q}^n : \langle \mathbf{u},\alpha\rangle \geq 0\}$$

Note that in general $\triangle_{\succcurlyeq} \subseteq \mathbb{R}^n$, and is the minimal set needed to generate the polyhedral cone $\mathbb{R}_+\langle \alpha_g^1 : g\in\mathcal{G}\rangle_{mon}$. Further we have the relations,

$$\bigcap_{\alpha\in\triangle_{\succcurlyeq}}\{\mathbf{u}\in\mathbb{Q}^n : \langle \mathbf{u},\alpha\rangle > 0\} \subseteq \mathfrak{A}^{\succcurlyeq}(Q) \subseteq \bigcap_{\alpha\in\triangle_{\succcurlyeq}}\{\mathbf{u}\in\mathbb{Q}^n : \langle \mathbf{u},\alpha\rangle \geq 0\} \quad (4.2.2)$$

Since $\mathfrak{A}^{\succcurlyeq}(Q)$ is n dimensional each of the two bound are also n dimensional.

4.2.2 Representation of Initial Algebra by Archemedean Order

For two distinct orders $\succcurlyeq, \succcurlyeq'\in\Omega$, we say both give the same initial algebra on $\mathbf{R}$ if and only if

$$in_{\succcurlyeq}(\mathbf{R}) = in_{\succcurlyeq'}(\mathbf{R})$$

which is equivalently express by the image of initial algebra map which gives the equality of the fundamental domain associated with these orders. i.e. $\mathfrak{A}^{\succcurlyeq} = \mathfrak{A}^{\succcurlyeq'}$ [Tesemma and Wang (2011), Lemma 3.2(ii)]. In such case both will have same SAGBI bases, the following theorem give the converse of [Tesemma and Wang (2011), Theorem 4.2].

Theorem 4.2.2. *Let $\mathcal{G}$ be a finite subgroup in $GL(n,\mathbb{Z})$ then the following are equivalent,*

1. *For each $\succcurlyeq\in\Omega$ there exist an archimedean order $\succcurlyeq_\omega$ such that*

$$\mathfrak{A}^{\succcurlyeq} = \mathfrak{A}^{\succcurlyeq_\omega} \quad \textit{equivalentlly} \quad in_{\succcurlyeq}(\mathbf{R}) = in_{\succcurlyeq_\omega}(\mathbf{R})$$

2. *$\mathcal{G}$ is reflection group.*

Proof. ($1 \implies 2$) Let consider the lemma,

Lemma 4.2.3. *Let $\mathcal{G}$ be a finite subgroup of $GL(n,\mathbb{Z})$ and for $\succcurlyeq\in\Omega$ there exists an archemedian order $\succcurlyeq_\omega$ such that $\mathfrak{A}^{\succcurlyeq} = \mathfrak{A}^{\succcurlyeq_\omega}$ then for each $\alpha_h^1\in\triangle_{\succcurlyeq}$ there exist $\alpha_g^\omega\in\triangle_{\succcurlyeq_\omega}$ and $r\in\mathbb{R}_+$ such that $\alpha_h^1 = r\alpha_g^\omega$ (hence have same rational dimension).*

Proof. By [proposition 2.4.2, 3]

$$\mathfrak{A}^{\succcurlyeq}(\mathbb{Q}) = \mathbb{Q}_{+}\mathfrak{A}^{\succcurlyeq} = \mathbb{Q}_{+}\mathfrak{A}^{\succcurlyeq_{\omega}} = \mathfrak{A}^{\succcurlyeq_{\omega}}(\mathbb{Q})$$

Let $\triangle_{\succcurlyeq}$ and $\triangle_{\succcurlyeq_\omega}$ be the minimal sets for $\mathcal{G}$ associated to the orders $\succcurlyeq$ and $\succcurlyeq_\omega$. Let consider the cones (in$\mathbb{Q}^n$),

$$\begin{aligned}
\mathbf{T} =& \{\mathbf{u} \in \mathbb{Q}^n : \langle \mathbf{u}, \alpha_g^1 \rangle > 0, \forall \alpha_g^1 \in \triangle_{\succcurlyeq}\}. \\
\mathbf{N} =& \{\mathbf{u} \in \mathbb{Q}^n : \langle \mathbf{u}, \alpha_g^\omega \rangle \geq 0, \alpha_g^\omega \in \triangle_{\succcurlyeq_\omega}\}. \\
\mathbf{L} =& \{\mathbf{u} \in \mathbb{Q}^n : \langle \mathbf{u}, \alpha_g^1 \rangle \geq 0, \alpha_g^1 \in \triangle_{\succcurlyeq}\}.
\end{aligned}$$

