



ADDIS ABABA UNIVERSITY
COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES
DEPARTMENT OF MATHEMATICS

**A GRADUATE THESIS ON POISSON EQUATION AND GREEN'S
FUNCTIONS IN 2D**

**(SUBMITTED IN PARTIAL FULFILLMENT OF THE M.Sc. DEGREE IN
MATHEMATICS)**

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The undersigned hereby certify that have read and recommended to the Graduate studies for acceptance a Thesis paper work entitled “**ON POISSON EQUATION AND GREEN’S FUNCTIONS IN 2D**” by Adefris Mamo in partial fulfillment of the requirements for the degree of **MSc** in Mathematics.

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Declaration

I hereby declare that the thesis work, titled “On Poisson Equation & Green’s Functions in 2D”, is an original piece of work carried out by me under the guidance of my advisor Dr.Tadesse Abdi at Addis Ababa University in the Department of Mathematics. This thesis has not been submitted previously for any degree or diploma at any other institution. All sources of information and references used in this thesis have been duly acknowledged.

Name _____Signature_____Date_____

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Abstract

This thesis explores the Poisson equation and its solutions through the application of Green's functions in two dimensional domains. The Poisson equation, a fundamental partial differential equation in mathematical physics, describes the potential field generated by a given charge distribution. We begin by deriving the Green's functions for the two dimensional Poisson equation emphasizing its role as a fundamental solution that encapsulates the boundary conditions & source terms. The thesis further investigates various methods for constructing Green's functions including integral transforms and separations of variables, and demonstrate their application to specific boundary conditions. Additionally, we discuss the implications of our findings in physical context such as electrostatics and heat conduction. The results highlight the versatility and power of Green's functions as a tool for solving linear differential equations, providing insights into theoretical aspects of the Poisson equation in two dimensions.

CHAPTER ONE

INTRODUCTION

The Poisson equation is a fundamental partial differential equation in mathematical physics, plays a crucial role in various fields such as electrostatics, mechanical engineering, and theoretical physics. It is expressed as the form:

$$\Delta u(x, y) = f(x, y)$$

where, Δ is the Laplace operator

$u(x, y)$ is the unknown function to be determined and

$f(x, y)$ is a given source function

The equation describes the potential field generated by a give charge or mass density distribution.

In two dimensions solving the Poisson equation can be particularly, challenging due to the complexity of boundary conditions and the nature of the domain .Traditionally, methods such as separation of variables or Fourier transforms, often fall short in providing efficient and general solutions for arbitrary domains and boundary conditions.

Green's functions offer a powerful and elegant method to tackle this problem. The British mathematician George Green developed what are known as Green's functions as part of his broader efforts to solve the partial differential equation by transforming into integral equation, and simplifying the process of finding the solutions, especially in contexts involving varying sources and boundary conditions. The Green's $G(x, y; \xi, \eta)$ for the two dimensional Poisson equation satisfies:

$$\Delta G(x, y; \xi, \eta) = \delta(x - \xi, y - \eta),$$

Where, δ is the Dirac delta function representing a point source at (ξ, η) .

The solution to the Poisson equation can then be expressed as:

$$u(x, y) = \int_D G(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta + \int_{\partial D} G(x, y; \xi, \eta) \frac{\partial u}{\partial n} ds$$

where, D is the domain of the problem and

∂D represents the boundary with boundary condition

This research delves into the interplay between the Poisson equation and Green's functions in two-dimensional spaces, on focusing theoretical aspects. By exploring these relationships, the

study aims to enhance the understanding of how Green's functions can be employed to solve the Poisson equation more effectively and to address various boundary value problems in two dimensions..

The significance of this research lies in its potential to enhance our understanding of partial differential equations and their solutions, offering insights that can be applied to a wide range of scientific and engineering problems .Through this study, we seek to contribute to the broader field of mathematical physics, providing tools and methodologies that can be utilized in both academic research and practical applications.

1.1 Notations & Symbols

Δ the Laplace operator or simply Laplacian

D the domain over which the integration is performed

∂D represents the boundary of the domain D

$\frac{\partial u}{\partial n}$ the derivative of u in the direction normal to the boundary ∂D

2D represents two dimensional spaces

\int_D is the integral over the domain D

$\int_{\partial D}$ the integral over the boundary domain

$\frac{\partial^2 u}{\partial x^2}$ the second partial derivative with respect to x

1.2 Definitions & concepts

- Partial differential equations (PDEs): A partial differential equation in two dimensions involves functions of two independent variables and their partial derivative with respect to these variables. Generally, a PDE in 2D can be written in the form:

$$F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}) = 0,$$

where $u = u(x, y)$ is the unknown function of the two independent variables x & y and the equations involves partial derivatives of u with respect to x and y .

- **Divergence:** $\text{div} u = \nabla \cdot u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$ in 2D
- **Laplace operator :** The Laplace operator is defined in 2D as $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, it measures the second spatial derivative of a function and is centered to the Poisson equation.
- **Domain D :** The spatial region in which the equation is defined and solved. It can be bounded or unbounded, and the shape and size of D affect the form of Green's function.
- **Boundary Conditions (BCs):** Conditions specified on the boundary of the domain.
- **Superposition Principle:** In the context of linear PDEs like the Poisson equation, the superposition principle states that the solution $u(x, y)$ can be constructed by integrating the product of the Green's function and the source term $f(x, y)$.
- **Eigenfunction Expansion:** A method of solving differential equation by expanding the solutions in terms of eigenfunctions of an operator.
- **Singularity:** A point where a function or an equation become undefined or behaves infinitely.
- **Fundamental Solution:** The Green's function for the Poisson equation in free space (without boundaries), often used as a building block for more complex solutions.
- **Smooth functions:** It is a type of function that is infinitely differentiable, meaning that you can take its derivative as many times as you want, and it will always be well-defined and continuous. In simpler terms, smooth functions are those that do not have sharp corners, breaks, or abrupt changes in slope.

1.3 Background and Motivation

1.3.1 Background of the study

The Poisson's equation is a fundamental second- order linear partial differential equation that has a general form of :

$$\Delta u = f ,$$

where, Δ is the Laplacian operator

u is the unknown function to be solved and

f is a given source term or forcing function

This equation is named after the French mathematician, Simeon Denis Poisson, who first derived it in 1812 while studying electromagnetic phenomena. The Poisson equation generalizes

Laplace's equation, when the source term $f(x, y)$ is zero. This generalization allows the Poisson equation to describe more complex physical situations where a source or sink potential exists, such as the distributions of electric charge in a plate or the distribution of heat in a thin plate.

Green's functions provide a powerful method for solving linear differential equation with boundary conditions, including the Poisson equation. The concept was introduced by George Green in the 1820s and its application to the Poisson equation has been the significant area of research. The Green's function $G(x, y; \xi, \eta)$ for the Poisson equation in two dimensions satisfies:

$$\Delta G(x, y; \xi, \eta) = \delta(x - \xi, y - \eta)$$

The solution to the Poisson equation can be then expressed as an integral involving the Green's function and the source term f :

$$u(x, y) = \iint_D G(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta$$

This formulation simplifies the solution of the Poisson equation, especially in domains with complex geometries or boundary conditions.

1.3.2 Analytical Solutions and Applications

Several analytical solutions for Poisson equation in 2D using Green's functions have been developed, particularly for simple geometries like infinite planes, disk, and rectangles. For example, in an infinite 2D plane, the Green's function has the form:

$$G(x, y; \xi, \eta) = -\frac{1}{2\pi} \ln \sqrt{(x - \xi)^2 + (y - \eta)^2}$$

This solution has been applied to problems in electrostatics, such as calculating the potential due to line charge, and in fluid dynamics, for modeling flow in porous media.

In domains with complex geometries, the Green's functions must satisfy boundary conditions, leading to more intricate solutions. Techniques methods such as methods of images, separation of methods, and conformal mapping have been employed to construct Green's functions for specific 2D problems.

1.3.3. Motivation

Analytical Insight: Using Green's functions to solve the Poisson equation provides deep analytical insight into the nature of the solution. It allows us to understand how the solution behaves in response to a different source terms and boundary conditions.

Versatility: Green's functions are versatile and can be adapted to various domains and boundary conditions. This makes them a valuable tool for solving partial differential equations in complex geometries where traditional methods might struggle.

Historical significance: The method of Green's functions has a rich history in mathematical physics and engineering. Understanding and applying this method connects modern research with classical techniques, providing a comprehensive view of the field.

Computational Efficiency: In numerical simulations, Green's functions can be used to develop efficient algorithms for solving partial differential equations. This is particularly useful in large scale simulations where computational resource and limiting factor.

Interdisciplinary Applications: The Poisson equation and Green's functions appear in various scientific and engineering disciplines. Research in this area can have broad applications, from designing better electrical circuits to improving models in fluid dynamics and beyond.

