Optimization Problems in Classes of Rearrangements

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Approval

This is to certify that the thesis prepared by Feyissa Kebede Bushu entitled: "*Optimization Problems in Classes of Rearrangements* " submitted in fulfillment of the requirements for the Degree of Doctor of Philosophy in Mathematics complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

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Declaration

I, **Feyissa Kebede Bushu**, with student ID GSR/2594/08, hereby declare that this thesis has been composed by myself and that it has never been submitted for completion of graduate qualification at any higher learning institution. I confirm that this is my own work under supervision of Professor Giovanni Porru and Dr.Tadesse Abdi. The use of all materials from other sources has been properly and fully acknowledged.

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Abstract

This work is concerned with maximization and minimization problems of functionals associated with solutions of elliptic differential equations depending on functions which belong to classes of rearrangements. We prove existence and uniqueness results, and present some features of optimal solutions.

We also discuss maximization and minimization of the energy integral associated with solutions to partial differential equations with coefficients depending on a suitable source or well. Under suitable geometrical conditions on the domain Ω we find the optimal configurations.

Basic Notations

- Ω: is bounded smooth domain in $\mathbb{R}^N$
- $|E|$: Lebesgue measure of measurable set E
- $\mathcal{F}, \mathcal{G}$: the classes of rearrangements
- $C_0^\infty(\Omega)$: space of smooth functions with compact support in Ω
- $\overline{\mathcal{F}}, \overline{\mathcal{G}}$: the closure of $\mathcal{F}, \mathcal{G}$ respectively
- $L_+^p(\Omega)$: the space of non-negative p-integerable functions
- $L_+^\infty(\Omega)$: the space of non-negative essentially bounded functions
- $H^{1,p}(\Omega)$: a Sobolev space
- $H_0^{1,p}(\Omega)$: the closure of $C_0^\infty(\Omega)$
- χ_E: Characteristic function of the measurable set E
- $\rightharpoonup$: Weak convergence
- $u^\sharp$: Steiner symmetric rearrangements of function u
- u^Δ: Schwarz symmetric rearrangements of function u
- $g \not\equiv 0$: g is not identically zero

Chapter 1

Introduction

1.1 Motivation

We describe some problems that can be solved by using our optimization results.

• **Heat diffusion.** Let $\Omega \subset \mathbb{R}^3$ be the region occupied by a body surrounded by a thermostat regulated at a zero temperature, and let D be a region inside the body where heat is produced at a constant density, say C. The corresponding heat equation is the following

$$\rho \frac{\partial u}{\partial t} = K\Delta u + C\chi_D(x) \text{ in } \Omega \times (0, T),$$

$$u(x, 0) = u_0(x), \quad u(x, t) = 0 \text{ on } \partial\Omega \times (0, T).$$

where u represents the temperature at the point $x \in \Omega$ and time t, ρ is a constant depending on the thermal characteristics of material. K is the (constant) diffusion coefficient (depending on the material under consideration), and $\chi_D(x)$ is the characteristic function of D. In a steady state situation we have $\frac{\partial u}{\partial t} = 0$, so the problem becomes

$$-\Delta u = \frac{C}{K}\chi_D, \quad u(x) = 0 \text{ on } \partial\Omega.$$

If $\frac{C}{K} = 1$, the corresponding energy integral is

$$I(D) = \int_D u(x)\, dx.$$

By changing the shape and location of D without changing the measure $|D|$, we shall investigate the maximum and the minimum of $I(D)$. This corresponds to the maximum or minimum quantity of heat accumulated inside the plate.

For some materials the diffusion coefficient K is not a constant: if it depends on the power of the gradient $|\nabla u|^{p-2}$, $1 < p$, the corresponding equation (in the steady state situation) is

$$-\Delta_p u = \chi_D,$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, the p-Laplace operator. The density of the heat produced may depend on some power of u. In this case the term χ_D must be replaced by $\chi_D u^{q-1}$, $1 \leq q < p$.

• **Equilibrium of a membrane.** Let $\Omega \subset \mathbb{R}^2$ be the region occupied by a uniform elastic membrane fixed at the boundary $\partial\Omega$ and subject to a uniform load (say of density one) on a region $D \subset \Omega$. The corresponding equation is the following

$$-\Delta u = \chi_D(x), \quad u(x) = 0 \text{ on } \partial\Omega,$$

where u represents the deformation of the membrane at the point x. The corresponding energy integral is

$$I(D) = \int_D u(x)\, dx.$$

By changing the shape and location of D without changing the measure $|D|$, we shall investigate the maximum and the minimum of $I(D)$. In other words, we investigate the location of the loads through Ω so as to maximize or minimize the global deformation of the membrane.

• **Vibration of a membrane.** Let $\Omega \subset \mathbb{R}^2$ be the region occupied by a uniform elastic membrane fixed at the boundary $\partial\Omega$ (as in the previous case). The vibration of the membrane is described by the hyperbolic equation

$$g(x)\frac{\partial^2 w}{\partial t^2} = \Delta w, \quad w(x,t) = 0 \text{ on } \partial\Omega \times (0,T),$$

where $g(x) > 0$, $w = w(x,t)$ represents the deformation of the membrane at the point x and time t, and Δw is the spatial Laplacian. A classical method for solving this problem is the following: we seek for a solution of the form $w(x,t) = v(t)u(x)$. The equation becomes

$$g(x)\ddot{v}(t)u(x) = v(t)\Delta u(x),$$

that can be written as

$$\frac{\ddot{v}(t)}{v(t)} = \frac{\Delta u(x)}{g(x)u(x)} = -\lambda.$$

Therefore we must solve the two equation

$$-\ddot{v} = \lambda v, \quad -\Delta u = \lambda g u.$$

The solution of the first equation is easy, provided one knows the value of λ. This value must be found from the second equation, that is

$$-\Delta u = \lambda g u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

This is a well understood eigenvalue problem. There is a sequence of positive values λ_n (with $\lambda_n \to \infty$ as $n \to \infty$) such that for each λ_n the previous boundary value problem has a nontrivial solution u_n. The vibration of the membrane is described by the sum

$$u(x,t) = a_1 \cos(\sqrt{\lambda_1} t) u_1(x) + a_2 \cos(\sqrt{\lambda_2} t) u_2(x) + a_3 \cos(\sqrt{\lambda_3} t) u_3(x) + \cdots .$$

The values of the constants a_n depend on the initial conditions. The values of λ_n depend on the geometry of Ω and on the function g. It is well known that the eigenfunction $u_1(x)$ is positive in Ω, whereas, all other eigenfunctions $u_n(x)$, $n \geq 2$, are sign changing. We concentrate on the first eigenvalue λ_1. Assume

$$g(x) = 1 + \chi_D,$$

where $D \subset \Omega$. Then, λ_1 depends on the shape and location of D (we suppose Ω fixed). We investigate the maximum and the minimum of $\lambda_1(D)$ when $D \subset \Omega$ varies with the condition that $|D|$ is fixed. Clearly λ_1 is related with frequency of the vibration of the membrane.

• **Chemical equilibrium.** Let $\Omega \subset \mathbb{R}^3$. For $1 < q < p$ and $1 \leq \alpha < q$ we consider the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \Delta_p u + \Delta_q u + g(x) u^{\alpha - 1}, \quad x \in \Omega,$$

where u represents a concentration of some chemical substance at the point x and time t, $\Delta_p u + \Delta_q u$ represents the spatial diffusion, whereas $g(x)u^{\alpha-1}$ is related to a source or a loss of the chemical substance. In the steady state situation we find the elliptic problem

$$-\Delta_p u - \Delta_q u = g(x) u^{\alpha-1} \ \text{ in } \ \Omega, \quad u(x) = 0 \ \text{ on } \ \partial\Omega.$$

Note that the same equation describes other phenomena in bio-physics [20], plasma-physics [44], and models of elementary particles [3]. We investigate the maximum and the minimum of the energy integral associated with the solution u to this problem as g varies in a class of rearrangements.

1.2 Background

A basic tool in this work is the notion of rearrangement of a measurable function. If f and g are real functions defined in bounded domains $\Omega \subset \mathbb{R}^N$ and $D \subset \mathbb{R}^M$ respectively, we say that f and g are rearrangements of each other if and only if $|\Omega|$=$|D|$ and

$$|\{x \in \Omega : f(x) \geq \beta\}| = |\{y \in D : g(y) \geq \beta\}| \quad \forall \beta \in \mathbb{R}.$$

As usual, if E is a measurable set in $\mathbb{R}^N$ we denote by $|E|$ its N-Lebesgue measure. Similarly, if F is a measurable set in $\mathbb{R}^M$ we denote by $|F|$ its M-Lebesgue measure.
We shall use functions that are non-negative. Given $f_0(x) \geq 0$, we denote by $\mathcal{F}$ the family of all functions $f(x)$ which are rearrangements of $f_0(x)$. It is well known that all functions $f \in \mathcal{F}$ have the same distribution function $\mu_f(t)$. As a consequence, recalling the definition of Lebesgue integral, all functions $f \in \mathcal{F}$ have the same integral. For this reason, the notion of rearrangement of functions is old as that of the notion of Lebesgue integral.
In applications, this notion is used in many contexts. For example, assume one wants to organize the location of some objects over a wood table. Changing the position of these objects, the total load remains unchanged. However, the inflection of the table depends not only from the total load of the objects but also from their distribution.
Another interesting example is the following. Assume one has a fixed quantity of food to be distributed in a large park to feed wild animals. A different distribution of the food may give different results concerning the benefit of the animals. A similar example concerns the distribution of a fixed quantity of pesticide in order to battle against damaging animals.
For the mathematical theory on rearrangements we refer to the papers by Burton [4, 5], the paper by Burton-McLeod [6], the paper by Alvino-Lions-Trombetti [2] and references therein.
For problems concerning optimization of the energy integral related to the Laplacian we refer to [4, 5] (Maximization), and to [6] (Minimization).
For problems concerning optimization of the principal eigenvalue related to the Laplacian we refer to [11] and [13].
For problems concerning optimization of the principal eigenvalue related to the p-Laplacian we refer to [17] and [16].
The theory of the (p, q)-Laplacian is relatively recent. For a survey we refer to [34] and references therein. As far as we know, there are no papers concerning optimization in classes of rearrangements for (p, q)-Laplace problems except [29]. Optimization problems for the principal eigenvalue related to rigid obstacles are discussed in the paper [23]. Optimization problems for the energy integral related to rigid obstacles are discussed in [30].

1.3 Plan of the work

In Chapter 3, we shall investigate two problems. Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and let $g_0(x)$ be a non-negative bounded function defined in Ω. We assume $g_0(x) > 0$ in a subset of positive measure. We denote by $\mathcal{G} = \mathcal{G}(g_0)$ the corresponding class of rearrangements. Moreover, we denote by $\overline{\mathcal{G}}$ the closure of $\mathcal{G}$ in the weak* topology of $L^\infty(\Omega)$.

First, we consider the boundary value problem

$$-\Delta_p u = g u^{q-1}, \quad u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.1}$$

where $1 < q < p, \quad g \in \mathcal{G}, \quad u \in H_0^{1,p}(\Omega)$. The energy integral corresponding to the solution of problem (1.1) is defined as

$$I(g) = \int_\Omega g u_g^q \, dx.$$

We shall discuss the optimization problems:

$$\max_{g \in \mathcal{G}} I(g), \qquad \min_{g \in \mathcal{G}} I(g).$$

In case of $p = 2$ and $q = 1$ these problems have been investigated in the papers [4, 5, 6]. For the case $p \neq 2$ and $q \neq 1$ we refer to the paper [19]. We report the results of these papers with some simplifications of the proofs, and some remarks.
After that words, we consider the eigenvalue problem

$$-\Delta_p u = \lambda g u^{p-1}, \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

where $p > 1$, $g \in \mathcal{G}$, $u \in H_0^{1,p}(\Omega)$. Here $\lambda = \lambda_g$ is the principal eigenvalue corresponding to problem (1.2).
We are interested in the investigation of the following optimization problems:

$$\max_{g \in \mathcal{G}} \lambda_g, \qquad \min_{g \in \mathcal{G}} \lambda_g.$$

Also these optimization problems have been investigated. For the case $p = 2$, see the papers [2] and [18]. For the general case we refer to the paper [17].

In Chapter 4, Section 4.1, we consider the boundary value problem

$$-\Delta_p u - \Delta_q u = g(x) u^{\alpha-1}, \quad u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.3}$$

where $g(x)$ is a bounded non-negative function, $1 \leq \alpha < q < p$. We will discuss the following items.

i) We prove that problem (1.3) has a solution.

ii) We prove that the solution of problem (1.3) is unique.

iii) We investigate the optimization problems:

$$\max_{g \in \mathcal{G}} J(g), \qquad \min_{g \in \mathcal{G}} J(g),$$

where

$$J(g) = \int_\Omega \Big(\frac{1}{\alpha} g u^\alpha - \frac{1}{p}|\nabla u|^p - \frac{1}{q}|\nabla u|^q\Big)\, dx.$$

These results are contained in the paper [29]. An important result is the proof of uniqueness for a class of (p, q)-Laplace equations. In view of this result, we investigate the energy integral corresponding to solutions of the boundary value problem (1.3). The crucial point is the proof of continuity, convexity and Gâteaux differentiability of the functional $J(g)$. Then, applying a well known theory developed by G. Burton, we find results of existence and representation of the maximum and the minimum of such a functional in the class of rearrangements $\mathcal{G}$.

In Chapter 4, Section 4.2, we assume $\Omega \subset \mathbb{R}^2$ and consider the boundary value problem

$$-\Delta u + gu = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.4}$$

where g and f are non-negative functions. The corresponding energy integral is the functional

$$I(g, f) = \int_\Omega (|\nabla u|^2 + gu^2)\, dxdy = \int_\Omega fu\, dxdy.$$

We discuss the following optimization problems. Fixed g and given $D_0 \subset \Omega$ we consider the family $\mathcal{D}$ of all $D \subset \Omega$ which are translations of D_0, and investigate the problems

$$\max_{\mathcal{D}} \int_\Omega \chi_D u_D\, dxdy, \qquad \min_{\mathcal{D}} \int_\Omega \chi_D u_D\, dxdy,$$

where u_D is the solution of problem (1.4) for $f = \chi_D$. Next we consider problem (1.4) with f fixed and $g = \chi_D$, and investigate the problems

$$\max_{\mathcal{D}} \int_\Omega f(x, y) u_D\, dxdy, \qquad \min_{\mathcal{D}} \int_\Omega f(x, y) u_D\, dxdy,$$

where u_D is the solution of problem (1.4) with $g = \chi_D$. These results have been published in [30].

Under this Chapter, we point out the main results of the thesis.

Chapter 2

Preliminaries

This Chapter is taken mainly from the papers [2, 4, 5, 6]. Definitions and basic results are reported. Some examples are provided and some proofs are simplified.

2.1 Rearrangements

Let Ω be a bounded smooth domain in $\mathbb{R}^N$. If $E \subset \mathbb{R}^N$ is a measurable set, we denote with $|E|$ its Lebesgue measure.

Definition 2.1. *Let f be defined in $\Omega \subset \mathbb{R}^N$ and let g be defined in $D \subset \mathbb{R}^M$, with $|\Omega| = |D|$. We say that $f(x)$ and $g(x)$ have the same rearrangement if and only if*

$$|\{x \in \Omega : f(x) \geq \beta\}| = |\{y \in D : g(y) \geq \beta\}|, \; \forall \beta \in \mathbb{R}.$$

In case of $N = M$ and $\Omega = D$ we have

$$|\{x \in \Omega : f(x) \geq \beta\}| = |\{x \in \Omega : g(x) \geq \beta\}|, \; \forall \beta \in \mathbb{R}.$$

Let f_0 be a bounded non-negative measurable function defined in Ω. Assume f_0 is not a constant. We denote by $\mathcal{F} = \mathcal{F}(f_0)$ the class of the functions f defined in Ω which are rearrangements of f_0.

Two functions $f, g \in \mathcal{F}$ have several properties in common. For example, we have

$$\inf_{\Omega} f(x) = \inf_{\Omega} g(x), \quad \text{and} \quad \sup_{\Omega} f(x) = \sup_{\Omega} g(x).$$

Furthermore, for any Borel measurable function φ we have

$$\int_{\Omega} \varphi(f(x))\, dx = \int_{\Omega} \varphi(g(x))\, dx.$$

Example 1. *Let* $N = 1$ *and* $\Omega = (0, 4)$. *If* $E \subset \Omega$ *is a measurable set, we define* $f_0(x) = \chi_E(x)$. *As usual, we have* $\chi_E(x) = 1$ *if* $x \in E$, *and* $\chi_E(x) = 0$ *if* $x \in \Omega \setminus E$.
Note that

$$\{x \in \Omega : \chi_E(x) \geq \beta\} = \begin{cases} \Omega & \text{if} \quad \beta = 0 \\ E & \text{if} \quad 0 < \beta \leq 1 \\ \emptyset & \text{if} \quad \beta > 1. \end{cases}$$

Hence the class of rearrangements generated by χ_E *is the family of all functions* χ_F *where* $F \subset \Omega$ *and* $|E|{=}|F|$. *If* $E = \{x \in \Omega : 0 \leq x \leq 2\}$ *we may have*

$$\begin{aligned} F_1 &= \{x \in \Omega : 0 \leq x \leq 2\}, \\ F_2 &= \{x \in \Omega : 2 \leq x \leq 4\}, \\ F_3 &= \{x \in \Omega : 1 \leq x \leq 2\} \cup \{x \in \Omega : 3 \leq x \leq 4\}, \\ F_4 &= \{x \in \Omega : 1 \leq x \leq 1.5\} \cup \{x \in \Omega : 2.5 \leq x \leq 4\} \end{aligned}$$

etc.

The class of rearrangements $\mathcal{F} = \mathcal{F}(f_0)$ is not convex. Indeed, if we consider the above Example 1, the function

$$\frac{1}{2}[\chi_{F_1}(x) + \chi_{F_2}(x)] = \frac{1}{2} \quad \forall x \in \Omega$$

is not a rearrangements of $\chi_E(x)$. We say that two functions are equivalent if they differ in a subset of zero measure only.
It is convenient to enlarge the class $\mathcal{F}$ as follows.

Definition 2.2. *Let* $\overline{\mathcal{F}}$ *be the closure of* $\mathcal{F}$ *in the weak** *topology on* $L^\infty(\Omega)$. *This means that a function* $f \in \overline{\mathcal{F}}$ *if and only if there is a sequence* $\{f_i\}, f_i \in \mathcal{F}$, *such that*

$$\lim_{i \to \infty} \int_\Omega f_i(x) g(x) \, dx = \int_\Omega f(x) g(x) \, dx \quad \forall g \in L^1(\Omega).$$

Consider the following example.
Let $\Omega = (0, 1)$, and let $f_0(x) = \chi_E(x), E = \{x \in \Omega : 0 \leq x \leq 0.5\}$. Given an integer i, the function

$$f_i(x) = \begin{cases} 1 \ \text{ if } \ 0 \leq x < \frac{1}{2i}, \ \ \frac{2}{2i} \leq x < \frac{3}{2i}, \ \ \cdots, \ \ \frac{2i-2}{2i} \leq x < \frac{2i-1}{2i} \\ \\ 0 \ \text{ if } \ \frac{1}{2i} \leq x < \frac{2}{2i}, \ \ \frac{3}{2i} \leq x < \frac{4}{2i}, \ \ \cdots, \ \ \frac{2i-1}{2i} \leq x \leq 1 \end{cases}$$

is a rearrangement of $f_0(x)$. For any function $g \in L^1(\Omega)$ we find

$$\lim_{i \to \infty} \int_\Omega f_i(x) g(x) \, dx = \int_\Omega f(x) g(x) \, dx$$

with $f(x) = 0.5$. The proof is trivial for $g(x) = \mathit{constant}$. Then one proves the result for functions like $g(x) = \sum_{i=1}^{n} c_i \chi_{E_i}$, with $E_i \cap E_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^{n} E_i = \Omega$. Finally, the general result follows by density. Note that $f(x)$ (in this particular example) is not a rearrangement of $f_0(x)$ and that

$$0 = \inf_{\Omega} f_0(x) < \inf_{\Omega} f(x) = 0.5 = \sup_{\Omega} f(x) < \sup_{\Omega} f_0(x) = 1.$$

In general, if $f \in \overline{\mathcal{F}}(f_0)$ we have

$$\inf_{\Omega} f_0(x) \leq \inf_{\Omega} f(x) \leq \sup_{\Omega} f(x) \leq \sup_{\Omega} f_0(x).$$

Let us give a proof of this fact. Let $f_i \rightharpoonup f$ (weak* convergence) with $f_i \in \mathcal{F}$. For $x_0 \in \Omega$ we use Definition 2.2 with $g = \chi_{B(x_0,\epsilon)}$, where $B(x_0, \epsilon)$ is a ball centered at x_0 and radius ϵ . We find

$$\lim_{i \to \infty} \int_{B(x_0,\epsilon)} f_i(x)\, dx = \int_{B(x_0,\epsilon)} f(x)\, dx.$$

Since

$$\inf_{\Omega} f_i(x) = \inf_{\Omega} f_0(x), \quad \sup_{\Omega} f_i(x) = \sup_{\Omega} f_0(x)$$

and

$$\int_{B(x_0,\epsilon)} f_i(x)\, dx \geq \int_{B(x_0,\epsilon)} \inf_{\Omega} f_i(x)\, dx = \int_{B(x_0,\epsilon)} \inf_{\Omega} f_0(x)\, dx$$

we find

$$\inf_{\Omega} f_0(x)|B(x_0,\epsilon)| \leq \int_{B(x_0,\epsilon)} f(x)\, dx \leq \sup_{\Omega} f_0(x)|B(x_0,\epsilon)|.$$

If we divide by $|B(x_0, \epsilon)|$ and let $\epsilon \to 0$ we find

$$\inf_{\Omega} f_0(x) \leq f(x_0) \leq \sup_{\Omega} f_0(x)$$

for almost all $x_0 \in \Omega$. The latter inequalities imply our result.

We claim that the integral over Ω of $f \in \overline{\mathcal{F}} = \overline{\mathcal{F}}(f_0)$ is the same as the integral of f_0. Indeed, let $f_i \rightharpoonup f$ with $f_i \in \mathcal{F}$. The integral of f_i is the same as the integral of f_0 because f_i and f_0 have the same rearrangement. By Definition 2.2 with $g = 1$ we find

$$\int_{\Omega} f_0(x)\, dx = \lim_{i \to \infty} \int_{\Omega} f_i(x)\, dx = \int_{\Omega} f(x)\, dx.$$

The claim follows.

The class $\overline{\mathcal{F}}$ is much larger than $\mathcal{F}$. To understand better the properties of $\overline{\mathcal{F}}$ we shall prove some standard results.

Definition 2.3. *Let Ω be a bounded domain in $\mathbb{R}^N$ and let $f \in L^p(\Omega)$, $(1 \leq p \leq \infty)$ be non-negative. We denote by μ_f the distribution function of f defined as*

$$\mu_f(t) = |\{x \in \Omega : f(x) > t\}|, \quad t \geq 0.$$

We denote by f^* the decreasing rearrangement of f defined as

$$f^*(s) = sup\{t > 0 : \mu_f(t) > s\}.$$

Similarly, we denote by f_* the increasing rearrangement of f defined as $f_*(s) = f^*(|\Omega| - s)$.

Recall that $|E|$ denotes the Lebesgue measure of E. We observe that, even if we say f^* is a decreasing rearrangement, it may happen that f^* is non-increasing. A similar remark holds for non-decreasing rearrangement f_*.
The distribution function $\mu_f(t)$ is clearly non-increasing. In case it is continuous and strictly decreasing then $f^*(s)$ is the inverse function of $\mu_f(t)$. In general f^* is the non-increasing function on $[0, \infty)$ such that f and f^* have the same distribution function. We also observe that f and g have the same rearrangement if and only if they have the same distribution function (or the same decreasing rearrangement).
If $f(x) = \chi_E$ then $f^*(s) = \chi_{\tilde{E}}$, where $\tilde{E}$ is the interval $[0, |E|]$.

Example 2. *Let*

$$f(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x), \quad \cup_{i=1}^{n} E_i = \Omega, \quad E_i \cap E_j = \emptyset, \quad i \neq j, \quad |E_i| = \frac{|\Omega|}{n}.$$

Then,

$$f^*(s) = \sum_{i=1}^{n} \bar{a}_i \chi_{\tilde{E}_i}(s),$$

where

$$\tilde{E_1} = \Big[0, \frac{|\Omega|}{n}\Big), \quad \tilde{E_2} = \Big[\frac{|\Omega|}{n}, 2\frac{|\Omega|}{n}\Big), \cdots, \quad \tilde{E_n} = \Big[\frac{(n-1)|\Omega|}{n}, |\Omega|\Big],$$

and where $(\bar{a}_1, \ldots, \bar{a}_n)$ is a permutation of $(a_1, \ldots, a_n)$ such that $\bar{a}_1 \geq \bar{a}_2 \geq \cdots \geq \bar{a}_n$.

Since $\mu_f = \mu_{f^*}$ we have

$$\int_\Omega f(x)\, dx = \int_0^{|\Omega|} f^*(s)\, ds. \tag{2.1}$$

Indeed,

$$\int_\Omega f(x)dx = \int_0^\infty \mu_f(t)dt, \quad \int_0^{|\Omega|} f^*(s)ds = \int_0^\infty \mu_{f^*}(t)dt.$$

More generally we have

$$\int_\Omega \varphi(f(x))dx = \int_0^{|\Omega|} \varphi(f^*(s))ds$$

for all Borel function φ.

Definition 2.4. *The* ***layer cake representation*** *of a non-negative, real-valued measurable function f defined on $\mathbb{R}^N$ is given by*

$$f(x) = \int_0^\infty \chi_{\{f(x) \geq t\}} dt.$$

Let us give a short proof of this result. We have

$$f(x) = \int_0^{f(x)} dt = \int_0^\infty \chi_{[0,f(x)]} dt = \int_0^\infty \chi_{\{f(x) \geq t\}} dt.$$

Indeed,

$$\chi_{[0,f(x)]} = \begin{cases} 1 & 0 < t \leq f(x), \\ 0 & t > f(x). \end{cases}$$

$$\chi_{\{f(x) \geq t\}} = \begin{cases} 1 & f(x) \geq t, \\ 0 & f(x) < t. \end{cases}$$

Lemma 2.1. *[4, Lemma 2] Let $U \subset \Omega$ be a measurable set. Then,*

$$\int_U f \ dx \leq \int_0^{|U|} f^* \ ds.$$

Proof. Write

$$\begin{aligned} G &= \{(x,s) \in \Omega \times R : 0 \leq s \leq f(x)\}, \\ F(s) &= \{x \in \Omega : f(x) > s\}. \end{aligned}$$

Note that

$$\int_U \chi_G dx = \begin{cases} |U| & \text{if } \ U \subset \{x \in \Omega : s \leq f(x)\} \\ |U \cap F| & \text{otherwise.} \end{cases}$$

Therefore, by Fubini's theorem we have

$$\begin{aligned} &\int_U f \ dx = \int_U \int_0^\infty \chi_G \ ds \ dx = \int_0^\infty \int_U \chi_G \ dx \ ds \\ &= \int_0^\infty |F \cap U| \ ds \leq \int_0^\infty |\{t \in (0,|U|) : f^*(t) \geq s\}| \ ds = \int_0^{|U|} f^* \ ds. \end{aligned}$$

The last equality holds because

$$\int_0^{|U|} f^* ds = \int_0^\infty \mu_{f^*}(t) dt = \int_0^\infty |\{t \in (0,|U|) : f^*(t) \geq s\}| \ ds.$$

This completes the proof. □

Lemma 2.2. *Recall that $f_*(s) = f^*(|\Omega| - s)$. If $U \subset \Omega$ is a measurable set then,*

$$\int_U f \ dx \geq \int_0^{|U|} f_* \ ds.$$

Proof. The proof is similar to that of the previous Lemma. Write

$$G = \{(x,t) \in \Omega \times R : 0 \leq s \leq f(x)\}$$

$$F(s) = \{x \in \Omega : f(x) > s\}.$$

Then by Fubini's theorem we have

$$\int_U f\ dx = \int_U \int_0^\infty \chi_G\ ds\ dx = \int_0^\infty \int_U \chi_G\ dx\ ds$$
$$= \int_0^\infty |F \cap U|\ ds \geq \int_0^\infty |\{t \in (0,|U|) : f_*(t) \geq s\}|\ ds = \int_0^{|U|} f_*\ ds.$$

This completes the proof. □

A very important result is the following:

Theorem 2.1. *[4, Theorem 1] (Hardy-Littlewood inequality)*
If $f \in L^1_+(\Omega)$ and $\phi \in L^\infty_+(\Omega)$ then

$$\int_\Omega f\phi\ dx \leq \int_0^{|\Omega|} f^*\phi^*\ ds. \tag{2.2}$$

Proof. Let us prove the theorem in case of

$$f = \sum_{i=1}^n a_i\chi_{E_i}, \quad \phi = \sum_{i=1}^n b_i\chi_{E_i}, \quad \cup_{i=1}^n E_i = \Omega, \quad E_i \cap E_j = \emptyset \quad i \neq j, \quad |E_i| = \frac{|\Omega|}{n}.$$

Note that $a_i\chi_{E_i}b_j\chi_{E_j} = 0$, for $i \neq j$ and $a_i\chi_{E_i}b_i\chi_{E_i} = a_ib_i\chi_{E_i}$, for $i = j$.
Then we have

$$\int_\Omega f\phi\ dx = \frac{|\Omega|}{n}\sum_{i=1}^n a_ib_i \leq \frac{|\Omega|}{n}\sum_{i=1}^n \bar{a}_i\bar{b}_i,$$

where $(\bar{a}_1, \cdots, \bar{a}_n)$ is a permutation of $(a_1, \cdots, a_n)$ such that $\bar{a}_1 \geq \bar{a}_2 \geq \cdots \geq \bar{a}_n$, and similarly for $(\bar{b}_1, \cdots, \bar{b}_n)$. The inequality in above can be proved easily as follows. We increase $\bar{a}_1 b_{i_1}$ by $\bar{a}_1\bar{b}_1$. Then, we increase $\bar{a}_2 b_{i_2}$ by $\bar{a}_2\bar{b}_2$, and so on.
Note that we have

$$f^*(s) = \sum_{i=1}^n \bar{a}_i\chi_{\tilde{E}_i}(s), \quad \phi^*(s) = \sum_{i=1}^n \bar{b}_i\chi_{\tilde{E}_i}(s),$$

where

$$\tilde{E}_1 = \Big[0, \frac{|\Omega|}{n}\Big), \quad \tilde{E}_2 = \Big[\frac{|\Omega|}{n}, 2\frac{|\Omega|}{n}\Big), \cdots, \quad \tilde{E}_n = \Big[\frac{(n-1)|\Omega|}{n}, |\Omega|\Big].$$

Hence,

$$\frac{|\Omega|}{n}\sum_{i=1}^n \bar{a}_i\bar{b}_i = \sum_{i=1}^n \bar{a}_i\bar{b}_i \int_{\tilde{E}_i} ds = \int_0^{|\Omega|} f^*\phi^*\ ds.$$

We have proved the theorem in this particular case. The general case follows by density. Recall that

$$f_*(s) = f^*(|\Omega| - s).$$

The counterpart of inequality (2.2) is the following

$$\int_\Omega f\phi \, dx \geq \int_0^{|\Omega|} f_*\phi^* \, ds = \int_0^{|\Omega|} f^*\phi_* \, ds. \tag{2.3}$$

The proof of inequality (2.3) is similar to that of inequality (2.2). Using the same notation, we have

$$\sum_{i=1}^n a_i b_i \geq \sum_{i=1}^n \bar{a}_i \bar{b}_{n+1-i}.$$

To prove this inequality, we decrease $\bar{a}_1 b_{i_1}$ by $\bar{a}_1 \bar{b}_n$. Then, we decrease $\bar{a}_2 b_{i_2}$ by $\bar{a}_2 \bar{b}_{n-1}$, and so on.

