Graduate Seminar Report on Methods of Conjugate Directions

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A C K N O W L E G E M E N T

This Seminar would have been impossible without the fatherly guidance and day to day follow up of my most respected advisor Prof.Dr. R.Deumlich. Who has helped me alot to make this report a reality. Furthermore,my gratitude also goas to W/t Bizuayehu Adafre,who has helped me to type and print this paper.

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CONTENTS

P R E F A C E

Methods of conjugate Directions is one of the methods of unconstrained function minimization. Studying minimization of unconstrained function is useful, since some of the most powerful and convenient methods of solving constrained minimization problems (such as Lagrange-Method) involve the transformation of the problem into one of unconstrained minimization

Since the main objective of optimization is to minimize (or maximize) optimization problems,one shoud have a way of tackling unconstrained function minimization,and so among the major methods of unconstrained function minimization Methods of Conjugate Directions is the one.

This Seminar is a compilation of the two Seminars I have delivered,for the qualification for M.Sc.in Mathematics.Thus in this Seminar Paper I have attempted to present the basic definitions and properties of Methods of Conjugate Directions.

I have divided the paper into two chapters:

I - Preliminaries - a review of definitions and properties of convex sets and strongly covex functions

II - Methods of Conjugate Directions - discusion on the Methods of Conjugate Directions and minimization of strictly & stongly convex functions.

CHAPTER ONE
PRELIMINARIES (REVIEW)

1.1. CONVEX SETS AND FUNCTIONS

DEFINITION 1.1.1: A set $C \subseteq \mathbb{R}^n$ is said to be convex if and only if $\lambda x + (1-\lambda)y \in C$ for each $x, y \in C$ and $\lambda \in [0,1]$.

Example: $\mathbb{R}^n$ is convex.

DEFINITION 1.1.2: A function $f:\mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$, not identically $+\infty$, is said to be convex if for each $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$ there holds

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

DEFINITION 1.1.3: An (n,n) - matrix A is said to be

a) Positive semi definite if and only if

$$\langle Ax, x\rangle \geq 0 \ \forall x \in \mathbb{R}^n,$$

b) Positive definite if and only if

$$\langle Ax, x\rangle > 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

DEFINITION 1.1.4: Let $U \subseteq \mathbb{R}^n$ be an open set, $f: U \text{—} \mathbb{R}$. Then

(i) f is said to be *Frechet-differentiable* at $x \in U$ if and only if there exists a linear and centinuous function $A:\mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\lim_{\|h\| \longrightarrow 0,\, h \neq 0} \frac{\|f(x+h) - f(x) - A(h)\|}{\|h\|} = 0.$$

A is said to be the *Frechet-differential* of f at x and it is denoted by $f'(x) = A$.

(ii) f is said to be Frechet-differentiable on U if and only if f is Frechet-differentiable at each $x \in U$.

(iii) The mapping

$$f': U \longrightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R})$$

is said to be the Frechet-derivative of f. Where $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ is the set of linear and continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$.

(iv) f is said to be continuous Frechet-differentiable if and only if f′ is continuous.

LEMMA 1.1.1: Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be twice continuously *Frechet-differentiable* . Then f is convex if and only if the matrix f"(x) is positive semi definite for each $x \in \mathbb{R}^n$.

1.2. STRICTLY AND STRONGLY CONVEX FUNCTIONS

DEFINITION 1.2.1: A function $f:\mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is said to be strictly convex if for each $x,y \in \mathbb{R}^n$, $x \neq y$ and $\lambda \in (0,1)$ there holds

$$f(\lambda x + (1-\lambda)y) < \lambda\, f(x) + (1-\lambda)\, f(y).$$

LEMMA 1.2.1: Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ be twice continuously Frechet-differentiable. Then f is strictly convex if and only if the matrix f"(x) is positive definite for each $x \in \mathbb{R}^n$.

COROLLARY 1.2.1: The quadratic function

$$f(x) = \frac{1}{2} \langle Ax, x\rangle + \langle b, x\rangle + c$$

is strictly convex if and only if matrix A is positive definite.

CRITERION OF HURWITZ: Let $A = (a_{ij})_{i,j=1,\ldots,n}$ be asymmetrical matrix. Then A is positive definite if and only if $\det(a_{ij})_{i,j = 1,\ldots,k} > 0$

$$\forall\, k \in \{1,\ldots,n\}.$$

We now consider a class of functions for which on any nonempty closed convex set there always exists a unique point of minimum.

DEFINITION 1.2.2: A function f(x) on $\mathbb{R}^n$ is said to be strongly convex if there is a constant $\gamma > 0$ such that for each $x, y \in \mathbb{R}^n$, $\lambda \in (0,1)$ there holds

$$f(\lambda x + (1-\lambda)y) \le \lambda\, f(x) + (1-\lambda)\, f(y) - \lambda(1-\lambda)\, \gamma \|x-y\|^2. \quad (1.2.1).$$

Note that astrongly convex function is also strictly convex.

LEMMA 1.2.2. Let $f:\mathbb{R}^n \longrightarrow \mathbb{R}$ be twice continuously Frechet-differentiable. Then the condition of strong convexity (1.2.2) is equivalent to the condition

$$m\|p\|^2 \le \langle f''(x)p, p\rangle \le M\|p\|^2, \quad M \ge m > 0 \qquad (1.2.2)$$

for each $x, p \in \mathbb{R}^n$.

LEMMA 1.2.3. If the matrix $f''(x)$ satisfies condition (1.2.2) then there exists the inverse matrix $f''^{-1}(x)$ which is bounded.

LEMMA 1.2.4. If $f(x)$ is a twice continuously differentiable strongly convex function, then for any $x_o \in \mathbb{R}^n$ the set

$$y = \{x : f(x) \le f(x_o)\}$$

is closed and bounded.

LEMMA 1.2.5. If $f(x)$ is twice continuously differentiable strongly convex function on a closed and convex set $K \subseteq \mathbb{R}^n$, then for each $x \in K$, $\partial > 0$ a constant, the following holds

a) $\|x - x_*\| \le \frac{2}{\partial} < f(x) - f(x_*)>$,

b) $\|x - x_*\| \le \frac{1}{\partial} \|f'(x)\|$,

c) $0 \le f(x) - f(x_*) \le \frac{1}{\partial} \|f'(x)\ ^2$.

1.3. SOME ADDITIONAL INFORMATION

1.3A. NORM AND ITS PROPERTIES

DEFINITION 1.3.1. A norm in $\mathbb{R}^n$ is defined by

$$\|x\| = \sqrt{\langle x\ ,\ x\rangle}\ ,$$

Properties: For each $x, y \in \mathbb{R}^n$ the following holds:

(i) $|\langle x, y\rangle| \le \|x\|.\|y\|$ (cauchy-Buniakowski's inequality)

(ii) $\|x + y\| \le \|x\| + \|y\|$ (Triangle inequality)

(iii) $|\,\|x\| - \|y\|\,| \le \|x + y\|$.

1.3B. PROPERTY OF OPERATORS

If $F(x)$ is a nonlinear differentiable operator, then for any $x, h, y \in \mathbb{R}^n$ the following formula is valid:

$$\langle F(x + h) - F(x), y\rangle = \langle F'(x + \theta h)h, y\rangle, \ \theta \in [0,1].$$

This formula is called Lagrange's formula for operators.

In the following chapters we shall have many occasions of using Taylor's formula with the remainder term in Lagrange's form. If $f(x)$ is a twice continuously differentiable function in a convex set $k \subseteq \mathbb{R}^n$, then for any $x, x + h \in k$ and $\alpha \in [0,1]$

$$f(x + \alpha h) - f(x) = \alpha \langle f'(x + \alpha\theta_1 h), h\rangle$$

and

$$f(x + \alpha h) = f(x) + \alpha \langle f'(x), h\rangle + \frac{\alpha^2}{2} \langle f''(x + \theta_2 h)h, h)$$

where θ_1, $\theta_2 \in [0,1]$.

CHAPTER TWO

METHEDS OF CONJUGATE DIRECTIONS

2.0. INTRODUCTION

This chapter is devoted to the problem of minimization of unconstrained function f(x) defined in an n-dimensional Euclidean space $\mathbb{R}^n$. Accordingly, in this chapter x is always an n-dimensional vector.

In solving the problem we shall use iterative processes of the type

$$x_{k+1} = x_{k+\alpha_k p_k}$$

Where p_k is a vector determing the direction of motin from point x_k and α_k is a numerical factor whose value determines the length of the step in the direction of p_k.

In order to get nearer to a minimum point x_* (such that $f(x_*)$ is minimum), one should naturaly move from point x_k in the direction of descent. If point x_k is not the point of minimum or a stationary point, then there is an infinite number of vectors p which determine the direction of descent from point x_k and each vector is defined by

$$<f'(x_k),\ p> < 0.$$

This is seen form the following argument.

Let $x = x_k + \alpha p$. Expansion of the function in Taylor's series about x_k gives

$$f(x) = f(x_k) + \alpha<f'(x_k), p> + \frac{\alpha^2}{2} < f''(x_{kc})p, p>$$

where $x_{kc} = x_k + \Theta(x - x_k),\ \Theta \in[0,1]$.

If $<f'(x_k), p > < 0$ then atleast with small value of α, $f(x)<f(x_k)$ since the sign of the right-hand is determined by a term which is linear with respect to α.

Now we apply methods of conjugate directions to choose the direction of the descent and factor α_k.

In what follows we use the notation

$f'_k := f'(x_k) = \nabla f(x_k)$.

2.1. MINIMIZATION OF QUADRATIC FUNCTIONS

2.1.A .CONJUGATE DIRECTIONS AND THEIR PROPERTIES

Let us see the problem of minimizing quadratic functions of the form

$$f(x) = \frac{1}{2} \langle Ax, x\rangle + \langle b, x\rangle + C \qquad (2.1.1)$$

where A is an (n,n)-symmetric, positive definite matrix, b is a vector and c is a scalar quantity.

DEFINITION 2.1.1. Let A be an (n,n)-symmetric matrix. A set of n vectors (or directions) $\{p_0, p_1, \ldots, p_{n-1}\}$ is said to be conjugate with respect to A (or A-orthogonals) if $\langle Ap_i, p_j\rangle = 0$ for each $i \neq j$, $i, j = 0,1, \ldots, n-1$.

THEOREM 2.1.1. Let A be an (n,n)-symmetric and positive definite matrix. A set of non zero A-orthogenal vectors $\{p_0, p_1, \ldots, p_{n-1}\}$ is linearly independent.

PROOF: Suppose that

$$\sum_{j=0}^{n-1} \lambda_j p_j = 0 \quad , \text{ for some scalar } \lambda_j.$$

If i is any one of the values of j, then multiplying both sides of the equality by Ap_i , we obtain

$$\sum_{j=0}^{n-1} \lambda_j \langle Ap_i, p_j\rangle = 0$$

or

$$\lambda_i \langle Ap_i, p_i\rangle = 0$$

because of the conjugacy, all the remaining terms vanish. But since A is positive definite and $p_i \neq 0$, then

$$\langle Ap_i, p_i\rangle \neq 0$$

which implies, $\lambda_i = 0$ for each i. Since i is arbitrary,

$$\sum_{j=0}^{n-1} \lambda_j p_j = 0$$

implies $\lambda_j = 0$ for each $j \in \{0, 1, \ldots, n-1\}$.

Thus $\{p_0 p_1, \ldots, p_{n-1}\}$ is linearly independent. //

Let x_* minimizes the quadratic function f(x). Then

$$\nabla f(x_*) = b + AX_* = 0 \qquad (2.1.2)$$

Now given a point x_o, and a set of (nonzero) A-conjugate directions $p_0, p_1, \dots, p_{n-1}$, then since $p_0, p_1, \dots, p_{n-1}$ is a basis of $\mathbb{R}^n$ point x_* can be represented in the form:

$$x_* - x_0 = \sum_{j=0}^{n-1} \alpha_j p_j$$

or
$$x_* = x_0 + \sum_{j=0}^{n-1} \alpha_j p_j \qquad (2.1.3)$$

using equation (2.1.2) we have

$$b + A(x_0 + \sum_{j=0}^{n-1} \alpha_j p_j) = 0, \text{ i.e.}$$

or
$$b + Ax_0 + A \sum_{j=0}^{n-1} \alpha_j p_j = 0$$

$$f'_0 + A \sum_{j=0}^{n-1} \alpha_j p_j = 0$$

Multiplying this equation by p_i , we obtain

$$<f'_0, p_i> + \alpha_i <Ap_i, p_i> = 0, \text{ i.e.}$$

$$\alpha_i = - \frac{<f'_0, p_i>}{<Ap_i, p_i>} . \qquad (2.1.4)$$

Thus if a certain system of conjugate directions is known, then the minimum point of a quadratic function (2.1.1) is easly found by using formulas (2.1.3) and (2.1.4).

The procedure of determining point x_* by formula (2.1.3) can be considered as a process of construction of successive points:

$$x_{i+1} = x_i + \alpha_i p_i , \quad i = 0, 1, \dots, n-1 \qquad (2.1.5)$$

where the parameter α_i are determined by formulas (2.1.4).

DEFINITION 2.1.2: Iterative processes of the type

$$x_{i+1} = x_i + \alpha_i p_i , \; i = 0, 1, \dots, n-1.$$

in which α_i are determined by formulas (2.1.4) is called methods of conjugate directions.

It follows that using the methods of conjugate directions one can solve the problem of quadratic function minimization after performing a finite number of steps not excceding n.

