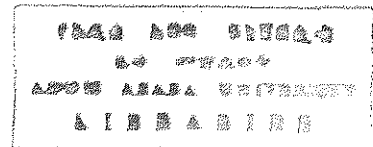


**THE PROPAGATOR FORMULATION
OF
PARAMETRIC OSCILLATION**



A Thesis

Submitted to the

School of Graduate Studies

Addis Ababa University

In Partial Fulfillment

of the Requirements for the

Degree of Master of Science in Physics

by

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June 1997

Addis Ababa

Acknowledgement

I want to express my gratitude to all of those whose help has been invaluable so that this thesis which needed a great deal of devotion, could finally come to light.

I can say nothing but the help of my adviser, Dr. Fesseha Kassahun, head, Department of physics, AAU, without whom this paper could have not been possible, is beyond my expression. He was not merely an adviser. It is only because of him that I could obtain a fund from ICTP and complete this thesis after six months interruption due to financial problem. Allowing easy access of his own materials, painstakingly careful followups and fatherly treatments are some but not all of his supports which are worth mentioning. Ato Sintayehu Tesfa, all the physics staffs of AAU, Applied and Natural Science Department staffs of NTC, my families, friends and intimates deserve my heart felt thanks.

Abstract

Employing the Q-function formalism, we analyze the squeezing spectrum and the photon number distribution for the signal mode as well as the quadrature fluctuations, the photon number distribution and the spectrum of the intensity-difference fluctuations for the signal-idler modes generated in parametric oscillators operating below threshold. The Q-function is obtained by solving the pertinent Fokker-Planck equation applying the method of evaluating the propagator developed by Fesseha [1]. The task of evaluating the propagator by means of this method essentially reduces to the problem of solving the pertinent Euler-Lagrange equations.

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1. Introduction

The nonclassical properties of light have generated a considerable interest for the past few years. One of these properties involves squeezing in which the fluctuations in one of the quadrature components is reduced below the vacuum level at the expense of enhanced fluctuations in the other component without violation of the Heisenberg uncertainty principle. Another nonclassical property of light is sub-Poissonian photon statistics in which the photon number distribution is narrower than the Poisson distribution.

One of the main reasons for the interest in a squeezed light is due to its potential applications in optical communications and gravitational wave detection, where signals close to or even below the quantum limit are expected.

Theoretical predictions of the generation of light with nonclassical features in various physical processes have been made by different authors. It has been shown that squeezing should be realizable in parametric oscillation [4-13], four-wave mixing [14,15], harmonic generation [16,17], resonance fluorescence [18,19] and the free-electron laser [20].

Because of the inherent two-photon nature of the interaction, the parametric processes have been studied as a source of a squeezed light. In parametric amplification, a coherent light of frequency 2ω interacts with a nonlinear medium and is down converted into two signal photons each of frequency ω , or into signal and idler photons with frequencies ω_1 and ω_2 such that $2\omega = \omega_1 + \omega_2$.

The down conversion of a coherent light into a single-mode squeezed light was predicted for the first time in a parametric amplifier by Takahashi [4]. In order to increase the gain the parametric medium is placed inside an optical cavity where it is coherently pumped and becomes a parametric oscillator [21].

A theoretical analysis of quantum fluctuations and photon statistics of the radiation generated in parametric oscillators has been made by a number of authors applying different techniques. And it has been found that a complete suppression of the noise at zero frequency in one of the quadrature components of the output signal mode produced in a degenerate parametric oscillator is possible [31, 32]. In addition, the spectrum of fluctuations in the intensity-difference of the output signal-idler modes produced in a nondegenerate parametric oscillator has been found to show a complete cancellation of the noise at zero frequency [49].

Experimentally, squeezing amounting to a noise reduction of 61% below the vacuum level has been achieved in a near degenerate parametric oscillator operating below threshold [22]. A quantum noise reduction of 69% in the intensity difference of the signal-idler modes has been observed in a NDPO operating below threshold [63].

In quantum optics, the analysis of a physical system is often carried out using a c-number formalism. This can be done applying the P-function or the Q-function. For systems with nonclassical features, the solution of the pertinent Fokker-Planck equation for the Glauber-Surdashan P-function is highly singular. This is because the diffusion coefficient in the Fokker-Planck equation for such systems becomes negative. On the other hand, we realize that the solution of the Fokker-Planck equation for the Q-function exists for systems with nonclassical features.

The main objective of this thesis is to analyze the squeezing spectrum as well as

the photon number distribution of the signal mode produced in a DPO and also to study the quadrature fluctuations, photon number distribution and the spectrum of the intensity-difference fluctuations of the signal-idler modes generated in a NDPO using the Q-function formalism.

The Q-function is expressible in terms of the Q-function propagator and the initial Q-function. In view of this, the problem of determining the Q-function reduces to the task of evaluating the Q-function propagator. The Q-function propagator may be determined using path integral method [2] or by directly solving the pertinent Fokker-Planck equation. However, we find it to be convenient to evaluate the Q-function propagator applying the method developed by Fesseha [1].

We would like to point out that in the parametric oscillations considered in this thesis the pump mode is treated classically and its amplitude is assumed to be constant. This is equivalent to a parametric oscillator operating below threshold. In addition, we would like to stress that our analysis assumes a single-cavity decay rate and no other losses such as intracavity absorption of the radiation are included.

The organization of the thesis is as follows. In chapter two, we present a systematic derivation of general expressions for a two-time correlation function and the photon number distribution of a single-mode light in terms of the Q-function, and with the application of these expressions the squeezing spectrum and the photon number distribution of the signal light produced in DPO are investigated. In chapter three, we analyze the quadrature fluctuations as well as the photon number distribution and the spectrum of the intensity-difference fluctuations of the signal-idler modes generated in a NDPO. Finally, in chapter four we discuss briefly the main results of this thesis and make certain remarks of interest.

2. The Degenerate Parametric Oscillator

The realization of a squeezed light generated in a degenerate parametric oscillator [21-24] and the theoretical explanation given by different authors [25-29] that a squeezed light can be used to enhance the sensitivity of interferometers beyond the shot-noise level made the DPO to receive a considerable attention for the past few years. In a DPO a coherent light of frequency 2ω interacts with a nonlinear medium inside a cavity and is down converted into two signal photons each of frequency ω .

The squeezing properties of the signal mode have been studied by several authors applying different methods [30-34,43,44]. Milburn and Walls [44] have calculated the quadrature squeezing of the intracavity signal mode below and above threshold. The squeezing spectrum of the output signal mode below threshold was determined by Collet and Gardiner [31] and Yurke [32]. On the other hand, Collet and Walls [33] as well as Savage and Walls [34] have calculated the squeezing spectrum above threshold. Experimentally, a squeezing amounting to a noise reduction of 61% below the vacuum level has been achieved in a near degenerate parametric oscillator operating below threshold [21].

Several authors have also discussed the statistical properties of the signal mode [35-43]. Vyas and Singh [35,39] have considered both the photon number and photon count distributions below threshold. Wolinsky [41] has obtained a complete description of the intracavity statistical properties.

Here we seek to analyze the squeezing spectrum and the intracavity photon num-

ber distribution of the signal mode generated by a DPO operating below threshold for the case in which the signal mode is initially in a vacuum state.

2.1 The Q-function

With the pump mode treated classically, the DPO is described in the interaction picture by the Hamiltonian ($\hbar = 1$)

$$\hat{H} = \frac{i\kappa\beta}{2}(a^2 - a^{\dagger 2}) + a\Gamma^\dagger + a^\dagger\Gamma,$$

where a is the annihilation operator for the signal mode, β (assumed to be real and constant) is the amplitude of the pump mode, κ is the coupling constant and Γ is a heat bath operator representing cavity losses by the signal mode .

Using standard techniques to eliminate the heat bath variables [3], it can be verified that the equation of evolution for the density operator of the signal mode is

$$\frac{\partial \rho}{\partial t} = i[\rho, \hat{H}'] + \frac{\gamma}{2} (2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a), \quad (2.1a)$$

where γ is the cavity damping rate and

$$\hat{H}' = \frac{i\kappa\beta}{2} (a^2 - a^{\dagger 2}). \quad (2.1b)$$

Substituting (2.1b) into (2.1a), we have

$$\frac{\partial \rho}{\partial t} = \frac{\kappa\beta}{2} (a^2 \rho - a^{\dagger 2} \rho - \rho a^2 + \rho a^{\dagger 2}) + \frac{\gamma}{2} (2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a). \quad (2.2)$$

Next we derive the Fokker-Planck equation for the Q-function by putting all the terms in normal ordering. To this end, using the relations

$$[a, f(a, a^\dagger)] = \frac{\partial f}{\partial a^\dagger} \quad (2.3a)$$

and

$$[a^\dagger, f(a, a^\dagger)] = -\frac{\partial f}{\partial a} \quad (2.3b)$$

one can easily show that

$$a^2 \rho = a \rho a + a \frac{\partial \rho}{\partial a^\dagger} = \rho a^2 + 2 \frac{\partial \rho}{\partial a^\dagger} a + \frac{\partial^2 \rho}{\partial a^\dagger^2}, \quad (2.4a)$$

$$\rho a^\dagger^2 = a^\dagger \rho a^\dagger + \frac{\partial \rho}{\partial a} a^\dagger = a^\dagger^2 \rho + 2 a^\dagger \frac{\partial \rho}{\partial a} + \frac{\partial^2 \rho}{\partial a^2}, \quad (2.4b)$$

$$2a \rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a = 2 \frac{\partial^2 \rho}{\partial a^\dagger \partial a} + a^\dagger \frac{\partial \rho}{\partial a^\dagger} + \frac{\partial \rho}{\partial a} a + 2\rho. \quad (2.4c)$$

Upon substituting (2.4) into (2.2), one finds

$$\frac{\partial \rho}{\partial t} = \frac{\kappa\beta}{2} \left(\frac{\partial^2 \rho}{\partial a^2} + \frac{\partial^2 \rho}{\partial a^\dagger^2} + 2a^\dagger \frac{\partial \rho}{\partial a} + 2 \frac{\partial \rho}{\partial a^\dagger} a \right) + \frac{\gamma}{2} \left(2 \frac{\partial^2 \rho}{\partial a^\dagger \partial a} + a^\dagger \frac{\partial \rho}{\partial a^\dagger} + \frac{\partial \rho}{\partial a} a + 2\rho \right),$$

where the density operator is assumed to be in normal order. Therefore, the Fokker-Planck equation for the Q-function is

$$\frac{\partial Q}{\partial t} = \frac{\kappa\beta}{2} \left(\frac{\partial^2 Q}{\partial \alpha^2} + \frac{\partial^2 Q}{\partial \alpha^{*2}} + 2\alpha^* \frac{\partial Q}{\partial \alpha} + 2 \frac{\partial Q}{\partial \alpha^*} \alpha \right) + \frac{\gamma}{2} \left(2 \frac{\partial^2 Q}{\partial \alpha^* \partial \alpha} + \alpha^* \frac{\partial Q}{\partial \alpha^*} + \frac{\partial Q}{\partial \alpha} \alpha + 2Q \right). \quad (2.5)$$

Using the relations

$$\alpha^* \frac{\partial Q}{\partial \alpha^*} = \frac{\partial (\alpha^* Q)}{\partial \alpha^*} - Q \quad (2.6a)$$

and

$$\frac{\partial Q}{\partial \alpha} \alpha = \frac{\partial (\alpha Q)}{\partial \alpha} - Q \quad (2.6b)$$

and noting that α and α^* are independent variables, Equation (2.5) can be put in the form

$$\begin{aligned} \frac{\partial Q(\alpha, \alpha^*, t)}{\partial t} = & \left[\frac{\kappa\beta}{2} \left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \alpha^{*2}} + 2 \frac{\partial}{\partial \alpha} \alpha^* + 2 \frac{\partial}{\partial \alpha^*} \alpha \right) + \frac{\gamma}{2} \left(2 \frac{\partial^2}{\partial \alpha^* \partial \alpha} + \frac{\partial}{\partial \alpha^*} \alpha^* \right. \right. \\ & \left. \left. + \frac{\partial}{\partial \alpha} \alpha \right) \right] Q(\alpha, \alpha^*, t). \end{aligned}$$

It proves to be convenient to introduce Cartesian coordinates defined by

$$\alpha = x + iy.$$

It then follows that

$$\begin{aligned} \frac{\partial Q(x, y, t)}{\partial t} = & \left[\left(\frac{\kappa\beta + \gamma}{4} \right) \frac{\partial^2}{\partial x^2} - \left(\frac{\kappa\beta - \gamma}{4} \right) \frac{\partial^2}{\partial y^2} + \left(\kappa\beta + \frac{\gamma}{2} \right) \frac{\partial}{\partial x} \right. \\ & \left. - \left(\kappa\beta - \frac{\gamma}{2} \right) \frac{\partial}{\partial y} \right] Q(x, y, t). \end{aligned}$$

In order to solve this differential equation using the propagator method, we need to transform it into a Schrödinger type equation. This can be achieved replacing $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x, y, Q(x, y, t) \right)$ by $(i\hat{p}_x, i\hat{p}_y, \hat{x}, \hat{y}, |Q(t)\rangle)$. One then finds

$$i \frac{d|Q(t)\rangle}{dt} = \left[-\frac{i\lambda_1}{4} \hat{p}_x^2 + \frac{i\lambda_2}{4} \hat{p}_y^2 - \lambda_3 \hat{p}_x \hat{x} + \lambda_4 \hat{p}_y \hat{y} \right] |Q(t)\rangle, \quad (2.7)$$

where

$$\lambda_1 = \kappa\beta + \gamma, \quad (2.8a)$$

$$\lambda_2 = \kappa\beta - \gamma, \quad (2.8b)$$

$$\lambda_3 = \kappa\beta + \frac{\gamma}{2}, \quad (2.8c)$$

$$\lambda_4 = \kappa\beta - \frac{\gamma}{2}. \quad (2.8d)$$

A formal solution of (2.7) can be put in the form

$$|Q(t)\rangle = \hat{U} |Q(0)\rangle, \quad (2.9)$$

in which

$$\hat{U} = \exp(-i\hat{H}t) \quad (2.10a)$$

and

$$\hat{H} = -\frac{i\lambda_1}{4} \hat{p}_x^2 + \frac{i\lambda_2}{4} \hat{p}_y^2 - \lambda_3 \hat{p}_x \hat{x} + \lambda_4 \hat{p}_y \hat{y}. \quad (2.10b)$$

Now multiplying (2.9) by $\langle x, y|$ from the left, we have

$$Q(x, y, t) = \langle x, y|\hat{U}|Q(0)\rangle, \quad (2.11)$$

where

$$Q(x, y, t) = \langle x, y|Q(t)\rangle.$$

Using the completeness relation

$$I = \int dx' dy' |y', x'\rangle\langle x', y'|,$$

equation (2.11) can be rewritten as

$$Q(x, y, t) = \int dx' dy' Q(x, y, t|x', y', 0) Q_0(x', y'), \quad (2.12)$$

where

$$Q(x, y, t|x', y', 0) = \langle x, y|\hat{U}|y', x'\rangle \quad (2.13a)$$

and

$$Q_0(x', y') = \langle x', y'|Q(0)\rangle. \quad (2.13b)$$

According to Fesseha [1], the propagator associated with a quadratic Hamiltonian

$$\hat{H} = a\hat{p}_x^2 + a'\hat{p}_y^2 + b(t)\hat{p}_x\hat{x} + b'(t)\hat{p}_y\hat{y} + c(t)\hat{x}^2 + c'(t)\hat{y}^2$$

is expressible in the form

$$K(x, y, t|x', y', 0) = \left[\frac{i}{2\pi} \frac{\partial^2 S_c}{\partial x' \partial x} \right]^{\frac{1}{2}} \left[\frac{i}{2\pi} \frac{\partial^2 S_c}{\partial y' \partial y} \right]^{\frac{1}{2}} \exp \left[-\xi \int_0^t [b(t') + b'(t')] dt' + iS_c \right] \quad (2.14)$$

where S_c is the classical action, ξ is a constant parameterizing operator ordering, and a and a' are constants different from zero.

