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Graduate Seminar Report
On
Non Commutative Noetherian Rings

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NON COMMUTATIVE NOETHERIAN RINGS

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Introduction:

After a review of the definition and basic properties of noetherian modules and rings, we introduce classes of examples of Non-Commutative Noetherian rings. We concentrate on module finite algebras over commutative rings, skew-Laurent rings and the corresponding skew polynomial rings twisted by automorphisms which will provide us examples on non-commutative noetherian rings.

Chapter 1

Preliminary Results

1.1. Properties of Rings

Definition1.1.1 (Ring): A ring is a non empty set R with two binary operations

+ and . called addition and multiplication respectively such that

(a) The elements of R from an abelian group under addition i.e.

1) $x+y \in R$ (closure)
2) $(x+y)+z = x+(y+z)$ (associativity)
3) $x+y = y+x$ (commutative)
4) There is an element $0 \in R$ called the zero element (or additive identity) such that $x + 0 = x$ for all $x \in R$.
5) Given $x \in R$ there is an element $-x \in R$ such that $x+(-x) = 0$ (existence of additive inverse) $\forall x, y, z \in R$

(b) R is closed under multiplication, associative, and commutative law also holds and R has identity i.e. $x, y, z \in R$, we have

1) $x\,y \in R$ (closure)
2) $x\,y = y\,x$ (commutative)
3) $x\,(y\,z) = (x\,y)\,z$ (associative)
4) There is an element $1 \in R$ called the (multiplicative) identity element such that $x.1 = x \ \forall\ x \in R$

(c) The distributive law holds: i.e. for all $x, y, z \in R$ we have

$x\,(y+z) = x\,y + y\,z$ & $(y+z)\,x = y\,x + z\,x$.

(In this case the ring R is a commutative ring).

Notation: (R, +, .)-ring R.

Examples: (z, +, .) - ring of integers, (R, +, .) Z_n (+, .) (Q,+, .),
C [x]-polynomial in x with coefficients in the complex numbers.

Examples of non-commutative ring:

-the set of 2 by 2 matrices with entries in the complex number.

-since every field F contains 0 and 1, M_2 (F) always has (among its elements)$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ and$\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$. $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$

While $\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$. Thus $M_2(F)$-2×2 matrices with entries in F is non-commutative ring. Since $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$ additive identity. This example shows there is divisor of zero in $M_2(F)$. But it is not domain. The same is true of $M_n(F)$ for $n\geq 2$, $n \in Z^+$.

Definition1.1.2 (Subring): A non empty subset S of ring R is subring of R if i) $0\in S$ ii) $x, y\in S \Rightarrow x-y\in S$ and iii) $x, y\in S\Rightarrow x\, y\in S$

Example: a) Z is subring of Q.

b) $Y=\left\{\begin{pmatrix}a & 0\\ b & c\end{pmatrix} \mid a,b,c\in Z\right\}$ is subring M_2 of all 2×2 matrices over Z (set of integer).

Definition1.1.3 (Ideal): The set I is ideal of ring R if i) $I\neq\varnothing$, ii) $I\subseteq R$, iii) a, $b\in I\Rightarrow a-b\in I$, iv)$r\in R$, $a\in I\Rightarrow r\,a$, $a\,r\in I$.

Example: Z is not an ideal of Q. since $1/2\in Q$ and $3\in Z$ but $½ \times 3 = 3/2\notin Z$ (Z is set of integer, Q is set of rational number)

Definition1.1.4 (**Noetherian ring**): A ring in which every strictly ascending chain of right (left) ideals finite is called a right (left) noetherian

Example: a) The ring of integer Z is P I R (every ideal is principal), any ascending chain of ideals of Z is of the form $(n) \subset (n_1) \subset (n_2) \subset\ldots$, where, n, n_1, n_2…are in Z. Since $(n_i) \subset (n_{i+1}) \Rightarrow n_{i+1}/n_i$, any ascending chain of ideals in Z starting with n can have at most n terms. This shows that Z as Z -module is noetherian.

b) Consider division ring D. Since the only right ideals of D are (0) and D itself. D is right noetherian. Similarly D is also left noetherian.

1.1.5 Opposite ring:

The opposite of a product of rings is the product of the opposite rings, associated with every ring R is the ring R^{OP} called the opposite ring. The underlying set of R^{OP} is the same as that of R. It is only multiplications they differ. For $r_1, r_2 \in R^{op}$, we define the product $r_1 r_2$ to be the element $r_2 r_1$ in R. Multiplication in R^{OP} is opposite in R.
$R^{OP}=R \Leftrightarrow$ R is a commutative ring. $(R^{OP})^{OP} = R$.

Remark1.1.5.1 M is a left R-module implies M is a right R^{op} module where R^{op} is the ring opposite to R.

Proof: we have a homomorphism of ringsψ: $R \to End_Z(M)$. Compose this with the identity map id: $R^{op} \to R$ which is anti-isomorphism to get anti-homomorphism $R^{op} \to End_Z(M)$ which means M is a right R^{op}-module. Conversely, suppose we have an anti-homomorphism of rings ψ: $R^{op} \to End_Z(M)$. Compose this with the identity map id: $R \to R^{op}$ which is an anti-isomorphism, to get a homomorphism $R \to End_Z(M)$. Therefore M is a left R-module.
Note: The map given by $r \to r$ is an anti-isomorphism $R \to R^{op}$. If $A = (a_{ij})$ and $B = (b_{ij})$ are n×n matrices over R, then A and B may also be considered to be matrices over R^{op}. In $Mat_n R$, $AB = (c_{ij})$ where $c_{ij} = \sum_{k=0}^{n} a_{ik} b_{kj}$, but in $Mat_n R^{op}$, $AB = (d_{ij})$ where $d_{ij} = \sum_{k=1}^{n} a_{ki} \circ b_{kj} = \sum_{k=1}^{n} b_{kj} a_{ik}$.

Definition1.1.6 (Center of ring R): If R is any ring, then the center of R is the set A= $\{a \in R \mid ar = ra \;\; for\ all\ r \in R\}$

Example: a) The center of the ring $M_n(R)$ consisting of all matrices of the form r I_n where r is in the center of R (every matrix in the center of $M_n(R)$ must commute with each of the matrices Br,s where Br,s has 1_R in position(r, s) and 0 else where.

b) The center of $M_n(R)$ is isomorphic to the center of R.

1.2. Module:

Definition1.2.1. A module over a ring R (or R- module) is an abelian group M (written additively) together with a map (r, m) → r m from R×M →M satisfying the conditions

M1.$r(m_1+ m_2) = rm_1+ rm_2$ M2.$(r_1+ r_2)m = r_1m + r_2m$

M3.$(r_1r_2)m = r_1(r_2m)$ M4. $1m = m$,for all $r_1,r_2,\in R$ and all $m, m_1\ m_2 \in M$

Example: a) A vector space over a field K- a K-module

b) Any abelian group A is a z-module

c) Any ring (with 1) B is a module over itself

Definition1.2.2 (Left R-module): Let R be any ring (with or with out 1 and commutative or not). By a left R-module M, we mean an abelian group (M, +) together with a map R×M→M: (a, x)→a x called scalar multiplication or structure map, such that

1. $a (x + y) = a x + a y\ \forall a\in R$ and $x, y\in M$
2. $(a + b) x = a x + b x, \forall a, b\in R$, and $x\in M$
3. $x (a b) = a(b x)\ \forall a, b\in R, x\in M$

Definition1.2.3 (Right R-module): An abelian group (M, +) is called a right

R-module if there is a map from M×R→M denoted (x, a) → x a

Such that 1. $(x + y) a = x a + y a$ for all $a\in R$ and $x, y \in M$

2. $x (a + b) = x a + x b\ \forall a, b\in R$ and $x\in M$

3. $x (a b) = (x a) b, \forall a, b\in R$ and $x\in M$

Definition1.2.4 (Submodule): Let M is an R-module. A non empty subset N of M is called an R-submodule (or simply submodule) of M if

i. $a - b\in N\ \forall a, b\in N$

ii, $ra \in N, \forall a \in N, r \in R$.

Example:

a) 0, M trival submodule

b) If M is an R-module and $x \in M$, then the set $R x = \{r x \mid r \in R\}$ is an R- submodule of M for $r_1x-r_2x = (r_1-r_2)x\in Rx$, $r_1(r_2x) = (r_1r_2)x\in Rx$ for all $r_1,r_2 \in R$.

Definition1.2.5 (Finitely generated (f.g) module): An R-module M is said to be finitely generated over R if there is a finite sub set X of M such that M is the submodule generated by X *i.e* if X = $\{x_1, \dots, xr\}$, then (assume R has 1 and M unitary) we have M = $\left\{\sum_{i=1}^{r} a_i x^i \mid a_i \in R\right\}$.

Definition1.2.6 (Free module): An R-module M is called a free module if M has a basis B i.e a linearily independent sub set B of M such that M is spanned by B over R.

Examples: a) A vector space is a free module i.e it has a basis.
b) Any finite abelian group is not free as a module over Z.

Definition1.2.7 A partial ordered set (poset) is a non empty set P together with
a relation R on P×P (called partially ordering of P) which is reflexive, transitive and antisymmetric ($a \leq b$ & $b \leq a \Rightarrow a = b$).

Definition1.2.8 let S a subset of P, $a \in S$ is maximal if there is no element $b \in S$ with $b > a$.

Definition1.2.9 $a \in S \subset P$ is minimal element if there is no $b \in S$ with $b < a$.

Example: $S \neq \emptyset$ and finite then it has maximal and minimal element.

Definition1.2.10 (Ascending chain condition (ACC)): A module M is to satisfy ACC on submodules(or to be noetherian) if for every ascending chain $M_1 \subset M_2 \subset \dots$ of submodules of M then there is an integer n such that $M_i = M_n \ \forall i \geq n$.

Definition1.2.11 (Noetherian module): A module M is called Noetherian if ACC (or equivaletly maximum condition) holds for M.

Exapmle: a)finite abelian groups are noetherian as a module over Z.
b) finite dimentional vector spaces are Noetherian.

An R module M is called cyclic module if M = (x) for some $x \in M$.

1.3. **Modular law**: let V be a vector space having subspaces A, B, C such that $B \subseteq A$. Then $A \cap (B + C) = B + (A \cap C)$

Proof: Let $x\in A\cap(B+C)$. Then $x = b + c$ ($b\in B$, $c\in C$) and $x\in A$. Then $c = x - b\in A$ So $c\in A\cap C$ and $x\in B+(A\cap C)$.The converse is similar.