Formulas (2.1.4) can be transformed as follows:

If x_i is determined by formula (2.1.5), then

$$\begin{aligned}\langle f'_0, p_i\rangle &= \langle f'_0 - f'_1 + f'_1 - \dots - f'_i + f'_i\,,\ p_i\rangle \\ &= \langle -\alpha_0 AP_0 - \alpha_1 AP_1 - \dots - \alpha_{i-1}AP_{i-1} + f'_i,\ p_i\rangle \\ &= -\alpha_0\langle AP_0, P_i\rangle - \alpha_1\langle AP_1, P_i\rangle - \dots - \alpha_{i-1}\langle AP_{i-1}, P_i\rangle + \langle f'_i, p_i\rangle \\ &= \langle f'_i, p_i\rangle, \text{ due to A-orthogonality of vectors } p_0, \dots, p_i.\end{aligned}$$

Consequently from (2.1.4)

$$\alpha_i = -\frac{\langle f'_i, p_i\rangle}{\langle AP_i, p_i\rangle}, \quad i = 0, 1, \dots, n-1. \tag{2.1.6}$$

THEOREM 2.1.2: If point x_i is reached after i - 1 steps while minimizing the quadratic function (2.1.1) and if $p_o, p_1, \dots p_{i-1}$ are (non zero) A-orthogonal vectors then

$$\langle f'_i\,,\ p_j\rangle = 0\,,\ j = 0, 1, \dots, i-1. \tag{2.1.7}$$

PROOF: Since x_i is reached after i-1 minimizing steps, it can be written as

$$\begin{aligned}x_i &= x_0 + \alpha_0 p_0 + \alpha_1 p_1 + \dots + \alpha_j p_j + \alpha_{j+1}p_{j+1} + \dots + \alpha_{i-1}p_{i-1}. \\ &= x_{j+1} + \alpha_{j+1}p_{j+1} + \dots + \alpha_{i-1}p_{i-1} \\ &= x_{j+1} + \sum_{k=j+1}^{i-1}\alpha_k p_k\end{aligned}$$

where α_k is a step length in the direction of p_k. But since

$$f'_i = \nabla f(x_i) = Ax_i + b,$$

then using the above we obtain

$$\begin{aligned}f'_i = Ax_i + b &= A\left(x_{j+1} + \sum_{k=j+1}^{i-1}\alpha_k p_k\right) + b \\ &= Ax_{j+1} + b + A\sum_{k=j+1}^{i-1}\alpha_k p_k \\ &= f'_{j+1} + \sum_{k=j+1}^{i-1}\alpha_k AP_k.\end{aligned}$$

Multiplying both sides by p_j, we obtain

$$\langle f'_i\,,\ p_j\rangle = \langle f'_{j+1}\,,\ p_j\rangle + \sum_{k=j+1}^{i-1}\alpha_k\langle AP_k,\ p_j\rangle.$$

But since α_j is the step length along the direction p_j, we obtain (using (2.1.6))

$$\langle f'_{j+1} , p_j \rangle = 0$$

and since $p_0, p_1, \ldots, p_{i-1}$ are A-orthogenals,

$$\sum_{k=j+1}^{i-1} \alpha_k \langle AP_k , p_j \rangle = 0$$

Hence

$$\langle f'_i , p_j \rangle = 0 , \; j = 0, 1, \ldots, i-1. \; //$$

If with a certain $0 \le i \le n - 1$ in formula (2.1.5) $\alpha_i = 0$ (i.e. $x_{i+1} = x_i$), then from (2.1.6) we obtain $\langle f'_i , p_i \rangle = 0$. Combining this with (2.1.7) we obtain

$$\langle f'_{i+1} , p_j \rangle = \langle f'_i , p_j \rangle = 0 , \; j = 0, 1, \ldots, i.$$

Thus the fact that the coefficient α_i becomes zero means that the corresponding point x_i provides the minimum of the quadratic function in the subspace formed by vectors $p_0, p_1, \ldots, p_i$ and passing through point x_0.

Finally, not that by (2.1.7) $\langle f'_i , p_{i-1} \rangle = 0$. This means that the choice of coefficients α_i by formulas (2.1.4) or (2.1.6) corresponds to choosing α_i under the condition

$$f(x_i + \alpha_i p_i) = \min_{\alpha} f(x_i + \alpha p_i).$$

2.1B. METHODS OF CONSTRUCTING CONJUGATE VECTORS

Now we turn to the study of methods of constructing conjugate vectors. Each of these methods determines one or other method of conjugate directions, which consists in the construction of successiue approximations to the solution of the problem of minimization of function (2.1.1) making use of formulas (2.1.5) and (2.1.4) (or (2.1.6)).

REMARKS: 1. The process of constructing conjugate vectors should use only calculations of the function and its gradient and should not use the calculation of second derivatives of the function.

2. For any of the methods of constructing conjugate vectors, the condition

$$\langle f'_i , p_i \rangle = 0 , \quad 0 \le i \le n - 1$$

is satisfied if and only if $f'_i = 0$. Indeed, if this condition is satisfied, then by (2.1.6) $\alpha_i = 0$ and there fore in sequence (2.1.5) $x_{i+1} = x_i$. This means that we shall not be able to construct vector $p_{i+1} \neq p_i$. The process will accordingly degenerate (stop) with out reaching the solution if $f'_i \neq 0$.

Thus for any of the methods of constructing conjugate vectors, the condition

$$<f'_i , p_i> \neq 0 \text{ if } f'_i \neq 0 \qquad (2.1.8)$$

must be satisfied.

Taking into account the above remarks, let us turn to the actual working out of the relations for the construction of A-orthegonal vectors.

In what follows we use the notations

$$r_i := x_{i+1} - x_i = \alpha_i p_i, \quad e_i := f'_{i+1} - f'_i = \alpha_i AP_i. \qquad (2.1.9)$$

An arbitrary direction of descent of function (2.1.1) may be chosen to be vector $p_0 = -H_o^T f'_o$, where H_0 is a symmetric, positive definite matrix.

Let us establish the requirements which vector p_k, $1 \leq k \leq n - 1$, must satisfy inorder to fulfill the conditions of A-orthogenality:

$$<p_k, AP_j> = 0 \ , \ 0 \leq j \leq k - 1. \qquad (2.1.10)$$

To this end, we make use of the fact that according to the properties of conjugate directions (see (2.1.7)) in choosing α_i in process of (2.1.5) by formula (2.1.6), conditions (2.1.10) and at the same time also the equality

$$<f'_k , p_j> = 0 \ , \ 0 \leq j \leq k - 1 \qquad (2.1.11)$$

must be satisfied. If we set

$$p_k = -H_k^T f'_k \qquad (2.1.12)$$

where H_k is an (n,n) square matrix, then conditions (2.1.10) can be written in the form:

$$<f'_k , H_k AP_j> = 0 \ , \quad 0 \leq j \leq k - 1$$

Comparision of the equalities obtained with (2.1.11) shows that if (2.1.11) is satisfied, then (2.1.10) will be also be satisfied, provided matrix H_k satisfies the relations

$$H_k AP_j = ap_j \ , \quad 0 \leq j \leq k - 1$$

where a is an arbitrary constant.

Since according to condition (2.1.5) and the strict convexity of function (2.1.1) we have $0 < |\alpha_i| < \infty$ with any $0 \leq i \leq n - 1$, equalities (2.1.10) and (2.1.11) can be written in the form:

$$<r_k , e_j> = <\alpha_k p_k , \alpha_j AP_j> = \alpha_k \alpha_j <p_k, AP_j> = 0, \text{ i.e.}$$

$$<r_k , e_j> = 0 , \quad 0 \le j \le k - 1 \qquad (2.1.13)$$

and

$$<f'_k , r_j> = <f'_k , \alpha_j p_j> = \alpha_j <f'_k , p_j> = 0 , \text{ i.e.}$$

$$<f'_k , r_j> = 0, \quad 0 \le j \le k - 1 \qquad (2.1.14)$$

and the condition for determining matrix H_k can be written as:

$$H_k e_j = ar_j , \quad 0 \le j \le k - 1. \qquad (2.1.15)$$

Thus the conditions of A-orthogenality (2.1.10) will be satisfied if matrix H_k which determines p_k by formula (2.1.12) satisfies equations (2.1.15).

With $k < n - 1$, the number of vector equations (2.1.15) will be less than n; it follows that matrix H_k is not uniquely defined. Besides, with different values of constant a the system of equations for defining matrix H_k will also be different. All this suggests the diversity of algorithms which can be used to construct conjugate directions as we have to use various methods of constructing differnt matrices H_k.

Since equations (2.1.15) must be satisfied with any $k = 1,2,..,n - 1$, it is natural to try and construct matrix H_k by recursive relations ($H_k = H_{k+1} + \Delta H_{k-1}$, $k > 1$ where ΔH_{k-1} is defined below).

Let us write (2.1.15) in the following form:

$$H_k e_j = (H_{k-1} + \Delta H_{k-1}) e_j = ar_j, \text{ i.e.}$$

$$(H_{k-1} + \Delta H_{k-1}) e_j = ar_j, \quad 0 \le j \le k - 1. \qquad (2.1.16)$$

But since matrix H_{k-1} must satisfy the equations

$$H_{k-1} e_j = ar_j , \quad 0 \le j \le k - 2,$$

it follows from (2.1.16) that matrix ΔH_{k-1} is defined by the following conditions:

$$\Delta H_{k-1} e_j = 0, \quad 0 \le j \le k - 2,$$

$$\Delta H_{k-1} e_{k-1} = ar_{k-1} - H_{k-1} e_{k-1}. \qquad (2.1.17)$$

The latter equality will evidently be satisfied if we assume

$$\Delta H_{k-1} = a \frac{r_{k-1} u_{k-1}^T}{<u_{k-1}, e_{k-1}>} - \frac{H_{k-1} e_{k-1} v_{k-1}^T}{<v_{k-1}, e_{k-1}>} \qquad (2.1.18)$$

where u_{k-1} , v_{k-1} are unknown vectors. It is necessary that the vectors be such that the first of the conditions (2.1.17) is satisfied. i.e.

$$\langle u_{k-1}, e_j\rangle = 0, \quad \langle v_{k-1}, e_j\rangle = 0, \quad 0 \le j \le k-2. \qquad (2.1.19)$$

clearly, vectors u_{k-1}, v_{k-1} must also satisfy conditions

$$\langle u_{k-1}, e_{k-1}\rangle \ne 0, \ \langle v_{k-1}, e_{k-1}\rangle \ne 0. \qquad (2.1.20)$$

Taking into account (2.1.13) it is clear that conditions (2.1.19) will be satisfied if we choose $u_{k-1} = v_{k-1} = r_{k-1}$, conditions (2.1.20) will also be satisfied since

$$\langle r_{k-1}, e_{k-1}\rangle = \langle r_{k-1}, Ar_{k-1}\rangle > 0 \qquad (2.1.21)$$

according to the positive definitness of matrix A.

Vectors U_{k-1}, V_{k-1} can also be chosen by using the follwoing considerations. If condition (2.1.10) is satisfied, then we have

$$\langle AP_{k-1}, P_j\rangle = \langle \frac{1}{\alpha_{k-1}} e_{k-1}, \frac{1}{\alpha_j} r_j\rangle = \frac{1}{\alpha_{k-1}\alpha_j} \langle e_{k-1}, r_j\rangle = 0, \text{ i.e.}$$

$$\langle AP_{k-1}, p_j\rangle = \frac{1}{\alpha_{k-1}\alpha_j} \langle e_{k-1}, r_j\rangle = 0, \quad 0 \le j \le k-2.$$

Making use of (2.1.15) we have then

$$a\langle e_{k-1}, r_j\rangle = \langle e_{k-1}, H_{k-1}e_j\rangle = \langle H^T_{k-1}e_{k-1}, e_j\rangle = 0, \quad 0 \le j \le k-2.$$

It follows that inorder to satisfy (2.1.19) we can assume

$$U_{k-1} = V_{k-1} = H^T_{k-1} e_{k-1}.$$

Ingeneral, if we choose vectors u_{k-1} and V_{k-1} in the form

$$u_{k-1} = t_{1,k} r_{k-1} + t_{2,k} H^T_{k-1} e_{k-1},$$

$$v_{k-1} = t_{3,k} r_{k-1} + t_{4,k} H^T_{k-1} e_{k-1} \qquad (2.1.22)$$

Where $t_{1,k}$, $t_{2,k}$, $t_{3,k}$, $t_{4,k}$ are arbitrary numbers (which can change with changing k), then conditions (2.1.19) and (2.1.20) (in particular with $t_{1,k} = t_{3,k} = 1$, $t_{2,k} = t_{4,k} = 0$) will be satisfied.

Thus, choosing vectors u_{k-1}, v_{k-1} in the form (2.1.22) we are able to construct matrix ΔH_{k-1} by formula (2.1.18) and consequently matrrix H_k that the vector p_k which it determines will satisfy the conditions of A-orthogonality (2.1.10).

To each pair of vectors u_{k-1}, v_{k-1} and constant a chosen there will correspond their particular matrix ΔH_{k-1} and, consequently, matrix H_k. In other words, with different vectors u_k, v_k and constant a we shall construct different methods of conjugate directions.

2.1C. GENERAL PROPERTIES OF THE METHODS

Let us establish the general properties of the methods of conjugate directions, which can be constructed in the manner described above.

THEOREM 2.1.3: If matrix H_0 is symmetric and positive definite, then condition (2.1.8), i.e.

$$\langle f'_j, p_j\rangle \neq 0 \text{ if } f'_j \neq 0$$

is satisfied by the methods of conjugate directions.