1.4 Objectives of the study:

- To derive the Green's function for the two dimensional Poisson equation
- To apply the derived Green's function to solve specific problems
- To analyze the effectiveness and importance of their method

CHAPTER TWO

MATHEMATICAL PRELIMINARIES

To understand the Poisson equation and Green's functions in two-dimensional spaces, it's important to review some foundational concepts. Here's the structured outline of the mathematical preliminaries.

Partial Differential Equations (PDEs): Understanding the basics of PDE theory, including classification of PDEs, boundary conditions, and solution techniques, is essential. Knowledge of concepts such as Laplace's equation, which is a special case of the Poisson equation including a source term.

Green's Functions Theory: Green's functions theory provides a powerful method for solving linear differential equation, including PDEs. Understanding the definition of Green's functions, their properties (such as symmetry, boundary conditions, and normalization), and their role in representing solutions to PDEs is fundamental. This theory also involves concepts from functional analysis of Green's functions in the context of PDEs.

Dirac delta Function: The Dirac delta function $\delta(x - \xi, y - \eta)$ is not a function in the traditional sense but a distribution that is zero everywhere except at $(x, y) = (\xi, \eta)$, where it is infinitely large, such that its integral over the entire space is 1.

Properties:

- $\int_{\mathbb{R}^2} \delta(x - \xi, y - \eta) dx dy = 1$
- For any smooth function $h(x, y)$,

$$\int_{\mathbb{R}^2} h(x, y) \delta(x - \xi, y - \eta) = h(\xi, \eta).$$

Vector Calculus: The Poisson equation in 2D involves vector calculus operations such as gradients, divergences, and Laplace. Proficiency in vector calculus helps for manipulating and transforming the Poisson equation into its integral form suitable for Green's functions solutions. Understanding vector fields, line integrals, surface integral, and the divergence theorem is essential for deriving and applying Green's function solution effectively.

Integral Equations: Green's function solutions often involve integral equations, where the solution at a point is expressed as an integral over a domain involving the Green's function and a

source term. Knowledge of integral equations, including methods for solving them analytically and numerically, is important for formulating Green's function solutions to the Poisson equation

Boundary Value Problems (BVPs): Solving Poisson equation using Green's functions typically involves boundary value problems, where the solution is sought subject to specified boundary conditions. Understanding concepts related to BVP including uniqueness and existence of solutions, boundary conditions classification (such as Dirichlet, Neumann, mixed) and methods for solving BVPs numerically (such as finite difference, finite element, or spectral methods) is essential for applying Green's function solutions effectively.

2.1 Poisson Equation in Two-Dimensions

Definition: The Poisson equation relates a function $u(x, y)$ to a source term $f(x, y)$ through the Laplace operator Δ which measures the rate at which value of u around a point differs from the value at the point itself. In 2Ds, the mathematical form of the Laplace equation is

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad (1)$$

Then the inhomogeneous form of the Laplace equation $\Delta u(x, y) = f(x, y)$ is called the **Poisson equation**.

2.1.1 Boundary Conditions and their Importance

Boundary conditions are essential in solving partial differential equations (PDEs) like the Poisson equation in 2D because they define the behavior of the solution on the boundary of the domain. Without boundary conditions, the solution to a PDE is not unique; boundary conditions are required to ensure that the solution is well-defined and physically meaningful.

Some of the different types of boundary conditions and their importance are the followings.

A. Dirichlet Boundary Conditions

Definition: Dirichlet boundary conditions specify the value of the solution $u(x, y)$ on the boundary of the domain. Its mathematical formula for the Dirichlet boundary condition for a domain D with boundary ∂D can be written as:

$$u(x, y) = g(x, y) \text{ on } \partial D$$

Example: If we have a rectangular domain, we might specify the potential $u(x, y)$ is zero on all sides of the rectangle.

The Dirichlet conditions ensure that the solution to the Poisson equation is unique. They provide specific values that the solution must take on the boundary, removing any ambiguity in the solution. And also, in physical problems, the Dirichlet conditions can represent fixed temperatures, or potentials on the boundary. They are used when the boundary values are known and fixed.

B. Neumann Boundary Conditions

Definition: Neumann boundary conditions specify the value of the derivative of the solution normal to the boundary. This often represents the flux or gradient of the solution.

Mathematically, the Neumann boundary conditions can be written as:

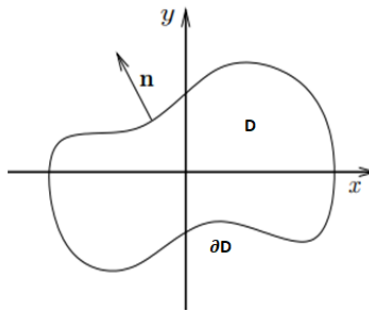
$$\frac{\partial u}{\partial n} = h(x, y) \text{ on } \partial D,$$

where, $\frac{\partial u}{\partial n}$ is the derivative in the direction normal to the boundary.

Example: For a heat conduction problem, this could represent the heat flux across the boundary. This condition is used when the solution $u(x, y)$ is known at the boundary. It is particularly useful in problems involving conservation laws or when the boundary condition describes how the solution changes across the boundary.

C. Mixed (Robin) Boundary Condition: This Condition is a linear combination of Dirichlet and Neumann conditions. Mathematically, it is expressed as;

$$au + b \frac{\partial u}{\partial n} = c \text{ on the boundary, where } a, b \text{ \& } c \text{ are constants}$$



This types of boundary condition is useful for modeling situations where both the value of the potential and its flux are relevant, such as in heat transfer problems where both temperature and heat flux might be specified.

Importance of Boundary Conditions

- Uniqueness and Existence: Boundary conditions ensure that the solution to the Poisson equation is unique and exists. Without appropriate boundary conditions, the problem may be ill-posed or may have infinitely many solutions.
- Physical Relevance: They provide physical constraints that model real-world scenarios, such as specifying the potential in electrostatic problems, temperature distribution in heat transfer, or fluid flow in various applications.
- Numerical Solutions: In computational methods, such as finite difference or finite element methods, boundary conditions are essential to correctly set up and solve the discrete version of the Poisson equation. They determined how the numerical domain intersects with the external constraints.

2.1.2 Green's Function for the Poisson Equation in 2D

Now we develop the Green's function approach to solve boundary value problems involving the two-dimensional Poisson equation. The Green's function is characterized as the solution to the homogenous boundary value problem in which the inhomogeneity is a concentrated unit impulse of a delta function. The solution to the general forced boundary value is then obtained via linear superposition, that is, as convolution integral with the Green's function.

2.2 The Green's Function

The Green's function is defined as the solution to the inhomogeneous differential equation when subject to a concentrated unit delta impulse at a prescribed point $(\xi, \eta) \in \mathbf{D}$ inside the domain. In this situation, the Poisson equation takes the form

$$\Delta u = -\delta_{(\xi, \eta)}, \text{ or, explicitly } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\delta(x - \xi, y - \eta).$$

The function $\mathbf{u}(\mathbf{x}, \mathbf{y})$ is also subject to some homogenous boundary conditions, e.g., the Dirichlet conditions $\mathbf{u} = \mathbf{0}$ on $\partial\mathbf{D}$. The resulting solution is called the Green's function for the boundary value problem, and written $G(\mathbf{x}, \mathbf{y}; \xi, \eta)$.

Once we know the Green's function, the solution to the general Poisson boundary value problem

$$\Delta u = f \text{ in } D \quad (2)$$

$$u = 0 \text{ on } \partial D$$

is reconstructed as follows. We regard the forcing function

$$\int \int_D \delta(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta \quad f(x, y) = f(x, y)$$

as a superposition of delta impulse, whose strength equals the value of f at the impulse point. Linearity implies that the solution to the boundary value problem is the corresponding superposition of Green's function responses to each of the constituent impulses. The net result is the fundamental superposition formula

$$u(x, y) = \int \int_D G(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta \quad (3)$$

for the solution to the boundary value problem. Indeed,

$$\begin{aligned} -\Delta u(x, y) &= \int \int_D -\Delta G(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta \\ &= \int \int_D \delta(x - \xi, y - \eta) f(\xi, \eta) d\xi d\eta = f(x, y), \end{aligned}$$

while the fact that $G(x, y; \xi, \eta) = 0$ for all $(x, y) \in \partial D$ implies that $u(x, y) = 0$ on ∂D .

2.2.1 Properties of Green's Functions

Green's functions are a powerful tool for solving differential equations, particularly boundary value problems. For the Poisson equation, Green's function $\mathbf{G}(\mathbf{x}, \mathbf{y}; \boldsymbol{\xi}, \boldsymbol{\eta})$ helps in expressing the solution in terms of an integral over a source term. Here are some key properties of Green's functions, followed by a detailed explanation of each.

