ADDIS ABABA UNIVERSITY
SCHOOL OF GRADUATE STUDIES
DEPARTMENT OF MATHEMATICS

GRADUATE SEMINAR REPORT

ON

ASYMPTOTIC SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS HAVING AN IRREGULAR SINGULAR POINT

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

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June, 2010

Addis Ababa

CONTENTS

PREFACE

The problem of finding a basis solution of linear homogeneous ordinary differential equations with variable coefficients is, in general difficult. Indeed a considerable effort was directed at constructing solutions of specific equations with non constant coefficients.

The solutions of differential equations with analytic coefficients were obtained from a theory called the method of power series. Most importantly is the case in which the equation has a singular point. In the special case of a so called regular (weak) singular point we can apply the method of Frobenius to construct convergent series solutions in a neighborhood of this point. In general this cannot be done for an irregular (strong) singular point. Treatment of this case is the main aim of this seminar.

The first chapter gives the background information about the problem under investigated. The second chapter constructs a method of approximating the solutions of differential equations in a neighborhood of an irregular singular point. The final chapter extends these approximations to asymptotic expansions.

AKNOWLEDGEMENT

I would like to express my gratitude to my advisor Dr. Tadesse Abdi for his encouragement, useful comments and valuable suggestions. I deeply appreciate his cooperation in providing materials.

Also, it is a pleasure to acknowledge the Science library, Digital library, and Mathematics post graduate library of Addis Ababa University for their assistance in providing the necessary resources.

Last, but not least, I thank my friends Abinet Kefeni, Arega Getaneh and Mulugeta for their cooperation.

CHAPTER ONE

INTRODUCTION

Ordinary differential equations play an important role in the solutions of many problems. This makes them an essential topic in Mathematics and /or Science. Even though most physical problems involve nonlinear differential equations, it is possible to approximate the solutions of many problems by replacing non linear by linear ones.

Consider the general linear differential equation of order n,

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0(x)y = f(x)$$

Since a detail discussion of techniques for solving differential equations of higher order involve much time and effort, we consider the second order equations which occur more frequently than others in applications.

The general second order linear differential equation is of the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(*)$$

Assuming that the functions $a_0, a_1, and\ a_2$ are continuous on some interval, say $a < x < b$, and also that the function $a_2(x)$ nowhere zero in this interval, equation $(*)$ can be rewritten to in the form

$$y'' + p(x)y' + q(x)y = r(x).$$

Let as introduce a classification of points of the x-axis of the equation

$$y'' + p(x)y' + q(x)y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Ordinary point:

The point x_0 is an ordinary point of (1) if both $p(x)\ and\ q(x)$ are analytic at x_0.

Singular point:

The point x_0 is a singular point of (1) if at least one of the functions $p(x)\ and\ q(x)$ is not analytic at x_0.

Furthermore, a singular point of (1) is weak (regular) if the functions

$(x - x_0)p(x)\ and\ (x - x_0)^2 q(x)$ are analytic at x_0. Otherwise, the singular point is strong (irregular).

When the coefficients are constants the solutions are obtained easily by elementary methods , in terms of elementary functions. For equations whose solutions cannnot be obtained easily a possible tool which comes to mind is expansion of the solution in series. If the coefficients of (1) are analytic, we can construct convergent series solutions by the method of power series. However, many differential equations that arise in practical problems have singular points, and it is a matter of practical importance studying solutions near a singular point. In the special case of a regular singular point, we use the method of Frobenius to construct convergent series solutions. Of course, such approaches confine us to equations and solutions that have convergent power series representations.

In general, constructing convergent series solutions can not be done if the singular point is irregular. Since a number of ordinary differential equations that arise in many applications are of interest primarily in the neighborhood of irregular singular points, and the choice of physically appropriate solutions is often determined by their behavior near these points, this assumption of convergence excludes many of the most significant situations in the theory of ordinary differential equations.

In this case some knowledge of asymptotic analysis of differential equations is needed. In an asymptotic analysis one tries to find analytical solutions which are approximately valid.

Computationally, a convergent series is not always useful because convergence is a concept relating to the behavior of the terms at the tail end. That is a series converges says nothing about how rapidly the terms will decrease in magnitude. On the other hand, in an asymptotic series, the terms will usually decrease rapidly with n at first for some point x. Sometimes, they may begin to increase with increasing n at some point after decreasing initially. When the terms are decreasing rapidly, if we sum just the fist few terms and we know the error incurred is of the order of the next term, we can get a good estimate of the sum. This is why asymptotic series, even when divergent, are practically useful. In addition, asymptotic series may be the only means of obtaining an analytical solution of a difficult problem, and are commonly used for this purpose.

The familiar techniques (the power series and Frobenius method) will not be a subject for further discussion in this seminar. Here, we shall be interested in more general methods and procedures which can be applied to a wider class of differential equations. More specifically, we shall show how a useful approximation to solutions of second order linear ordinary differential equations can be obtained by expanding in an asymptotic series in a neighborhood of an irregular singular point.