Now, let us give a different prove of the above inequalities.

Theorem 2.2. *[4, Theorem 1] Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f, g \geq 0$ then*

$$\int_\Omega fg \, dx \leq \int_0^{|\Omega|} f^* g^* \, ds.$$

Proof. Write

$$G = \{(x, s) \in \Omega \times R : 0 \leq s \leq f(x)\}$$

$$F(s) = \{x \in \Omega : f(x) > s\}, \quad \sigma(s) = |F(s)|.$$

Then by Fubini's theorem and Lemma 2.1 we have

$$\begin{aligned}\int_\Omega fg \, dx &= \int_\Omega \int_0^\infty \chi_G \; g \; ds \; dx = \int_0^\infty \int_\Omega \chi_G \; g \; dx \; ds \\ &= \int_0^\infty \int_{F(s)} g \; dx \; ds \leq \int_0^\infty \int_0^{\sigma(s)} g^* \; dt \; ds = \int_0^{|\Omega|} f^* g^* \; ds.\end{aligned}$$

To prove the latter equality it is convenient to define χ_{G^*} with

$$G^* = \{(s, t) \in (0, |\Omega|) \times \mathbb{R} : 0 \leq s \leq f^*(t)\}$$

and

$$\tilde{F}(s) = \{t \in (0, |\Omega|) : f^*(t) > s\}.$$

Note that $|\tilde{F}(s)| = |F(s)| = \sigma(s)$ (by definition of decreasing rearrangement).

$$\begin{aligned}\int_0^{|\Omega|} f^* g^* dt &= \int_0^{|\Omega|} g^* \int_0^\infty \chi_{G^*} ds dt = \int_0^\infty \int_0^{|\Omega|} \chi_{G^*} \; g^* \; dt \; ds \\ &= \int_0^\infty ds \int_0^{|\tilde{F}(s)|} g^* \; dt \; = \int_0^\infty \int_0^{\sigma(s)} g^* \; dt \; ds.\end{aligned}$$

This completes the proof. □

Theorem 2.3. *[29, Colloraly 1, page 16] Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f, g \geq 0$ then*

$$\int_0^{|\Omega|} f_* g^* \ ds \leq \int_\Omega fg \ dx \quad \textit{and} \quad \int_0^{|\Omega|} f^* g_* \ ds \leq \int_\Omega fg \, ds.$$

Proof. We prove the first inequality only. The proof of the second inequality can be obtained by changing f with g.
Write, as before,

$$G = \{(x, s) \in \Omega \times R : 0 \leq s \leq f(x)\}$$

$$F(s) = \{x \in \Omega : f(x) > s\}, \quad \sigma(s) = |F(s)|.$$

Then by Fubini's theorem and Lemma 2.2 we have

$$\begin{aligned}\int_\Omega fg \ dx &= \int_\Omega \int_0^\infty \chi_G \ g \ ds \ dx = \int_0^\infty \int_\Omega \chi_G \ g \ dx \ ds \\ &= \int_0^\infty \int_{F(s)} g \ dx \ ds \geq \int_0^\infty \int_0^{\sigma(s)} g_* \ dt \ ds = \int_0^{|\Omega|} f^* g_* \ ds.\end{aligned}$$

The latter inequality can be proved as in the previous theorem. We shall use the fact that, if

$$\tilde{F}(s) = \{t \in (0, |\Omega|) : f_*(t) > s\}$$

then $|\tilde{F}(s)| = |F(s)| = \sigma(s)$ (by definition of increasing rearrangement).

$$\begin{aligned}\int_0^{|\Omega|} f^* g_* dt &= \int_0^{|\Omega|} g_* \int_0^\infty \chi_{G^*} ds dt = \int_0^\infty \int_0^{|\Omega|} \chi_{G^*} \ g_* \ dt \ ds \\ &= \int_0^\infty ds \int_0^{|\tilde{F}(s)|} g_* \ dt \ = \int_0^\infty \int_0^{\sigma(s)} g_* \ dt \ ds.\end{aligned}$$

This completes the proof. □

Let us prove the following result.

Theorem 2.4. *[2, Proposition 2.1] Let $f, g \in L^1_+(\Omega)$. Then, the following conditions are equivalent:*

(i) $\int_0^{|\Omega|} f^* \phi^* \, ds \leq \int_0^{|\Omega|} g^* \phi^* \, ds \quad \forall \phi \in L^\infty_+(\Omega),$ and $\int_\Omega f \, dx = \int_\Omega g \, dx,$

(ii) $\int_\Omega f \phi \, dx \leq \int_0^{|\Omega|} g^* \phi^* \, ds \quad \forall \phi \in L^\infty_+(\Omega),$ and $\int_\Omega f \, dx = \int_\Omega g \, dx,$

(iii) $\int_0^r f^*(s) \, ds \leq \int_0^r g^*(s) \, ds \quad 0 < r < |\Omega|,$ *and* $\int_0^{|\Omega|} f^*(s) \, ds = \int_0^{|\Omega|} g^*(s) \, ds.$

Furthermore, any of the previous conditions imply

$$\int_\Omega F(f) \ dx \leq \int_\Omega F(g) \ dx \quad \text{for all convex, Lipschitz functions } F. \tag{2.4}$$

Proof. We observe that (i) implies (ii) from Theorem 2.1. To prove that (ii) implies (iii), we use the following fundamental relation

$$\int_0^r f^* \, ds = \max\Big\{ \int_E f \, dx, \; E \text{ Borel}, \; E \subset \Omega, \; |E| = r \Big\}. \tag{2.5}$$

If $\phi = \chi_E$ and $|E| = r$ then $\phi^* = \chi_{[0,r)}$. Hence, by (ii) and (2.5) we find

$$\int_\Omega f\phi \, dx = \int_\Omega f\chi_E \, dx = \int_E f \, dx = \int_0^r f^* \, ds = \int_0^{|\Omega|} f^*\chi_{[0,r)} \le \int_0^{|\Omega|} g^*\chi_{[0,r)} = \int_0^r g^* \, ds.$$

It follows that (using also equality in (ii) and (2.1))

$$\int_0^r f^* ds \le \int_0^r g^* ds \quad \text{and} \quad \int_0^{|\Omega|} f^* \, ds = \int_0^{|\Omega|} g^* \, ds.$$

Let us prove that (iii) implies (i).
Equation in (i) follows by the equation in (iii) and (2.1). Moreover, using the inequality in (iii) we find

$$\int_0^{|\Omega|} f^*\phi^* \, ds = \int_0^{|\Omega|} \Big(\int_0^s f^*(t) \, dt \Big)' \phi^* ds = - \int_0^{|\Omega|} \Big(\int_0^s f^*(t) \, dt \Big) d\phi^*(s) + \phi^*(|\Omega|) \int_0^{|\Omega|} f^*(s) \, ds$$
$$\le - \int_0^{|\Omega|} \Big(\int_0^s g^*(t) \, dt \Big) d\phi^*(s) + \phi^*(|\Omega|) \int_0^{|\Omega|} g^*(s) \, ds = \int_0^{|\Omega|} g^*\phi^* \, ds.$$

Note that

$$\int_0^{|\Omega|} g^*\phi^*(s) \, ds = \int_0^{|\Omega|} \Big(\int_0^s g^*(t) \, dt \Big)' \phi^* ds = - \int_0^{|\Omega|} \Big(\int_0^s g^*(t) \, dt \Big) d\phi^*(s) + \phi^*(|\Omega|) \int_0^{|\Omega|} g^*(s) \, ds.$$

Therefore, $\int_0^{|\Omega|} f^*\phi^* \, ds \le \int_0^{|\Omega|} g^*\phi^* \, ds$.
The first part of the theorem is proved.

To prove (2.4) we use (i). Assuming F is C^1 (and convex), since $F'(f)^* = F'(f^*)$, by (i) we have

$$\int_0^{|\Omega|} f^*F'(f^*) \, ds \le \int_0^{|\Omega|} g^*F'(f^*) \, ds.$$

Using the convexity of F and the latter inequality we find

$$\begin{aligned} \int_\Omega \big(F(g) - F(f)\big) \, dx &= \int_0^{|\Omega|} \big(F(g^*) - F(f^*)\big) \, ds \\ &\ge \int_0^{|\Omega|} F'(f^*)(g^* - f^*) ds \\ &= \int_0^{|\Omega|} g^*F'(f^*) - \int_0^{|\Omega|} f^*F'(f^*)) ds \ge 0. \end{aligned}$$

Therefore, $\int_\Omega \big(F(g) - F(f)\big) \, dx \ge 0$, from which (2.4) follows. $\square$

Putting $g = f_0$ in Theorem 2.4, condition (iii) read as

$$\int_0^r f^*(s)\,ds \le \int_0^r f_0^*(s)\,ds \quad \forall r \in [0, |\Omega|] \quad \text{and} \quad \int_0^{|\Omega|} f^*(s)\,ds = \int_0^{|\Omega|} f_0^*(s)\,ds. \tag{2.6}$$

We denote by $\mathcal{K} = \mathcal{K}(f_0)$ the class of functions which satisfy (2.6). We will see that $\mathcal{K} = \overline{\mathcal{F}}$. We start proving the following easy result.

Theorem 2.5. *[2, Theorem 2.1] $\overline{\mathcal{F}}(f_0) \subset \mathcal{K}(f_0)$.*

Proof. If $f \in \overline{\mathcal{F}}$, then there is a sequence $\{f_n\}$, $n = 1, 2, \cdots$, of functions in $\mathcal{F}$ such that

$$\lim_{n\to\infty} \int_\Omega f_n g\,dx = \int_\Omega f g\,dx \quad \forall g \in L^1(\Omega). \tag{2.7}$$

If $E \subset \Omega$ is a Borel set and $g = \chi_E$, using Lemma (2.1) we find

$$\int_E f\,dx = \lim_{n\to\infty} \int_E f_n\,dx \le \int_0^r f_0^*(s)\,ds,$$

where $r = |E|$. If we choose E as in (2.5) we have

$$\int_E f\,dx = \int_0^r f^*(s)\,ds.$$

Hence,

$$\int_0^r f^*(s)\,ds \le \int_0^r f_0^*(s)\,ds,$$

which is the inequality in (2.6). To prove the equality in (2.6), we use (2.7) with $g = 1$.
We find

$$\int_\Omega f_0(x)\,dx = \int_\Omega f_n(x)\,dx = \int_\Omega f(x)\,dx.$$

Note that the first equality in above holds because $f_n \in \mathcal{F}$. Moreover, the equality of the first and the last term can be written as

$$\int_0^{|\Omega|} f^*(s)\,ds = \int_0^{|\Omega|} f_0^*(s)\,ds,$$

so the equality in (2.6) holds for f. This completes the proof. □

In order to prove the reverse inclusion in the latter theorem, we show some results on $\mathcal{K}$.

Theorem 2.6. *$\mathcal{K}$ is sequentially compact and convex.*

Proof. Let $\{f_n\}$ be a sequence of functions in $\mathcal{K}$. By (2.5), for all $r \in (0, |\Omega|)$ we have

$$\max_{E\subset\Omega, |E|=r} \int_E f_n\,dx = \int_0^r f_n^*(s)\,ds \le \int_0^r f_0^*(s)\,ds. \tag{2.8}$$

Note that to write the last inequality in (2.8) we have used the inequality in (2.6) for f_n. Now, since

$$\lim_{r\to 0}\int_0^r f_0^*(s)\,ds = 0,$$

given $\epsilon > 0$ we find $\delta(\epsilon) > 0$ such that

$$\int_0^r f_0^*(s)\,ds < \epsilon \quad \forall r < \delta(\epsilon).$$

From the latter inequality and (2.8) we get

$$\int_E f_n(x)\,dx < \epsilon \quad \text{for all Borel} \quad E \subset \Omega \quad \text{with} \quad |E| < \delta(\epsilon).$$

Since $\delta(\epsilon)$ is independent of n, the latter inequality gives a uniform integrability of $f_n(x)$. Then by Dunford-Pettis Theorem, $\{f_n\}$ has a subsequence (that we call again $\{f_n\}$) which converges weakly to some f. To finish our proof we have to show that this f satisfies conditions (2.6). Since $f_n \rightharpoonup f$(weak convergence) we have, for any Borel E with $|E| = r$

$$\int_E f\,dx = \int_\Omega f\chi_E\,dx = \lim_{n\to\infty}\int_\Omega f_n\chi_E\,dx = \lim_{n\to\infty}\int_E f_n\,dx \le \int_0^r f_0^*\,ds.$$

Note that the latter inequality follows from the inequality in (2.6) applied to f_n. If we choose a Borel set $E \subset \Omega$ with $|E| = r$ which gives the maximum of the integral $\int_E f dx$, from the inequality above, using (2.5) we find

$$\int_0^r f^*(s)\,ds \le \int_0^r f_0^*(s)\,ds, 0 < r < |\Omega|,$$

which yields the inequality in (2.6) for f. The equality in (2.6) for f holds since

$$\int_0^{|\Omega|} f^*(s)\,ds = \int_\Omega f\,dx = \lim_{n\to\infty}\int_\Omega f_n\,dx = \lim_{n\to\infty}\int_0^{|\Omega|} f_n^*(s)\,ds = \int_0^{|\Omega|} f_0^*(s)\,ds.$$

The compactness is proved.

Let us prove convexity.

If $f_1, f_2 \in \mathcal{K}$ we put

$$f(x) = \lambda f_1(x) + (1-\lambda)f_2(x), \quad 0 < \lambda < 1.$$

By (2.5) we have, for some E with $|E| = r$,

$$\int_0^r f^*(s)\,ds = \int_E f(x)\,dx = \lambda\int_E f_1(x)\,dx + (1-\lambda)\int_E f_2(x)\,dx.$$

By Lemma 2.1 we also have

$$\int_E f_1(x)\,dx \le \int_0^r f_1^*(s)\,ds \quad \text{and} \quad \int_E f_2(x)\,dx \le \int_0^r f_2^*(s)\,ds.$$

It follows that

$$\int_0^r f^*(s)\,ds \le \lambda\int_0^r f_1^*(s)\,ds + (1-\lambda)\int_0^r f_2^*(s)\,ds.$$

By (2.6) we know that

$$\int_0^r f_1^*(s)\,ds \le \int_0^r f_0^*(s)\,ds \quad \text{and} \quad \int_0^r f_2^*(s)\,ds \le \int_0^r f_0^*(s)\,ds.$$

Hence,

$$\int_0^r f^*(s)\,ds \le \lambda \int_0^r f_0^*(s)\,ds + (1-\lambda)\int_0^r f_0^*(s)\,ds = \int_0^r f_0^*(s)\,ds,$$

which yields the first condition of (2.6) for $f(x)$.
We have

$$\begin{aligned}\int_\Omega f(x)\,dx &= \int_\Omega (\lambda f_1(x) + (1-\lambda) f_2(x))\,dx. \\ &= \lambda \int_\Omega f_1(x) + (1-\lambda)\int_\Omega f_2(x)\,dx. \\ &= \lambda \int_\Omega f_0(x) + (1-\lambda)\int_\Omega f_0(x)\,dx = \int_\Omega f_0(x)\,dx.\end{aligned}$$

The equality of the first and the last integrals can be rewritten as

$$\int_0^{|\Omega|} f^*(s)\,ds = \int_0^{|\Omega|} f_0^*(s)\,ds,$$

which yields the second part of (2.6). Therefore, $f \in \mathcal{K}$ and $\mathcal{K}$ is convex.
This completes the proof. □

Definition 2.5. *We say that $f \in \mathcal{K}$ is an extreme point of $\mathcal{K}$ if there does not exist $f_1, f_2 \in \mathcal{K}$ with $f_1 \neq f_2$ such that $f = (f_1 + f_2)/2$.*

Theorem 2.7. *[2, Theorem 2.2] $\mathcal{F} = \mathcal{F}(f_0)$ is the set of extreme points of $\mathcal{K} = \mathcal{K}(f_0)$.*

Proof. If f is not an extreme point of $\mathcal{K}$ we can write $f = \frac{f_1+f_2}{2}$ with $f_1, f_2 \in \mathcal{K}$, $f_1 \neq f_2$. Using the strictly convex Lipschitz function $G(t) = \sqrt{1+t^2}$), we find

$$G(f) = G\Big(\frac{f_1+f_2}{2}\Big) < \frac{1}{2}\big(G(f_1) + G(f_2)\big) \quad \text{in a set of positive measure.} \tag{2.9}$$

By (2.9) and (2.4) with $g = f_0$ we find

$$\int_\Omega G(f)\,dx < \frac{1}{2}\Big(\int_\Omega G(f_1)\,dx + \int_\Omega G(f_2)\,dx\Big) \le \int_\Omega G(f_0)\,dx.$$

It follows that f is not a rearrangement of f_0 (otherwise, the integrals of $G(f)$ and $G(f_0)$ would have the same value). We conclude that if f is not an extreme point of $\mathcal{K}$ then f cannot belong to $\mathcal{F}$.
Conversely, if $f \in \mathcal{K} \setminus \mathcal{F}$, one finds $f_1, f_2 \in \mathcal{K}$ with $f_1 \neq f_2$ such that $f = \dfrac{f_1+f_2}{2}$. This complete the proof. □

Now we recall the following theorem by Krein-Milman.

Theorem 2.8. *[38, page 67] Let X be a locally convex linear topological vector space. Let $\mathcal{K}$ be convex and compact in X. Then*

- *The set of extreme points is not empty.*
- *$\mathcal{K}$ is the closure of the convex hull of its extreme points.*

Theorem 2.9. *[4, Lemma 6] $\overline{\mathcal{F}}(f_0) = \mathcal{K}(f_0)$.*

Proof. Since $L^p(\Omega)$, $1 \leq p \leq \infty$, is a locally convex linear topological vector space, the assertion follows by Definition 2.5, and Theorems 2.7 and 2.8. □

2.2 Maximization and Minimization of linear functionals

Let $f_0 \in L^p_+(\Omega)$ and $\frac{1}{p} + \frac{1}{q} = 1$. With $g \in L^q_+(\Omega)$ fixed, we define the linear functional

$$L = L(f) = \int_\Omega fg \; dx, \quad f \in \mathcal{F}.$$

Here $\mathcal{F}$=$\mathcal{F}(f_0)$. We investigate the maximization problem

$$\max_{f \in \mathcal{F}} \int_\Omega fg \; dx, \tag{2.10}$$

as well as the minimization problem

$$\min_{f \in \mathcal{F}} \int_\Omega fg \; dx. \tag{2.11}$$

To investigate problems (2.10) and (2.11) we need some preparation. We make use of a function φ which is non-decreasing and a function ψ which is non-increasing.

Lemma 2.3. *[4, Threorem 3] Let $1 \leq p \leq \infty$, let q be the conjugate of p. Suppose there is a non-decreasing function φ such that $\hat{f} = \varphi(g) \in \mathcal{F}$. Then*

$$L(f) = \int_\Omega fg \, dx \leq \int_\Omega \hat{f} g \, dx \;\; \forall f \in \overline{\mathcal{F}}.$$

Moreover, $\hat{f} = \varphi(g)$ is the unique maximizer of the functional L relative to $\overline{\mathcal{F}}$.

Proof. Writing $\phi(s) := \varphi(s)s$ we have

$$\int_\Omega \hat{f} g \; dx = \int_\Omega \phi(g) \; dx.$$

On the other hand, since $\varphi(g^*)$ is a rearrangement of $\varphi(g)$ we have $\hat{f}^* = \varphi(g^*)$ and

$$\int_0^{|\Omega|} \hat{f}^* g^* \; ds = \int_0^{|\Omega|} \phi(g^*) \; ds.$$

Since

$$\int_0^{|\Omega|} \phi(g^*) \; ds = \int_\Omega \phi(g) \; dx$$

it follows that

$$\int_\Omega \hat{f} g \; dx = \int_0^{|\Omega|} \hat{f}^* g^* \; ds.$$

If $f \in \mathcal{F}$ then $\hat{f}^* = f^*$ (since both are in $\mathcal{F}$). It follows that, for $f \in \mathcal{F}$,

$$\int_\Omega \hat{f} g \, dx = \int_0^{|\Omega|} f^* g^* \, ds.$$

Recalling Theorem 2.1, from the latter result we get

$$\int_\Omega fg \, dx \leq \int_0^{|\Omega|} f^* g^* \, ds = \int_\Omega \hat{f} g \, dx \quad \forall f \in \mathcal{F}.$$

If $f \in \overline{\mathcal{F}}$, we can take a sequence $\{f_i\}$ with $f_i \in \mathcal{F}$ such that $f_i \rightharpoonup f$ weakly. Then, from

$$\int_\Omega f_i g \, dx \leq \int_\Omega \hat{f} g \, dx,$$

using the continuity of $L(f)$ we get (as $i \to \infty$)

$$\int_\Omega fg \, dx \leq \int_\Omega \hat{f} g \, dx \quad \forall f \in \overline{\mathcal{F}}.$$

The first assertion of the lemma is proved.
Let us prove uniqueness. For $s \geq 0$ we define

$$A_s = \{f \in \mathcal{F} \quad \text{such that} \quad \|\hat{f} - f\|_p \geq s\},$$

and

$$\psi(s) = \inf_{f \in A_s} \int_\Omega (\hat{f} - f) g \, dx.$$

Since

$$\int_\Omega fg \, dx \leq \int_\Omega \hat{f} g \, dx$$

we have $\psi(s) \geq 0$. Also, if $s = 0$ then $A_0 = \mathcal{F}$ and $\psi(0) = 0$. Moreover, since the set A_s does not increase as s increases, $\psi(s)$ is non-decreasing. To prove that $\psi(s) > 0$ for $s > 0$, let $f_i \in A_s$ such that

$$\psi(s) = \lim_{i \to \infty} \int_\Omega (\hat{f} - f_i) g \, dx.$$

If this limit were zero, we would have

$$\lim_{i \to \infty} \int_\Omega f_i g \, dx = \int_\Omega \hat{f} g \, dx,$$

and by Theorem 2 of [4],

$$\lim_{i\to\infty} \|\hat{f} - f_i\|_p = 0,$$

contradicting the assumption that $f_i \in A_s$. Hence, $\psi(s) > 0$.
At this point we can prove the uniqueness in $\mathcal{F}$. Indeed, if $\hat{f}_1$ is the maximizer different from $\hat{f}$ then

$$\int_\Omega \hat{f} g\, dx = \int_\Omega \hat{f}_1 g\, dx$$

and

$$\|\hat{f} - \hat{f}_1\|_p = s_1 > 0.$$

So, $\hat{f}_1 \in A_{s_1}$ and

$$0 < \psi(s_1) \leq \int_\Omega (\hat{f} - \hat{f}_1) g\, dx = 0.$$

The latter inequality implies $s_1 = 0$, a contradiction.
To prove uniqueness in $\overline{\mathcal{F}}$, we introduce a lower semi-continuous convex function $\bar{\psi}(s)$ such that $0 \leq \bar{\psi}(s) \leq \psi(s)$ and such that $\bar{\psi}(s)$ is the largest of these functions. Clearly $\bar{\psi}(0) = 0$, and we claim that $\bar{\psi}(s) > 0$ for $s > 0$. We make the following remark. If $f \in \mathcal{F}$ we have $\|\hat{f} - f\|_p \leq \|\hat{f}\|_p + \|f\|_p \leq a$ for some $a > 0$. Hence, if $s > a$ the set A_s is empty, and $\psi(s) = \infty$ (any inferior in an empty set is ∞ by definition). Choose $0 < \sigma < s$ such that $\psi(\sigma) < \infty$ and define

$$\theta(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t < \sigma \\ \psi(\sigma)\dfrac{t-\sigma}{a-\sigma} & \text{for} \quad t \geq \sigma. \end{cases}$$

Clearly $\theta(t)$ is continuous and convex, so $\theta(t) \leq \bar{\psi}(t) \leq \psi(t)$ for all t (by definition). Note that $\theta(s) = \psi(\sigma)\dfrac{s-\sigma}{a-\sigma} > 0$, hence $\bar{\psi}(s) > 0$ as claimed. Now, since for $f \in \mathcal{F}$ we have (recall the definition of ψ)

$$\psi\Big(\|\hat{f} - f\|_p\Big) \leq \int_\Omega (\hat{f} - f) g\, dx,$$

and since $\bar{\psi}(s) \leq \psi(s)$ we also have

$$\bar{\psi}\Big(\|\hat{f} - f\|_p\Big) - \int_\Omega (\hat{f} - f) g\, dx \leq 0.$$

The latter inequality holds for $f \in \mathcal{F}$, and we want to extend it to $\overline{\mathcal{F}}$. Let $\bar{f} \in \overline{\mathcal{F}}$ and $f_i \in \mathcal{F}$ such that $f_i \rightharpoonup \bar{f}$ in the weak topology. We have

$$\bar{\psi}\Big(\|\hat{f} - f_i\|_p\Big) - \int_\Omega (\hat{f} - f_i) g\, dx \leq 0.$$

Since the map $f \to \|\hat{f} - f\|_p$ is lower semi-continuous (known theorem for the norm) and since $s \to \bar{\psi}(s)$ is lower semi-continuous (by definition), the term $\bar{\psi}\Big(\|\hat{f} - f\|_p\Big)$ is a lower semi-continuous and convex functional in $f \in \mathcal{F}$. Therefore, we have

$$\bar{\psi}\Big(\|\hat{f} - \bar{f}\|_p\Big) \leq \liminf_{i\to\infty} \bar{\psi}\Big(\|\hat{f} - f_i\|_p\Big).$$

Hence, from the above inequality we have

$$\bar{\psi}\Big(\|\hat{f}-\bar{f}\|_p\Big) - \int_\Omega (\hat{f}-\bar{f})g\,dx \le 0 \quad \forall \bar{f} \in \overline{\mathcal{F}}.$$

Now, if $\hat{f}_1 \in \overline{\mathcal{F}}$ is a second maximizer we have $\int_\Omega (\hat{f}-\hat{f}_1)g\,dx = 0$ and (by the previous inequality)

$$\bar{\psi}\Big(\|\hat{f}-\hat{f}_1\|_p\Big) \le 0,$$

which implies $\|\hat{f}-\hat{f}_1\|_p = 0$ and $\hat{f} = \hat{f}_1$. This completes the proof. □

Lemma 2.4. *Let $1 \le p \le \infty$, let q be the conjugate of p. Suppose there is a non-increasing function ψ such that $\check{f} = \psi(g) \in \mathcal{F}$. Then*

$$L(f) = \int_\Omega fg\ dx \ge \int_\Omega \check{f}g\ dx \ \ \forall f \in \overline{\mathcal{F}}.$$

Moreover, $\check{f}$ is the unique minimizer of the functional L relative to $\overline{\mathcal{F}}$.

Proof. Writing $\Phi(s) := \psi(s)s$ we have

$$\int_\Omega \check{f}g\,dx = \int_\Omega \Phi(g)\,dx.$$

Note that $\check{f}^* = \psi(g_*)$. Thus,

$$\int_0^{|\Omega|} \check{f}^* g_*\,ds = \int_0^{|\Omega|} \Phi(g_*)\,ds.$$

Let $f \in \mathcal{F}$. Since

$$\int_\Omega \Phi(g)\,dx = \int_0^{|\Omega|} \Phi(g_*)\,ds$$

and since $\check{f}^* = f^*$ it follows that

$$\int_\Omega \check{f}g\,ds = \int_0^{|\Omega|} \check{f}^* g_*\,ds = \int_0^{|\Omega|} f^* g_*\,ds.$$

By the latter result and Theorem (2.3) we find

$$\int_\Omega fg\,dx \ge \int_\Omega \check{f}g\,dx \quad \forall f \in \mathcal{F}.$$

If $f \in \overline{\mathcal{F}}$, we can take a sequence $\{f_i\}$ with $f_i \in \mathcal{F}$ such that $f_i \rightharpoonup f$ weakly. Then, from

$$\int_\Omega f_i g\,dx \ge \int_\Omega \check{f}g\,dx,$$

using the continuity of $L(f)$ we get (as $i \to \infty$)

$$\int_\Omega fg\,dx \ge \int_\Omega \check{f}g\,dx \quad \forall f \in \overline{\mathcal{F}}.$$

The first assertion of the lemma is proved.
Concerning the uniqueness, we can invoke the previous lemma for the maximum. Indeed, if $\check{f}$ and $\check{f}_1$ are two minima for $L(f)$ then they are two maxima for the functional

$$\bar{L}(f) = \int_\Omega f(-g)\,dx,$$

therefore $\check{f} = \check{f}_1$. This completes the proof. $\square$

Remark 2.1. *Without any assumption on g we do not have, in general, the uniqueness of maximizer or minimizer. For example, if $g = 1$ and $f \in L^1(\Omega)$ then*

$$\int_\Omega fg\,dx = \int_\Omega f\,dx = constant$$

for any $f \in \overline{\mathcal{F}}$.