Linearity: Green's function is linear with respect to the source term. That is, if L is a linear differential operator, and $G(x, y; \xi, \eta)$ is the Green's function associated with L , then for any linear combination of the solutions corresponding to each individual source term. This means that if you have two source terms $f_1(x, y)$ and $f_2(x, y)$, and their corresponding solutions

$u_1(x, y)$ and $u_2(x, y)$, then the solution for a linear combination $af_1(x, y) + bf_2(x, y)$ will be $au_1(x, y) + bu_2(x, y)$. This property follows from the linearity of the differential operator.

Symmetry (for self-adjoint operators): For a self-adjoint operator L , the Green's function is symmetric in its arguments. That means, if L is self-adjoint, then $L = L^*$, and the Green's function $G(x, y; \xi, \eta)$ satisfies $G(x, y; \xi, \eta) = G(\xi, \eta; x, y)$. This symmetry arises because the adjointness implies that the differential operator's kernel is symmetric, leading to symmetric Green's functions.

Boundary Conditions: Green's function satisfies the boundary conditions of the problem. That is, for a given boundary value problem, Green's function $G(x, y; \xi, \eta)$ is constructed to satisfy the same boundary conditions as the solution $u(x, y)$. For example, if the boundary conditions are Dirichlet (fixed value) or Neumann (fixed gradient) conditions, $G(x, y; \xi, \eta)$ will be chosen or constructed to ensure it satisfies these conditions on the boundary.

Singularity at Source Point: Green's function typically has a singularity at the source point. At the point where the source is located (i.e., $(x, y) = (\xi, \eta)$), Green's function $G(x, y; \xi, \eta)$ generally exhibits a singular behavior. For example, in 3D space, $G(\mathbf{x}, \mathbf{x}')$ for the Poisson equation typically has a $1/|\mathbf{x} - \mathbf{x}'|$ singularity. This singularity reflects the fact that the Green's function is essentially capturing the effect of a point source.

Integral representation: The solution to the differential equation can be expressed as an integral involving the Green's function. That means, for a linear operator L , if $Lu(x, y) = f(x, y)$ with appropriate boundary conditions, the solution $u(x, y)$ can be written as an integral over the green's function:

$$u(x, y) = \int_D G(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta + \int_{\partial D} G(x, y; \xi, \eta) \frac{\partial u}{\partial n} ds \quad (4)$$

where, $\frac{\partial u}{\partial n}$ is the normal derivative on the boundary. This integral representation is powerful, because it transforms the differential equations into an integral equation, which can be easier to solve in many cases.

Green's Second Identity: This identity is particularly useful in two dimensions. It states that for functions u & v that are twice continuously differentiable:

$$\int_D (u\Delta v - v\Delta u) dA = \int_{\partial D} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds \quad (5)$$

This identity helps in deriving the integral form of the solution and ensuring that the boundary conditions are properly incorporated.

Normalization: Green's function is normalized according to the differential operator and boundary conditions. The normalization of Green's function ensures that it correctly represents the influence of a unit source at (ξ, η) on the point (x, y) . The exact form of normalization depends on the specific differential operator and the problem's boundary conditions. For example, for the Poisson equation in a bounded domain, normalization ensures that Green's function satisfies the appropriate boundary conditions and contributes correctly to the solution.

Each of these properties contributes to the utility of Green's functions in solving boundary value problems by providing a structured and systemic approach to finding solutions.

2.2.2 Green's Function Derivation for Different Boundary Conditions

In the study of partial differential equations, particularly the Poisson equation, Green's functions play an important role in solving boundary value problems. Let's delve into how Green's functions are derived for the Poisson equation in 2D for different boundary conditions.

A. Dirichlet Boundary Condition

Problem: Solve $\Delta u = f$ in a domain D with $u = 0$ on ∂D .

For Dirichlet boundary conditions, the Green's function $G(x, y; \xi, \eta)$ should vanish on the boundary of D .

- For a rectangular domain $D = [0, a] \times [0, b]$, the Green's function is given by:

$$G(x, y; \xi, \eta) = \frac{1}{4\pi} \log \frac{1}{\sqrt{(x - \xi)^2 + (y - \eta)^2}}$$

And then adjusted to satisfy the boundary conditions using the method of images or separation of variables. For more complex domain, an integral representation or series solution may be used. The method of images involves reflecting the domain over its boundary to construct G such that it vanishes on the boundary.

B. Neumann Boundary Condition

Problem: Solve $\Delta u = f$ in a domain D with $\frac{\partial u}{\partial n} = 0$ on ∂D , where n is outward normal to the boundary.

For Neumann boundary condition, the Green's function $G(x, y; \xi, \eta)$ should satisfy:

$$\frac{\partial G}{\partial n} = 0 \text{ on } \partial D$$

For Circular domain of radius R , the Green's function can be derived using separation of variables and Fourier series. The form might include terms involving Bessel functions and series expansions.

For a rectangular domain, the method of images is modified to insure that the normal derivative of G is zero on the boundary. This often involves solving a boundary value problem using Fourier series or eigenfunction expansions.

C. Mixed Boundary Condition

Problem: Solve $\Delta u = f$ with a combination of Dirichlet and Neumann conditions on different parts of ∂D .

In this case, the Green's function needs to be constructed to satisfy the specific boundary conditions on different parts of the boundary.

For a domain with mixed boundary conditions, the Green's function is usually derived by solving a more complex boundary value problem. Combining methods for Dirichlet and Neumann conditions appropriately.

Techniques such as the method of images or integral equations are adapted to meet the specific conditions on each segment of the boundary.

In general, Green's functions for the Poisson equation in 2D are tailored to meet the specific boundary conditions of the problem. The Dirichlet problem often uses images methods to ensure the function vanishes on the boundary, while the Neumann problem involves adjusting the function to satisfy zero normal derivative conditions. Mixed boundary conditions require a combination methods or more intricate boundary problem solutions. Each derivation depends on

the geometry of the domain and the nature of the boundary conditions, and various mathematical techniques are employed to construct the appropriate Green's functions.

2.2.3 General procedures for Derivation

Identify the boundary Conditions: Determine if they are Dirichlet, Neumann, or mixed.

Choose an appropriate Method: Use methods like separation of variables, Fourier series, or method of images based on boundary conditions.

Solve the PDE: Ensure that the solution satisfies the boundary conditions and the source item given by the Dirac delta function.

Iv. Verify Solution: Check the Green's function satisfies the original PDE with the appropriate boundary condition.

By applying the specific boundary conditions, we can simplify the integrals accordingly.

CHAPTER THREE

DERIVATION OF GREEN'S FUNCTIONS FOR TWO DIMENSIONAL POISSON EQUATION

3.1 The Fundamental Solution

In two dimensions, the fundamental solution of the Laplace equation which has a special case of the Poisson equation with $\Delta u(x, y) = 0$ can be used to derive the Green's function for the Poisson equation in free space.

3.1.1 Derivation of the Fundamental Solution for the Poisson Equation in 2D

Derive the fundamental solution (Green's) function for Laplace equation in free space (without boundaries) in 2D.

The Laplace equation in 2D is:

$$\Delta u(x, y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Problem setup: To derive Green's function $G(x, y; \xi, \eta)$ for the Laplace equation we need to find:

$$\Delta G(x, y; \xi, \eta) = \delta(x - \xi, y - \eta) \quad (6)$$

$$\frac{\partial^2 G}{\partial r^2} + \frac{\partial^2 G}{\partial r^2} = \delta(x - \xi, y - \eta)$$

Radial Symmetry

For simplicity, we use the fact that the Poisson equation is symmetric in x and y . Hence, it is often convenient to use polar coordinates centered at (ξ, η) , where r is the distance from the point (ξ, η) :

$$r^2 = (x - \xi)^2 + (y - \eta)^2$$

In polar coordinates, the Laplace Δ of G becomes:

$$\Delta G = \frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} + \frac{1}{r^2} \frac{\partial^2 G}{\partial \theta^2} \quad (7)$$

Solving the Radial Equation

Since G depend on r (due to radial symmetry) the equation becomes:

$$\Delta G = \frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} \quad (8)$$

The PDE simplifies to ODE as the form:

$$\frac{d^2 G}{dr^2} + \frac{1}{r} \frac{dG}{dr} = \delta(r) \quad (9)$$

To find $G(r)$, consider the behavior in different regions of r . For $r \neq 0$, the right hand side of the equation is zero, so,

$$\frac{d^2 G}{dr^2} + \frac{1}{r} \frac{dG}{dr} = 0 \quad (10)$$

since $\delta(r) = 0$ everywhere except $(x, y) = (\xi, \eta)$ in the distribution sense.