CHAPTER TWO

ASYMPTOTIC APPROXIMATIONS TO SOLUTIONS OF DIFFERENTIAL EQUATIONS

Since most equations cannot be solved, it is natural to seek approximate solutions. It is straight forward to know the leading order behavior (asymptotic representation) of solutions of differential equations in a neighborhood of an ordinary point or regular singular point; we simply use a Taylor expansion or Frobenius method. However, if we are considering an irregular singular point, we use a more creative and heuristic approach. In this chapter we study how to construct the approximate solutions of differential equations near an irregular singular point. First we introduce a classification of singularity of equation (1) at infinity.

2.1 Large Values of the Independent Variable

It is often desirable to study the solutions of equation (1) for large values of the independent variable. To discuss the solutions in a neighborhood of infinity, we change the independent variable from x to $t = \frac{1}{x}$ then large x's correspond to small t's.

Set: $t = \frac{1}{x}$

Then using chain rule,

$$y' = \frac{dy}{dx}$$

$$= \frac{dy}{dt}\frac{dt}{dx}$$

$$= \frac{dy}{dt}\left(-\frac{1}{x^2}\right)$$

$$= -t^2\frac{dy}{dt}$$

and

$$y'' = \frac{d}{dx}\left(-t^2\frac{dy}{dt}\right)$$

$$= \frac{d}{dt}\left(-t^2\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$= \left[-2t\frac{dy}{dt} - t^2\frac{d^2y}{dt^2}\right](-t^2)$$

$$= 2t^3\frac{dy}{dt} + t^4\frac{d^2y}{dt^2}$$

Substituting $y'and\ \ y''$ in equation (1), we obtain

$$t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt} - t^2p\left(\frac{1}{t}\right)\frac{dy}{dt} + q\left(\frac{1}{t}\right)y = 0$$

$$\Longrightarrow \frac{d^2y}{dt^2} + \left(\frac{2}{t} - \frac{p\left(\frac{1}{t}\right)}{t^2}\right)\frac{dy}{dt} + \frac{q\left(\frac{1}{t}\right)}{t^4}y = 0$$

Thus equation (1) is transformed to

$$\ddot{y} + \left(\frac{2}{t} - \frac{p\left(\frac{1}{t}\right)}{t^2}\right)\dot{y} + \frac{q\left(\frac{1}{t}\right)}{t^4}y = 0 \ldots\ldots\ldots\ldots(2)$$

The singularity of (1) at infinity is classified according to the nature of the singularity of (2) at $t = 0$.The rank of the irregular singularity is $m + 1$, where m is the least nonnegative integer such that $x^{-m}p(x)$ and $x^{-2m}q(x)$ are analytic at infinity.

2.2 Asymptotic Approximations to Solutions of Differential Equations near an Irregular Singular Point

The standard form of a linear homogeneous second order differential equation is taken to be

$$y'' + p(x)y' + q(x)y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Our aim here is to approximate the solutions of (1) near an irregular singular point. First we have the following preliminary lemmas.

Lemma 1. Using the transformation

$$y = e^{\frac{-1}{2}\int p(x)dx}u$$

the differential equation

$$y'' + p(x)y' + q(x)y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

is reduced to the equation(called normal form)

$$u'' - \left(\frac{1}{4}p^2(x) + \frac{1}{2}p'(x) - q(x)\right)u = 0 \ldots\ldots\ldots\ldots. (2)$$

Proof:

we set $y = uv$ where $v = e^{-\frac{1}{2}\int p(x)dx}$

Now,

$$v' = -\frac{1}{2}p(x)e^{-\frac{1}{2}\int p(x)dx}$$

$$= -\frac{1}{2}pv$$

$$v'' = -\frac{1}{2}p'v + \frac{1}{4}p^2 v$$

Thus

$$y' = (u' - \frac{1}{2}pu)v$$

$$y'' = u''v + 2u'v' + uv''$$

$$= u''v + 2u'\left(-\tfrac{1}{2}pv\right) + u\left(-\tfrac{1}{2}p' + \tfrac{1}{4}p^2\right)v$$

$$= (u'' - pu' - \frac{1}{2}p'u + \frac{1}{4}p^2 u)v$$

Substituting $y, y' and\ y''$ in (1),

$$\left(u'' - pu' - \frac{1}{2}p'u + \frac{1}{4}p^2 u\right)v + \left(pu' - \frac{1}{2}p^2 u\right)v + quv = 0$$

$$\Rightarrow u'' - \left(\frac{1}{4}p^2(x) + \frac{1}{2}p'(x) - q(x)\right)u = 0$$

Hence by appropriate change of the independent or dependent variable, every linear homogeneous differential equation of second order can be put in the form

$$y'' - f(x)y = 0 \dots\dots\dots\dots\dots\dots\dots..(3)$$

Lemma 2: Consider the differential equation

$$u'' - f(x)u = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots(3)$$

Using the substitution $t = t(x)$ and its inverse $x = x(t)$, equation (3) is transformed to the equation

$$\frac{d^2u}{dt^2} - \frac{\ddot{x}}{\dot{x}}\frac{du}{dt} - \dot{x}^2 f(x)u = 0 \dots\dots\dots\dots\dots\dots\dots(4)$$

,where dots signify derivatives with respect to t.