PROOF: Using expressions (2.1.18), (2.1.22) and the recursiue formula for matrix $H_j (H_j = H_{j-1} + \Delta H_{j-1})$ we have

$$-p_j = H_j^T f'_j = (H_{j-1} + \Delta H_{j-1})^T f'_j = H_{j-1}^T f'_j + \Delta H_{j-1}^T f'_j.$$

Making use of (2.1.18) and (2.1.14) we can write

$$\Delta H_{j-1}^T f'_j = - \frac{v_{j-1}\langle e_{j-1}, H_{j-1}^T f'_j\rangle}{\langle v_{j-1}, e_{j-1}\rangle} = - \frac{\left[t_{3,j} r_{j-1} + t_{4j} H_{j-1}^T e_{j-1}\right]\langle e_{j-1}, H_{j-1}^T f'_j\rangle}{\langle v_{j-1}, e_{j-1}\rangle}.$$

But since

$$\begin{aligned} H_{j-1}^T e_{j-1} &= H_{j-1}^T (f'_j - f'_{j-1}) \\ &= H_{j-1}^T f'_j - H_{j-1}^T f'_{j-1} \\ &= H_{j-1}^T f'_j + p_{j-1}, \end{aligned}$$

then vector $-p_j$ can be written in the form:

$$-p_j = H_j^T f'_j = \partial_j \left[I - \frac{r_{j-1} e_{j-1}^T}{\langle r_{j-1}, e_{j-1}\rangle}\right] H_{j-1}^T f'_j \qquad (2.1.23)$$

where $\partial_j := 1 - t_{4,j} \dfrac{\langle e_{j-1}, H_{j-1}^T f'_j\rangle}{\langle v_{j-1}, e_{j-1}\rangle}$.

If vector v_{j-1} satisfies condition (2.1.20) and $\alpha_{j-1} t_{3,j} \neq - t_{4,j}$, with any $j = 1, 2, \ldots$ factor $\partial_j \neq 0$ since

$$\frac{t_{4,j}\langle e_{j-1}, H_{j-1}^T f'_j\rangle}{\langle v_{j-1}, e_{j-1}\rangle} \neq 1.$$

Suppose that factors $t_{3,j}$ and $t_{4,j}$ are such that with $j \geq 1$ conditions $\langle v_{j-1}, e_{j-1}\rangle \neq 0$ and $\partial_j \neq 0$ be satisfied. Then we obtain

$$H_{i+1}^T f'_j = H_i^T f'_j \text{ for } 0 \leq i \leq j - 2, \text{ i.e.}$$

$$H_0f'_j = H_1^Tf'_j = \dots = H_{j-2}^Tf'_j = H_{j-1}^Tf'_j . \quad (2.1.24)$$

Taking into account these equalities, we can write expression (2.1.23) in the following form:

$$-p_j = H_j^Tf'_j = \partial_j\left[I - \frac{r_{j-1}e_{j-1}^T}{\langle r_{j-1}, e_{j-1}\rangle}\right]H_0f'_j. \quad (2.1.25)$$

Now multiplying both sides of equation (2.1.25) by f'_j and using condition (2.1.14), we obtain

$$-\langle f'_j, p_j\rangle = \partial_j\langle f'_j, H_0f'_j\rangle , \; j \geq 0. \quad (2.1.26)$$

But since H_0 is a positive definite matrix, then for $f'_j \neq 0$, $\langle f'_j, H_0f'_j\rangle > 0$. Consequently, if $\partial_j \neq 0$, then it follows from (2.1.26) that $\langle f'_j, p_j\rangle \neq 0$. //

REMARK: The successive approximations to the solution of the problem of minimization of a quadratic function are the same for different methods of conjugate directions.

2.1D. CONCRETE ALGORITHMS

Let us now consider formulas which can be used in constructing conjugate directions. Each of such formulas determines a method of conjugate directions consisting in constructing successive approximations to the solution by formulas

$$x_{k+1} = x_k + \alpha_kp_k, \; p_k = - H_k^Tf'_k, \; k = 0, 1, \dots, n-1 \quad (2.1.27)$$

where α_k is chosen under the condition $f(x_k + \alpha_kp_k) = \min_\alpha f(x_k + \alpha p_k)$

Now let us construct two methods:

Method 1: Set in (2.1.18) $a = 1$, $u_{k-1} = r_{k-1}$, $v_{k-1} = H_{k-1}^T e_{k-1}$ (i.e. in formulas (2.1.22) $t_{1,k} = t_{4,k} = 1$, $t_{2,k} = t_{3,k} = 0$). Then

$$H_k = H_{k-1} + \Delta H_{k-1} = H_{k-1} + \frac{r_{k-1}r_{k-1}^T}{\langle r_{k-1}, e_{k-1}\rangle} - \frac{H_{k-1}e_{k-1}e_{k-1}^TH_{k-1}}{\langle H_{k-1}e_{k-1}, e_{k-1}\rangle}. \quad (2.1.28)$$

Let us study some properties of matrix H_k obtained by this method.

Properties: (i) H_k is symmetric.

(ii) H_k is positive definite.

(iii) $H_n = A^{-1}$.

Proof: (i) This fact is easily established by induction. Matrix H_0 is symmetric. The two matrices which form ΔH_0 are symmetric too. Therefor, H_1 is a symmetric matrix. Similar arguments hold for any k = 2, ..., n. //

(ii) We prove by induction. Matrix H_o is positive definite. Let H_k be a positive definite matrix. Then for any $x \in \mathbb{R}^n$

$$\langle H_{k+1}x, x\rangle = \langle H_k x, x\rangle + \frac{\langle r_k, x\rangle^2}{\langle r_k, e_k\rangle} - \frac{\langle H_k e_k, x\rangle^2}{\langle H_k e_k, e_k\rangle}$$

$$= \frac{\langle H_k x, x\rangle\langle H_k e_k, e_k\rangle - \langle H_k e_k, x\rangle^2}{\langle H_k e_k, e_k\rangle} + \frac{\langle r_k, x\rangle^2}{\langle r_k, e_k\rangle}.$$

But since H_k is apositive definite, there is a square root $H_k^{1/2}$. Consequently taking into account the summetry of matrix H_k, we have

$$\langle H_k x, x\rangle = \langle H_k^{1/2} H_k^{1/2} x, x\rangle = \langle H_k^{1/2} x, H_k^{1/2} x\rangle =: \langle y, y\rangle;$$

similarly

$$\langle H_k x, x\rangle = \langle H_k^{1/2} e_k, H_k^{1/2} e_k\rangle = : \langle z, z\rangle,$$

$$\langle H_k e_k, x\rangle = \langle H_k^{1/2} e_k, H_k^{1/2} x\rangle = : \langle z, y\rangle.$$

Now using these relations and applying cauchy-Buniakowski's inequality we conclude that the following holds:

$$\langle H_k x, x\rangle\langle H_k e_k, e_k\rangle - \langle H_k e_k, x\rangle^2 = \langle y, y\rangle\langle z, z\rangle - \langle z, y\rangle^2 \geq 0$$

and this inequality holds only if z = y, i.e. Since H_k is non singular, only if $x = e_k$. But in this case

$\langle r_k, x\rangle = \langle r_k, e_k\rangle = \langle r_k, Ar_k\rangle > 0$, since A is positive definite.

Thus for any $x \neq o$, we have

$$\langle H_{k+1}x, x\rangle = \frac{\langle y, y\rangle\langle z, z\rangle - \langle z, y\rangle^2}{\langle H_k e_k, e_k\rangle} + \frac{\langle r_k, x\rangle^2}{\langle r_k, e_k\rangle} > 0$$

and this proves that our reasoning by induction holds. //

(iii) To show $H_n = A^{-1}$, since H_k satisfies (2.1.15) with a = 1, then we have

$H_n e_j = r_j$, j = 0, 1, ..., n - 1, or making use of (2.1.9)

$$H_n A r_j = r_j,\ j = 0, 1, \ldots, n-1.$$

or $(H_n A - I) r_j = 0$, $j = 0, 1, \ldots, n-1$, But since $r_0, r_1, \ldots, r_{n-1}$ are linearly independent, then we have

$$H_n A = I, \text{ i.e.}$$

$$A^{-1} = H_n. \quad //$$

Method 2: Another method of constructing H_k is obtained if we take $a = 1$ in (2.1.18) and choose $u_{k-1} = v_{k-1} = r_{k-1}$ (i.e. in fermulas (2.1.22) $t_{1,k} = t_{3,k} = 1$ and $t_{2,k} = t_{4,k} = 0$). Then

$$H_k = H_{k-1} + \Delta H_{k-1} = H_{k-1} + (r_{k-1} - H_{k-1} e_{k-1}) \frac{r_{k-1}^T}{\langle r_{k-1}, e_{k-1} \rangle}. \qquad (2.1.29)$$

Let us see some properties of matrix H_k obtained by this method.

Properties: (i) H_k is not symmetric

(ii) $H_n = A^{-1}$

Proof: (i) trivial

(ii) it can be demonstrated just in the same way as for method (2.1.28). //

But we can write (2.1.29) in the form:

$$H_k = H_0 + \sum_{i=0}^{k-1} (r_i - H_i e_i) \frac{r_i^T}{\langle r_i, e_i \rangle}. \qquad (2.1.30)$$

From (2.1.9) and (2.1.10), we have $\langle r_i, e_k \rangle = 0$, $0 \le i \le k-1$. Consequently, it follows from (2.1.30) that

$$H_k e_k = H_0 e_k,\ k = 0, 1, \ldots, n-1. \qquad (2.1.31)$$

Thus formula (2.1.29) can be written as follows:

$$H_k = H_{k-1} + (r_{k-1} - H_0 e_{k-1}) \frac{r_{k-1}^T}{\langle r_{k-1}, e_{k-1} \rangle}. \qquad (2.1.32)$$

If $H_0 = I$, this formula is simpler than (2.1.29).

The constructing of methods of conjugate directions can be continued by choosing various combinations of constant a and vectors u_k, v_k by formulas (2.1.22) but we shall not do so.

Remark: In each of the methods treated above conditions (2.1.20) were satisfied by vectors u_k, v_k.

For, in method (2.1.28)(i.e. in the case $V_k = H_k^T e_k$), since matrix H_k is positive definite, we have $\langle v_k, e_k\rangle = \langle H_k^T e_k, e_k\rangle > 0$. In method (2.1.29) (i.e. in the case $u_k = v_k = r_k$), it was mentioned by equation (2.1.21). Thus, in accordance with the results of chapter 2.1C, condition (2.1.8) is satisfied by the methods discussed, i.e. the methods are guaranted to be non degenerated.

Let us now derive formulas directly applicable to the calculation of vectors p_k defined by different matrices H_k. This is easily done by using formula (2.1.25). Since $r_{k-1} = \alpha_{k-1} p_{k-1}$, we have from (2.1.25)

$$p_k = \partial_k (H_0 f'_k - \beta_k p_{k-1}) \tag{2.1.33}$$

$$\beta_k = \frac{\langle H_0 f'_k, e_{k-1}\rangle}{\langle p_{k-1}, e_{k-1}\rangle} \tag{2.1.34}$$

a) Consider **Method 1**, in which matrix H_k is constructed by using vector $v_{k-1} = H_{k-1}^T e_{k-1}$. Then using (2.1.11) and (2.1.2.4) we obtain

$$\partial_k = 1 - \frac{\langle H_0 f'_k, e_{k-1}\rangle}{\langle H_0 f'_k, f'_k\rangle - \langle p_{k-1}, f'_{k-1}\rangle}. \tag{2.1.35}$$

But from (2.1.25), because of (2.1.11) and (2.1.14), we obtain

$$\langle H_0 f'_k, e_{k-1}\rangle = \langle H_0 f'_k, f'_k\rangle \tag{2.1.36}$$

and then using this equation and (2.1.35) we find that

$$\partial_k = - \frac{\langle p_{k-1}, f'_{k-1}\rangle}{\langle H_0 f'_k, f'_k\rangle - \langle p_{k-1}, f'_{k-1}\rangle} \tag{2.1.37}$$

Note also that $\langle f'_k, p_k\rangle = \langle f'_k, p_k\rangle - \langle f'_{k+1}, p_k\rangle = - \langle e_k, p_k\rangle$. (2.1.38)

Comparing formulas (2.1.34) and (2.1.37) and taking into account (2.1.36) and (2.1.38) we have $\partial_k = \frac{1}{1 + \beta_k}$. Hence $\partial_k \beta_k = 1 - \partial_k$. Consequently, formula (2.1.33) which determines vector p_k, in the case where in constructing matrix H_k we use vector $v_{k-1} = H_{k-1}^T e_{k-1}$, can be written in the form

$$p_k = -\partial_k H_0 f'_k + (1 - \partial_k) p_{k-1} \tag{2.1.39}$$

were ∂_k is determined by one of the formulas (2.1.35) or (2.1.37).

b) Consider **Method 2**, in which matrix H_k is constructed by using vectors $u_{k-1} = v_{k-1} = r_{k-1}$. In this case $t_{4,k} = 0$, therefore, from (2.1.23) $\partial_k = 1$ and from (2.1.33) we obtain

$$p_k = H_0 f'_k + \beta_k p_{k-1} \tag{2.1.40}$$

where

$$\beta_k = \frac{\langle H_0 f'_k,\ e_{k-1}\rangle}{\langle p_{k-1},\ e_{k-1}\rangle}.$$

If we use equalities (2.1.36),(2.1.38) and (2.1.26)((2.1.26) has the form $\langle p_k, f'_k\rangle = -\langle H_0 f'_k, f'_k\rangle$) then for determining coefficient β_k one of the follwing formulas can be obtained:

$$\beta_k = -\frac{\langle H_0 f'_k,\ e_{k-1}\rangle}{\langle p_{k-1},\ f'_{k-1}\rangle} = -\frac{\langle H_0 f'_k,\ f'_k\rangle}{\langle H_0 f'_{k-1},\ f'_{k-1}\rangle} = \frac{\langle H_o f'_k,\ f'_k\rangle}{\langle H_o f'_{k-1},\ f'_{k-1}\rangle}. \quad (2.1.41)$$

Expressions (2.1.39) and (2.1.40) which determine vector p_k in their turn can be given the form $p_k = -H_k^T f'_k$, where H_k depends on the coefficients ∂_k and β_k.