Here we have the inclusion and equality of cones (from eq:4.2.2 follows that)

$$\mathbf{T} \subseteq \mathfrak{A}^{\succcurlyeq}(\mathbb{Q}) = \mathfrak{A}^{\succcurlyeq_\omega}(\mathbb{Q}) = \mathbf{N} \subseteq \mathbf{L}$$

where the second equality is due to the fact each g in $\mathcal{G}$ have only one inequality to check, i.e. $A_g(\mathbb{Q}) = \mathbb{R}_+(A_g(\mathbb{Q})) \cap \mathbb{Q}^n = \{\mathbf{u} \in \mathbb{Q}^n : \langle \mathbf{u}, \alpha_g^\omega \rangle \geq 0\}$ and lemma,4.2.1 1. Further taking the positive hull of each over the reals, same inclusion as above persists and the cone $\mathbb{R}_+(\mathbf{N})$ and $\mathbb{R}_+(\mathbf{L})$, need not be closed subset of $\mathbb{R}^n$. Taking the closure of these cones with the euclidean topology,

$$\mathbb{R}_+(\mathbf{N}) \subseteq \{v \in \mathbb{R}^n : \langle v, \alpha_g^\omega \rangle \geq 0, \alpha_g^\omega \in \triangle_{\succcurlyeq_\omega}\} = \overline{\mathbb{R}_+(\mathbf{N})} = \overline{\mathbf{N}}$$

Similarly

$$\mathbb{R}_+(\mathbf{L}) \subseteq \{v \in \mathbb{R}^n : \langle v, \alpha_g^1 \rangle \geq 0, \alpha_g^1 \in \triangle_{\succcurlyeq}\} = \overline{\mathbb{R}_+(\mathbf{L})} = \overline{\mathbf{L}}$$

Since $\mathbf{T} \subseteq \mathbf{N} \subseteq \mathbf{L}$ and $\overline{\mathbf{T}} = \overline{\mathbf{L}}$ we have $\overline{\mathbf{N}} = \overline{\mathbf{L}}$ further more $\overline{\mathbf{L}} \cap \mathbb{Q}^n = \mathbf{L}$ and $\overline{\mathbf{N}} \cap \mathbb{Q}^n = \mathbf{N}$ implies $\mathbf{L} = \mathbf{N}$. Since the rationals, $\mathbb{Q}^n$ is dense over $\mathbb{R}^n$, the finite set $\triangle_{\succcurlyeq_\omega}$ and $\triangle_{\succcurlyeq}$ are minimal, i.e. there is no $\triangle \subsetneq \triangle_{\succcurlyeq}$ such that $\{v \in \mathbb{R}^n : \langle v, \alpha_g^1 \rangle \geq 0, \alpha_g^1 \in \triangle\} = \overline{\mathbf{L}}$, this will contradict the minimality over $\mathbb{Q}^n$ of $\triangle_{\succcurlyeq}$. Since the cones $\overline{\mathbf{N}} = \overline{\mathbf{L}}$. i.e. both polyhedral cones are finite intersection of half spaces, implies for each $\alpha_h^1 \in \triangle_{\succcurlyeq}$ there exist $\alpha_g^\omega \in \triangle_{\succcurlyeq_\omega}$ and $r \in \mathbb{R}_+$ such that $\alpha_h^1 = r\alpha_g^\omega$. Further both sets $\triangle_{\succcurlyeq_\omega}$ and $\triangle_{\succcurlyeq}$ have equal number elements. □

(proof of theorem) Let $\succcurlyeq \in \Omega$ be an order of type n, each ω_i in the lex representation of $\succcurlyeq$ are all in $\mathbb{Q}^n$. Then,

$$\Phi_{\succcurlyeq}^{+} = \{\alpha_g^1 \in \mathbb{R}^n : g \in \mathcal{G}\} \subseteq \mathbb{Q}^n$$