Lemma 2.5. *[5, Lemma 2.9] Let $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ be measurable functions, and suppose that every level set of g has measure zero. Then there exists a non-decreasing function φ such that $\varphi(g)$ is a rearrangement of f. Moreover, there exists a non-increasing function ψ such that $\psi(g)$ is a rearrangement of f.*

Proof. Since the level sets of g have zero measure, g^* is strictly decreasing. The right inverse for g^* is defined by

$$(g^*)^{-1}(s) = |\{t \in (0, |\Omega|) : g^*(t) \geq s\}| = |\{x \in \Omega : g(x) \geq s\}|,$$

that is, $(g^*)^{-1}(g^*(t)) = t$ for $0 < t < |\Omega|$. Define

$$\varphi(s) = f^*((g^*)^{-1}(s)),$$

which is an increasing function (the composition of two decreasing functions is increasing). Then,

$$\varphi(g^*) = f^*((g^*)^{-1}(g^*(s))) = f^*.$$

Hence, $\varphi(g)$ is a rearrangement of f.
The proof of the second assertion is similar. The function g_* is strictly increasing. Therefore, the function

$$\psi(s) = f^*((g_*)^{-1}(s)),$$

is a decreasing function. Then,

$$\psi(g_*) = f^*((g_*)^{-1}(g_*(s))) = f^*.$$

Hence, $\psi(g)$ is a rearrangement of f.
This completes the proof. $\square$

Remark 2.2. *By Lemma 2.3 and Lemma 2.5 it follows that problem (2.10) has a unique solution in case every level set of g has measure zero. In case some level set of g has a positive measure, the second assertion of Lemma 2.3 may hold or may not hold. However, we have the following lemma:*

Lemma 2.6. *[4, Theorem 5] If the linear functional*

$$L(f) = \int_\Omega f g \, dx$$

has only one maximizer $\hat{f} \in \mathcal{F}$, then there is a non-decreasing function φ such that $\varphi(g)$ is a rearrangement of $\hat{f}$.

It is possible to prove the existence of a maximizer for problem (2.10) without assuming that every level set of g has measure zero. Indeed, the following result holds.

Theorem 2.10. *Let $L : \overline{\mathcal{F}} \to \mathbb{R}$ be linear and weakly continuous. Then there exist $\tilde{f} \in \mathcal{F}$ and $\check{f} \in \mathcal{F}$ such that*

$$L(\tilde{f}) = \sup_{\overline{\mathcal{F}}} L(f), \quad L(\check{f}) = \inf_{\overline{\mathcal{F}}} L(f).$$

Proof. Let us prove the result for superior. Let $\{f_i\}, f_i \in \overline{\mathcal{F}}$ be a sequence such that

$$\lim_{i\to\infty} L(f_i) = \sup_{\overline{\mathcal{F}}} L(f) \quad \text{(superior always exist).}$$

Since $\overline{\mathcal{F}}$ is weakly compact, there is a subsequence (again denoted by $\{f_i\}$) and an element $\tilde{f} \in \overline{\mathcal{F}}$ such that

$$f_i \rightharpoonup \tilde{f} \quad \text{(weakly).}$$

Since L is weakly continuous we have

$$L(\tilde{f}) = \lim_{i\to\infty} L(f_i) = \sup_{\overline{\mathcal{F}}} L(f).$$

To show that $\tilde{f} \in \mathcal{F}$ we denote

$$H = \{f \in \overline{\mathcal{F}} : L(f) = L(\tilde{f})\}.$$

We are going to prove that $H \cap \mathcal{F} \neq \emptyset$. First we show that H is convex. Let $f_1, f_2 \in H$ then $L(f_1) = L(\tilde{f})$ and $L(f_2) = L(\tilde{f})$,

$$\begin{aligned} L(\lambda f_1 + (1-\lambda) f_2) &= \lambda L(f_1) + (1-\lambda) L(f_2) \\ &= \lambda L(\tilde{f}) + (1-\lambda) L(\tilde{f}) = L(\tilde{f}). \end{aligned}$$

Hence $\lambda f_1 + (1-\lambda) f_2 \in H$, so, H is convex.
We now show H is closed. Let $f_i \in H$ be a sequence such that $f_i \rightharpoonup f$. We have $L(f_i) = L(\tilde{f})$.

Since L is continuous we find $L(f) = L(\tilde{f})$, therefore $f \in H$.
Now, let us show that H is an extreme set in $\overline{\mathcal{F}}$. This means that if $f \in H$ and $f = \dfrac{f_1 + f_2}{2}$ with $f_1, f_2 \in \overline{\mathcal{F}}$, then necessarily $f_1, f_2 \in H$. To show this, we write

$$L(\tilde{f}) = L(f) = \frac{L(f_1) + L(f_2)}{2} \leq \frac{L(\tilde{f}) + L(\tilde{f})}{2} = L(\tilde{f}).$$

Since equality must hold, we find

$$L(f_1) = L(\tilde{f}) \quad \text{and} \quad L(f_2) = L(\tilde{f}), \quad \text{so} \quad f_1, f_2 \in H.$$

On the other hand, since H is weakly compact ($H \subset \overline{\mathcal{F}}$) and convex, we can apply the Krein-Milman theorem (see Theorem 2.8) to deduce that $ext(H) \neq \emptyset$. Next, we show that

$$ext(H) \subset \mathcal{F} = ext(\overline{\mathcal{F}}).$$

To this end, let us assume the inclusion is false. Then, there exists $f \in ext(H)$ which is not in $\mathcal{F}$. Therefore, we have $f = tf_1 + (1-t)f_2$ for some $f_1, f_2 \in \overline{\mathcal{F}}$ and some $t \in (0,1)$. Since $ext(H) \subset H$ and H is an extreme set, we deduce that $f_1, f_2 \in H$. But this contradicts $f \in ext(H)$. We conclude that $ext(H) \subset \mathcal{F}$. Furthermore, we have $H \cap \mathcal{F} \neq \emptyset$ as desired.
This completes the proof. $\square$

2.3 Maximization of convex functionals

In what follows, $1 \leq p \leq \infty$ and q is the conjugate of p, that is $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 2.11. *[4, Theorem 7] Let $\overline{\mathcal{F}} = \overline{\mathcal{F}}(f_0)$ with $f_0 \in L^p_+(\Omega)$. Let $I : \overline{\mathcal{F}} \to \mathbb{R}$ be convex, sequentially continuous with respect to the L^q topology on $L^p(\Omega)$, and let*

$$\lim_{t \to 0^+} \frac{I(f + t(h-f)) - I(f)}{t} = \int_\Omega G(f)(h-f)\,dx, \quad \forall h, f \in \overline{\mathcal{F}}, \tag{2.12}$$

where $G(f) \in L^q(\Omega)$. Then

(i) there is $\hat{f} \in \mathcal{F}$ such that $I(f) \leq I(\hat{f})$ for all $f \in \overline{\mathcal{F}}$;

(ii) if I is strictly convex, if $\hat{f} \in \mathcal{F}$ is a maximizer of I and if (2.12) holds at $\hat{f}$ then $\hat{f} = \varphi(G(\hat{f}))$ a.e. in Ω for some non-decreasing function φ.

Proof. By Theorems 2.6 and 2.9, $\overline{\mathcal{F}}$ is sequentially compact. Hence, there is a maximizer $f_1 \in \overline{\mathcal{F}}$ of I. With $f = f_1$ and $0 < t < 1$ the left hand side of (2.12) is non-positive, hence

$$0 \geq \int_\Omega G(f_1)(h - f_1)\,dx \quad \forall h \in \overline{\mathcal{F}}.$$

On the other hand, by Theorem 2.10 the linear functional

$$L(h) = \int_\Omega G(f_1) h\,dx$$

has a maximizer $\hat{f} \in \mathcal{F}$ in $\overline{\mathcal{F}}$. It follows that

$$0 = \int_\Omega G(f_1)(\hat{f} - f_1)\,dx.$$

In view of this equality, using (2.12) with $f = f_1$ and $h = \hat{f}$ we find

$$I(f_1 + t(\hat{f} - f_1)) - I(f_1) = o(t).$$

Finally, recalling that I is convex we find

$$I(f_1 + t(\hat{f} - f_1)) = I(t\hat{f} + (1-t)f_1)) \leq tI(\hat{f}) + (1-t)I(f_1)$$

and

$$t[I(\hat{f}) - I(f_1)] \geq o(t).$$

By the convexity of I we have $o(t) \geq 0$. Hence $I(\hat{f}) \geq I(f_1)$. Therefore, $\hat{f} \in \mathcal{F}$ is a maximizer and (i) is proved.

Let us prove (ii). Let $\hat{f} \in \mathcal{F}$ be a maximizer and let $h \in \overline{\mathcal{F}}$ such that $h \neq \hat{f}$. By (2.12) we have

$$I(\hat{f} + t(h - \hat{f})) - I(\hat{f}) = t \int_\Omega G(\hat{f})(h - \hat{f})\,dx + o(t).$$

By using strict convexity of I we find

$$t[I(h) - I(\hat{f})] > t \int_\Omega G(\hat{f})(h - \hat{f})\, dx + o(t) \geq t \int_\Omega G(\hat{f})(h - \hat{f})\, dx.$$

Since $I(h) \leq I(\hat{f})$ we must have

$$\int_\Omega G(\hat{f}) h\, dx < \int_\Omega G(\hat{f}) \hat{f}\, dx.$$

It follows that $\hat{f}$ is the unique maximizer of the linear functional

$$L(h) = \int_\Omega G(\hat{f}) h\, dx$$

for $h \in \overline{\mathcal{F}}$. The assertion follows from Lemma 2.6. $\square$

Chapter 3

The p-Laplace equations and optimization problems

In this section we discuss some classical problems involving partial differential equations with coefficients belonging to classes of rearrangements. Next, we consider functionals like energy integrals or eigenvalues and investigate their maximum or minimum in classes of rearrangements.

3.1 Optimization of the energy integral

Let $1 \leq q < p < \infty$ and let Ω be a bounded domain in $\mathbb{R}^n$. With $g \in L^\infty_+(\Omega)$, we consider the boundary value problem

$$-\Delta_p u = gu^{q-1}, \quad u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega. \tag{3.1}$$

Here $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$. The solution $u \in H_0^{1,p}(\Omega)$ satisfies

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\eta \, dx = \int_\Omega gu^{q-1}\eta \, dx \quad \forall \eta \in H_0^{1,p}(\Omega). \tag{3.2}$$

In case of $p = 2$ and $q = 1$, the left hand side of equation (3.2) is a scalar product in the Hilbert space $H_0^1(\Omega)$, and the right hand side $\int_\Omega g\eta \, dx$ is a linear functional which is continuous on $H_0^1(\Omega)$. Hence, the existence and uniqueness of a solution $u = u_g$ is given by the Riesz representation Theorem in a Hilbert space.
For the general case we start proving on existence result.

To prove existence for problem (3.2), for $w \in H_0^{1,p}(\Omega)$ we introduce the functional

$$B(g, w) = \frac{q}{p-q} \int_\Omega \left(\frac{p}{q} g|w|^q - |\nabla w|^p \right) dx. \tag{3.3}$$

Note that if $u \geq 0$ is a solution to problem (3.2) putting $\eta = u$ we find

$$\int_\Omega |\nabla u|^p dx = \int_\Omega g u^q \, dx.$$

Using the latter equation we get

$$B(g,u) = \int_\Omega g u^q \, dx.$$

The superior $u \geq 0$ of $B(g,w)$ (here g is fixed) for $w \in H_0^{1,p}(\Omega)$ satisfies problem (3.2). Indeed we must have

$$\frac{d}{dt} B(g, u+t\eta)\Big|_{t=0} = 0$$

for each $\eta \in H_0^{1,p}(\Omega)$. We have

$$\frac{d}{dt}\int_\Omega \Big(\frac{p}{q} g|u+t\eta|^q - |\nabla(u+t\eta)|^p\Big)\,dx\Big|_{t=0} = \int_\Omega \Big(pgu^{q-1}\eta - p|\nabla u|^{p-2}\nabla u\cdot\nabla\eta\Big)\,dx = 0.$$

Hence

$$\int_\Omega g u^{q-1}\eta \, dx = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \eta \, dx \quad \forall \eta \in H_0^{1,p}(\Omega)$$

which is equation (3.2).

To investigate the maximum of $w \to B(g,w)$, we define

$$W_g^+ = \{w \in H_0^{1,p}(\Omega) : \int_\Omega g|w|^q \, dx > 0\}.$$

If $w \in W_g^+$, for $t > 0$ we have

$$B(g,tw) = \frac{q}{p-q}\Big[t^q \int_\Omega \frac{p}{q} g|w|^q \, dx - t^p \int_\Omega |\nabla w|^p \, dx\Big].$$

If we compute the derivative with respect to t, we find

$$\frac{dB(g,tw)}{dt} = \frac{q}{p-q}\Big(t^{q-1}\int_\Omega pg|w|^q \, dx - pt^{p-1}\int_\Omega |\nabla w|^p \, dx\Big).$$

After simplification we find

$$\frac{dB(g,tw)}{dt} = \frac{pq}{p-q} t^{q-1}\Big(\int_\Omega \Big(g|w|^q \, dx - t^{p-q}\int_\Omega |\nabla w|^p \, dx\Big).$$

Therefore,

$$\frac{dB(g,tw)}{dt} = 0$$

for

$$t_0^{p-q} = \frac{\int_\Omega g|w|^q \, dx}{\int_\Omega |\nabla w|^p \, dx}. \tag{3.4}$$

The function $t \longmapsto B(g,tw)$ is equal to zero at $t = 0$, and since $q < p$, it approaches to minus infinity as t goes to plus infinity. Since the derivative vanishes at t_0 only, $B(g, t_0 w)$ must be the maximum. Hence,

$$B(g,tw) \leq B(g,t_0 w).$$

Indeed, computing $B(g, t_0 w)$ we find

$$
\begin{aligned}
B(g, t_0 w) &= \frac{q}{p-q}\left[\Big(\frac{\int_\Omega g|w|^q\,dx}{\int_\Omega |\nabla w|^p\,dx}\Big)^{\frac{q}{p-q}}\ \frac{p}{q}\int_\Omega g|w|^q\,dx - \Big(\frac{\int_\Omega g|w|^q\,dx}{\int_\Omega |\nabla w|^p\,dx}\Big)^{\frac{p}{p-q}}\int_\Omega |\nabla w|^p\,dx\right] \\
&= \frac{q}{p-q}\left[\frac{p}{q}\ \frac{\left[\int_\Omega g|w|^q\,dx\right]^{\frac{p}{p-q}}}{\left(\int_\Omega |\nabla w|^p\,dx\right)^{\frac{q}{p-q}}} - \frac{\left[\int_\Omega g\,|w|^q\,dx\right]^{\frac{p}{p-q}}}{\left(\int_\Omega |\nabla w|^p\,dx\right)^{\frac{q}{p-q}}}\right] \\
&= \left[\frac{\left(\int_\Omega g\,|w|^q\,dx\right)^{\frac{p}{q}}}{\int_\Omega |\nabla w|^p\,dx}\right]^{\frac{q}{p-q}}.
\end{aligned}
$$

Thus,

$$
B(g, tw) \le \frac{\left(\int_\Omega g\,|w|^q dx\right)^{\frac{p}{p-q}}}{\left(\int_\Omega |\nabla w|^p\,dx\right)^{\frac{q}{p-q}}}.
$$

As a consequence,

$$
\sup_{w\in H_0^{1,p}} B(g,w) = \sup_{w\in W_g^+} B(g,w) = \sup_{w\in W_g^+}\left[\frac{\left(\int_\Omega g\,|w|^q\,dx\right)^{\frac{p}{q}}}{\int_\Omega |\nabla w|^p\,dx}\right]^{\frac{q}{p-q}}. \tag{3.5}
$$

Note that if $w = u$ is a maximizer for the functional

$$
\frac{\left(\int_\Omega g\,|w|^q\,dx\right)^{\frac{p}{q}}}{\int_\Omega |\nabla w|^p dx} \tag{3.6}
$$

then also νu, $\nu \neq 0$, is a maximizer for the same functional. A maximizer $u \in W_g^+$ for (3.6) is also a maximizer for $B(g, w)$ if $t_0 = 1$ in (3.4), that is if

$$
\int_\Omega g|u|^q\,dx = \int_\Omega |\nabla u|^p\,dx. \tag{3.7}
$$

A function $u_g \ge 0$ is a maximizer of $B(g, w)$ in W_g^+ if and only if it is a minimizer of

$$
\inf_{w\in W_g^+} \frac{\int_\Omega |\nabla w|^p\,dx}{\left(\int_\Omega g|w|^q\,dx\right)^{\frac{p}{q}}}, \tag{3.8}
$$

normalized as in (3.7). The minimization problem (3.8) has been discussed in the paper [28], where it is proved that it admits a non-negative minimizer u_g. Therefore, the maximization problem (3.5) has a non-negative solution which satisfies problem (3.2). By [43, Theorem 5] we know that $u_g(x) > 0$ in Ω.

Let us prove a result of uniqueness.

Theorem 3.1. *[19, Theorem 3.2] Let Ω be a bounded smooth domain in $\mathbb{R}^N$, and let $g \ge 0, g \not\equiv 0$, be a bounded function. Let $1 \le q < p$, and let $u, v \in C^1(\Omega) \cap C^0(\bar\Omega)$ be positive solutions to problem (3.2). Then $v(x) = u(x)$.*

Proof. If $q = 1$ the result follows from a classical comparison principle, see [22, Theorem 10.7] or [37, Theorem 2.4.1]. Let $q > 1$ and $A = \{x \in \Omega : u(x) > v(x)\}$. If we prove that A is empty, we have $u(x) \leq v(x)$ in Ω. If we change u with v we find $v(x) \leq u(x)$ in Ω and assertion of the theorem follows. We argue by contradiction, assuming A is not empty. For $\epsilon > 0$, define $u_\epsilon = u + \epsilon$ and $v_\epsilon = v + \epsilon$. Note that $A = \{x \in \Omega : u_\epsilon(x) > v_\epsilon(x)\}$. We define

$$\eta_1(x) = \max\Big[\frac{u_\epsilon^q(x) - v_\epsilon^q(x)}{u_\epsilon^{q-1}(x)}, 0\Big] \quad \text{and} \quad \eta_2(x) = \min\Big[\frac{v_\epsilon^q(x) - u_\epsilon^q(x)}{v_\epsilon^{q-1}(x)}, 0\Big].$$

We put $\eta = \eta_1$ in (3.2) to obtain

$$\int_A |\nabla u|^{p-2}\nabla u.\nabla\Big(\frac{u_\epsilon^q - v_\epsilon^q}{u_\epsilon^{q-1}}\Big)\,dx = \int_A g(u_\epsilon^q - v_\epsilon^q)\Big(\frac{u}{u_\epsilon}\Big)^{q-1}\,dx.$$

Next we consider (3.27) with v instead of u and we put $\eta = \eta_2$ to obtain

$$\int_A |\nabla v|^{p-2}\nabla v.\nabla\Big(\frac{v_\epsilon^q - u_\epsilon^q}{v_\epsilon^{q-1}}\Big)\,dx = \int_A g(v_\epsilon^q - u_\epsilon^q)\Big(\frac{v}{v_\epsilon}\Big)^{q-1}\,dx.$$

Adding both equations we get

$$\int_A |\nabla u|^{p-2}\nabla u.\nabla\Big(\frac{u_\epsilon^q - v_\epsilon^q}{u_\epsilon^{q-1}}\Big)\,dx + \int_A |\nabla v|^{p-2}\nabla v.\nabla\Big(\frac{v_\epsilon^q - u_\epsilon^q}{v_\epsilon^{q-1}}\Big)\,dx = I_\epsilon, \tag{3.9}$$

where we set

$$I_\epsilon = \int_A g(u_\epsilon^q - v_\epsilon^q)\Big(\frac{u}{u_\epsilon}\Big)^{q-1}\,dx + \int_A g(v_\epsilon^q - u_\epsilon^q)\Big(\frac{v}{v_\epsilon}\Big)^{q-1}\,dx. \tag{3.10}$$

Since

$$\nabla\Big(\frac{u_\epsilon^q - v_\epsilon^q}{u_\epsilon^{q-1}}\Big) = \nabla u + (q-1)\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^q\nabla u - q\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{q-1}\nabla v,$$

the first integral of left-hand side of (3.9) becomes

$$\int_A \Big\{|\nabla u|^p\Big(1 + (q-1)\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^q\Big) - q|\nabla u|^{p-2}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{q-1}\nabla u.\nabla v\Big\}\,dx.$$

The second integral of the left-hand side of (3.9) is obtained by the first one by changing u_ϵ and v_ϵ; thus, (3.9) yields

$$\begin{aligned}\int_A \Big\{|\nabla u|^p\Big[1 + (q-1)\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^q\Big] - q|\nabla u|^{p-2}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{q-1}\nabla u.\nabla v \\ +|\nabla v|^p\Big[1 + (q-1)\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^q\Big] - q|\nabla v|^{p-2}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{q-1}\nabla u.\nabla v\Big\}\,dx = I_\epsilon.\end{aligned} \tag{3.11}$$

Now, for $X, Y \in \mathbb{R}^N$, we recall the inequality

$$|X|^{p-2}X.Y \leq \frac{1}{r}|X|^p + \frac{1}{p}|Y|^p \quad \text{with} \quad \frac{1}{p} + \frac{1}{r} = 1.$$

Replacing the vector X by λX, where λ is a positive real number, we get

$$|X|^{p-2}\lambda^{p-1}X.Y \leq \frac{1}{r}\lambda^p|X|^p + \frac{1}{p}|Y|^p.$$

With $\lambda = \Big(\dfrac{v_\epsilon}{u_\epsilon}\Big)^{\dfrac{q-1}{p-1}}, X = \nabla u, Y = \nabla v$ we find

$$|\nabla u|^{p-2}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{q-1}\nabla u.\nabla v \leq \frac{1}{r}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{r(q-1)}|\nabla u|^p + \frac{1}{p}|\nabla v|^p.$$

Similarly, with $\lambda = \Big(\dfrac{u_\epsilon}{v_\epsilon}\Big)^{\dfrac{q-1}{p-1}}, X = \nabla v, Y = \nabla u$ we find

$$|\nabla v|^{p-2}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{q-1}\nabla v.\nabla u \leq \frac{1}{r}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{r(q-1)}|\nabla v|^p + \frac{1}{p}|\nabla u|^p.$$

In view of these inequalities, by (3.11) we get

$$\begin{aligned} I_\epsilon \geq \int_A \Big\{ &|\nabla u|^p\Big[1 + (q-1)\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^q - \frac{q}{r}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{r(q-1)} - \frac{q}{p}\Big] \\ &+|\nabla v|^p\Big[1 + (q-1)\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^q - \frac{q}{r}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{r(q-1)} - \frac{q}{p}\Big]\Big\}\, dx. \end{aligned}$$

Putting

$$\varphi(t) = 1 + (q-1)t^q - \frac{q}{r}t^{r(q-1)} - \frac{q}{p},$$

the latter inequality reads as

$$I_\epsilon \geq \int_A \Big\{|\nabla u|^p\varphi\Big(\frac{v_\epsilon}{u_\epsilon}\Big) + |\nabla v|^p\varphi\Big(\frac{u_\epsilon}{v_\epsilon}\Big)\Big\}\, dx.$$

As $\epsilon \to 0$, by (3.10) we find

$$\lim_{\epsilon\to 0} I_\epsilon = 0.$$

Therefore, we find

$$0 \geq \int_A \Big\{|\nabla u|^p\varphi\Big(\frac{v}{u}\Big) + |\nabla v|^p\varphi\Big(\frac{u}{v}\Big)\Big\}\, dx. \tag{3.12}$$

We have $\varphi(1) = 0$ and $\varphi'(t) = q(q-1)t^{q-1}(1 - t^{\dfrac{q-p}{p-1}})$. Hence, $\varphi(t)$ attains its minimum value at $t = 1$ and $\varphi(t) > 0$ for $t \neq 1$. Since $\dfrac{u}{v} > 1$ in A, by (3.12) we must have $|\nabla u| = |\nabla v| = 0$ in A. Hence, $\nabla(u - v) = 0$ in A and $u - v = 0$ on ∂A. Then, $u(x) = v(x)$, contradicting the definition of A. The theorem follows. □

If u and v are two maximizers of $B(g, w)$ then they are solutions of our problem (3.2). By the uniqueness Theorem 3.1 we find $u = v$ in Ω.

Now we investigate optimization problems associated with the energy integral. Given $g_0 \in L^\infty_+(\Omega)$, define $\mathcal{G}$ as the class of rearrangements of g_0, and $\overline{\mathcal{G}}$ as the weak* closure of $\mathcal{G}$. For $g \in \overline{\mathcal{G}}$, we define

$$I(g) = \sup_{w\in H_0^{1,p}(\Omega)} B(g, w). \tag{3.13}$$

If u_g is the non-negative maximizer of $B(g,w)$ then, by using (3.7), we find

$$I(g) = B(g,u_g) = \int_\Omega g\, u_g^q dx = \int_\Omega |\nabla u_g|^p dx. \tag{3.14}$$

We investigate the problem

$$\max_{g\in\mathcal{G}} I(g).$$

We find a result of existence and a result of representation of a maximizer.
Furthermore, we investigate the problem

$$\min_{g\in\mathcal{G}} I(g).$$

We find results of existence and uniqueness, and a result of representation of the minimizer. In case Ω is a ball we find precise configurations for the maximizer and for the minimizer. The present results are contained in the paper [19] and generalize those of [4], [5] and [6] relative to the case $p=2$ and $q=1$.

Proposition 3.1. *[19, Proposition 2.1] Let $g_0 \geq 0$ be a bounded function, positive in a subset of positive measure. Let $\overline{\mathcal{G}}$ be the weak* closure of $\mathcal{G}=\mathcal{G}(g_0)$ in $L^\infty(\Omega)$, and let $I(g)$ be defined as in (3.13) for $g\in\overline{\mathcal{G}}$.*

(i) The functional $I(g)$ is continuous with respect to the weak topology in $L^\infty(\Omega)$.*

(ii) The functional $I(g)$ satisfies condition (2.12) with $G(g)=\frac{p}{p-q}u_g^q$, where u_g is the maximizer of $B(g,w)$ for $w\in H_0^{1,p}(\Omega)$.

(iii) The functional $I(g)$ is strictly convex.

Proof. (i) Let $g_i \rightharpoonup g$ in the weak* topology of $L^\infty(\Omega)$. Since $g(x)>0$ in a subset of positive measure, $g_i(x)>0$ in subsets of positive measure for i large. Let u_{g_i} be the non-negative maximizer of $B(g_i,w)$, and let u_g be the non-negative maximizer of $B(g,w)$ in $H_0^{1,p}(\Omega)$. Then

$$\begin{aligned}
&I(g)+\frac{q}{p-q}\int_\Omega \frac{p}{q}(g_i-g)u_g^q\,dx\\
&=\frac{q}{p-q}\int_\Omega\Big(\frac{p}{q}g_iu_g^q-|\nabla u_g|^p\Big)dx\\
&\leq \frac{q}{p-q}\int_\Omega\Big(\frac{p}{q}g_iu_{g_i}^q-|\nabla u_{g_i}|^p\Big)dx = I(g_i)\\
&=\frac{q}{p-q}\int_\Omega\Big(\frac{p}{q}\big((g_i-g)+g\big)u_{g_i}^q-|\nabla u_{g_i}|^p\Big)dx\\
&\leq \frac{q}{p-q}\int_\Omega \frac{p}{q}(g_i-g)u_{g_i}^q\,dx + I(g).
\end{aligned} \tag{3.15}$$

By assumption, we have

$$\lim_{i\to\infty}\int_\Omega (g_i-g)u_g^q\,dx = 0. \tag{3.16}$$

Let us prove that

$$\lim_{i\to\infty}\int_\Omega (g_i - g)u_{g_i}^q\ dx = 0. \tag{3.17}$$

By the normalization (3.7) we have

$$\int_\Omega |\nabla u_{g_i}|^p dx = \int_\Omega g_i\ u_{g_i}^q dx.$$

Since g_0 is bounded and since $\inf_\Omega g_0 \le g_i \le \sup_\Omega g_0$, by using Poincarè inequality we find

$$\int_\Omega g_i u_{g_i}^q dx \le \tilde{C}\int_\Omega u_{g_i}^q dx \le C\int_\Omega |\nabla u_{g_i}|^q dx.$$

Hence,

$$\int_\Omega |\nabla u_{g_i}|^p dx \le C\int_\Omega |\nabla u_{g_i}|^q dx \le C\Big(\int_\Omega |\nabla u_{g_i}|^p dx\Big)^{\frac{q}{p}}|\Omega|^{\frac{p-q}{p}}. \tag{3.18}$$

Recalling that $q < p$, by estimate (3.18) we infer that the norm $\|\nabla u_{g_i}\|_{L^p(\Omega)}$ is bounded by a constant independent of i. Therefore, a sub-sequence of u_{g_i} (denoted again u_{g_i}) converges weakly in $H^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ to some function $z \in H_0^{1,p}(\Omega)$, $z(x) \ge 0$. Since $q < p$, u_{g_i} converges to z strongly in $L^q(\Omega)$. We claim that

$$\lim_{n\to\infty}\int_\Omega (g_i - g)(u_{g_i}^q - z^q)\ dx = 0,$$

which implies (3.17). This claim follows from the inequality

$$|(g_i - g)(u_{g_i}^q - z^q)| \le C|u_{g_i} - z|q(u_{g_i} + z)^{q-1}$$

and Hölder's inequality

$$\Big|\int_\Omega |u_{g_i} - z|(u_{g_i} + z)^{q-1}dx\Big| \le \|u_{g_i} - z\|_{L^q(\Omega)}\|u_{g_i} + z\|_{L^q(\Omega)}^{q-1}.$$

From (3.15), (3.16) and (3.17) we infer

$$\lim_{i\to\infty} I(g_i) = I(g). \tag{3.19}$$

Thus part *(i)* of the proposition is proved.
We claim that the function z in above is equal to u_g, the maximizer of $B(g, w)$ for $w \in H_0^{1,p}(\Omega)$. To this end, we recall the inequality for vectors $a, b \in \mathbb{R}^N$

$$|b|^p \ge |a|^p + p(|a|^{p-2}a, b - a),$$

expressing the convexity of $|x|^p$. It follows that

$$\int_\Omega |\nabla u_{g_i}|^p\ dx \ge \int_\Omega |\nabla z|^p\ dx + p\int_\Omega |\nabla z|^{p-2}\nabla z(\nabla u_{g_i} - \nabla z)\ dx.$$

Since $u_{g_i} \rightharpoonup z$ weakly in $H^{1,p}(\Omega)$ we have

$$\lim_{i\to\infty}\int_\Omega |\nabla z|^{p-2}\nabla z(\nabla u_{g_i} - \nabla z)\ dx = 0.$$

Hence,

$$\liminf_{i\to\infty} \int_\Omega |\nabla u_{g_i}|^p \, dx \ge \int_\Omega |\nabla z|^p \, dx.$$

Using this inequality and (3.19), from

$$I(g_i) = \frac{q}{p-q}\Big[\int_\Omega \frac{p}{q} g_i u_{g_i}^q \, dx - \int_\Omega |\nabla u_{g_i}|^p \, dx\Big]$$

we find

$$I(g) \le \frac{q}{p-q}\Big[\int_\Omega \frac{p}{q} g z^q \, dx - \int_\Omega |\nabla z|^p \, dx\Big] = B(g,z) \le I(g).$$

By the uniqueness of the maximizer of $B(g,w)$, we have $z = u_g$, as claimed.