1.4. Matrices:

Definition1.4.1 (Matrix): A matrix is a rectangular array of scalars. The scalars may be real or complex numbers.

The rectangular array takes the form$A_{m,n}=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$. The matrix A has m rows and n columns. Any element in A has a representation a_{ij} where the subscrpit i refers to the row and j to the column where the element lies.

Example: $A=\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 4 \end{pmatrix}$ - 2×3 matrix.

Definition1.4.2 (Diagonal matrix): An n×n matrix $A = (a_{ij})$ is called a diagonal matrix if $a_{ij} = 0$ when ever $i \neq j$.

Definition1.4.3 (Upper (Lower) triangular matrix): An n ×n matrix $A = (a_{ij})$ is called an upper (lower) triangular matrix if $a_{ij} = 0$, $i > j$ (respectively $i< j$) we use the notation $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix}$ for upper triangular matrix and $\begin{pmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ for lower triangular matrix.

Definition1.4.4 (Trace of a matrix): let $A = (a_{ij})$ be an n×n matrix. Then sum $a_{11}+a_{22}+a_{33}+\ldots\ldots.+ a_{nn}$ of diagonal elements of A is called trace of A.

Definition1.4.5 (Rank of a matrix): Let A be an m×n matrix over a field F. Then the row rank of A is equal to the maximum number of linearly independent (L.I) row vectors of A. Row rank of an m×n matrix is at most n as
$\dim F^{(n)} = n$. Let A be an m×n matrix over a field F .Then the column rank of A is the maximum number of L.I columns of A. The column ran of A is at most m, since $\dim F^{(m)} = m$.

Definition1.4.6 A matrix $A \in M_n(R)$ is: - **symmetric** if $A = A^t$ where A^t is transpose of matrix A. -**Anti symmetric (skew symmetric)** if $A = -A^t$. Only square matrices can be symmetric or antisymmetric.

Examples: 1.$A = A^t = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -3 & 2 \\ 1 & 2 & 1 \end{pmatrix}$, A is symmetric.

2. $A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix}$, $A^t = -A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix}$, A is anti symmetric

1.5. Algebras

Definition1.5.1 Let M, N & P be A-modules. A function $f: M \times N \to P$ is called bilinear if $f(x + x', y) = f(x, y) + f(x' + y)$

$f(x, y + y') = f(x, y) + f(x, y')$

$f(\alpha x, y) = \alpha f(x, y) = f(x, \alpha y)$

Definition1.5.2 Let A be a commutative ring. An A-algebra (or algebra over A) is a module M together with a bi linear map $M \times M \to M$. Let $f : A \to B$ be a ring homomorphism such that f (A) is contained in the center of B i.e. f (a) commutes with every element of B for every $a \in A$ ($x \in B$: $x f(a) = f(a) x$,

$\forall a \in A$) then we may view B as an A-module defining the operation of A on B by the map $(a, b) \to f(a) b$ for all $a \in A$ and $b \in B$. The axiom of module is trivially satisfied, and the multiplicative law of composition $B \times B \to B$ is clearly bi- linear (i.e. A-linear). Algebra over A is by definition, a ring B together with a ring homomorphism $f : A \to B$. (Thus unless specified, by algebra over A, mean ring homomorphism). We say that algebra is finitely generated if B is finitely generated as a ring over f (A).

Example: a) A it self an A-algebra in a natural way

b) Any ring may be regarded as a Z-algebra by putting

$n a = a + a + \ldots + a$ (n times)

$(-n) a = -n a$, $a \in R$, $n \in N$.

c) K-algebra (K is a field) is a ring containing K as a subring. Let A be any ring with identity element, then there is a unique homomorphism of the ring integers Z into A, namely $n \to n.1$. Thus every ring is a Z-algebra.

A ring B is a finitely generated A-algebra, if there is a finite set of

elements $x_1,\ldots,x_n$ in B such that every element of B can be written as a polynomial in $x_1,\ldots,x_2$ with coefficients in f(A). Or equivalently if there is an A-algebra homomorphism from a polynomial $A[y_1,\ldots y_n]$ on to B.
A ring A is finitely generated if it is finitely generated as a Z-algebra.

1.5.3 Group algebra:

One of the earliest stimuli to the modern development of non commutative ring theory came from the study of group representations. The key idea was to study a group G by "representing" it in terms of linear transformations on a vector space V, namely studying a group homomorphism ϕ from G to the group of invertible linear transformations on V. In the case of finite group G, the group algebra K[G] (representation of G on a vector space over K) is finite dimensional, and theory of finite dimensional algebras has much to say about representation of G. Noetherian group algebra is known when G is poly-cyclic by finite.

1.5.3.1 Group ring of poly cyclic by finite group:

G is said to be poly cyclic by finite if it has a finite (chain) series $1= G_o\subset G_1\subset\ldots\subset G_m= G$ with G_{j-1} normal in G_j and G_j/G_{j-1} either finite or infinite cyclic, one can arrange that the only finite factor is the last, G_j/G_{j-1}. It is possible to push all the finite layers to the very top factor and assume that G_j/G_{j-1} is infinite for $j < m$. The group algebra $k[G_{m-1}]$ then looks much like an iterated twisted polynomial ring.
There are only two poly-cyclic by finite groups:
1. H is the "discrete Heisenberg group". It is generated by x, y & z subject to the relation that z be central and $[x, y] = z$. It can be represented as the group of 3×3 unit triangular matrices with integer entries. Obviously H has a normal series $1\subset\langle z\rangle\subset\langle z,x\rangle\subset\langle z,x,y\rangle = H$ with infinite cyclic factors. We set $A = \langle Z\rangle$, the center of H. Notice that the group is nilpotent. Every finitely generated nilpotent group is poly cyclic.
2. B is the simplest interesting non nilpotent poly cyclic group (It will be to illustrate the ubiquity of Bergman's Theorem).It is the semi direct product $(Z\oplus Z)\rtimes\langle w\rangle$ where w acts on $Z\oplus Z$ like the matrix$\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}$. We will find it useful to that the eigen values of $\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}$ are real irrational number $1\pm\sqrt{2}$ (recall to calculate eigen value i.e. $\begin{vmatrix}\lambda-1 & -2\\ -1 & \lambda-1\end{vmatrix} = 0\Rightarrow(\lambda-1)^2 = 2\Rightarrow\lambda= 1\pm\sqrt{2}$).

When we speak of B, the notion A will designate the copy Z⊕Z which is the normal abelian sub group at "the bottom".

1.5.4 Quantum group:

Quantum groups are not groups at all but certain algebras that arose in 1980 s in connection with research on some problems in quantum statistical mechanics. Quantum groups are algebras of functions on non-existent groups.

Special linear group $SL_2(K)$,the group of 2×2 matrices over a field K having determinant 1.First of all,$SL_2(K)$ lies inside the 4-dimensional vector space $M_2(K)$,where it can be described as the set of zeros of polynomial $X_{11}X_{22}-X_{12}X_{21}-1$here $X_{11},X_{12},X_{21},X_{22}$ are just four independent indeterminate conveniently labeled for application to the entries of 2×2 matrices.Thus,$SL_2(K)$ is an affine algebraic variety over K, and its coordinate ring is the algebra $Q(SL_2(K)) = K[X_{11},X_{12},X_{21},X_{22}] / \langle x_{11},x_{22}-x_{12}x_{21}-1\rangle$. The group structure is encoded in certain algebra homomorphism. In particular, the group operation viewed as a map $SL_2(K) \times SL_2(K) \rightarrow SL_2(K)$,induces(by composition of functions) a k-algebra homomorphism .$\Delta: Q(SL_2(K)) \rightarrow Q(SL_2(K) \times SL_2(K)) \xrightarrow{\cong} Q(SL_2(K)) \oplus Q(SL_2(K))$ called co-multiplication. There are also a K-algebra homomorphism $t: Q(SL_2(K)) \rightarrow K$, $S: Q(SL_2(K)) \rightarrow Q(SL_2(K))$ corresponding the identity and imply certain relation among these maps. The algebra $Q(SL_2(k))$ together with the three maps Δ, t, s form a structure called a Hopf algebra, which we will not define here. $Q_q(SL_2(K))$ is a Hopf algebra, and it has four generators that satisfy a relation very similar to the equation "determinant =1" which characterizes $SL_2(K)$. The only drawback is that this new algebra is not commutative, and so it cannot be algebra of k-valued functions on any thing. Nonetheless, thinking of this algebra as if it consisted of functions, and it became known as the coordinate ring of quantum SL_2 (K). Thus there is no "quantum group" SL_2 (K); the group has disappeared and only the algebra of "functions" on it remains.

Regardless (or because of) its origin, this new algebra is an interesting object of study, and among other properties it turns out to be Noetherian. Here is a brief description of the construction:

First, pick a non zero scalar q (the "quantum parameter") in K. (originally q was taken to be e^h, where h is Planck's constant, so that q was a real number very close to 1).

Next, one forms a k-algebra with four generators X_{11}, X_{12}, X_{21}, X_{22} and six relations; for instance, $X_{11}X_{12}=qX_{12}X_{11}$ and $X_{11}X_{22}-X_{22}X_{11}=(q-q^{-1})X_{12}X_{21}$. This algebra is the coordinate ring of quantum 2×2 matrices, $Q_q(M_2(K))$. $D_q=X_{11}X_{22}-qX_{12}X_{21}$-quantum determinant which lies in the center of $Q_q(M_2(K))$. Finally the coordinate ring of quantum $SL_2(K)$ is the algebra $Q_q(SL_2(K))=Q_q(M_2(K))/\langle D_q-1\rangle.Q_q(M_2(K))$ is an iterated skew polynomial ring. Consequently $Q_q(M_2(K))$ is noetherian and therefore $Q_q(SL_2(K))$ noetherian.

1.6. Automorphism & Inner automorphism:

Definition1.6.1 (Automorphism): An isomorphism of $f: G\rightarrow G$ (G is ring) is called an automorphism of G.

Example: A is abelian group, and then the map $a\rightarrow a^{-1}$ is an automorphism of A.

Remark: - If G is non abelian, we might wish to consider for some $g\in G$, the set $C_G(g)=\{x\in G \mid xg=gx\}$ of all elements that commute with g in G. This set is the centralizer of g in G. It is subgroup of G.