By the lemma 4.2.3,

$$\mathfrak{A}^{\succcurlyeq}(\mathbb{Q}) = \mathfrak{A}^{\succcurlyeq_\omega}(\mathbb{Q}) = \mathbf{N} = \mathbf{L} = \{\mathbf{u} \in \mathbb{Q}^n : \langle \mathbf{u}, \alpha_g^1 \rangle \geq 0 \quad \alpha_g^1 \in \triangle_{\succcurlyeq}\}$$

In this case $\mathbb{R}_+(\mathbf{N}) = \overline{\mathbf{N}}$ and hence $\mathbb{R}_+(\mathfrak{A}^{\succcurlyeq}) = \overline{\mathbf{N}}$ is finite intersection of half spaces of rational hyperplanes, by Gordans' lemma ([Günter (1996) 4, lemma 3.4] and or [Fulton

(1993),Proposition 1]) $\mathfrak{A}^{\succcurlyeq} = \mathbb{Z}^n \cap \overline{N}$ is finitely generated saturated subsemigroup of $\mathbb{Z}^n$. Hence Reichstein [Reichstein (2003), theorem 1.6], $\mathcal{G}$ is reflection group.
($2 \implies 1$) Let $\mathcal{G}$ be a reflection group and $\succcurlyeq \in \Omega$, be any order and let Φ be the root system of $\mathcal{G}$, by lemma 4.2.1,2. For each $g \in R_{\mathcal{G}}, \alpha_g^1 = r_g \mathbf{w}_g$ where $\mathbf{w}_g \succ \mathbf{0}$ and $r > 0$. where $R_{\mathcal{G}}$ contains all reflection in $\mathcal{G}$. Hence $\{\alpha_g^1 : g \in R_{\mathcal{G}}\} = *\Phi^+_{\succcurlyeq}$, where $= *$ is to denote the equality is up to positive scalar multiple. Here $\Phi^+_{\succcurlyeq}$ is the positive root associated with $\succcurlyeq$. Now,

$$\mathfrak{A}^{\succcurlyeq}(\mathbb{Q}) = \{\mathbf{u} \in \mathbb{Q}^n : \langle \mathbf{u}, \mathbf{w}_g \rangle \geq 0 \quad \forall \mathbf{w}_g \in \Phi^+_{\succcurlyeq}\}$$

There exists $\triangle_{\succcurlyeq} = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n\}$, the associated base for $\Phi^+_{\succcurlyeq}$. Following theorem 2.6.2,

$$\mathfrak{A}^{\succcurlyeq}(Q) = \{u \in \mathbb{Q}^n : \langle \mathbf{u}, \mathbf{w}_i \rangle \geq 0 \quad \forall \mathbf{w}_i \in \triangle_{\succcurlyeq}\} = \mathbb{Q}_+ < u_1, \cdots, u_n >_{mon}$$

where $\{u_1, u_2, \cdots, u_n\}$ is the associated fundamental weight to $\triangle_{\succcurlyeq}$. Now consider $\omega = \sum_{i=1}^n \gamma_i u_i$ where for each $i \quad \gamma_i \in \mathbb{R}_{>0}$ and the set $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ is $\mathbb{Q}$ independent (hence $\mathfrak{rdim}(\omega) = n$), then we have

$$\Phi^+_{\succcurlyeq} = \Phi^+(\omega).$$

There for the archeamedian order associated with ω give the same chambers and,

$$\mathfrak{A}^{\succcurlyeq} = \mathfrak{A}^{\succcurlyeq_\omega}$$

hence the proof. □

Example 4.2.1. Let G be the 2-reflection (fix $n-2$ dimensional subspace instead of $n-1$) group in $GL(3, \mathbb{Z})$

$$\left\{ s_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \; s_y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \; s_z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad I_3 \right\}$$

Consider lexicography order $\{e_1, e_2, e_3\}$ then we have,

$$\begin{vmatrix} \alpha_g^1 : e_i & e_1 & e_2 & e_3 \\ e_i - s_x(e_i) & 0 & \underline{2e_2} & 2e_3 \\ e_i - s_y(e_i) & \underline{2e_1} & 0 & 2e_3 \\ e_i - s_z(e_i) & \underline{2e_1} & 2e_2 & 0 \end{vmatrix},$$

where α_g^1 See the plot fig 3, If there exist ω with rational dimension 3, then $\{\omega - s_i(\omega) i = x, y, z\}$ and the fact $\mathfrak{rdim}(\omega - s_i(\omega)) = 2$ and $\mathcal{H}_{\omega - s_i(\omega)}$ will have dimension 1 which shows that the two dimensional faces $F_1 = \{(x, 0, z) \in \mathbb{Q}^n : x, z \geq 0\}$ and