(ii) Let $t_i > 0$ be a sequence such that $t_i \to 0$ as $i \to \infty$. Let $g, h \in \overline{\mathcal{G}}$ and let $g_{t_i} = g + t_i(h-g)$. Then, by (3.15) we find

$$I(g) + t_i \frac{p}{p-q} \int_\Omega (h-g)\, u_g^q \, dx \le I(g + t_i(h-g)) \le I(g) + t_i \frac{p}{p-q} \int_\Omega (h-g) u_{g_{t_i}}^q \, dx.$$

It follows that

$$\frac{p}{p-q} \int_\Omega (h-g) u_g^q \, dx \le \frac{I(g+t_i(h-g)) - I(g)}{t_i} \le \frac{p}{p-q} \int_\Omega (h-g) u_{g_{t_i}}^q \, dx. \tag{3.20}$$

Since $t_i \to 0$ as $i \to \infty$, we have $g_{t_i} \to g$ as $i \to \infty$ in the norm of $L^\infty(\Omega)$. As a consequence, $u_{g_{t_i}} \to u_g$ in the $L^p(\Omega)$ norm and, since $q < p$, we have

$$\lim_{i\to\infty} \int_\Omega (h-g) u_{g_{t_i}}^q \, dx = \int_\Omega (h-g) u_g^q \, dx.$$

Since the sequence t_i is arbitrary, from the latter equation and (3.20) we find

$$\lim_{t\to 0^+} \frac{I(g+t(h-g)) - I(g)}{t} = \frac{p}{p-q} \int_\Omega (h-g) u_g^q \, dx. \tag{3.21}$$

Thus part *(ii)* of the proposition follows.

(iii) Let $0 < t < 1$. If $g_1,\ g_2 \in \overline{\mathcal{G}}$ and $g_t = tg_1 + (1-t)g_2$, we have

$$B(g_t, w) = tB(g_1, w) + (1-t)B(g_2, w),$$

$$\sup_{w\in H_0^{1,p}} B(g_t, w) \le t \sup_{w\in H_0^{1,p}} B(g_1, w) + (1-t) \sup_{w\in H_0^{1,p}} B(g_2, w).$$

It follows that

$$I(g_t) \le tI(g_1) + (1-t)I(g_2), \tag{3.22}$$

that is, the convexity of $I(g)$. Now, let us prove strict convexity. Recall that $g_0(x) \ge 0$ is the function which generates $\mathcal{G}$ and $\overline{\mathcal{G}}$. In particular, every function $g \in \overline{\mathcal{G}}$, satisfies $g(x) \ge 0$. If equality holds in (3.22) for some $0 < t < 1$, then

$$B(g_t, u_{g_t}) = tB(g_1, u_{g_1}) + (1-t)B(g_2, u_{g_2}),$$

where u_{g_t}, u_{g_1}, u_{g_2} are respectively maximizers of $B(g_t, w)$, $B(g_1, w)$, $B(g_2, w)$ for $w \in H_0^{1,p}$. Since $B(g_t, u_{g_t}) = tB(g_1, u_{g_t}) + (1-t)B(g_2, u_{g_t})$, it follows that

$$B(g_1, u_{g_t}) = B(g_1, u_{g_1}) \quad \text{and} \quad B(g_2, u_{g_t}) = B(g_2, u_{g_2}).$$

By the uniqueness of the maximizer we must have $u_{g_t} = u_{g_1} = u_{g_2}$. Furthermore, we have

$$-\Delta_p u_{g_t} = g_1 \ u_{g_t}^{q-1} \ \text{ a.e. in } \ \Omega$$

and

$$-\Delta_p u_{g_t} = g_2 \ u_{g_t}^{q-1} \ \text{ a.e. in } \ \Omega.$$

Since $g_1(x) \geq 0$ and $g_1(x) \not\equiv 0$, by the strong maximum principle ([43] Theorem 5), $u_{g_t}(x) > 0$. It follows that $g_1(x) = g_2(x)$ a.e. in Ω.
The proof of the proposition is completed. □

Theorem 3.2. *[19, Theorem 2.2] Let $g_0 \geq 0$ be bounded and positive in a subset of positive measure, let $\mathcal{G}$ be the class of rearrangements generated by g_0, and let $I(g)$ be defined as in* (3.13)*. Then*

(I) problem

$$\max_{g \in \mathcal{G}} I(g)$$

has a solution $\hat{g} \in \mathcal{G}$; furthermore, if $\hat{g}$ is a maximizer of $I(g)$ and if $u_{\hat{g}}$ is the non-negative maximizer of $w \to B(\hat{g}, w)$ then $\hat{g} = \varphi\big(u_{\hat{g}}\big)$ for some non-decreasing function $\varphi(t)$.

(II) problem

$$\min_{g \in \mathcal{G}} I(g)$$

has a unique solution $\check{g} \in \mathcal{G}$; furthermore, if $\check{g}$ is the minimizer of $I(g)$, and if $u_{\check{g}}$ is the non-negative maximizer of $w \to B(\check{g}, w)$, then $\check{g} = \psi\big(u_{\check{g}}\big)$ for some non-increasing function $\psi(t)$.

Proof. By Proposition 3.1, the functional $I(g)$ is continuous with respect to the weak* topology of $L^\infty(\Omega)$, is convex on $\overline{\mathcal{G}}$ and condition (2.12) holds with $G(g) = \frac{p}{p-q} u_g^q$. Then, by Theorem 2.11 (i), there is a function $\hat{g} \in \mathcal{G}$ such that $I(g) \leq I(\hat{g})$ for all $g \in \overline{\mathcal{G}}$. Furthermore, by Proposition 3.1, $I(g)$ is strictly convex. Therefore by Theorem 2.11 (ii), there is a non-decreasing function $\tilde{\varphi}$ such that $\hat{g} = \tilde{\varphi}\big(\frac{p}{p-q} u_{\hat{g}}^q\big)$. The conclusion of part (I) follows with $\varphi(t) = \tilde{\varphi}\big(\frac{p}{p-q} t^q\big)$.
Let us prove part (II). Since $I(g)$ is continuous with respect to the weak* topology of $L^\infty(\Omega)$ and since $\overline{\mathcal{G}}$ is compact, we know that a minimizer $\bar{g}$ of $I(g)$ exists on $\overline{\mathcal{G}}$, and is unique by the strict convexity (proved in Proposition 3.1). We have to show that $\bar{g} \in \mathcal{G}$. If $0 < t < 1$ and if $g_t = tg + (1-t)\bar{g}$, using the minimality of $\bar{g}$ and (2.12) we find

$$I(\bar{g}) \leq I(g_t) = I(\bar{g}) + t \int_\Omega (g - \bar{g}) \frac{p}{p-q} u_{\bar{g}}^q dx + o(t),$$

where
$$\lim_{t\to 0}\frac{o(t)}{t}=0.$$
It follows that
$$\int_\Omega (g-\bar{g})\frac{p}{p-q}u_{\bar{g}}^q dx \geq 0.$$
Equivalently,
$$\int_\Omega g\; u_{\bar{g}}^q dx \geq \int_\Omega \bar{g}\; u_{\bar{g}}^q dx \quad \forall g\in\overline{\mathcal{G}}. \tag{3.23}$$
By the equation
$$-\Delta_p u_{\bar{g}} = \bar{g}\; u_{\bar{g}}^{q-1}, \tag{3.24}$$
it follows that the function $u_{\bar{g}}$ cannot have flat zones in the set
$$F=\{x\in\Omega : \bar{g}(x)>0\}.$$
If $|F|<|\Omega|$, since $\bar{g}\in\overline{\mathcal{G}}$, by Lemma 5 of [4] we have
$$|F|\geq |\{x\in\Omega : g_0(x)>0\}|.$$
Then there is $g_1\in\mathcal{G}$ such that its support is contained in F. By Lemma 2.5, there is a non-increasing function $\psi_1(t)$ such that $\psi_1(u_{\bar{g}}^q)$ is a rearrangement of g_1 on F.
Define
$$\alpha=\inf_{x\in\Omega\setminus F} u_{\bar{g}}^q(x).$$
We claim that $u_{\bar{g}}^q(x)\leq\alpha$ in F. Arguing by contradiction suppose the claim is false. Therefore there exist a number $S_1>\alpha$ and a subset A of F with $|A|>0$ such that $u_{\bar{g}}^q(x)>S_1$ a.e. on A. Now let $\alpha<S_2<S_1$. We can find a set D of positive measure contained in $\Omega\setminus F$ such that $u_{\bar{g}}^q(x)<S_2$ a.e. on D. We can assume $|A|=|D|$. Using a measure preserving bijection $T:A\to D$ we define a particular rearrangement of $\bar{g}$, denoted by h, as follows.
$$h(x)=\begin{cases}\bar{g}(Tx), & x\in A\\ \bar{g}(T^{-1}x) & x\in D\\ \bar{g}(x) & x\in\Omega\setminus(A\cup D).\end{cases}$$
Thus
$$\begin{aligned}&\int_\Omega h\,u_{\bar{g}}^q\,dx-\int_\Omega \bar{g}\,u_{\bar{g}}^q\,dx=\int_{A\cup D} h\,u_{\bar{g}}^q\,dx-\int_{A\cup D}\bar{g}\,u_{\bar{g}}^q\,dx\\ &=\int_A h\,u_{\bar{g}}^q\,dx+\int_A \bar{g}\,u_{\bar{g}}^q\circ T\,dx-\int_A \bar{g}\,u_{\bar{g}}^q\,dx-\int_A h\,u_{\bar{g}}^q\circ T\,dx\\ &=\int_A \bar{g}\,\left(u_{\bar{g}}^q\circ T-u_{\bar{g}}^q\right)\,dx<(S_2-S_1)\int_A \bar{g}\,dx<0.\end{aligned}$$

Therefore $\int_\Omega h\, u_{\bar g}^q \ \ dx < \int_\Omega \bar g\, u_{\bar g}^q\, dx$, which contradicts (3.23), and the claim follows.
By using equation (3.24) we see that the set $\{x \in F : u_{\bar g}^q(x) = \alpha\}$ has measure zero. Therefore we find that $u_{\bar g}^q(x) < \alpha$ a.e. in F. Now define

$$\tilde\psi(t) = \begin{cases} \psi_1(t) & \text{if } \ 0 \le t < \alpha \\ 0 & \text{if } \ t \ge \alpha. \end{cases}$$

The function $\tilde\psi(t)$ is non-increasing. Let us show that $\tilde\psi(u_{\bar g}^q) \in \mathcal{G}$. Indeed, the functions g_1 and $\tilde\psi(u_{\bar g}^q)$ have the same rearrangement on F, and both vanish on $\Omega \setminus F$. Therefore, $\tilde\psi(u_{\bar g}^q)$ is a rearrangement of $g_1 \in \mathcal{G}$. For $\breve g = \tilde\psi(u_{\bar g}^q)$, by Lemma 2.4 we have

$$\int_\Omega g\ u_{\bar g}^q\ dx \ge \int_\Omega \breve g\ u_{\bar g}^q\ dx \quad \forall g \in \overline{\mathcal{G}}.$$

By the latter result, the uniqueness of the minimizer and (4.25) we must have $\breve g = \bar g$. The theorem is proved with $\psi(t) = \tilde\psi(t^q)$.
If $|F| = |\Omega|$, the proof is easier and we do not need the introduction of the function g_1. This completes the proof. □

3.1.1 Energy integral in symmetric domains

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain Steiner symmetric with respect to some hyperplane, say $\{x_1 = 0\}$. Recall that Ω is Steiner symmetric with respect to $\{x_1 = 0\}$ if and only if the intersection of Ω with any straight line orthogonal to this hyperplane is either empty or a segment centered in the hyperplane. In this case we write $\Omega = \Omega^\sharp$. A function u defined in a Steiner symmetric domain Ω is Steiner symmetric if and only if all sets $\{x \in \Omega : u(x) > t\}$, $t \in \mathbb{R}$, are Steiner symmetric (with respect to the same hyperplane $\{x_1 = 0\}$). In this case we write $u = u^\sharp$. If u is not Steiner symmetric we always can associate to u a rearrangement $u^\sharp$ which is Steiner symmetric. The following results are classical.

Lemma 3.1. *[26, page 23 and page 26] Let $\Omega = \Omega^\sharp$.*

i) If $f(x)$ and $g(x)$ belong to $L_+^\infty(\Omega)$ then

$$\int_\Omega f(x)g(x)dx \le \int_\Omega f^\sharp(x)g^\sharp(x)dx. \tag{3.25}$$

ii) If $u \in H_0^{1,p}(\Omega)$, $u(x) \ge 0$ then $u^\sharp \in H_0^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u|^p dx \ge \int_\Omega |\nabla u^\sharp|^p dx. \tag{3.26}$$

Proposition 3.2. *[19, Proposition 2.4] Let $\Omega = \Omega^\sharp$ and let $I(g)$ be defined as in (3.13). We have*

$$I(g) \leq I(g^\sharp) \quad \forall g \in \mathcal{G}.$$

Proof. If $g \in \mathcal{G}$ and if u_g is the corresponding non-negative maximizer of $w \to B(g, w)$, using (3.25) and (3.26) we find

$$\begin{aligned} I(g) &= \frac{q}{p-q}\Big[\frac{p}{q}\int_\Omega g\, u_g^q\, dx - \int_\Omega |\nabla u_g|^p\, dx\Big] \\ &\leq \frac{q}{p-q}\Big[\frac{p}{q}\int_\Omega g^\sharp (u_g^\sharp)^q\, dx - \int_\Omega |\nabla u_g^\sharp|^p dx\Big] \\ &\leq \frac{q}{p-q}\Big[\frac{p}{q}\int_\Omega g^\sharp u_{g^\sharp}^q\, dx - \int_\Omega |\nabla u_{g^\sharp}|^p\, dx\Big] = I(g^\sharp). \end{aligned}$$

This completes the proof. □

Now let $\Omega = B_R$ be a ball centered at the origin and radius R. A function u defined in B_R is Schwarz symmetric if and only if all sets $\{x \in \Omega : u(x) > t\}$, $t \in \mathbb{R}$, are balls centered in the origin. In this case we write $u = u^\Delta$. If u is not Schwarz symmetric we always can associate to u a rearrangement u^Δ which is Schwarz symmetric. The function u^Δ is also named the Schwarz decreasing rearrangement of u. We also use the function u_Δ, a rearrangement of u which is radially symmetric and increasing with respect to $|x|$.
Note that if $u^*(t)$ defined in $[0, |\Omega|]$ is the usual decreasing rearrangement of $u(x)$ then we have

$$u^\Delta(x) = u^*(\omega_N |x|^N), \quad u_\Delta(x) = u_*(\omega_N |x|^N).$$

Here, ω_N denotes the measure of the unit ball in $\mathbb{R}^N$.
We shall use the following classical result.

Lemma 3.2. *[26, page 23 and page 26] Let $\Omega = B_R$. If $f(x)$ and $g(x)$ are non-negative and bounded in Ω then*

$$\int_\Omega f^\Delta(x) g_\Delta(x) dx \leq \int_\Omega f(x) g(x) dx.$$

Proposition 3.3. *[19, Proposition 2.6] Let $\Omega = B_R$ and let $I(g)$ be defined as in (3.13). We have*

$$I(g) \leq I(g^\Delta) \quad \forall g \in \mathcal{G}$$

and

$$I(g_\Delta) \leq I(g) \quad \forall g \in \mathcal{G}.$$

Proof. The first statement follows from Proposition 3.2, since a ball is Steiner symmetric with respect to any hyperplane passing through the origin.

Let us come to the second statement. Since $g(x) \geq 0$ we have $g_\Delta(x) \geq 0$. Hence, by the uniqueness for the problem

$$-\Delta_p u = g_\Delta u^{q-1} \quad \text{in} \quad B_R, \quad u = 0 \quad \text{on} \quad \partial B_R,$$

one finds that $u = u_{g_\Delta}$ is radially symmetric. Therefore we can write the previous equation as

$$-(r^{N-1}(|u'|^{p-2}u'))' = r^{N-1} g_\Delta u^{q-1}.$$

Integrating we find

$$-r^{N-1}(|u'|^{p-2}u') = \int_0^r s^{N-1} g_\Delta u^{q-1}\, ds.$$

It follows that u is non-increasing with respect to r and that $u^q_{g_\Delta} = (u^q_{g_\Delta})^\Delta$. Hence using Lemma 3.2 we find

$$\begin{aligned} I(g) &= \frac{q}{p-q}\Big[\frac{p}{q}\int_\Omega g\, u_g^q\, dx - \int_\Omega |\nabla u_g|^p\, dx\Big] \\ &\geq \frac{q}{p-q}\Big[\frac{p}{q}\int_\Omega g u^q_{g_\Delta}\, dx - \int_\Omega |\nabla u_{g_\Delta}|^p\, dx\Big] \\ &\geq \frac{q}{p-q}\Big[\frac{p}{q}\int_\Omega g_\Delta u^q_{g_\Delta}\, dx - \int_\Omega |\nabla u_{g_\Delta}|^p\, dx\Big] = I(g_\Delta). \end{aligned}$$

This completes the proof. □

Particular cases. Let $g_0(x) = \chi_{E_0}$, where $E_0 \subset B_R$. By Proposition 3.3 , $I(g)$ attains its maximum value when $g = \chi_{\hat{E}}$, $\hat{E}$ being the ball centered in the origin with $|\hat{E}| = |E_0|$. Moreover, $I(g)$ attains its minimum value when $g = \chi_{\check{E}}$, where $B_R \setminus \check{E}$ is a suitable ball centered in the origin.

Let B be a disc (in the plane) with radius 1, and let $g(x) = \chi_{E_0}$, where E_0 is a disc concentric with B and radius $0 < a < 1$. Consider the boundary value problem for ($p = 2$ and $q = 1$)

$$-\Delta u = \chi_{E_0}, \quad \text{in} \quad B, \quad u = 0 \quad \text{on} \quad \partial B.$$

Since E_0 is radially symmetric, the solution u must be radially symmetric. We write $u(x) = u(r)$ where $r = |x|$. Then we get

$$-(ru')' = \begin{cases} r & \text{if } 0 \leq r < a, \\ 0 & \text{if } a \leq r < 1 \end{cases} \qquad u'(0) = u(1) = 0.$$

We find

$$-ru' = \begin{cases} \frac{r^2}{2} & \text{if } 0 \leq r < a, \\ \frac{a^2}{2} & \text{if } a \leq r < 1 \end{cases}$$

and

$$-u' = \begin{cases} \frac{r}{2} & \text{if } 0 \le r < a, \\ \frac{a^2}{2r} & \text{if } a \le r < 1. \end{cases}$$

The corresponding energy integral is

$$\begin{aligned} \int_B |\nabla u|^2 dxdy = 2\pi \int_0^1 (u')^2 r dr &= 2\pi \Big[\int_0^a \frac{r^3}{4} dr + \int_a^1 \frac{a^4}{4} \frac{1}{r} dr \Big] \\ &= 2\pi \Big[\frac{a^4}{16} - \frac{a^4}{4} \log a \Big] = \frac{\pi a^4}{4} \log \frac{\sqrt{e}}{a^2}. \end{aligned}$$

Hence, if $\beta = \pi a^2$ we have

$$I(\beta) = \frac{\beta^2}{4\pi} \log\Big(\frac{\pi\sqrt{e}}{\beta}\Big).$$

We consider now plane domains different from a disc.

Let $\Omega = B_1 \cup B_2$, where B_1 and B_2 are disjoint unit discs, and let $E_0 = E_1 \cup E_2$, where E_1 and E_2 are discs concentric with B_1 and B_2 respectively. Let $|E_1| = s$ and $|E_2| = \beta - s$. For $0 \le s \le \beta \le \pi$, the total energy in Ω is

$$J(s) = I(s) + I(\beta - s).$$

Let us solve the inequality $J(\beta/2) < J(\beta)$. We have

$$\begin{aligned} &\frac{\beta^2}{8\pi} \log\Big(\frac{2\pi\sqrt{e}}{\beta}\Big) < \frac{\beta^2}{4\pi} \log\Big(\frac{\pi\sqrt{e}}{\beta}\Big), \\ &\log\Big(\frac{2\pi\sqrt{e}}{\beta}\Big) < \log\Big(\frac{\pi\sqrt{e}}{\beta}\Big)^2, \\ &2 < \frac{\pi\sqrt{e}}{\beta}, \\ &\beta < \frac{\pi\sqrt{e}}{2}. \end{aligned}$$

Therefore, when $\beta < \frac{\pi\sqrt{e}}{2}$, the maximum of $J(s)$ cannot correspond to $s = \beta/2$ (the symmetric configuration). This phenomenon is known as symmetry breaking. We can show that the maximum corresponds to either $s = 0$ or $s = \beta$. Indeed, we have

$$J(s) = \frac{1}{4\pi} \Big[s^2 \log \frac{\pi\sqrt{e}}{s} + (\beta - s)^2 \log \frac{\pi\sqrt{e}}{\beta - s} \Big].$$

$$J'(s) = \frac{1}{2\pi} \Big[s \log \frac{\pi}{s} - (\beta - s) \log \frac{\pi}{\beta - s} \Big].$$

Hence $J'(\beta/2) = 0$ and $J'(0) < 0$. Note that we write $J'(0) = \lim_{s\to 0} J'(s)$. Moreover we find

$$J''(s) = \frac{1}{2\pi} \log \frac{\pi^2}{e^2 s(\beta - s)}.$$

We have $J''(s) > 0$ for $\pi^2 > e^2 s(\beta - s)$. The latter inequality holds for $\pi^2 > e^2(\frac{\beta}{2})^2$ that is, $\beta < \frac{2\pi}{e}$. Therefore, if $\beta < \frac{2\pi}{e}$ we have $J''(s) > 0$ on $(0, \beta/2)$, hence, the maximum of $J(s)$ is attained at $s = 0$. If $\frac{2\pi}{e} < \beta \leq \frac{\pi\sqrt{e}}{2}$ we have $J''(s) > 0$ on $(0, s_0)$ and $J''(s) < 0$ on $(s_0, \frac{\pi\sqrt{e}}{2})$ (for a suitable value of s_0). Also in this case the maximum is attained at $s = 0$.
Since $J(s) = J(\beta - s)$, $J(\beta)$ is another point of maximum. Therefore, we do not have uniqueness for the maximum in this situation.
In case of $\frac{\pi\sqrt{e}}{2} < \beta < \pi$ one finds $J'(0) < 0$, $J'(\beta/2) = 0$, $J''(s) > 0$ on $(0, s_1)$ and $J''(s) < 0$ on $(s_1, \beta/2)$ (for some s_1). It follows that $s = \beta/2$ is the maximum value of $J(s)$ in this case.

We note that the union of two disjoint discs is not connected. For $h > 0$ small, let us define the dumbbell

$$\Omega_h = B(-2, 0) \cup ((-2, 2) \times (-h, h)) \cup B(2, 0),$$

where $B(-2, 0)$ and $B(2, 0)$ are unit discs centered at $(-2, 0)$ and $(2, 0)$ respectively. If u_h is the solution of our maximization problem in Ω_h, by using [22, Theorem 8.27] one finds

$$\max_{dist(x,\partial\Omega)\leq 3h} u_h(x) \leq Ch^\beta ||u_h||_{L^2(\Omega)},$$

for some $\beta \in (0, 1)$. By using this estimate one can show that the limit of the energy integral corresponding to Ω_h converges to the energy integral corresponding to Ω (Ω is the union of $B(-2, 0)$ and $B(2, 0)$). Since we have proved that

$$2I\Big(\frac{\beta}{2}\Big) < I(\beta)$$

for $0 < \beta < \frac{\pi\sqrt{e}}{2}$, it follows that for h small enough, the previous inequality must hold in Ω_h. Therefore, we have symmetry breaking for a suitable dumbbell.

Let $\Omega = B_1 \cup B_2 \cup B_3$, where B_i $(i = 1, 2, 3)$ are disjoint unit discs, and let $E_0 = E_1 \cup E_2 \cup E_3$ where E_i are discs concentric with B_i. Let $|E_i| = s_i$ with $s_1 + s_2 + s_3 = \beta$. The total energy in Ω is

$$I(s_1) + I(s_2) + I(s_3), \quad s_1 + s_2 + s_3 = \beta.$$

Let us solve the inequality $3I(\beta/3) < I(\beta)$. We have

$$\begin{aligned}
&3\frac{\beta^2}{9\pi}\log\Big(\frac{3\pi\sqrt{e}}{\beta}\Big) < \frac{\beta^2}{4\pi}\log\Big(\frac{\pi\sqrt{e}}{\beta}\Big),\\
&\frac{1}{3}\log\Big(\frac{3\pi\sqrt{e}}{\beta}\Big) < \log\Big(\frac{\pi\sqrt{e}}{\beta}\Big),\\
&\log\Big(\frac{3\pi\sqrt{e}}{\beta}\Big) < \log\Big(\frac{\pi\sqrt{e}}{\beta}\Big)^3,\\
&\beta < \frac{\pi\sqrt{e}}{\sqrt{3}}.
\end{aligned}$$

Therefore, when $\beta < \frac{\pi\sqrt{e}}{\sqrt{3}}$, the maximum cannot correspond to the symmetric configuration and we have a symmetry breaking. Note that we do not have uniqueness for the maximum in this situation. Indeed, if $(\beta_1, \beta_2, \beta_3)$ corresponds to a maximum, we can assume $\beta_1 \neq \beta_2$. Then also $(\beta_2, \beta_1, \beta_3)$ corresponds to another point of maximum.

For $n \geq 4$ let us take n disjoint unit discs and $0 < \beta \leq \pi$. Let us compute

$$nI(\beta/n) < I(\beta).$$

We have

$$\begin{aligned}
&n\frac{\beta^2}{n^2\pi}\log\Big(\frac{n\pi\sqrt{e}}{\beta}\Big) < \frac{\beta^2}{4\pi}\log\Big(\frac{\pi\sqrt{e}}{\beta}\Big),\\
&\frac{1}{n}\log\Big(\frac{n\pi\sqrt{e}}{\beta}\Big) < \log\Big(\frac{\pi\sqrt{e}}{\beta}\Big),\\
&\log\Big(\frac{n\pi\sqrt{e}}{\beta}\Big) < \log\Big(\frac{\pi\sqrt{e}}{\beta}\Big)^n,\\
&\beta < \frac{\pi\sqrt{e}}{n^{\frac{1}{n-1}}}.
\end{aligned}$$

Note that, for $n \geq 4$ we have

$$\beta < \frac{\pi\sqrt{e}}{n^{\frac{1}{n-1}}}.$$

Therefore, when $\beta \leq \pi$, the maximum cannot correspond to the symmetric configuration and we have a symmetry breaking.

Let B and E_0 be the same as before. Consider the p-Laplacian boundary value problem

$$-\Delta_p u = \chi_{E_0}, \quad \text{in} \quad B, \quad u = 0 \quad \text{on} \quad \partial B.$$

Since E_0 is radially symmetric, and since the p-Laplace operator is invariant for rotations, the solution u must be radially symmetric. We write $u(x) = u(r)$ where $r = |x|$. Then we get

$$\Big(r(-u')^{p-1}\Big)' = \begin{cases} r & \text{if } 0 \leq r < a, \\ 0 & \text{if } a \leq r < 1 \end{cases} \qquad u'(0) = u(1) = 0.$$

We find

$$(-u')^{p-1} = \begin{cases} \frac{r}{2} & \text{if } 0 \leq r < a, \\ \frac{a^2}{2r} & \text{if } a \leq r < 1 \end{cases}$$

and

$$-u' = \begin{cases} (\frac{1}{2})^{\frac{1}{p-1}} r^{\frac{1}{p-1}} & \text{if } 0 \le r < a, \\ (\frac{a^2}{2})^{\frac{1}{p-1}} r^{\frac{-1}{p-1}} & \text{if } a \le r < 1. \end{cases}$$

We have

$$\begin{aligned}
\frac{1}{2\pi}\int_B |\nabla u|^p \, dxdy = \int_0^1 |u'|^p r \, dr &= \Big[\int_0^a \Big(\frac{1}{2}\Big)^{\frac{p}{p-1}} r^{\frac{p}{p-1}+1} \, dr\Big] + \Big[\int_a^1 \Big(\frac{a^2}{2}\Big)^{\frac{p}{p-1}} r^{1-\frac{p}{p-1}} \, dr\Big] \\
&= \Big(\frac{1}{2}\Big)^{\frac{p}{p-1}} \Big[a^{\frac{3p-2}{p-1}} \frac{p-1}{3p-2} + a^{\frac{2p}{p-1}} \frac{p-1}{p-2}\Big(1 - a^{\frac{p-2}{p-1}}\Big)\Big] \\
&= \Big(\frac{1}{2}\Big)^{\frac{p}{p-1}} (p-1) a^{\frac{2p}{p-1}} \Big[a^{\frac{p-2}{p-1}} \frac{1}{3p-2} + \frac{1}{p-2} - \frac{1}{p-2} a^{\frac{p-2}{p-1}}\Big] \\
&= \Big(\frac{1}{2}\Big)^{\frac{p}{p-1}} \frac{p-1}{p-2} a^{\frac{2p}{p-1}} \Big[1 - \frac{2p}{3p-2} a^{\frac{p-2}{p-1}}\Big].
\end{aligned}$$

Hence, if $\beta = \pi a^2$, the corresponding energy integral is

$$I(\beta) = 2\pi\Big(\frac{1}{2}\Big)^{\frac{p}{p-1}} \frac{p-1}{p-2}\Big(\frac{\beta}{\pi}\Big)^{\frac{p}{p-1}} \Big[1 - \frac{2p}{3p-2}\Big(\frac{\beta}{\pi}\Big)^{\frac{p-2}{2(p-1)}}\Big].$$

Let $\Omega = B_1 \cup B_2$, where B_1 and B_2 are disjoint unit discs, and let $E_0 = E_1 \cup E_2$, where E_1 and E_2 are discs concentric with B_1 and B_2 respectively. Let $|E_1| = s$ and $|E_2| = \beta - s$. For $0 \le s \le \beta$, the total energy in Ω is the following

$$J(s) = I(s) + I(\beta - s).$$

We have

$$J\Big(\frac{\beta}{2}\Big) = 2I\Big(\frac{\beta}{2}\Big) = 4\pi\Big(\frac{1}{2}\Big)^{\frac{p}{p-1}} \frac{p-1}{p-2}\Big(\frac{\beta}{2\pi}\Big)^{\frac{p}{p-1}} \Big[1 - \frac{2p}{3p-2}\Big(\frac{\beta}{2\pi}\Big)^{\frac{p-2}{2(p-1)}}\Big].$$

Let us solve the inequality $J(\beta/2) < J(\beta)$. To solve this inequality we consider the following two cases.