- $1\in C_G(g)$, it is closed under multiplication
- $x\in C_G(g)$, then $xg=gx$ i.e. $x^{-1}xgx^{-1}=x^{-1}gxx^{-1}$, Thus $gx^{-1}=x^{-1}g$ and $x^{-1}\in C_G(g)$ as required.
- $Z(G)=C_G(G)=\{y\in G \mid xy=yx \text{ for all } x\in G\}$ is a subgroup
- $Z(G)$ is abelian $\Rightarrow G=Z(G)$

–identity map on G is an automorphism but most groups have many other automorphisms too. -since isomorphism carries center to center it follows that every automorphism of G maps Z(G) to it self.

An important example of automorphism of G is the inner automorphism θ_g induced by an element $g\in G$. This is the map $\theta_g(x)=g^{-1}xg$

Definition1.6.2 (Inner automorphism): If u is a unit in a ring R with identity then the map $R\rightarrow R$ given by $r\rightarrow uru^{-1}$ is an automorphism of the ring R. It is called inner automorphism induced by u. The inner automorphism of $R=S[x,x^{-1},\sigma]$ induced by x extends σ from S to R, and denoted byσ.

A derivation:

A derivationδ: F→K (F is a field, K an extension field of F) is a map which satisfies $\delta(x+y) = x\delta(y) + \delta(x)y$. If E is a subfield of F,δ is called an E-derivation if in addition$\delta(x) = 0$ for all $x \in E$ (so δ is E-linear).

Inner derivation: Given any element a of a ring R there is a derivation of R, namely r→r a –a r =[r, a]; this is called an inner derivation and dented by ad a. for example if R=S[x;δ], then ad x is an inner derivation of R which, since it extends δ from S to R, and denoted byδ.

1.7. Power & Laurent series:

Definition 1.7.1(power series): A power series in a variable x is an infinite sum of the form $\sum_{i=0}^{\infty} a_i x^i$ where $i \geq 0$ and a_i are integers, real numbers complex numbers
or any other quantities of a given type.

Definition1.7.2 Let R be a ring and σ an endomorphism, then R $[[x;\sigma]]$ denotes the ring of power series $\sum_{i=0}^{\infty} x^i a_i$ subject only to the relation $ax = x\sigma(a)$. This is called the skew power series ring.
R $[[x_1 \ldots x_n]]$ - ring of ring of formal power series in $x_1,\ldots,x_n$ with coefficients in R. The elements of this ring are formal sums of the form $\sum_{i=0}^{\infty} f_i$ where f_i is, for each $i \in \mathrm{N_O}$, a homogeneous polynomials in R$[x_1\ldots,x_n]$ which is either zero or of degree i. Two such formal power series $\sum_{i=0}^{\infty} f_i$ and $\sum_{i=0}^{\infty} g_i$ are equal when $f_i = g_i$ for all $i \in \mathrm{N_O}$. The operation addition and multiplication are given by $(\sum_{i=0}^{\infty} f_i + \sum_{i=0}^{\infty} g_i) = \sum_{i=0}^{\infty}(f_i + g_i)$

$(\sum_{i=0}^{\infty} f_i)(\sum_{i=0}^{\infty} g_i) = \sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} f_i g_{i-j})$ for all, $\sum_{i=0}^{\infty} f_i, \sum_{i=0}^{\infty} g_i \in$ R $[[x_1,\ldots,x_n]]$.

Definition1.7.3 (**Laurent series):** A Laurent series centered about a is a series of the form $\sum_{-\infty}^{\infty} c_n (x-a)^n$ where c_n, a, $x \in$ C where C is complex number.

Definition 1.7.4 Let R be a ring and σ an automorphism. Then R$[x; x^{-1}, \sigma]$ denotes ring of polynomials over R in x and x^{-1} subject to $ax = x\sigma(a)$. This is the ring of skew Laurent polynomials. Each element has a unique representation in the form $\sum_{i\in Z} x^i a_i$ with all but finitely many coefficients being zero.

Note R$[x; \sigma]$ is a sub ring of R$[x, x^{-1}; \sigma]$.

Chapter 2

A few Noetherian Rings

2.1. The Noetherian condition.

Notation: $A > B$ ($B < A$) - means that B is a proper submodule of A. ($A \leq B$ means A is a sub module of B).

Proposition 2.1 For a module A, the following conditions are equivalent:
a) A has the ACC on submodules
b) Every non empty family of submodules of A has a maximal element.
c) Every submodule of A is finitely generated.

Proof: (a) => (b): suppose that **A** is a non empty family of submodules of A with out maximal element. Choose $A_1 \in \mathbf{A}$. since A_1 is not maximal, there exists $A_2 \in \mathbf{A}$ such that $A_2 > A_1$, continuing in this manner we obtain a properly ascending infinite chain $A_1 < A_2 < A_3 ...$ of submodules of A contradicting the ACC .Hence the result holds.
b) =>(c): Let B be a submodule of A, and let β be the family of all finitely generated submodules of B. Since empty set ($\varnothing$) generate the zero module, $0 \in \beta \Rightarrow \beta \neq \varnothing$. By **(b)** there is a maximal element $c \in \beta$ .If $C \neq B$, choose an element $x \in B / C$ and let C' be the submodule of B generated by c and x .Then $C' \in \beta$ and $c' > c$, contradicting maximality of c. Thus $C = B$, hence B is finitely generated. (c) => (a): let $B_1 \leq B_2 \leq \ldots$ an ascending chain of submodules of A .let B the union of the B_n. By (c) there exists a finite set X of generators for B. Since X is finite it is contained in some B_n, hence $B_n = B$. Thus $B_m = B_n$ for all $m \geq n$, establishing the ACC on finitely generated submodules.

Definition 2.1.1 A module A is noetherian iff the equivalent condition of proposition 2.1 is satisfied.

Example: a)Any finite dimensional vector space V over a field K is noetherian k-module since a properly ascending chain of submodules (subspaces) of V can not contain more than $\dim_k(V) + 1$ term. For let

W is proper subspace of V, then $\dim W < \dim V = n$ thus any properly ascending chain of submodules can not have more than n+1 terms.

Definition 2.1.2 A ring R is right (left) noetherian iff the right module R_R (left module $_RR$) is noetherian. If both condition hold, R is noetherian.
A ring R is right (left) noetherian ⇔ R has the ACC on right (left) ideal
⇔ all right (left) ideals of R are f.g.

Examples: a) Z is noetherian ring because all its ideals are principal (singly generated)

b) A polynomial ring K[x] in one indeterminate over a field K is noetherian. Since K[x] is PID. It follows that K[x] is a noetherian ring.

c) 2x2 matrices over Q the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $a \in Z$ and $b, c \in Q$ is a right noetherian but not left noetherian (See the example under proposition 2.8.)

d) Any finite direct product (sum) of a right (left) noetherian rings is right (left) noetherian (see corollary 2.3 below)

e) If G is finite group and F a field, the group algebra F (G) is both noetherian and artinian.

f) The m ×m matrix ring $R_{m\times m}$ over a field F is also noetherian and artinian.

g) The ring of upper (lower) triangular matrices over a field F is both noetherian and artinian.

Proposition 2.2 let B be a submodule of a module A. Then A is noetherian if and only if B and A / B are both noetherian.

Proof (⇒) : assume that A is noetherian .Since any ascending chain of sub modules of B is also an ascending chain of submodules of A, this implies that B is Noetherian .If $C_1 < C_2 < \ldots$ is an ascending chain of submodules of A /B, each C_i is of the form A_i /B for submodules A_i of A that contains B, and $A_1 \le A_2 \le \ldots$since A is noetherian there is some n such that $A_i = A_n$ for all $i \ge n$, and then $C_i = C_n$ for all $i \ge n$ thus A/B is Noetherian .
(⇐) :assume B and A/B are Noetherian and let $A_1 \le A_2 \le \ldots$ be an ascending chain of submodules A. There are ascending chains of submodules $A_1 \cap B \le A_2 \cap B \le \ldots$ & $(A_1+ B)/B \le (A_2+B)/B \le \ldots$in B and in A/B respectively Hence there is some n such that $A_i \cap B = A_n \cap B$ and $(A_i + B)/B = (A_n+ B)/B$ for all $i \ge n$ .The later equation yields $A_i + B = A_n + B$ $\forall i \ge n$, we concluded that $A_i = A_i \cap (A_i + B) = A_i \cap (A_n+ B) = A_n+ (A_i \cap B) = A_n+ (A_n \cap B) = A_n$

(using modular law for the third equality). Preposition 2.2 shows any factor ring of a right noetherian ring is a right noetherian.

Corollary 2.3 Any finite direct sum of noetherian modules is noetherian.

Proof: wts: direct sum of finite Noetherian modules is noetherian. It is enough to proof for two i.e. A_1 and A_2 are Noetherian module $\Rightarrow A = A_1 \oplus A_2$ noetherian module. The module $A=A_1 \oplus A_2$ has a submodule $B=A_1 \oplus 0$ such that $B \cong A_1$ and $A/B \cong A_2$. B and A/B are noetherian as they are isomorphic to noetherian module A_1 and A_2 respectively. Hence A is noetherian module by proposition 2.2 (A noetherian $\Leftrightarrow$ B and A/B are notherian).

Corollary 2.4 If R a right Noetherian ring, all finitely generated right R-modules are Noetherian.

Proof: If A is a finitely generated right R-module, then $A \cong F/K$ for some finitely generated free right R-module F and some submodule $K \leq F$. Since F is isomorphic to a finite direct sum of copies of the noetherian module R_R (R_R- right R-module). It is noetherian by corollary 2.3.Then, by proposition 2.2, A must be noetherian (Or A is f.g right R- module $\Rightarrow$ there is f.g free right R module F with finite basis and an epimorphism $\pi: F \rightarrow A$. Since F is isomorphism to a direct sum of a finite number of copies of noetherian module R_R, F is noetherian by corollary 2 .3. Then $A \cong F/\ker\pi$ (module homomorphism i.e.π: $F \rightarrow$ A is an epimorphism, then $F/\ker\pi \cong \mathrm{Im}(\pi) = A$). Therefore A is noetherian by proposition 2.2.

Corollary 2.5 let S be a subring of a ring R. If S is a right noetherian and R is finitely generated as a right S-module, and then R is a right noetherian.