We wish to obtain the Q-function propagator employing this relation. We see from (2.10b) that $b = -\lambda_3$, $b' = \lambda_4$ and we recall that for the antistandard form of

ordering $\xi = \frac{1}{2}$ [1]. Thus, the Q-function propagator associated with the Hamiltonian (2.10b) can be written as

$$Q(x, y, t | x', y', 0) = e^{(\frac{\lambda_3 - \lambda_4}{2})t} \left[\frac{i}{2\pi} \frac{\partial^2 S_c}{\partial x' \partial x} \right]^{\frac{1}{2}} \left[\frac{i}{2\pi} \frac{\partial^2 S_c}{\partial y' \partial y} \right]^{\frac{1}{2}} e^{iS_c}. \quad (2.15)$$

We now proceed to calculate the classical action. To this end, we note that the “Hamiltonian function” associated with the quantum Hamiltonian (2.10b) is

$$H(x, y, p_x, p_y) = -\frac{i\lambda_1}{4} p_x^2 + \frac{i\lambda_2}{4} p_y^2 - \lambda_3 p_x x + \lambda_4 p_y y \quad (2.16)$$

and hence the corresponding “Lagrangian” is

$$L = \dot{x} p_x + \dot{y} p_y - H(x, y, p_x, p_y).$$

Using the Hamilton equations

$$\dot{x} = \frac{\partial H}{\partial p_x}$$

and

$$\dot{y} = \frac{\partial H}{\partial p_y}$$

one finds

$$p_x = -\frac{2(\dot{x} + \lambda_3 x)}{i\lambda_1}$$

and

$$p_y = \frac{2(\dot{y} - \lambda_4 y)}{i\lambda_2}$$

so that the Lagrangian takes the form

$$L = \frac{i(\dot{x} + \lambda_3 x)^2}{\lambda_1} - \frac{i(\dot{y} - \lambda_4 y)^2}{\lambda_2}. \quad (2.17)$$

Now applying the Euler-Lagrange equations along with (2.17), one easily gets

$$\ddot{x} - \lambda_3^2 x = 0$$

and

$$\ddot{y} - \lambda_4^2 y = 0.$$

The solution of these differential equations can be written as

$$x(t) = Ae^{\lambda_3 t} + Be^{-\lambda_3 t} \quad (2.18a)$$

and

$$y(t) = De^{\lambda_4 t} + Ee^{-\lambda_4 t}. \quad (2.18b)$$

Substitution of (2.18) and their time derivatives into (2.17) gives

$$L = \frac{4i\lambda_3^2}{\lambda_1} A^2 e^{2\lambda_3 t} - \frac{4i\lambda_4^2}{\lambda_2} E^2 e^{-2\lambda_4 t},$$

so that the classical action

$$S_c = \int_0^T L(t) dt$$

takes the form

$$S_c = \frac{2i\lambda_3}{\lambda_1} A^2 (e^{2\lambda_3 T} - 1) + \frac{2i\lambda_4}{\lambda_2} E^2 (e^{-2\lambda_4 T} - 1).$$

Setting $x(T) = x''$, $y(T) = y''$, $x(0) = x'$ and $y(0) = y'$, one easily obtains from (2.18)

$$A = - \left(\frac{x' - x'' e^{\lambda_3 T}}{e^{2\lambda_3 T} - 1} \right)$$

and

$$E = - \left(\frac{y' - y'' e^{-\lambda_4 T}}{e^{-2\lambda_4 T} - 1} \right)$$

consequently

$$S_c = \frac{2i\lambda_3}{\lambda_1} \left[\frac{x'^2 - 2x'x''e^{\lambda_3 T} + x''^2 e^{2\lambda_3 T}}{e^{2\lambda_3 T} - 1} \right] + \frac{2i\lambda_4}{\lambda_2} \left[\frac{y'^2 - 2y'y''e^{-\lambda_4 T} + y''^2 e^{-2\lambda_4 T}}{e^{-2\lambda_4 T} - 1} \right], \quad (2.19)$$

where we have replaced (x'', y'', T) by (x, y, t) . It then follows that

$$\frac{\partial^2 S_c}{\partial x' \partial x} = - \frac{4i\lambda_3 e^{\lambda_3 t}}{\lambda_1 (e^{2\lambda_3 t} - 1)} \quad (2.20a)$$

and

$$\frac{\partial^2 S_c}{\partial y' \partial y} = -\frac{4i\lambda_4 e^{-\lambda_4 t}}{\lambda_2 (e^{-2\lambda_4 t} - 1)}. \quad (2.20b)$$

Now combination of (2.15), (2.19) and (2.20) results in

$$\begin{aligned} Q(x, y, t | x', y', 0) &= \frac{1}{\pi} \left[\frac{2\lambda_3 e^{2\lambda_3 t}}{\lambda_1 (e^{2\lambda_3 t} - 1)} \frac{2\lambda_4 e^{-2\lambda_4 t}}{\lambda_2 (e^{-2\lambda_4 t} - 1)} \right]^{\frac{1}{2}} \\ &\times \exp \left[-\frac{2\lambda_3}{\lambda_1} \left(\frac{x'^2 - 2x'x e^{\lambda_3 t} + x^2 e^{2\lambda_3 t}}{e^{2\lambda_3 t} - 1} \right) - \frac{2\lambda_4}{\lambda_2} \left(\frac{y'^2 - 2y'y e^{-\lambda_4 t} + y^2 e^{-2\lambda_4 t}}{e^{-2\lambda_4 t} - 1} \right) \right] \end{aligned} \quad (2.21a)$$

This represents the Q-function propagator for the signal mode. Although we are interested in the case for which the signal mode is initially in a vacuum state, we also need the Q-function satisfying the initial condition

$$Q_0(x', y') = \frac{1}{\pi} \exp \left[-(x' - x_o)^2 - (y' - y_o)^2 \right]. \quad (2.21b)$$

Thus combining (2.12) and (2.21), we have

$$\begin{aligned} Q(x, y, t) &= \frac{1}{\pi^2} \left[\frac{4\lambda_3 \lambda_4 \exp \left[-\frac{\lambda_3}{\lambda_1 (1 - e^{-2\lambda_3 t})} x^2 - \frac{\lambda_4}{\lambda_2 (1 - e^{2\lambda_4 t})} y^2 - \frac{x_o^2 + y_o^2}{2} \right]}{\lambda_1 \lambda_2 (1 - e^{-2\lambda_3 t}) (1 - e^{2\lambda_4 t})} \right]^{\frac{1}{2}} \\ &\times \int_{-\infty}^{\infty} \exp \left[-\left[\frac{2\lambda_3 e^{-2\lambda_3 t}}{\lambda_1 (1 - e^{-2\lambda_3 t})} + 1 \right] x'^2 + \left[\left(\frac{4\lambda_3 e^{-\lambda_3 t}}{\lambda_1 (1 - e^{-2\lambda_3 t})} \right) x + 2x_o \right] x' \right] dx' \\ &\times \int_{-\infty}^{\infty} \exp \left[-\left[\frac{2\lambda_4 e^{2\lambda_4 t}}{\lambda_2 (1 - e^{2\lambda_4 t})} + 1 \right] y'^2 + \left[\left(\frac{4\lambda_4 e^{\lambda_4 t}}{\lambda_2 (1 - e^{2\lambda_4 t})} \right) y + 2y_o \right] y' \right] dy' \end{aligned}$$

and using the relation

$$\int_{-\infty}^{\infty} e^{-px^2 + qx'} dx' = \left(\frac{\pi}{p} \right)^{\frac{1}{2}} e^{\frac{q^2}{4p}} \quad (2.22)$$

one readily gets the result

$$\begin{aligned} Q(x, y, t) &= \frac{2}{\pi} \left[\frac{\lambda_3 \lambda_4}{\lambda_1 \lambda_2 (1 - e^{-2\lambda_3 t}) (1 - e^{2\lambda_4 t})} \right]^{\frac{1}{2}} \exp \left[\frac{-2\lambda_3 (x - x_o e^{-\lambda_3 t})^2}{(2\lambda_3 - \lambda_1) e^{-2\lambda_3 t} + \lambda_1} \right. \\ &\quad \left. + \frac{-2\lambda_4 (y - y_o e^{\lambda_4 t})^2}{(2\lambda_4 - \lambda_2) e^{2\lambda_4 t} + \lambda_2} \right]. \end{aligned}$$

This can also be expressed as

$$Q(\alpha, \alpha^*, t) = \frac{A(t)}{\pi} \exp \left[-B(t) \alpha^* \alpha + \frac{C(t)}{2} (\alpha^{*2} + \alpha^2) + D(t) \alpha + D^*(t) \alpha^* \right], \quad (2.23)$$

where

$$A(t) = \sqrt{B^2(t) - C^2(t)} \exp \left[- \left[\frac{[B+C] e^{-2(\frac{\gamma}{2}-\kappa\beta)t} + [B-C] e^{-2(\frac{\gamma}{2}+\kappa\beta)t}}{2} \right] \alpha_o^* \alpha_o \right. \\ \left. + \left[\frac{[B(t)+C(t)] e^{-2[\frac{\gamma}{2}-\kappa\beta]t} - [B(t)-C(t)] e^{-2(\frac{\gamma}{2}+\kappa\beta)t}}{4} \right] (\alpha_o^{*2} + \alpha_o^2) \right], \quad (2.24a)$$

$$B(t) = \frac{1}{2} \left[\frac{\gamma - 2\kappa\beta}{\gamma - \kappa\beta (1 + e^{-2(\frac{\gamma}{2}-\kappa\beta)t})} + \frac{\gamma + 2\kappa\beta}{\gamma + \kappa\beta (1 + e^{-2(\frac{\gamma}{2}+\kappa\beta)t})} \right], \quad (2.24b)$$

$$C(t) = \frac{1}{2} \left[\frac{\gamma - 2\kappa\beta}{\gamma - \kappa\beta (1 + e^{-2(\frac{\gamma}{2}-\kappa\beta)t})} - \frac{\gamma + 2\kappa\beta}{\gamma + \kappa\beta (1 + e^{-2(\frac{\gamma}{2}+\kappa\beta)t})} \right], \quad (2.24c)$$

$$D(t) = \frac{1}{2} \left[[B(t) + C(t)] e^{-(\frac{\gamma}{2}-\kappa\beta)t} (\alpha_o^* - \alpha_o) + [B(t) - C(t)] e^{-(\frac{\gamma}{2}+\kappa\beta)t} (\alpha_o^* + \alpha_o) \right]. \quad (2.24d)$$

We are interested in the case for which the signal mode is initially in a vacuum state and hence setting $\alpha_o = \alpha_o^* = 0$ results in

$$Q(\alpha, \alpha^*, t) = \frac{\sqrt{B^2(t) - C^2(t)}}{\pi} \exp \left[-B(t) \alpha^* \alpha + \frac{C(t)}{2} (\alpha^{*2} + \alpha^2) \right] \quad (2.25)$$

2.2 The squeezing spectrum of the signal mode

In this section we seek to determine the squeezing spectrum of the signal mode employing the Q-function obtained in the previous section. The squeezing spectrum is defined by

$$S_j(\omega) = 1 + \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \langle : a_j^{out}(t+\tau), a_j^{out}(t) : \rangle, \quad (2.26)$$

where

$$\langle : a_j^{out}(t + \tau), a_j^{out}(t) : \rangle = \langle : a_j^{out}(t + \tau) a_j^{out}(t) : \rangle - \langle a_j^{out}(t + \tau) \rangle \langle a_j^{out}(t) \rangle, \quad (2.27a)$$

:: stands for normal ordering and a_j^{out} represents the output quadrature operators for the signal mode which is related to the input mode by

$$a_j^{out}(t) = \sqrt{\gamma} a_j(t) - a_j^{in}(t). \quad (2.27b)$$

Employing (2.27b) in (2.27a) one can easily verify that for a vacuum input

$$\langle : a_j^{out}(t + \tau), a_j^{out}(t) : \rangle = \gamma \langle : a_j(t + \tau), a_j(t) : \rangle, \quad (2.27c)$$

where a_j represents the intracavity quadrature operators of the signal mode defined by

$$a_1 = a^\dagger(t) + a(t) \quad (2.28a)$$

and

$$a_2 = i [a^\dagger(t) - a(t)]. \quad (2.28b)$$

Then applying (2.28a), we can write

$$\begin{aligned} \langle : a_1(t + \tau), a_1(t) : \rangle &= \langle a(t + \tau) a(t) \rangle + \langle a^\dagger(t + \tau) a(t) \rangle + \langle a^\dagger(t) a(t + \tau) \rangle \\ &+ \langle a^\dagger(t + \tau) a^\dagger(t) \rangle - [\langle a^\dagger(t + \tau) \rangle + \langle a(t + \tau) \rangle] [\langle a^\dagger(t) \rangle + \langle a(t) \rangle]. \end{aligned}$$

Using the Q-function (2.25), one can easily verify that

$$\langle a(t) \rangle = \langle a^\dagger(t) \rangle = 0,$$

so that

$$\langle : a_1(t + \tau), a_1(t) : \rangle = \langle a(t + \tau) a(t) \rangle + \langle a^\dagger(t + \tau) a(t) \rangle + \langle a^\dagger(t) a(t + \tau) \rangle$$

$$+\langle a^\dagger(t+\tau)a^\dagger(t)\rangle. \quad (2.29)$$

A two-time correlation function is expressible in the Schrödinger picture as

$$\langle a^\dagger(t)a(t+\tau)\rangle = Tr \left[a^\dagger(0)a(\tau)\rho(0) \right].$$

Upon expanding the density operator in normal ordering and introducing the completeness relation

$$I = \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| \quad (2.30)$$

the above expression can be put in the form

$$\langle a^\dagger(t)a(t+\tau)\rangle = \int \frac{d^2\alpha}{\pi} \sum_{l,m} C_{lm}(t) Tr \left[a^\dagger(0)a(\tau)|\alpha\rangle\langle\alpha|a^\dagger(0)a^m(0) \right].$$

Now applying the identity

$$|\alpha\rangle\langle\alpha|a^n = \left(\alpha + \frac{\partial}{\partial\alpha^*} \right)^n (|\alpha\rangle\langle\alpha|), \quad (2.31)$$

we have

$$\langle a^\dagger(t)a(t+\tau)\rangle = \int \frac{d^2\alpha}{\pi} \sum_{l,m} C_{lm}(t) \alpha^{*l} \left(\alpha + \frac{\partial}{\partial\alpha^*} \right)^m Tr \left[a^\dagger(0)a(\tau)|\alpha\rangle\langle\alpha| \right]$$

or

$$\langle a^\dagger(t)a(t+\tau)\rangle = \int d^2\alpha Q \left(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t \right) Tr \left[a^\dagger(0)a(\tau)|\alpha\rangle\langle\alpha| \right]. \quad (2.32)$$

We note that

$$Tr \left[a^\dagger(0)a(\tau)|\alpha\rangle\langle\alpha| \right] = \alpha^* Tr [a(\tau)\rho(0)] = \alpha^* Tr [a(0)\rho(\tau)],$$

in which

$$\rho(0) = |\alpha\rangle\langle\alpha|.$$

Therefore one can write that

$$\text{Tr} \left[a^\dagger(0)a(\tau)|\alpha\rangle\langle\alpha| \right] = \alpha^* \int d^2\lambda \lambda Q(\lambda^*, \lambda, \tau), \quad (2.33)$$

where $Q(\lambda^*, \lambda, \tau)$ reduces to

$$Q(\lambda^*, \lambda, \tau) = \frac{1}{\pi} \exp [-(\lambda - \alpha)^2]$$

at the initial time. Substitution of (2.33) into (2.32) leads to

$$\langle a^\dagger(t)a(t+\tau) \rangle = \int d^2\alpha d^2\lambda Q \left(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t \right) \alpha^* Q(\lambda^*, \lambda, \tau) \lambda \quad (2.34a)$$

It can also be established in a similar manner that

$$\langle a^\dagger(t+\tau)a(t) \rangle = \int d^2\alpha d^2\lambda Q \left(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t \right) \alpha Q(\lambda^*, \lambda, \tau) \lambda^* \quad (2.34b)$$

$$\langle a^\dagger(t+\tau)a^\dagger(t) \rangle = \int d^2\alpha d^2\lambda Q \left(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t \right) \alpha^* Q(\lambda^*, \lambda, \tau) \lambda^*, \quad (2.34c)$$

$$\langle a(t+\tau)a(t) \rangle = \int d^2\alpha d^2\lambda Q \left(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t \right) \alpha Q(\lambda^*, \lambda, \tau) \lambda \quad (2.34d)$$

so that combination of (2.29) and (2.34) yields

$$\langle : a_1(t+\tau), a_1(t) : \rangle = \int d^2\alpha d^2\lambda Q \left(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t \right) (\alpha + \alpha^*) Q(\lambda^*, \lambda, \tau) (\lambda + \lambda^*). \quad (2.35)$$