The simplest formula for calculating A-orthogenal vectors can be obtained by choosing $H_0 = I$ in (2.1.40). In this case

$$p_k = -f'_k + \beta_k p_{k-1},\ k = 1,\ 2,\ \dots\ n-1,\ p_0 = -f'_0. \quad (2.1.42)$$

where β_k is determined by one of the follwoing formulas:

$$\beta_k = -\frac{\langle f'_k,\ e_{k-1}\rangle}{\langle p_{k-1},\ f'_{k-1}\rangle} = -\frac{\langle f'_k,\ f'_k\rangle}{\langle p_{k-1}\ f'_{k-1}\rangle} = \frac{\langle f'_k, f'_k\rangle}{\langle f_{k-1}, f_{k-1}\rangle} \quad (2.1.43)$$

Method (2.1.27) in which conjugate vectors are constructed by (2.1.42) and (2.1.43) is widely known as the method of conjugate gradients.

EXAMPLE: Minimize $f(x_1, x_2) = 2x_1^2 + x_2^2 + 2x_1x_2 + x_1 - x_2$, starting from point $(0, 0)^T$,

Solution: $f(x) = \frac{1}{2}\langle Ax,\ x\rangle + \langle b, x\rangle$, where $A = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$ $b = (1,-1)^T$, $x = (x_1,\ x_2) \in \mathbb{R}^2$. Clearly A is symmetric and since $4 > 0$, $\begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} = 4 > 0$, then $\langle Ax,\ x\rangle > 0\ \forall x \in \mathbb{R}^2\setminus\{0\}$, i.e. $f(x)$ is strictly convex function.

Now we use method (2.1.27) in which conjugate vectors are constructed by formulas (2.1.42) and (2.1.43), i.e.

$$x_{k+1} = x_k + \alpha_k p_k,\ k = 0,1$$

where

$$p_k = -f'_k + \beta_k p_{k-1},\quad p_0 = -f'_0,\ \beta_k = \frac{f'^T_k f'_k}{f'^T_{k-1} f'_{k-1}},\quad \alpha_k = -\frac{p_k^T f'_0}{p_k^T A p_k}.$$

Iteration 1:

$$f(x) = \nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right)^T = (4x_1 + 2x_2 + 1,\ 2x_1 + 2x_2 - 1)^T.$$

Hence

$f'_o = f'(x_o) = (1,-1)^T$ where $x_o = (0,0)^T$ is a starting point. But since $f'_0 = (1,-1)^T \neq (0,0)^T$, then $x_0 = (0,0)^T$ is not a minimum point. And since then

$$p_0 = -f'_0 = (-1,1)^T$$

$$\alpha_0 = -\frac{p_0^T f'_0}{p_0^T A p_0} = -\frac{(-1,1)(1,-1)^T}{(-1,1)\begin{bmatrix}4 & 2\\ 2 & 2\end{bmatrix}(-1,1)^T} = 1,$$

$$\therefore x_1 = x_0 + \alpha_0 p_0$$

$$= (0,0)^T + 1.(-1,1)^T = (-1,1)^T.$$

But since $f'_1 = f'(x_1) = (-1,-1)^T \neq (0,0)^T$, then $x_1 = (-1,1)^T$ is not a minimum point. Infact we have $f(x_1) = -1$.

Interation 2:

$$x_2 = x_1 + \alpha_1 p_1$$

where $\quad p_1 = -f'_1 + \beta_1 p_0, \quad \beta_1 = \dfrac{f'^T_1 f'^T_1}{f'^T_0 f'^T_0} = \dfrac{(-1,-1)(-1,-1)^T}{(1,-1)(1,-1)^T} = \dfrac{2}{2} = 1.$

But then

$$p_1 = -(-1,-1)^T + 1(-1,1)^T = (0,2)^T,$$

and

$$\alpha_1 = -\frac{p_1^T f'_0}{p_1^T AP_1} = \frac{(0,2)(1,-1)^T}{(0,2)\begin{bmatrix}4 & 2\\ 2 & 2\end{bmatrix}(0,2)^T} = \frac{1}{4}.$$

$$\therefore \quad x_2 = x_1 + \alpha_1 p_1$$

$$= (-1,1)^T + \frac{1}{4}(0,2)^T = (-1,\ 3/2)^T.$$

Since $f'_2 = f'(x_2) = (0,0)^T$, then $x_* = x_2 = (-1,3/2)^T$ is a minimum point such that

$$f(x_*) = -5/4 < -1 = f(x_1).$$

Even if we don't know the point $(-1,\ 3/2)^T$ to be minimum, we will not be able to move from this point, since

$$f'_2 = f'(x_2) = (0,0)^T, \ \beta_2 = \frac{f'^T_2 f'_2}{f'^T_1 f'_1} = \frac{(0,0)(0,0)^T}{2} = 0$$

$$\therefore \quad p_2 = -f'_2 + \beta_2 p_1 = (0,0)^T.$$

This shows that there is no conjugate direction to reduce f further and hence x_2 is a minimum point.

2.1E SUMMARY

We have considered a general schem of constructing methods of conjugate directions and on its basis obtained concerete algorithms. Any of these methods make it possible to find the minimum of a strictly convex quadratic function after a number of steps in process (2.1.27) not excedding n.

If algorithms are judge by the amount of calculations periteration, then algorithms (2.1.39) and (2.1.40) should certainly be perfered. These methods are specially easy to implement if the choice. $H_0 = I$ is made.

The difference in properlies of algorithms tells considerably when they are used for minimization of non quadratic function; this will be discussed in the next section.

2.2. MINIMIZATION OF ARBITRARY FUNCTIONS

2.2A. CONSIDARATIONS ABOOUT THE APPLICABILITY OF THE METHODS

DEFINITION 2.2.1: Iterative processes of the type

$$x_{k+1} = x_k + \alpha_k p_k, \; p_k = -H_k^T f'_k, \; k = 0, 1, \dots, \quad (2.2.1)$$

in which vector p_k (or matrix H_k) is constructed by algorithms of chapter 2.1D and the value of α_k is chosen on condition that

$$f(x_k + \alpha_k p_k) = \min_{\alpha} f(x_k + \alpha p_k),$$

is called methods of conjugate directions.

Suppose theat we intend to use process (2.2.1) for the minimization of an arbitrary (not quadratic) convex function $f(x)$. In this case, matrix $f''(x)$ will have different elements at different points of sequence (2.2.1); by virtue of this fact vectors $p_0, \dots, p_k$ constructed by any of the methods of chapter 2.1D will not be conjugate ($f''(x)$ - orthogenal). However, if the intial point x_0 is in a close neighbourhood of the minimum point, x_*, of a smooth convex function $f(x)$, then at any point of this region matrix $f''(x)$ is close enough to matrix $f''(x_*)$, i.e. the quadratic function

$$\varphi(x) = \frac{1}{2} < f''(x_*)(x-x_*), \; x-x_*> + f(x_*)$$

is a good approximation to the function $f(x)$.

To see this consider Taylor's series expansiev of $f(x)$ about its minimum point, x_*.

$$f(x) = f(x_*) + <f'(x_*), \; x - x_*> + \frac{1}{2} < f''(x_*)(x - x_*), \; x - x_*> + \dots .$$

But since $f'(x_*) = 0$ and the terms involving higher derivatives will be dominated by the quadratic terms then $f(x)$ approaches the quadratic function $\varphi(x)$, in the close neighbour hood of x_*.

Further more, if the matrix $f''(x_*)$ is positive definite, then $\varphi(x)$ will have its minimum at x_*. Thus, we can expect that the properties of vectros $p_0, \dots, p_k$ determined by methods of chapter 2.1D will be close enough to the properties of conjugate vectors ($f''(x_*)$ orthegenal) and therefore the methods of chapter 2.1D prove sufficiently effective inminimizing non quadratic functions too. But in general the methods will no mor yield the result after a finite number of steps since the conditions

$<f''(x_*)p_i, p_j> = 0, \ i \neq j$

will not be strictly satisfied with any initial point x_0.

Note that the condition under which paramenter α_k is chosen can be written also in the form:

$$<f'_{k+1}, p_k> = 0$$

The objective of this section is to substantiate the convergence of methods of conjugate directions in the minimization of nonquadratic funcitons and to obtain bounds on the rate of convergence.

2.2B. THEOREM ON CONVERGENCE OF THE METHODS

In what follows we shall assume that f(x) is a strongly convex twice continueusly differentiable funcition, i.e. the conditions

$$m\|y\|^2 \leq <f''(x)y, y> \leq M\|y\|^2, \ M \geq m > 0 \qquad (2.2.3)$$

are satisfied for all x, $y \in \mathbb{R}^n$, and that a symmetric, positive definite matrix has been chosen as H_0 i.e.

$$m_0\|y\|^2 \leq <H_0 y, y> \leq M_0\|y\|^2, \ M_0 \geq m_0 > 0 \qquad (2.2.4)$$

for all $y \in \mathbb{R}^n$.

DEFINITION 2.2.2: A matrix H_k is said to be restored after afinite number of steps, say n, if with any $\xi = 0, 1, \ldots,$ $H_{\xi n} = H_0$ where H_0 is symmetric and positive definite matrix.

Processes of type (2.2.1) can be realized either with restoration of matrix H_k after afinite number of steps, or with out such a reinitialization.

REMARK: If a process with restoration of matrix H_k after a finite number of steps is being ralized, then for any of the methods of conjugate directions the condition

$$\lim_{k \to \infty} \|f'(x_k)\| = 0 \qquad (2.2.5)$$

is fulfilled.

The fulfilment of condition (2.2.5) for astrictly convex function means that any of the methods discussed in chapter 2.1D, if realized with restoration of matrix H_k after afinite number of steps, converges to the solution x_*.

However, if process are realized with out restoration of H_k, then their convergence must be substantiated. Besides, it is also necessary to eastimate their rate of convergence.

DEFINITION 2.2.3: We say that a sequence (x_k) converges to a point x_* at a super linear rate if the inequality

$$\|x_{k+1} - x_*\| \le q_k\|x_k - x_*\|$$

is satisfied, where $q_k \to 0$ as $k \to \infty$.

REMARKS: 1. In what follows for simplicity we shall often in using vectors and parameters $r_{\xi n+i}$, $f'_{\xi n+i}$, $e_{\xi n+i}$, $\alpha_{\xi n+i}$, $\beta_{\xi n+i}$ etc. $i = 0, 1, \ldots, n-1$ omit index ξn and operates with r_i, f'_i, e_i, α_i, β_i etc. Note that this is done only to simplfy the written form and the real index is $\xi n + 1$.

2. In what follows we shall use the notation: $g = o(h)$ if and only if $\frac{g}{h} \to 0$, for each g, $h \in R^n$.

Let us now formulate the theorem whose contents are the main result of this section.

THEOREM 2.2.1: For the minimization of function $f(x)$ which satisfies conditions (2.2.3) let there be applied process (2.2.1) in which the construction of matrix H_k is performed by one of the methods of chapter 2.1D ((2.1.28), (2.1.29), (2.1.32)) with restoration fo H_k after n steps. If the value of α_k is chosen under the condition that the minimum of the function be in the direction of p_k, then the sequence $(x_{\xi n})$ whatever the initial point x_o chosen converges to the solution at a supper linear rate.

PROOF: Suppose that it is not true, i.e. assume that for the iterative processes described the condition:

$$\|x_{k+1} - x_*\| \ge \lambda\|x_k - x_*\| \qquad (2.2.6)$$

is satisfied with any k were $\lambda > 0$ is a constant. But, using the inequality $\|x_{k+1} - x_*\| \le \frac{1}{\delta}\|f'_{k+1}\|$, $\delta > 0$ and the expression

$$\|f'(x)\| = \|f'(x) - f'(x_*\| \le M\|x - x_*\| \qquad (2.2.7)$$

which hold for a function which satisfies condition (2.2.3), we find that condition (2.2.6) is equivalent to:

$$\|f'_{k+1}\| \geq \delta\|f'_k\| \qquad (2.2.8)$$

where $\delta > 0$ is a constant.

Now then studying the properties of process (2.2.1) and assumming that condition (2.2.8) is fulfilled we find that the following estimates hold:

$$C\|f'_k\| \leq \|r_k\| \leq N\|f'_k\| \qquad (2.2.9)$$

where C,N are constants independent of k, $C > 0$ and

$$\langle e_i, r_j\rangle = O(\|e_i\|\|r_j\|), \quad i \neq j, \quad 0 \leq i, \ j \leq n - 1. \qquad (2.2.10)$$

(The proof that these estimates hold for different algorithms will be given in the next subsection)

Now then using Lagrange's formula:

$$\langle f'(x_i + r_i) - f'(x_i), r_j\rangle = \langle f''(x_i + \theta r_i)r_i, r_j\rangle, \ \theta\in[0,1], \ x_i, \ r_i, \ r_j \in \mathbb{R}^n$$

we obtain

$$\langle e_i, r_j\rangle = \langle f'(x_i+r_i) - f'(x_i), \ r_j\rangle = \langle f''(x_i+\theta r_i)r_i, r_j\rangle = \langle f''_{ic}r_i, r_j\rangle =$$

$$= \langle (f''_i+f''_{ic} - f''_i)r_i, r_j\rangle = \langle f''_i r_i, r_j\rangle + \langle f''_i - f''_{ic})r_i, r_j\rangle$$

i.e. $\langle e_i, r_j\rangle = \langle f''_{ic}r_i, r_j\rangle = \langle f''_i r_i, r_j\rangle + \langle (f''_i - f''_{ic})r_i, \ r_j\rangle \qquad (2.2.11)$

where $x_{ic} = x_i + \theta r_i, \theta\in[0,1]$.