Proof

Let $x = x(t)$

Then using chain rule,

$$u' = \frac{du}{dx}$$

$$= \frac{du}{dt}\frac{dt}{dx}$$

$$= \frac{1}{\dot{x}}\frac{du}{dt}$$

$$u'' = \frac{d}{dx}\left(\frac{du}{dx}\right)$$

$$= \frac{d}{dt}\left[\frac{1}{\dot{x}}\frac{du}{dt}\right]\frac{dt}{dx}$$

$$= \left[\frac{1}{\dot{x}}\frac{d^2u}{dt^2} - \frac{\ddot{x}}{\dot{x}^2}\frac{du}{dt}\right]\frac{1}{\dot{x}}$$

$$= \frac{1}{\dot{x}^2}\frac{d^2u}{dt^2} - \frac{\ddot{x}}{\dot{x}^3}\frac{du}{dt}$$

Substituting u'' in (3) we get

$$\frac{d^2u}{dt^2} - \frac{\ddot{x}}{\dot{x}}\frac{du}{dt} - \dot{x}^2 f(x)u = 0$$

,where $f(x)$ is expressed in terms of t.

Now, to find the asymptotic approximation of the solutions of the differential equation

$$y'' + p(x)y' + q(x)y = 0 \dots\dots\dots\dots\dots\dots\dots\dots..(1)$$

near an irregular singular point ,we proceed as follows.

Without loss of generality assume that (1) has an irregular singularity at infinity (this is to follow historical precedent, and to acknowledged that infinity is the natural distinguished point in many physical applications).

Using the change of variable

$$y = e^{\frac{-1}{2}\int p(x)dx} u$$

equation (1) is reduced to normal form

$$u'' - (\tfrac{1}{4}p^2 + \tfrac{1}{2}p' - q)u = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2) \text{ (by lemma 1)}$$

Hence equation (1) is reduced to equation of the form

$$u'' - f(x)u = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

We will exclude functions $f(x)$ which change sign as x increases (i.e. transition points) and for convenience, assume that $f(x) > 0$.

First we made a preliminary transformation of (3) in to a differential equation of the same type but $f(x)$ replaced by a function that varies more slowly.

Using the substitution

$$x = x(t)$$

equation (3) is transformed to the equation (by lemma 2)

$$\frac{d^2u}{dt^2} - \frac{\ddot{x}}{\dot{x}}\frac{du}{dt} - \dot{x}^2 f(x)u = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

where f is expressed in terms of the new independent variable t.

Again, the change of variable

$$u = e^{\frac{1}{2}\int\frac{\ddot{x}}{\dot{x}}dt} w$$

$$= e^{\frac{1}{2}\log \dot{x}} w$$

$$= \dot{x}^{\frac{1}{2}} w$$

, called Liouville transformation, yields

$$\frac{d^2w}{dt^2} - \dot{x}^2 f(x) w = \left(\frac{3\ddot{x}^2}{4\dot{x}^2} - \frac{\dddot{x}}{2\dot{x}}\right) w \ldots\ldots\ldots\ldots.(5)$$

It is not less difficult to arrange that the coefficient of w in (5) be constant, than to solve the original differential equation(3). Hence for the left hand side of (5) to be identical with the comparison differential equation

$$\frac{d^2w}{dt^2} - w = 0$$

we choose

$$\dot{x}^2 f(x) = 1$$

$$\frac{dt}{dx} = \pm f^{\frac{1}{2}}$$

$$t(x) = \pm \int f^{\frac{1}{2}}\, dx$$

Thus substituting

$\dot{x} = f^{\frac{-1}{2}}$ in (5), we obtain

$$\frac{d^2w}{dt^2} = \left(1 + \frac{4ff'' - 5f'^2}{16f^3}\right) w \ldots\ldots\ldots\ldots..(6)$$

, provided that f is twice differentiable.

So far, the analysis is exact .Now, if $|\varphi| \ll 1$ as $x \to \infty$, where $\varphi(t) = \frac{4ff'' - 5f'^2}{16f^3}$

then the approximate solutions of (6) are

$$w = e^{\pm t} \text{ for large } x.$$

Consequently, the approximate solutions of (3) are

$$u = \dot{x}^{\frac{1}{2}} w \sim f^{\frac{-1}{4}} e^{\pm \int f^{\frac{1}{2}} dx} \qquad (x \to \infty)$$

Hence the asymptotic solutions of (1) are

$$y_1(x) \sim e^{\frac{-1}{2}\int p dx} f^{\frac{-1}{4}} e^{\int f^{\frac{1}{2}} dx}$$

and

$$y_2(x) \sim e^{\frac{-1}{2}\int p dx} f^{\frac{-1}{4}} e^{-\int f^{\frac{1}{2}} dx} \qquad (x \to \infty) \ldots\ldots\ldots\ldots\ldots..(7)$$

Therefore, where $f(x)$ positive and real in a neighborhood of an irregular singularity, the solutions of (1) has exponential behavior.

Note also that if we include intervals containing zeros of f (called turning points or transition points of the differential equation (3)), then φ becomes infinite and the approximation fails at this points. Throughout this seminar it is supposed that all domains under consideration are free from turning points.