Replacing $(\alpha, \alpha^*, \alpha_o, \alpha_o^*, t)$ by $(\lambda, \lambda^*, \alpha, \alpha^*, \tau)$ in the Q-function (2.23), we have

$$Q(\lambda, \lambda^*, \tau) = \frac{A'(\tau)}{\pi} \exp \left[-B'(\tau)\lambda^*\lambda + \frac{C'(\tau)}{2} (\lambda^{*2} + \lambda^2) + D'(\tau)\lambda + D'^*(\tau)\lambda^* \right]$$

where A', B', C' and D' are described by expressions (2.24) with $(\alpha_o, \alpha_o^*, t)$ replaced

by (α, α^*, τ) . It then follows that

$$\int d^2\lambda (\lambda + \lambda^*) Q(\lambda^*, \lambda, \tau) = A'(\tau) \int \frac{d^2\lambda}{\pi} (\lambda + \lambda^*) \exp \left[-B'(\tau)\lambda^*\lambda + \frac{C'(\tau)}{2} (\lambda^{*2} + \lambda^2) + D'(\tau)\lambda + D'^*(\tau)\lambda^* \right]$$

or

$$\int d^2\lambda (\lambda + \lambda^*) Q(\lambda^*, \lambda, \tau) = \left[\frac{\partial}{\partial D'} + \frac{\partial}{\partial D'^*} \right] A'(\tau) \int \frac{d^2\lambda}{\pi} \exp[-B'(\tau)\lambda^*\lambda + \frac{C'(\tau)}{2}(\lambda^{*2} + \lambda^2) + D'(\tau)\lambda + D'^*(\tau)\lambda^*].$$

In view of the relation

$$\int \frac{d^2\alpha}{\pi} \exp[-a\lambda^*\lambda + b\lambda^2 + c\lambda^{*2} + d\lambda + e\lambda^*] = \frac{\exp\left[\frac{ade+be^2+cd^2}{a^2-4bc}\right]}{\sqrt{a^2-4bc}}, \quad (2.36)$$

we find that

$$\int d^2\lambda (\lambda + \lambda^*) Q(\lambda^*, \lambda, \tau) = \left[\frac{\partial}{\partial D'} + \frac{\partial}{\partial D'^*} \right] \left[\frac{A' \exp\left[\frac{B'D'D'^* + \frac{C'}{2}(D'^2 + D'^{*2})}{B'^2 - C'^2}\right]}{\sqrt{B'^2 - C'^2}} \right]$$

Now performing the differentiation and using the explicit forms of A' , B' , C' , D' and D'^* , one gets

$$\int d^2\lambda (\lambda + \lambda^*) Q(\lambda^*, \lambda, \tau) = e^{-(\frac{\gamma}{2} + \kappa\beta)\tau} (\alpha + \alpha^*), \quad (2.37)$$

so that combination of (2.35) and (2.37) leads to

$$\langle : a_1(t + \tau), a_1(t) : \rangle = e^{-(\frac{\gamma}{2} + \kappa\beta)\tau} \int d^2\alpha Q\left(\alpha^*, \alpha + \frac{\partial}{\partial \alpha^*}, t\right) (\alpha^2 + \alpha^{*2} + 2\alpha\alpha^*). \quad (2.38)$$

We recall that

$$\langle \hat{A} \rangle = Tr(\rho(t)\hat{A}(0))$$

expanding the density operator in normal ordering and introducing the completeness relation (2.30), one can write that

$$\langle \hat{A} \rangle = \int \frac{d^2\alpha}{\pi} \sum_{l,m} C_{lm}(t) Tr[|\alpha\rangle\langle\alpha| a^{\dagger l} a^m \hat{A}(a^\dagger, a)].$$

On account of the relation (2.31), this expression takes the form

$$\langle \hat{A} \rangle = \int \frac{d^2\alpha}{\pi} Q\left(\alpha^*, \alpha + \frac{\partial}{\partial \alpha^*}, t\right) A_n(\alpha^*, \alpha), \quad (2.39)$$

where $A_n(\alpha^*, \alpha)$ is the c-number equivalent of $\hat{A}(a^\dagger, a)$ for the normal ordering.

Therefore, in view of (2.39), expression (2.38) can be put in the form

$$\langle : a_1(t + \tau), a_1(t) : \rangle = e^{-(\frac{\gamma}{2} + \kappa\beta)\tau} \langle a^\dagger{}^2 + a^2 + 2a^\dagger a \rangle \quad (2.40)$$

so that applying the Q-function (2.25), we have

$$\langle a^\dagger{}^2 + a^2 + 2a^\dagger a \rangle = \sqrt{B^2(t) - C^2(t)} \int \frac{d^2\alpha}{\pi} (\alpha^2 + \alpha^{*2} + 2\alpha\alpha^* - 2) \exp[-B(t)\alpha^*\alpha + \frac{C(t)}{2}(\alpha^{*2} + \alpha^2)]$$

or

$$\langle a^\dagger{}^2 + a^2 + 2a^\dagger a \rangle = 2\sqrt{B^2(t) - C^2(t)} \left[\frac{\partial}{\partial C} - \frac{\partial}{\partial B} - 1 \right] \int \frac{d^2\alpha}{\pi} \exp[-B(t)\alpha^*\alpha + \frac{C(t)}{2}(\alpha^{*2} + \alpha^2)].$$

Upon carrying out the integration with the aid of the relation (2.36), we get

$$\langle a^\dagger{}^2 + a^2 + 2a^\dagger a \rangle = 2\sqrt{B^2 - C^2} \left(\frac{\partial}{\partial C} - \frac{\partial}{\partial B} - 1 \right) \frac{1}{\sqrt{B^2 - C^2}}$$

and using the explicit forms of B and C from (2.24), we find

$$\langle a^\dagger{}^2 + a^2 + 2a^\dagger a \rangle = -\frac{2\kappa\beta [1 - e^{-(\gamma+2\kappa\beta)t}]}{\gamma + 2\kappa\beta}. \quad (2.41)$$

Finally substitution of (2.41) into (2.40) results in

$$\langle : a_1(t + \tau), a_1(t) : \rangle = -\frac{2\kappa\beta [1 - e^{-(\gamma+2\kappa\beta)t}]}{\gamma + 2\kappa\beta} e^{-(\frac{\gamma}{2} + \kappa\beta)\tau}. \quad (2.42a)$$

Following a similar procedure, one can readily verify that

$$\langle : a_2(t + \tau), a_2(t) : \rangle = \frac{2\kappa\beta [1 - e^{-(\gamma-2\kappa\beta)t}]}{\gamma - 2\kappa\beta} e^{-(\frac{\gamma}{2} - \kappa\beta)\tau}. \quad (2.42b)$$

Now setting $\tau = 0$ and recalling that

$$(\Delta a_1)^2 = 1 + \langle : a_1(t), a_1(t) : \rangle,$$

with a similar expression for $(\Delta a_2)^2$, we see that

$$(\Delta a_1)^2 = 1 - \frac{2\kappa\beta [1 - e^{-(\gamma+2\kappa\beta)t}]}{\gamma + 2\kappa\beta} \quad (2.43a)$$

and

$$(\Delta a_2)^2 = 1 + \frac{2\kappa\beta [1 - e^{-(\gamma-2\kappa\beta)t}]}{\gamma - 2\kappa\beta}. \quad (2.43b)$$

At steady state ($t \rightarrow \infty$) these expressions take the form

$$(\Delta a_1)^2 = \frac{\gamma}{\gamma + 2\kappa\beta} \quad (2.44a)$$

and

$$(\Delta a_2)^2 = \frac{\gamma}{\gamma - 2\kappa\beta} \quad (2.44b)$$

and at threshold ($\kappa\beta \rightarrow \frac{\gamma}{2}$), we see that

$$(\Delta a_1)^2 = \frac{1}{2} \quad (2.45a)$$

and

$$(\Delta a_2)^2 \rightarrow \infty. \quad (2.45b)$$

Equations (2.43), (2.44) and (2.45) represent the variances of the quadrature operators. We note from these equations that the fluctuations in the first quadrature is below the vacuum level with enhanced fluctuations in the second quadrature. This verifies that the signal mode is in a squeezed state.

Anwar and Zubiary [43] have evaluated the variances of these two quadrature operators with the DPO coupled with a squeezed vacuum. When r set equal to zero, the results they obtained reduce to (2.43) and (2.45). In addition, the variances described by (2.43) and (2.45) are in a complete agreement with the results found by Milburn and Walls [44] and Lugiato as well as Strini [45].

Now combining (2.26), (2.27c) and (2.42), we put the spectrum of the quadratures fluctuations as

$$S_1(\omega) = 1 - \frac{2\kappa\beta\gamma [1 - e^{-(\gamma+2\kappa\beta)t}]}{\gamma + 2\kappa\beta} \int_{-\infty}^{\infty} d\tau e^{-(\frac{\gamma}{2} + \kappa\beta + i\omega)\tau} \quad (2.46a)$$

and

$$S_2(\omega) = 1 + \frac{2\kappa\beta\gamma [1 - e^{-(\gamma-2\kappa\beta)t}]}{\gamma - 2\kappa\beta} \int_{-\infty}^{\infty} d\tau e^{-(\frac{\gamma}{2} - \kappa\beta + i\omega)\tau}. \quad (2.46b)$$

Then applying the stationarity property for the integral (2.46a), we have

$$S_1(\omega) = 1 - \frac{2\kappa\beta\gamma [1 - e^{-(\gamma+2\kappa\beta)t}]}{\gamma + 2\kappa\beta} \left[\int_0^{\infty} d\tau e^{-(\frac{\gamma}{2} + \kappa\beta + i\omega)\tau} + \int_0^{\infty} d\tau e^{-(\frac{\gamma}{2} + \kappa\beta - i\omega)\tau} \right]$$

so that

$$S_1(\omega) = 1 - \frac{8\kappa\beta\gamma [1 - e^{-(\gamma+2\kappa\beta)t}]}{(\gamma + 2\kappa\beta)^2 + (2\omega)^2}. \quad (2.47a)$$

Similarly for the second quadrature operator, one finds

$$S_2(\omega) = 1 + \frac{8\kappa\beta\gamma [1 - e^{-(\gamma-2\kappa\beta)t}]}{(\gamma - 2\kappa\beta)^2 + (2\omega)^2}. \quad (2.47b)$$

At steady state ($t \rightarrow \infty$) expressions (2.47) take the form

$$S_1(\omega) = \frac{(\gamma - 2\kappa\beta)^2 + (2\omega)^2}{(\gamma + 2\kappa\beta)^2 + (2\omega)^2} \quad (2.48a)$$

and

$$S_2(\omega) = \frac{(\gamma + 2\kappa\beta)^2 + (2\omega)^2}{(\gamma - 2\kappa\beta)^2 + (2\omega)^2} \quad (2.48b)$$

and at threshold ($\kappa\beta \rightarrow \frac{\gamma}{2}$) these reduce to

$$S_1(\omega) = \frac{\omega^2}{\gamma^2 + \omega^2} \quad (2.49a)$$

and

$$S_2(\omega) = \frac{\gamma^2 + \omega^2}{\omega^2}. \quad (2.49b)$$

Expression (2.49) is the squeezing spectrum of the signal mode at steady state and at threshold. From (2.49a) we note that the squeezing spectrum is a Lorentzian with a width of 2γ . Moreover, one can see that at $\omega = 0$, there is a complete suppression of the noise in the first quadrature with infinitely enhanced fluctuations in the second quadrature. The results given in expressions (2.48) and (2.49) are in a complete agreement with the results obtained by Collet and Gardiner [31].

An experiment demonstrating the generation of a squeezed light in a DPO operating below threshold has been performed by Wu et al. [21]. At a fixed frequency ω they have obtained a maximum noise reduction of 61% relative to the vacuum level.

2.3 The photon number distribution

Next we wish to calculate the photon number distribution for the signal mode applying the Q-function (2.25). To this end, the photon number distribution for a single-mode light is defined as

$$P(n) = \langle n | \rho(a^\dagger, a) | n \rangle. \quad (2.50)$$

Introducing the completeness relation (2.30) twice in this expression one can write

$$P(n) = \int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi} \langle n | \alpha \rangle \langle \alpha | \rho(a^\dagger, a) | \beta \rangle \langle \beta | n \rangle$$

or

$$P(n) = \frac{1}{n!} \int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi} (\alpha\beta^*)^n \exp \left[- \left(\frac{\alpha^*\alpha + \beta^*\beta}{2} \right) \right] \langle \alpha | \rho(a^\dagger, a) | \beta \rangle, \quad (2.51)$$

where we have employed the relations

$$\langle n | \alpha \rangle = e^{-\frac{\alpha^*\alpha}{2}} \frac{\alpha^n}{\sqrt{n!}}$$

and

$$\langle \beta | n \rangle = e^{-\frac{\beta^* \beta}{2}} \frac{\beta^{*n}}{\sqrt{n!}}.$$

Now expanding the density operator in normal ordering, we have

$$P(n) = \frac{1}{n!} \sum_{l, m} C_{lm}(t) \int \frac{d^2 \alpha}{\pi} \frac{d^2 \beta}{\pi} (\alpha \beta^*)^n \exp \left[- \left(\frac{\alpha^* \alpha + \beta^* \beta}{2} \right) \right] \langle \alpha | a^{\dagger l} a^m | \beta \rangle$$

or

$$P(n) = \frac{1}{n!} \sum_{l, m} C_{lm}(t) \int \frac{d^2 \alpha}{\pi} \frac{d^2 \beta}{\pi} (\alpha \beta^*)^n \alpha^{*l} \beta^m \exp \left[- \left(\frac{\alpha^* \alpha + \beta^* \beta}{2} \right) \right] \langle \alpha | \beta \rangle$$

so that application of the relation

$$\langle \alpha | \beta \rangle = \exp \left[- \left(\frac{\alpha^* \alpha + \beta^* \beta}{2} \right) + \alpha^* \beta \right]$$

yields

$$P(n) = \frac{1}{n!} \sum_{l, m} C_{lm}(t) \int \frac{d^2 \alpha}{\pi} \frac{d^2 \beta}{\pi} (\alpha \beta^*)^n \alpha^{*l} \beta^m \exp [-\alpha^* \alpha - \beta^* \beta + \alpha^* \beta]. \quad (2.52)$$

Then rewriting (2.52) as

$$P(n) = \frac{1}{n!} \sum_{l, m} C_{lm}(t) \int \frac{d^2 \alpha}{\pi} \alpha^n \alpha^{*l} \exp [-\alpha^* \alpha + \lambda^* \alpha] \int \frac{d^2 \beta}{\pi} \beta^{*n} \beta^m \\ \times \exp [-\beta^* \beta + \alpha^* \beta + \lambda \beta^*] \Big|_{\lambda = \lambda^* = 0},$$

one can put the photon number distribution in the form

$$P(n) = \frac{1}{n!} \sum_{l, m} C_{lm}(t) \frac{\partial^{2n}}{\partial \lambda^{*n} \partial \lambda^n} \left[\int \frac{d^2 \alpha}{\pi} \alpha^{*l} \exp [-\alpha^* \alpha + \lambda^* \alpha] \right. \\ \left. \times \frac{\partial^m}{\partial \alpha^{*m}} \int \frac{d^2 \beta}{\pi} \exp [-\beta^* \beta + \alpha^* \beta + \lambda \beta^*] \right] \Big|_{\lambda = \lambda^* = 0}.$$