This simplifies (10) to:

$$\frac{d}{dr} \left(r \frac{dG}{dr} \right) = 0 \quad (11)$$

Integrate (11) once:

$$r \frac{dG}{dr} = C, \text{ where } C \text{ is a constant.}$$

Thus:

$$\frac{dG}{dr} = \frac{C}{r}$$

Integrating again:

$$G(r) = C \ln r + B, \text{ where } B \text{ is another constant}$$

For $r \rightarrow 0$, to ensure that $G(r)$ remains finite at $r = 0$, we must have $C = 0$, because $\ln r$ diverges as $r \rightarrow 0$.

Hence:

$$G(r) = B$$

To find B , we use the fact that the Laplacian of G must produce a delta function introduces a discontinuity in the derivative of G .

Specifically:

$$\left. \frac{\partial G}{\partial r} \right|_{r=0^+} - \left. \frac{\partial G}{\partial r} \right|_{r=0} = 1 \quad (12)$$

Given the solution

$$G(r) = -\frac{1}{2\pi} \ln r, \text{ this matches the condition because the Laplacian of } G(r)$$

results in a delta function.

Thus, the Green's function for the 2D Poisson equation is:

$$G(x, y; \xi, \eta) = -\frac{1}{2\pi} \ln \sqrt{(x - \xi)^2 + (y - \eta)^2} \quad (13)$$

The Green's function $G(x, y; \xi, \eta)$ serves as the fundamental solution to the 2D Poisson equation, providing a way to solve the equation with various source terms $f(x, y)$. By employing the radial symmetry and solving the Laplacian in polar coordinates, we derived that G is proportional to the logarithm of the distance between the points.

3.1.2 Green's Function for Dirichlet Problem in the plane

Consider the Dirichlet Problem for the Poisson equation

$$\Delta u(x, y) = f(x, y), \quad (x, y) \in D, \quad (14)$$

$$u(x, y) = g(x, y) \text{ on } \partial D$$

The Green's function $G(x, y; \xi, \eta)$ for the Poisson equation satisfies:

$$\Delta G(x, y; \xi, \eta) = \delta(x - \xi, y - \eta) \quad \text{in } D \quad (15)$$

$$G(x, y; \xi, \eta) = 0 \text{ on } \partial D.$$

Apply Green's 2nd identity to the functions $u(x, y)$ and $G(x, y)$:

$$\int_D (u \Delta G - G \Delta u) dA = \int_{\partial D} \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) ds \quad (16)$$

Substitute $\Delta u = f$ & $\Delta G = \delta$ in (34):

$$\int_D (u \delta(x - \xi, y - \eta) - G f) dA = \int_{\partial D} \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) ds$$

Simplifying the LHS term involving the delta function to $u(\xi, \eta)$:

$$u(\xi, \eta) - \int_D G(x, y; \xi, \eta) f(x, y) dA = \int_{\partial D} \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) ds \quad (17)$$

Since $G = 0$ on ∂D the boundary term involving G vanishes:

$$u(\xi, \eta) = \int_D G(x, y; \xi, \eta) f(x, y) dA + \int_{\partial D} \left(u \frac{\partial G}{\partial n} \right) ds \quad (18)$$

The **representation formula** for u in terms of the Green's function is:

$$u(x, y) = \int_D G(x, y; \xi, \eta) f(x, y) dA + \int_{\partial D} \left(g(x, y) \frac{\partial G}{\partial n} \right) ds \quad (19)$$

This formula expresses the solution u in terms of the Green's function G , the source term f , the boundary data g , The Green's function u encapsulates the influence of the BCs the geometry of the domain.

3.1.3 Mathematical interpretation

The derivation of Green's functions for the 2-dimensional Poisson equation can be interpreted as finding a fundamental solution that represents the effect of a point source on the potential in a given domain.

1. In fundamental solution:

Singular Behavior: The Green's function $\mathbf{G}(\mathbf{x}, \mathbf{y}; \xi, \eta)$ serves as the fundamental solution to the Laplace equation in 2D (when $f = 0$). This means it characterizes the response of the system to a point source.

Singularity at Source: At the source location (ξ, η) , the Green's function is singular, reflecting the infinite potential at the exact location of the point source.

2. Superposition Principle:

Response to Source Distribution: For a general source term $f(\mathbf{x}, \mathbf{y})$, the solution to the Poisson equation can be expressed as a superposition of the effect of all point sources distributed according to $f(\mathbf{x}, \mathbf{y})$. Specifically, the solution is given by:

$$u(x, y) = \int \int_D G(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta$$

3. Boundary Conditions:

Incorporation into Solutions: The Green's function helps incorporate boundary conditions by modifying the fundamental solution appropriately. For example, for Dirichlet boundary conditions, we can adjust the Green's function to satisfy these conditions at the boundary of the domain.

4. Example of 2D Green's Function

In 2D, the Green's function for the Poisson equation is:

$$G(x, y; \xi, \eta) = -\frac{1}{2\pi} \ln \sqrt{(x - \xi)^2 + (y - \eta)^2}.$$

This function reflects the logarithmic potential due to a point source in a plane, highlighting the nature of potential fields in two dimensions.

Example: If we have a unit source (0, 0), the Green's function at a point (x, y) is:

$$G(x, y; 0, 0) = \frac{1}{2\pi} \ln \sqrt{x^2 + y^2}$$

This function describe how the potential at (x, y) is influenced by a unit source at the origin.

The solution $u(x, y)$ is obtained by integrating the product of the Green's function and the source function $f(\xi, \eta)$ over the domain plus boundary conditions.

3.2 Green's Function to Specific Domains

3.2.1 Derivation for Rectangular and Circular Domains

Rectangular Domain

Consider the 2-dimensional Poisson equation

$$\Delta u(x, y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -f(x, y). \quad (20)$$

This equation arises in equilibrium problems, such as the static deflection of a rectangular membrane. In that case, $f(x, y)$ represents the external load per unit area, divided by the tension in the membrane. The solution $u(x, y)$ must satisfy certain boundary conditions. For present, let's choose $u(0, y) = u(a, y) = 0$ and $u(x, 0) = u(x, b) = 0$,

To find the Green's function for (5.2.1) we must solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\delta(x - \xi, y - \eta), 0 < x, \xi < a, 0 < y, \eta < b \quad (21)$$

Subject to the boundary conditions

$$G(0, y; \xi, \eta) = G(a, y; \xi, \eta) = G(x, 0; \xi, \eta) = G(x, b; \xi, \eta) = 0 \quad (22)$$

$$u(x, y) = \int_0^a \int_0^b G(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta \quad (23)$$

One approach to finding the Green's function to expand it in terms of eigenfunctions $\psi(x, y)$

of the differential equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\mu \psi \quad (24)$$

and the boundary conditions (21). The eigenvalues are

$$\mu_{nm} = \frac{n^2 \pi^2}{a^2} + \frac{m^2 \pi^2}{b^2} \quad (25)$$

When $n = 1, 2, \dots$, $m = 1, 2, \dots$ and the corresponding eigenfunctions are

$$\psi_{nm}(x, y) = \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \quad (26)$$

Therefore, we seek $\mathbf{G}(x, y; \xi, \eta)$ in the form

$$G(x, y; \xi, \eta) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \quad (27)$$

Because the delta function can be written

$$\delta(x - \xi) \delta(y - \eta) = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi \xi}{a}\right) \sin\left(\frac{m\pi \eta}{b}\right), \quad (28)$$

We find that

$$\left(\frac{n^2 \pi^2}{a^2} + \frac{m^2 \pi^2}{b^2}\right) A_{nm} = \frac{4}{ab} \sin\left(\frac{n\pi \xi}{a}\right) \sin\left(\frac{m\pi \eta}{b}\right) \quad (29)$$

After substituting (28) and (29) into the PDE (23) and setting the corresponding harmonics equal to each other.