The main results of this section can be summarized by the following theorem.

Theorem 1.Let $f(x)$ be positive, real, twice continuously differentiable function on the interval $(a, \infty), a > 0$. Then the differential equation

$$y'' - f(x)y = 0$$

has twice continuously differentiable solutions whose asymptotic representation is

$$y_1(x) \sim f^{\frac{-1}{4}} e^{\int f^{\frac{1}{2}} dx}$$

$$\text{and} \quad y_2(x) \sim f^{\frac{-1}{4}} e^{-\int f^{\frac{1}{2}} dx} \qquad (x \to \infty)$$

Provided that $\left|\frac{4ff''-5f'^2}{16f^3}\right| \ll 1$ as $x \to \infty$.

Remark:

For the equation

$$y'' + f(x)y = 0 \dots\dots\dots\dots\dots\dots\dots\dots (*),$$

where $f(x)$ is positive; we choose

$$\dot{x}^2 f(x) = -1$$

$$\left(\frac{dt}{dx}\right)^2 = -f$$

$$\frac{dt}{dx} = \pm i f^{\frac{1}{2}}$$

$$\Longrightarrow t = \pm i \int f^{\frac{1}{2}}\, dx$$

Hence the asymptotic solutions are

$$y \sim f^{\frac{-1}{4}} e^{\pm i \int f^{\frac{1}{2}} dx} \quad \text{as } x \to \infty.$$

Since the linear combination of the solutions of a homogeneous linear differential equations is also a solution, the real asymptotic approximation of solutions of $(*)$ are

$$y_1(x) \sim f^{\frac{-1}{4}} \cos\left(\int f^{\frac{1}{2}}\, dx\right)$$

and

$$y_2(x) \sim f^{\frac{-1}{4}} \sin\left(f^{\frac{1}{2}} dx\right) \text{ as } x \to \infty.$$

Although much of the subsequent analysis carries straightforwardly to the case $f(x)$ is real and negative, in the interest of clarity we confine attention for the case $f(x)$ is real and positive.

2.3/ Example: The Modified Bessel Equation

$$x^2y'' + xy' - (x^2+v^2)y = 0 \dots\dots\dots\dots (1.1)$$

We would like to know how the solutions of this equation behave as $x \to \infty$.

Now, (1.1) can be written in the form

$$y'' + \frac{1}{x}y' - \left(1 + \frac{v^2}{x^2}\right)y = 0 \dots\dots\dots\dots (1.2)$$

First we need to classify the point at infinity. The change of variable

$x = \frac{1}{t}$ yields

$$\ddot{y} + \frac{1}{t}\dot{y} - \left(\frac{1}{t^4} + \frac{v^2}{t^2}\right)y = 0 \dots\dots\dots\dots\dots (1.3)$$

Since (1.3) has an irregular singular point at $t = 0$, (1.2) has an irregular singular point at infinity.

Now put

$$y = e^{\frac{-1}{2}\int\frac{1}{x}dx}u$$

$$= e^{\frac{-1}{2}\log x}u$$

$$= x^{\frac{-1}{2}}u$$

Then (1.2) becomes

$$u'' - \left(1 - \frac{1}{4x^2} + \frac{v^2}{x^2}\right)u = 0 \dots\dots\dots\dots\dots\dots\dots\dots (1.4)$$

Clearly, $f(x) = 1 - \frac{1}{4x^2} + \frac{v^2}{x^2} > 0$ in the neighborhood of infinity.

Since $v^2 \ll x^2$ and $\frac{1}{4x^2} \ll 1$as $x \to \infty$,the solutions of (1.4) can be approximated by the solutions the equation

$$u'' - u = 0 \text{ for large } x.$$

However, this equation has solutions

$$u_1(x) = e^x$$

and

$$u_2(x) = e^{-x}$$

Thus (1.2) have approximate solutions

$$y_1(x) \sim x^{\frac{-1}{2}} e^x \ \ and \ \ y_2(x) \sim x^{\frac{-1}{2}} e^{-x} \qquad (x \to \infty)$$

Therefore, two linearly independent solutions of the modified Bessel equation are the modified Bessel functions, $j_v(x)\ and\ \ k_v(x)$ which have asymptotic property

$$j_v(x) \sim (const) x^{\frac{-1}{2}} e^x$$

and

$$k_v(x) \sim (const) x^{\frac{-1}{2}} e^{-x} \qquad (x \to \infty)$$

2.4/ Asymptotic Approximations (Continued)

To find the asymptotic solutions of

$$y'' = \big(f(x) + g(x)\big)y \dots\dots\dots\dots\dots..(8)$$

we apply the transformation

$$y = \dot{x}^{\frac{1}{2}} w \; and \;\; t(x) = \int f^{\frac{1}{2}}(x)dx$$

then (8) becomes

$$\frac{d^2w}{dt^2} = \left(1 + \frac{g}{f} + \frac{4ff'' - 5f'^2}{16f^3}\right)w \dots\dots\dots\dots..(9)$$