Case 1. If $p > 2$

$$2\Big(\frac{\beta}{2\pi}\Big)^{\frac{p}{p-1}} \Big[1 - \frac{2p}{3p-2}\Big(\frac{\beta}{2\pi}\Big)^{\frac{p-2}{2(p-1)}}\Big] < \Big(\frac{\beta}{\pi}\Big)^{\frac{p}{p-1}} \Big[1 - \frac{2p}{3p-2}\Big(\frac{\beta}{\pi}\Big)^{\frac{p-2}{2(p-1)}}\Big]$$

we have

$$\Big(\frac{1}{2}\Big)^{\frac{1}{p-1}} \Big[1 - \frac{2p}{3p-2}\Big(\frac{\beta}{2\pi}\Big)^{\frac{p-2}{2(p-1)}}\Big] < \Big[1 - \frac{2p}{3p-2}\Big(\frac{\beta}{\pi}\Big)^{\frac{p-2}{2(p-1)}}\Big]$$

and

$$\begin{aligned}
1 - \Big(\frac{1}{2}\Big)^{\frac{1}{p-1}} &> \frac{2p}{3p-2}\Big[\Big(\frac{\beta}{\pi}\Big)^{\frac{p-2}{2(p-1)}} - \Big(\frac{1}{2}\Big)^{\frac{1}{p-1}}\Big(\frac{\beta}{2\pi}\Big)^{\frac{p-2}{2(p-1)}}\Big] \\
&= \frac{2p}{3p-2}\Big[\Big(\frac{\beta}{\pi}\Big)^{\frac{p-2}{2(p-1)}} - \Big(\frac{1}{2}\Big)^{\frac{p}{2(p-1)}}\Big(\frac{\beta}{\pi}\Big)^{\frac{p-2}{2(p-1)}}\Big] \\
&= \frac{2p}{3p-2}\Big(\frac{\beta}{\pi}\Big)^{\frac{p-2}{2(p-1)}} \Big[1 - \pi\Big(\frac{1}{2}\Big)^{\frac{p}{2(p-1)}}\Big].
\end{aligned}$$

Then we find

$$\left(\frac{\beta}{\pi}\right)^{\frac{p-2}{2(p-1)}} < \frac{3p-2}{2p}\frac{1-\left(\frac{1}{2}\right)^{\frac{1}{p-1}}}{1-\left(\frac{1}{2}\right)^{\frac{p}{2(p-1)}}}.$$

Case 2. If $p < 2$

$$2\left(\frac{\beta}{2\pi}\right)^{\frac{p}{p-1}}\left[1-\frac{2p}{3p-2}\left(\frac{\beta}{2\pi}\right)^{\frac{p-2}{2(p-1)}}\right] > \left(\frac{\beta}{\pi}\right)^{\frac{p}{p-1}}\left[1-\frac{2p}{3p-2}\left(\frac{\beta}{\pi}\right)^{\frac{p-2}{2(p-1)}}\right]$$

we have

$$\left(\frac{1}{2}\right)^{\frac{1}{p-1}}\left[1-\frac{2p}{3p-2}\left(\frac{\beta}{2\pi}\right)^{\frac{p-2}{2(p-1)}}\right] > \left[1-\frac{2p}{3p-2}\left(\frac{\beta}{\pi}\right)^{\frac{p-2}{2(p-1)}}\right]$$

and

$$\begin{aligned}
1-\left(\frac{1}{2}\right)^{\frac{1}{p-1}} &< \frac{2p}{3p-2}\left[\left(\frac{\beta}{\pi}\right)^{\frac{p-2}{2(p-1)}} - \left(\frac{1}{2}\right)^{\frac{1}{p-1}}\left(\frac{\beta}{2\pi}\right)^{\frac{p-2}{2(p-1)}}\right] \\
&= \frac{2p}{3p-2}\left[\left(\frac{\beta}{\pi}\right)^{\frac{p-2}{2(p-1)}} - \left(\frac{1}{2}\right)^{\frac{p}{2(p-1)}}\left(\frac{\beta}{\pi}\right)^{\frac{p-2}{2(p-1)}}\right] \\
&= \frac{2p}{3p-2}\left(\frac{\beta}{\pi}\right)^{\frac{p-2}{2(p-1)}}\left[1-\left(\frac{1}{2}\right)^{\frac{p}{2(p-1)}}\right].
\end{aligned}$$

Then we find

$$\left(\frac{\beta}{\pi}\right)^{\frac{2-p}{2(p-1)}} < \frac{2p}{3p-2}\frac{1-\left(\frac{1}{2}\right)^{\frac{p}{2(p-1)}}}{1-\left(\frac{1}{2}\right)^{\frac{1}{p-1}}}.$$

3.2 Optimization of the principal eigenvalue

Let us consider the case $q = p$. This leads to the theory of eigenvalues. We recall the definition of principal eigenvalue. Consider $g \geq 0$, $g \not\equiv 0$. Let $\lambda > 0$ such that the boundary value problem

$$-\Delta_p u = \lambda g u^{p-1}, \quad u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega \tag{3.27}$$

has a solution. In this case λ is the principal eigenvalue. The meaning of the latter equation is the following:

$$u \in H_0^{1,p}(\Omega), \quad \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\eta \, dx = \lambda \int_\Omega g u^{p-1}\eta \, dx \quad \forall \eta \in H_0^{1,p}(\Omega).$$

The value of λ depends on g, p and on Ω, but when p and Ω are fixed we write $\lambda = \lambda_g$. The following variational characterization of λ_g is well known.

$$\lambda_g = \inf_{v \in H_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla v|^p \, dx}{\int_\Omega g|v|^p \, dx}. \tag{3.28}$$

Let us show that a minimizer $u \geq 0$ of (3.28) satisfies the above equation. We have

$$\lambda_g = \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega g|u|^p \, dx},$$

and

$$\frac{\int_\Omega |\nabla(u + t\eta)|^p \, dx}{\int_\Omega g|u + t\eta|^p \, dx} \geq \lambda_g \quad \forall \eta \in H_0^{1,p}(\Omega), \quad \forall t \in (-1, 1).$$

As a consequence we have

$$\frac{d}{dt}\frac{\int_\Omega |\nabla(u + t\eta)|^p \, dx}{\int_\Omega g|u + t\eta|^p \, dx}\Big|_{t=0} = 0 \quad \forall \eta \in H_0^{1,p}(\Omega).$$

Therefore,

$$\frac{p\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\eta \, dx}{\int_\Omega g|u|^p \, dx} - p\frac{\int_\Omega |\nabla u|^p \, dx}{\Big(\int_\Omega g|u|^{p-1}\eta \, dx\Big)^2}\int_\Omega g|u|^p \, dx = 0 \quad \forall \eta \in H_0^{1,p}(\Omega).$$

Then after simplification we get

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\eta \, dx = \lambda \int_\Omega g u^{p-1}\eta \, dx \quad \forall \eta \in H_0^{1,p}(\Omega).$$

Sometimes, the previous definition is written as

$$\lambda_g = \inf_{v \in H_0^{1,p}(\Omega)} \Big\{\int_\Omega |\nabla v|^p \, dx : \int_\Omega g|v|^p \, dx = 1\Big\}.$$

We claim that the following characterization of λ_g is equivalent to (3.28).

$$\frac{1}{\lambda_g^2} = \sup_{v \in H_0^{1,p}(\Omega)} \Big\{\int_\Omega 2g|v|^p \, dx - \Big(\int_\Omega |\nabla v|^p \, dx\Big)^2\Big\}. \tag{3.29}$$

Indeed, if we define

$$A(g,v) = \int_\Omega 2g|v|^p\ dx - \Big(\int_\Omega |\nabla v|^p\ dx\Big)^2$$

and replace v by tv, $t>0$, we find

$$A(g,tv) = t^p \int_\Omega 2g|v|^p\ dx - t^{2p}\Big(\int_\Omega |\nabla v|^p\ dx\Big)^2.$$

Let us show that $A(g,tv) \le A(g,t_0 v)$ with

$$t_0^p = \frac{\int_\Omega g|v|^p\ dx}{\Big(\int_\Omega |\nabla v|^p\ dx\Big)^2}.$$

Indeed, the derivative with respect to t of $A(g,tv)$ vanishes at $t=t_0$. Moreover, computing $A(g,t_0v)$ we find

$$\begin{aligned} A(g,t_0v) &= 2\frac{\Big(\int_\Omega g|v|^p\ dx\Big)}{\Big(\int_\Omega |\nabla v|^p\ dx\Big)^2}\Big(\int_\Omega g|v|^p\ dx\Big) - \Big(\frac{\int_\Omega g|v|^p\ dx}{(\int_\Omega |\nabla v|^p\ dx)^2}\Big)^2\Big(\int_\Omega |\nabla v|^p\ dx\Big)^2 \\ &= \Big(\frac{\int_\Omega g|v|^p\ dx}{\int_\Omega |\nabla v|^p\ dx}\Big)^2. \end{aligned}$$

Thus,

$$A(g,tv) \le \Big(\frac{\int_\Omega g|v|^p\ dx}{\int_\Omega |\nabla v|^p\ dx}\Big)^2.$$

Since

$$\sup_{H_0^{1,p}(\Omega)} \frac{\int_\Omega g|v|^p\ dx}{\int_\Omega |\nabla v|^p\ dx} = \Big(\inf_{H_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla v|^p\ dx}{\int_\Omega g|v|^p\ dx}\Big)^{-1}$$

the claim follows.
Note that a non-negative maximizer $u=u_g$ of (3.29) is normalized so that $t_0=1$, that is,

$$\int_\Omega g u_g^p\ dx = \Big(\int_\Omega |\nabla u_g|^p\ dx\Big)^2. \tag{3.30}$$

Therefore, the maximizer $u=u_g$ of (3.29) satisfies (3.30).

The minimization problem (3.28) has been investigated in [28] for a wide class of function g including the case $g\ge 0$ in Ω. In [28, Theorem 1.3], it is proved that a non-negative minimizer of (3.28) exists and is unique up to a constant factor. Since

$$\Delta_p u = -\lambda g(x) u^{p-1} \le 0$$

and $u\ge 0$, by [43, Theorem 5] we must have $u_g > 0$ in Ω. It follows that the eigenvalue problem (3.27) has a solution.

Recalling that

$$A(g,v)=\int_\Omega 2g|v|^p\,dx-\Big(\int_\Omega |\nabla v|^p\,dx\Big)^2$$

we define

$$J(g)=\sup_{v\in H_0^{1,p}(\Omega)} A(g,v). \tag{3.31}$$

If u_g is the positive maximizer of $A(g,w)$ then we find

$$J(g)=\Big(\frac{\int_\Omega gu_g^p\,dx}{\int_\Omega |\nabla u_g|^p\,dx}\Big)^2. \tag{3.32}$$

Of course, $J(g)=1/\lambda_g^2$.

Let $\mathcal{G}$ be as in the previous part. We investigate the problem

$$\max_{g\in\mathcal{G}} J(g).$$

We find a result of existence and a result of representation of a maximizer.
Furthermore, we investigate the problem

$$\min_{g\in\mathcal{G}} J(g).$$

We find results of existence and uniqueness, and a result of representation of the minimizer. In case Ω is a ball we find precise configurations for the maximizer and for the minimizer. The present results are contained in the papers [17, 18] and generalize previous results relative to the case $p=2$ and $q=1$.

Proposition 3.4. *[36, Lemma 3.1] Let $g_0\ge 0$ be a bounded function, positive in a subset of positive measure. Let $\overline{\mathcal{G}}$ be the weak* closure of $\mathcal{G}=\mathcal{G}(g_0)$ in $L^\infty(\Omega)$, and let $J(g)$ be defined as in (3.32) for $g\in\overline{\mathcal{G}}$.*

(i) The functional $J(g)$ is continuous with respect to the weak topology in $L^\infty(\Omega)$.*

(ii) The functional $J(g)$ satisfies condition (2.12) with $G(g)=2u_g^p$, where u_g is the maximizer of $w\to A(g,w)$ for $w\in H_0^{1,p}(\Omega)$.

(iii) The functional $J(g)$ is strictly convex.

Proof. Let $g_i\rightharpoonup g$ in the weak* topology of $L^\infty(\Omega)$. Since $g(x)>0$ in a subset of positive measure, $g_i(x)>0$ in subsets of positive measure for i large. Let u_{g_i} be the non-negative maximizer of $w\to A(g_i,w)$, and let u_g be the non-negative maximizer of $w\to A(g,w)$ in

$H_0^{1,p}(\Omega)$. Then

$$\begin{aligned}
&J(g) + \int_\Omega 2(g_i - g)u_g^p \, dx \\
&= \int_\Omega 2g_i u_g^p \, dx - \Big(\int_\Omega |\nabla u_g|^p \, dx\Big)^2 \\
&\le \int_\Omega 2g_i u_{g_i}^p \, dx - \Big(\int_\Omega |\nabla u_{g_i}|^p \, dx\Big)^2 = J(g_i) \\
&= \int_\Omega 2(g_i - g)u_{g_i}^p \, dx + \int_\Omega 2g u_{g_i}^p \, dx - \Big(\int_\Omega |\nabla u_{g_i}|^p \, dx\Big)^2 \\
&\le \int_\Omega 2(g_i - g)u_{g_i}^p \, dx + J(g).
\end{aligned} \tag{3.33}$$

By assumption, we have

$$\lim_{i\to\infty} \int_\Omega (g_i - g)u_g^p \, dx = 0. \tag{3.34}$$

Let us prove that

$$\lim_{i\to\infty} \int_\Omega (g_i - g)u_{g_i}^p \, dx = 0. \tag{3.35}$$

By the normalization (3.30) we have

$$\Big(\int_\Omega |\nabla u_{g_i}|^p \, dx\Big)^2 = \int_\Omega g_i \, u_{g_i}^p \, dx.$$

Since g_0 is bounded and since $0 \le g_i \le \sup_\Omega g_0$, by using Poincarè inequality we find

$$\int_\Omega g_i u_{g_i}^p dx \le \tilde{C} \int_\Omega u_{g_i}^p dx \le C \int_\Omega |\nabla u_{g_i}|^p dx.$$

It follows that

$$\int_\Omega |\nabla u_{g_i}|^p dx \le C. \tag{3.36}$$

By estimate (3.36) we infer that the norm $\|\nabla u_{g_i}\|_{L^p(\Omega)}$ is bounded by a constant independent of i. Therefore, a sub-sequence of u_{g_i} (denoted again u_{g_i}) converges weakly in $H^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ to some function $z \in H_0^{1,p}(\Omega)$, $z(x) \ge 0$. We claim that

$$\lim_{n\to\infty} \int_\Omega (g_i - g)(u_{g_i}^p - z^p) \, dx = 0,$$

which implies (3.35). This claim follows from the inequality

$$|(g_i - g)(u_{g_i}^p - z^p)| \le C|u_{g_i} - z| p(u_{g_i} + z)^{p-1}$$

and Hölder's inequality

$$\int_\Omega |u_{g_i} - z|(u_{g_i} + z)^{p-1} dx \le \|u_{g_i} - z\|_{L^p(\Omega)} \|u_{g_i} + z\|_{L^p(\Omega)}^{p-1}.$$

From (3.33), (3.34) and (3.35) we infer

$$\lim_{i\to\infty} J(g_i) = J(g). \tag{3.37}$$

Part (i) of the proposition is proved.

We claim that the function z just mentioned equals u_g, the maximizer of $v \to A(g,v)$ for $v \in H^{1,p}(\Omega)$. Indeed, from

$$J(g_i) = \int_\Omega 2g_i u_{g_i}^p \, dx - \Big(\int_\Omega |\nabla u_{g_i}|^p \, dx\Big)^2$$

and

$$\liminf_{i\to\infty} \int_\Omega |\nabla u_{g_i}|^p dx \geq \int_\Omega |\nabla z|^p dx,$$

using (3.37) we find

$$J(g) \leq \int_\Omega 2gz^q \, dx - \Big(\int_\Omega |\nabla z|^p \, dx\Big)^2 \leq J(g).$$

By the uniqueness of the maximizer of $A(g,v)$, we must have $z = u_g$, as claimed.

Let $t_i > 0$ be a sequence such that $t_i \to 0$ as $i \to \infty$. Let $g, h \in \overline{\mathcal{G}}$ and let $g_{t_i} = g + t_i(h-g)$. Then, by (3.33) we find

$$J(g) + t_i \int_\Omega 2(h-g)\, u_g^p \, dx \leq J(g + t_i(h-g)) \leq J(g) + t_i \int_\Omega (h-g) 2u_{g_{t_i}}^p \, dx.$$

It follows that

$$\int_\Omega 2(h-g)u_g^p \, dx \leq \frac{J(g+t_i(h-g)) - J(g)}{t_i} \leq \int_\Omega 2(h-g)u_{g_{t_i}}^p \, dx. \tag{3.38}$$

Since $t_i \to 0$ as $i \to \infty$, we have $g_{t_i} \to g$ as $i \to \infty$ in the norm of $L^\infty(\Omega)$. As a consequence, $u_{g_{t_i}} \to u_g$ in the $L^p(\Omega)$ norm, and

$$\lim_{i\to\infty} \int_\Omega (h-g)u_{g_{t_i}}^p \, dx = \int_\Omega (h-g)u_g^p \, dx.$$

Since the sequence t_i is arbitrary, from the latter equation and (3.38) we find

$$\lim_{t\to 0^+} \frac{J(g+t(h-g)) - J(g)}{t} = \int_\Omega 2(h-g)u_g^p \, dx. \tag{3.39}$$

Part (ii) of the proposition follows. Let $0 < t < 1$. If $g_1,\ g_2 \in \overline{\mathcal{G}}$ and $g_t = tg_1 + (1-t)g_2$, we have

$$A(g_t, v) = tA(g_1, v) + (1-t)A(g_2, v),$$

$$\sup_{v\in H_0^{1,p}(\Omega)} A(g_t, w) \leq t \sup_{v\in H_0^{1,p}(\Omega)} A(g_1, w) + (1-t) \sup_{v\in H_0^{1,p}(\Omega)} A(g_2, w).$$

It follows that

$$J(g_t) \leq tJ(g_1) + (1-t)J(g_2), \tag{3.40}$$

that is, the convexity of $J(g)$. Now, let us prove strict convexity. Recall that $g_0(x) \geq 0$ is the function which generates $\mathcal{G}$ and $\overline{\mathcal{G}}$. In particular, every function $g \in \overline{\mathcal{G}}$, satisfies $g(x) \geq 0$ and $g \not\equiv 0$. If equality holds in (3.40) for some $0 < t < 1$, then

$$A(g_t, u_{g_t}) = tA(g_1, u_{g_1}) + (1-t)A(g_2, u_{g_2}),$$

where u_{g_t}, u_{g_1}, u_{g_2} are respectively maximizers of $A(g_t, w)$, $A(g_1, w)$, $A(g_2, w)$ for $w \in H_0^{1,p}(\Omega)$. Since $A(g_t, u_{g_t}) = tA(g_1, u_{g_t}) + (1-t)A(g_2, u_{g_t})$, it follows that

$$A(g_1, u_{g_t}) = A(g_1, u_{g_1}) \ \text{ and } \ A(g_2, u_{g_t}) = A(g_2, u_{g_2}).$$

By the uniqueness of the maximizer we must have $u_{g_t} = u_{g_1} = u_{g_2}$. Furthermore we have

$$-\Delta_p u_{g_t} = \lambda(g_1)\ g_1 u_{g_t}^{p-1} \ \text{ a.e. in } \ \Omega$$

and

$$-\Delta_p u_{g_t} = \lambda(g_2)\ g_2 u_{g_t}^{p-1} \ \text{ a.e. in } \ \Omega.$$

Since $g_1(x) \geq 0$ and $g_1(x) \not\equiv 0$, by the strong maximum principle ([43] Theorem 5), $u_{g_t}(x) > 0$. It follows that $\lambda(g_1)g_1(x) = \lambda(g_2)g_2(x)$ a.e. in Ω. Integration over Ω yields $\lambda(g_1) = \lambda(g_2)$. Hence, $g_1(x) = g_2(x)$ a.e. in Ω.
The proof of the proposition is completed. □

Theorem 3.3. *[36, Theorem 3.2] Let $g_0 \geq 0$ be bounded and positive in a subset of positive measure, let $\mathcal{G}$ be the class of rearrangements generated by g_0, and let $J(g)$ be defined as in (3.32). Then*

(I) problem

$$\max_{g\in\mathcal{G}} J(g)$$

has a solution $\hat{g} \in \mathcal{G}$; furthermore, if $\hat{g}$ is a maximizer of $J(g)$ and if $u_{\hat{g}}$ is the positive maximizer of $v \to A(\hat{g}, v)$ then $\hat{g} = \varphi\big(u_{\hat{g}}\big)$ for some increasing function $\varphi(t)$.

(II) problem

$$\min_{g\in\mathcal{G}} J(g)$$

has a unique solution $\check{g} \in \mathcal{G}$; furthermore, if $\check{g}$ is the minimizer of $J(g)$, and if $u_{\check{g}}$ is the positive maximizer of $v \to A(\check{g}, v)$, then $\check{g} = \psi\big(u_{\check{g}}\big)$ for some decreasing function $\psi(t)$.

Proof. By Proposition 3.4, the functional $J(g)$ is continuous with respect to the weak* topology of $L^\infty(\Omega)$, and convex on $\overline{\mathcal{G}}$. Then, by Theorem 2.11 (i), there is a function $\hat{g} \in \mathcal{G}$ such that $J(g) \leq J(\hat{g})$ for all $g \in \overline{\mathcal{G}}$. Since $g_0(x) \geq 0$, by Proposition 3.4 (ii) and (iii), $J(g)$ is strictly convex and satisfies condition (2.12) with $G(g) = 2u_g^p$. By Theorem 2.11 (ii), there is a non-decreasing function $\tilde{\varphi}$ such that $\hat{g} = \tilde{\varphi}\big(2u_{\hat{g}}^p\big)$. The conclusion of part (I) follows with $\varphi(t) = \tilde{\varphi}\big(2t^p\big)$.
Let us prove part (II). Since $J(g)$ is continuous with respect to the weak* topology of $L^\infty(\Omega)$ and since $\overline{\mathcal{G}}$ is weakly compact, we know that a minimizer $\bar{g}$ of $J(g)$ exists on $\overline{\mathcal{G}}$, and is unique

by the strict convexity (proved in Proposition 3.4). We have to show that $\bar{g} \in \mathcal{G}$. If $0 < t < 1$ and if $g_t = tg + (1-t)\bar{g}$, since $J(g)$ satisfies condition (2.12) we have

$$J(\bar{g}) \leq J(g_t) = J(\bar{g}) + t \int_{\Omega} (g - \bar{g}) 2u_{\bar{g}}^p \, dx + o(t) \quad \text{as} \quad t \to 0.$$

Hence,

$$\int_{\Omega} (g - \bar{g}) 2u_{\bar{g}}^p \, dx \geq 0.$$

Equivalently,

$$\int_{\Omega} g \, u_{\bar{g}}^p \, dx \geq \int_{\Omega} \bar{g} \, u_{\bar{g}}^p \, dx \quad \forall g \in \overline{\mathcal{G}}. \tag{3.41}$$

By the equation

$$- \Delta_p u_{\bar{g}} = \lambda \bar{g} \, u_{\bar{g}}^{p-1}, \tag{3.42}$$

it follows that the function $u_{\bar{g}}$ cannot have flat zones in the set

$$F = \{x \in \Omega : \bar{g}(x) > 0\}.$$

If $|F| < |\Omega|$, since $\bar{g} \in \overline{\mathcal{G}}$, by Lemma 5 of [4] we have

$$|F| \geq |\{x \in \Omega : g_0(x) > 0\}|.$$

Then there is $g_1 \in \mathcal{G}$ such that its support is contained in F. By Lemma 2.5, there is a non-increasing function $\psi_1(t)$ such that $\psi_1(u_{\bar{g}}^p)$ is a rearrangement of g_1 on F.
Define

$$\alpha = \inf_{x \in \Omega \setminus F} u_{\bar{g}}^p(x).$$

From now on the proof continues as in that of Theorem 3.2. □

3.2.1 Principal eigenvalue in symmetric domains

If Ω has some symmetry as in the previous case we can prove analogous results also for the functional J. We use the same notations as before.

Proposition 3.5. *[17] Let $\Omega = \Omega^\sharp$ and let $J(g)$ be defined as in (3.32). We have*

$$J(g) \leq J(g^\sharp) \quad \forall g \in \mathcal{G}.$$

Proof. If $g \in \mathcal{G}$ and if u_g is the corresponding non-negative maximizer of $v \to A(g, v)$, using Lemma 2.1 we find

$$J(g) = \int_{\Omega} 2g \, u_g^p \, dx - \Big(\int_{\Omega} |\nabla u_g|^p \, dx \Big)^2$$

$$\leq \int_\Omega 2g^\sharp (u_g^\sharp)^p \, dx - \Big(\int_\Omega |\nabla u_g^\sharp|^p dx\Big)^2$$
$$\leq \int_\Omega 2g^\sharp u_{g^\sharp}^p \, dx - \Big(\int_\Omega |\nabla u_{g^\sharp}|^p dx\Big)^2 = J(g^\sharp).$$

This completes the proof. □

Proposition 3.6. *[17, Theorem 3.3] Let $\Omega = B_R$ and let $J(g)$ be defined as in (3.32). We have*

$$J(g) \leq J(g^\Delta) \quad \forall g \in \mathcal{G}$$

and

$$J(g_\Delta) \leq J(g) \quad \forall g \in \mathcal{G}.$$

Proof. The first statement follows from Proposition 3.5, since a ball is Steiner symmetric with respect to any hyperplane passing through the origin.
Let us prove the second statement. Since $g(x) \geq 0$ we have $g_\Delta(x) \geq 0$. Hence, from the equation

$$-\Delta_p u = \lambda_g g_\Delta u^{p-1}$$

by uniqueness one finds that $u = u_{g_\Delta}$ is radially symmetric and decreasing with respect to $|x|$. It follows that $u_{g_\Delta}^p = (u_{g_\Delta}^p)^\Delta$. Hence, using Lemma 3.2 we find

$$\begin{aligned} J(g) &= \int_\Omega 2g \, u_g^p \, dx - \Big(\int_\Omega |\nabla u_g|^p dx\Big)^2 \\ &\geq \int_\Omega 2g \, u_{g_\Delta}^p \, dx - \Big(\int_\Omega |\nabla u_{g_\Delta}|^p dx\Big)^2 \\ &\geq \int_\Omega 2g_\Delta u_{g_\Delta}^p \, dx - \Big(\int_\Omega |\nabla u_{g_\Delta}|^p dx\Big)^2 = J(g_\Delta). \end{aligned}$$

This completes the proof. □

Particular case. Let $g_0(x) = \chi_{E_0}$, where $E_0 \subset B_R$. By Proposition 3.6, $J(g)$ attains its maximum value when $\hat{g} = \chi_{\hat{E}}$, $\hat{E}$ being the ball centered in the origin with $|\hat{E}| = |E_0|$. By Theorem 3.2 we know that $\chi_{\hat{E}} = \varphi(u_{\hat{E}})$ for some increasing φ. In our case we have $\varphi(s) = H(s-t)$. Here $H(s-t) = 1$ if $s \geq t$ and $H(s-t) = 0$ if $s < t$, where t is a suitable number depending on measure of E_0. Indeed we have

$$\chi_{\hat{E}} = H(u_{\hat{E}}(x) - t).$$

Similarly, $J(g)$ attains its minimum value when $\check{g} = \chi_{\check{E}}$, where $\check{E} = B_R \setminus B_0$. Here B_0 is a disc concentric with B_R such that $|B_0| = |B_R| - |E_0|$. By Theorem 3.2 we know that $\chi_{\check{E}} = \psi(u_{\check{E}})$ for some decreasing ψ. In our case we have $\psi(s) = H(\tau - s)$. Here $H(\tau - s) = 1$ if $s \leq \tau$ and

$H(\tau - s) = 0$ if $s > \tau$, where τ is a suitable number depending on measure of E_0. Indeed we have

$$\chi_{\check{E}} = H(\tau - u_{\check{E}}(x)).$$

If Ω is a general domain, we may have multiple solutions for the maximum of $J(g)$, which means multiple solutions for the minimum of λ_g. Let us describe the following eigenvalue problem in a plane domain Ω:

$$-\Delta u + \chi_E u = \lambda_E u, \quad u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

This problem has been discussed in [11]. Even if this eigenvalue problem is not the same as our problem, in [11] it is proved that they are equivalent. In [11, Theorem 7] it is proved that in case Ω is a suitable dumbbell, the problem

$$\min_{|E|=\beta} \lambda_E,$$

for β small has non-symmetric solutions. It follows that the solution of minimizer may not be unique in general.

Chapter 4

The (p, q)-Laplace equation

4.1 The (p, q)-Laplace equation and optimization problems

Now we discuss optimization problems in classes of rearrangements for (p, q)-Laplace equations. A simplified version of these results are contained in paper [29].

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, and let $1 < q < p$. We consider the boundary value problem

$$-\Delta_p u - \Delta_q u = f(x, u) \quad \text{in} \ \ \Omega, \quad u = 0 \ \ \text{on} \ \ \partial\Omega.$$

The non-homogeneous differential operator $\Delta_p + \Delta_q$ is called (p, q)-Laplacian. As observed in [35], it stems from a wide range of important applications including biophysics [20], plasma physics [44], reaction-diffusion equations [12], as well as models of elementary particles [3]. In the last decades there has been a great interest in the investigation of these problems mainly concerning existence and multiplicity of solutions, eigenvalues, ground-state solutions [1, 7, 21].