Therefore, the bilinear formula for Green's function of Poisson's equation is

$$G(x, y; \xi, \eta) = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi \xi}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{m\pi \eta}{b}\right)}{n^2 \pi^2 / a^2 + m^2 \pi^2 / b^2} \quad (30)$$

Thus, the solution to the Poisson equation can be now written as

$$u(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_{nm}}{n^2 \pi^2 / a^2 + m^2 \pi^2 / b^2} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right), \quad (31)$$

where,

$$a_{nm} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) dx dy$$

The derivation of the Green's function with Neumann boundary conditions can be obtained by considering each direction separately. To satisfy the boundary conditions along the edges $y = 0$ and $y = b$, we write the Green function as the Fourier series

$$G(x, y; \xi, \eta) = \sum_{m=1}^{\infty} G_m(x; \xi) \sin\left(\frac{m\pi y}{b}\right) \quad (32)$$

where the coefficient $G_m(x; \xi)$ left as undermined functions of x, ξ , and η . Substituting this series into the PDE for G , multiplying by $2\sin(n\pi y/b)/b$, and integrating over y , we find that

$$\frac{d^2 G_n}{dx^2} - \frac{n^2 \pi^2}{b^2} G_n = -\frac{2}{b} \sin\left(\frac{n\pi y}{b}\right) \delta(x - \xi). \quad (33)$$

The DE shows that the expansion coefficients $G_n(x; \xi)$ are one-dimensional Green's functions; we can find by piecing together homogeneous solutions to (31) that are valid over various intervals. For the region $0 \leq x \leq \xi$ the solution to (5.2.15) vanishes at $x = 0$ is

$$G_n(x; \xi) = A_n \sinh\left(\frac{n\pi x}{b}\right), \quad (34)$$

where A_n is presently arbitrary. The corresponding solution for $\xi \leq x \leq a$ is

$$G_n(x; \xi) = B_n \sinh\left[\frac{n\pi(a-x)}{b}\right] \quad (35)$$

Note that this solution vanishes that at $x = a$. because the Green's function must be continuous at $x = \xi$,

$$A_n \sinh\left(\frac{n\pi \xi}{b}\right) = B_n \sinh\left[\frac{n\pi(a-\xi)}{b}\right] \quad (36)$$

On the other hand, the appropriate jump discontinuity of $G'_n(x; \xi)$ yields

$$-\frac{n\pi}{b} B_n \cosh\left[\frac{n\pi(a-\xi)}{b}\right] - \frac{n\pi}{b} A_n \cosh\left(\frac{n\pi \xi}{b}\right) = -\frac{2}{b} \sin\left(\frac{n\pi \eta}{b}\right), \quad (37)$$

Solving for A_n and B_n ,

$$A_n = \frac{2}{n\pi} \sin\left(\frac{n\pi \eta}{b}\right) \frac{\sinh[\pi(a-\xi)/b]}{\sinh(n\pi a/b)} \quad (38)$$

and

$$B_n = \frac{2}{n\pi} \sin\left(\frac{n\pi \eta}{b}\right) \frac{\sinh(n\pi \xi/b)}{\sinh(n\pi a/b)} \quad (39)$$

This yields the Green's function

$$G(x, y; \xi, \eta) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sinh[n\pi(a-x>)/b] \sinh(n\pi x</b)}{n \sinh(n\pi a/b)} \times \sin\left(\frac{n\pi \eta}{b}\right) \sin\left(\frac{n\pi y}{b}\right) \quad (40)$$

If we began with a Fourier expansion in the y direction, we would have obtained

$$G(x, y, \xi, \eta) = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\sinh[m\pi(b-y>)/a] \sinh(m\pi y</a)}{m \sinh(m\pi b/a)} \times \sin\left(\frac{m\pi\xi}{a}\right) \sin\left(\frac{m\pi x}{a}\right). \quad (41)$$

Circular Domain

Derive fundamental solution (Green's function) for Poisson equation in circular domain.

Derivation Steps: For a circular domain of radius R , the Poisson equation is:

$$\Delta u(r, \theta) = f(r, \theta) \quad (42)$$

with boundary conditions $u = 0$ or $x^2 + y^2 = R^2$ on the boundary $r = R$.

Transform the problem into polar coordinates (r, θ) , where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

The Laplacian in polar coordinates is:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \quad (43)$$

The Green's function $G(r, \theta; \rho, \phi)$ satisfies:

$$\Delta G(r, \theta; \rho, \phi) = \delta(r - \rho, \theta - \phi)$$

Substitute (46) into $\Delta G(r, \theta; \rho, \phi)$, we get

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}\right) G(r, \theta; \rho, \phi) = \frac{1}{r} \delta(r - \rho, \theta - \phi) \quad (44)$$

By separations of variables, and assuming that a solution has of the form:

$$G(r, \theta; \rho, \phi) = \sum_{n=-\infty}^{\infty} g_n(r, \theta) e^{in(\theta - \phi)} \quad (45)$$

Substitute this into the Poisson equation and separate variables to get:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2}\right) g_n(r, \rho) = \frac{1}{r} \delta(r - \rho)$$

Solving the radial equation using Bessel function J_n and Y_n :

$$g_n(r, \rho) = \begin{cases} A_n J_n(kr), & \text{for } r < \rho \\ B_n Y_n(kr), & \text{for } r > \rho \end{cases} \quad (46)$$

Apply boundary conditions at $r = \rho$ to determine A_n & B_n and Construct the Green's function.

Then, combine the solution in (48) & (49) to construct the full Green's function to get:

$$G(r, \theta; \rho, \phi) = \sum_{n=-\infty}^{\infty} [A_n J_n(kr) H(\rho - r) + B_n Y_n(kr) H(r - \rho)] e^{in(\theta - \phi)} \quad (47)$$

where H is the Heaviside Step function

Then, the final form of the Green's function for the circular domain is :

$$G(r, \theta; \rho, \phi) = \frac{1}{2\pi} \ln \left(\frac{R^2 - r\rho \cos(\theta - \phi)}{r^2 + \rho^2 - 2r\rho \cos(\theta - \phi)} \right) \quad (48)$$

This Green's function satisfies the BCs & provides the fundamental solution to the Poisson equation in a circular domain.

Thus, the solution to the Poisson equation is :

$$u(r, \theta) = \int_0^R \int_0^{2\pi} G(r, \theta; \rho, \phi) f(\rho, \phi) d\rho d\phi \quad (49)$$

3.2.2 Illustrative Examples with Specific Functions & Boundary Conditions

Constant Source Term with Dirichet BCs

1. Solve the Poisson equation in a unit square domain $0 \leq x, y \leq 1$ with $f(x, y) = 1$ & Dirichlet BCs $u(x, y) = 0$ on ∂D .

Solution: The Poisson equation in 2D is:

$$\Delta u(x, y) = f(x, y) \text{ in } D$$

For a unit square with $f(x, y) = 1$. The Green's function $G(x, y; \xi, \eta)$ for the 2D Poisson equation satisfies

$$\Delta G(x, y; \xi, \eta) = \delta(x - \xi, y - \eta) \text{ in } D$$

$$G(x, y; \xi, \eta) = 0 \text{ on } \partial D \quad (50)$$

The Green's function for the unit square with Dirichlet BCs is:

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \sqrt{(x - \xi)^2 + (y - \eta)^2} + H(x, y; \xi, \eta), \quad (51)$$

where $H(x, y; \xi, \eta)$ is a harmonic function that ensures the BCs are satisfied.

Since $u(x, y) = 0$ on ∂D , we need $G(x, y; \xi, \eta) = 0$ on ∂D . This implies that $H(x, y; \xi, \eta)$ must cancel out the logarithmic singularity on the boundary.

Then, using Green's 2nd identity $\int_D u \Delta G - G \Delta u dA = \int_{\partial D} (u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n}) ds$ and substitute the BCS into Green's 2nd identity, we get the solution $u(x, y)$

$$\int_D u \delta(x - \xi, y - \eta) G f d\xi d\eta = 0, \quad (52)$$

Since $G = 0, u = 0$ on ∂D , the boundary integrals vanishes & the solution becomes

$$\begin{aligned} u(\xi, \eta) - \int_D G(x, y; \xi, \eta) f d\xi d\eta &= 0 \\ u(x, y) &= \int_D G(x, y; \xi, \eta) f d\xi d\eta \end{aligned} \quad (53)$$

Substituting $f(x, y) = 1$ & the Green's function into the integral equation (60):

$$u(x, y) = \int_0^1 \int_0^1 \left(-\frac{1}{2\pi} \ln(\sqrt{(x-\xi)^2 + (y-\eta)^2}) + H(x, y; \xi, \eta) \right) d\xi d\eta \quad (54)$$

The harmonic function $H(x, y; \xi, \eta)$ is chosen such that the BCs are satisfied, but its exact form can be complex to derive. For simplicity, we focus on the logarithmic term.

This integral can be evaluated numerically or symmetry arguments to simplify the calculation.

The result will give the potential $u(x, y)$ inside the unit square.

2. Solve the Poisson equation in a circular domain of radius R with a point source at the center

$f(x, y) = \delta(x, y)$ and Neumann BCs $\frac{\partial u}{\partial n} = 0$ on ∂D .