Now performing the integration over β with the aid of (2.36) leads to

$$P(n) = \frac{1}{n!} \sum_{l, m} C_{lm}(t) \frac{\partial^{2n}}{\partial \lambda^{*n} \partial \lambda^n} \left[\int \frac{d^2 \alpha}{\pi} \alpha^{*l} \exp [-\alpha^* \alpha + \lambda^* \alpha] \frac{\partial^m}{\partial \alpha^{*m}} [e^{\lambda \alpha^*}] \right] \Big|_{\lambda = \lambda^* = 0}$$

and upon differentiating with respect to α^* , one finds

$$P(n) = \frac{1}{n!} \sum_{l, m} C_{lm}(t) \frac{\partial^{2n}}{\partial \lambda^{*n} \partial \lambda^n} \left[\lambda^m \int \frac{d^2 \alpha}{\pi} \alpha^{*l} \exp[-\alpha^* \alpha + \lambda^* \alpha + \lambda \alpha^*] \right]_{\lambda=\lambda^*=0}$$

or

$$P(n) = \frac{1}{n!} \sum_{l, m} C_{lm}(t) \frac{\partial^{2n}}{\partial \lambda^{*n} \partial \lambda^n} \left[\lambda^m \frac{\partial^l}{\partial \lambda^l} \int \frac{d^2 \alpha}{\pi} \exp[-\alpha^* \alpha + \lambda^* \alpha + \lambda \alpha^*] \right]_{\lambda=\lambda^*=0}$$

Next on integrating over α , we obtain

$$P(n) = \frac{1}{n!} \sum_{l, m} C_{lm}(t) \frac{\partial^{2n}}{\partial \lambda^{*n} \partial \lambda^n} \left[\lambda^m \frac{\partial^l e^{\lambda^* \lambda}}{\partial \lambda^l} \right]_{\lambda=\lambda^*=0}$$

and differentiating l times with respect to λ , we find

$$P(n) = \frac{1}{n!} \sum_{l, m} C_{lm}(t) \frac{\partial^{2n}}{\partial \lambda^{*n} \partial \lambda^n} [\lambda^{*l} \lambda^m e^{\lambda^* \lambda}]_{\lambda=\lambda^*=0}. \quad (2.53)$$

Expression (2.53) can also be put in the form

$$P(n) = \frac{\pi}{n!} \frac{\partial^{2n} [Q(\lambda^*, \lambda, t) e^{\lambda^* \lambda}]}{\partial \lambda^{*n} \partial \lambda^n} \Big|_{\lambda=\lambda^*=0}, \quad (2.54)$$

where

$$Q(\lambda^*, \lambda, t) = \frac{1}{\pi} \sum_{l, m} C_{lm}(t) \lambda^{*l} \lambda^m. \quad (2.55)$$

Employing the Q-function (2.25) in (2.54), one can then write the photon number distribution for the signal mode as

$$P(n) = \frac{\sqrt{B^2 - C^2} \partial^{2n} \exp \left[(1-B) \alpha^* \alpha + \frac{C}{2} (\alpha^2 + \alpha^{*2}) \right]}{n! \partial \alpha^n \partial \alpha^{*n}} \Big|_{\alpha=\alpha^*=0}. \quad (2.56)$$

Now, expanding the exponential functions in power series, we have

$$e^{(1-B)\alpha^* \alpha} = \sum_{l=0}^{\infty} \frac{(1-B)^l}{l!} (\alpha^* \alpha)^l,$$

$$e^{\frac{C}{2} \alpha^2} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{C}{2} \right)^k \alpha^{2k}$$

and

$$e^{\frac{C}{2}\alpha^{*2}} = \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{C}{2}\right)^m \alpha^{*2m}$$

so that (2.56) takes the form

$$P(n) = \frac{\sqrt{B^2 - C^2}}{n!} \sum_{l,k,m=0}^{\infty} \frac{(1-B)^l C^{k+m} \partial^{2n} (\alpha^{l+2k} \alpha^{*l+2m})}{2^{k+m} k! l! m! \partial \alpha^n \partial \alpha^{*n}} \Big|_{\alpha=\alpha^*=0}$$

or

$$P(n) = \frac{\sqrt{B^2 - C^2}}{n!} \sum_{l,k,m=0}^{\infty} \frac{(1-B)^l C^{k+m} \partial^n (\alpha^{l+2k}) \partial^n (\alpha^{*l+2m})}{2^{k+m} k! l! m! \partial \alpha^n \partial \alpha^{*n}} \Big|_{\alpha=\alpha^*=0}$$

On account of the relation

$$\frac{\partial(x^m)}{\partial x^n} = \frac{m!}{(m-n)!} x^{m-n} \quad (2.57)$$

the above expression takes the form

$$P(n) = \frac{\sqrt{B^2 - C^2}}{n!} \sum_{l,k,m=0}^{\infty} \frac{(1-B)^l C^{k+m} (l+2k)! (l+2m)! \alpha^{l+2k-n} \alpha^{*l+2m-n}}{2^{k+m} k! l! m! (l+2k-n)! (l+2m-n)!} \Big|_{\alpha=\alpha^*=0} \quad (2.58)$$

We note that (2.58) is different from zero if

$$l + 2k - n = 0$$

and

$$l + 2m - n = 0.$$

From these equations we see that

$$m = k$$

and

$$l = n - 2k.$$

Hence, Expression (2.58) reduces to

$$P(n) = \frac{\sqrt{B^2 - C^2}}{n!} \sum_{k=0}^{\infty} \frac{n!^2 (1-B)^{n-2k} C^{2k}}{2^{2k} k!^2 (n-2k)!} \quad (2.59)$$

Since $(n-2k)!$ is defined for zero or positive integer, we see that

$$n - 2k \geq 0$$

so that

$$k \leq \frac{n}{2}.$$

Therefore, one can put (2.59) in the form

$$P(n) = \frac{\sqrt{B^2 - C^2}}{n!} \sum_{k=0}^{[n]} \frac{n!^2 (1-B)^{n-2k} C^{2k}}{2^{2k} k!^2 (n-2k)!}, \quad (2.60)$$

where $[n] = \frac{n}{2}$ (for even n) and $[n] = \frac{n-1}{2}$ (for odd n). This equation represents the photon number distribution for the signal mode. In the absence of cavity damping ($\gamma = 0$), one finds from (2.24b) and (2.24c)

$$B = 1$$

and

$$C = -\tanh(\kappa\beta t).$$

then, substituting these values into (2.60), in the absence of cavity damping the photon number distribution takes the form

$$P(n) = \begin{cases} 0 & \text{for odd } n \\ \frac{n! \tanh^n(\kappa\beta t)}{2^{n(\frac{n}{2})!} 2 \cosh(\kappa\beta t)} = \frac{\tanh^n(\kappa\beta t) H_n^2(0)}{n! 2^n \cosh(\kappa\beta t)} & \text{for even } n \end{cases} \quad (2.61)$$

The probability of finding odd number of photons is zero due to the fact that the signal photons are generated in pairs.

Next we seek to obtain the photon number distribution in terms of the mean photon number at steady state. To this end, we express the mean photon number as

$$\bar{n} = \int d^2\alpha \alpha \alpha^* Q(\alpha, \alpha, t) - 1$$

so that application of (2.25) results in

$$\bar{n} = \frac{B}{B^2 - C^2} - 1.$$

At steady state (2.24b) and (2.24c) take the form

$$B = \frac{\gamma^2 - 2(\kappa\beta)^2}{\gamma^2 - (\kappa\beta)^2} \quad (2.62a)$$

and

$$C = -\frac{\kappa\beta\gamma}{\gamma^2 - (\kappa\beta)^2}. \quad (2.62b)$$

Therefore, at steady state the mean photon number is put in the form

$$\bar{n}_{ss} = \frac{1}{2} \left(\frac{(2\kappa\beta)^2}{\gamma^2 - (2\kappa\beta)^2} \right).$$

In terms of the mean photon number at steady state, (2.62) can be rewritten as

$$B = 1 - \frac{\bar{n}_{ss}}{3\bar{n}_{ss} + 2}$$

and

$$C = \frac{\sqrt{2\bar{n}_{ss}(2\bar{n}_{ss} + 1)}}{3\bar{n}_{ss} + 2}$$

so that in view of these expressions (2.60) takes the form

$$P_{ss}(n) = \frac{\sqrt{2} \bar{n}_{ss}^n}{(3\bar{n}_{ss} + 2)^{n+\frac{1}{2}}} \sum_{k=0}^{[n]} \frac{n!}{k!^2(n-2k)!} \left(\frac{2\bar{n}_{ss} + 1}{2\bar{n}_{ss}} \right)^k. \quad (2.63)$$

This is the intracavity photon number distribution of the signal mode at steady state. The form of the intracavity photon number distribution obtained by Vyas and Singh [35], applying the positive P-function, is different from (2.63). However, both distribution functions have identical values.

3. The Nondegenerate Parametric Oscillator

The nondegenerate parametric oscillator is a quantum optical system which produces a two-mode squeezed light. In a NDPO a pump photon of frequency ω interacts with a nonlinear medium inside a cavity and is down converted into highly correlated signal and idler photons with frequencies ω_1 and ω_2 such that $\omega_1 + \omega_2 = \omega$. This phenomenon for the first time was observed in parametric amplification by Burnham and Weinberg [46] and later by Friberg et al. [47].

A theoretical analysis of quantum fluctuations in intensity difference and quadrature components for signal-idler modes has been made by a number of authors for the past few years [49,52,54-56,58,61]. The spectrum of the intensity-difference fluctuations in the signal-idler modes for equal cavity decay rates has been calculated for below threshold by Reynaud et al. [49]. On the other hand, taking into consideration different cavity decay rates and intracavity losses Lane et al. [61] have analyzed the spectrum of intensity-difference fluctuations. In addition, Reid and Drummond have obtained the spectrum of quadrature component-difference fluctuations above threshold [52].

Due to the strong correlation between the signal and idler modes it has been shown that the intensity-difference fluctuations is below the shot-noise level. This has been confirmed experimentally by Heidmann et al. [50] and Debuisschert et al. [63].

Several groups have also analyzed the photon statistics of the signal-idler modes. The mean and the variance of the photons have been calculated by Collet and Loudon [53] and photon counting statistics has been analyzed by Vyas [59].

In this chapter we wish to analyze the intracavity fluctuations in the quadrature components as well as the photon number distribution and the spectrum of the intensity-difference fluctuations for the signal-idler modes generated in a NDPO operating below threshold.

3.1 The Q-function

With the pump mode treated classically, the NDPO in the interaction picture is described by the Hamiltonian ($\hbar = 1$)

$$\hat{H} = i\kappa\gamma_o(ab - a^\dagger b^\dagger) + a^\dagger\Gamma_a^\dagger + a^\dagger\Gamma_a + b^\dagger\Gamma_b^\dagger + b^\dagger\Gamma_b, \quad (3.1)$$

where κ is the coupling constant, $a(b)$ is the annihilation operator for the signal (idler) mode and Γ_a, Γ_b are heat bath operators and γ_o is the amplitude of the pump mode.

Using standard techniques to eliminate the heat-bath variables [3] we find equation of evolution for the density operator to be

$$\frac{\partial\rho}{\partial t} = i[\rho, \hat{H}'] + \gamma_a[2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a] + \gamma_b[2b\rho b^\dagger - b^\dagger b\rho - \rho b^\dagger b], \quad (3.2)$$

where

$$\hat{H}' = i\kappa\gamma_o(ab - a^\dagger b^\dagger). \quad (3.3)$$

Applying (3.3) expression (3.2) can be put in the form

$$\begin{aligned} \frac{\partial\rho}{\partial t} = & -\kappa\gamma_o(\rho ab - \rho a^\dagger b^\dagger - ab\rho + a^\dagger b^\dagger\rho) + \gamma_a(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \\ & + \gamma_b(2b\rho b^\dagger - b^\dagger b\rho - \rho b^\dagger b). \end{aligned} \quad (3.4)$$

To obtain the Fokker-Planck equation for the Q-function one should put all terms in expression (3.4) in normal ordering. To this end, we note that a and b are independent operators and $\rho = \rho(a, b, a^\dagger, b^\dagger, t)$ so that applying the relations (2.3), we have

$$\begin{aligned}\rho a^\dagger &= a^\dagger \rho + \frac{\partial \rho}{\partial a}, \\ \rho b^\dagger &= b^\dagger \rho + \frac{\partial \rho}{\partial b}, \\ \frac{\partial \rho}{\partial a} b^\dagger &= b^\dagger \frac{\partial \rho}{\partial a} + \frac{\partial^2 \rho}{\partial a \partial b}.\end{aligned}$$

It then follows that

$$\rho a^\dagger b^\dagger = a^\dagger b^\dagger \rho + a^\dagger \frac{\partial \rho}{\partial b} + b^\dagger \frac{\partial \rho}{\partial a} + \frac{\partial^2 \rho}{\partial a \partial b}. \quad (3.5a)$$

Similarly

$$ab\rho = \rho ab + \frac{\partial \rho}{\partial b^\dagger} a + \frac{\partial \rho}{\partial a^\dagger} b + \frac{\partial^2 \rho}{\partial b^\dagger \partial a^\dagger}. \quad (3.5b)$$

In view of expression (2.4c), one can write

$$2b\rho b^\dagger - b^\dagger b\rho - \rho b^\dagger b = 2\frac{\partial^2 \rho}{\partial b^\dagger \partial b} + b^\dagger \frac{\partial \rho}{\partial b^\dagger} + \frac{\partial \rho}{\partial b} b + 2\rho \quad (3.5c)$$

so that substitution of (3.5) along with (2.4c) into (3.4) leads to

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= \kappa \gamma_o \left[\frac{\partial^2 \rho}{\partial a \partial b} + \frac{\partial^2 \rho}{\partial a^\dagger \partial b^\dagger} + \frac{\partial \rho}{\partial b^\dagger} a + \frac{\partial \rho}{\partial a^\dagger} b + a^\dagger \frac{\partial \rho}{\partial b} + b^\dagger \frac{\partial \rho}{\partial a} \right] + \gamma_a \left[2\frac{\partial^2 \rho}{\partial a^\dagger \partial a} \right. \\ &\quad \left. + a^\dagger \frac{\partial \rho}{\partial a^\dagger} + \frac{\partial \rho}{\partial a} a + 2\rho \right] + \gamma_b \left[2\frac{\partial^2 \rho}{\partial b^\dagger \partial b} + b^\dagger \frac{\partial \rho}{\partial b^\dagger} + \frac{\partial \rho}{\partial b} b + 2\rho \right]. \quad (3.6)\end{aligned}$$

All terms in this expression are in normal ordering (assuming that ρ is also in normal ordering). Now upon replacing each operator by its c-number equivalent, the Fokker-Planck equation takes the form

$$\frac{\partial Q}{\partial t} = \kappa \gamma_o \left[\frac{\partial^2 Q}{\partial \alpha \partial \beta} + \frac{\partial^2 Q}{\partial \alpha^* \partial \beta^*} + \frac{\partial Q}{\partial \beta^*} \alpha + \frac{\partial Q}{\partial \alpha^*} \beta + \alpha^* \frac{\partial Q}{\partial \beta} + \beta^* \frac{\partial Q}{\partial \alpha} \right] + \gamma_a \left[2\frac{\partial^2 Q}{\partial \alpha^* \partial \alpha} \right.$$

$$+ \alpha^* \frac{\partial Q}{\partial \alpha^*} + \frac{\partial Q}{\partial \alpha} \alpha + 2Q \Big] + \gamma_b \left[2 \frac{\partial^2 Q}{\partial \beta^* \partial \beta} + \beta^* \frac{\partial Q}{\partial \beta^*} + \frac{\partial Q}{\partial \beta} \beta + 2Q \right], \quad (3.7)$$

where $\alpha(\beta)$ is the c-number equivalent for $a(b)$. Recalling that α , α^* , β and β^* are independent variables and applying the relations (2.6), we can put (3.7) in the form

$$\begin{aligned} \frac{\partial Q}{\partial t} = \kappa \gamma_o \left[\frac{\partial^2 Q}{\partial \alpha \partial \beta} + \frac{\partial^2 Q}{\partial \alpha^* \partial \beta^*} + \frac{\partial(\alpha Q)}{\partial \beta^*} + \frac{\partial(\beta Q)}{\partial \alpha^*} + \frac{\partial(\alpha^* Q)}{\partial \beta} + \frac{\partial(\beta^* Q)}{\partial \alpha} \right] + \gamma_a \left[2 \frac{\partial^2 Q}{\partial \alpha^* \partial \alpha} \right. \\ \left. + \frac{\partial(\alpha^* Q)}{\partial \alpha^*} + \frac{\partial(\alpha Q)}{\partial \alpha} \right] + \gamma_b \left[2 \frac{\partial^2 Q}{\partial \beta^* \partial \beta} + \frac{\partial(\beta^* Q)}{\partial \beta^*} + \frac{\partial(\beta Q)}{\partial \beta} \right] \end{aligned}$$

or

$$\begin{aligned} \frac{\partial Q(\alpha^*, \alpha, \beta^*, \beta, t)}{\partial t} = \left[\kappa \gamma_o \left(\frac{\partial^2}{\partial \alpha \partial \beta} + \frac{\partial^2}{\partial \alpha^* \partial \beta^*} + \frac{\partial}{\partial \beta^*} \alpha + \frac{\partial}{\partial \alpha^*} \beta + \frac{\partial}{\partial \beta} \alpha^* + \frac{\partial}{\partial \alpha} \beta^* \right) \right. \\ \left. + \gamma_a \left(2 \frac{\partial^2}{\partial \alpha^* \partial \alpha} + \frac{\partial}{\partial \alpha^*} \alpha^* + \frac{\partial}{\partial \alpha} \alpha \right) + \gamma_b \left(2 \frac{\partial^2}{\partial \beta^* \partial \beta} \right. \right. \\ \left. \left. + \frac{\partial}{\partial \beta^*} \beta^* + \frac{\partial}{\partial \beta} \beta \right) \right] Q(\alpha^*, \alpha, \beta^*, \beta, t). \quad (3.8) \end{aligned}$$