If $\|r_i\| \longrightarrow 0$, then because of the uniform continuity of second derivatives of f(x) on (closed and bounded) set $S = \{x{:}f(x) \leq f(x_0)\}$ we have $\|f''_{ic} - f''_i\| \longrightarrow 0$ and it follows from (2.2.11) that, if (2.2.10) is satisfied, estimates

$$\langle f''_i r_i, \ r_j\rangle = O(\|r_i\|\|r_j\|) + O(\|e_i\|\|r_j\|),$$

$$i \neq j, \quad 0 \leq i, \ j \leq n - 1$$

hold too.

Under conditions (2.2.3) (or using (2.2.7))

$\|e_i\| = \|f'_{i+1} - f'_i\| \leq M\|x_{i+1} - x_i\| = M\|r_i\|$, consequently $\|e_i\|$ and $\|r_i\|$ are of the same order of smallness. Taking this into account, we have

$$\langle f''_i r_i, r_j\rangle = O(\|r_i\|\|r_j\|), \ i \neq j, \ 0 \leq i, \ j \leq n - 1. \qquad (2.2.12)$$

If estimates (2.2.12) are fulfilled, then there are vectors

$$\bar{r}_i = r_i + \omega_i \quad , \ i = 0, \ 1, \ldots, \ n - 1, \qquad (2.2.13)$$

where $\|\omega_i\| = O(\|r_i\|)$, such that

$$\langle f''_{\xi n} \ \bar{r}_i, \ r_j\rangle = 0, \ i \neq j, \ 0 \leq i, \ j \leq n - 1. \qquad (2.2.14)$$

But since vectors $\bar{r}_i$, $i = 0, 1, \ldots, n - 1$, with sufficiently large ξ

are linearly independent (in $\mathbb{R}^n$) then $\bar{r}_o, \bar{r}_1, \ldots, \bar{r}_{n-1}$ is abasis of $\mathbb{R}^n$.

Let $Z_{\xi n}$ be the minimum of the quadratic funciton

$$\varphi(x) = \langle f'_{\xi n}, x - x_{\xi n}\rangle + \frac{1}{2}\langle f''_{\xi n}(x - x_{\xi n}), x - x_{\xi n}\rangle + f_{\xi n}.$$

let us write vectors $z_{\xi n} - x_{\xi n} = \sum_{i=0}^{n-1} a_i \bar{r}_i$. ''''' (2.2.15)

since $\varphi'(z_{\xi n}) = f'_{\xi n} + f''_{\xi n}(z_{\xi n} - x_{\xi n}) = 0$, then using (2.2.15) we obtain

$$\sum_{i=0}^{n-1} a_i f''_{\xi n}\bar{r}_i = - f'_{\xi n}.$$

Hence, taking into account (2.2.14), we get

$$a_i \langle f''_{\xi n}\bar{r}_i, r_i\rangle = -\langle f'_{\xi n}, r_i\rangle$$

i.e. $$a_i = -\frac{\langle f'_{\xi n}, r_i\rangle}{\langle f''_{\xi n}\bar{r}_i, r_i\rangle}, \quad i = 0, 1, \ldots, n-1$$

But $\langle f'_{\xi n}, r_i\rangle = \langle f'_0 - f'_1 + f'_1 - \cdots - f'_{i+1} + f'_{i+1}, r_i\rangle$

$= \langle -e_0 - e_1 - \cdots e_i + f'_{i+1}, r_i\rangle$

$= -\langle e_0, r_i\rangle - \langle e_1, r_i\rangle - \ldots - \langle e_i, r_i\rangle$, since, by (2.2.2), $\langle f'_{i+1}, r_i\rangle = 0$

$= -\langle e_i, r_i\rangle - \sum_{j=0}^{i-1}\langle e_j, r_i\rangle.$

Hence, having in mind estimates (2.2.10), it follows that

$$-\langle f'_{\xi n}, r_i\rangle = \langle e_i, r_i\rangle + \sum_{j=0}^{i-1} O(\|e_j\|\|r_i\|) \qquad (2.2.16)$$

Remarks: 1. According to conditions (2.2.8) and (2.2.9) all of the vectors $r_0, \ldots, r_{n-1}$ are of the same order of smallness

2. Since $\|e_i\| \le M\|r_i\|$, vectors $e_0, \ldots, e_{n-1}$ are of the same order of smallness.

Taking into account the above remarks the equalities (2.2.16) can be written in the following forms:

$$-\langle f'_{\xi n}, r_i\rangle = \langle e_i, r_i\rangle + O(\|r_i\|^2), \quad i = 0, 1, \ldots, n-1.$$

Further, taking into account (2.2.13), we find that

$$\langle f''_{\xi n}\bar{r}_i, r_i\rangle = \langle f''_{\xi n}(r_i+\omega_i), r_i\rangle = \langle (f''_{ic}+f''_{\xi n}-f''_{ic})r_i, r_i\rangle + \langle f''_{\xi n}\omega_i r_i\rangle$$

$$= \langle f''_{ic}r_i, r_i\rangle + \langle (f''_{\xi n}-f''_{ic})r_i, r_i\rangle + \langle f''_{\xi n}\omega_i, r_i\rangle = \langle e_i, r_i\rangle + O_1(\|r_i\|^2)$$

Thus

$$a_i = -\frac{\langle f'_{\xi n}, r_i\rangle}{\langle f''_{\xi n}\bar{r}_i, r_i\rangle} = \frac{\langle e_i, r_i\rangle + O(\|\bar{r}_i\|^2)}{\langle e_i, r_i\rangle + O_1(\|\bar{r}_i\|^2)}.$$

By (2.2.3) and (2.2.11), we have

$$\langle e_i, r_i\rangle = \langle f''_{ic} r_i, r_i\rangle \geq m\|r_i\|^2, \quad i = 0, 1, \ldots, n-1. \tag{2.2.17}$$

consequently, as $\xi \to \infty$ (i.e. as $\|r_i\| \to 0$)

$$a_i \to 1, \quad i = 0, 1, \ldots, n-1. \tag{2.2.18}$$

Since $x_{(\xi+1)n} - x_{\xi n} = x_{\xi n+n} - x_{\xi n} = x_n - x_0 = \sum_{i=0}^{n-1} r_i$, we have, from (2.2.15),

$$x_{(\xi+1)n} - z_{\xi n} = (x_{(\xi+1)n} - x_{\xi n}) - (z_{\xi n} - x_{\xi n}) = \sum_{i=0}^{n-1} (r_i - a_i\bar{r}_i).$$

Hence, taking into account (2.2.13) and (2.2.18), we obtain

$$\|x_{(\xi+1)n} - z_{\xi n}\| = \sum_i O(\|r_i\|)$$

or using (2.2.8) and (2.2.9), we obtain

$$\|x_{(\xi+1)n} - z_{\xi n}\| = O(\|f'_{\xi n}\|). \tag{2.2.19}$$

since $z_{\xi n} - x_{\xi n} = -(f''_{\xi n})^{-1} f'_{\xi n}$ and taking into account (2.2.19), we have

$$x_{(\xi+1)n} - x_{\xi n} = (z_{\xi n} - x_{\xi n}) + (x_{(\xi+1)n} - z_{\xi n}) = -(f''_{\xi n})^{-1} f'_{\xi n} + \eta_{\xi n}$$

where $\|\eta_{\xi n}\| = O(\|f'_{\xi n}\|)$.

It follows that there is a sequence of matrices $D^{-1}_{\xi n} f'_{\xi n} \to (f''_{\xi n})^{-1}$ such that

$$x_{(\xi+1)n} - x_{\xi n} = -D^{-1}_{\xi n} f'_{\xi n}.$$

(we can take, for instance, that

$$D^{-1}_{\xi n} = (f''_{\xi n})^{-1} + \frac{z_{\xi n} - x_{(\xi+1)n}}{\langle f'_{\xi n}, f'_{\xi n}\rangle} (f'_{\xi n})^T.)$$

Equality (2.2.20) shows that sequence $(x_{\xi n})$, $\xi = 0, 1, \ldots$ converges at asuper linear rate to the solution.

Thus assuming that condition (2.2.6) is satisfied and taking estimates (2.2.10) and (2.2.9) to be satisfied, we have demonstrated that for $(x_{\xi n})$ inequality

$$\|x_{(\xi+1)n} - x_*\| \leq \lambda_{\xi n}\|x_{\xi n} - x_*\| \tag{2.2.21}$$

Where $\lambda_{\xi n} \to 0$ as $\xi \to \infty$, holds. However, if condition (2.2.6) holds, the inequality (2.2.21) can not be satisfied. Thus we have come to contradiction. This means that condition (2.2.6) can not be fulfilled for process (2.2.1). It follows that the sequence $(x_{\xi n})$ converges to the solution at a super linear rate. //

2.2C. STUDY OF PROPERTIES OF DIFFERENT ALGORITHMS.

Weturn now to the proof of the validity of estimates (2.2.9), (2.2.10) for different methods of conjugate directions with restoration of matrix H_k after n-steps, assuming that inequality (2.2.6) (or (2.2.8)) isfulfilled.

The fact that for any of these methods the estimates hold is established by induction; it is demonstrated that estimates (2.2.9), (2.2.10) take place with $i \neq j$, $i,j = 0, 1$, and then supposing that these estimates take place with $0 \leq i, j \leq \tau < n - 1$ we prove that they remain valid also with $0 \leq i, j \leq \tau + 1$.

THEOREM 2.2.2: If inequality (2.2.6) (or(2.2.8)) is fulfilled, then estimates (2.2.9), (2.2.10) are valid for method (2.1.28) (compare chapter 2.1D) with restoration of matrix H_k after n steps.

PROOF: If restoration of matrix (2.1.28), i.e.

$$H_k = H_{k-1} + \frac{r_{k-1} r_{k-1}^T}{\langle r_{k-1}, e_{k-1}\rangle} - \frac{H_{k-1} e_{k-1} e_{k-1}^T H_{k-1}}{\langle H_{k-1}^T e_{k-1}, e_{k-1}\rangle}.$$

is performed after a finite number of steps, then with any k matrix H_k is bounded:

$$\|H_k\| \leq L, \ L < \infty \tag{2.2.22}$$

To see this, by (2.2.2), $\langle H_k f'_k, f'_{k+1}\rangle = -\langle p_k, f'_{k+1}\rangle = 0$

There fore

$$\begin{aligned}\langle H_k e_k, e_k\rangle &= \langle H_k (f'_{k+1} - f'_k), f'_{k+1} - f'_k\rangle \\ &= \langle H_k f'_k, f'_k\rangle + \langle H_k f'_{k+1}, f'_{k+1}\rangle.\end{aligned} \tag{2.2.23}$$

since H_k is positive definite (comp. chapter 2.1D), we obtain, from (2.2.2) and (2.2.23), $\langle H_k e_k, e_k\rangle \geq \langle H_k f'_k, f'_k\rangle = -\langle p_k, f'_{k+1} - e_k\rangle$

$$= \langle p_k, e_k\rangle = \frac{1}{\alpha_k} \langle e_k, r_k\rangle.$$

Hence, according to (2.2.17)

$$\langle H_k e_k, e_k\rangle \geq \frac{m}{\alpha_k} \|r_k\|^2. \tag{2.2.24}$$

Taking into account more over that $\|e_k\| \leq M\|r_k\|$, we obtain from (2.1.28) that

$$\|H_{k+1}\| \le \|H_k\| + \frac{\|r_k\|^2}{m\|r_k\|^2} + \frac{\alpha_k\|H_k\|^2 M^2\|r_k\|^2}{m\|r_k\|^2}$$

Using condition (2.2.4) we obtain that $\alpha_0 := \alpha_{\xi n} \le \bar{\alpha} < \infty$ and on the strength of this it follows from the recursive inequality for $\|H_{k+1}\|$ that estimate (2.2.22) hold for $H_{\xi n+1} (=: H_1)$. On this ground we shall prove

below by induction that $\alpha_{\xi n+i} \le \bar{\alpha} < \infty$ with any $i = 1, \ldots, n-1$. Taking this into account we find that (2.2.22) holds.