Now, if $|\varphi| \ll 1 \; and \;\; \int f^{\frac{-1}{2}} g < \infty$, then (7) approximate the solutions. That is g does not affect the gross asymptotic behavior at all. However, if $\int f^{\frac{-1}{2}} g$ diverge then we regard the coefficient $f(x) + g(x)$ as a single function of x and use(7) . That is

$$y_1 \sim (f+g)^{\frac{-1}{4}} e^{\int (f+g)^{\frac{1}{2}} dx}$$

$$y_1 \sim (f+g)^{\frac{-1}{4}} e^{-\int (f+g)^{\frac{1}{2}} dx}$$

Note:$(f+g)^{\frac{1}{2}} = f^{\frac{1}{2}} + \frac{1}{2} f^{\frac{-1}{2}} g - \frac{1}{8} f^{\frac{-3}{2}} g^2 + \cdots$

Example: Find the asymptotic approximation of the solutions of the equation

$$y'' - (x + \log x)y = 0 \dots\dots\dots\dots\dots.(2.1) \text{ as } x \to \infty.$$

Set $t = \frac{1}{x}$ then equation (2.1) is transformed to the equation

$$\ddot{y} + \frac{2}{t^3}\dot{y} - \left(\frac{1}{t^5} + \frac{1}{t^4}\log\left({}^{1}\!/_{t}\right)\right)y = 0 \dots\dots\dots\dots\dots.(2.2).$$

Since (2.2) has an irregular singular point at $t = 0$, equation (2.1) has an irregular singular point at infinity.

Now, if we take

$$f = x \; and \; g = \log x,$$

Then

$$\int f^{\frac{-1}{2}} g \text{ would diverge.}$$

Hence we set $f = x + \log x$,

From the binomial expansion ,we have

$$(x + \log x)^{\frac{1}{2}} = x^{\frac{1}{2}} + \frac{1}{2} x^{-\frac{1}{2}} \log x + O(x^{-\frac{3}{2}}((\log x)^2)$$

$$\text{,and } (x + \log x)^{-\frac{1}{4}} = x^{\frac{-1}{4}} + O(x^{-\frac{5}{4}} \log x) \qquad \text{as } x \to \infty \, .$$

Thus

$$\int (x + \log x)^{\frac{1}{2}} \, dx = \int x^{\frac{1}{2}} dx + \frac{1}{2} \int x^{\frac{1}{2}} \log x \, dx + const + o(1)$$

$$= \frac{2}{3} x^{\frac{3}{2}} + x^{\frac{1}{2}} \log x - 2x^{\frac{1}{2}} + const + o(1) \;\; , x \to \infty$$

Therefore, the asymptotic solutions are

$$y_1 \sim x^{\frac{-1}{4} + x^{\frac{1}{2}}} e^{\left(\frac{2}{3} x^{\frac{3}{2}} - 2x^{\frac{1}{2}}\right)}$$

and

$$y_2 \sim x^{\frac{-1}{4} - x^{\frac{1}{2}}} e^{(-\frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}})} \qquad (x \to \infty)$$

CHAPTER THREE

ASYMPTOTIC EXPANSIONS OF DIFFERENTIAL EQUATIONS WITH IRREGULAR SINGULAR POINT

In the preceding chapter we saw that how the solutions of a linear second order differential equation are represented asymptotically in a neighborhood of an irregular singularity. In this chapter we show how to extend these approximations in to asymptotic expansions. We first study a general method for a singularity of unit rank, the commonest case in applications.

3.1/ Differential Equations with Irregular Singularity of Unit Rank

Consider the equation

$$y'' + p(x)y' + q(x)y = 0 \dots\dots\dots\dots\dots\dots\dots\dots..(1).$$

Without loss of generality assume that the singularity is located at infinity. That is there exist $a > 0$ in which $p(x)$ and $q(x)$ can be expanded in convergent series of the form

$$p(x) = \sum_{n=0}^{\infty} p_n\, x^{-n} \quad , q(x) = \sum_{n=0}^{\infty} q_n\, x^{-n} \qquad\qquad , |x| > a$$

Not all of the coefficients $p_0\,, q_0\ \ and\ q_1$ vanish, otherwise the singularity would be regular.

Using $y = e^{\frac{-1}{2}\int pdx}u$ equation (1) is transformed in to

$$u'' - f(x)u = 0 \dots\dots\dots\dots..(2)$$

where

$$f(x) = \frac{1}{4}p^2(x) + \frac{1}{2}p'(x) - q(x)$$

Now,

$$p^2(x) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} p_k p_{n-k}\right] x^{-n}$$

$$p'(x) = \sum_{n=1}^{\infty} p_n(-n)x^{-n-1}$$

Thus

$$f(x) = \left(\frac{{p_0}^2}{4} - q_0\right) + \left(\frac{1}{2}p_0p_1 - q_1\right)x^{-1} + \left(\frac{1}{2}p_2p_0 - p_1 - q_2\right)x^{-2} + \cdots$$

for $|x| > a$ … … … … … … … … … … … … … … (3)

In chapter two we showed that with appropriate restrictions equation (2) has solutions with properties

$$u \sim f^{\frac{-1}{4}}(x)e^{\pm\int f^{\frac{1}{2}}(x)dx} \qquad (x \to \infty)$$