Introducing Cartesian coordinates defined by

$$\alpha = x_1 + iy_1 \quad (3.9a)$$

and

$$\beta = x_2 + iy_2, \quad (3.9b)$$

the above Fokker-Planck equation can be put in the form

$$\begin{aligned} \frac{\partial Q(x_1, x_2, y_1, y_2, t)}{\partial t} = \left[\frac{\gamma_a}{2} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} \right) + \frac{\gamma_b}{2} \left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2} \right) \right. \\ \left. + \frac{\kappa \gamma_o}{2} \left(\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{\partial y_1 \partial y_2} \right) + \kappa \gamma_o \left(\frac{\partial}{\partial x_1} x_2 + \frac{\partial}{\partial x_2} x_1 - \frac{\partial}{\partial y_1} y_2 - \frac{\partial}{\partial y_2} y_1 \right) \right. \\ \left. + \gamma_a \left(\frac{\partial}{\partial x_1} x_1 + \frac{\partial}{\partial y_1} y_1 \right) + \gamma_b \left(\frac{\partial}{\partial x_2} x_2 + \frac{\partial}{\partial y_2} y_2 \right) \right] Q(x_1, x_2, y_1, y_2, t). \quad (3.10) \end{aligned}$$

Assuming equal cavity damping rates for both the signal and idler modes ($\gamma_a = \gamma_b = \gamma$), we have

$$\frac{\partial Q(x_1, x_2, y_1, y_2, t)}{\partial t} = \left[\frac{\gamma}{2} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2} \right) + \frac{\kappa \gamma_o}{2} \left(\frac{\partial^2}{\partial x_1 \partial x_2} \right. \right.$$

$$\begin{aligned}
& - \frac{\partial^2}{\partial y_1 \partial y_2} \Big) + \kappa \gamma_0 \left(\frac{\partial}{\partial x_1} x_2 + \frac{\partial}{\partial x_2} x_1 - \frac{\partial}{\partial y_1} y_2 - \frac{\partial}{\partial y_2} y_1 \right) + \gamma \left(\frac{\partial}{\partial x_1} x_1 \right. \\
& \left. + \frac{\partial}{\partial y_1} y_1 + \frac{\partial}{\partial x_2} x_2 + \frac{\partial}{\partial y_2} y_2 \right) \Big] Q(x_1, x_2, y_1, y_2, t). \tag{3.11}
\end{aligned}$$

In order to obtain the solution of this differential equation applying the propagator method, we introduce the transformation defined by

$$x_1 = x + u, \tag{3.12a}$$

$$x_2 = x - u, \tag{3.12b}$$

$$y_1 = v + y, \tag{3.12c}$$

$$y_2 = v - y \tag{3.12d}$$

so that with the aid of the relation

$$\frac{\partial}{\partial x_1} = \frac{\partial x}{\partial x_1} \frac{\partial}{\partial x} + \frac{\partial u}{\partial x_1} \frac{\partial}{\partial u}, \tag{3.13}$$

one readily obtains

$$\frac{\partial}{\partial x_1} = \frac{1}{2} \left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \right], \tag{3.14a}$$

$$\frac{\partial}{\partial x_2} = \frac{1}{2} \left[\frac{\partial}{\partial x} - \frac{\partial}{\partial u} \right], \tag{3.14b}$$

$$\frac{\partial}{\partial y_1} = \frac{1}{2} \left[\frac{\partial}{\partial v} + \frac{\partial}{\partial y} \right], \tag{3.14c}$$

$$\frac{\partial}{\partial y_2} = \frac{1}{2} \left[\frac{\partial}{\partial v} - \frac{\partial}{\partial y} \right]. \tag{3.14d}$$

It then follows that

$$\frac{\partial^2}{\partial x_1^2} = \frac{1}{4} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial u^2} + 2 \frac{\partial^2}{\partial x \partial u} \right], \tag{3.15a}$$

$$\frac{\partial^2}{\partial x_2^2} = \frac{1}{4} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial u^2} - 2 \frac{\partial^2}{\partial x \partial u} \right], \tag{3.15b}$$

$$\frac{\partial^2}{\partial y_1^2} = \frac{1}{4} \left[\frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial y^2} + 2 \frac{\partial^2}{\partial v \partial y} \right], \tag{3.15c}$$

$$\frac{\partial^2}{\partial y_2^2} = \frac{1}{4} \left[\frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial y^2} - 2 \frac{\partial^2}{\partial v \partial y} \right], \quad (3.15d)$$

$$\frac{\partial^2}{\partial x_1 \partial x_2} = \frac{1}{4} \left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial u^2} \right], \quad (3.15e)$$

$$\frac{\partial^2}{\partial y_1 \partial y_2} = \frac{1}{4} \left[\frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial y^2} \right]. \quad (3.15f)$$

Then combining (3.11), (3.12), (3.14) and (3.15), we find

$$\begin{aligned} \frac{\partial Q(x, y, u, v, t)}{\partial t} = & \left[\frac{\gamma}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + \frac{\kappa\gamma_0}{8} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right. \right. \\ & \left. \left. - \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) + \gamma \left(\frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial u} u + \frac{\partial}{\partial v} v \right) + \kappa\gamma_0 \left(\frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y \right. \right. \\ & \left. \left. - \frac{\partial}{\partial u} u - \frac{\partial}{\partial v} v \right) \right] Q(x, y, u, v, t) \end{aligned}$$

or

$$\begin{aligned} \frac{\partial Q(x, y, u, v, t)}{\partial t} = & \left[\left(\frac{\kappa\gamma_0 + 2\gamma}{8} \right) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \left(\frac{\kappa\gamma_0 - 2\gamma}{8} \right) \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \right. \\ & \left. + (\kappa\gamma_0 + \gamma) \left(\frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y \right) - (\kappa\gamma_0 - \gamma) \left(\frac{\partial}{\partial u} u + \frac{\partial}{\partial v} v \right) \right] Q(x, y, u, v, t). \quad (3.16) \end{aligned}$$

Now replacing $(x, y, u, v, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial u}, \frac{\partial}{\partial v}, Q(x, y, u, v, t))$ by $(\hat{x}, \hat{y}, \hat{u}, \hat{v}, i\hat{p}_x, i\hat{p}_y, i\hat{p}_u, i\hat{p}_v,$

$|Q(t)\rangle)$, Equation (3.16) can be put in the form

$$\begin{aligned} i \frac{d|Q(t)\rangle}{dt} = & i \left[-\frac{\lambda_1}{8} (\hat{p}_x^2 + \hat{p}_y^2) + \frac{\lambda_2}{8} (\hat{p}_u^2 + \hat{p}_v^2) + i\lambda_3 (\hat{p}_x \hat{x} + \hat{p}_y \hat{y}) \right. \\ & \left. - i\lambda_4 (\hat{p}_u \hat{u} + \hat{p}_v \hat{v}) \right] |Q(t)\rangle, \quad (3.17) \end{aligned}$$

where

$$\lambda_1 = \kappa\gamma_0 + 2\gamma, \quad (3.18a)$$

$$\lambda_2 = \kappa\gamma_0 - 2\gamma, \quad (3.18b)$$

$$\lambda_3 = \kappa\gamma_0 + \gamma, \quad (3.18c)$$

$$\lambda_4 = \kappa\gamma_0 - \gamma. \quad (3.18d)$$

The solution of (3.17) is expressible as

$$|Q(t)\rangle = \hat{U}|Q(0)\rangle, \quad (3.19)$$

in which

$$\hat{U} = \exp(-i\hat{H}t) \quad (3.20)$$

and

$$\hat{H} = -i\frac{\lambda_1}{8}(\hat{p}_x^2 + \hat{p}_y^2) + i\frac{\lambda_2}{8}(\hat{p}_u^2 + \hat{p}_v^2) - \lambda_3(\hat{p}_x\hat{x} + \hat{p}_y\hat{y}) + \lambda_4(\hat{p}_u\hat{u} + \hat{p}_v\hat{v}). \quad (3.21)$$

Multiplying (3.19) by $\langle x, y, u, v|$ from the left, we have

$$Q(x, y, u, v, t) = \langle x, y, u, v, |\hat{U}|Q(o)\rangle$$

and introducing the four dimensional completeness relation for the position eigenstates

$$I = \int dx' dy' du' dv' |v', u', x', y'\rangle \langle x', y', u', v'|, \quad (3.22)$$

we obtain

$$Q(x, y, u, v, t) = \int dx' dy' du' dv' Q(x, y, u, v, t|x', y', u', v', 0) Q_o(x', y', u', v'), \quad (3.23)$$

where

$$Q_o(x', y', u', v') = \langle x', y', u', v'|Q(o)\rangle \quad (3.24)$$

and

$$Q(x, y, u, v, t|x', y', u', v', 0) = \langle x, y, u, v|\hat{U}|v', u', y', x'\rangle. \quad (3.25)$$

Next we wish to find the Q-function propagator (3.25) applying the method we employed in section 2.1. The propagator for a quadratic Hamiltonian of the form

$$\hat{H} = a_x\hat{p}_x + a_y\hat{p}_y + a_u\hat{p}_u + a_v\hat{p}_v + b_x(t)\hat{p}_x\hat{x} + b_y(t)\hat{p}_y\hat{y} + b_u(t)\hat{p}_u\hat{u} + b_v(t)\hat{p}_v\hat{v}$$

$$+ c_x(t)\hat{x}^2 + c_y(t)\hat{y}^2 + c_u(t)\hat{u}^2 + c_v(t)\hat{v}^2$$

is expressible as [1]

$$K(x, y, u, v, t | x', y', u', v', t) = \left[\frac{i}{2\pi} \frac{\partial^2 S_c}{\partial x' \partial x} \right]^{\frac{1}{2}} \left[\frac{i}{2\pi} \frac{\partial^2 S_c}{\partial y' \partial y} \right]^{\frac{1}{2}} \left[\frac{i}{2\pi} \frac{\partial^2 S_c}{\partial u' \partial u} \right]^{\frac{1}{2}} \left[\frac{i}{2\pi} \frac{\partial^2 S_c}{\partial v' \partial v} \right]^{\frac{1}{2}} \\ \times \exp \left[-\xi \int_0^t \left[b_x(t') + b_y(t') + b_u(t') + b_v(t') \right] dt' + iS_c \right]. \quad (3.26)$$

Since from (3.21) $b_x = b_y = -\lambda_3$, $b_u = b_v = \lambda_4$ and recalling that $\xi = \frac{1}{2}$, the Q-function propagator associated with the Hamiltonian (3.21) can be written as

$$Q(x, y, u, v, t | x', y', u', v', 0) = e^{(\lambda_3 - \lambda_4)t} \left[\frac{i}{2\pi} \frac{\partial^2 S_c}{\partial x' \partial x} \right]^{\frac{1}{2}} \left[\frac{i}{2\pi} \frac{\partial^2 S_c}{\partial y' \partial y} \right]^{\frac{1}{2}} \left[\frac{i}{2\pi} \frac{\partial^2 S_c}{\partial u' \partial u} \right]^{\frac{1}{2}} \\ \times \left[\frac{i}{2\pi} \frac{\partial^2 S_c}{\partial v' \partial v} \right]^{\frac{1}{2}} e^{iS_c}. \quad (3.27)$$

We now proceed to get the classical action. To this end, we note that the ‘‘Hamiltonian function’’ associated with the quantum Hamiltonian (3.21) is

$$H = -i \frac{\lambda_1}{8} (p_x^2 + p_y^2) + i \frac{\lambda_2}{8} (p_u^2 + p_v^2) - \lambda_3 (p_x x + p_y y) + \lambda_4 (p_u u + p_v v) \quad (3.28)$$

and hence the corresponding ‘‘Lagrangian’’ is

$$L = \dot{x}p_x + \dot{y}p_y + \dot{u}p_u + \dot{v}p_v - H. \quad (3.29)$$

Applying Hamilton’s equation

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

one readily gets

$$p_x = \frac{4i(\dot{x} + \lambda_3 x)}{\lambda_1}, \\ p_y = \frac{4i(\dot{y} + \lambda_3 y)}{\lambda_1}, \\ p_u = -\frac{4i(\dot{u} - \lambda_4 u)}{\lambda_2},$$

$$p_v = -\frac{4i(\dot{v} - \lambda_4 v)}{\lambda_2}.$$

Hence, employing these expressions along with (3.28) in (3.29), one gets

$$L = \frac{2i}{\lambda_1} [(\dot{x} + \lambda_3 x)^2 + (\dot{y} + \lambda_3 y)^2] - \frac{2i}{\lambda_2} [(i - \lambda_4 u)^2 + (\dot{v} - \lambda_4 v)^2]. \quad (3.30)$$

Upon using (3.30) and the Euler-Lagrange equations, we obtain

$$\ddot{x} - \lambda_3^2 x = 0,$$

$$\ddot{y} - \lambda_3^2 y = 0,$$

$$\ddot{u} - \lambda_4^2 u = 0,$$

$$\ddot{v} - \lambda_4^2 v = 0.$$

The solutions for these equations can be written

$$x = Ae^{\lambda_3 t} + Be^{-\lambda_3 t}, \quad (3.31a)$$

$$y = Ge^{\lambda_3 t} + He^{-\lambda_3 t}, \quad (3.31b)$$

$$u = Ce^{\lambda_4 t} + De^{-\lambda_4 t}, \quad (3.31c)$$

$$v = Ee^{\lambda_4 t} + Fe^{-\lambda_4 t} \quad (3.31d)$$

respectively. Then substitution of these expressions along with their time derivatives into (3.30) leads to

$$L = \frac{8i\lambda_3^2}{\lambda_1} (A^2 + G^2)e^{2\lambda_3 t} - \frac{8i\lambda_4^2}{\lambda_1} (D^2 + F^2)e^{-2\lambda_4 t}. \quad (3.32)$$

Now employing (3.32) in the relation

$$S_c = \int_0^T L(t) dt$$

and carrying out the integration, the classical action takes the form

$$S_c = \frac{4i\lambda_3}{\lambda_1} (e^{2\lambda_3 T} - 1) (A^2 + G^2) + \frac{4i\lambda_4}{\lambda_2} (e^{-2\lambda_4 T} - 1) (D^2 + F^2). \quad (3.33)$$