Solution: The Green's function for a circular domain with Neumann BCs is:

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \left(\frac{R}{(x-\xi)^2 + (y-\eta)^2} \right) \quad (55)$$

Using Green's 2nd identity & Neumann BCs $\frac{\partial u}{\partial n} = 0$ on ∂D & $G = 0$

$$\begin{aligned} \int_D (u\Delta G - G\Delta u) dA &= \int_{\partial D} \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) ds \\ \int_D (u\delta(x-\xi, y-\eta) - Gf) dA &= \int_{\partial D} \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) ds \end{aligned} \quad (56)$$

The RHS of the boundary integral vanishes,

$$\begin{aligned} \int_D (u\delta(x-\xi, y-\eta) - Gf) dA &= 0 \\ \int_D u\delta(x-\xi, y-\eta) dA &= \int_D Gf dA \\ u(\xi, \eta) &= \int_D Gf dA \\ u(x, y) &= \int_D G(x, y; \xi, \eta) \delta(\xi, \eta) d\xi d\eta \end{aligned} \quad (57)$$

Since $f(x, y) = \delta(x, y)$ is delta function the integral simplifies to:

$$u(x, y) = G(x, y; 0, 0) = \frac{1}{2\pi} \ln \left(\frac{R}{\sqrt{x^2 + y^2}} \right) \quad (58)$$

These examples illustrate the Process of solving the Poisson equation using Green's function for different sources & boundary conditions.

CHAPTER FOUR

APPLICATION OF GREEN'S FUNCTIONS

Green's functions are powerful tools used to solve inhomogeneous differential equations. They are particularly useful in solving boundary value problems. In this context, we will explore how Green's functions can be applied to solve the Poisson equation in two dimensions.

4.1 Green's Function For 2D the Poisson Equation

In this section we consider the two dimensional Poisson equation of Dirichlet boundary conditions with the problem

$$\begin{aligned} \Delta u &= f, \text{ in } D \\ u &= g, \text{ on } C \end{aligned} \quad (59)$$

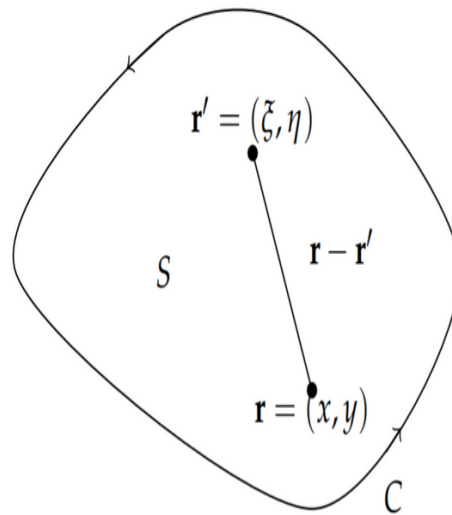


Figure 4.1(a) Domain for solving Poisson equation

To solve this problem using Green's function $G(x, y; \xi, \eta)$, the Green's function satisfies the differential equation and homogenous boundary conditions that associated with the problem is given by:

$$\Delta G(x, y; \xi, \eta) = \delta(x - \xi, y - \eta) \text{ in } D \quad (60)$$

$$G(x, y; \xi, \eta) = 0 \text{ on } \partial D$$

Assuming that the Green's function is symmetric in its arguments. However, this is not always the case and depends on things such as the self adjointness of the problem. Thus, we will assume that the Green's function satisfies

$$\Delta_r G = \delta(\xi - x, \eta - y),$$

where the notation ∇_r , means differentiation w.r.t. the variables ξ & η .

Thus,

$$\Delta G = \frac{\partial^2 G}{\partial \xi^2} + \frac{\partial^2 G}{\partial \eta^2}$$

Now we apply Green's second identity for two dimensions. We have

$$\int_D (u \Delta_r G - G \Delta_r u) dA' = \int_{\partial D} \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) dS' \quad (61)$$

Inserting the differential equations the LHS of the equation becomes

$$\begin{aligned} \int_D (u \Delta_r G - G \Delta_r u) dA' &= \int_D (u(\xi, \eta) \delta(\xi - x, \eta - y) - G(x, y; \xi, \eta) f(\xi, \eta)) d\xi d\eta \\ &= u(x, y) - \int_D G(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta \end{aligned} \quad (62)$$

Using the boundary conditions,

$$u(\xi, \eta) = g(\xi, \eta) \text{ on } \partial D \text{ and } G(x, y; \xi, \eta) = 0 \text{ on } \partial D$$

The RHS of the equation becomes

$$\int_{\partial D} \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) dS' = \int_{\partial D} g(\xi, \eta) \frac{\partial G}{\partial n} dS' \quad (63)$$

Solving for $u(x, y)$, we have the solution written in terms of the Green's function,

$$u(x, y) = \int_D G(x, y; \xi, \eta) f(\xi, \eta) + \int_{\partial D} g(\xi, \eta) \frac{\partial G}{\partial n} dS'$$

Example

Solve the Dirichlet problem for Laplace's equation in a disc of radius a ,

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \text{ in } r < a \text{ with } u = f(\theta) \text{ on } r = a. \quad (64)$$

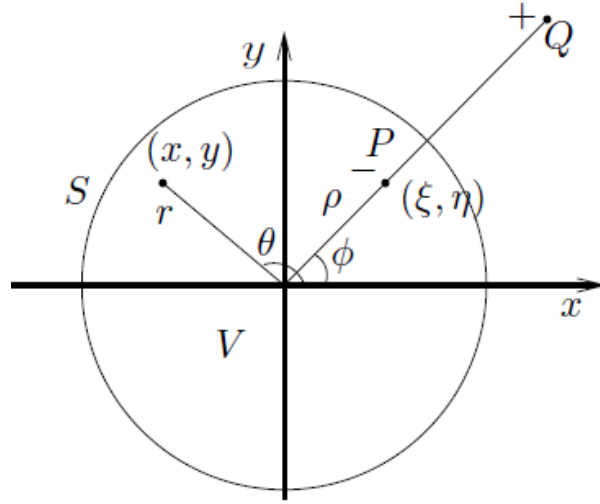


Figure 4.1(b)

Consider image of point P at inverse point Q

$$P = (\rho \cos \Phi, \rho \sin \Phi),$$

$$Q = (q \cos \Phi, q \sin \Phi),$$

With $pq = a^2$ (i.e. $OP \cdot OQ = a^2$).

$$G(x, y, \xi, \eta) = -\frac{1}{4\pi} \ln((x - \xi)^2 + (y - \eta)^2) \quad (65)$$

$$+ \frac{1}{4\pi} \ln\left(\left(x - \frac{a^2}{\rho} \cos \Phi\right)^2 + \left(y - \frac{a^2}{\rho} \sin \Phi\right)^2\right) + h(x, y, \xi, \eta) \quad (\text{with } \xi^2 + \eta^2 = \rho^2).$$

We need to consider the function $h(x, y, \xi, \eta)$ to make G symmetric and zero on the boundary. We can express this in polar coordinates, $x = r \cos \theta$, $y = r \sin \theta$,

$$\begin{aligned} G(r, \theta, \rho, \Phi) &= \frac{1}{4\pi} \ln \left(\frac{(r \cos \theta - a^2/\rho \cos \Phi)^2 + (r \sin \theta - a^2/\rho \sin \Phi)^2}{(r \cos \theta - \rho \cos \Phi)^2 + (r \sin \theta - \rho \sin \Phi)^2} \right) + h, \\ &= \frac{1}{4\pi} \ln \left(\frac{r^2 + a^4/\rho^2 - 2a^2 r/\rho \cos(\theta - \Phi)}{r^2 + \rho^2 - 2r\rho \cos(\theta - \Phi)} \right) + h. \end{aligned} \quad (66)$$

Choose h such that $G = 0$ on $r = a$,

$$G|_{r=a} = \frac{1}{4\pi} \ln \left(\frac{a^2 + a^4/\rho^2 - 2a^3/\rho \cos(\theta - \Phi)}{a^2 + \rho^2 - 2a\rho \cos(\theta - \Phi)} \right) + h, \quad (67)$$

$$= \frac{1}{4\pi} \ln \left(\frac{a^2}{\rho^2} \frac{\rho^2 + a^2 - 2a\rho \cos(\theta - \Phi)}{\rho^2 + a^2 - 2a\rho \cos(\theta - \Phi)} \right) + h = 0 \Rightarrow h = \frac{1}{4\pi} \ln \left(\frac{\rho^2}{a^2} \right).$$

Note that,

$$\begin{aligned} w(r, \theta, \rho, \Phi) &= \frac{1}{4\pi} \ln \left(r^2 + \frac{a^4}{\rho^2} - 2 \frac{a^2 r}{\rho} \cos(\theta - \Phi) \right) + \frac{1}{4\pi} \ln \left(\frac{\rho^2}{a^2} \right) \\ &= \frac{1}{4\pi} \ln \left(a^2 + \frac{r^2 \rho^2}{a^2} - 2r \rho \cos(\theta - \Phi) \right) \end{aligned} \quad (68)$$

is symmetric, regular and solution of $\Delta w = 0$ in V .