Let us prove now that with $i = 1$ the following relations hold:

$$\langle r_1, e_0\rangle = 0, \quad \langle e_1, r_0\rangle = O(\|r_1\|\|r_0\|),$$

$$C_1\|f'_1\| \le \|r_1\| \le N_1\|f'_1\| \tag{2.2.25}$$

where the constants N_1, c_1 are independent of k and $c_1 > 0$. The first of these estimates is found as follows:

$$\langle r_1, e_0\rangle = \langle \alpha_1 p_1, e_0\rangle = -\alpha_1\langle H_1 f_1, e_o\rangle = -\alpha_1\langle f'_1, H_1 e_0\rangle.$$

But, using (2.1.15), we obtain $H_1 e_0 = r_0$, therefore $\langle r_1, e_0\rangle = -\alpha_1\langle f'_1, r_0\rangle = 0$, using (2.1.14). Further, using (2.2.11) we obtain

$$\langle e_1, r_0\rangle = \langle f_{1c} r_1, r_0\rangle = = \langle r_1, e_0\rangle + O(\|r_1\|\|r_0\|) = O(\|r_1\|\|r_0\|).$$

Let us now show that the estimates hold for $\|r_1\|$. It follows from (2.1.28), taking into account (2.2.2) and (2.2.23), that

$$\langle H_1 f'_1, f'_1\rangle = \langle H_0 f'_1, f'_1\rangle + \frac{\langle f'_1, r_0\rangle^2}{\langle r_0, e_0\rangle} - \frac{\langle H_0 e_0, f'_1\rangle^2}{\langle H_0 e_0, e_0\rangle}$$

$$= \langle H_0 f'_1, f'_1\rangle - \frac{\langle H_0 e_0, f'_1\rangle^2}{\langle H_0 e_0, e_0\rangle} = \langle H_0 f'_1, f'_1\rangle - \frac{\langle H_0 e_0, f'_1\rangle^2}{\langle H_0 f'_0, f'_0\rangle + \langle H_0 f'_1, f'_1\rangle}$$

But since $\langle H_0 e_0, f_i\rangle = \langle H_0(f'_1 - f'_0), f'_1\rangle = \langle H_0 f'_1, f'_1\rangle + \langle p_0, f'_1\rangle = \langle H_0 f'_1, f'_1\rangle$,

Then $$\langle H_1 f'_1, f'_1\rangle = \frac{\langle H_0 f'_1, f'_1\rangle\langle H_0 f_0; f'_0\rangle}{\langle H_0 f'_0, f'_0\rangle + \langle H_0 f'_1, f'_1\rangle} = \langle H_0 f'_1, f'_1\rangle \frac{1}{1 + \frac{\langle H_0 f'_1, f'_1\rangle}{\langle H_0 f'_0, f'_0\rangle}}$$

Using estimate (2.2.7) we deduce that for a function which satisfies (2.2.3)

$$m\left(1 + \frac{m}{M}\right)(f(x) - f(x_*)) \le \|f'(x)\|^2 \le \frac{2M^2}{m}(f(x) - f(x_*)). \tag{2.2.26}$$

Taking into account estimates (2.2.4) and (2.2.26) we have on set $S_o := \{x : f(x) \le f(x_0)\}$,

$$\frac{\langle H_0 f'_1, f'_1\rangle}{\langle H_0 f'_0, f'_0\rangle} \le \frac{M_0\|f'_1\|^2}{m_0\|f'_0\|^2} \le \frac{d_1(f'_1 - f_*)}{d_2(f_0 - f_*)} \le \frac{d_1}{d_2}$$

where d_1, d_2 are constants independent of ξ.

To see this, from (2.2.4) we obtain $\langle H_0 f'_1, f'_1\rangle \le M_0\|f'_1\|$ and

$$\frac{1}{\langle H_0 f'_0, f'_0\rangle} \le \frac{1}{m_0\|f'_0\|^2}. \quad \text{Therefore, } \frac{\langle H_0 f'_1, f'_1\rangle}{\langle H_0 f'_0, f'_0\rangle} \le \frac{M_0\|f'_1\|^2}{m_0\|f'_0\|^2}.$$

But from (2.2.26) $\|f'_1\|^2 \le l_1(f_1 - f_*)$, $l_1 := \frac{2M^2}{m}$ and $l_2(f_0 - f_*) \le \|f'_0\|^2$,

$l_2 := m(1+\frac{m}{M})$. Thus $\frac{1}{\|f_o\|} \le \frac{1}{l_2(f_0-f_*)}$ and therefore, $\frac{M_0\|f'_1\|^2}{m_0\|f'_0\|^2} \le \frac{M_0 l_1(f_1-f_*)}{m_0 l_2(f_0-f_*)}$

$$= \frac{d_1(f_1-f_*)}{d_2(f_0-f_*)}$$

where $d_1 := M_0 l_1 = \frac{2M_0M^2}{m} > 0$, $d_2 := m_0 l_2 = m_0 m(1 + \frac{m}{M}) > 0$.

And since $f_0 > f_1 > f_*$ we get $0 < f_1-f_* < f_0-f_*$, i.e. $0 < \frac{f_1-f_*}{f_0-f_*} < 1$

Hence $\frac{\langle H_0 f'_1, f'_1\rangle}{\langle H_0 f'_0, f'_0\rangle} \le \frac{M_0\|f'_1\|^2}{m_0\|f'_0\|^2} \le \frac{d_1(f_1-f_*)}{d_2(f_1-f_*)} \le \frac{d_1}{d_2}$.

By virtue of this,

$$\langle H_1 f'_1, f'_1\rangle \ge \frac{\langle H_0 f'_1, f'_1\rangle}{1 + \frac{d_1}{d_2}} \ge \frac{m_0}{1 + \frac{d_1}{d_2}}\|f'_1\|^2 = a_1\|f'_1\|^2. \tag{2.2.27}$$

where $a_1 := m_0 \big/ \left(1 + \frac{d_1}{d_2}\right) > 0$ is independent of ξ.

Let us use now inequality (2.2.27) in order to estimate the value of parameter $\alpha_{\xi n+1}$. Since, by Taylor's series expansion,

$$f_2 = f_1 + \alpha_1\langle f'_1, p_1\rangle + \frac{\alpha_1^2}{2}\langle f''_{1c}p_1, p_1\rangle$$

then $-\langle f'_1, p_1\rangle = \alpha_1\langle f''_1 p_1, p_1\rangle$, i.e. $-\frac{\langle f'_1, p_1\rangle}{\langle f''_1 p_1, p_1\rangle} = \alpha_1$.

But using (2.2.3) and the fact that $-\langle f'_1, p_1\rangle = \langle H_1 f'_1, f'_1\rangle > 0$ we obtain

$$-\frac{\langle f'_1, p_1\rangle}{M\|p_1\|^2} \le \alpha_1 \le -\frac{\langle f'_1, p_1\rangle}{m\|p_1\|^2}.$$

Now by (2.2.27), we have $-\langle f'_1, p_1\rangle = \langle H_1 f'_1, f'_1\rangle \ge a_1\|f'_1\|^2$ and by (2.2.22),

$\|p_1\| = \|H_1 f'_1\| \le L\|f'_1\|$; taking these estimates into account, we have

$$\alpha_1 \ge -\frac{\langle f'_1, p_1\rangle}{M\|p_1\|^2} \ge \frac{a_1\|f'_1\|^2}{M L^2\|f'_1\|^2} = \frac{a_1}{ML^2} =: \alpha > 0.$$

At the same time it follows from (2.2.27) that $\|p_1\| \ge a_1\|f'_1\|$. Using this estimate we obtain

$$\text{that } \alpha_1 \le -\frac{\langle f'_1, p_1\rangle}{m\|p_1\|^2} = \frac{\langle H_1 f'_1, f'_1\rangle}{m\|p_1\|^2} \le \frac{L\|f'_1\|^2}{ma_1^2\|f'_1\|^2} = \frac{L}{ma_1^2} =: \alpha < \infty.$$

Thus we find that

$$N_1\|f'_1\| = \bar{\alpha}L\|f'_1\| \ge \|r_1\| = \alpha_1\|p_1\| \ge \alpha\, a_1\|f'_1\| = c_1\|f'_1\|$$

where constants $N_1 := \alpha a_1, C_1 := \alpha a_1$ are independent of ξ. Thus estimates (2.2.25) hold.

Suppose that the estimates

$$\langle r_i, e_j\rangle = O(\|r_i\|\|r_j\|),\ i \ne j,\ 0 \le i, j \le \tau < n - 1, \tag{2.2.28}$$

$$C_i\|f'_i\| \le \|r_i\| \le N_i\|f'_i\|,\ 0 \le i \le \tau \tag{2.2.29}$$

where constants N_i, $C_i > 0$ are independent of ξ, hold. Let us show that similar estimates take place also with $0 \le i, j \le \tau + 1$

$$\langle f'_{\tau+1}, r_j\rangle = \langle f'_{j+1}, r_j\rangle + \langle e_{j+1} + \dots + e_\tau, r_j\rangle,\ 0 \le j < \tau. \tag{2.2.30}$$

According to condition (2.2.8) and estimates (2.2.29) quantities $\|f'_{\tau+1}\|, \|f'_\tau\|$ and $\|r_i\|$ with any $0 \le i < \tau$ are of the same order of smallness. Taking this into account and using conditions (2.2.2) and (2.2.28) we find according to (2.2.30) that

$$\langle f'_{\tau+1}, r_j\rangle = O(\|f'_{\tau+1}\|\|r_j\|) = O(\|r_j\|),\ 0 \le j < \tau.$$

since $\langle f'_{\tau+1}, r_\tau\rangle = 0$, according to (2.2.2), we obtain finally

$$\langle f'_{\tau+1}, r_j\rangle = O(\|r_j\|^2),\quad 0 \le j \le \tau. \tag{2.2.31}$$

Let us estimate now the quantity $\langle H_{\tau+1} f'_{\tau+1}, f'_{\tau+1}\rangle$. using formula (2.1.28) and taking into account (2.2.23) we obtain for any $0 \le j \le \tau$:

$$\langle H_{j+1} f'_{\tau+1}, f'_{\tau+1}\rangle = \langle H_j f'_{\tau+1}, f'_{\tau+1}\rangle + \frac{\langle r_j, f'_{\tau+1}\rangle^2}{\langle r_j, e_j\rangle} - \frac{\langle H_j e_j, f'_{\tau+1}\rangle^2}{\langle H_j e_j, e_j\rangle} \ge$$

$$\ge \frac{1}{\langle H_j e_j, e_j\rangle}\left[\langle H_j f'_{\tau+1}, f'_{\tau+1}\rangle\langle H_j e_j, e_j\rangle - \langle H_j e_j, f'_{\tau+1}\rangle^2\right]$$

$$= \frac{1}{\langle H_j e_j, e_j\rangle}\left[\langle H_j f'_{\tau+1}, f'_{\tau+1}\rangle\langle H_j f'_{j+1}, f'_{j+1}\rangle + \langle H_j f'_{\tau+1}, f'_{\tau+1}\rangle\langle H_j f'_j, f'_j\rangle\right.$$

$$-\langle H_j f'_{j+1}, f'_\tau\rangle^2 - \langle H_j f'_j, f'_{\tau+1}\rangle^2 + 2\langle H_j f'_{j+1}, f'_{\tau+1}\rangle\langle H_j f'_j, f'_{\tau+1}\rangle].$$

On the right-hand side of this in equality the difference between the first and the third terms of the numerator, by Cauchy-Buniakowski's in equality, is nonnegative. Taking into account estimates (2.2.31), (2.2.24) and (2.2.22) and that α_j, $j \le \tau$ is bounded, we obtain that the ratio of the last two terms of the numerator to the denominator is of the order of $O(\|r_j\|\|f'_{\tau+1}\| = O(\|f'_{\tau+1}\|^2)$. Hence

$$\langle H_{j+1} f'_{\tau+1}, f'_{\tau+1}\rangle \ge \frac{\langle H_j f'_{\tau+1}, f'_{\tau+1}\rangle\langle H_j f'_j, f'_j\rangle}{\langle H_j e_j, e_j\rangle} - O(\|f'_{\tau+1}\|^2).$$

Estimates (2.2.29) imply that there are constants a_i independent of ξ such that $\langle H_j f'_j, f_j\rangle = -\langle p_j, f_j\rangle \ge a_j \|f'_j\|^2$. Making use of this fact and of (2.2.22) we have

$$\langle H_{j+1} f'_{\tau+1}, f'_{\tau+1}\rangle \ge \frac{a_j \|f'_j\|^2}{M\, L \|r_j\|^2} \langle H_j f'_{\tau+1}, f'_{\tau+1}\rangle$$

$$- O(\|f'_{\tau+1}\|^2) \ge \tilde{a}_j \langle H_j f'_{\tau+1}, f'_{\tau+1}\rangle - O(\|f'_{\tau+1}\|^2) \qquad (2.2.32)$$

where $\tilde{a}_j > 0$ and is independent of ξ (by (2.2.29)).

It was noted in the preceding subsection that for processes with restoration of H_k as $k \to \infty$ we have $\|f'_k\| \to 0$. Therefore, it follows from inequalities (2.2.32), taking into account that matrix H_k is positive definite, that if with any ξ we have $\langle H_j f'_{\tau+1}, f'_{\tau+1}\rangle \ge \partial_{j+1}\|f'_{\tau+1}\|^2$ where $\partial_j > o$ and is independent of ξ, then there is aconstant $\partial_{j+1} > o$ such that with any ξ we shall have $\langle H_{j+1} f'_{\tau+1}, f_{\tau+1}\rangle \ge \partial_{j+1}\|f_{\tau+1}\|^2$. But in estimating the quantity $\langle H_1 f'_{\tau+1}, f'_{\tau+1}\rangle$ we find, since

$$\langle H_0 f'_{\tau+1}, f'_{\tau+1}\rangle \ge m_0 \|f'_{\tau+1}\|^2,$$

that there is a constant ∂_1 such that $\langle H_1 f'_{\tau+1}, f'_{\tau+1}\rangle \ge \partial_1 \|f'_{\tau+1}\|^2$ with any ξ. Taking this into account, our argument by induction shows that there is a constant $a_{\tau+1}$ independent of ξ and such that $\langle H_{\tau+1} f'_{\tau+1}, f'_{\tau+1}\rangle \ge a_{\tau+1}\|f'_{\tau+1}\|^2$. We establish now just as we did above that

$$\frac{a_{\tau+1}}{ML^2} \le -\frac{\langle f'_{\tau+1}, p_{\tau+1}\rangle}{M\|p_{\tau+1}\|^2} \le \alpha_{\tau+1} \le -\frac{\langle f'_{\tau+1}, p_{\tau+1}\rangle}{m\|p_{\tau+1}\|^2} \le \frac{L}{ma^2_{\tau+1}}.$$

There fore, we have

$$N_{\tau+1}\|f'_{\tau+1}\| \geq \|r_{\tau+1}\| = \alpha_{\tau+1}\|H_{\tau+1}f'_{\tau+1}\| \geq C_{\tau+1}\|f'_{\tau+1}\| \qquad (2.2.33)$$

Let us show now that

$$H_{\tau+1}\, e_j = r_j + \eta_j, \quad 0 \leq j \leq \tau \qquad (2.2.34)$$

where $\|\eta_j\| = 0\,(\|r_j\|)$.