Proof: by corollary 2.4 R is noetherian as a right S-module. Since all right ideals of R are also right S-module (S_R), the ACC on the right ideals follows. Using corollary 2.5 we obtain some easy examples of non-commutative rings:

Examples: a) Rings of quaternion H is subring of $R = M_2(C)$

b) Subring of $M_n(R)$ in the context of PIR (principal ideal ring)

Consider the ring of all upper triangular matrices,

$T_n(R) = \{(a_{ij} | a_{ij} \in R, a_{ij} = 0 \text{ for } i>j\}$ & $T_n(R)$ is a subring of $M_n(R)$. Let $M_n(R)$ is f.g as $T_n(R)$-module and let $T_n(R)$ be a right noetherian module ($T_n(R)$ is a right noetherian $\Leftrightarrow$ R is a right noetherian), then $M_n(R)$ is a right

Noetherian. As an R-module $M_n(R) \cong R^{n^2}$ so if R is right noetherian, $M_n(R)$ is right notherian as a right R-module and so as a right $M_n(R)$.

Remark 2.5.1 $R \subseteq S \subseteq M_n(R)$ are rings, and then S is a right Noetherian if and only if R is a right Noetherian.

Proof: R is a right Noetherian => $M_n(R)$ is a right Noetherian => S is a subring of $M_n(R)$ is a right Noetherian.
Note. If $\{I_j \mid j = 1, 2\ldots\}$ is a strictly ascending chain of right ideals of R, then $\{M_n(I_j)\}$ is similar chain in $M_n(R)$.

Proposition 2.6 If R is a module -finite algebra over a commutative noetherian rings, and then R is a noetherian ring.

Proof: The image of S in R is a noetherian subring S′ of the center of R such that R is finitely generated (right or left) S′-module. Thus R is noetherian ring by corollary 2.5.

Examples: a) let S=Z + Zi + Zj + Zk a subring of division ring **H**(ring of quaternion: four dimensional real vector space with basis {1, i, j, k,})
Non-commutative division ring .$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ Since S is finitely generated module over the commutative noetherian ring Z, then proposition 2.6 shows that S is a noetherian ring.

b) For another example, proposition 2.6 shows that for any positive integer n, the ring of n × n matrices over a commutative noetherian ring is noetherian.
For example let $K = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in Z\right\}$ K is a 2×2 matrices over a commutative Noetherian ring Z. $\therefore$K is noetherian.
This also holds for matrix rings over non- commutative Noetherian rings.

Definition2.6.1 Given a ring R and positive integer n, $M_n(R)$-the ring of all n ×n matrices over R. The standard n×n matrix units in $M_n(R)$ are matrices e_{ij} (for i, j = 1,…) such that e_{ij} has 1for i, j entry and 0 for the other entries.

Proposition 2.7 let R be a right noetherian ring and S a subring of a matrix ring M_n (R). If S contains the subring $R'=\left\{\begin{pmatrix} r & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r \end{pmatrix} \mid r\in R\right\}$ of all scalar matrices, then S is a right noetherian. In particular, M_n (R) is a right noetherian ring.

Proof: Clearly $R'\cong R$, hence R' is a right noetherian ring .$M_n(R)$ is generated as a right R'-module by the standard n×n matrix units .Hence corollary 2.4 implies that M_n (R) is a right R'-module as all right ideals of S are also right R'-sub modules of M_n (R). We conclude that S is a right noetherian.

2.2. Formal triangular matrix rings

Definition 2.2.1 (Bimodule): Let S and T be rings .An (S,T) –bimodule structure is an abelian group B with a left S-module structure and a right T-module structure such that s(b t) = (s b)t for all s∈S, b∈B, t∈T .It is denoted by ${}_S B_T$ –An (S,T) bimodule ,B- abelian group and s(b t) =(s b)t for s∈S ,b∈B, and t∈T.

Definition 2.2.2(Sub bimodule): An (S,T) sub bimodule of B is any subgroup of B which is both a left S-submodule and a right T-submodule. Note: If C is a submodule of B, the factor group B /C is a bi- module.

Example1) If S is a ring and T a subring, then S itself (or an ideal of S) can be regard as an (S, T) bimodule (or as a (T, S) bimodule).

2) If B is a right module over a ring T and S is a subring of End_T (B) (End_T (B) = {f: B→B is homomorphism and B is abelian group}).Then B is an (S, T)-bi module. If $I \subseteq J$ is ideals in a ring S, then J/I are an (S, S) bimodule. Let ${}_S B_T$ be bimodule and write $\begin{pmatrix} S & B \\ O & T \end{pmatrix}$ for the abelian group S⊕B⊕T where $(s, b, t) \in$ S⊕B⊕T written as formal 2×2 matrices$\begin{pmatrix} s & b \\ o & t \end{pmatrix}$. B must be closed under addition, left multiplication by elements of S and right multiplication by elements of T.

$R=\begin{pmatrix} S & B \\ O & T \end{pmatrix} =\left\{\begin{pmatrix} s & b \\ o & t \end{pmatrix} \mid s\in S, b\in B, t\in T\right\}$ is a ring. i. e.

i) $\langle R,+\rangle$ is abelian group

ii) R is closed, associative and distributive under the operation multiplication.

∴(R, +, .) is a ring. Similarly $\begin{pmatrix} T & O \\ B & S \end{pmatrix}$ is a ring for the abelian group T⊕B⊕S where (t, b, s)$\in$T$\oplus$B$\oplus$S be written as a formal 2×2 matrices $\begin{pmatrix} t & o \\ b & s \end{pmatrix}$. $R' = \begin{pmatrix} T & O \\ B & S \end{pmatrix} = \left\{ \begin{pmatrix} t & o \\ b & s \end{pmatrix} \mid t \in T, b \in B, s \in S \right\}$ (R′, +, .) is a ring (lower triangular matrices ring).

Example: Show that $R' = \begin{pmatrix} T & O \\ B & S \end{pmatrix} \cong \begin{pmatrix} S & B \\ O & T \end{pmatrix} = R$.

Proof: define the map f: R→R′ by f (A) = A′ where A$\in$R, A′,$\in$R′, 1)f is well defined i.e. $a_1 = a_2 \Rightarrow f(a_1) = f(a_2)$ for all $a_1, a_2 \in R$, 2) f is homomorphism i.e. $f(a+b) = f(a) + f(b)$ & $f(ab) = f(a)\, f(b)$ $\forall a, b \in R$ &
3) $\forall a \in R' \,\exists\, b \in R \ni f(b) = a \Rightarrow$ f is epimorphism
$\therefore$ R/kerf $\cong$ R′ by FTH (Homomorphism Theorem: if f: R→R′ is a ring homomorphism then R/kerf $\cong$ R′) $\Rightarrow$ R$\cong$ R′, since kerf $= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

The symbol $\begin{pmatrix} s & b \\ o & t \end{pmatrix}$ will form a ring under matrix addition and multiplication provided only that B is simultaneously a left S-module and a right T-module satisfying an associative law connecting its left and right module structures. Clearly the set $\begin{pmatrix} O & B \\ O & O \end{pmatrix}$ of matrices of the form $\begin{pmatrix} o & b \\ o & o \end{pmatrix}$ is an ideal of $\begin{pmatrix} S & B \\ O & T \end{pmatrix}$.

Definition 2.2.3 A formal triangular matrix ring is any of the form $\begin{pmatrix} S & B \\ O & T \end{pmatrix}$ or $\begin{pmatrix} T & O \\ B & S \end{pmatrix}$ where S and T are rings and B an (S, T)-bi module.

If S and T are subring of U, and B is (S, T) Submodule of U, and the formal triangular matrix ring $\begin{pmatrix} S & B \\ O & T \end{pmatrix}$ is isomorphic to the subring of $M_2(U)$ consisting of all matrices of the form $\begin{pmatrix} s & b \\ o & t \end{pmatrix}$ with $s \in S$, $b \in B$, $t \in T$.

Proposition 2.8. Let $R = \begin{pmatrix} S & B \\ O & T \end{pmatrix}$ be a formal triangular matrix ring. Then R is a right noetherian if and only if S and T are right noetherian and B_T is finitely generated (f.g). Similarly R is a left Noetherian if S and T are left noetherian and ${}_SB$ is f.g.

Proof: Assume first that S and T are right noetherian and B_T f.g. As S & T are right noetherian, by corollary 2.3 S×T is right noetherian $\Rightarrow$ S$\oplus$0$\oplus$T is right noetherian. This implies that the diagonal subring $\begin{pmatrix} S & O \\ O & T \end{pmatrix}$ is isomorphic to S×T (i.e. S$\oplus$0$\oplus$T for finite case) and let the elements $b_1, \ldots, b_n$ generate B as a right T-module ,then the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} o & b_1 \\ o & o \end{pmatrix}, \begin{pmatrix} o & b_2 \\ o & o \end{pmatrix}, \ldots \begin{pmatrix} o & b_n \\ o & o \end{pmatrix}$

generate R as a right $\begin{pmatrix} S & O \\ O & T \end{pmatrix}$ module consequently corollary 2.5 Shows R is a right Noetherian. Conversely, assume R is a right Noetherian. The projection maps f:$\begin{pmatrix} s & b \\ o & t \end{pmatrix} \rightarrow$ s and g: $\begin{pmatrix} s & b \\ o & t \end{pmatrix} \rightarrow$ t are surjective ring homomorphism of R on to S and of R on to T, Since the map is ring homomorphism R/ker f $\cong S$ and R/ kerg $\cong T$ (by homomorphism theorem) $\therefore$ S and T are right noetherian. We see that $\begin{pmatrix} O & B \\ O & O \end{pmatrix}$ is a right ideal of R and must have a finite list of generators$\begin{pmatrix} o & b_1 \\ o & o \end{pmatrix}, \begin{pmatrix} o & b_2 \\ o & o \end{pmatrix}, \dots, \begin{pmatrix} o & b_n \\ o & o \end{pmatrix}$ from which we conclude that the elements $b_1, \dots, b_n$ generate B_T.

Example: The ring R = $\begin{pmatrix} Z & Q \\ O & Q \end{pmatrix}$ is a right noetherian but not left noetherian. Since ${}_ZQ$ is not Noetherian.