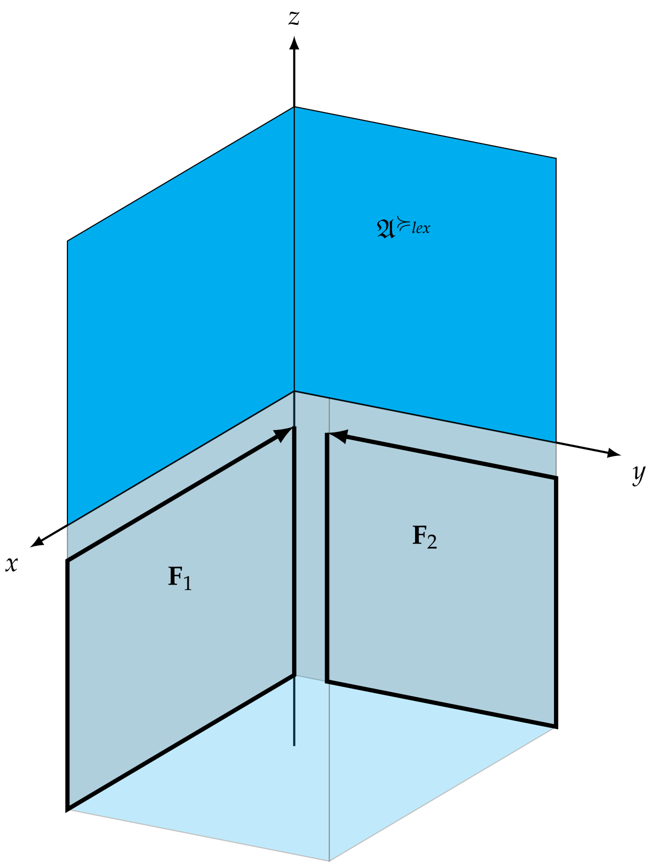


Fig.3

$F_2 = \{(0, y, z) \in \mathbb{Q}^n : y, z \geq 0\}$, can't be contained in $\{\mathbf{u} \in \mathbb{Q}^n : \langle \mathbf{u}, \omega - s_i(\omega) \rangle \geq 0 \quad i = x, y, z\}$. Note interchanging $e_2 \leftrightarrow e_1$ which is different order give same $\mathfrak{A}^{\succcurlyeq}$.

Example 4.2.2. Let $\mathcal{G}$ be a reflection generated by $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ here the associated inner product possess $M = \frac{1}{2}(gTg + I) = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 3 & 1 \end{pmatrix}$ and the usual lex ordering $e_1 \succ e_2 \succ e_3$ we have the $\Phi = \{\pm(0, 1, 0)\}$ and $w = (0, 1, 0) \succcurlyeq_{lex} 0$ hence

$$\mathbb{R}_+(\mathfrak{A}^{\succcurlyeq}) = \{\mathbf{v} \in \mathbb{R}^3 : \langle \mathbf{v}, (0, 1, 0) \rangle \geq 0\} = \{(x, y, z) \in \mathbb{R}^3 : 2y + z \geq 0\}$$

take any $\omega \in \mathbb{R}_+(\mathfrak{A}^{\succcurlyeq})$ of rational dimension 3, with the $\mathcal{G}$ invariant inner product system or take $M^{-1}\omega$ for the usual inner product system, gives the same initial algebra for the invariant ring.

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Appendix A

Annex-1

A.1 Linearization of non-reflection groups in $GL(2,\mathbb{Z})$

Here we give the linearization, for all non trivial finite group of $GL(2,\mathbb{Z})$. We will use the same notation as in the book, Lorenz (2005), § 1.10.1, table 1.2. The reflection groups have a unique branching hence, we only investigate the non reflection groups,

$$d = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \textit{and} \quad x = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

Fig 4, below gives the hyperplane associated with each of the reflection formed, by $\{s, d, x\}$. All of the linearization semigroups lie on the intersection half spaces bounded by these (hyperplanes) lines. We also give a canonical ordering $\succcurlyeq$ of type n, so that