Setting

$$[x(0), y(0), u(0), v(0)] = [x', y', u', v']$$

and

$$[x(T), y(T), u(T), v(T)] = [x'', y'', u'', v'']$$

from (3.31), one easily finds

$$A = - \left[\frac{x' - x'' e^{\lambda_3 T}}{e^{2\lambda_3 T} - 1} \right],$$

$$G = - \left[\frac{y' - y'' e^{\lambda_3 T}}{e^{2\lambda_3 T} - 1} \right],$$

$$D = - \left[\frac{u' - u'' e^{-\lambda_4 T}}{e^{-2\lambda_4 T} - 1} \right],$$

$$F = - \left[\frac{v' - v'' e^{-\lambda_4 T}}{e^{-2\lambda_4 T} - 1} \right].$$

On account of these expressions the classical action (3.33) can then be written as

$$S_c = \frac{4i\lambda_3}{\lambda_1} \left[\frac{(x' - x e^{\lambda_3 t})^2}{e^{2\lambda_3 t} - 1} + \frac{(y' - y e^{\lambda_3 t})^2}{e^{2\lambda_3 t} - 1} \right] + \frac{4i\lambda_4}{\lambda_2} \left[\frac{(u' - u e^{-\lambda_4 t})^2}{e^{-2\lambda_4 t} - 1} + \frac{(v' - v e^{-\lambda_4 t})^2}{e^{-2\lambda_4 t} - 1} \right], \quad (3.34)$$

where we replaced (x'', y'', u'', v'', T) by (x, y, u, v, t) . Now applying (3.34), we have

$$\frac{\partial^2 S_c}{\partial x' \partial x} = \frac{\partial^2 S_c}{\partial y' \partial y} = \frac{-8i\lambda_3}{\lambda_1} \frac{e^{\lambda_3 t}}{e^{2\lambda_3 t} - 1} \quad (3.35a)$$

and

$$\frac{\partial^2 S_c}{\partial u' \partial u} = \frac{\partial^2 S_c}{\partial v' \partial v} = \frac{-8i\lambda_4}{\lambda_2} \frac{e^{-\lambda_4 t}}{e^{-2\lambda_4 t} - 1} \quad (3.35b)$$

so that combination of (3.27), (3.34) and (3.35) leads to

$$\begin{aligned}
Q(x, y, u, v, t | x', y', u', v', 0) &= \left[\frac{16\lambda_3\lambda_4}{\pi^2\lambda_1\lambda_2(1 - e^{-2\lambda_3 t})(1 - e^{2\lambda_4 t})} \right] \\
&\times \exp \left[\frac{-4\lambda_3}{\lambda_1(1 - e^{-2\lambda_3 t})} (x^2 + x'^2 e^{-2\lambda_3 t} - 2x'x e^{-\lambda_3 t} + y^2 + y'^2 e^{-2\lambda_3 t} - 2y'y e^{-\lambda_3 t}) \right. \\
&\left. + \frac{-4\lambda_4}{\lambda_2(1 - e^{2\lambda_4 t})} (u^2 + u'^2 e^{2\lambda_4 t} - 2u'u e^{\lambda_4 t} + v^2 + v'^2 e^{2\lambda_4 t} - 2v'v e^{\lambda_4 t}) \right]. \quad (3.36)
\end{aligned}$$

Equation (3.36) is the Q-function propagator for the signal-idler modes. Although we are interested in the case for which the signal-idler modes are initially in a vacuum state, we also need the Q-function satisfying the initial condition

$$Q_o(\alpha', \beta') = \frac{1}{\pi^2} \exp \left[-|\alpha' - \alpha_o|^2 - |\beta' - \beta_o|^2 \right]$$

or, in terms of the Cartesian variables (3.9)

$$Q_o(x'_1, x'_2, y'_1, y'_2) = \frac{1}{\pi^2} \exp \left[-(x'_1 - x_{1o})^2 - (x'_2 - x_{2o})^2 - (y'_1 - y_{1o})^2 - (y'_2 - y_{2o})^2 \right].$$

This can also be expressed in terms of x', y', u' and v' such that

$$\int dx'_1 dx'_2 dy'_1 dy'_2 Q_o(x'_1, x'_2, y'_1, y'_2) = \int dx' dy' du' dv' Q_o(x', y', u', v'),$$

where

$$Q_o(x', y', u', v') = |J| \exp \left[-2(x' - x_o)^2 - 2(y' - y_o)^2 - 2(u' - u_o)^2 - 2(v' - v_o)^2 \right] \quad (3.37)$$

and J is the Jacobian of the transformation given by

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial x} & \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial x} & \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \\ \frac{\partial y_1}{\partial x} & \frac{\partial y_1}{\partial y} & \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial v} \\ \frac{\partial y_2}{\partial x} & \frac{\partial y_2}{\partial y} & \frac{\partial y_2}{\partial u} & \frac{\partial y_2}{\partial v} \end{vmatrix}$$

so that using (3.12) one finds $|J| = 4$. Consequently, we see that

$$Q_o(x', y', u', v') = \frac{4}{\pi^2} \exp \left[-2(x' - x_o)^2 - 2(y' - y_o)^2 - 2(u' - u_o)^2 - 2(v' - v_o)^2 \right]. \quad (3.38)$$

Now substituting (3.36) along with (3.38) into (3.23), we have

$$\begin{aligned} Q(x, y, u, v, t) = & \left[\frac{64\lambda_3\lambda_4}{\pi^4\lambda_1\lambda_2(1-e^{-2\lambda_3t})(1-e^{2\lambda_4t})} \right] \exp \left(\frac{-4\lambda_3x^2}{\lambda_1(1-e^{-2\lambda_3t})} - 2x_o^2 \right) \\ & \times \int_{-\infty}^{\infty} dx' \exp \left[- \left(\frac{-4\lambda_3e^{-2\lambda_3t}}{\lambda_1(1-e^{-2\lambda_3t})} - 2 \right) x'^2 + \exp \left(\frac{8\lambda_3e^{-\lambda_3t}}{\lambda_1(1-e^{-2\lambda_3t})} + 4x_o \right) x' \right] \\ & \times \exp \left(\frac{-4\lambda_3y^2}{\lambda_1(1-e^{-2\lambda_3t})} - 2y_o^2 \right) \int_{-\infty}^{\infty} dy' \exp \left[- \left(\frac{-4\lambda_3e^{-2\lambda_3t}}{\lambda_1(1-e^{-2\lambda_3t})} - 2 \right) y'^2 \right. \\ & \left. + \exp \left(\frac{8\lambda_3e^{-\lambda_3t}}{\lambda_1(1-e^{-2\lambda_3t})} + 4y_o \right) y' \right] \exp \left(\frac{-4\lambda_4u^2}{\lambda_2(1-e^{2\lambda_4t})} - 2u_o^2 \right) \\ & \times \int_{-\infty}^{\infty} du' \exp \left[- \left(\frac{-4\lambda_4e^{2\lambda_4t}}{\lambda_2(1-e^{2\lambda_4t})} - 2 \right) u'^2 + \exp \left(\frac{8\lambda_4e^{\lambda_4t}}{\lambda_2(1-e^{2\lambda_4t})} + 4u_o \right) u' \right] \\ & \times \exp \left(\frac{-4\lambda_4v^2}{\lambda_2(1-e^{2\lambda_4t})} - 2v_o^2 \right) \int_{-\infty}^{\infty} dv' \exp \left[- \left(\frac{-4\lambda_4e^{2\lambda_4t}}{\lambda_2(1-e^{2\lambda_4t})} - 2 \right) v'^2 \right. \\ & \left. + \exp \left(\frac{8\lambda_4e^{\lambda_4t}}{\lambda_2(1-e^{2\lambda_4t})} + 4v_o \right) v' \right] \end{aligned}$$

and carrying out the integrations with the aid of the relation (2.22), we find

$$\begin{aligned} Q(x, y, u, v, t) = & \frac{16\lambda_3\lambda_4}{\pi^2 [\lambda_1 + (2\lambda_3 - \lambda_1)e^{-2\lambda_3t}] [\lambda_2 + (2\lambda_4 - \lambda_2)e^{2\lambda_4t}]} \\ & \times \exp \left[\frac{-4\lambda_3 \left[(x - x_o e^{-\lambda_3t})^2 + (y - y_o e^{-\lambda_3t})^2 \right]}{\lambda_1 + (2\lambda_3 - \lambda_1)e^{-2\lambda_3t}} \right. \\ & \left. + \frac{-4\lambda_4 \left[(u - u_o e^{\lambda_4t})^2 + (v - v_o e^{\lambda_4t})^2 \right]}{\lambda_2 + (2\lambda_4 - \lambda_2)e^{2\lambda_4t}} \right]. \end{aligned}$$

Applying the inverse of the transformation described by (3.12) one can put the Q-function in the form

$$\begin{aligned} Q'(x_1, x_2, y_1, y_2, t) = & \frac{16\lambda_3\lambda_4}{\pi^2 [\lambda_1 + (2\lambda_3 - \lambda_1)e^{-2\lambda_3t}] [\lambda_2 + (2\lambda_4 - \lambda_2)e^{2\lambda_4t}]} \\ & \times \exp \left[\frac{-\lambda_3 \left[(x_1 + x_2 - (x_{1o} + x_{2o})e^{-\lambda_3t})^2 + (y_1 - y_2 - (y_{1o} - y_{2o})e^{-\lambda_3t})^2 \right]}{\lambda_1 + (2\lambda_3 - \lambda_1)e^{-2\lambda_3t}} \right] \end{aligned}$$

$$+ \frac{-\lambda_4 \left[(x_1 - x_2 - (x_{1o} - x_{2o}) e^{\lambda_3 t})^2 + (y_1 + y_2 - (y_{1o} + y_{2o}) e^{\lambda_4 t})^2 \right]}{\lambda_2 + (2\lambda_4 - \lambda_2) e^{2\lambda_4 t}} \Big]. \quad (3.39)$$

It can be easily verified that the Jacobian of the inverse transformation is $J = \frac{1}{4}$. We can then write

$$\int dx dy du dv Q(x, y, u, v, t) = \int dx_1 dx_2 dy_1 dy_2 Q(x_1, x_2, y_1, y_2, t),$$

where

$$Q(x_1, x_2, y_1, y_2, t) = \frac{1}{4} Q'(x_1, x_2, y_1, y_2, t). \quad (3.40)$$

Now taking into account (3.9), (3.39) and (3.40), we get

$$Q(\alpha, \beta, t) = \frac{A(t)}{\pi^2} \exp[-B(t)(\alpha^* \alpha + \beta^* \beta) + C(t)(\alpha \beta + \alpha^* \beta^*) + D(t)\alpha + E(t)\beta + D^*(t)\alpha^* + E^*(t)\beta^*], \quad (3.41)$$

where

$$A(t) = [B^2(t) - C^2(t)] \exp[-(be^{2\lambda_4 t} + ae^{-2\lambda_3 t})(\alpha_o^* \alpha_o + \beta_o^* \beta_o) + (be^{2\lambda_4 t} - ae^{-2\lambda_3 t})(\alpha_o \beta_o + \alpha_o^* \beta_o^*)], \quad (3.42a)$$

$$B(t) = b + a, \quad (3.42b)$$

$$C(t) = b - a, \quad (3.42c)$$

$$D(t) = (be^{\lambda_4 t} + ae^{-\lambda_3 t}) \alpha_o^* - (be^{\lambda_4 t} - ae^{-\lambda_3 t}) \beta_o, \quad (3.42d)$$

$$E(t) = (be^{\lambda_4 t} + ae^{-\lambda_3 t}) \beta_o^* - (be^{\lambda_4 t} - ae^{-\lambda_3 t}) \alpha_o, \quad (3.42e)$$

with

$$a = \frac{\lambda_3}{\lambda_1 + (2\lambda_3 - \lambda_1) e^{-2\lambda_3 t}} \quad (3.43a)$$

and

$$b = \frac{\lambda_4}{\lambda_2 + (2\lambda_4 - \lambda_2) e^{2\lambda_4 t}}. \quad (3.43b)$$

Since we are interested in the case for which the signal-idler modes are initially in a vacuum state, setting $\alpha_o = \alpha_o^* = \beta_o = \beta_o^* = 0$ in (3.41), yields

$$Q(\alpha, \beta, t) = \left[\frac{B^2(t) - C^2(t)}{\pi^2} \right] \exp[-B(t)(\alpha^* \alpha + \beta^* \beta) + C(t)(\alpha \beta + \alpha^* \beta^*)]. \quad (3.44)$$

Applying (3.44) the Q-function for the signal mode only is expressible as

$$Q(\alpha, \alpha^*, t) = \left[\frac{B^2(t) - C^2(t)}{\pi} \right] e^{-B(t)(\alpha^* \alpha)} \int \frac{d^2 \beta}{\pi} \exp[-B(t)\beta^* \beta + C(t)\alpha \beta + C(t)\alpha^* \beta^*]$$

with the aid of the relation (2.36), one then finds

$$Q(\alpha, \alpha^*, t) = \frac{1}{\pi} \left[\frac{B^2(t) - C^2(t)}{B(t)} \right] \exp \left[- \left(\frac{B^2(t) - C^2(t)}{B(t)} \right) \alpha^* \alpha \right]. \quad (3.45)$$

3.2 Quadrature squeezing of the signal-idler modes

In this section we wish to study the quadrature fluctuations of the signal-idler modes. To this end, we define the intracavity quadrature operators for the signal-idler modes as

$$c_1 = \frac{1}{\sqrt{2}}(a_1 + b_1) \quad (3.46a)$$

and

$$c_2 = \frac{1}{\sqrt{2}}(a_2 + b_2), \quad (3.46b)$$

where (a_1, a_2) and (b_1, b_2) are the quadrature operators for the signal and idler modes defined according to (2.28b), respectively. Then one can easily verify that the variances of these quadrature operators are expressible in the form

$$(\Delta c_1)^2 = \frac{1}{2}(\Delta a_1)^2 + \frac{1}{2}(\Delta b_1)^2 + \langle a_1, b_1 \rangle \quad (3.47a)$$

and

$$(\Delta c_2)^2 = \frac{1}{2}(\Delta a_2)^2 + \frac{1}{2}(\Delta b_2)^2 + \langle a_2, b_2 \rangle, \quad (3.47b)$$

in which

$$(\Delta a_1)^2 = 1 + 2\langle a^\dagger a \rangle + \langle a^{\dagger 2} \rangle + \langle a^2 \rangle - \langle a \rangle^2 - \langle a^\dagger \rangle^2 - 2\langle a^\dagger \rangle \langle a \rangle, \quad (3.48a)$$

$$(\Delta a_2)^2 = 1 + 2\langle a^\dagger a \rangle - \langle a^{\dagger 2} \rangle - \langle a^2 \rangle + \langle a \rangle^2 + \langle a^\dagger \rangle^2 - 2\langle a^\dagger \rangle \langle a \rangle, \quad (3.48b)$$

$$\langle a_1, b_1 \rangle = \langle a_1 b_1 \rangle - \langle a_1 \rangle \langle b_1 \rangle \quad (3.48c)$$

with similar expressions for $(\Delta b_1)^2$, $(\Delta b_2)^2$ and $\langle a_2, b_2 \rangle$.