Solution: Z and Q are noetherian. Q is f.g over Q.
$\therefore$R is right Noetherian.
Z and Q are Noetherian but ${}_ZQ$ is not f.g.
$\therefore$ R is not left Noetherian.
To show it clearly, for non negative integer k consider the set
$A_k = \left\{ \begin{pmatrix} 0 & m/2^k \\ 0 & 0 \end{pmatrix} | m \in Z \right\}$.Then A_k is a left ideal of R (1. $A_k \neq \varnothing$, 2.A_1, $A_2 \in A_k \Rightarrow A_1 - A_2 \in A_k$ 3. $rA_1 \in A_k, \forall r \in R, A_1 \in A_K$ $\therefore A_k$ is a left ideal of R) further $A_K < A_{K+1}$ as $m/2^k = 2m/2^{k+1}$ and $\begin{pmatrix} 0 & 1/2^{k+1} \\ 0 & 0 \end{pmatrix} \notin A_k$.Thus we get a non terminating strictly ascending chain $A_O < A_1 < A_2 < A_3 < \dots$ of left ideals of R. This shows that R is not left noetherian.
To prove that R is right noetherian it is sufficient to establish that each non zero right ideal of R is finitely generated. Let A be a non zero right ideal of R. let X = $\left\{ n \in Z \mid \begin{pmatrix} n & x \\ o & y \end{pmatrix} \in A \text{ for some } x, y \in Q \right\}$. It is clear that X is an ideal in Z. Hence X = (n_o) for some $n_o \in$Z, because Z is a principal ideal ring. If A is a non trivial ideal in R=$\begin{pmatrix} Z & Q \\ O & Q \end{pmatrix}$ then A is one of the following:

i) A = $\begin{pmatrix} n_o & 1 \\ 0 & 1 \end{pmatrix}$R = $\begin{pmatrix} n_o Z & Q \\ 0 & Q \end{pmatrix}$($n_o$ a non zero fixed element of Z)

ii) A = $\begin{pmatrix} n_o & 1 \\ 0 & 0 \end{pmatrix}$R = $\begin{pmatrix} n_o Z & Q \\ 0 & 0 \end{pmatrix}$

iii) A= $\begin{pmatrix} o & \beta \\ o & \gamma \end{pmatrix}$R = $\left\{ \begin{pmatrix} o & \beta y \\ o & \gamma y \end{pmatrix} \mid y \in Q \right\}$

(β, γ are fixed elements of Q).

In particular, A= $\begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}$ according to whether $\gamma = 0, \beta = 0$.

iv) A = $\begin{pmatrix} 0 & Q \\ 0 & Q \end{pmatrix}$. The first three are principal ideals, the last is not principal but can be generated by two elements. Thus every right ideal A of R is f. g $\therefore$ R is right noetherian.

2.3. The Hilbert Basis Theorem

A large class of examples of Noetherian rings (particularly commutative ones) is revealed by this famous theorem.

Theorem 2.9 [Hilbert Basis Theorem]: Let S = R[x] be a polynomial ring in one indeterminate. If the coefficient ring R is right (left) noetherian, then so is S. (let R be a commutative noetherian ring, then the polynomial ring R[x] is also noetherian)

Proof: The two cases are symmetric .let R be a right noetherian and S = R[x]. Wts: any right ideal I of S is finitely generated. Let $I \neq 0$.

Step1. Let J be the set of leading coefficients of elements of I together with 0 i.e $J=\{r\in R / rx^{d} + r_{d-1} x^{d-1} + \ldots\ldots + r_0 \in I \text{ for some } r_{d-1},\ldots\ldots, r_0 \in R\}$
Claim: J is a right ideal of R. 1) $J \neq \varnothing$ 2) $J \subseteq R$ 3) Let r, $r' \in J$ are leading coefficients of elements s, $s' \in I$ with degree d, d′ respectively, replacing s by s $x^{d'}$ and s′ by $s'x^d$ and assume s and s′ have same degree
$\Rightarrow r x^{d+d'} - r' x^{d+d'} = (r - r') x^{d+d'} \in I \Rightarrow r - r' \in J$.
4) $r\in J$, $a \in R \Rightarrow ra \in R$ and $rax^d + \ldots \in I$
$\Rightarrow r a \in J$
$\therefore$ J is a right ideal of R.
Step2. Since R is right noetherian J is finitely generated. Let $r_1,\ldots,r_k$ be a finite list of generators for J; assume they are all non zero. Each r_i occurs as the leading coefficient of a polynomial $p_i \in I$ of some degree n_i. Set n = max $\{n_1,\ldots,n_k\}$and replace each p_i by $p_i x^{n-ni}$. Thus wlog assume degree of p_i is n.
Step3. Set $N= R+ Rx+\ldots+Rx^{n-1}=R+ x R+\ldots+x^{n-1}R$, the set of elements of S with degree < n. This is not ideal of S; but it is a left and right R-submodule viewed as a right R-module, N is f.g and so is noetherian by corollary 2.4 (since R is noetherian and all f.g right R-modules are noetherian).Now $I\cap N$ is a right R-submodule of N, consequently it must be f.g. Let $q_1,\ldots\ldots,q_t$ be a finite list of right R module generators for $I\cap N$.

Step4. claim: $p_1,\dots,p_k,q_1\dots,q_t$ generate I. I_0 denote the right ideal of S generated by these polynomials ,then $I_0 \subseteq I$ and if remains to show that any polynomials $p\in I$ lies in I_0 (i.e. I_0 is maximal) .If p has degree $< n$, since in that case $p\in I\cap N$ and $p=q_1 a_1+\dots q_t a_t$ for some $a_j\in R$.
Step5. Suppose that $p\in I$ has degree $m > n$ and that I_0 contains all elements of I with degree less than m, let r be leading coefficient of p .then $r\in J$, and so $r=r_1a_1+\dots\dots+r_k a_k$ for some $a_i \in R$. Set $q = (p_1a_1+\dots+ p_k a_k)x^{m-n}$, an element of I_0 with degree m and leading coefficient r. Now p-q is an element of I with degree less than m. By induction hypothesis $p- q\in Io$, and thus $p\in Io$. Therefore I=Io and we are done.

Note. Any polynomial ring $R[x_1,\dots,x_n]$ in a finite number of in determinates over a right (left) noetherian ring R is right (left) noetherian. Since we may view $R[x_1,\dots,x_n]$ as a polynomial ring in the single indeterminate x_n, with coefficients from the ring $R[x_1\dots x_{n-1}]$. Because $R[x_1,\dots, x_n] = R'[x_n]$ where $R' = R[x_1,\dots,x_{n-1}]$ it clearly suffices to prove for the n =1.

Example: Consider Q[x], as Q is noetherian, Q[x] is also notherian

Corollary 2.10: let R be algebra over a field K. If R is commutative and finitely generated as a k-algebra, then R is noetherian.

Proof: Let $x_1, x_2\dots, x_n$ generated R as a k-algebra and let $S = k[y_1,\dots y_n]$ be a polynomial ring over k in n independent indeterminate. Since R is commutative, there exists a k-algebra map $\phi: S\rightarrow R$ such that $\phi(y_i) = x_i$ for each i and ϕ is surjective because x_i generators R. Hence, $R\cong S/\ker\phi$ is a noetherian ring and therefore R is noetherian. Note: Non commutative finitely generated algebras need not be noetherian.

Example: let V be a countable infinite dimensional vector space over K with a basis $\{v_1,v_2,\dots\}$.Define $s, t\in \mathrm{End}_k(v)$ so that $s(v_i) = v_{i+1}$for all $i>1$ and $t(v_1) = 0$, and let R be K-algebra of $\mathrm{End}_k(v)$ generated by s and t. show that R is neither right nor left noetherian.
Proof: define $e_1, e_2 ,\dots$ in $\mathrm{End}_k(v)$ so that $e_i(v_i)$ for all i while $e_j(v_j) = 0$ for all $i \neq j$ and each $e_i\in R$. Then $\sum_i e_i R$ and $\sum_i Re_i$ are not f.g

$\therefore$ R is neither right nor left noetherian.

2.4. Skew polynomial rings twisted by automorphisms

This section concerns polynomials over a ring R in a variable x which is not assumed to commute with the elements of R. Let R be a ring α an automorphism of R, and x an indeterminate. Let S be the set of all formal expression $a_o + a_1 x + \ldots + a_n x^n$ where n is a non negative integer and then $a_i \in$R, $\sum_i a_i x^i$, $i \in \mathrm{N}_0$, $a_i \in$ R as an infinite sum in which almost all of the coefficients a_i is zero. We define addition operation in S as: $\sum_i a_i x^i + \sum_i b_i x^i = \sum a_i + b_i)x^i$. We define $x\,a = \alpha(a)\,x$ and iterate that rule to obtain $x^i\, a = \alpha^i(a)x^i$. This leads to define the following multiplication rule in S:

$$(\sum_i a_i x^i)(\sum_i b_i x^i) = \sum_{i,j} a_i \alpha^i (b_j) x^{i+j}$$

$$= \sum_k \sum_{i+j=k} a_i \alpha^i (b_j) x^k$$

When two formal expressions define the same elements of S, two elements of S are same only if there coefficients are same i.e. $\sum_i a_i x^i = \sum_i b_i x^i$ if and only if $a_i = b_i$ for all i. The elements 1, x, x^2,… in S are linearly independent over R .Since every elements of S is a linear combination of these powers S is thus a free left R-module with powers of x forming a basis. This leads to the following definition:

Definition 2.4.1 Let R be a ring and α an automorphism of R.
S = R[x;α] to mean that
a) S is a ring containing R as a subring;
b) x is an element of S
c) S is a free left R-module with basis $\{1, x, x^2 \ldots\}$
d) $x\,r = \alpha(r)\,x$ for all $r \in$R. Whenever S = R[x;α],then S is a skew polynomial ring over R. Thus, the expression S =R[x;α] can be used either to introduce a new ring S (constructed as above) or to say that a given ring S and an element x satisfy conditions (a)-(d).

Remark 2.4.1.1 The element x in R[x; α] is just a ring element with certain special property not "indeterminate". The advantage of such definition is that to say some ring equals a skew polynomial ring S = R[x;α] rather than having to say that it is isomorphic to R[x; α]. This is useful in the context of ordinary polynomial rings.

Proposition 2.4. 1.2 Let R be a ring, α is an automorphism of R, then S = R[x; α] of skew polynomials over R is a ring.