In what follows we shall consider $f(x, u) = g(x)|u|^{\alpha-1}$, where $1 \leq \alpha < q$ and $g(x)$ is a measurable bounded non-negative function which is positive in a subset with a positive measure. So, we consider the boundary value problem

$$-\Delta_p u - \Delta_q u = g(x)|u|^{\alpha-1} \quad \text{in} \ \ \Omega, \quad u = 0 \ \ \text{on} \ \ \partial\Omega. \tag{4.1}$$

A solution to this problem is a function $u \in H_0^{1,p}(\Omega)$ such that

$$\int_\Omega \big(|\nabla u|^{p-2} + |\nabla u|^{q-2}\big)\nabla u \cdot \nabla \phi \, dx = \int_\Omega g(x)|u|^{\alpha-1}\phi \, dx \quad \forall \phi \in H_0^{1,p}(\Omega).$$

We define
$$I(v)=\int_\Omega\Big(\frac{1}{\alpha}g|v|^\alpha-\frac{1}{p}|\nabla v|^p-\frac{1}{q}|\nabla v|^q\Big)dx,\quad v\in H_0^{1,p}(\Omega).$$
Clearly, since $1\le\alpha<q<p$, $I(v)$ is finite for any $v\in H_0^{1,p}(\Omega)$. We consider the classical maximization problem
$$\max_{v\in H_0^{1,p}(\Omega)} I(v). \tag{4.2}$$
To prove the existence of a solution to problem (4.2), let
$$\hat{I}:=\sup_{v\in H_0^{1,p}(\Omega)} I(v)$$
and let $v_i\in H_0^{1,p}(\Omega)$, $i=1,2,\cdots$, be a sequence such that
$$\hat{I}=\lim_{i\to\infty} I(v_i).$$
We have
$$I(v_i)\le\frac{1}{\alpha}\int_\Omega g|v_i|^\alpha dx-\frac{1}{p}\int_\Omega|\nabla v_i|^p dx.$$
By using Poincaré and Hölder inequalities we find
$$\frac{1}{\alpha}\int_\Omega g|v_i|^\alpha dx\le C\int_\Omega|\nabla v_i|^\alpha dx\le C\Big(\int_\Omega|\nabla v_i|^p dx\Big)^{\frac{\alpha}{p}}.$$
Here and in what follows we denote by C constants (possibly different) independent of i. It follows that
$$I(v_i)\le C\Big(\int_\Omega|\nabla v_i|^p\,dx\Big)^{\frac{\alpha}{p}}-\frac{1}{p}\int_\Omega|\nabla v_i|^p\,dx. \tag{4.3}$$
Recalling the well known inequality
$$B=(\epsilon B)\frac{1}{\epsilon}\le\frac{\alpha}{p}(\epsilon B)^{\frac{p}{\alpha}}+\frac{p-\alpha}{p}\Big(\frac{1}{\epsilon}\Big)^{\frac{p}{p-\alpha}},$$
for
$$B=C\Big(\int_\Omega|\nabla v_i|^p dx\Big)^{\frac{\alpha}{p}}$$
and
$$\epsilon=\frac{1}{C}\Big(\frac{1}{2\alpha}\Big)^{\frac{\alpha}{p}}$$
we find
$$C\Big(\int_\Omega|\nabla v_i|^p dx\Big)^{\frac{\alpha}{p}}\le\frac{1}{2p}\int_\Omega|\nabla v_i|^p dx+C.$$
(Clearly, the two constants C in above are different.) Insertion of the latter inequality into (4.3) yields
$$I(v_i)\le\frac{1}{2p}\int_\Omega|\nabla v_i|^p\,dx+C-\frac{1}{p}\int_\Omega|\nabla v_i|^p\,dx=-\frac{1}{2p}\int_\Omega|\nabla v_i|^p\,dx+C.$$

It follows that the value of $\hat{I}$ is finite. From the latter result we also find

$$\int_\Omega |\nabla v_i|^p dx \le C.$$

Hence, a subsequence of $\{v_i\}$ (denoted again $\{v_i\}$) converges weakly in $H^{1,p}(\Omega)$ and in $H^{1,q}(\Omega)$ to some function $u \in H_0^{1,p}(\Omega)$. By Rellich's Theorem, $\{v_i\}$ converges to u strongly in $L^p(\Omega)$ as well as in $L^q(\Omega)$ and in $L^\alpha(\Omega)$. Therefore, we have

$$\begin{aligned}
\hat{I} &\ge \int_\Omega \Big(\frac{1}{\alpha} g|u|^\alpha - \frac{1}{p}|\nabla u|^p - \frac{1}{q}|\nabla u|^q\Big) dx \\
&\ge \lim_{i\to\infty} \int_\Omega \frac{1}{\alpha} g|v_i|^\alpha \, dx - \liminf_{i\to\infty} \int_\Omega \frac{1}{p}|\nabla v_i|^p dx - \liminf_{i\to\infty} \int_\Omega \frac{1}{q}|\nabla v_i|^q dx \\
&\ge \lim_{i\to\infty} \int_\Omega \Big(\frac{1}{\alpha} g|v_i|^\alpha - \frac{1}{p}|\nabla v_i|^p dx - \frac{1}{q}|\nabla v_i|^q\Big) dx = \hat{I}.
\end{aligned}$$

Hence, u is a maximizer in (4.2). Since $I(u) = I(|u|)$, if u is a maximizer so is $|u|$. Thus, we have proved that a non-negative maximizer of problem (4.2) exists.

It is easy to show that a non-negative maximizer u of problem (4.2) is a solution to the boundary value problem (4.1), that is

$$\int_\Omega \big((|\nabla u|^{p-2} + |\nabla u|^{q-2}) \nabla u \cdot \nabla \phi - g u^{\alpha-1} \phi \big) dx = 0 \quad \forall \phi \in H_0^{1,p}(\Omega). \tag{4.4}$$

Indeed, let $u \ge 0$ be a maximizer of problem (4.2). If $t \in \mathbb{R}$ and $\phi \in H_0^{1,p}(\Omega)$ we must have

$$I(u) \ge I(u + t\phi) \quad \forall t \in (-1, 1).$$

Hence,

$$\frac{dI}{dt}\Big|_{t=0} = 0.$$

By computation we find

$$\int_\Omega \Big(g u^{\alpha-1}\phi - |\nabla u|^{p-2} \nabla u \cdot \nabla \phi - |\nabla u|^{q-2} \nabla u \cdot \nabla \phi \Big) dx = 0,$$

which is equivalent to (4.4).

Under our assumptions (in particular $\alpha < q < p$), a maximizer $u \ge 0$ of (4.2) is non-trivial. Indeed, if $v \in H^{1,p}(\Omega)$, $v > 0$, and $\epsilon > 0$ small enough we have

$$I(\epsilon v) = \epsilon^\alpha \int_\Omega \Big(\frac{1}{\alpha} g(x) v^\alpha - \frac{\epsilon^{p-\alpha}}{p} |\nabla v|^p - \frac{\epsilon^{q-\alpha}}{q} |\nabla v|^q \Big) dx > 0.$$

It follows that problem (4.1) has a non-trivial solution.

Problem (4.1) has been discussed in [32, 33]. In particular, since g is non-negative and bounded, any solution u belongs to $C^{1,\sigma}(\Omega)$ for some $0 < \sigma < 1$.
Let us prove the following comparison principle.

Theorem 4.1. *[Weak comparison principle] Let Ω be a bounded domain in $\mathbb{R}^N$ and let $1 < q < p$. Let $u, v \in H^{1,p}(\Omega)$ satisfy*

$$u(x) - v(x) \leq 0 \quad on \quad \partial\Omega$$

and let

$$\int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u \cdot \nabla\phi \, dx \leq 0 \quad \forall \phi \in H_0^{1,p}(\Omega), \quad \phi(x) \geq 0, \tag{4.5}$$

$$\int_\Omega (|\nabla v|^{p-2} + |\nabla v|^{q-2})\nabla v \cdot \nabla\phi \, dx \geq 0 \quad \forall \phi \in H_0^{1,p}(\Omega), \quad \phi(x) \geq 0. \tag{4.6}$$

Then, $u(x) - v(x) \leq 0$ a.e. in Ω.

Proof. Let $A = \{x \in \Omega : u(x) > v(x)\}$. We must prove that $|A| = 0$. Subtracting (4.6) from (4.5) we find

$$\int_\Omega \Big((|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u - (|\nabla v|^{p-2} + |\nabla v|^{q-2})\nabla v \Big) \cdot \nabla\phi \, dx \leq 0.$$

Putting $\phi = (u(x) - v(x))^+$ we have

$$\int_A \Big((|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u - (|\nabla v|^{p-2} + |\nabla v|^{q-2})\nabla v \Big) \cdot (\nabla u - \nabla v) \, dx \leq 0.$$

We can also write

$$\begin{aligned} &\int_A \Big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \Big) \cdot (\nabla u - \nabla v) \, dx \\ &+ \int_A \Big(|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v \Big) \cdot (\nabla u - \nabla v) \, dx \leq 0. \end{aligned} \tag{4.7}$$

By using the inequality

$$\Big(|X|^{q-2}X - |Y|^{q-2}Y \Big) \cdot (X - Y) \geq 0 \quad \text{for} \quad X, Y \in \mathbb{R}^N,$$

with $X = \nabla u$ and $Y = \nabla v$ we find

$$\int_A \Big(|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v \Big) \cdot (\nabla u - \nabla v) \, dx \geq 0.$$

Hence, from (4.7) it follows that

$$\int_A \Big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \Big) \cdot (\nabla u - \nabla v) \, dx \leq 0. \tag{4.8}$$

Now we use the inequality

$$\Big(|X|^{p-2}X - |Y|^{p-2}Y \Big) \cdot (X - Y) > 0 \quad \text{for} \quad X \neq Y.$$

With $X = \nabla u$ and $Y = \nabla v$ we find

$$(|\nabla u|^{p-2}\nabla u - (|\nabla v|^{p-2}\nabla v \Big) \cdot (\nabla u - \nabla v) > 0 \quad \text{for} \quad \nabla u \neq \nabla v.$$

In view of this inequality, if $|\nabla u - \nabla v| \neq 0$ in a subset of A with a positive measure, (4.8) yields a contradiction. If $\nabla u = \nabla v$ almost everywhere then $\nabla(u-v) = 0$ in A. Since $u - v = 0$ on ∂A, we have $u(x) = v(x)$, so the measure of A must be zero. Hence, $u(x) \le v(x)$ in Ω and the theorem is proved.

Now we prove the following version of a strong maximum principle.

Theorem 4.2. *[Strong comparison principle] Let $1 < q < p$ and let $u \in C^1(\Omega)$ such that $u(x) \ge 0$ and $\Delta_p u + \Delta_q u \le 0$ a.e. in Ω. Then either $u(x) \equiv 0$ or $u(x) > 0$ in Ω.*

Proof. If $u(x_0) = 0$ for some $x_0 \in \Omega$ and $u(x) \not\equiv 0$ we shall get a contradiction. We may assume that there is a ball $B \subset \Omega$ centered at x_1 with radius R such that $x_0 \in \partial B$ and $u(x) > 0$ in B. Let

$$a = \min_{|x|=R/2} u(x).$$

Clearly, $a > 0$. For $k > 0$ (that will be fixed later on), let

$$v(r) = c\big(e^{kr} - 1\big), \quad 0 \le r \le R/2, \quad v(R/2) = a.$$

We have $v(0) = 0$ and $v'(r) > 0$ for $r \ge 0$.
In the annulus

$$G = \big\{x \in B : R/2 < |x - x_1| < R\big\}$$

we define

$$z(x) = v(R - |x - x_1|).$$

We have $z(x) = 0$ on $|x - x_1| = R$ and $z(x) = a \le u(x)$ on $|x - x_1| = R/2$. Hence,

$$z(x) - u(x) \le 0 \quad \text{ on } \ \partial G.$$

On the other hand, since z is radial with respect to $r = |x - x_1|$ we find

$$\begin{aligned}
\Delta_p z &= r^{1-N}\Big(r^{N-1}|z'|^{p-2}z'\Big)' \\
&= -r^{1-N}\Big(r^{N-1}(v'(R-r))^{p-1}\Big)' \\
&= (p-1)(v'(R-r))^{p-2}v''(R-r) - \frac{N-1}{r}(v'(R-r))^{p-1} \\
&= v'(R-r))^{p-2}\Big[(p-1)v''(R-r) - \frac{N-1}{r}v'(R-r))\Big] \\
&= v'(R-r))^{p-1}\Big[(p-1)k - \frac{N-1}{r}\Big] \ge 0 \ \text{ for } \ k \ge \frac{2(N-1)}{R(p-1)}.
\end{aligned}$$

Similarly, we find

$$\Delta_q z = v'(R-r))^{q-1}\Big[(q-1)k - \frac{N-1}{r}\Big] \ge 0 \ \text{ for } \ k \ge \frac{2(N-1)}{R(q-1)}.$$

Hence, for $k = \frac{2(N-1)}{R(q-1)}$ we have

$$\Delta_p z + \Delta_q z \geq 0 \quad \text{in} \quad G.$$

By assumption, we also have

$$\Delta_p u + \Delta_q u \leq 0 \quad \text{in} \quad G.$$

Recalling that

$$z(x) - u(x) \leq 0 \ \text{ on } \ \partial G,$$

by Theorem 4.1 with $\Omega = G$ and $v = z$, we must have $z(x) \leq u(x)$ for all $x \in G$.
At this point we find the following contradiction. Since x_0 is a point of minimum for u, the gradient of u at x_0 vanishes. On the other hand, for $x \in G$ we have

$$\frac{u(x) - u(x_0)}{|x - x_0|} \geq \frac{z(x) - z(x_0)}{|x - x_0|} = \frac{v(R - |x - x_1|)}{|x - x_0|}.$$

If x belongs to the segment $[x_1, x_0]$ we have $|x - x_0| = R - |x - x_1|$. Therefore,

$$\frac{u(x) - u(x_0)}{|x - x_0|} \geq \frac{v(R - |x - x_1|)}{R - |x - x_1|}.$$

It follows that

$$\begin{aligned}
\liminf_{x \to x_0} \frac{u(x) - u(x_0)}{|x - x_0|} &\geq \lim_{x \to x_0} \frac{v(R - |x - x_1|)}{R - |x - x_1|} \\
&= \lim_{t \to R} \frac{v(R - t)}{R - t} \\
&= v'(0) > 0,
\end{aligned}$$

contradicting the fact that the gradient of u vanishes at $x = x_0$. The theorem follows.

The following uniqueness result is crucial for our purposes.

Theorem 4.3. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $g \in L^\infty_+(\Omega)$ and $1 \leq \alpha < q < p$. If $u, v \in H^{1,p}_0(\Omega) \cap C^0(\overline{\Omega})$ are positive solutions to problem (4.4) then $u(x) = v(x)$ in Ω.*

Proof. Define $A = \{x \in \Omega : u(x) > v(x)\}$. If we prove that A is empty, the assertion of the theorem follows. We argue by contradiction, assuming A is not empty. For $\epsilon > 0$, define $u_\epsilon = u + \epsilon$ and $v_\epsilon = v + \epsilon$. Note that in A we have $u_\epsilon(x) > v_\epsilon(x)$. Using

$$\phi_1(x) = \max\Big[\frac{u_\epsilon^\alpha(x) - v_\epsilon^\alpha(x)}{u_\epsilon^{\alpha-1}(x)}, 0\Big]$$

as test function in the equation for u we obtain

$$\begin{aligned}
&\int_A |\nabla u|^{p-2} \nabla u \cdot \nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big) dx + \int_A |\nabla u|^{q-2} \nabla u \cdot \nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big) dx \\
&= \int_A g(x)(u_\epsilon^\alpha - v_\epsilon^\alpha)\Big(\frac{u}{u_\epsilon}\Big)^{\alpha-1} dx.
\end{aligned}$$

Using

$$\phi_2(x) = \max\Big[\frac{u_\epsilon^\alpha(x) - v_\epsilon^\alpha(x)}{v_\epsilon^{\alpha-1}(x)}, 0\Big]$$

as test function in the equation for v we obtain

$$\int_A |\nabla v|^{p-2}\nabla v \cdot \nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx + \int_A |\nabla v|^{q-2}\nabla v \cdot \nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx$$
$$= \int_A g(x)(u_\epsilon^\alpha - v_\epsilon^\alpha)\Big(\frac{v}{v_\epsilon}\Big)^{\alpha-1} dx.$$

Subtracting the latter equality from the first one we get

$$\begin{aligned}&\int_A |\nabla u|^{p-2}\nabla u \cdot \nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big)dx + \int_A |\nabla u|^{q-2}\nabla u \cdot \nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big)dx \\ &+ \int_A |\nabla v|^{p-2}\nabla v \cdot \nabla\Big(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx + \int_A |\nabla v|^{q-2}\nabla v \cdot \nabla\Big(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx \\ &= L_\epsilon,\end{aligned} \tag{4.9}$$

where

$$L_\epsilon = \int_A g(x)(u_\epsilon^\alpha - v_\epsilon^\alpha)\Big(\frac{u}{u_\epsilon}\Big)^{\alpha-1}dx + \int_A g(x)(v_\epsilon^\alpha - u_\epsilon^\alpha)\Big(\frac{v}{v_\epsilon}\Big)^{\alpha-1}dx. \tag{4.10}$$

We claim that the sum of the second and the fourth integrals in the left hand side of (4.9) is non-negative. Indeed, since

$$\nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big) = \nabla u + (\alpha-1)\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^\alpha \nabla u - \alpha\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{\alpha-1}\nabla v$$

and

$$\nabla\Big(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big) = \nabla v + (\alpha-1)\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^\alpha \nabla v - \alpha\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{\alpha-1}\nabla u$$

we find

$$\begin{aligned}&\int_A |\nabla u|^{q-2}\nabla u \cdot \nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big)dx + \int_A |\nabla v|^{q-2}\nabla v \cdot \nabla\Big(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx \\ &= \int_A \Big\{|\nabla u|^q\Big(1 + (\alpha-1)\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^\alpha\Big) - \alpha|\nabla u|^{q-2}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{\alpha-1}\nabla u\cdot\nabla v\Big\}dx \\ &+ \int_A \Big\{|\nabla v|^q\Big(1 + (\alpha-1)\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^\alpha\Big) - \alpha|\nabla v|^{q-2}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{\alpha-1}\nabla v\cdot\nabla u\Big\}dx.\end{aligned} \tag{4.11}$$

Now, for $X, Y \in \mathbb{R}^N$, we recall the inequality

$$|X|^{q-2}X\cdot Y \le \frac{1}{s}|X|^q + \frac{1}{q}|Y|^q \quad \text{with} \quad \frac{1}{s} + \frac{1}{q} = 1,$$

and replace the vector X by λX, where λ is a positive real number. We get

$$|X|^{q-2}\lambda^{q-1}X\cdot Y \le \frac{1}{s}\lambda^q|X|^q + \frac{1}{q}|Y|^q.$$

With $\lambda = \big(\frac{v_\epsilon}{u_\epsilon}\big)^{\frac{\alpha-1}{q-1}}$, $X = \nabla u$, $Y = \nabla v$ we find

$$|\nabla u|^{q-2}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{\alpha-1}\nabla u\cdot\nabla v \le \frac{1}{s}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{s(\alpha-1)}|\nabla u|^q + \frac{1}{q}|\nabla v|^q.$$

Similarly, with $\lambda = \left(\frac{u_\epsilon}{v_\epsilon}\right)^{\frac{\alpha-1}{q-1}}$, $X = \nabla v$, $Y = \nabla u$ we find

$$|\nabla v|^{q-2}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{\alpha-1}\nabla v\cdot\nabla u \le \frac{1}{s}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{s(\alpha-1)}|\nabla v|^q + \frac{1}{q}|\nabla u|^q.$$

In view of these inequalities, by (4.11) we get

$$\begin{aligned}&\int_A |\nabla u|^{q-2}\nabla u\cdot\nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big)dx + \int_A |\nabla v|^{q-2}\nabla v\cdot\nabla\Big(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx\\ &\ge \int_A\Big\{|\nabla u|^q\Big[1+(\alpha-1)\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^\alpha - \frac{\alpha}{s}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{s(\alpha-1)} - \frac{\alpha}{q}\Big]\\ &+ |\nabla v|^q\Big[1+(\alpha-1)\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^\alpha - \frac{\alpha}{s}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{s(\alpha-1)} - \frac{\alpha}{q}\Big]\Big\}dx.\end{aligned}$$

Using the function

$$\psi(t) = 1 + (\alpha-1)t^\alpha - \frac{\alpha}{s}t^{s(\alpha-1)} - \frac{\alpha}{q},$$

the latter inequality reads

$$\begin{aligned}&\int_A |\nabla u|^{q-2}\nabla u\cdot\nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big)dx + \int_A |\nabla v|^{q-2}\nabla v\cdot\nabla\Big(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx\\ &\ge \int_A\Big\{|\nabla u|^q\psi\Big(\frac{v_\epsilon}{u_\epsilon}\Big) + |\nabla v|^q\psi\Big(\frac{u_\epsilon}{v_\epsilon}\Big)\Big\}dx.\end{aligned}$$

Note that $\psi(t)$ is non-negative for $t > 0$. Indeed we have $\psi(1) = 0$ and $\psi'(t) = \alpha(\alpha - 1)t^{\frac{\alpha q-2q+1}{q-1}}\big(t^{\frac{q-\alpha}{q-1}} - 1\big)$, hence $\psi(t) > 0$ for $t \neq 1$. Therefore,

$$\int_A |\nabla u|^{q-2}\nabla u\cdot\nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big)dx + \int_A |\nabla v|^{q-2}\nabla v\cdot\nabla\Big(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx \ge 0$$

as claimed. Therefore, from (4.9) we find

$$\int_A |\nabla u|^{p-2}\nabla u\cdot\nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big)dx + \int_A |\nabla v|^{p-2}\nabla v\cdot\nabla\Big(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx \le L_\epsilon, \tag{4.12}$$

where L_ϵ is defined as in (4.10). The left hand side of (4.12) can be rewritten as in (4.11) with p in place of q, that is

$$\begin{aligned}&\int_A |\nabla u|^{p-2}\nabla u\cdot\nabla\Big(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\Big)dx + \int_A |\nabla v|^{p-2}\nabla v\cdot\nabla\Big(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\Big)dx\\ &= \int_A\Big\{|\nabla u|^p\Big(1+(\alpha-1)\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^\alpha\Big) - \alpha|\nabla u|^{p-2}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{\alpha-1}\nabla u\cdot\nabla v\Big\}dx\\ &+ \int_A\Big\{|\nabla v|^p\Big(1+(\alpha-1)\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^\alpha\Big) - \alpha|\nabla v|^{p-2}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{\alpha-1}\nabla v\cdot\nabla u\Big\}dx.\end{aligned} \tag{4.13}$$

Recalling the inequality

$$|X|^{p-2}\lambda^{p-1}X\cdot Y \le \frac{1}{r}\lambda^p|X|^p + \frac{1}{p}|Y|^p, \quad \frac{1}{r}+\frac{1}{p} = 1$$

we find

$$|\nabla u|^{p-2}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{\alpha-1}\nabla u\cdot\nabla v \le \frac{1}{r}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{r(\alpha-1)}|\nabla u|^p + \frac{1}{p}|\nabla v|^p$$

and

$$|\nabla v|^{p-2}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{\alpha-1}\nabla v\cdot\nabla u \le \frac{1}{r}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{r(\alpha-1)}|\nabla v|^p + \frac{1}{p}|\nabla u|^p.$$

Using these inequalities, from (4.12) and (4.13) we find

$$\begin{aligned}&\int_A \Big\{|\nabla u|^p\Big[1+(\alpha-1)\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^\alpha - \frac{\alpha}{r}\Big(\frac{v_\epsilon}{u_\epsilon}\Big)^{r(\alpha-1)} - \frac{\alpha}{p}\Big]\\ &+|\nabla v|^p\Big[1+(\alpha-1)\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^\alpha - \frac{\alpha}{r}\Big(\frac{u_\epsilon}{v_\epsilon}\Big)^{r(\alpha-1)} - \frac{\alpha}{p}\Big]\Big\}dx \le L_\epsilon.\end{aligned}$$

Putting

$$\varphi(t) = 1+(\alpha-1)t^\alpha - \frac{\alpha}{r}t^{r(\alpha-1)} - \frac{\alpha}{p},$$

we have

$$\int_A \Big\{|\nabla u|^p\varphi\Big(\frac{v_\epsilon}{u_\epsilon}\Big) + |\nabla v|^p\varphi\Big(\frac{u_\epsilon}{v_\epsilon}\Big)\Big\}dx \le L_\epsilon. \tag{4.14}$$

Since

$$\lim_{\epsilon\to 0} u_\epsilon = u, \quad \lim_{\epsilon\to 0} v_\epsilon = v, \quad \lim_{\epsilon\to 0} L_\epsilon = 0,$$

as $\epsilon \to 0$, (4.14) yields

$$\int_A \Big\{|\nabla u|^p\varphi\Big(\frac{v}{u}\Big) + |\nabla v|^p\varphi\Big(\frac{u}{v}\Big)\Big\}dx \le 0. \tag{4.15}$$

We have $\varphi(1) = 0$ and $\varphi'(t) = \alpha(\alpha-1)t^{\frac{\alpha p-2p+1}{p-1}}\big(t^{\frac{p-\alpha}{p-1}} - 1\big)$, hence $\varphi(t) > 0$ for $t \ne 1$. Since $\frac{u}{v} > 1$ in A, by (4.15) we must have $|\nabla u| = |\nabla v| = 0$ in A. Hence, $\nabla(u-v) = 0$ in A and $u - v = 0$ on ∂A. Then, $u(x) = v(x)$, contradicting the definition of A. The theorem follows. $\square$

4.1.1 Optimization problems

Recall that two measurable functions $g_1(x)$ and $g_2(x)$ defined in Ω have the same rearrangement if and only if

$$|\{x\in\Omega : g_1(x)\ge t\}| = |\{x\in\Omega : g_2(x)\ge t\}| \quad \forall t\in\mathbb{R}.$$

Let $g_0(x)$ be a non-negative bounded function defined in Ω. We assume that $g_0(x) \ge 0$ is not a constant in Ω. As in the previous part of this work, we denote by $\mathcal{G} = \mathcal{G}(g_0)$ the class of functions g which have the same rearrangement as g_0. Moreover, we denote by $\overline{\mathcal{G}}$ the closure of $\mathcal{G}$ in the weak* topology of $L^\infty(\Omega)$.

With $g \in \overline{\mathcal{G}}$, we consider the functional

$$J(g) = \int_\Omega \Big(\frac{1}{\alpha}gu^\alpha - \frac{1}{p}|\nabla u|^p - \frac{1}{q}|\nabla u|^q\Big)dx, \tag{4.16}$$

where u is the positive solution to problem (4.1). Observe that this function u satisfies

$$\int_\Omega \Big(\frac{1}{\alpha} g u^\alpha - \frac{1}{p}|\nabla u|^p - \frac{1}{q}|\nabla u|^q\Big)dx = \max_{v\in H_0^1(\Omega), v\geq 0} \int_\Omega \Big(\frac{1}{\alpha} g v^\alpha - \frac{1}{p}|\nabla v|^p - \frac{1}{q}|\nabla v|^q\Big)dx,$$

and that, by Theorem 4.3, the maximizer is unique.
We are interested in the maximization and the minimization of the functional $J(g)$ for $g \in \mathcal{G}$.

Let us prove the following result.

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^N$, g_0 be a measurable bounded non-negative function which is positive in a subset with a positive measure, $\mathcal{G}$ be the class of rearrangements of g_0, and $\overline{\mathcal{G}}$ be the closure of $\mathcal{G}$ in the weak* topology of $L^\infty(\Omega)$. Further let*

$$J(g) = \int_\Omega \Big(\frac{1}{\alpha} g u^\alpha - \frac{1}{p}|\nabla u|^p - \frac{1}{q}|\nabla u|^q\Big)dx,$$

where $g \in \overline{\mathcal{G}}$ and u is the positive solution to problem (4.4) corresponding to g. Then:
i) $J(g)$ is weakly continuous with respect to the weak topology of $L^\infty(\Omega)$;*
ii) $J(g)$ is Gâteaux differentiable with derivative $\frac{1}{\alpha}u^\alpha$ (here u is the positive solution to problem (4.4) *corresponding to g);*
iii) $J(g)$ is strictly convex on $\overline{\mathcal{G}}$.