Applying the Q-function for the signal mode only (3.45), we can write

$$\langle a \rangle = \left[\frac{B^2(t) - C^2(t)}{B(t)} \right] \int \frac{d^2\alpha}{\pi} \alpha \exp \left[- \left(\frac{B^2(t) - C^2(t)}{B(t)} \right) \alpha^* \alpha \right],$$

$$\langle a^2 \rangle = \left[\frac{B^2(t) - C^2(t)}{B(t)} \right] \int \frac{d^2\alpha}{\pi} \alpha^2 \exp \left[- \left(\frac{B^2(t) - C^2(t)}{B(t)} \right) \alpha^* \alpha \right]$$

and

$$\langle a^\dagger a \rangle = \left[\frac{B^2(t) - C^2(t)}{B(t)} \right] \int \frac{d^2\alpha}{\pi} \alpha^* \alpha \exp \left[- \left(\frac{B^2(t) - C^2(t)}{B(t)} \right) \alpha^* \alpha \right] - 1$$

so that on account of the relation (2.36), we find

$$\langle a \rangle = \langle a^2 \rangle = 0 \quad (3.49a)$$

and

$$\langle a^\dagger a \rangle = \frac{B(t)}{B^2(t) - C^2(t)} - 1 \quad (3.49b)$$

Similarly, one finds

$$\langle a^\dagger \rangle = \langle a^{\dagger 2} \rangle = 0 \quad (3.49c)$$

In view of the above results, expressions (3.48a) and (3.48b) take the form

$$(\Delta a_1)^2 = (\Delta a_2)^2 = \frac{2B(t)}{B^2(t) - C^2(t)} - 1. \quad (3.50)$$

Since the Q-function for the idler mode is identical to that of the signal mode, we see that

$$(\Delta b_1)^2 = (\Delta b_2)^2 = \frac{2B(t)}{B^2(t) - C^2(t)} - 1. \quad (3.51)$$

Employing (2.28a) in (3.48c) and taking into account the results (3.49a) and (3.49c), we have

$$\langle a_1, b_1 \rangle = \langle (a + a^\dagger)(b + b^\dagger) \rangle$$

and using the Q-function (3.44), we can then write that

$$\langle a_1, b_1 \rangle = (B^2 - C^2) \int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi} (\alpha + \alpha^*)(\beta + \beta^*) \exp[-B(\alpha^*\alpha + \beta^*\beta) + C(\alpha\beta + \alpha^*\beta^*)].$$

Replacing $C\alpha$ by C' in the exponential part, we can put this integral in the form

$$\langle a_1, b_1 \rangle = (B^2(t) - C^2(t)) \int \frac{d^2\alpha}{\pi} (\alpha + \alpha^*) e^{-B(t)(\alpha^*\alpha)} \left[\frac{\partial}{\partial C'} + \frac{\partial}{\partial C'^*} \right] \times \int \frac{d^2\beta}{\pi} \exp[-B\beta^*\beta + C'\beta + C'^*\beta^*]$$

so that integration over β using (2.36) yields

$$\langle a_1, b_1 \rangle = (B^2(t) - C^2(t)) \int \frac{d^2\alpha}{\pi} (\alpha + \alpha^*) e^{-B\alpha^*\alpha} \left[\frac{\partial}{\partial C'} + \frac{\partial}{\partial C'^*} \right] \frac{e^{\frac{C'C'^*}{B}}}{B}.$$

Differentiating with respect to C' , C'^* and replacing (C', C'^*) by $(C\alpha, C\alpha^*)$, one then finds

$$\langle a_1, b_1 \rangle = \left[\frac{C(B^2 - C^2)}{B^2} \right] \int \frac{d^2\alpha}{\pi} (\alpha^2 + 2\alpha^*\alpha + \alpha^{*2}) \exp \left[- \left(\frac{B^2 - C^2}{B} \right) \alpha^*\alpha \right]. \quad (3.52)$$

The result of carrying out this integration, taking into account (3.49), leads to

$$\langle a_1, b_1 \rangle = \frac{2C(t)}{B^2(t) - C^2(t)}. \quad (3.53)$$

Similarly one readily verify that

$$\langle a_2, b_2 \rangle = -\frac{2C(t)}{B^2(t) - C^2(t)}. \quad (3.54)$$

Finally, combining (3.47), (3.50), (3.51), (3.53) and (3.54), we have

$$(\Delta c_1)^2 = \frac{2}{B(t) - C(t)} - 1$$

and

$$(\Delta c_2)^2 = \frac{2}{B(t) + C(t)} - 1$$

so that using the explicit forms of $B(t)$ and $C(t)$ from (3.42b) and (3.42c) along with (3.43), we arrive at

$$(\Delta c_1)^2 = \frac{\gamma + \kappa\gamma_0 e^{-2(\gamma+\kappa\gamma_0)t}}{\gamma + \kappa\gamma_0} \quad (3.55a)$$

and

$$(\Delta c_2)^2 = \frac{\gamma - \kappa\gamma_0 e^{-2(\gamma-\kappa\gamma_0)t}}{\gamma - \kappa\gamma_0}. \quad (3.55b)$$

At steady state (3.55) takes the form

$$(\Delta c_1)^2 = \frac{\gamma}{\gamma + \kappa\gamma_0} \quad (3.56a)$$

and

$$(\Delta c_2)^2 = \frac{\gamma}{\gamma - \kappa\gamma_0}. \quad (3.56b)$$

When the amplitude of the pump mode $\gamma_0 \rightarrow \frac{\gamma}{\kappa}$, we find

$$(\Delta c_1)^2 = \frac{1}{2} \quad (3.57a)$$

and

$$(\Delta c_2)^2 \rightarrow \infty. \quad (3.57b)$$

Equations (3.55), (3.56) and (3.57) show that for $t > 0$ the fluctuations in the first quadrature is below the vacuum level with enhanced fluctuations in the second quadrature. This indicates the signal-idler modes in a NDPO are in a squeezed state. Furthermore, from these results we also note that the Heisenberg uncertainty principle is not violated.

3.3 The photon number distribution of the signal-idler modes

Next we seek to obtain the photon number distribution for the signal-idler modes applying the Q-function obtained in section 3.1. To this end, we generalize Expression (2.54) for a two-mode light as

$$P(n, n) = \left(\frac{\pi}{n!}\right)^2 \frac{\partial^{4n} [Q(\alpha, \alpha^*, \beta, \beta^*, t) e^{\alpha^* \alpha + \beta^* \beta}]}{\partial \alpha^{*n} \partial \alpha^n \partial \beta^{*n} \partial \beta^n} \Big|_{\alpha^* = \alpha = \beta^* = \beta = 0}$$

so that in view of (3.44) the photon number distribution for signal-idler modes can be put in the form

$$P(n, n) = \left[\frac{B^2(t) - C^2(t)}{n!^2} \right] \frac{\partial^{4n} e^{[R(t)(\alpha^* \alpha + \beta^* \beta) + C(t)(\alpha \beta + \alpha^* \beta^*)]}}{\partial \alpha^{*n} \partial \alpha^n \partial \beta^{*n} \partial \beta^n} \Big|_{\alpha^* = \alpha = \beta^* = \beta = 0}, \quad (3.58)$$

in which

$$R(t) = 1 - B(t). \quad (3.59)$$

Upon expanding the exponential functions involved in expression (3.58) in a power series, we have

$$P(n, n) = \left[\frac{B^2(t) - C^2(t)}{n!^2} \right] \sum_{i, k, l, m=0}^{\infty} \frac{R^{k+l}(t) C^{i+m}(t)}{i! k! l! m!} \\ \times \frac{\partial^{4n} [\alpha^{*i+k} \alpha^{k+m} \beta^{*i+l} \beta^{l+m}]}{\partial \alpha^{*n} \partial \alpha^n \partial \beta^{*n} \partial \beta^n} \Big|_{\alpha^* = \alpha = \beta^* = \beta = 0}$$

so that performing the differentiation using the relation (2.57) leads to

$$P(n, n) = \left[\frac{B^2(t) - C^2(t)}{n!^2} \right] \sum_{i,k,l,m=0}^{\infty} \left[\frac{R^{k+l}(t) C^{i+m}(t) (k+m)! (l+m)!}{i! k! l! m! (k+m-n)! (l+m-n)!} \right. \\ \left. \times \frac{(i+k)! (i+l)!}{(i+k-n)! (i+l-n)!} \alpha^{*i+k-n} \alpha^{k+m-n} \beta^{*i+l-n} \beta^{l+m-n} \right] \Big|_{\alpha^*=\alpha=\beta^*=\beta=0}. \quad (3.60)$$

This expression is different from zero provided that

$$k + m - n = 0,$$

$$l + m - n = 0,$$

$$i + k - n = 0,$$

$$i + l - n = 0.$$

It then follows that

$$i = m, \quad (3.61a)$$

$$k = n - m, \quad (3.61b)$$

$$l = n - m. \quad (3.61c)$$

Substitution of (3.61) into (3.60) yields

$$P(n, n) = [B^2(t) - C^2(t)] \sum_{m=0}^{\infty} \left[\frac{n!}{m! (n-m)!} \right]^2 R^{2(n-m)}(t) C^{2m}(t)$$

Finally, noting that $(n-m)!$ is defined for $m \leq n$ and using (3.59), this expression can be put in the form

$$P(n, n) = [B^2(t) - C^2(t)] \sum_{m=0}^n \left[\frac{n!}{m! (n-m)!} \right]^2 [1 - B(t)]^{2(n-m)} C^{2m}(t). \quad (3.62)$$

This is the photon number distribution for the signal-idler modes generated in a NDPO. In the absence of cavity damping, (3.43a) and (3.43b) reduces to

$$a = \frac{1}{1 + e^{-\kappa\gamma_0 t}}$$

and

$$b = \frac{1}{1 + e^{\kappa\gamma_0 t}}$$

so on account of these relations (3.42a) and (3.42b) take the form

$$B(t) = 1$$

and

$$C(t) = -\tanh(\kappa\gamma_0 t).$$

Therefore the photon number distribution (3.62) turns out to be

$$P(n, n) = \left[\frac{\tanh^n(\kappa\gamma_0 t)}{\cosh(\kappa\gamma_0 t)} \right]^2. \quad (3.63)$$

This expression is identical to the photon number distribution of a two-mode squeezed vacuum [60]

3.4 The spectrum of the intensity-difference fluctuations

Here we wish to calculate the spectrum of the intensity-difference fluctuations of the signal-idler modes. To this end, the spectrum of intensity-difference fluctuations is defined as

$$S_d(\omega) = \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \langle I_d^{\text{out}}(t + \tau), I_d^{\text{out}}(t) \rangle, \quad (3.64)$$

where

$$\langle I_d^{\text{out}}(t + \tau), I_d^{\text{out}}(t) \rangle = \langle I_d^{\text{out}}(t + \tau) I_d^{\text{out}}(t) \rangle - \langle I_d^{\text{out}}(t + \tau) \rangle \langle I_d^{\text{out}}(t) \rangle \quad (3.65a)$$

and

$$I_d^{\text{out}}(t) = a_{\text{out}}^\dagger(t) a_{\text{out}}(t) - b_{\text{out}}^\dagger(t) b_{\text{out}}(t), \quad (3.65b)$$

in which a_{out} (b_{out}) is the operator describing the output signal (idler) mode and is related to the input mode according to

$$a_{out} = \sqrt{2\gamma}a(t) - a_{in}(t). \quad (3.65c)$$

Combining (3.65) and the relation

$$[a(t + \tau), a^\dagger(t)] = \delta(\tau),$$

it can be easily shown that the output intensity-difference fluctuations for a vacuum input is expressible as

$$\begin{aligned} \langle I_d^{out}(t + \tau), I_d^{out}(t) \rangle = & 4\gamma^2 \left[\langle a^\dagger(t)a^\dagger(t + \tau)a(t + \tau)a(t) \rangle - \langle a^\dagger(t)b^\dagger(t + \tau)b(t + \tau)a(t) \rangle \right. \\ & \left. + \langle b^\dagger(t)b^\dagger(t + \tau)b(t + \tau)b(t) \rangle - \langle b^\dagger(t)a^\dagger(t + \tau)a(t + \tau)b(t) \rangle \right] \\ & + 2\gamma\delta(\tau) \left[\langle a^\dagger(t)a(t + \tau) \rangle + \langle b^\dagger(t)b(t + \tau) \rangle \right] - 4\gamma^2 \left[\langle a^\dagger(t + \tau)a(t + \tau) \rangle \right. \\ & \left. - \langle b^\dagger(t + \tau)b(t + \tau) \rangle \right] \left[\langle a^\dagger(t)a(t) \rangle - \langle b^\dagger(t)b(t) \rangle \right]. \end{aligned}$$

so that in view of the fact that

$$\langle a^\dagger(t)a(t) \rangle = \langle b^\dagger(t)b(t) \rangle,$$

the above equation reduces to

$$\begin{aligned} \langle I_d^{out}(t + \tau), I_d^{out}(t) \rangle = & 4\gamma^2 \left[\langle a^\dagger(t)a^\dagger(t + \tau)a(t + \tau)a(t) \rangle - \langle a^\dagger(t)b^\dagger(t + \tau)b(t + \tau)a(t) \rangle \right. \\ & \left. + \langle b^\dagger(t)b^\dagger(t + \tau)b(t + \tau)b(t) \rangle - \langle b^\dagger(t)a^\dagger(t + \tau)a(t + \tau)b(t) \rangle \right] \\ & + 2\gamma\delta(\tau) \left[\langle a^\dagger(t)a(t + \tau) \rangle + \langle b^\dagger(t)b(t + \tau) \rangle \right]. \quad (3.66) \end{aligned}$$

Now making use of (3.66) one can express (3.64) as

$$S_d(\omega) = 4\gamma^2 \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \langle \hat{f}(t + \tau) \rangle + 2\gamma \left[\langle a^\dagger(t)a(t) \rangle + \langle b^\dagger(t)b(t) \rangle \right], \quad (3.67)$$

in which

$$\begin{aligned}\hat{f}(t + \tau) &= \langle a^\dagger(t)a^\dagger(t + \tau)a(t + \tau)a(t) \rangle - \langle a^\dagger(t)b^\dagger(t + \tau)b(t + \tau)a(t) \rangle \\ &\quad + \langle b^\dagger(t)b^\dagger(t + \tau)b(t + \tau)b(t) \rangle - \langle b^\dagger(t)a^\dagger(t + \tau)a(t + \tau)b(t) \rangle.\end{aligned}\quad (3.68)$$

Applying a similar method we used in section 2.2, one can easily verify that the two-time correlation functions involved in (3.68) are expressible as

$$\begin{aligned}\langle a^\dagger(t)a^\dagger(t + \tau)a(t + \tau)a(t) \rangle &= \int d^2\alpha d^2\beta d^2\lambda d^2\eta Q\left(\alpha^*, \beta^*, \alpha + \frac{\partial}{\partial\alpha^*}, \beta + \frac{\partial}{\partial\beta^*}, t\right) \\ &\quad \times \alpha^* \alpha (\lambda^* \lambda - 1) Q(\lambda^*, \eta^*, \lambda, \eta, \tau)\end{aligned}\quad (3.69a)$$

$$\begin{aligned}\langle a^\dagger(t)b^\dagger(t + \tau)b(t + \tau)a(t) \rangle &= \int d^2\alpha d^2\beta d^2\lambda d^2\eta Q\left(\alpha^*, \beta^*, \alpha + \frac{\partial}{\partial\alpha^*}, \beta + \frac{\partial}{\partial\beta^*}, t\right) \\ &\quad \times \alpha^* \alpha (\eta^* \eta - 1) Q(\lambda^*, \eta^*, \lambda, \eta, \tau)\end{aligned}\quad (3.69b)$$

$$\begin{aligned}\langle b^\dagger(t)b^\dagger(t + \tau)b(t + \tau)b(t) \rangle &= \int d^2\alpha d^2\beta d^2\lambda d^2\eta Q\left(\alpha^*, \beta^*, \alpha + \frac{\partial}{\partial\alpha^*}, \beta + \frac{\partial}{\partial\beta^*}, t\right) \\ &\quad \times \beta^* \beta (\eta^* \eta - 1) Q(\lambda^*, \eta^*, \lambda, \eta, \tau)\end{aligned}\quad (3.69c)$$

and

$$\begin{aligned}\langle b^\dagger(t)a^\dagger(t + \tau)a(t + \tau)b(t) \rangle &= \int d^2\alpha d^2\beta d^2\lambda d^2\eta Q\left(\alpha^*, \beta^*, \alpha + \frac{\partial}{\partial\alpha^*}, \beta + \frac{\partial}{\partial\beta^*}, t\right) \\ &\quad \times \beta^* \beta (\lambda^* \lambda - 1) Q(\lambda^*, \eta^*, \lambda, \eta, \tau),\end{aligned}\quad (3.69d)$$

where $Q(\lambda^*, \eta^*, \lambda, \eta, \tau)$ is the Q-function for the signal-idler modes which takes the form

$$Q_o(\lambda^*, \eta^*, \lambda, \eta) = \exp[-|\lambda - \alpha|^2 - |\eta - \beta|^2]$$

at $\tau = 0$. Thus combination of (3.68) and (3.69) yields

$$\langle \hat{f}(t + \tau) \rangle = \int d^2\alpha d^2\beta d^2\lambda d^2\eta Q\left(\alpha^*, \beta^*, \alpha + \frac{\partial}{\partial\alpha^*}, \beta + \frac{\partial}{\partial\beta^*}, t\right)$$

$$\times (\alpha^* \alpha - \beta^* \beta) (\lambda^* \lambda - \eta^* \eta) Q(\lambda^*, \eta^*, \lambda, \eta, \tau). \quad (3.70)$$