Proof: Let f $=\sum_i a_i x^i$, g $=\sum_i b_i x^i$ and h $=\sum_i c_i x^i$ for all $a_i, b_i, c_i \in$ R and $i = 0, 1, \ldots$. Then i) $\sum_i a_i x^i + \sum_i b_i x^i = \sum_i (a_i + b_i) x^i \Rightarrow a_i + b_i \in$ R(since R is a ring)

$\therefore \sum_i (a_i + b_i) x^i \in$ R[x; α]

$\Rightarrow$ R[x; α] is closed under addition.

ii) $[\sum_i a_i x^i + \sum_i b_i x^i] + (\sum_i c_i x^i) = \sum_i (a_i + b_i) x^i + \sum_i c_i x^i$

$= \sum_i ((a_i + b_i) + c_i) x^i = \sum_i (a_i + (b_i + c_i)) x^i$ since R is ring.

$= \sum_i a_i x^i + \sum_i (b_i + c_i) x^i = \sum_i a_i x^i + [\sum_i b_i x^i + \sum_i c_i x^i]$

$\therefore$ Associative law of addition holds in S = R[x; α]

iii) There exists 0(x) $\in$ S, $\forall$ f$\in$S such that 0(x) + f(x) = f(x) + 0(x), 0(x) is the identity (zero polynomials of S under +.

iv) $\forall$ f(x)$\in$S, $\exists$ -f(x)$\in$S $\ni$ f(x) + (-f(x)) = 0(x) = -f(x) + f(x)

-f(x) is the additive inverse of f(x)

$\Rightarrow$ Every element of S has additive inverse

v) f (x) + g(x) $= \sum_i a_i x^i + \sum_i b_i x^i = \sum_i (a_i + b_i) x^i = \sum_i (b_i + a_i) x^i$ since R is a ring

$= \sum_i b_i x^i + \sum_i a_i x^i =$ g(x) + f(x)

$\therefore$ (S, +) is an abelian group

vi)$\forall a_i, b_j, c_k$, we have $[(\sum_{i=0}^{\infty} a_i x^i)(\sum_{j=0}^{\infty} b_j x^j)][\sum_{k=0}^{\infty} c_k x^k]$

$$= [\sum_{i=0}^{\infty} (\sum_{i=0}^{n} a_i b_{n-i}) x^n][\sum_{k=0}^{\infty} c_k x^k]$$

$$= \sum_{s=0}^{\infty} [\sum_{n=0}^{s} (\sum_{i=0}^{n} a_i b_{n-i}) c_{s-n}] x^s$$

$$= \sum_{s=0}^{\infty} (\sum_{i+j+k=s} a_i b_j c_k) x^s$$

$$= (\sum_{i=0}^{\infty} a_i x^i)[\sum_{m=0}^{\infty} (\sum_{j=0}^{m} b_j c_{m-s}) x^m]$$

$$= (\sum_{i=0}^{\infty} a_i x^i)[\sum_{j=0}^{\infty} b_j x^j \sum_{k=0}^{\infty} c_k x^k]$$

$\therefore$. is associative in S.

Vii) $\sum_{i=0}^{\infty} a_i x^i)[\sum_{j=0}^{\infty} b_j x^j + \sum_{k=0}^{\infty} c_k x^k] = (\sum_{i=0}^{\infty} a_i x^i \sum_{j=0}^{\infty} b_j x^j) + (\sum_{i=0}^{\infty} a_i x^i \sum_{k=0}^{\infty} c_k x^k)$

$$=\sum_{i,j} a_i\alpha^i(b_j)x^{i+j}+\sum_{i,k} a_i\alpha^i(c_k)x^{i+k}$$

$$=\sum_{m}\Big(\sum_{i+j=m} a_i\alpha^i(b_j)\Big)x^m+\sum_{n}\Big(\sum_{i+k=n} a_i\alpha^i(c_k)\Big)x^n$$. Since

$\sum_{i+j=m} a_i\alpha^i(b_j), \sum_{i=k=n} a_i\alpha^i(c_k)\in R$, each of the summand is in S.

$\therefore$ distributive law holds in S.

Hence S = R[x; α] is a ring (by steps i, ii, iii, iv, v , vi & vii). Since α is a ring automorphism, we have $\alpha(1) = 1$ and so 1 serves as multiplicative identity element of R[x;α]. When R is identified set of elements of S involving no positive powers of x. it becomes a subring (i.e. a_i, $b_i \in$ R$\Rightarrow$ $a_i + b_i \in$ R& $a_i - b_i \in$ R) of S. This proof shows that, Given ring R& automorphism α of R, a skew polynomial ring S = R[x;α] does exist.

Example: Given field C of complex number and α be an automorphism of complex conjugate. In C[x;α], $a\in$ C. Then for complex number i, we have $(ia)^2 = (ia)\,(ia) = i(ai)\,a = i(-i)a^2 = -i^2\,a^2 = a^2$

Definition2.4.3 (Universal mapping property): A pair (U,ε) is called universal for a group G (with respect to abelian epimorphism images) if U is an abelian groupε: G→U is an epimorphism and if given any abelian group A and homomorphism f: G→A, then there is a unique homomorphism g: U→A For Which f = gε. i.e. the diagram

$$\begin{tikzcd}
G \arrow[rr, "f"] \arrow[dr, "\varepsilon"'] & & A \\
& U \arrow[ur, "g"'] &
\end{tikzcd}$$

is commutative. We say that f can be "factored through" U.

Proposition 2.4.3.1 If a universal pair (U,ε) exists for a group G then U is unique (up to isomorphism). Proof: Let (U$_1$,ε$_1$) be an other universal pair for G. We have

$$\begin{tikzcd}
G \arrow[rr, "\varepsilon_1"] \arrow[dr, "\varepsilon"'] & & U_1 \\
& U \arrow[ur, "g_1"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
G \arrow[rr, "\varepsilon"] \arrow[dr, "\varepsilon_1"'] & & U \\
& U_1 \arrow[ur, "g_2"'] &
\end{tikzcd}$$

Or $\varepsilon_1 = g_1\varepsilon$ and $\varepsilon = g_2\varepsilon_1$. Thus $\varepsilon_1 = g_1g_2\varepsilon_1$ and $\varepsilon = g_2g_1\varepsilon$. But then we have

$$\begin{tikzcd}
G \arrow[rr, "\varepsilon"] \arrow[dr, "\varepsilon"'] & & U \\
& U \arrow[ur, "1_U"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
G \arrow[rr, "\varepsilon"] \arrow[dr, "\varepsilon"'] & & U \\
& U \arrow[ur, "g_2g_1"'] &
\end{tikzcd}$$

and by uniqueness in the definition of universal pair we see that $g_2g_1 = 1_U$, the identity map on U. Similarly g_1g_2 is the identity map on U_1 and so g_1 and g_2 are inverse isomorphism.

Lemma 2.11: Let R be a ring, α an automorphism of R and $S = R[x;\alpha]$. Suppose that we have a ring T, a ring homomorphism $\phi: R \to T$ and an element $y \in T$ such that $y\,\phi(r) = \phi\,\alpha(r)\,y$ for all $r \in R$. Then there is a unique ring homomorphism $\psi: S \to T$ such that $\psi|R = \phi$ and $\psi(x) = y$.

The diagram

$$\begin{tikzcd}
S = R[x;\alpha] \arrow[rr] \arrow[dr, "C"'] & & T \\
& R \arrow[ur] &
\end{tikzcd}$$

is commutative.

Proof: Let a map $\psi: S \to T$ be given by the rule $\psi(\sum_i a_i x^i) = \sum_i \phi(a_i) y^i$

$$\begin{aligned}
\sum_i a_i x^i = \sum_i b_i x^i &\Rightarrow \sum_i a_i x^i - \sum_i b_i x^i = 0 \\
&\Rightarrow \sum_i (a_i - b_i) x^i = 0 \\
&\Rightarrow a_i - b_i = 0 \text{ (since } x^{i}\text{'s are linearly independent).}
\end{aligned}$$

$$\Rightarrow \phi(a_i - b_i) = \phi(0)$$
$$\Rightarrow \phi(a_i) - \phi(b_i) = 0 \text{ (because } \phi \text{ is a ring homomorphism)}$$
$$\Rightarrow \phi(a_i) = \phi(b_i)$$

Hence $\sum_i \phi(a_i) y^i = \sum_i \phi(b_i) y^i$

$$\Rightarrow \psi(\sum_i a_i x^i) = \psi(\sum_i b_i x^i)$$

Hence ψ is well defined.

Let ψ': S→T be an other map given by the rule $\psi'(\sum_i a_i x^i) = \sum_i \phi(a_i) y^i$,

then $\psi'(\sum_i a_i x^i) = \sum_i \phi(a_i) y^i = \psi(\sum_i a_i x^i) \Rightarrow \psi = \psi'$. Hence ψ is unique.

Define ψ: S→T such that $y\phi(r) = \phi\,\alpha(r) y$ and by induction
$y^i \phi(r) = \phi\alpha^i(r)\, y^i$ for all $i \in Z^+$ and $r \in R$.
Hence$[\psi(\sum_i aix^i)]\,[\psi(\sum_j b_j x^j)] = [\sum_i \phi(a_i) y^i]\,[\sum_j \phi(b_j) y^j]$
$= \sum_{i,j} \phi(a_i)\phi\alpha^i(b_j) y^{i+j} = \sum_k (\sum_{i+j=k} \phi(a_i)\phi\alpha^i(b_j)) y^k$
$= \psi[\sum_k (\sum_{i+j=k} a_i \alpha^i(b_j)) x^k]$
$= \psi[(\sum_i a_i x^i)(\sum_j b_j x^j)]$ for all elements $\sum_i a_i x^i$ and $\sum_j b_j x^j$ in S.
Let us assume t, s∈S, $\psi(s+t) = \psi(\sum_i a_i x^i + \sum_i b_i x^i) = \psi(\sum (a_i + b_i) x^i)$

$$= \sum_i \phi(a_i + b_i) y^i = \sum_i (\phi(a_i) + \phi(b_i)) y^i \quad (\because \phi \text{ is a homomorphism})$$
$$= \sum_i \phi(a_i) y^i + \sum_i \phi(b_i) y^i = \psi(\sum_i a_i x^i) + \psi(\sum_i a_i x^i)$$
$$= \psi(s) + \psi(t), \text{ where}$$

$s = \sum_i a_i x^i$, $t = \sum_i a_i x^i$. Therefore ψ is a ring homomorphism as required.

Hence ψ is a unique ring homomorphism.

Corollary 2.12: Let R be a ring and α be an automorphism of R. Suppose that $S = R[x; \alpha]$ and $S' = R[x'; \alpha]$.Then there is a unique ring isomorphism ψ: S→S′ such that $\psi(x) = x'$ and ψ|R is identity on R.