So,

$$G(r, \theta, \rho, \Phi) = v + w = \frac{1}{4\pi} \ln \left(\frac{a^2 + r^2 \rho^2 / a^2 - 2r \rho \cos(\theta - \Phi)}{r^2 + \rho^2 - 2r \rho \cos(\theta - \Phi)} \right), \quad (69)$$

G is symmetric and zero on the boundary. This enables us to get the result for Dirichlet problem for a circle,

$$u(\rho, \Phi) = - \int_0^{2\pi} f(\theta) \frac{\partial G}{\partial r} \Big|_{r=\rho} a d\theta, \quad (70)$$

Where

$$\frac{\partial G}{\partial r} = \frac{1}{4\pi} \left(\frac{2r \rho^2 / a^2 - 2\rho \cos(\theta - \Phi)}{a^2 + r^2 \rho^2 / a^2 - 2r \rho \cos(\theta - \Phi)} - \frac{2r - 2\rho \cos(\theta - \Phi)}{r^2 + \rho^2 - 2r \rho \cos(\theta - \Phi)} \right),$$

$$\begin{aligned} \text{So } \frac{\partial G}{\partial n} \Big|_S &= \frac{\partial G}{\partial r} \Big|_{r=a} = \frac{1}{2\pi} \left(\frac{\rho^2 / a - \rho \cos(\theta - \Phi)}{a^2 + \rho^2 - 2a\rho \cos(\theta - \Phi)} - \frac{a - \rho \cos(\theta - \Phi)}{a^2 + \rho^2 - 2a\rho \cos(\theta - \Phi)} \right), \\ &= \frac{1}{2\pi a} \frac{\rho^2 - a^2}{a^2 + \rho^2 - 2\rho \cos(\theta - \Phi)}. \end{aligned} \quad (71)$$

Then

$$u(\rho, \Phi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2 - \rho^2}{a^2 + \rho^2 - 2a\rho \cos(\theta - \Phi)} f(\theta) d\theta, \quad (72)$$

and relabeling,

$$u(r, \theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\Phi)}{a^2 + r^2 - 2ar \cos(\theta - \Phi)} d\Phi. \quad (73)$$

4.1.1 Case Study Examples

1. Electric Potential in a Rectangular Domain

Consider a rectangular domain D with boundary ∂D and solve the Poisson equation.

Solution: The Poisson equation is:

$$\Delta\phi = -\rho \text{ in } D, \text{ where } \phi \text{ is electric potential \& } \rho \text{ is the charge density}$$

with Dirichlet BCs, $\phi = -\rho$ on ∂D .

The Green's function $G(x, y; \xi, \eta)$ for the Poisson equation in 2D is:

$$\Delta G(x, y; \xi, \eta) = \delta(x - \xi, y - \eta)$$

with BCs $G = 0$ on ∂D .

For a rectangle domain, the Green's function can be constructed using the methods of images or by solving the BVP directly.

The Green's function for a rectangle with sides of the length a & b is:

$$G(x, y; \xi, \eta) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4}{ab\lambda_{mn}} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{m\pi\xi}{a}\right) \sin\left(\frac{n\pi\eta}{b}\right) \quad (74)$$

Where $\lambda_{mn} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

Then, the solution for the potential ($\phi(x, y)$) can be written as

$$\phi(x, y) = \int_D G(x, y; \xi, \eta) \rho(\xi, \eta) d\xi d\eta$$

This integral represents the convolution of the Green's function with the charged density (ρ).

For Dirichlet BCs, ϕ must be zero on ∂D . Let's consider a specific example where the charge density ρ is uniform $\rho(x, y) = \rho_0$.

Then, the Potential ϕ becomes:

$$\phi(x, y) = \rho_0 \int_0^a \int_0^b G(x, y; \xi, \eta) d\xi d\eta \quad (75)$$

Substituting the Green's function:

$$\phi(x, y) = \rho_0 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4}{ab\lambda_{ab}} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \int_0^a \int_0^b \sin\left(\frac{m\pi\xi}{a}\right) \sin\left(\frac{n\pi\eta}{b}\right) d\xi d\eta \quad (76)$$

Evaluating the integrals,

$$\int_0^a \sin\left(\frac{m\pi\xi}{a}\right) d\xi = \frac{a}{m\pi} [1 - \cos(m\pi)] = \frac{2a}{m\pi} \text{ for odd } m$$

$$\int_0^b \sin\left(\frac{n\pi\eta}{b}\right) d\eta = \frac{b}{n\pi} [1 - \cos(n\pi)] = \frac{2b}{n\pi} \text{ for odd } n$$

Thus, the potential simplifies to:

$$\phi(x, y) = \rho_0 \sum_{m=1,3,5,\dots}^{\infty} \sum_{n=1,3,5,\dots}^{\infty} \frac{16}{n^2 m n \lambda_{mn}} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad (77)$$

This series converges to give the electric potential $\phi(x, y)$ in the rectangular domain.

2. Heat Distribution in Circular Domain

Solve the Poisson equation for heat distribution in circular domain of radius R.

Solution: The Poisson equation in a circular domain D with radius of heat distribution is given by:

$$\Delta U(x, y) = -Q \text{ in } r < R \text{ With BCs } U = 0 \text{ on } r = R,$$

where U is the temperature distribution & Q is the heat source function

The Green's function $G(x, y; \xi, \eta)$ for the Poisson equation in 2D satisfies:

$$\Delta G(x, y; \xi, \eta) = \delta(x - \xi, y - \eta) \text{ with BCs } G = 0 \text{ on } \partial D$$

For a circular domain, the Green's function can be expressed using the method of images or by solving the BVPs directly.

The fundamental solution in free space (without boundary) is:

$$G_0(r) = \frac{1}{2\pi} \ln r, \text{ Where } r = \sqrt{(x - \xi)^2 + (y - \eta)^2}$$

To satisfy the BCs $G = 0$ on ∂D , we modify the fundamental solution of Green's function in a circular domain and we get:

$$G(x, y; \xi, \eta) = \frac{-1}{2\pi} \left[\ln r - \ln\left(\frac{R^2}{r}\right) \right] \quad (78)$$

Where $r = \sqrt{(x - \xi)^2 + (y - \eta)^2}$ and R is the radius of the circular domain

Then, the solution to the Poisson equation using Green's function is:

$$U(x, y) = \int_D G(x, y; \xi, \eta) Q(\xi, \eta) d\xi d\eta$$

Substituting the Green's function,

$$U(x, y) = \frac{-1}{2\pi} \int_D \left[\ln\left(\frac{r}{R^2/r}\right) \right] Q(\xi, \eta) d\xi d\eta \quad (79)$$

$$= \frac{-1}{2\pi} \int_D \left[\ln r - \ln\left(\frac{R^2}{r}\right) \right] Q(\xi, \eta) d\xi d\eta$$

$$\begin{aligned}
&= \frac{-1}{2\pi} \int_D [\ln r + \ln r - \ln R^2] Q(\xi, \eta) d\xi d\eta \\
&= \frac{-1}{2\pi} \int_D (2\ln r) Q(\xi, \eta) d\xi d\eta + \frac{1}{2\pi} \ln R^2 \int_D Q(\xi, \eta) d\xi d\eta \quad (80)
\end{aligned}$$

Thus, the final solution for the temperature distribution $U(x, y)$ in the circular domain is:

$$U(x, y) = \frac{-1}{\pi} \int_D \ln r Q(\xi, \eta) d\xi d\eta + \frac{\ln R^2}{2\pi} \int_D Q(\xi, \eta) d\xi d\eta \quad (81)$$

This solution provides the temperature distribution in the circular domain based on the given heat source function f or Q .

3. In electrostatic, the Poisson equation describe the potential ϕ due to charge distribution ρ

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}, \text{ where } \epsilon_0 \text{ is the permittivity of free space}$$

Using Green's function, the potential ϕ at a point (x, y) due to a point charge at (ξ, η) is:

$$\phi(x, y) = \int_D G(x, y; \xi, \eta) \frac{\rho(\xi, \eta)}{\epsilon_0} d\xi d\eta \quad (82)$$

For a point charge q at (ξ, η) , $\rho(\xi, \eta) = q\delta(x - \xi, y - \eta)$, and the potential simplifies:

$$\phi(x, y) = \frac{q}{\epsilon_0} G(x, y; \xi, \eta) \quad (83)$$

This approach can be extended to more complex charge distribution and boundary conditions.