Multiplying both sides of formula (2.1.28) by e_j we obtain

$$H_{s+1}e_j = H_s e_j + \frac{r_s\langle r_s, e_j\rangle}{\langle r_s, e_s\rangle} - \frac{\langle H_s e_s, e_j\rangle H_s e_s}{\langle H_s e_s, e_s\rangle}. \qquad (2.2.35)$$

If we assume that with a certain s, $j+1 \leq s \leq \tau$, equalities $H_s e_j = r_j + \tilde{\eta}_j$ take place where $\|\tilde{\eta}_j\| = O(\|r_j\|)$, then using estimates (2.2.28), (2.2.24), (2.2.22) and taking into account that all of the quantities $\|r_s\|$ are of the same order of smallness we also have by (2.2.35) that $H_{s+1}\, e_j = r_j + \eta_j$, where $\|\eta_j\| = O(\|r_j\|)$. But $H_{j+1}\, e_j = r_j$, and we establish by induction that equalities (2.2.34) hold true.

Taking into account (2.2.34) we have

$$\langle r_{\tau+1}, e_j\rangle = -\alpha_{\tau+1}\langle H_{\tau+1}f'_{\tau+1}, e_j\rangle = -\alpha_{\tau+1}\langle f'_{\tau+1}, r_j + \eta_j\rangle.$$

Therefore by (2.2.31), we find that

$$\langle r_{\tau+1}, e_j\rangle = O(\|r_j\|^2) + O(\|f'_{\tau+1}\|\|r_j\|), \quad 0 \leq j \leq \tau.$$

In equalities (2.2.8) and (2.2.33) show that $\|r_{\tau+1}\|$ is of the same order of smallness as $\|f'_{\tau+1}\|$ and consequently as $\|r_j\|$, $0 \leq j \leq \tau$. It follows that

$$\langle r_{\tau+1}, e_j\rangle = O(\|r_{\tau+1}\|\|r_j\|) = O(\|r_{\tau+1}\|^2), \quad 0 \leq j \leq \tau. \qquad (2.2.36)$$

Taking this into account we establish in a manner analogous to that used before with i = 1 that also

$$\langle e_{\tau+1}, r_j\rangle = O(\|r_{\tau+1}\|^2) \quad 0 \leq j \leq \tau. \qquad (2.2.37)$$

The relations (2.2.33), (2.2.36) and (2.2.37) show that estimates (2.2,28) and (2.2.29) realy take place with $\tau+1$ too.

Thus it has been established that estimates (2.2.9), (2.2.10) hold for method (2.1.28) if it is assumed that process (2.2.1) is realized with restoration of matrix H_k after afinite number of steps. //

THEOREM 2.2.3: If inequality (2.2.6) (or(2.2.8)) is fulfilled, then estimates (2.2.9), (2.2.10) are valid for method (2.1.29) (comp. chapter 2.1D) with restoration of matrix H_k after n steps.

PRROF: If matrix (2.1.29), i.e.

$$H_k = H_{k-1} + (r_{k-1} - H_{k-1}e_{k-1})\frac{r_{k-1}^T}{<r_{k-1}, e_{k-1}>}.$$

is restored after afinite number of steps, then with any k the matrix H_k has a bound. This follows from inequality

$$\|H_{k+1}\| \le \|H_k\| + \frac{\|r_k\|^2}{m\|r_k\|^2} + \frac{\|H_k\| M \|r_k\|^2}{m\|r_k\|^2}.$$

With i = 1, $<H_1^T f'_1, f'_1> = <H_0 f'_1, f'_1> + (r_0 - H_0 e_0)\frac{<r_0, f'_1>}{<r_0, e_0>}$.

But since by (2.2.2) $<r_0, f'_1> = \alpha_0 <p_0, f'_1> = 0$, then

$$<H_1^T f'_1, f'_1> = <H_0 f'_1, f'_1> \ge m_0\|f'_1\|^2.$$

Making use of these relations and reasoning as in studying method (2.1.28) we establish that estimates (2.2.25) hold and then assuming that estimates (2.2.28) and (2.2.29) hold we demonstrate that estimate (2.2.31) holds.

Further we have that, using (2.1.30) (comp. chapter 2.1D),

$$<H_{\tau+1}^T f'_{\tau+1}, f'_{\tau+1}> = <H_0 f'_{\tau+1}, f'_{\tau+1}> + \sum_{i=0}^{\tau} \frac{<f'_{\tau+1}, r_i><r_i - H_i e_i, f'_{\tau+1}>}{<r_i, e_i>}.$$

Using estimates (2.2.17), (2.2.31) and the fact that H_k has a bound and taking into account that all the quantities $\|r_i\|$, $\|e_i\|$, $\|f'_{\tau+1}\|$ are of the same order of smallness we find that

$$<H_{\tau+1}^T f'_{\tau+1}, f'_{\tau+1}> \ge m_0\|f'\|^2 + O(\|f'_{\tau+1}\|^2).$$

Consequently with sufficiently small values of $\|f'_{\tau+1}\|$ (i.e. with sufficiently large ξ) we have

$$<H_{\tau+1}^T f'_{\tau+1}, f'_{\tau+1}> \ge a_{\tau+1}\|f'_{\tau+1}\|^2$$

where $a_{\tau+1} > 0$ and is independent of ξ. This being so, we find that

$\bar{\alpha} \ge \alpha_{\tau+1} \ge \alpha > 0$ and $C_{\tau+1}\|f'_{\tau+1}\| \le \|r_{\tau+1}\| \le N_{\tau+1}\|f'_{\tau+1}\|$.

Using equalities $H_{s+1}e_i = H_s e_i + (r_s - H_s e_s)\frac{<r_s, e_i>}{<r_s, e_s>}$, $i + 1 \le s \le \tau$,

taking into account estimates (2.2.28), (2.2.17), the fact that H_k has a bound and reasoning as we did in studing (2.1.28), we ascertain that matrix $H_{\tau+1}$ satisfies equations (2.2.34) and consequently estimates (2.2.36) and (2.2.37) remain valid.

This completes the proof that our reasoning by induction holds. //

REMARK: The study of method (2.1.32) is carried out in a similar way.

2.2D. FURTHER STUDY OF THE RATE OF CONVERGENCE

1. Suppose now that matrix $f''(x)$ besides conditions (2.2.3) satisfies Lipschitz condition:

$$\|f''(x_1) - f''(x_2)\| \le R\|x_1 - x_2\|, \tag{2.2.38}$$

where $x_1, x_2 \in \mathbb{R}^n$ and $R > 0$ is aconstant.
In this case it is possible to obtain a more precise bound on the rate of convergence of sequence $(x_{\xi n})$.

To make refering more convenient, we shall give the different relations which hold if (2.2.3) does (many of them were often used before):

$$m\|x - x_*\| \le \|f'(x)\| \le M\|x - x_*\| \tag{2.2.39}$$

$$d_1[f(x) - f(x_*)] \le \|f'(x)\|^2 \le d_2[f(x) - f(x_*)] \tag{2.2.40}$$

(the constants d_1 and d_2 are independent of the choice of point x)

$$m\|r_k\| \le \|e_k\| \le M\|r_k\| \tag{2.2.41}$$

Let x, y be arbitrary points such that

$$f(y) \le f(x). \tag{2.2.42}$$

Then making use of (2.2.40) we establish that

$$\|f'(y)\| \le C\|f'(x)\|. \tag{2.2.43}$$

Here and further on in this subsection C denotes various constants (not equal to zero) which are independent of the choice of points $x, y \in \mathbb{R}^n$.

If (2.2.42) is satisfied we have by (2.2.39) and (2.2.43)

$$\|y - x_*\| \le C\|x - x_*\|. \tag{2.2.44}$$

2. Suppose that for the iterative processes being studied the following estimate holds:

$$\|f'_{\xi n+i}\|\|f'_{\xi n+i+1}\| \le \lambda_\xi\|f'_{\xi n+j}\|, \quad 0 \le i < j \le n - 1. \tag{2.2.45}$$

Here and further on, λ_ξ will denote different variables tending to zero as $\xi \to \infty$.

In what follows we shall limit our selves to the study of the properties of method (2.1.28). However, the results obtained (lemma 2.2.1, theorem 2.2.4) hold also for other algorithms of conjugate directions.

LEMMA: 2.2.1: Let process (2.2.1) be used for the minimization of the twice continuously differentiable function f(x) which also satisfies conditions (2.2.3) and (2.2.38); in this process the construction of matrix H_k is performed by formula (2.1.28). Then if in equalities (2.2.45) hold, estimates (2.2.9) also hold and more over

$$|\langle r_{\xi n+i}, e_{\xi n+j}\rangle| \le C\|r_{\xi n+t}\|^2\|r_{\xi n+t+1}\|,$$

$$t = \min\{i,j\},\ i \ne j,\ \ 0 \le i,j \le n-1. \qquad (2.2.46)$$

PROOF: This lemma is proved in the same way as estimates (2.2.9) and (2.2.10) form method (2.1.28) in Theorem 2.2.2; only the order of smallness of some quantities is determined maore precisely. There fore, we shall concentrate only on the changes involved.

With i = 1

$$\langle e_1, r_0\rangle = \langle r_1, e_0\rangle + \langle r_1, (f''_{1c} - f''_c)r_0\rangle.$$

Taking (2.2.38) into account, we obtain

$$\|f''_{1c} - f''_{0c}\| = \|f''(x_1+\theta r_1) - f''(x_0+\theta r_0)\|$$

$\le R(\|x_1-x_0\| + \|r_1\| + \|r_0\|) \le R(2\|r_0\| + \|r_1\|)$. Since $x_1-x_0 = r_0$, $\|\theta\| \le 1$.

By (2.2.44) and (2.2.39), we have

$$\|r_k\| \le \|x_k-x_*\| + \|x_{k+1}-x_*\| \le C\|f'_k\|. \qquad (2.2.47)$$

Using (2.2.47) and (2.2.43) we establish that $\|r_1\| \le C\|f'_1\| \le C\|f'_0\|$. Taking into account also that $\|r_0\| \ge C\|f'_0\|$ we obtain $\|r_1\| \le C\|r_0\|$. Consequently, $\|f''_{1c}-f''_{0c}\| \le C\|r_0\|$. Using this we find that

$$|\langle e_1, r_0\rangle| \le \|r_1\|\|f''_{1c}-f''_{0c}\|\|r_0\| \le C\|r_0\|^2\|r_1\|.$$

suppose that estimates (2.2.9) and (2.2.46) hold with $0 \le i,j \le \tau < n-1$. Then since $\langle f'_{j+1}, r_j\rangle = 0$, we have

$$|\langle f'_{\tau+1}, r_j\rangle| = |\langle e_\tau + \ldots + e_{j+1}, r_j\rangle| \le c\|r_j\|^2\|r_{j+1}\|,\ 0 \le j < \tau. \qquad (2.2.48)$$

Hence using (2.2.47),

$$|\langle f'_{\tau+1}, r_j\rangle| \le C\|r_j\|\|f'_j\|\|f'_{j+1}\|. \qquad (2.2.49)$$

If (2.2.45) is satisfied, then $\|f'_j\|\|f'_{j+1}\| \le \lambda_\xi\|f'_{\tau+1}\|$. Making use of this inequality we obtain from (2.2.49) that

$$|\langle f'_{\tau+1}, f_j\rangle| \le \lambda_\xi\|r_j\|\|f'_{\tau+1}\|,\ \ 0 \le j < \tau.$$

since $\langle f'_{\tau+1}, r_\tau\rangle = 0$, we finally obtain

$$|\langle f'_{\tau+1}, r_j\rangle| \le \lambda_\xi \|r_j\| \|f'_{\tau+1}\|, \quad 0 \le j < \tau. \tag{2.2.50}$$

using estimates (2.2.24), (2.2.50) and (2.2.22) we obtain also:

$$\frac{\langle H_j f'_j , f'_{\tau+1}\rangle^2}{\langle H_j e_j , e_j\rangle} \le \frac{\langle f'_{\tau+1} , r_j\rangle^2}{m\|r_j\|^2} \le \lambda_\xi \|f'_{\tau+1}\|^2, \ 0 \le j \le \tau , \tag{2.2.51}$$

$$\frac{|\langle H_j f'_{j+1}, f'_{\tau+1}\rangle\langle H_j f'_j, f'_{\tau+1}\rangle}{\langle H_j e_j, e_j\rangle} \le \frac{\lambda_\xi \|f'_{\tau+1}\|^2 \|f_{j+1}\|}{\|r_j\|} . \tag{2.2.52}$$

Taking into account that $\|r_j\| \ge C\|f'_j\|$, $0 \le j \le \tau$ and using (2.2.47), (2.2.43) we obtain

$$\|r_i\| \le C\|f'_i\| \le C\|f'_j\| \le C\|r_j\|, \ 0 \le j \le i \le \tau. \tag{2.2.53}$$

Making use of (2.2.53) we have

$$\frac{\lambda_\xi \|f'_{\tau+1}\|^2 \|f'_{j+1}\|}{\|r_j\|} \le \lambda_\xi \|f'_{\tau+1}\|^2, \quad 0 \le j \le \tau. \tag{2.2.54}$$