Proof. We start proving *i)*. Let $g_i \rightharpoonup g$ as $i \to \infty$ in the weak* topology of $L^\infty(\Omega)$. We shall prove that

$$\lim_{i\to\infty} J(g_i) = J(g). \tag{4.17}$$

Indeed, if u_i is the solution to (4.4) corresponding to g_i, putting $\phi = u_i$ in that equation we find

$$\int_\Omega \big(|\nabla u_i|^p + |\nabla u_i|^q\big)dx = \int_\Omega g_i u_i^\alpha \, dx. \tag{4.18}$$

Since $0 \leq g_i \leq \sup g_0(x)$, using Poincaré and Hölder inequalities we find

$$\int_\Omega g_i u_i^\alpha \, dx \leq C \int_\Omega |\nabla u_i|^\alpha \leq C\Big(\int_\Omega |\nabla u_i|^p \, dx\Big)^{\frac{\alpha}{p}}.$$

The latter result and (4.18) imply

$$\int_\Omega |\nabla u_i|^p dx \leq C\Big(\int_\Omega |\nabla u_i|^p \, dx\Big)^{\frac{\alpha}{p}},$$

from which we get

$$\int_\Omega |\nabla u_i|^p \, dx \leq C. \tag{4.19}$$

Recall that we denote by C constants independent of i. By (4.19), it follows that a subsequence (denoted again $\{u_i\}$) converges weakly in $H^{1,p}(\Omega)$ (as well as in $H^{1,q}(\Omega)$) and strongly in $L^\alpha(\Omega)$

to some function $z \in H_0^{1,p}(\Omega)$.
We claim that

$$\lim_{i\to\infty} \int_\Omega (g_i - g)(u_i^\alpha - z^\alpha)\; dx = 0.$$

Indeed, since

$$|(g_i - g)(u_i^\alpha - z^\alpha)| \le C|u_i - z| \cdot |u_i + z|^{\alpha-1}$$

we have

$$\Big| \int_\Omega (g_i - g)(u_i^\alpha - z^\alpha)\; dx \Big| \le C\|u_i - z\|_{L^\alpha(\Omega)} \|u_i + z\|^{\alpha-1}_{L^\alpha(\Omega)}.$$

Since $u_i \to z$ strongly in $L^\alpha(\Omega)$, we have

$$\lim_{i\to\infty} \|u_i - z\|_{L^\alpha(\Omega)} = 0,$$

and

$$\|u_i + z\|^{\alpha-1}_{L^\alpha(\Omega)} \le C.$$

Hence,

$$\lim_{i\to\infty} \|u_i - z\|_{L^\alpha(\Omega)} \|u_i + z\|^{\alpha-1}_{L^\alpha(\Omega)} = 0.$$

Therefore,

$$\lim_{i\to\infty} \Big| \int_\Omega (g_i - g)(u_i^\alpha - z^\alpha)\; dx \Big| = 0,$$

and claim follows.
Since (by assumption)

$$\lim_{i\to\infty} \int_\Omega (g_i - g) z^\alpha \; dx = 0,$$

and

$$0 = \lim_{i\to\infty} \int_\Omega (g_i - g)(u_i^\alpha - z^\alpha)\; dx = \lim_{i\to\infty} \int_\Omega (g_i - g)u_i^\alpha \; dx - \lim_{i\to\infty} \int_\Omega (g_i - g) z^\alpha \; dx$$

we find

$$\lim_{i\to\infty} \int_\Omega (g_i - g)u_i^\alpha \; dx = 0. \tag{4.20}$$

Obviously, if u is the positive solution to problem (4.4) corresponding to g we also have

$$\lim_{i\to\infty} \int_\Omega (g_i - g)u^\alpha \; dx = 0. \tag{4.21}$$

Now we write

$$\begin{aligned}
&J(g) + \int_\Omega \frac{1}{\alpha}(g_i - g)u^\alpha \; dx \\
&= \int_\Omega \Big(\frac{1}{\alpha} g_i u^\alpha - \frac{1}{p}|\nabla u|^p - \frac{1}{q}|\nabla u|^q \Big) dx \\
&\le \int_\Omega \Big(\frac{1}{\alpha} g_i u_i^\alpha - \frac{1}{p}|\nabla u_i|^p - \frac{1}{q}|\nabla u_i|^q \Big) dx = J(g_i) \\
&= \int_\Omega \frac{1}{\alpha}(g_i - g)u_i^\alpha \; dx + \int_\Omega \Big(\frac{1}{\alpha} g u_i^\alpha - \frac{1}{p}|\nabla u_i|^p - \frac{1}{q}|\nabla u_i|^q \Big) dx \\
&\le \int_\Omega \frac{1}{\alpha}(g_i - g)u_i^\alpha \; dx + J(g).
\end{aligned} \tag{4.22}$$

By (4.20), (4.21) and (4.22) we find (4.17), and assertion i) of the theorem is proved.

To prove assertion *ii)*, we first claim that the function z mentioned above is equal to u, the positive solution to problem (4.4) corresponding to g. Indeed, since $u_i \rightharpoonup z$ weakly in $H^{1,p}(\Omega)$ we have

$$\int_\Omega |\nabla z|^p \, dx \leq \liminf_{i\to\infty} \int_\Omega |\nabla u_i|^p \, dx$$

and

$$\int_\Omega |\nabla z|^q \, dx \leq \liminf_{i\to\infty} \int_\Omega |\nabla u_i|^q \, dx.$$

Moreover, since $u_i \to z$ strongly in $L^\alpha(\Omega)$, by (4.20) we get

$$\lim_{i\to\infty} \int_\Omega g_i u_i^\alpha \, dx = \lim_{i\to\infty} \int_\Omega g u_i^\alpha \, dx = \lim_{i\to\infty} \int_\Omega g z^\alpha \, dx.$$

From (4.22), using the results in above and (4.17) we find

$$\begin{aligned}
J(g) &= \int_\Omega \Big(\frac{1}{\alpha} g u^\alpha - \frac{1}{p}|\nabla u|^p - \frac{1}{q}|\nabla u|^q\Big) dx \\
&= \lim_{i\to\infty} J(g_i) = \lim_{i\to\infty} \int_\Omega \Big(\frac{1}{\alpha} g_i u_i^\alpha - \frac{1}{p}|\nabla u_i|^p - \frac{1}{q}|\nabla u_i|^q\Big) dx \\
&\leq \lim_{i\to\infty} \int_\Omega \frac{1}{\alpha} g_i u_i^\alpha \, dx - \liminf_{i\to\infty} \int_\Omega \frac{1}{p}|\nabla u_i|^p dx - \liminf_{i\to\infty} \int_\Omega \frac{1}{q}|\nabla u_i|^q dx \\
&\leq \int_\Omega \Big(\frac{1}{\alpha} g z^\alpha - \frac{1}{p}|\nabla z|^p - \frac{1}{q}|\nabla z|^q\Big) dx \leq J(g).
\end{aligned}$$

It follows that

$$\int_\Omega \Big(\frac{1}{\alpha} g u^\alpha - \frac{1}{p}|\nabla u|^p - \frac{1}{q}|\nabla u|^q\Big) dx = \int_\Omega \Big(\frac{1}{\alpha} g z^\alpha - \frac{1}{p}|\nabla z|^p - \frac{1}{q}|\nabla z|^q\Big) dx.$$

By the uniqueness of the maximizer of

$$I(v) = \int_\Omega \Big(\frac{1}{\alpha} g v^\alpha - \frac{1}{p}|\nabla v|^p - \frac{1}{q}|\nabla v|^q\Big) dx$$

for $v > 0$, $v \in H_0^{1,p}(\Omega)$, we must have $u = z$, as claimed.
In view of the latter result, (4.20) implies

$$\lim_{i\to\infty} \int_\Omega g_i u_i^\alpha \, dx = \lim_{i\to\infty} \int_\Omega g u_i^\alpha \, dx = \int_\Omega g u^\alpha dx.$$

Now we finally prove assertion *ii)*. Let $t_i > 0$ be a sequence of real numbers such that $t_i \to 0$ as $i \to \infty$. Let $h \in \overline{\mathcal{G}}$ and $g_i = g + t_i(h - g)$. Then, by (4.22) we find

$$J(g) + t_i \int_\Omega \frac{1}{\alpha}(h-g)\, u^\alpha \, dx \leq J(g + t_i(h-g)) \leq J(g) + t_i \int_\Omega \frac{1}{\alpha}(h-g) u_i^\alpha \, dx.$$

It follows that

$$\int_\Omega \frac{1}{\alpha}(h-g)u^\alpha\, dx \le \frac{J(g+t_i(h-g))-J(g)}{t_i} \le \int_\Omega \frac{1}{\alpha}(h-g)u_i^\alpha\, dx. \tag{4.23}$$

Since $g_i \to g$ as $i \to \infty$, $u_i \to u$ in $L^\alpha(\Omega)$ we have,

$$\lim_{i\to\infty}\int_\Omega \frac{1}{\alpha}(h-g)u_i^\alpha\, dx = \int_\Omega \frac{1}{\alpha}(h-g)u^\alpha\, dx.$$

Since the sequence t_i is arbitrary, from the latter equation and (4.23) we find

$$\lim_{t\to 0^+}\frac{J(g+t(h-g))-J(g)}{t} = \int_\Omega \frac{1}{\alpha}(h-g)u^\alpha\, dx.$$

This proves assertion *ii)*.

To prove *iii)*, let $0<t<1$. If $g_1,\ g_2 \in \overline{\mathcal{G}}$ and $g_t = tg_1+(1-t)g_2$, we have

$$\begin{aligned}&\int_\Omega\Big(\frac{1}{\alpha}g_t v^\alpha - \frac{1}{p}|\nabla v|^p - \frac{1}{q}|\nabla v|^q\Big)dx\\ &= t\int_\Omega\Big(\frac{1}{\alpha}g_1 v^\alpha - \frac{1}{p}|\nabla v|^p - \frac{1}{q}|\nabla v|^q\Big)dx\\ &+(1-t)\int_\Omega\Big(\frac{1}{\alpha}g_2 v^\alpha - \frac{1}{p}|\nabla v|^p - \frac{1}{q}|\nabla v|^q\Big)dx.\end{aligned}$$

If we take the superior for $v \in H_0^{1,p}(\Omega)$ in both sides of this equation we find

$$J(g_t) \le tJ(g_1)+(1-t)J(g_2).$$

This yields the convexity of $J(g)$. To prove strict convexity, assume equality holds in the latter inequality for some $0<t<1$. If u_t, u_1 and u_2 are the solutions of (4.4) corresponding to g_t, g_1 and g_2 respectively, then we have

$$\begin{aligned}&t\int_\Omega\Big(\frac{1}{\alpha}g_1 u_t^\alpha - \frac{1}{p}|\nabla u_t|^p - \frac{1}{q}|\nabla u_t|^q\Big)dx\\ &+(1-t)\int_\Omega\Big(\frac{1}{\alpha}g_2 u_t^\alpha - \frac{1}{p}|\nabla u_t|^p - \frac{1}{q}|\nabla u_t|^q\Big)dx\\ &= t\int_\Omega\Big(\frac{1}{\alpha}g_1 u_1^\alpha - \frac{1}{p}|\nabla u_1|^p - \frac{1}{q}|\nabla u_1|^q\Big)dx\\ &+(1-t)\int_\Omega\Big(\frac{1}{\alpha}g_2 u_2^\alpha - \frac{1}{p}|\nabla u_2|^p - \frac{1}{q}|\nabla u_2|^q\Big)dx.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_\Omega\Big(\frac{1}{\alpha}g_1 u_t^\alpha - \frac{1}{p}|\nabla u_t|^p - \frac{1}{q}|\nabla u_t|^q\Big)dx\\ &= \int_\Omega\Big(\frac{1}{\alpha}g_1 u_1^\alpha - \frac{1}{p}|\nabla u_1|^p - \frac{1}{q}|\nabla u_1|^q\Big)dx\end{aligned}$$

and

$$\int_{\Omega}\Big(\frac{1}{\alpha}g_2u_t^{\alpha}-\frac{1}{p}|\nabla u_t|^p-\frac{1}{q}|\nabla u_t|^q\Big)dx$$
$$=\int_{\Omega}\Big(\frac{1}{\alpha}g_2u_2^{\alpha}-\frac{1}{p}|\nabla u_2|^p-\frac{1}{q}|\nabla u_2|^q\Big)dx.$$

By uniqueness of the maximizer of $I(v)$ we must have $u_t = u_1 = u_2$. The corresponding equations for u_t are

$$-\Delta_p u_t - \Delta_q u_t = g_1\ u_t^{\alpha-1} \ \text{ a.e. in } \ \Omega$$

and

$$-\Delta_p u_t - \Delta_q u_t = g_2\ u_t^{\alpha-1} \ \text{ a.e. in } \ \Omega.$$

Hence,

$$g_1\ u_t^{\alpha-1} = g_2\ u_t^{\alpha-1} \ \text{ a.e. in } \ \Omega.$$

Since $g_1(x) \geq 0$ and $g_1(x) \not\equiv 0$, by the strong maximum principle $u_t(x) > 0$. It follows that $g_1(x) = g_2(x)$ a.e. in Ω, which yields strict convexity of $J(g)$.
The theorem is proved. □

Theorem 4.5. *Let Ω, g_0, $\mathcal{G}$, $\overline{\mathcal{G}}$ and $J(g)$ be as in Theorem 4.4. Then*
i) the problem

$$\max_{g\in\mathcal{G}} J(g)$$

has (at least) a solution $\hat{g} \in \mathcal{G}$. Moreover, if $\hat{g}$ is any maximizer and if $\hat{u} = u_{\hat{g}}$ is the solution to problem (4.4) corresponding to $\hat{g}$ then we have $\hat{g} = \varphi(\hat{u})$, where φ is some non-decreasing function.
ii) the problem

$$\min_{g\in\mathcal{G}} J(g)$$

has a unique solution $\check{g} \in \mathcal{G}$. Moreover, if $\check{u} = u_{\check{g}}$ is the solution to problem (4.4) corresponding to $\check{g}$ we have $\check{g} = \psi(\check{u})$, where ψ is some non-increasing function.

Proof. By Theorem 4.4, $J(g)$ is weakly continuous and strictly convex. Therefore, assertion i) follows by Theorem 2.11.

To prove existence of the minimizer, let

$$\check{J} = \inf_{g\in\mathcal{G}} J(g),$$

and let $\{g_i\}$ be a sequence such that

$$\check{J} = \lim_{i\to\infty} J(g_i).$$

Since $\overline{\mathcal{G}}$ is weakly compact we can assume that for some subsequence of $\{g_i\}$ (again denoted $\{g_i\}$) there is $\bar{g} \in \overline{\mathcal{G}}$ with $g_i \rightharpoonup \bar{g}$ in the weak* topology of $L^\infty(\Omega)$. By Theorem 4.4 J is continuous, therefore

$$\check{J} = \lim_{i\to\infty} J(g_i) = J(\bar{g}).$$

Let us prove uniqueness. Assume $g_1, g_2 \in \overline{\mathcal{G}}$, $g_1 \neq g_2$ are minimizers for J. Let $J(g_1) = J(g_2) = m$. By the strict convexity of J we have

$$m \leq J\Big(\frac{g_1+g_2}{2}\Big) < \frac{1}{2}(J(g_1) + J(g_2)) = m,$$

a contradiction. Hence, uniqueness holds in $\overline{\mathcal{G}}$.

Let us show that $\bar{g} \in \mathcal{G}$. If $g \in \overline{\mathcal{G}}$, $0 < t < 1$ and $g_t = tg + (1-t)\bar{g}$, since $J(g)$ is Gâteaux differentiable (by Theorem 4.4) we have, for $u = u_{\bar{g}}$,

$$J(\bar{g}) \leq J(g_t) = J(\bar{g} + t(g-\bar{g})) = J(\bar{g}) + t\int_\Omega (g-\bar{g})\frac{1}{\alpha}u^\alpha \, dx + o(t), \tag{4.24}$$

where $o(t)$ is a quantity such that

$$\lim_{t\to 0}\frac{o(t)}{t} = 0.$$

Hence, from (4.24) we find

$$\int_\Omega (g-\bar{g})\frac{1}{\alpha}u^\alpha \, dx \geq 0.$$

Equivalently,

$$\int_\Omega g\, u^\alpha \, dx \geq \int_\Omega \bar{g}\, u^\alpha \, dx \quad \forall g \in \overline{\mathcal{G}}. \tag{4.25}$$

Recall that $u = u_{\bar{g}}$.

On the other hand, by equation (4.4) (written as equation (4.1)) it follows that the function u (and the function u^α) cannot have flat zones in the set

$$F = \{x \in \Omega : \bar{g}(x) > 0\}.$$

Consider first the case $|F| = |\Omega|$. Then, by Lemma 2.5 there is a non-increasing function $\bar{\psi}$ such that $\bar{\psi}(u^\alpha) \in \mathcal{G}$. For $\check{g} = \bar{\psi}(u^\alpha)$, by Lemma 2.4 we must have

$$\int_\Omega g\, u^\alpha \, dx \geq \int_\Omega \check{g}\, u^\alpha \, dx \quad \forall g \in \overline{\mathcal{G}},$$

and $\check{g}$ is the unique minimizer. By the latter result and (4.25) we must have $\check{g} = \bar{g}$. When $|F| = |\Omega|$ the proof of the theorem is complete with $\psi(t) = \bar{\psi}(t^\alpha)$.

If $|F| < |\Omega|$, we argue as in the proof of Theorem 3.2 . Since $\bar{g} \in \overline{\mathcal{G}}$, by Lemma 5 of [4] we have

$$|F| \geq |\{x \in \Omega : g_0(x) > 0\}|.$$

Then there is $g_1 \in \mathcal{G}$ such that its support is contained in F. By Lemma 2.5 there is a non-increasing function $\psi_1(t)$ such that $\psi_1(u^\alpha)$ is a rearrangement of g_1 on F.
Define

$$m = \inf_{x \in \Omega \setminus F} u^\alpha(x).$$

By using (4.25), and arguing as in the proof of Theorem 3.2, one proves that $u^\alpha(x) < m$ in F.
Now define

$$\tilde{\psi}(t) = \begin{cases} \psi_1(t) & \text{if } \ 0 \le t < m \\ 0 & \text{if } \ t \ge m. \end{cases}$$

The function $\tilde{\psi}(t)$ is non-increasing and $\tilde{\psi}(u^\alpha)$ is a rearrangement of $g_1(x)$ in Ω. Indeed, the functions g_1 and $\tilde{\psi}(u^\alpha)$ have the same rearrangement on F, and both vanish on $\Omega \setminus F$. For $\breve{g} = \tilde{\psi}(u^\alpha)$, by Lemma 2.4 we have

$$\int_\Omega g\, u^\alpha\, dx \ge \int_\Omega \breve{g}\, u^\alpha\, dx \quad \forall g \in \overline{\mathcal{G}}.$$

By the latter result, the uniqueness of the minimizer and (4.25) we must have $\breve{g} = \bar{g}$. The theorem is proved with $\psi(t) = \tilde{\psi}(t^\alpha)$. □

The case $\alpha = q$. The problem

$$-\Delta_p u - \Delta_q u = \lambda g(x) u^{q-1} \quad \text{in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega, \tag{4.26}$$

is an eigenvalue problem. Given $g(x)$, λ is an eigenvalue if and only if problem (4.26) has a non-trivial solution.
Let us describe some results from [41]. Let λ_1 be the first eigenvalue of the problem

$$-\Delta_q u = \lambda_1 g(x) u^{q-1}, \quad u > 0 \quad \text{in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega. \tag{4.27}$$

By Theorem 1.1 of [41], if $\lambda > \lambda_1$, problem (4.26) has a unique positive solution. This means that $g(x)$ does not identify a unique principal eigenvalue. In other words, we cannot define a functional $\lambda(g)$. As a consequence, we cannot investigate optimization problems related to principal eigenvalues of (4.26).

4.2 Optimization in case of rigid obstacles

A simplified version of these results has been published in [30].

Let Ω be a bounded domain in $\mathbb{R}^2$, and let $f = f(x, y)$ and $g = g(x, y)$ be non-negative bounded functions. We assume f to be positive in a subset with a positive measure. Consider the boundary value problem

$$-\Delta u + gu = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

The variational formulation of this problem is the following:

$$u \in H_0^1(\Omega): \quad \int_\Omega \Big(\nabla u \cdot \nabla \phi + gu\phi\Big)\,dxdy = \int_\Omega f\phi\,dxdy \quad \forall \phi \in H_0^1(\Omega).$$

Let us show that this Dirichlet problem has a unique solution. Since g is non-negative function, the quantity

$$((u,\phi)) := \int_\Omega \Big(\nabla u \cdot \nabla \phi + gu\phi\Big)\,dxdy$$

is a scalar product in the Hilbert space in $H_0^1(\Omega)$. On the other hand, since f is bounded, the functional

$$F(\phi) := \int_\Omega f\phi\,dxdy$$

is linear and continuous in $H_0^1(\Omega)$. Therefore, the existence and uniqueness of a solution u for our problem follows from Riesz representation theorem on Hilbert spaces. By standard results, $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ and is positive on Ω.
The corresponding energy integral is the following:

$$I = \int_\Omega (|\nabla u|^2 + gu^2)dxdy = \int_\Omega fu\,dxdy.$$

Let $\mathcal{F}$ be the class of rearrangements of a given function f_0. A typical problem is the investigation of the maximum or the minimum of I for $f \in \mathcal{F}$. Again, let $\mathcal{G}$ be the class of rearrangements of a given function g_0. One can investigate the maximum or the minimum of I for $g \in \mathcal{G}$. These problems have been discussed in many papers. We refer to [2, 4, 5] and references therein.

We consider subclasses of $\mathcal{F}$ and $\mathcal{G}$. More precisely, let $f_0 = \chi_{D_0}$, where D_0 is a given subset of Ω. We shall investigate the maximum and the minimum of $I(D)$ for g fixed and $f = \chi_D$ where $D \subset \Omega$ is any translation of D_0. Furthermore, let $g_0 = \chi_{D_0}$, where D_0 is a given subset of Ω. We shall investigate the maximum and the minimum of $I(D)$ for f fixed and $g = \chi_D$. These problems are inspired by the paper [23], where the case of eigenvalues is discussed. However, the situation here is different for the simultaneous presence of f and g.
Let D be a domain with a piecewise C^1 boundary and Steiner symmetric with respect to some straight line. Our main effort will be the localization of D throughout Ω so as to maximize or minimize the corresponding energy integral $I(D)$. This will be possible under suitable symmetry assumptions on Ω.

The equation $-\Delta u + gu = f$ models the temperature u in Ω in case of a steady state situation. The term gu corresponds to the density of heat absorbed (well), while f corresponds to the density of the heat produced (source). The energy integral $\int_\Omega fu \; dxdy$ is related with the average temperature in Ω. One may be interested in the maximization or minimization of the average temperature acting either on the data g or the data f.

4.2.1 Placement of a source so as to optimize energy integral

Let Ω be a bounded plane domain, and D be a smooth subset of Ω. Consider the Dirichlet problem

$$-\Delta u + gu = \chi_D \;\text{ in }\; \Omega, \quad u = 0 \;\text{ on }\; \partial\Omega, \tag{4.28}$$

where $g = g(x,y)$ is a non-negative bounded function. The corresponding energy integral is

$$I(D) = \int_\Omega (|\nabla u|^2 + gu^2) dxdy = \int_\Omega \chi_D u \; dxdy.$$

When the location of D (that we call source) changes through Ω, $I(D)$ may change. We are interested in finding the locations of D which realize the maximum or the minimum value of $I(D)$.

We first discuss the effect of a translation of the source. If D_ϵ is the domain D shifted at a distance ϵ in the x direction (assuming this is allowed), the new problem reads

$$-\Delta u_\epsilon + gu_\epsilon = \chi_{D_\epsilon} \;\text{ in }\; \Omega, \quad u_\epsilon = 0 \;\text{ on }\; \partial\Omega, \tag{4.29}$$

and the corresponding energy integral is

$$I(D_\epsilon) = \int_\Omega (|\nabla u_\epsilon|^2 + gu_\epsilon^2) dxdy = \int_\Omega \chi_{D_\epsilon} u_\epsilon \; dxdy.$$

So, we are interested in finding the sign of

$$I(D_\epsilon) - I(D) = \int_\Omega \big[\chi_{D_\epsilon} u_\epsilon - \chi_D u\big] \; dxdy$$

for a small $\epsilon > 0$.
From the equations in (4.28) and (4.29) we find

$$-\Delta(u_\epsilon - u) + g(u_\epsilon - u) = \chi_{D_\epsilon} - \chi_D. \tag{4.30}$$

The Alexandrov-Bakelman-Pucci maximum principle (see [22], Theorem 9.1) applied to (4.30) yields

$$\sup_\Omega |u_\epsilon - u| \leq C \|\chi_{D_\epsilon} - \chi_D\|_{L^2(\Omega)}, \tag{4.31}$$

where the constant C depends on Ω, but it is independent of ϵ and g. Note that

$$|\chi_{D_\epsilon} - \chi_D|^2 = \begin{cases} 1 & x \in (D \setminus D_\epsilon) \cup (D_\epsilon \setminus D) \\ \\ 0 & \text{elsewhere in } \ \Omega. \end{cases}$$

Moreover, if P is the perimeter of D we have $|D \setminus D_\epsilon| = |D_\epsilon \setminus D| < P\epsilon$. Therefore, we can find a constant C such that

$$\|\chi_{D_\epsilon} - \chi_D\|_{L^2(\Omega)} \leq C\epsilon^{\frac{1}{2}}. \tag{4.32}$$

Here and in what follows we denote by C constants (sometimes different from line to line) independent of ϵ. From (4.31) and (4.32) we find

$$\sup_\Omega |u_\epsilon - u| \leq C\epsilon^{\frac{1}{2}}. \tag{4.33}$$

Note that in addition to (4.32) we have

$$\int_\Omega |\chi_{D_\epsilon} - \chi_D| \; dxdy \leq C\epsilon.$$

From the latter estimate and (4.33) we find

$$\int_\Omega (\chi_{D_\epsilon} - \chi_D)(u_\epsilon - u) \; dxdy = o(\epsilon). \tag{4.34}$$

Moreover, multiplying (4.28) by u_ϵ, (4.29) by u and integrating over Ω we find

$$\int_\Omega \chi_D u_\epsilon \; dxdy = \int_\Omega \chi_{D_\epsilon} u \; dxdy. \tag{4.35}$$

Hence,

$$\begin{aligned} \frac{1}{\epsilon}\big[I(D_\epsilon) - I(D)\big] =& \frac{1}{\epsilon}\int_\Omega \big[\chi_{D_\epsilon} u_\epsilon - \chi_D u\big] dxdy \quad \text{(by using (4.35))} \\ =& \frac{1}{\epsilon}\int_\Omega \big[\chi_{D_\epsilon} u_\epsilon - \chi_D u_\epsilon + \chi_{D_\epsilon} u - \chi_D u\big] dxdy \quad \text{(by using (4.34))} \\ =& \frac{2}{\epsilon}\int_\Omega \big[\chi_{D_\epsilon} u - \chi_D u\big] dxdy + \frac{o(\epsilon)}{\epsilon} \\ =& 2\int_D \frac{u(x+\epsilon, y) - u(x,y)}{\epsilon} dxdy + \frac{o(\epsilon)}{\epsilon}. \end{aligned}$$

As $\epsilon \to 0$ we find

$$\frac{dI_{D_\epsilon}}{d\epsilon}\Big|_{\epsilon=0} = 2\int_D \frac{\partial u}{\partial x}(x,y) \; dxdy.$$

Recall the Green formula

$$\int_D \frac{\partial u}{\partial x}(x,y) \; dxdy = \int_{\partial D} u(x,y)\cos(n,x) ds,$$

where n is the exterior normal to ∂D. Hence,

$$\frac{dI_{D_\epsilon}}{d\epsilon}\bigg|_{\epsilon=0} = 2\int_{\partial D} u(x,y)\cos(n,x)ds. \tag{4.36}$$

To find the sign of the derivative in (4.36) we need some conditions on Ω and on the function $g(x,y)$.

Theorem 4.6. *Assume Ω to be Steiner symmetric with respect to the line $\{x = 0\}$. The function $g(x,y)$ is assumed to satisfy $g(x,y) = g(-x,y)$ and to be non-increasing with respect to x for $x < 0$. Assume that $D \subset \Omega$ is Steiner symmetric with respect to some line $\{x = a\}$. Then the energy integral $I(D)$ associated with problem* (4.28) *increases as the domain D approaches the line $\{x = 0\}$.*

Proof. We note that the cosine of the angle between the exterior normal n to the boundary ∂D and the x-axis is non-negative on $\partial D \cap \{x > a\}$. Assuming that $a < 0$, the line $\{x = a\}$ divides Ω into Ω_1 (the small part) and Ω_2. Since Ω is Steiner symmetric, the reflection of Ω_1 with respect to the line $\{x = a\}$ is strictly contained in Ω_2. In Ω_1, define

$$w(x,y) = u(2a - x, y) - u(x,y).$$

Of course, we have $w = 0$ for $x = a$. Furthermore, recalling that $u(x,y) > 0$ in Ω and $u(x,y) = 0$ on $\partial\Omega$ we have $w(x,y) > 0$ on the part of the boundary $\partial\Omega_1$ with $x < a$. Since D is symmetric with respect to $\{x = a\}$, by (4.28) we find

$$-\Delta u(2a - x, y) + g(2a - x, y)u(2a - x, y) = \chi_D(2a - x, y), \quad (x,y) \in \Omega_1.$$

Subtracting (4.28) from the latter equation we find

$$-\Delta w + g(2a - x, y)u(2a - x, y) - g(x,y)u(x,y) = \chi_D(2a - x, y) - \chi_D(x,y).$$

It is easy to check that $(x,y) \in D$ if and only if $(2a - x, y) \in D$. Therefore,

$$-\Delta w + g(2a - x, y)u(2a - x, y) - g(x,y)u(x,y) = 0 \ \text{ in } \ \Omega_1. \tag{4.37}$$

We claim that $g(2a - x, y) \leq g(x,y)$ for $x < a$. Indeed, if $2a - x \leq 0$ (since $2a - x \geq x$ in Ω_1) the claim follows from the assumption that $g(x,y)$ is non-increasing with respect to x for $x < 0$. If $2a - x > 0$, using the assumption $g(x,y) = g(-x,y)$ we rewrite the inequality as $g(x - 2a, y) \leq g(-x,y)$ and apply the condition that $g(x,y)$ is non-decreasing with respect to x for $x > 0$. The claim is proved.
Hence, from (4.37) we find

$$-\Delta w + g(x,y)w(x,y) \geq 0 \ \text{ in } \ \Omega_1. \tag{4.38}$$

Since $w(x,y) \geq 0$ on $\partial\Omega_1$, by the weak maximum principle (see [22], Theorem 8.1), we have $w(x,y) \geq 0$ in Ω_1. Since $w(x,y) > 0$ on one part of $\partial\Omega_1$, we have $w(x,y) \not\equiv 0$. Then, by the strong maximum principle (see [22], Theorem 9.6), yields $w(x,y) > 0$, that is,

$$u(x,y) < u(2a - x, y) \quad \text{in} \quad \Omega_1.$$

In particular, $u(x,y)$ computed at the left hand side of ∂D is less than $u(x',y)$ computed at the right hand side of ∂D.
If we denote $(\partial D)^r = \partial D \cap \{x > a\}$ and $(\partial D)^l = \partial D \cap \{x < a\}$ we have

$$\int_{\partial D} u(x,y)\cos(n,x)ds = \int_{(\partial D)^r} u(x,y)\cos(n,x)ds + \int_{(\partial D)^l} u(x,y)\cos(nx)ds.$$

By assumption $\cos(n,x)$ is non-negative at each point $(x,y) \in (\partial D)^r$. Clearly, we must have $\cos(n,x) > 0$ on a subset of positive measure. Moreover, if we take $(x,y) \in (\partial D)^l$ and $(x',y) \in (\partial D)^r$, the values of the corresponding $\cos(n,x)$ are opposite to each other. This fact, coupled with the information that the value of $u(x,y)$ computed at $(x,y) \in (\partial D)^l$ is less than the value of $u(x',y)$ computed at $(x',y) \in (\partial D)^r$ (for the same value of y) yields

$$\frac{dI_{D_\epsilon}}{d\epsilon}\bigg|_{\epsilon=0} = 2\int_{\partial D} u(x,y)\cos(n,x)ds > 0. \tag{4.39}$$

Therefore, the energy integral increases moving D in the x direction until the symmetry axis $\{x = a\}$ of D coincide with the y axis.
By symmetry, if $a > 0$, the energy integral increases moving D in the opposite direction of x until the symmetry axis $\{x = a\}$ of D coincide with the y axis. The theorem is proved. □

Note that if D is a disc, the conditions required by Theorem 4.6 for D are fulfilled. Therefore, we can solve the optimization problems in the following cases.