Replacing $(\alpha^*, \alpha, \beta^*, \beta, \alpha_o^*, \alpha_o, \beta_o^*, \beta_o, t)$ by $(\lambda^*, \lambda, \eta^*, \eta, \alpha^*, \alpha, \beta^*, \beta, \tau)$ in expression (3.41), we have

$$Q(\lambda^*, \lambda, \eta^*, \eta, \tau) = \frac{A'(\tau)}{\pi^2} \exp[-B'(\tau)(\lambda^* \lambda + \eta^* \eta) + C'(\tau)(\lambda \eta + \lambda^* \eta^*) + D'(\tau)\lambda + E'(\tau)\eta + D'^*(\tau)\lambda^* + E'^*(\tau)\eta^*], \quad (3.71)$$

where A', B', C', D' and E' are given by expressions (3.42) and (3.43) with $(\alpha_o^*, \alpha_o, \beta_o^*, \beta_o, t)$ replaced by $(\alpha^*, \alpha, \beta^*, \beta, \tau)$. It then follows that

$$\int d^2 \lambda d^2 \eta (\lambda^* \lambda - \eta^* \eta) Q(\lambda^*, \lambda, \eta^*, \eta, \tau) = A' \int \frac{d^2 \lambda}{\pi} \frac{d^2 \eta}{\pi} (\lambda^* \lambda - \eta^* \eta) \exp[-B'(\lambda^* \lambda + \eta^* \eta) + C'(\lambda \eta + \lambda^* \eta^*) + D'\lambda + E'\eta + D'^*\lambda^* + E'^*\eta^*]. \quad (3.72)$$

Expressing this equation in the form

$$\begin{aligned} \int d^2 \lambda d^2 \eta (\lambda^* \lambda - \eta^* \eta) Q(\lambda^*, \lambda, \eta^*, \eta, \tau) &= \left[\frac{\partial^2}{\partial D' \partial D'^*} - \frac{\partial^2}{\partial E' \partial E'^*} \right] \\ &\times \left[A' \int \frac{d^2 \lambda}{\pi} \exp(-B' \lambda^* \lambda + D'\lambda + D'^* \lambda^*) \right. \\ &\times \left. \int \frac{d^2 \eta}{\pi} \exp(-B' \eta^* \eta + (C'\lambda + E')\eta + (C'\lambda^* + E'^*)\eta^*) \right] \end{aligned}$$

and integrating over η applying (2.36), we get

$$\begin{aligned} \int d^2 \lambda d^2 \eta (\lambda^* \lambda - \eta^* \eta) Q(\lambda^*, \lambda, \eta^*, \eta, \tau) &= \left[\frac{\partial^2}{\partial D' \partial D'^*} - \frac{\partial^2}{\partial E' \partial E'^*} \right] \frac{A' \exp[E' E'^* / B']}{B'} \\ &\times \int \frac{d^2 \lambda}{\pi} \exp \left[- \left(\frac{B'^2 - C'^2}{B'} \right) \lambda^* \lambda + \left(D' + \frac{C'}{B'} E'^* \right) \lambda + \left(D'^* + \frac{C'}{B'} E' \right) \lambda^* \right] \end{aligned}$$

so that performing the integration over λ leads to

$$\begin{aligned} \int d^2 \lambda d^2 \eta (\lambda^* \lambda - \eta^* \eta) Q(\lambda^*, \lambda, \eta^*, \eta, \tau) &= \left[\frac{\partial^2}{\partial D' \partial D'^*} - \frac{\partial^2}{\partial E' \partial E'^*} \right] \frac{A'}{B'^2 - C'^2} \\ &\times \exp \left[\frac{B'}{B'^2 - C'^2} (D'^* D' + E'^* E') + \frac{C'}{B'^2 - C'^2} (D' E' + D'^* E'^*) \right]. \end{aligned}$$

After carrying out the differentiation, one then finds

$$\int d^2\lambda d^2\eta (\lambda^*\lambda - \eta^*\eta) Q(\lambda^*, \lambda, \eta^*, \eta, \tau) = \left[\frac{D'^*D' - E'^*E'}{B'^2 - C'^2} \right] \exp \left[\frac{B'}{B'^2 - C'^2} (D'^*D' + E'^*E') + \frac{C'}{B'^2 - C'^2} (D'E' + D'^*E'^*) \right].$$

Now employing the explicit forms of B', C', D', E' along with their complex conjugate in this expression, we obtain

$$\int d^2\lambda d^2\eta (\lambda^*\lambda - \eta^*\eta) Q(\lambda^*, \lambda, \eta^*, \eta, \tau) = e^{(\lambda_4 - \lambda_3)\tau} [\alpha^*\alpha - \beta^*\beta]. \quad (3.73)$$

and substitution of (3.73) into (3.70) yields

$$\begin{aligned} \langle \hat{f}(t + \tau) \rangle &= e^{(\lambda_4 - \lambda_3)\tau} \int d^2\alpha d^2\beta Q \left(\alpha^*, \beta^*, \alpha + \frac{\partial}{\partial \alpha^*}, \beta + \frac{\partial}{\partial \beta^*}, t \right) \\ &\times [\alpha^{*2}\alpha^2 + \beta^{*2}\beta^2 - 2\alpha^*\alpha\beta^*\beta]. \end{aligned} \quad (3.74)$$

On account of the relation (2.39), this can be expressed as

$$\langle \hat{f}(t + \tau) \rangle = e^{(\lambda_4 - \lambda_3)\tau} \left[\langle a^{\dagger 2} a^2 \rangle + \langle b^{\dagger 2} b^2 \rangle - 2 \langle a^\dagger a b^\dagger b \rangle \right].$$

Noting that

$$a^{\dagger 2} a^2 = a^2 a^{\dagger 2} - 4a a^\dagger + 2,$$

$$a^\dagger a b^\dagger b = a a^\dagger b b^\dagger - a a^\dagger - b b^\dagger + 1$$

and recalling that the Q-function for the signal and idler modes are identical, the above expression can be put in the form

$$\langle \hat{f}(t + \tau) \rangle = 2e^{(\lambda_4 - \lambda_3)\tau} \left[\langle a^2 a^{\dagger 2} - 2a a^\dagger \rangle - \langle a a^\dagger b b^\dagger \rangle + 1 \right]. \quad (3.75)$$

Next applying the Q-functions (3.44) and (3.45), one can write that

$$\langle a^2 a^{\dagger 2} - 2a a^\dagger \rangle = \left[\frac{B^2(t) - C^2(t)}{B(t)} \right] \int \frac{d^2\alpha}{\pi} [(\alpha\alpha^*)^2 - 2\alpha\alpha^*] \exp(-F\alpha^*\alpha)$$

and

$$\begin{aligned} \langle a a^\dagger b b^\dagger \rangle &= [B(t) - C(t)] \int \frac{d^2\alpha}{\pi} \alpha\alpha^* \exp[-B(t)\alpha^*\alpha] \int \frac{d^2\beta}{\pi} \beta\beta^* \\ &\quad \times \exp[-B(t)\beta^*\beta + C(t)\alpha\beta + C(t)\alpha^*\beta^*] \end{aligned}$$

or

$$\langle a^2 a^{\dagger 2} - 2a a^\dagger \rangle = \left[\frac{B^2(t) - C^2(t)}{B(t)} \right] \left[\frac{\partial^2}{\partial F^2} + 2 \frac{\partial}{\partial F} \right] \int \frac{d^2\alpha}{\pi} \exp(-F\alpha^*\alpha) \quad (3.76)$$

and

$$\begin{aligned} \langle a a^\dagger b b^\dagger \rangle &= - [B^2(t) - C^2(t)] \int \frac{d^2\alpha}{\pi} \alpha\alpha^* \exp[-B(t)\alpha^*\alpha] \frac{\partial}{\partial B} \\ &\quad \times \int \frac{d^2\beta}{\pi} \exp[-B(t)\beta^*\beta + C(t)\alpha\beta + C(t)\alpha^*\beta^*] \\ &= \left[\frac{B^2(t) - C^2(t)}{B(t)} \right] \int \frac{d^2\alpha}{\pi} \left[\left[\frac{C(t)}{B(t)} \alpha\alpha^* \right]^2 + \frac{\alpha\alpha^*}{B(t)} \right] \exp \left[- \left[\frac{B^2(t) - C^2(t)}{B(t)} \right] \alpha^*\alpha \right] \\ &= \left[\frac{B^2(t) - C^2(t)}{B(t)} \right] \left[\frac{C^2}{B^2} \frac{\partial^2}{\partial F^2} - \frac{1}{B} \frac{\partial}{\partial F} \right] \int \frac{d^2\alpha}{\pi} \exp(-F\alpha^*\alpha), \quad (3.77) \end{aligned}$$

where

$$F = \frac{B^2 - C^2}{B}.$$

After carrying out the integration and then the differentiation involved in (3.76) and (3.77), we get

$$\langle a^2 a^{\dagger 2} - 2a a^\dagger \rangle = \frac{2B^2(t)}{[B^2(t) - C^2(t)]^2} - \frac{2B(t)}{B^2(t) - C^2(t)} \quad (3.78a)$$

and

$$\langle a a^\dagger b b^\dagger \rangle = \frac{B^2(t) + C^2(t)}{[B^2(t) - C^2(t)]^2}. \quad (3.78b)$$

Now combining (3.75) and (3.78), we have

$$\langle \hat{f}(t + \tau) \rangle = 2e^{(\lambda_4 - \lambda_3)\tau} \left[\frac{1 - 2B(t)}{B^2(t) - C^2(t)} + 1 \right] \quad (3.79)$$

so that in view of this result, expression (3.67) takes the form

$$S_d(\omega) = 8\gamma^2 \left[\frac{1-2B}{B^2-C^2} + 1 \right] \int_{-\infty}^{\infty} d\tau e^{-(i\omega-\lambda_4-\lambda_3)\tau} + 2\gamma \left[\langle a^\dagger(t)a(t) \rangle + \langle b^\dagger(t)b(t) \rangle \right]. \quad (3.80)$$

Noting that

$$\langle a^\dagger(t)a(t) \rangle = \langle b^\dagger(t)b(t) \rangle$$

and using (3.49b), one can write

$$S_d(\omega) = 8\gamma^2 \left[\frac{1-2B}{B^2-C^2} + 1 \right] \int_{-\infty}^{\infty} d\tau e^{-(i\omega-\lambda_4+\lambda_3)\tau} + 4\gamma \left[\frac{B}{B^2-C^2} - 1 \right]. \quad (3.81)$$

Employing the stationarity property for the integral, we put (3.81) in the form

$$S_d(\omega) = 8\gamma^2 \left[\frac{1-2B}{B^2-C^2} + 1 \right] \left[\int_0^{\infty} d\tau e^{-(i\omega-\lambda_4+\lambda_3)\tau} + \int_0^{\infty} d\tau e^{-(-i\omega-\lambda_4+\lambda_3)\tau} \right] + 4\gamma \left[\frac{B}{B^2-C^2} - 1 \right] \quad (3.82)$$

so that integration over τ results in

$$S_d(\omega) = 4\gamma \left[\frac{B}{B^2-C^2} - 1 \right] + \left[\frac{1-2B}{B^2-C^2} + 1 \right] \frac{32\gamma^3}{\omega^2 + (2\gamma)^2}, \quad (3.83)$$

where we have replaced λ_3 and λ_4 by their explicit forms. Combining (3.42b), (3.42c)

and (3.43), one finds at steady state

$$B = \frac{4\gamma^2 - 2(\kappa\gamma_0)^2}{4\gamma^2 - (\kappa\gamma_0)^2}$$

and

$$C = -\frac{2\kappa\gamma_0\gamma}{4\gamma^2 - (\kappa\gamma_0)^2}$$

so that the expression for the spectrum reduces

$$S_d(\omega) = S_o - \left[\frac{8(\kappa\gamma_0)^2\gamma^3}{\gamma^2 - (\kappa\gamma_0)^2} \right] \frac{1}{\omega^2 + 4\gamma^2}, \quad (3.84a)$$

where

$$S_o = \frac{2(\kappa\gamma_o)^2\gamma}{\gamma^2 - (\kappa\gamma_o)^2} \quad (3.84b)$$

is the frequency independent term representing the contribution of the shot-noise from each mode. Thus to quantify the degree of reduction below the shot-noise level we define the ‘normalized’ intensity-difference fluctuations spectrum

$$\bar{S}_d(\omega) = \frac{S_d(\omega)}{S_o}. \quad (3.85)$$

Then combining (3.84) and (3.85), we obtain

$$\bar{S}_d(\omega) = \frac{\omega^2}{\omega^2 + 4\gamma^2}. \quad (3.86)$$

This is a simple inverted Lorentzian with a width of 4γ . We observe that at zero frequency there is a perfect noise suppression in the intensity difference for the signal and idler modes.

Our result assumes no additional cavity losses beyond those corresponding to the equal transmitivities at the output mirror. This is in complete agreement with the result obtained by Reynaud et al. [49]. For the case in which additional losses are included, Lane et al. [61] have indicated that perfect suppression of the noise is no longer possible as one of the pair of photons may be lost other than through the output mirror.

The prediction of noise suppression in the intensity difference has been confirmed experimentally by Haidmann et al [50] for a NDPO operating above threshold. In this experiment a maximum noise reduction of $30\% \pm 5\%$ is observed at a frequency of 8MHz. The noise reduction is limited due to other losses inside the cavity and various detector inefficiencies. In refinement of this experiment Debuissechert et al [63] have reported an improved quantum noise reduction of 69% in the intensity difference.

4. Conclusion

We have analyzed the statistical and squeezing properties of the signal and the signal-idler modes produced in parametric oscillators. In addition, we have studied the spectrum of the intensity-difference fluctuations for the signal-idler modes. Our analysis has been carried out using the Q-function obtained applying the method of evaluating the propagator developed by Fesseha [1].

We have calculated the quadrature fluctuations and the squeezing spectrum for the signal mode produced in a DPO operating below threshold. Expression (2.45) shows that at steady state 50% squeezing of the signal mode inside the cavity can be achieved when operating near threshold. However, the result described by (2.49) indicates that it is possible to obtain 100% steady-state squeezing in the output signal mode at zero frequency and at threshold.

We have also obtained the photon number distribution of the signal mode. The time dependent as well as the steady state forms of the photon number distributions described by expressions (2.60) and (2.63) indicate that there is a finite probability of finding odd number of photons inside the cavity. This is due to the fact that, although the signal photons are generated in pairs, there is a possibility for an odd number of photons to pass through the output mirror. However, in the absence of cavity damping, the probability of finding odd number of photons is zero as can be easily seen from expression (2.61).

On the other hand, our analysis shows that the intracavity mixed radiation of

the signal-idler modes is in a squeezed state. Moreover, expression (3.57) shows that 50% suppression of the noise below the vacuum level can be achieved in the first quadrature at steady state and at threshold with infinitely enhanced fluctuations in the second quadrature. These results are valid for equal cavity decay rates and in the absence of other losses.

We have also determined the photon number distribution of the signal-idler radiation inside the cavity. In the absence of cavity damping, this result is found to have the same form as the photon number distribution of a two-mode squeezed vacuum [60]. Unlike the case of the signal mode produced in a DPO, we see from expressions (3.62) and (3.63) that there is a finite probability of finding n -signal and n -idler photons, where n can be an even or an odd number. This is a consequence of pair generation of signal and idler photons and the assumption of equal cavity decay rates.

The other interesting nonclassical property that we found in the signal-idler radiation is a 100% reduction of the noise below the shot-noise level exhibited in the output spectrum of the intensity-difference fluctuations of these modes. The scaled steady-state spectrum of the intensity-difference fluctuations (3.86) is found to be a simple inverted Lorentzian with a width of 4γ . We have also seen that there is a perfect cancellation of the noise in the output intensity difference at zero frequency. However, these results are not realistic since the analysis assumes no additional cavity losses beyond those corresponding to the equal transmitivities at the output mirror. When additional losses are included, the correlation between the signal and idler is no longer perfect since one of the pair of photons may be lost other than through the output mirror. In that case there is no complete suppression of the noise at zero frequency [61,62].

Finally, we would like to point out that one major task in this thesis has been to determine the solution of the pertinent Fokker-Planck equation for the Q-function. And this has been done by transforming such equation into a Schrödinger-type equation and then evaluating the associated Q-function propagator applying the method developed by Fesseha [1].

This method can be used to evaluate the propagator associated with a quadratic Hamiltonian of arbitrary form. The task of evaluating the propagator using this method essentially reduces to the problem of solving the pertinent Euler-Lagrange equations. We therefore maintain the standpoint that this method provides a convenient means of solving second order partial differential equations.

4. References

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