By the use of (3) to simplify these representations, we have

$$f^{\frac{-1}{4}} = \left(\frac{{p_0}^2}{4} - q_0\right)^{\frac{-1}{4}} + O(x^{-1}) \text{ ,and}$$

$$f^{\frac{1}{2}} = \left(\frac{{p_0}^2}{4} - q_0\right)^{\frac{1}{2}} + \frac{1}{2}\left[\left(\frac{{p_0}^2}{4} - q_0\right)^{\frac{-1}{2}}\left(\frac{1}{2}p_0p_1 - q_1\right)x^{-1}\right] + O(x^{-2}) \qquad (x \to \infty).$$

Thus

$$\int f^{\frac{1}{2}}(x)dx = \left(\frac{{p_0}^2}{4} - q_0\right)^{\frac{1}{2}} x + \left(\frac{{p_0}^2}{4} - q_0\right)^{\frac{-1}{2}} \left(\left(\frac{p_0 p_1}{4} - \frac{q_1}{2}\right)\log x\right) + const + O(x^{-1})$$

$$\therefore u \sim (const) e^{\pm\left\{\left(\frac{{p_0}^2}{4}-q_0\right)^{\frac{1}{2}} x + \frac{\left(\frac{p_0 p_1}{4}-\frac{q_1}{2}\right)\log x}{\left(\frac{{p_0}^2}{4}-q_0\right)^{\frac{1}{2}}}\right\}} \qquad (x \to \infty)$$

That is

$$u \sim (const) e^{\pm(\rho x + \sigma \log x)} \ldots \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

where

$$\rho = \left(\frac{{p_0}^2}{4} - q_0\right)^{\frac{1}{2}} \text{,and}$$

$$\sigma = \frac{\left(\frac{p_0 p_1}{4} - \frac{q_1}{2}\right)}{\rho}$$

Now (4) is true unless $\rho = 0$.that is ${p_0}^2 = 4q_0$.Hence we have the following two cases.

Correspond to case 1 (i.e. ${p_0}^2 \neq 4q_0$),we have the following theorem.

Theorem 2. Let $p(x)\ and\ q(x)$ be analytic functions of the real variable x having convergent series expansion

$$p(x) = \sum_{n=0}^{\infty} p_n\, x^{-n} \quad , q(x) = \sum_{n=0}^{\infty} q_n\, x^{-n} \quad , |x| > a$$

with ${p_0}^2 \neq 4q_0$.Then the differential equation

$$y'' + p(x)y' + q(x)y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

has solutions with asymptotic expansions

$$y_i \sim e^{\lambda_i} x^{\mu_i} \sum_{n=0}^{\infty} a_{n,i}\, x^{-n} \qquad\qquad x \to \infty \quad , i = 1,2$$

Proof:

Assume ${p_0}^2 \neq 4q_0$. Using the above procedures and returning to the original differential equation we have

$$y \sim (const) e^{\frac{-1}{2}\int p(x)dx} e^{\pm(\rho x + \sigma \log x)} \qquad (x \to \infty)$$

But

$$\frac{-1}{2} p(x) = \frac{-1}{2} p_0 + \frac{-1}{2} p_1 x^{-1} + O(x^{-2})$$

and

$$e^{\frac{-1}{2}\int p(x)dx} = e^{\frac{-1}{2} p_0 x - \frac{1}{2} p_1 \log x} + o(1) + const \text{ as } (x \to \infty)$$

$$\therefore y \sim (const) e^{(\lambda x + \mu \log x)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (5),$$

where

$$\lambda = \pm\rho - \frac{1}{2} p_0,$$

$$\mu = \pm\sigma - \frac{1}{2} p_1$$

are the asymptotic representations of the solutions.

Since $p(x)$ and $q(x)$ have expansions in descending powers of x, it is natural to extend (5) in to formal series solutions of the form

$$y = e^{\lambda x} x^{\mu} \sum_{n=0}^{\infty} a_n x^{-n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (6)$$

Now, write $y = e^{\lambda x} x^{\mu} \sum_{n=0}^{\infty} a_n x^{-n}$ in the form

$$y = e^{\lambda x} \sum_{n=0}^{\infty} a_n x^{\mu - n}$$

$$= e^{\lambda x} \psi(x)$$

Then

$$y' = e^{\lambda x} (\lambda \psi(x) + \psi'(x))$$

$$y'' = e^{\lambda x} [\lambda^2 \psi(x) + 2\lambda \psi'(x) + \psi''(x)]$$

Substituting y, y', and y'' in (1), we obtain

$$\psi''(x) + \big(2\lambda + p(x)\big)\psi'(x) + \big(\lambda^2 + \lambda p(x) + q(x)\big)\psi(x) = 0 \ldots\ldots\ldots\ldots..(*)$$