Proof: First apply lemma 2.11 with ϕ: R→S′ being the inclusion map; we obtain a unique ring homomorphismψ: S→S′ such that $\psi(x) = x'$ and ψ|R= ϕ (ϕ|R is the identity on R). By symmetry, lemma 2.11 also provides a ring homomorphismψ′: S′→S such that $\psi(x') = x$ and ψ′|R is the identity on R. Nowψ′ψ: S′→S is a ring homomorphism such that $(\psi'\psi)(x) = x$ and (ψ′ψ)|R is identity on R. Hence, the uniqueness part of lemma 2.11 (where now T = S and $y = x$) implies that ψ′ψ equals the identity maps. Similarly,

$\psi\psi'$ equals the identity maps on S′. Therefore ψ and ψ' are mutually inverse isomorphism.

Example: Let R = K[y] is an ordinary polynomial ring over a field K. Given a non- zero q∈K. We define a k-algebra automorphism α on R such that $\alpha(y)$ = q(y) (In function notation,$\alpha(p(y)) = p(q\,y)$ for $p(y) \in K[y]$).Now let S= R[X;α]. Then $x\,y = \alpha(y)\,x = q\,y\,x$, the basic ″computational rule ″ in S. Since the polynomials in R are k-linear combination of y, elements of S can be written in the form $\sum_{i,j} \lambda_{ij} y^i x^j$ for scalars λ_{ij} (all but finitely many of which are zero), multiplication in S follows the rule

$$(\sum_{i,j} \delta_{ij} y^i x^j)(\sum_{s,t} \mu_{st} y^s x^t) = \sum_{i,j,s,t} \lambda_{i,j} \mu_{s,t} q^{js} y^{i+s} x^{j+t}$$

$$= \sum_{l,m} (\sum_{i+s=l,j+t=m} \lambda ij \mu st q^{js}) y^l x^m.$$

Definition2.12.1 Let K be a field and q∈K^X (multiplicative group of non zero elements) .The quantized coordinate ring of k^2(corresponding to the choice of q) is a k-algebra, denoted by $Q_q(k^2)$, presented by Two generators x, y and the relations x y = q x y. In short $Q_q(k^2) = k\langle x, y / xy = qxy\rangle$ in algebraic geometry, K^2 is the affine pane over k. Hence $Q_q(k^2)$ is also known as a coordinate ring of a quantum Plane (over k)

Examples: a)The polynomial ring K[x] is generated as a K-algebra by x and 0 and certainly $x\,0 = q0x$.

b) The basic field k itself is generated as a k-algebra by1 and 0, and 1 0=q01

c) If $k\langle x, y\rangle$ is a free algebra on two letters x and y (which satisfies no relation at all), by x y - q y x, we are declaring that
$Q_q(k^2) = k\langle x, y\rangle / \langle xy - qyx\rangle$.
-The elements x and y in the definition $Q_q(k^2)$ are the cosets of x and y.
It follows from this description that $Q_q(k^2)$ satisfies a universal mapping property and therefore uniquely determined up to isomorphism of k-algebra.
- $Q_q(k^2) \cong S \Rightarrow Q_q(k^2)$ is a skew polynomial ring.

Proposition 2.13: Let K be a field and q∈k^x. Then $Q_q(k^2)$ = k[y] [x:α], where k[y] is a polynomial ring and α is a k-algebra automorphism of k[y] such that α(y) = q y, $Q_q(k^2)$ = k [x] [y;β], where β is a k-algebra automorphism of the polynomial ring k[x] such that $\beta(x) = q^{-1}x$.

Definition 2.13.1 let k be a field. A multiplicatively anti symmetric matrix over k is an n×n matrix $q = (q_{ij})$ with entries $q_{ij} \in k^x$ such that $q_{ii} = 1$ for i and $q_{ji} = qij$ for all i, j. Given such a matrix, the corresponding multiparameter quantized coordinate ring of affine n-space, or a multiparameter quantum n-space is the k-algebra $Q_q(k^n)$ presented by generators $x_1, \ldots x_n$ and relations $x_i x_j = q_{ij} x_j x_i$ for all i, j . In short, we write, $Q_q(k^n) = k \langle x_1, \ldots, x_n \mid x_i x_j = q_{ij} x_j x_i for\ 1 \leq i,j \leq n \rangle$. As a special case, fix $q \in k^v$ and q be a unique multiplicative antisymmetric n×n matrix with $q_{ij} = q$ for all $i < j$. We use the subscript q in place of **q.** thus, $Q_q(k^n)$ is the k-algebra with generators $x_1, \ldots x_n$ and relations $x_i x_j = q x_j x_i$ for all i< j. It is called single parameter quantum n-space.

2.5. Skew Laurent Rings

Definition 2.5.1 Let R be a ring and α an automorphism of R. We write $T = R[x^{\pm 1}; \alpha]$ to mean that

a) T is a ring containing R as a subring;

b) x is an invertible element of T

c) T is a free left R-module with basis$\{1, x, x^{-1}, x^2, x^{-2} \ldots\}$

d) $x\, r = \alpha(r) x$ for all $r \in R$. When $T = R[x^{\pm 1}; \alpha]$, we say that T is a skew Laurent ring over R, or a skew Laurent extension of R.

Examples: let α an automorphism of a ring R.

a) The skew Laurent ring $T = R[x^{\pm 1}, \alpha]$ exists

Sol. i) Let $f = \sum_{i \epsilon Z} a_i x^i$, $g = \sum_{i \epsilon Z} b_i x^i$, $h = \sum_{i \epsilon Z} c_i x^i \in T$, then we can easily verify that (T,+) is abelian group.

ii) To show associativity: $[(\sum_{i \epsilon Z} a_i x^i) \sum_{j \epsilon Z} b_j x^j)] [(\sum_{k \epsilon Z} c_k x^k]$ $= \sum_{i,j,k} a_i \alpha^i(b_j) \alpha^{i+j+k}(c_k) x^{i+j+k} = [(\sum_{i \epsilon Z} a_i x^i)][\sum_{j \epsilon Z} b_j x^j)(\sum_{k \epsilon Z} c_k x^k]$.

iii) Obviously distributive law holds,

∴T is a ring. $\alpha(1) = 1$ since α is an automorphism. Therefore given a ring R and an automorphism α skew Laurent ring T exists.

b) If $T = R[x^{\pm 1}; \alpha]$ and $S = \sum_{i=0}^{\infty} R x^i \subseteq T$, then S is a subring of T and that $S = R[x; \alpha]$.

proof: i) $0 \in S$, ii) $\sum_{i=0}^{\infty} (a_i + b_i) x^i \in S$, since $a_i + b_i \in R$

iii) $(\sum_{i=0}^{\infty} a_i x^i)(\sum_{j=0}^{\infty} b_j x^j) = \sum_{i,j=0}^{\infty} a_i \alpha^i(b_j) x^{i+j} \in S$.

-let α automorphisms of a ring R. α^{-1} is an automorphism of R^{op} and that $R[x; \alpha]^{op} = R^{OP}[x; \alpha^{-1}]$

- Skew Laurent ring satisfy universal mapping property and are unique up to isomorphism (refer the proposition under universal mapping property & lemma 2.11)

Example: Let K be a field and let H be the Heisenberg group, which is presented by three generators x, y, z & the relations $x\, yx^{-1}\, y^{-1} = z$, $x\, z\ =\ z\, x$,yz=zy, z is central. Elements of H can be uniquely written as products $z^i\, y^j x^m$ for integers i, j, m, and so these products form a basis for the group algebra k[x].since y and z commute, the subalgebra of k[x] generate by $y^{\pm1}$ and $z^{\pm1}$ is an ordinary Laurent polynomial ring k $[y^{\pm1}, z^{\pm1}]$ since xyx^{-1}=$z\ y$ and $xz\, x^{-1}$= z we have x k $[y^{\pm1}, z^{\pm1}]x^{-1}$=k $[y^{\pm1}, z^{\pm1}]$. In fact conjugating elements of k$[y^{\pm1}, z^{\pm1}]$ by x has the same effect as applying the k-algebra automorphism α such that α(y) = z y and α(z) = z that is x r =α(r) x for all r∈k $[y^{\pm1}, z^{\pm1}]$, since the products $z^i y^j x^m$ form a basis for k[H] over k, the powers x^m form a basis for k[H] as a free left module over k$[y^{\pm1}, z^{\pm1}]$. Therefore we conclude that k[H] = k$[y^{\pm}, z^{\pm1}]$[$x^{\pm1}$,α] a skew Laurent extension of Laurent polynomial ring.

Definition 2.5.2 let k be a field and q∈k^x. The quantized coordinate ring of $(k^x)^2$ (corresponding to the choice of q) is a k-algebra Qq$(K^x)^2$ presented by generators x, x',y, y' and $xx' = x'x\ =\ y\, y' = y'y = 1$, $x\, y =$ q $y\, x$.
In brief we may say that $Q_q(k^x)^2$ is presented by generators $x^{\pm1}$ and $y^{\pm1}$ satisfying $x\ y$ = q $y\ x$. In algebraic geometry $(k^x)^2$ is known as an algebraic torus (of rank 2),and hence $Q_q(k^x)^2$ pick up nick name quantum torus.

Example: For k a field and q∈k^x, we have $Q_q(k^x)^2$ = k$[y^{\pm1}][x^{\pm1}; \alpha]$, where k$[y^{\pm1}]$ is an ordinary Laurent polynomial ring and α is the k-algebra automorphism of k$[y^{\pm1}]$ such that $\alpha(y) = qy$.In particular, the subalgebra of Qq$(k^x)^2$ generated by x and y coincides with $Q_q(k^2)$.

Definition2.5.3 Let k be a field and q = (q_{ij}) a multiplicatively antisymmetric
n×n matrix over k. The corresponding multi-parameter quantum torus is the k-algebra $Q_q((k^x)^n)$ presented by generators $x_1^{\pm1.}\, x_2^{\pm1}, \ldots x_n^{\pm1}$ and $x_i\, x_j = q_{ij}\, x_j\, x_i$ for all i, j. The single parameter version $Q_q\, ((k^x)^n)$ for k^x is the special case when q_{ij} = q for i< j.

2.6. A Skew Hilbert Basis Theorem

We derive a version of the Hilbert basis theorem for the skew polynomial rings discussed above, an analogous result will follow as corollary.