CHAPTER FIVE

ANALYSIS, DISCUSSION, AND CONCLUSION

5.1 Advantage of Using Green's Functions

5.1.1 Simplification of Complex Boundary Value Problems (BVPs)

- Using Green's functions to solve the Poisson equation in 2D offers several advantages, in simplifying complex BVPs;
- **Reduction to Integral Form:** Green's functions transform the differential Poisson equation into an integral equation. This change of perspective can simplify solving BVPs, especially in cases with complex geometries or boundary conditions.
- **Direct Solution:** By using Green's functions we can construct the solution directly from the known Green's function for the domain. This method bypasses the need for solving differential equations in the traditional sense, which can be particularly advantageous in complex domains.
- **Handling Complex Boundaries:** For complex boundary conditions, the Green's functions method allows the solution to be expressed in terms of integrals over the boundary and domain. This is often simpler than solving differential equations with complex boundary conditions directly.
- **Flexibility with different Boundary Conditions:** Green's function can be adapted to accommodate various boundary conditions (Dirichlet, Neumann, or Mixed), making them a versatile tool in tackling different problems.
- **Insight into Problem Structure:** The Green's function provides insight into how point sources affect the solution, allowing for a better understanding of the problem's structure and behavior under various conditions.
- **Superposition Principle:** Green's functions exploit the superposition principle, which allows complex problems to be decomposed into simpler problems involving point sources. The overall solution can then be constructed from these simpler solutions, simplifying the problem-solving process.

5.1.2 Analytical Insights & Physical Interpretations

A. Analytical Insights

- **Explicit solution Representation:** Green's functions offer an explicit analytical form for the solution of the Poisson equation. This representation simplifies understanding how the solution depends on the source term and boundary conditions.
- **Decomposition of Solutions:** The solution to the Poisson equation can be decomposed into contributions from point sources. This decomposition helps in understanding the overall effect of distributed sources and simplifies the analysis of complex problems.

- **Symmetry & Properties:** The Green's function often reveals symmetries & special properties of the problem, such as how the solution behaves at infinity or near the boundaries. This can provide deeper insights into the nature of the problem.

B. Physical Interpretations

- **Response to Point Sources:** Green's functions represent the response of the system to a point source. This allows for a clear physical interpretation of how localized source affect the potential field. Understanding this helps in visualizing the influence of each source & how they combine. The superposition principle facilitated by Green's functions, illustrates how the overall solution is built from the effects of individual point sources. This principle simplifies understanding the cumulative impact of multiple sources on the potential field.
- **Boundary Effects:** By incorporating the Green's function one can gain insights into how BCs influence the solution. For instance, Green's functions help visualize how changes in BCs affect the potential field throughout the domain

5.2 Limitations and Challenges

5.2.1 Green's Function Calculation

- **While Green's functions are a powerful tool for solving the Poisson equation in 2D, they come with certain limitations & challenges, particularly concerning computational complexity.**
- **Integral Computations:** The solution using Green's functions involves evaluating integrals, which can be computationally intensive, especially for large domains or complex boundary shapes. Numerical integration may be required adding to computational costs.
- **Large Number of Sources:** For problems involving many point sources or distributed sources, the Green's function approach requires summing over a large number of terms. This summation can become computationally expensive and require efficient algorithms to handle large domain.
- **Non-Analytical Green's Functions:** For domains with irregular shapes or complex BCs, obtaining an analytical form of the Green's function may be challenging or even

impossible. Numerical methods to approximate the Green's functions can be computationally demanding and may introduce approximation error.

- Boundary Conditions: Incorporating complex BCs into the Green's function framework can be intricate. Exact analytical forms might not be available, necessitating numerical approach that increases computational complexity.

5.2.2. Applicability to Non-Standard Domains (or Irregular Domains)

- For non-standard or irregular domains, obtaining an analytical Green's function can be difficult or impossible. Green's functions are often straightforward to derive for simple geometries like rectangle or circles, but can become very complex or even undefined for irregular shapes.
- Irregular boundaries complicate the formulation of Green's functions. For non-standard domains, specifying & incorporating BCs into the Green's function approach can be challenging and may require sophisticated numerical technique.
- For irregular domains, it may be necessary to decompose the domain into simpler subdomains where Green's functions can be applied more easily. This fragmentation adds complexity to both the problem formulation and the solution

5.3. Conclusion

5.3.1 Summary of Key concepts & Contributions

Solving the Poisson equation on using Green's function in 2D has several key steps and methods. Some of these and the concepts are of the followings:

1. Poisson Equation in 2D

The Poisson equation in 2D is given by:

$$\nabla^2 u(x, y) = f(x, y)$$

2. Green's Function Concept

Green's function ($G(x, y; \xi, \eta)$) is a fundamental solution to the Poisson equation with a delta function as the source of function that satisfies:

$$\nabla^2 G(x, y; \xi, \eta) = \delta(x - \xi, y - \eta)$$

3. Green's function for 2D Poisson equation

For the Poisson equation in 2D, the Green's function $G(x, y; \xi, \eta)$ is given by;

$$G(x, y; \xi, \eta) = -\frac{1}{2\pi} \ln(\sqrt{(x - \xi)^2 + (y - \eta)^2})$$

$$G(x, y; \xi, \eta) = \delta(x - \xi, y - \eta)$$

4. Solution using Green's function

The general solution $u(x, y)$ to the Poisson equation with source term $f(x, y)$ is obtained by integrating the product of $f(\xi, \eta)$ and the Green's function:

$$= \frac{1}{2\pi} \iint_D \ln(\sqrt{(x - \xi)^2 + (y - \eta)^2})$$

- **Fundamental Solution:** The Green's function provides a clear & practical way to solve the Poisson equation by converting it into integral problems, where the integral involves the source distribution & the Green's function.
- **Singularity Handling:** The logarithmic form of the Green's function in 2D reflects the fundamental nature of the solution to the Laplace equation and handles the singularity at the source point effectively.
- **Integral Representation:** The solution to the Poisson equation using Green's functions is represented as an integral involving the source term and the Green's function, which simplifies solving the equation especially in complex geometries or with irregular boundary conditions.
- **Physical Interpretation:** The Green's function provides physical insights into the potential field generated by a point source and extends to more complex source distributions through superposition.
- **Boundary Conditions:** While the Green's function method is very effective in infinite domains or simple boundary conditions, additional techniques (like using Green's functions for bounded domains or incorporating boundary conditions) are required for more complex geometries.

5.3.2 Implication for Mathematical Physics & Engineering

In 2D, the Poisson equation $\Delta u = f$ is a fundamental PDE that arises in various contexts within mathematical physics & engineering. Solving it using Green's functions provides significant insights & practical tools. Some of its implication for:

Mathematical Physics

- **Electrostatics:** In electrostatics, the Green's function approach provides the potential due to a given charge distribution. The logarithmic nature of the Green's function corresponds to the behavior of the potential in 2D electrostatic problems.

Heat conduction; for heat condition problems in 2D, the Green's function provides the temperature distribution resulting from a localized heat sources. This is crucial for understanding heat propagation & designing thermal systems.

Engineering

- **Boundary Element Method (BEM):** The Green's function approach is fundamental for BEM, which simplifies the sum of PDEs by reducing the problem to boundary integrals. This method is particularly useful in structural analysis & fluid mechanics.
- **Design & Optimization:** In engineering design, particularly in thermal & electrostatics applications, the insight from Green's functions help optimize system performance by predicating the effects of various configurations & loadings.

In general, from this research the following has been found:

- **Effectiveness of Green's Function:** Green's function offers a powerful method for solving the Poisson equation especially in complex domain. It transforms the problem into an integral equation, simplifying the solution process for various boundary conditions.
- **Versatility in Applications:** This method is highly versatile and can be applied to problems in electrostatics, fluid dynamics and heat conduction, among others. It helps in understanding the behavior of physical systems under various conditions.
- **Analytical Solutions:** The approach allows analytical solutions, making it adaptable to different problem settings and computational resources.

- **Boundary Conditions Handling:** Green's function method effectively handle different types of boundary conditions (Dirichlet, Neumann & Mixed), providing a comprehensive solution framework.

5.3.3 Future Research Direction

- The study of Poisson equation using Green's functions in 2D has established a robust framework for understanding potential theory & various BVPs. As research continues, several promising directions for extending this work emerge.
- **Higher-Dimensional Extensions:** While 2D problems are well-studied, extending Green's function methods to 3D & higher dimensional spaces presents challenges and opportunities. This includes tackling more complex geometries & BCs, which can impact the behavior of Green's functions & their applications.
- **Numerical methods & Algorithms:** Improved numerical techniques for approximating Green's functions, especially for irregular domains or complicated BCs, can enhance the accuracy & efficiency of simulations. This includes developing algorithms that can handle large scale problems effectively.
- **Nonlinear Poisson –like Equations:** Extending the Green's function approach to nonlinear versions of the Poisson equation or related PDEs can offer new insights into complex physical systems where linearity no longer holds.
- **Time-Dependent Problems;** while Green's functions are typically used for steady-state problems, adapting these techniques to time-dependent scenarios, such as wave propagation or diffusion processes, could be beneficial. This might involve developing time-dependent Green's functions or exploring transient behavior in physical systems.
- **Applications to physical Systems;** Exploring applications in various fields like fluid dynamics, electromagnetism, and materials science can reveal new use for Green's function techniques. For instance, applying these methods to study complex systems like turbulence or nonlinear optics could yield novel results.

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