Differentiating

$$\psi(x) = \sum_{n=0}^{\infty} a_n x^{\mu - n},$$

$$\psi'(x) = \sum_{n=0}^{\infty} (\mu - n) a_n x^{\mu - n - 1}$$

$$= \sum_{n=1}^{\infty} (\mu - n + 1) a_{n-1} x^{\mu - n}$$

$$\psi''(x) = \sum_{n=0}^{\infty} (\mu - n)(\mu - n - 1) a_n x^{\mu - n - 2}$$

$$= \sum_{n=1}^{\infty} (\mu - n + 1)(\mu - n) a_{n-1} x^{\mu - n - 1}$$

Thus

$$p(x)\psi^{'}(x) = (\sum_{n=0}^{\infty} p_n x^{-n})(\sum_{n=0}^{\infty} (\mu - n) a_n x^{\mu-n-1})$$

$$= p_0 \sum_{n=0}^{\infty} (\mu - n) a_n x^{\mu-n-1} + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} p_{n+1-k} (\mu - k + 1) a_{k-1} \right) x^{\mu-n-1}$$

and

$$\big(p(x) + q(x)\big)\psi(x) = \left(\sum_{n=0}^{\infty} (p_n + q_n) x^{-n} \right) \left(\sum_{n=0}^{\infty} a_n x^{\mu-n} \right)$$

$$= \sum_{n=0}^{\infty} (p_0 + q_0 a_n) x^{\mu-n} + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (p_{n+1-k} + q_{n+1-k})\, a_k \right) x^{\mu-n-1}$$

Substituting these and the expansions of $p(x)\ and\ q(x)$ in equation$(*)$, we obtain

$$= \sum_{n=1}^{\infty} (\mu - n + 1)(\mu - n) a_{n-1} x^{\mu-n-1} + 2\lambda \sum_{n=0}^{\infty} (\mu - n) a_n x^{\mu-n-1}$$

$$+ p_0 \sum_{n=0}^{\infty} (\mu - n) a_n x^{\mu-n-1} + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} p_{n+1-k} (\mu - k + 1) a_{k-1} \right) x^{\mu-n-1}$$

$$+ \lambda^2 \sum_{n=0}^{\infty} a_n x^{\mu-n} + \sum_{n=0}^{\infty} (\lambda p_0 + q_0) a_n x^{\mu-n}$$

$$+ \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (p_{n+1-k} + q_{n+1-k})\, a_k \right) x^{\mu-n-1} = 0.$$

$$\Rightarrow \left\{\lambda^2 \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} (\lambda p_0 + q_0) a_n\right\} x^{\mu-n}$$

$$+ \left\{\sum_{n=0}^{\infty} p_0(\mu-n)a_n + \sum_{n=1}^{\infty} (\mu-n+1)(\mu-n)a_{n-1} + 2\lambda \sum_{n=0}^{\infty} (\mu-n)a_n\right.$$

$$+ \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} p_{n+1-k}(\mu-k+1)a_{k-1}\right)$$

$$\left.+ \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} (p_{n+1-k} + q_{n+1-k})\, a_k\right)\right\} x^{\mu-n-1} = 0.$$

Equating the coefficient of x^{μ} to zero we obtain

$$(\lambda^2 + p_0\lambda + q_0)a_0 = 0$$

$$\Rightarrow \lambda^2 + p_0\lambda + q_0 = 0 \ldots\ldots\ldots\ldots\ldots..(6.01)$$

Equating the coefficient of $x^{\mu-1}$ to zero,

$$(p_0 + 2\lambda)\mu a_0 + (p_1\lambda + q_1)a_0 = 0$$

$$\Rightarrow (p_0 + 2\lambda)\mu = -(p_1\lambda + q_1) \ldots\ldots\ldots.(6.02)$$

,and equating the coefficient of $x^{\mu-n-1}$ $n \geq 1$ to zero, we have

$$\{(\lambda^2 + p_0\lambda + q_0) + (p_0 + 2\lambda)(\mu-n) + (p_1\lambda + q_1)\}a_n$$
$$+ \{(\mu-n+1)(\mu-n) + (\lambda p_2 + q_2 + p_1(\mu-n+1))\}a_{n-1} + \cdots$$
$$+ \{\lambda p_{n+1} + q_{n+1} + \mu p_n\}a_0 = 0$$

$$\Rightarrow \Big((p_0 + 2\lambda)(\mu-n) - \big((p_0 + 2\lambda)\mu\big)\Big)\, a_n$$
$$+ \{(\mu-n+1)(\mu-n) + (\lambda p_2 + q_2 + p_1(\mu-n+1))\}a_{n-1} + \cdots$$
$$+ \{\lambda p_{n+1} + q_{n+1} + \mu p_n\}a_0 = 0$$

$$\Rightarrow (p_0 + 2\lambda)(-n)a_n + \{(\mu-n+1)(\mu-n) + (\lambda p_2 + q_2 + p_1(\mu-n+1))\}a_{n-1} + \cdots$$
$$+ \{\lambda p_{n+1} + q_{n+1} + \mu p_n\}a_0 = 0.$$

$$\begin{aligned}\Rightarrow (p_0+2\lambda)na_n & \\ &= (n-\mu)(n-1-\mu)a_{n-1} + \{\lambda p_2 + q_2 - (n-1-\mu)p_1\}a_{n-1} \\ &+ \{\lambda p_3 + q_3 - (n-2-\mu)p_2\}a_{n-2} + \cdots \\ &+ \{\lambda p_{n+1} + q_{n+1} + \mu p_n\}a_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.03)\end{aligned}$$