Theorem 2.14: Let α be an automorphisms of a ring R and $S = R[x;\alpha]$. If R is right (left) noetherian, then so is S.

Proof: case I. Let us first assume that R is a right noetherian and any non zero ideal I of S is finitely generated.

Step1. Let J be the set of leading coefficients of elements of I, together with zero; $J = \{r \in R \mid rx^d + r_{d-1}x^{d-1} + \cdots + r_0 \in I$ for some $r_{d-1}, \cdots r_0 \in R\}$.J is an additive subgroup of R. Now consider elements $r \in J$ and $a \in R$; since J is the subgroup of R (i.e. $a, b \in J \Rightarrow ab^{-1} \in J$).wts: $r\,a \in J$. There is some skew polynomial of the form $p = r\,x^d$ + [lower terms] in I. $pa \in I$ and by replacing a by $\alpha^{-d}(a)$ we have $p\alpha^{-d}(a) \in I$ and $p\alpha^{-d}(a) = r\,ax^d$+[lower terms] hence $r\,a \in J$. This shows that J is a right ideal of R.

Step2. Since R is right noetherian, J is finitely generated, say $r_1, r_2, \ldots, r_k$ is a finite list of non zero generators for J. There exists $p_1, \ldots, p_k \in I$ such that p_i has leading coefficients r_i and some degree n_i. Set $n = \max\{n_1 \ldots, n_k\}$,and note that $p_i x^{n-ni}$ is an element of I with leading coefficients r_i but with degree n, Thus , there is no loss of generality in assuming that all the p_i have same degree n, that is, $p_i = r_i x^n$ + [lower terms]

Step3. Set $N = R + Rx + \ldots + Rx^{n-1}$, the set of elements of S with degree less than n. Observe that $N = R + xR + \ldots + x^{n-1}R$, since $bo + b_1x + \ldots + b_{n-1}x^{n-1} = b_0 + x\alpha^{-1}(b_1) + \ldots + \alpha^{1-n}(b_{n-1})$, $co + xc_1 + \ldots + x^{n-1}c_{n-1} = c_0 + \alpha(c_1)x + \ldots + \alpha^{n-1}(c_{n-1})x^{n-1}$ for all $b_j, c_j \in R$. Consequently, N is a right (as well a left) R sub module of S. Viewed as a right R-module, N is finitely generated, and so it is noetherian by corollary 2.4.Hence its submodule $I \cap N$ is finitely generated R-module; say $q_1, \ldots, q_t$ generate $I \cap N$.

Step4. Let Io be a right ideal of S generated by $p_1, \ldots, p_k, q_1, \ldots, q_t$. Then $Io \subseteq I$, and we claim that they are equal. If $p \in I$ with degree less than n, then $p \in I \cap N$ and $p = q_1a_1 + \ldots + q_t$ at for some $a_i \in R$, hence $p \in Io$.

Step5. Now consider some $p \in I$ with degree $m \geq n$, and suppose that all elements of I with degree less than m lies in Io. Let r be the leading coefficients of p; thus $p = rx^m$+ [lower terms]. Since $p \in I$, its leading coefficient r is in J, and so $r = r_1 a_1 + \ldots + r_k a_k$ for some $a_i \in R$. To construct an element of Io which also has degree m and leading coefficient r, we apply negative powers of α to the a_i(same in step1).more precisely, $p_i \alpha^{-n}(a_i) = r_i a_i x^n$+[lower terms] for all I, consequently, if $q = (p_1 \alpha^{-n}(a_1) + \ldots + p_k \alpha^{-n}(a_k)) x^{m-n}$,then $q \in I_o$ and $q = rx^m$+[lower terms]. Now p-q is an element of I with degree less than m. By induction hypothesis, $p - q \in Io$, and thus $p \in Io$. This induction has shown as I=Io, so that I is finitely generated. Therefore S is a right noetherian.

Case II: assume that R is a left noetherian and let I be a non- zero ideal of S. wts. I is f .g to prove the second case, we use the fact that α is an automorphisms of $R \Rightarrow \alpha^{-1}$ is an automorphisms of R^{op} and $R[x; \alpha]^{op} = R^{op}[x; \alpha^{-1}]$ with this change analogous of steps1-5 carried out. So I is f.g and therefore S is a left noetherian.

Immediate consequence of theorem 2.14 are that the quantum planes $Q_q(k^2)$ are noetherian and (by induction) the quantum n-spaces $Q_q(k^n)$ are noetherian.

Corollary 2.15. Let α be an automorphism of a ring R and $T = R[x^{\pm 1}; \alpha]$. If R is right (left) noetherian, then so is T.

Proof: Let $S = R[x; \alpha]$ and S is a subring of T. we proceed by relating the right ideals of T to those of S as follows,

Claim: If I is a right ideal of T, then $I \cap S$ is a right ideal of S and $I = (I \cap S)T$.

It is clear that $I \cap S$ is a right ideals of S and that $(I \cap S)$ I since $(I \cap S)T \subseteq IT \subseteq I \cap T \subseteq I$. If $p \in I$, then $p = a_m x^m + a_{m+1} x^{m+1} + \ldots + a_n x^n$ some integers $M \leq n$ and coefficients $a_i \in R$. $p \in (I \cap S)T$, and the claim is proved. Now suppose that R is right noetherian, and let $I_1 \subseteq I_2 \subseteq \ldots$ be an ascending chain of right ideals of T. Then $I_1 \cap S \subseteq I_2 \cap S \subseteq \ldots$ is an ascending chain of right ideals of S. since S is right noetherian by theorem 2.14, there is an index n such that $I_m \cap S = I_n \cap S$ for all $m \geq n$. Thus $I_m = (I_m \cap S)T = (I_n \cap S)T = I_n$ for all $m \geq n$, which establishes the ACC for right ideal of T. Therefore T is right noetherian. From this corollary we immediately obtain that all quantum tori $Q_q((k^x)^n)$ are noetherian.

Theorem 2.16 [Hall]: If k is a field and G a poly-cyclic by finite group, then the group algebra k[G] is a noetherian ring.

Proof: By assumption there exist subgroups $Go = (1) \subset G_1 \subset \ldots \subset G_n \subset G_{n+1} = G$ such that each G_{i-1} is a normal subgroup of G_i and G_i/G_{i-1} is infinite cyclic for $i = 1,\ldots,n$ while G/G_n is finite. There is a corresponding ascending sequence of subalgebras $[G_o] = k \subset k[G_1] \subset \ldots \subset k[G_n] \subset k[G]$, and we shall prove that each $k[G_i]$ noetherian. This is clear for $i = 0$. Now let $1 \leq i \leq n$ and assume that $k[G_{i-1}]$ is noetherian. Choose a coset $G_{i-1}x$ which generates the infinite cyclic group G_i/G_{i-1}. Then G i is the disjoint union of the cosets $G_{i-1}x^j$ for $j \in Z$, and so the rule $(g, j) \rightarrow gx^j$ gives a bijection $G_{i-1} \times Z \rightarrow G_i$. Consequently, $k[G_i] = \bigoplus_{j\in Z} \bigoplus_{g\in G_{i-1}} kgx^j = \bigoplus_{j\in Z}(\bigoplus_{g\in Gi-1} Kg)x^j = \bigoplus_{j\in Z} K[G_{i-1}]x^j$ That is, $k[G_i]$ is a free left module over $k[G_{i-1}]$ with basis $\{x^j/j \in Z\}$.Since G_{i-1} is a normal subgroup of G_i, we have $x G_{i-1} x^{-1} = k[G_{i-1}]$.AS a result, the rule $\alpha(r) = x r x^{-1}$ defines an automorphism α of $k[G_{i-1}]$. By definition of α, we have $x r = \alpha(r) x$ for all $r \in K[G_{i-1}]$, and thus $k[G_i] = k[G_{i-1}][x^{\pm 1}; \alpha]$ corollary 2.5 now shows that $k[G_i]$ is noetherian. Thus by induction, we conclude that $k[G_n]$ is noetherian. Now G is finite union of cosets $G_n y_1,\ldots,G_n y_t$, and so $k[G] = \sum_{j=0}^{t} K[G_n]y_j$ that is k[G] is finitely generated as a left $k[G_n]$-module.

Therefore k [G] is left noetherian by corollary 2.5, and by symmetry it is noetherian.

2.7. Simplicity in skew Laurent Rings

We conclude the chapter with a few considerations about (two sided ideals) in skew polynomials and skew Laurent rings. In theory of commutative ring, ideals are Ubiquitous (present every where being every where), but this is not longer true in the non-commutative theory because non-commutative noetherian rings need not have very many ideals. In the extreme case, there may be no ideals other than the two obvious ones.

Definition2.7.1 A simple ring is any non zero ring R such that the only ideals of R are 0 and R.(This terminology is only supposed to suggest that ideal theory of R is simple, not that the structure of R is necessarily simple in any other aspect). The only commutative simple rings are fields, but we shall soon find non- commutative noetherian simple rings which are not division rings .The example most immediately accessible to us at this point are skew Laurent rings of the following type:

Let T = K $[x^{\pm 1}; \alpha]$ where K is a field and α is an automorphism of K with infinite order (i.e. no non zero powers of α is identity). T is a simple ring. Proof: Let I be a non zero ideal of T, and pick a non zero p$\in I \cap k[x;\ \alpha]$ of minimal degree, say degree n. The constant term of p is non zero (for other wise p would be replaced by px^{-1}). Observe that for any r $\in K$, the difference pr- α^n(r) p is an element of I∩K$[x;\alpha]$ with degree at most n-1, and so pr - α^n(r) p = 0 compare the constant terms and conclude that α^n(r) = r for all r ∈ K. consequently, n = 0, hence p is a scalar, and therefore I = T.

Definition 2.7.2 let α be an automorphism of a ring R. An α-ideal of R is any ideal of I of R that is stable under α i.e. α (I) = I. The ring R is said to be α-simple provided R is non zero and its only α- ideals are 0 and R.

A Skew polynomial ring S = R$[x;\alpha]$ has no chance to be simple, since Sx is always a non trivial ideal of S (as are Sx^2, Sx^3, …). If a skew Laurent ring T = R$[x^{\pm 1}, \alpha]$ is to be simple, R needs to be α-simple and no positive power (equivalently no non zero power) of α can be inner(some positive powers of α is an inner automorphism of R; say there is a unit u∈R ∋ $\alpha^n(r) = u\, r\, u^{-1}$ for all r∈R). These are only conditions that need to be satisfied.

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