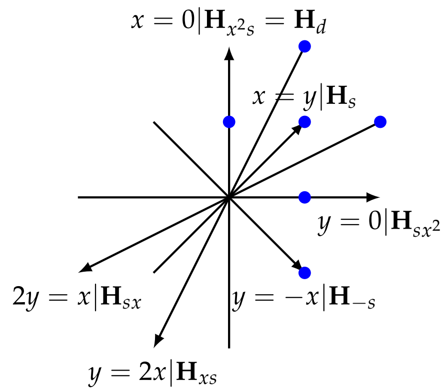

Fig.4 hyperplanes of reflections $\{s, d, x\}$.

the semigroups $\mathfrak{A}^{\succcurlyeq}$ has simple presentation. There are only four such groups (up to conjugate) these are, $\{\mathcal{G}_7 =< x >, \mathcal{G}_8 =< sd >, \mathcal{G}_9 =< x^2 >, \mathcal{G}_{10} =< x^3 = -I >\}$ each cyclic group of order 6, 4, 3 and 2 respectively. The $\mathcal{G}$-invariant bilinear form $\langle x, y \rangle = x^T M y$, where

$$M = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

for $\mathcal{G}_7$ and $\mathcal{G}_9$, and $M = I_2$ for the other two, used for defining the canonical order.

Group	$\mathcal{G}_7$	$\mathcal{G}_8 \vert \mathcal{G}_9$	$\mathcal{G}_{10}$
Generators	x	$sd \vert x^2$	$x^3 = -I_2$
Linear O'S	$\mathfrak{o}(2e_1+e_2), \mathfrak{o}(e_1+e_2)$	$\mathfrak{o}(e_1), \mathfrak{o}(e_1+e_2)$	$\mathfrak{o}(e_2), \mathfrak{o}(e_1)$
$\succcurlyeq$	$\{\varphi_1 = \langle\, , 2e_1+e_2\rangle,$ $\varphi_2 = \langle\, , e_1+e_2\rangle\}$	$\{\varphi_1 = \langle\, , e_1\rangle,$ $\varphi_2 = \langle\, , e_1+e_2\rangle\}$	$\{\varphi_1 = \langle\, , e_1\rangle,$ $\varphi_2 = \langle\, , e_2\rangle\}$
$\mathfrak{A}^{\succcurlyeq}$	$T \sqcup [(2,3)+s(T)]$ $T =< (2,1),(1,1) >$	$T \sqcup [(1,2)+s(T)]$ $T =< (1,0),(1,1) >$	$T \sqcup [(1,-1)+-d(T)]$ $T =< (1,0),(0,1) >$
$\phi_l(\succcurlyeq, \succcurlyeq') \in$	$< s, x >= \mathcal{G}_1$	$< s, d \vert x^2 >= \mathcal{G}_2 \vert \mathcal{G}_4$	$< d, -d >= \mathcal{G}_5$
Branching O'S	$\mathfrak{o}(2e_1+3e_2),$ $\mathfrak{o}(3e_1+2e_2)$	$\mathfrak{o}(e_1+2e_2),$ $\mathfrak{o}(2e_1+e_2)$	$\mathfrak{o}(e_1+e_2),$ $\mathfrak{o}(e_1-e_2)$

Table 1. linearization of non-reflection groups in $GL(2,\mathbb{Z})$.

In table 1, $\phi_l(\succcurlyeq, \succcurlyeq')$, refers to the semigroup homomorphism $\mathfrak{A}_l^{\succcurlyeq} \to \mathfrak{A}_l^{\succcurlyeq'}$ for any two $\succcurlyeq, \succcurlyeq' \in \Omega$. Note each of this fall in to some reflection group in which the group is embedded. i.e. $\mathcal{G}_7 \hookrightarrow \mathcal{G}_1, \mathcal{G}_8 \hookrightarrow \mathcal{G}_2, \mathcal{G}_9 \hookrightarrow \mathcal{G}_4$ and $\mathcal{G}_{10} \hookrightarrow \mathcal{G}_5$. Further more the invariant algebra is given by

$$\mathbf{R} = \mathbb{K}[f,g] \oplus h\mathbb{K}[f,g]$$

where f, g are the linear orbit and h any one of the two branching orbit sum (the two possible lead orbit sum for the product of linear orbits fg.) (Branching O'S, on Table 1).