- Let Ω be a disc (larger than D) centered at the origin $(0,0)$. If $g(x,y) \geq 0$ is radially symmetric and non-decreasing with respect to the distance from (x,y) to $(0,0)$, the maximum of $I(D)$ is attained when D is concentric with Ω, and the minimum is attained when D is (internally) tangent to $\partial\Omega$.

- Let Ω be the rectangle $(-a_1, a_1) \times (-a_2, a_2)$. Let $g(x,y) \geq 0$ satisfy $g(x,y) = g(-x,y) = g(x,-y)$, and let $g(x,y)$ be non-increasing with respect to x for $x < 0$ and non-increasing with respect to y for $y < 0$. Then the maximum of $I(D)$ occurs when D is centered at $(0,0)$, and the minimum occurs when D is located at one corner of the rectangle (there are four locations for the minimum).

- Let Ω be a regular polygon of n sides centered at the origin. If $g(x,y) \geq 0$ is radially symmetric and non-decreasing with respect to the distance from (x,y) to $(0,0)$, the maximum of $I(D)$ is attained when D is concentric with Ω, and the minimum occurs when D is located at one corner (there are n locations for the minimum).

• By the proof of Theorem 4.6 we can find indications for the maximum even if Ω is not Steiner symmetric as in the following example. Let Ω be the union of the half disc $\{x^2+y^2 < 1,\ x \leq 0\}$ and the square $\{0 < x < 2,\ -1 < y < 1\}$. Note that Ω is Steiner symmetric with respect to the x axis, but it is not symmetric with respect to any line $\{x = a\}$. However, the reflection of the half disc $\{x^2+y^2 < 1,\ x \leq 0\}$ with respect to $\{x = 0\}$ is contained in Ω, and the reflection of the rectangle $\{1 \leq x < 2,\ -1 < y < 1\}$ with respect to the line $\{x = 1\}$ is contained in Ω. If $g(x, y)$ satisfy suitable conditions (for example if $g(x, y)$ is a constant) one can conclude that if $I(D)$ is maximum then the center of D is located on $(a, 0)$, where a has a suitable value such that $0 < a < 1$.

Now, we note that if Ω is a disc, it is Steiner symmetric with respect to any straight line passing through its center. Therefore, we can solve the maximization problems in the following cases.

• Let Ω be a disc centered at the origin $(0, 0)$, and let D be a rectangle with sides parallel to x, y axes. Let $g(x, y) \geq 0$ satisfy $g(x, y) = g(-x, y) = g(x, -y)$, and let $g(x, y)$ be non-increasing with respect to x for $x < 0$ and non-increasing with respect to y for $y < 0$. By applying Theorem 4.6 one finds that the maximum of $I(D)$ occurs when D is centered at $(0, 0)$.

• Let Ω be a disc centered at the origin $(0, 0)$, and let D be a rhombus with diagonals parallel to x, y axes. Let $g(x, y) \geq 0$ satisfy $g(x, y) = g(-x, y) = g(x, -y)$, and let $g(x, y)$ be non-increasing with respect to x for $x < 0$ and non-increasing with respect to y for $y < 0$. By applying Theorem 4.6 one finds that the maximum of $I(D)$ occurs when D is centered at $(0, 0)$.

• Let Ω be a disc centered at the origin $(0, 0)$, and let D be an ellipse with axes parallel to x, y axes. Let $g(x, y) \geq 0$ satisfy $g(x, y) = g(-x, y) = g(x, -y)$, and let $g(x, y)$ be non-increasing with respect to x for $x < 0$ and non-increasing with respect to y for $y < 0$. By applying Theorem 4.6 one finds that the maximum of $I(D)$ occurs when D is centered at $(0, 0)$.

• Let Ω be a disc centered at $(0, 0)$ and let D be a regular polygon of n sides. If $g(x, y) \geq 0$ is radially symmetric and non-decreasing with respect to the distance from (x, y) to $(0, 0)$, the maximum of $I(D)$ is attained when D is concentric with Ω.

Theorem 4.6 also holds for $N = 1$. Let us show that without any assumption on $g(x)$ the result may not hold. Consider the following example (corresponding to the case the segment D is centered at the origin)

$$-u'' + \lambda^2 \chi_{[-1,1]} u = \chi_{[-1,1]}, \quad u(-2) = u(2) = 0.$$

Here λ is any positive real number. Note that the function $g(x) = \lambda^2 \chi_{[-1,1]}(x)$ does not satisfy the monotonicity condition required by Theorem 4.6. The solution of this problem can be written as

$$u(x) = \begin{cases} A(e^{\lambda x} + e^{-\lambda x}) + \lambda^{-2}, & |x| \leq 1 \\ \\ B(|x| - 2) & 1 < |x| < 2, \end{cases}$$

where A and B satisfy

$$A(e^{\lambda} + e^{-\lambda}) + \lambda^{-2} = -B, \quad \lambda A(e^{\lambda} - e^{-\lambda}) = B.$$

We find

$$A\big(e^{\lambda} + e^{-\lambda} + \lambda(e^{\lambda} - e^{-\lambda})\big) + \lambda^{-2} = 0.$$

It is clear that $A < 0$. Hence,

$$I = \int_{-1}^{1} u(x)\ dx = \int_{-1}^{1} \big[A(e^{\lambda x} + e^{-\lambda x}) + \lambda^{-2}\big] dx < \frac{2}{\lambda^2}.$$

Now we consider the following problem (note that D is not centered at the origin)

$$-u'' + \lambda^2 \chi_{[-1,1]} u = \chi_{[0,2]}, \quad u(-2) = u(2) = 0.$$

Let us show that for $1 < x < 2$ we have $u(x) > v(x)$, where

$$-v'' = 1, \quad v(1) = v(2) = 0.$$

This fact follows from the familiar comparison principle in that, for $1 < x < 2$ we have

$$-u'' = 1, \quad u(1) > 0, \quad u(2) = 0.$$

Therefore,

$$J = \int_{0}^{2} u(x)\ dx > \int_{1}^{2} v(x)\ dx = \int_{1}^{2} \Big(-\frac{x^2}{2} + \frac{3}{2}x - 1\Big) dx = \frac{1}{12}.$$

As a consequence, for λ such that $\frac{2}{\lambda^2} \leq \frac{1}{12}$ the energy integral I is less than the energy integral J. Hence, the maximum of the energy integral does not correspond to the case D is centered at the origin.

Let us give a simple model described by problem (4.28) in case of $g(x, y) \equiv 0$. Let Ω be a plane thermal conductor subject to a source of heat with density $\chi_D(x, y)$. In other words, we have a stove which occupies D and produces heat with density one (in D only). Assume the temperature is equal to zero on the boundary $\partial\Omega$. Then, in a steady state situation, the solution $u(x,y)$ to problem (4.28) yields the temperature in Ω. The energy integral

$$I(D) = \int_D u_D(x,y)\, dxdy$$

is related with the average of the temperature in D, and it depends on the location of our stove in Ω. By our previous results, for particular domains we can find the exact location of D which maximizes $I(D)$. This result agrees with the physical intuition in that, for maximizing the energy integral, D has to be located as far as possible from the boundary $\partial\Omega$.

4.2.2 Placement of a well so as to optimize energy integral

Now we discuss a new problem which is, in some sense, complementary to the previous one. Using the previous notations, consider the boundary value problem

$$-\Delta u + \chi_D u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{4.40}$$

where $f = f(x,y) \geq 0$ is a bounded function, positive in a subset of positive measure. Physically $f(x,y)$ represents the density of the heat produced (source) and $\chi_D u$ represents the density of the heat absorbed (well). The associated energy integral is

$$I(D) = \int_\Omega f(x,y) u_D \, dxdy.$$

As in previous case D is assumed to have a piecewise C^1 boundary and to be Steiner symmetric. We shall investigate the minimum and the maximum of $I(D)$ for $D \subset \Omega$.

Let us investigate the effect of a translation of D. If D_ϵ is the domain D shifted at a distance ϵ in the x direction (assuming this is allowed), the new problem reads

$$-\Delta u_\epsilon + \chi_{D_\epsilon} u_\epsilon = f \quad \text{in} \quad \Omega, \quad u_\epsilon = 0 \quad \text{on} \quad \partial\Omega, \tag{4.41}$$

and the corresponding energy integral is

$$I(D_\epsilon) = \int_\Omega f(x,y) u_\epsilon \, dxdy.$$

So, we are interested in finding the sign of

$$I(D_\epsilon) - I(D) = \int_\Omega f(x,y)(u_\epsilon - u)\, dxdy$$

for a small $\epsilon > 0$. Here we have denoted with u_ϵ the solution to problem (4.41), and with u the solution to problem (4.40).
From equation (4.40) we find

$$\int_\Omega \big(\nabla u \cdot \nabla u_\epsilon + \chi_D u u_\epsilon\big) dxdy = \int_\Omega f(x,y) u_\epsilon \, dxdy.$$

Similarly, from equation (4.41) we find

$$\int_\Omega \big(\nabla u_\epsilon \cdot \nabla u + \chi_{D_\epsilon} u_\epsilon u\big) dxdy = \int_\Omega f(x,y) u \, dxdy.$$

It follows that

$$\int_\Omega (\chi_D - \chi_{D_\epsilon}) u_\epsilon u \, dxdy = \int_\Omega f(x,y)(u_\epsilon - u) \, dxdy = I(D_\epsilon) - I(D). \tag{4.42}$$

From equations (4.40) and (4.41) we also find

$$-\Delta(u_\epsilon - u) + \chi_D(u_\epsilon - u) = (\chi_D - \chi_{D_\epsilon}) u_\epsilon.$$

The Alexandrov-Bakelman-Pucci maximum principle (see [22], Theorem 9.1) applied to the latter equation gives

$$\sup_\Omega |u_\epsilon - u| \leq C \|(\chi_D - \chi_{D_\epsilon}) u_\epsilon\|_{L^2(\Omega)} \leq C \|\chi_D - \chi_{D_\epsilon}\|_{L^2(\Omega)} \sup_\Omega u_\epsilon. \tag{4.43}$$

The Alexandrov-Bakelman-Pucci maximum principle applied to (4.40) and (4.41) also yields

$$\sup_\Omega u \leq C \|f\|_{L^2(\Omega)} \quad \text{and} \quad \sup_\Omega u_\epsilon \leq C \|f\|_{L^2(\Omega)}. \tag{4.44}$$

Moreover, as already observed, it is easy to find a constant C such that

$$\int_\Omega |\chi_{D_\epsilon} - \chi_D| \, dxdy \leq C\epsilon \quad \text{and} \quad \|\chi_D - \chi_{D_\epsilon}\|_{L^2(\Omega)} \leq C\epsilon^{\frac{1}{2}}. \tag{4.45}$$

Hence, (4.43) implies

$$\sup_\Omega |u_\epsilon - u| \leq C\epsilon^{\frac{1}{2}}. \tag{4.46}$$

Since

$$\int_\Omega (\chi_D - \chi_{D_\epsilon}) u_\epsilon u \, dxdy = \int_\Omega (\chi_D - \chi_{D_\epsilon}) u^2 \, dxdy + \int_\Omega (\chi_D - \chi_{D_\epsilon})(u_\epsilon - u) u \, dxdy,$$

using (4.45) and (4.46) we find

$$\int_\Omega (\chi_D - \chi_{D_\epsilon}) u_\epsilon u \, dxdy = \int_\Omega (\chi_D - \chi_{D_\epsilon}) u^2 \, dxdy + o(\epsilon).$$

Now, by (4.42) and the latter equation we have

$$I(D_\epsilon) - I(D) = \int_\Omega (\chi_D - \chi_{D_\epsilon}) u^2 \, dxdy + o(\epsilon).$$

Finally, we get

$$\lim_{\epsilon\to 0}\frac{I(D_\epsilon)-I(D)}{\epsilon}=-\lim_{\epsilon\to 0}\int_\Omega\frac{\chi_{D_\epsilon}-\chi_D}{\epsilon}u^2\,dxdy$$
$$=-\lim_{\epsilon\to 0}\int_D\frac{u^2(x+\epsilon,y)-u^2(x,y)}{\epsilon}\,dxdy=-\int_D\frac{\partial u^2}{\partial x}\,dxdy.$$

Hence,

$$\frac{dI_{D_\epsilon}}{d\epsilon}\Big|_{\epsilon=0}=-\int_D\frac{\partial u^2}{\partial x}\,dxdy.$$

By using Green's formula we find

$$\frac{dI_{D_\epsilon}}{d\epsilon}\Big|_{\epsilon=0}=-\int_{\partial D}u^2\;\cos(n,x)ds. \tag{4.47}$$

Theorem 4.7. *Assume Ω to be Steiner symmetric with respect to the line $\{x=0\}$. The function $f(x,y)$ is assumed to satisfy $f(x,y)=f(-x,y)$ and to be non-decreasing with respect to x for $x<0$. Assume that $D\subset\Omega$ is Steiner symmetric with respect to some line $\{x=a\}$. Then the energy integral $I(D)$ associated with problem (4.40) decreases as the domain D approaches the line $\{x=0\}$.*

Proof. To find the sign of the derivative (4.47), we proceed as in the previous case. Assuming that $a<0$, the line $\{x=a\}$ divides Ω into Ω_1 (the small part) and Ω_2, and the reflection of Ω_1 with respect to the line $\{x=a\}$ is strictly contained in Ω_2. In Ω_1, define

$$w(x,y)=u(2a-x,y)-u(x,y).$$

We have $w=0$ for $x=a$ and $w(x,y)>0$ on the part of the boundary $\partial\Omega_1$ with $x<a$.
Since D is symmetric with respect to $\{x=a\}$ we have $\chi_D(x,y)=\chi_D(2a-x,y)$, and equation (4.40) at the point $(2a-x,y)$ reads

$$-\Delta u(2a-x,y)+\chi_D u(2a-x,y)=f(2a-x,y).$$

Subtracting the equation (4.40) from the latter equation we find

$$-\Delta w+\chi_D w=f(2a-x,y)-f(x,y). \tag{4.48}$$

We claim that for $x<a$ we have

$$f(2a-x,y)-f(x,y)\geq 0.$$

Indeed, if $2a-x\leq 0$ (recall that $2a-x>x$ in Ω_1) the inequality holds since $f(x,y)$ is non-decreasing for $x<0$. If $2a-x>0$, recalling that $f(-x,y)=f(x,y)$ we can rewrite the inequality as

$$f(x-2a,y)-f(-x,y)\geq 0,$$

and this holds since $x - 2a < -x$ and $f(x,y)$ is non-increasing for $x > 0$. The claim follows. Hence, (4.48) yields

$$-\Delta w + \chi_D w \geq 0 \quad \text{in} \quad \Omega_1. \tag{4.49}$$

Since $w(x,y) \geq 0$ on $\partial\Omega_1$, by the weak maximum principle we have $w(x,y) \geq 0$ in Ω_1. Since $w(x,y) > 0$ on one part of $\partial\Omega_1$, we have $w(x,y) \not\equiv 0$. Then, by strong maximum principle we have $w(x,y) > 0$ and, equivalently,

$$u(2a - x, y) > u(x,y) \quad \text{in} \quad \Omega_1.$$

Hence, $u(x,y)$ computed at the right hand side of ∂D is larger than $u(x',y)$ computed at the left hand side of ∂D (for the same value of y). Arguing as in the previous case we find

$$\left.\frac{dI_{D_\epsilon}}{d\epsilon}\right|_{\epsilon=0} = -\int_{\partial D} u^2(x,y) \ \cos(n,x)ds < 0.$$

Therefore, the energy integral decreases moving D in the x direction until the symmetry axis $\{x = a\}$ of D coincide with the y axis.
By symmetry, if $a > 0$, the energy integral decreases moving D in the opposite direction of x until the symmetry axis $\{x = a\}$ of D coincide with the y axis. The theorem is proved. □

Remarks. We underline that in this second problem $f(x,y)$ is non-decreasing with respect to x for $x < 0$, whereas, in the first problem, $g(x,y)$ was non-increasing.
The conclusions concerning the location of the maximum and the minimum are reversed.
Theorem 4.7 allows one to solve the optimization problems for the special domains Ω discussed in the previous case by changing the word maximum with minimum.

Theorem 4.7 can be easily extended to the case when χ_D is replaced by $\lambda^2\chi_D$ for any $\lambda > 0$. However, it fails to hold even in dimension $N = 1$ without appropriate assumptions on $f(x)$. Indeed, let

$$-u'' + \lambda^2\chi_{[-1,1]}u = \chi_{[0,2]}, \quad u(-2) = u(2) = 0.$$

As already noticed, the corresponding energy integral J satisfies

$$J(\lambda) = \int_0^2 u(x)\ dx > \frac{1}{12} \quad \text{for any} \quad \lambda > 0.$$

On the other hand, let

$$-u'' + \lambda^2\chi_{[0,2]}u = \chi_{[0,2]}, \quad u(-2) = u(2) = 0.$$

One finds

$$u(x) = \begin{cases} A(x+2) & \text{if } \; -2 < x < 0 \\ \\ Be^{\lambda x} + Ce^{-\lambda x} + \frac{1}{\lambda^2} & \text{if } \; 0 < x < 2, \end{cases}$$

with

$$2A = B + C + \frac{1}{\lambda^2}, \quad A = \lambda(B - C), \quad Be^{2\lambda} + Ce^{-2\lambda} + \frac{1}{\lambda^2} = 0.$$

By easy computation one finds

$$B = -\frac{(2\lambda+1)e^{2\lambda} - 1}{\lambda^2\big(2\lambda - 1 + e^{4\lambda}(2\lambda+1)\big)},$$
$$C = -\frac{(2\lambda-1)e^{2\lambda} + e^{4\lambda}}{\lambda^2\big(2\lambda - 1 + e^{4\lambda}(2\lambda+1)\big)}.$$

Since B and C have a negative sign, the corresponding energy integral satisfies

$$I(\lambda) = \int_0^2 \Big(Be^{\lambda x} + Ce^{-\lambda x} + \frac{1}{\lambda^2}\Big) dx < \frac{2}{\lambda^2}.$$

As a consequence, for λ such that $\frac{2}{\lambda^2} \leq \frac{1}{12}$ we have $I(\lambda) < J(\lambda)$. Therefore, the configuration corresponding to $D = [-1, 1]$ cannot be a minimum of the energy integral.

We have a physical interpretation also for problem (4.40). As before, let Ω be a plane thermal conductor subject to a source of heat with density $f(x, y) \equiv 1$. The term $\chi_D u$ in the left hand side of the equation corresponds to a device which absorbs heat (like any fire extinguisher) located in D. Assume the temperature is equal to zero on the boundary $\partial\Omega$ and that we are in a steady state situation. The solution $u(x, y)$ to problem (4.40) yields the temperature in Ω, and the energy integral

$$I(D) = \int_\Omega f(x, y) u_D(x, y)\, dxdy$$

is related to the average temperature in Ω, and it depends on the location of D. By our results, for special domains Ω we can find the exact location of D which minimizes $I(D)$. This result agrees with the physical intuition in that, for minimizing the energy integral (when $f(x, y) \equiv 1$), D has to be located as far as possible from the boundary $\partial\Omega$.

To complete this part, let us consider the boundary value problem

$$-\Delta u = -\chi_D \;\text{ in }\; \Omega, \quad u = T \;\text{ on }\; \partial\Omega. \tag{4.50}$$

Here T is a positive constant. Physically this problem describes the temperature in Ω in case there is device which produces cool at density one in D.

Multiplying (4.50) by u and integrating over Ω we find

$$-\int_\Omega u\Delta u \, dxdy = -\int_D u \, dxdy,$$

$$-\int_{\partial\Omega} u\frac{\partial u}{\partial n}\,ds + \int_{\Omega} |\nabla u|^2\,dxdy = -\int_D u\,dxdy,$$
$$\int_{\Omega} |\nabla u|^2\,dxdy = T\int_{\partial\Omega} \frac{\partial u}{\partial n}\,ds - \int_D u\,dxdy. \tag{4.51}$$

On the other hand, integrating (4.50) over Ω we find

$$-\int_{\Omega} \Delta u\,dxdy = -\int_D dxdy,$$
$$\int_{\partial\Omega} \frac{\partial u}{\partial n}\,ds = |D|.$$

Insertion of the latter equation in the (4.51) yields

$$I(D) = \int_{\Omega} |\nabla u|^2\,dxdy = T|D| - \int_D u\,dxdy.$$

To minimize the temperature u in Ω (which corresponds to maximize the cool) we must maximize the energy integral $I(D)$. To solve this problem, let us put $u = T - v$ in (4.50). We find

$$-\Delta v = \chi_D \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega.$$

Since,

$$I(D) = \int_{\Omega} |\nabla u|^2\,dxdy = \int_{\Omega} |\nabla v|^2\,dxdy,$$

we can apply Theorem 4.6.

4.2.3 The case of general rearrangements in a ball

We consider $\Omega = B$, a ball in $\mathbb{R}^N$. Define $\mathcal{F}$ as the class of rearrangements of a bounded non-negative function f_0, and $\mathcal{G}$ as the class of rearrangements of a bounded non-negative function g_0. For $f \in \mathcal{F}$ and $g \in \mathcal{G}$, let

$$-\Delta u + gu = f \text{ in } B, \quad u = 0 \text{ on } \partial B. \tag{4.52}$$

We know that this problem has a unique solution $u \in H_0^1(B)$. We note that the solution u is a maximizer of the functional

$$I(v) = \int_B (2fv - |\nabla v|^2 - gv^2)\,dx \quad \forall v \in H_0^1(B).$$

Indeed, if $j(t) = I(u + tv)$, from $j(t) \le j(0)$, $-1 < t < 1$ it follows that

$$\frac{dj(t)}{dt}\Big|_{t=0} = 0.$$

Since

$$j(t) = \int_B \left(2f(u + tv) - |\nabla(u + tv)|^2 - g(u + tv)^2\right) dx$$

we have

$$\frac{dj(t)}{dt}\Big|_{t=0} = 2\int_B \big(fv - \nabla u\cdot\nabla v - guv\big)\,dx = 0,$$

that is

$$\int_B \big(\nabla u\cdot\nabla v + guv\big)\,dx = \int_B fv\,dx \quad \forall v\in H_0^1(B),$$

which is the weak form of equation (4.52). Since problem (4.52) has a unique solution, the functional $I(v)$ has a unique maximizer u. Therefore,

$$\int_B fu\,dx = \int_B \big(2fu - |\nabla u|^2 - gu^2\big)dx = \sup_{v\in H_0^1(B)}\int_B \big(2fv - |\nabla v|^2 - gv^2\big)dx. \tag{4.53}$$

We are interested in the following optimization problems:

$$\max_{f\in\mathcal{F}}\int_B fu_f\,dx, \quad \min_{f\in\mathcal{F}}\int_B fu_f\,dx, \quad (g \text{ fixed}), \tag{4.54}$$

and

$$\max_{g\in\mathcal{G}}\int_B fu_g\,dx, \quad \min_{g\in\mathcal{G}}\int_B fu_g\,dx, \quad (f \text{ fixed}). \tag{4.55}$$

In case of $f_0 = \chi_D$, the class $\mathcal{F}$ is the family of all functions χ_E with $|E| = |D|$ (so, E is not necessarily a translation of D). A similar remark holds for $g_0 = \chi_D$.
Let f^* be the radially symmetric non-increasing rearrangement of f, and let f_* be the radially symmetric non-decreasing rearrangement of f. We shall use the following classical results.

$$\int_B f_* g^*\,dx \le \int_B fg\,dx \le \int_B f^* g^*\,dx. \tag{4.56}$$

If $u \ge 0$ and $u\in H_0^1(B)$ then $u^*\ge 0$ and $u^*\in H_0^1(B)$. Furthermore,

$$\int_B |\nabla u^*|^2 dx \le \int_B |\nabla u|^2 dx. \tag{4.57}$$

For a proof of (4.56) and (4.57) we refer to [26].

Let us consider the first problem in (4.54). If $g = g_*$ then

$$\begin{aligned}
&\int_B fu_f\,dx = \int_B (2fu_f - |\nabla u_f|^2 - gu_f^2)\,dx \ \ (\text{using (4.56) and (4.57)})\\
&\le \int_B \big(2f^*u_f^* - |\nabla u_f^*|^2 - g_*(u_f^*)^2\big)\,dx \ \ (\text{using (4.53)})\\
&\le \int_B \big(2f^*u_{f^*} - |\nabla u_{f^*}|^2 - gu_{f^*}^2\big)\,dx = \int_B f^*u_{f^*}\,dx.
\end{aligned}$$

Therefore, f^* is a maximizer for $\int_B fu_f\,dx$ in $\mathcal{F}$. If $f_0 = \chi_D$ then $f^* = \chi_{\hat{D}}$, where $\hat{D}$ is a ball concentric with B and $|\hat{D}| = |D|$. This result is in accordance with Theorem 4.6 concerning the maximum.

Let us show that the situation is different for the minimum (second problem of (4.54)). In addition to the condition $g = g^*$ we suppose that the solution $u(r)$ to the problem

$$-r^{1-N}(r^{N-1}u')' + gu = f_*, \quad u'(0) = u(R) = 0$$

satisfies $u'(r) \le 0$ in $(0, R)$. Then $u = u_{f_*} = u^*_{f_*}$ and

$$\begin{aligned}
\int_B f u_f \, dx &= \int_B (2fu_f - |\nabla u_f|^2 - gu_f^2) \, dx \quad \text{(using (4.53))} \\
&\ge \int_B \left(2fu_{f_*} - |\nabla u_{f_*}|^2 - gu_{f_*}^2\right) dx \ \ \text{(using (4.56)) and (4.57)} \\
&\ge \int_B \left(2f_*u_{f_*} - |\nabla u_{f_*}|^2 - gu_{f_*}^2\right) dx = \int_B f_* u_{f_*} \, dx.
\end{aligned}$$

Therefore, f_* is a minimizer for $\int_B fu_f \, dx$ in $\mathcal{F}$. If $f_0 = \chi_D$ then $f_* = \chi_{\check{D}}$, where $\check{D} = B \setminus E$. Here E is a ball concentric with B such that $|B \setminus E| = |D|$.

Let us consider the first problem in (4.55). If $f = f^*$ then

$$\begin{aligned}
\int_B f u_g \, dx &= \int_B (2fu_g - |\nabla u_g|^2 - gu_g^2) \, dx \ \ \text{(using (4.56) and (4.57))} \\
&\le \int_B \left(2f^*u_g^* - |\nabla u_g^*|^2 - g_*(u_g^*)^2\right) dx \ \ \text{(using (4.53))} \\
&\le \int_B \left(2f^*u_{g_*} - |\nabla u_{g_*}|^2 - g_*u_{g_*}^2\right) dx = \int_B f^* u_{g_*} \, dx.
\end{aligned}$$

Therefore, if $f = f^*$ we have

$$\int_B fu_g \, dx \le \int_B fu_{g_*} \, dx.$$

So, g_* corresponds to a maximum.

To investigate the minimum, in addition to the condition $f = f_*$ we suppose that the solution $u(r)$ to the problem

$$-r^{1-N}(r^{N-1}u')' + g^*u = f, \quad u'(0) = u(R) = 0$$

satisfies $u'(r) \le 0$ in $(0, R)$. Then $u = u_{g^*} = u^*_{g^*}$ and,

$$\begin{aligned}
\int_B f u_g \, dx &= \int_B (2fu_g - |\nabla u_g|^2 - gu_g^2) \, dx \quad \text{(using (4.53))} \\
&\ge \int_B \left(2fu_{g^*} - |\nabla u_{g^*}|^2 - gu_{g^*}^2\right) dx \ \ \text{(using (4.56) and (4.57))} \\
&\ge \int_B \left(2f_*u_{g^*} - |\nabla u_{g^*}|^2 - g^*u_{g^*}^2\right) dx = \int_B f_* u_{g^*} \, dx.
\end{aligned}$$

Therefore

$$\int_B fu_g \, dx \ge \int_B fu_{g^*} \, dx.$$

So, g^* corresponds to a minimum.

Chapter 5

Open problems and future works

- Let $1 \le \alpha < p$. For the solution $u = u_g$ of the boundary value problem

$$-\Delta_p u = g(x)u^{\alpha-1} \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega,$$

we have discussed optimization problems for the energy integral

$$I(g) = \int_\Omega g u_g^\alpha \, dx, \quad g \in \mathcal{G}.$$

One could discuss optimization problems for other functionals, for example,

$$I(g) = \int_\Omega f(x) u_g^\alpha \, dx, \quad f(x) \ge 0 \ \text{ fixed},$$

$$I(g) = \int_\Omega u_g^r \, dx, \quad 1 \le r,$$

$$I(g) = \sup_{x \in \Omega} u_g(x).$$

- Let $1 \le \alpha < q < p$. For the solution $u = u_g$ of the boundary value problem

$$-\Delta_p u - \Delta_q u = g(x)u^{\alpha-1} \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega,$$

we have discussed optimization problems for the energy integral

$$I(g) = \int_\Omega \Big(\frac{1}{\alpha} g u_g^\alpha - \frac{1}{p}|\nabla u|^p - \frac{1}{q}|\nabla u|^q \Big) \, dx, \quad g \in \mathcal{G}.$$

One could discuss optimization problems for the functional

$$I(g) = \int_\Omega g u_g^\alpha \, dx, \quad g \in \mathcal{G},$$

or other functionals.

- In our work we made the assumption $g(x) \geq 0$. One could consider all the previous problems for sign changing functions g.

- All the equations we have considered are of variational type. One could consider non-variational equations like

$$-\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} = g(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where the functions $a_{ij} = a_{ij}(x)$ are continuous and the matrix $[a_{ij}]$ is positive definite in Ω. If $u = u_g$ is the solution to this problem one could discuss optimization problems for the functional

$$I(g) = \int_{\Omega} g u_g \, dx, \quad g \in \mathcal{G},$$

or other functionals.

- In Chapter 4 of our work we have investigated optimization problems considering translation of rigid obstacles. We have found some results under strong geometric assumptions on the domain Ω. Can one prove similar results by relaxing such assumptions?

- In Chapter 4 the operator under consideration was linear. Can one find similar results in case of nonlinear operators like the p-Laplacian or the (p, q)-Laplacian?

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