Using (2.2.51),(2.2.52) and (2.2.54) we establish in the same way as for method (2.1.28) that

$$\langle H_{\tau+1} f'_{\tau+1}, f'_{\tau+1}\rangle \ge a_{\tau+1} \|f'_{\tau+1}\|^2.$$

Let us show now that

$$H_{\tau+1} e_j = r_j + \eta_{j,\tau+1}, \quad 0 \le j \le \tau \tag{2.2.55}$$

Where $\|\eta_{j,\tau+1}\| \le C \sum_{v=j+1}^{\tau} \frac{\|r_j\|^2 \|r_{j+1}\|}{\|r_v\|}$, $0 \le j < \tau$, $\|\eta_{\tau,\tau+1}\| = 0$ (2.2.56)

Indeed,

$$H_{s+1} e_j = H_s e_j + \frac{r_s \langle r_s, e_j\rangle}{\langle r_s, e_s\rangle} - \frac{\langle e_s, H_s e_j\rangle H_s e_s}{\langle H_s e_s, e_s\rangle} , \tag{2.2.57}$$

we have $H_{j+1} e_j = r_j$, using estimates (2.2.46), which hold by assumption with $0 \le s, j \le \tau$, (2.2.22), (2.2.24), (2.2.41) and taking into account that $|\alpha_i| \le C$, $0 \le i \le \tau$, we obtain with $s = j+1$:

$$\frac{\|\langle e_{j+1}, H_{j+1} e_j\rangle H_{j+1} e_{j+1}\|}{\langle H_{j+1} e_{j+1}, e_{j+1}\rangle} = \frac{\|\langle e_{j+1}, r_j\rangle H_{j+1} e_{j+1}\|}{\langle H_{j+1} e_{j+1}, e_{j+1}\rangle} \le C \frac{\|r_j\|^2 \|r_{j+1}\|}{\|r_{j+1}\|}.$$

Noting also that, by (2.1.13), $\langle r_{j+1}, e_j\rangle = 0$ we obtain from (2.2.57):

$$H_{j+2}e_j = r_j + \eta_{j,i+2}, \quad \|\eta_{j,j+2}\| \le C \frac{\|r_j\|^2\|r_{j+1}\|}{\|r_{j+1}\|}$$

Suppose that with acertain $j+1 < s \le \tau$ we have

$$H_s e_j = r_j + \eta_{j,s}, \quad \|\eta_{j,s}\| \le C \sum_{v=j+1}^{s-1} \frac{\|r_j\|^2\|r_{j+1}\|}{\|r_v\|}.$$

Then because of the same conditions used with $s = j+1$ we have:

$$\frac{\|<e_s, H_s e_j>H_s e_s\|}{<H_s e_s, e_s>} \le C \frac{\|r_j\|^2\|r_{j+1}\|}{\|r_s\|} = C \sum_{v=j+1}^{s-1} \frac{\|r_j\|^2\|r_{j+1}\|}{\|r_v\|},$$

$$\frac{\|r_s<r_s, e_j>\|}{<r_s, e_s>} \le C \frac{\|r_j\|^2\|r_{j+1}\|}{\|r_s\|}.$$

using these estimates in (2.2.57) we establish that

$$H_{s+1} e_j = r_j + \eta_{j,s+1}, \quad \|\eta_{j,s+1}\| \le C \sum_{v=j+1}^{s} \frac{\|r_j\|^2 \|r_{j+1}\|}{\|r_v\|}.$$

Thus, by induction, (2.2.55) holds.

We can prove now that estimates (2.2.46) hold with $i = \tau+1$. Making use of (2.2.55) we have

$$<r_{\tau+1}, e_j> = -\alpha_{\tau+1} <H_{\tau+1}f'_{\tau+1}, e_j> = -\alpha_{\tau+1}<f'_{\tau+1}, r_j+\eta_{j,\tau+1}>. \quad (2.2.58)$$

By (2.2.53) and (2.2.43), we have

$$\|r_v\| \ge C\|f'_\tau\| \ge C\|f'_{\tau+1}\|, \; 0 \le v \le \tau, \quad 0 \le v \le \tau. \quad (2.2.59)$$

Taking this into account and using (2.2.56), we obtain $|<f'_{\tau+1}, \eta_{j,\tau+1}>| \le C\|r_j\|^2\|r_{j+1}\|$. Using this inequalities and also estimates (2.2.48) in (2.2.58) and taking into account that $\alpha_{\tau+1} \le C$, we find that

$$|<r_{\tau+1}, e_j>| \le C\|r_j\|^2\|r_{j+1}\|, \; 0 \le j \le \tau. \quad (2.2.60)$$

Further,

$$<e_{\tau+1}, r_j> = <r_{\tau+1}, e_j> + <r_{\tau+1}, (f''(x_{\tau+1} + \theta_{\tau+1} r_{\tau+1}) - f''(x_j+\theta_j r_j))r_j>$$

By (2.2.59) and (2.2.47),

$$\|r_{\tau+1}\| \le C\|r_j\|, \quad 0 \le j \le \tau. \quad (2.2.61)$$

Consequently, using (2.2.38) we have

$$\|f''(x_{\tau+1}+\theta_{\tau+1}r_{\tau+1}) - f''(x_j+\theta_j r_j)\| \le R(\|x_{\tau+1} - x_j\| + \|r_{\tau+1}\|+\|r_j\|) \le C\|r_j\|.$$

using this estimate and (2.2.60) in (2.2.61) we obtain:

$$|\langle e_{\tau+1}, r_j\rangle| \le C\|r_j\|^2\|r_{j+1}\|, \quad 0 \le j \le \tau.$$

Thus it has been established that estimates (2.2.9) and (2.2.46) hold. Theproof of the lemma is completed. //

THEOREM 2.2.4: Let $f(x)$ be a twice continuously differentiable function and matrix $f''(x)$ satisfies (2.2.3) and (2.2.38). If $f(x)$ is minimized by algorithm {(2.2.1), (2.1.28)}, then with any suffeciently large ξ the following estimate holds:

$$\|x_{(\xi+1)n}-x_*\| \le C\|x_{\xi n+1}-x_*\| \, \|x_{\xi n}-x_*\|. \tag{2.2.62}$$

PROOF: By (2.2.39), estimate (2.2.62) is equivalent to

$$\|f'_n\| \le C\|f'_1\|\|f'_0\|. \tag{2.2.63}$$

Suppose that estimate (2.2.63) with sufficiently large ξ does not hold. Then there must exist an infinite subsequence (ξ_m) such that for corresponding points the following inequalities hold:

$$\|f'_0\|\|f'_1\| \le \lambda_{\xi m}\|f'_n\|, \ \lambda_{\xi m}\to 0 \text{ as } \xi_m\to\infty. \tag{2.2.64}$$

without loss of generality, it can be assumed that the subsequence (ξ_m) coincides with the whole sequence $\xi = 0, 1, \ldots$ Taking into account (2.2.43) if (2.2.64) holds, estimates (2.2.45) hold too. Consequently, if we assume estimates (2.2.64) to be satisfied, then the requirements of theorem 2.2.4 provide for the fulfilment of the condition of lemma 2.2.1. Thus, if (2.2.64) is fulfilled, the estimates (2.2.9) and (2.2.46) hold.

Taking this into account we have

$$|\langle f'_n, r_j\rangle| = |\langle e_{n-1} + \ldots + e_{j+1}, r_j\rangle| \le C\|r_j\|^2\|r_{j+1}\|, \ 0 \le j \le n-2.$$

Now in a way analogous to that used in establishing (2.2.50) we can show that

$$|\langle f'_n, r_j\rangle| \le \lambda_\xi\|f'_n\|\|r_j\|, \ \lambda_\xi\to 0, \ 0 \le j \le n-1. \tag{2.2.65}$$

Let us demonstrate now that if (2.2.45) holds, then the system $r_0, \ldots, r_{n-1}$ is linearly independent. Note first of all that due to estimate (2.2.9), it follows from (2.2.45) that

$$\|r_i\|\|r_{i+1}\| \le \lambda_\xi\|r_j\|, \ 0 \le i < j \le n-1. \tag{2.2.66}$$

Making use of (2.2.66) and (2.2.41), estimates (2.2.46) can take the following form

$$|\langle r_i, e_j\rangle| \le \lambda_\xi \|r_i\|\|r_j\| \le \lambda_\xi \|r_i\| \ \|e_j\|, \quad \lambda_\xi \longrightarrow 0,$$
$$0 \le i \ne j \le n-1. \qquad (2.2.67)$$

Let $\bar{r}_{i:} = \dfrac{r_i}{\|r\|}$ and suppose that there is atleast one index $j \in \{0, 1, \ldots, n-1\}$

such that $B_j \ne 0$ and $\varphi := \sum_{i=0}^{n-1} \beta_i \bar{r}_i$, then

$$\langle\varphi, e_j\rangle = \beta_j \langle\bar{r}_j, e_j\rangle + \sum_{i \ne j} \beta_i \langle\bar{r}_i, e_j\rangle \quad , \quad i, j = 0, 1, \ldots, n-1.$$

or, $$|\langle\varphi, e_j\rangle| \ge ||\beta_j \langle\bar{r}_j, e_j\rangle| - |\sum_{i \ne j} \beta_i \langle\bar{r}_i, e_j\rangle||. \qquad (2.2.68)$$

But since $|\beta_j| > 0$ for at least one index $j \in \{0, 1, \ldots, n-1\}$, then by (2.2.17) and (2.2.41) we have

$$|\beta_j \langle\bar{r}_j, e_j\rangle| = \left|\frac{\beta_j}{\|r_j\|} \langle r_j, e_j\rangle\right| \ge \frac{|\beta_j|}{\|r_j\|} m\|r_j\|^2 = C\|r_j\| \ge C\|e_j\|.$$

With $i = j$ making use of (2.2.67) we have

$$|\beta_i \langle\bar{r}_i, e_j\rangle| \le \lambda_\xi |\beta_i| \|\bar{r}_i\| \|e_j\| = \lambda_\xi |\beta_i| \|e_j\|, \quad \lambda_\xi \longrightarrow 0 \text{ as } \xi \longrightarrow \infty.$$

Using the inequalities obtained in (2.2.68) we have with sufficiently large ξ $|\langle\varphi, e_j\rangle| \ge C\|e_j\|$, i.e.

$$\|\varphi\| \ge C > 0. \qquad (2.2.69)$$

Hence by contrapositive, it follows that the system of vectors $r_0, \ldots, r_{n-1}$ is linearly independent. Besides, if $\psi_0, \ldots, \psi_{n-1}$ is a system biorthogonal to $r_0, \ldots, r_{n-1}$, then with sufficiently large ξ we have

$$\|r_i\|\|\psi_i\| \le C, \quad 0 \le i \le n-1. \qquad (2.2.70)$$

Finally, under conditions (2.2.65) and (2.2.70) we obtain that the system of vectors $f'_n, r_0, \ldots, r_{n-1}$ is also linearly independent. To see this, suppose that

$$f'_n = \sum_{i=0}^{n-1} \partial_i \psi_i = \sum_{i=0}^{n-1} \langle f'_n, r_i\rangle \psi_i.$$

Then by (2.2.65) and (2.2.70) we obtain

$$\|f'_n\| \le C\lambda_\xi \|f'_n\|.$$

Since $\lambda_\xi \longrightarrow 0$ and $\xi \longrightarrow \infty$, the last inequality can not be satisfied with sufficiently large ξ. Hence, it follows that the system $f_n, r_o, \ldots, r_{n-1}$ is linearly independent.

Thus having assumed that estimate (2.2.62) doesnot hold with any $\xi \ge \xi_0$ (where ξ_0 is a certain sufficently great number), we have proved that a system of n+1 vectors f'_n, r_0, ..., r_{n-1} in $\mathbb{R}^n$ is linearly independent. However, this is impossible. Thus the initial assumption was wrong, i.e. estimate (2.2.62) holds. Thus the theorem is proved. //

2.2E. SUMMARY

Thus we have made it clear that all of the methods studied in chapter 2.1D can be applied for minimization of non quadratic functions. The rate of convergence of methods of conjugate direcions was established. The sequence considered was $(x_{\xi n})$, i.e. We considered as one iteration a unified group of n usual iterations of the process $x_{\xi n}$, $x_{\xi n+1}$, ..., $x_{\xi n+n-1}$.

Using the results obtained, we now compare the properties of different algorithms in the minimizaiton of nonquadratic functions.

The results of theorem 2.2.4 (estimate (2.2.62)) show that the rate of convergence of sequence $(X_{\xi n})$ depends considerably on the properties of matrix $H_{\xi n}$.

If, as $\xi \longrightarrow \infty$,

$$H_{\xi n} \longrightarrow (f''_{\xi n})^{-1}, \tag{2.2.71}$$

then

$$\frac{\|x_{\xi n+1} - x_*\|}{\|x_{\xi n} - x_*\|} \longrightarrow 0$$

and the rate of convergence increases. This fact is practically of the greatest interest for algorithms having the property that in minimizing a quadratic function we have

$$H_n = A^{-1}. \tag{2.2.72}$$

Algorithms (2.1.28) and (2.1.29) belong to methods of this group. If in implementing one of such algorithms condition (2.2.71) is fulfilled, then, by the above considerations, it is expedient not to restore matrix H_k.

The most effect methods of the class of methods of conjugate directions from the view point of the rate of convergence in the minimization of strictly convex functions should be methods that have property (2.2.72).

In methods of conjugate directions the step length is chosen under the condition that the minimum of the funcition is in the direction of motion. //

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