From equation (6.01) we get two possible values $\lambda_1, \lambda_2 = -\frac{1}{2}p_0 \pm (\frac{1}{4}{p_0}^2 - q_0)^{\frac{1}{2}}$ for λ

and from (6.02) the corresponding values μ_1, μ_2 of μ. These values are consistent with the values we get in equation (5).The values of a_0 $, say$ $a_{0,1}$ and $a_{0,2}$ in the two cases may be assigned arbitrarily. Higher coefficients $a_{n,1}$ and $a_{n.2}$ are then determined recursively by (6.03).The process fails if and only if $p_0 + 2\lambda = 0$; this is the excepted case ${p_0}^2 = 4q_0$. Equation (6.01) is called the characteristic equation, and its roots the characteristic values of the singularity. The expansions of the form (6) is sometimes called normal series or normal solution to distinguish it from expansions of Laurent type for y.

Remark:

If the expansion (6) converges for all sufficiently large $|x|$, then term wise differentiation would be valid and the series would define a solution of the differential equation. However, this is not the usual state of affairs. For instance, when the terms beyond the first are neglected on the right of (6.03), we have

$$\frac{a_n}{a_{n-1}} \sim \frac{n}{p_0 + 2\lambda} \qquad (n \to \infty)$$

Which implies that (6) diverges. Only in cases in which the first term on the right of (6.03) is largely cancelled by the contribution of other terms is there any possibility of convergence. Hence the most that can be hopped of (6), in general, is that it provides the asymptotic expansion of a solution in a certain real interval. Meaning, although (6) diverges it yields information about the solution y in the sense that for any integer $n \geq 0$,

$$y(x) = e^{\lambda x} x^{\mu} \sum_{i=0}^{n} a_i x^{-i} + O(x^{-n-1}) \qquad (x \to \infty)$$

<u>CASE</u> <u>2</u>. $\boldsymbol{p_0{}^2 = 4q_0}$.

The above theorem (theorem 2) fails to answer the case when $p_0{}^2 = 4q_0$.Hence correspond to this we use an alternative procedure to yield similar asymptotic forms for solutions. The procedure which leads to the same result is the transformation of Fabry.

Using the transformation of Fabry:

$$y = e^{-p_0\frac{x}{2}} w \ , t = x^{\frac{1}{2}}$$

equation (1) is transformed in to

$$\frac{d^2w}{dt^2} + F(t)\frac{dw}{dt} + G(t)w = 0 \ldots\ldots\ldots\ldots (7)$$

where

$$F(t) = 2tp(t^2) - 2p_0 t - t^{-1}$$

$$G(t) = t^2\{4q(t^2) - p_0{}^2 - 2p_0 p(t^2)\}$$

To show:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

$$= \frac{1}{2t}\frac{dy}{dt}$$

$$= \frac{1}{2t}\left\{\frac{-p_0}{2} e^{\frac{-p_0 x}{2}}(2t)w + e^{\frac{-p_{0x}}{2}}\frac{dw}{dt}\right\}$$

$$= \left[-\frac{p_0}{2}w + \frac{1}{2t}\frac{dw}{dt}\right] e^{\frac{-p_{0x}}{2}}$$

$$\frac{d^2w}{dt^2} = \left\{\frac{d}{dt}\left[\left(-\frac{p_0}{2}w + \frac{1}{2t}\frac{dw}{dt}\right)\right]e^{\frac{-p_{0x}}{2}}\right\}\left(\frac{1}{2t}\right)$$

$$= \left\{\frac{1}{4t^2}\frac{d^2w}{dt^2} + \left(\frac{-2p_0t^2 - 1}{4t^3}\right)\frac{dw}{dt} - \frac{{p_0}^2}{4}w\right\}e^{\frac{-p_{0x}}{2}}$$

Substituting these values in equation(1), we obtain

$$\frac{d^2w}{dt^2} + (2tp(t^2) - 2p_0t - t^{-1})\frac{dw}{dt} + t^2\{4q(t^2) - {p_0}^2 - 2p_0p(t^2)\}w = 0$$

, which is the desired result.

Equation (7) has the same form as (1). For $|t| > a^{\frac{1}{2}}$ its coefficients may be expanded in series

$$F(t) = \frac{2p_1 - 1}{t} + \frac{2p_2}{t^3} + \frac{2p_3}{t^5} + \cdots$$

$$G(t) = (4q_1 - 2p_0p_1) + \frac{4q_2 - 2p_op_1}{t^2} + \cdots$$

Here

$$F_0 = 0, F_1 = 2p_1 - 1 \text{ ,and}$$

$$G_0 = 4q_1 - 2p_0p_1, G_1 = 0$$

If $4q_1 = 2p_0p_1$,then (7) has regular singularity at $t = \infty$, and therefore admits of solutions in convergent power series.

Alternatively, if $4q_1 \neq 2p_0p_1$,

then

$$\rho = \left(\frac{1}{4}{F_0}^2 - G_0\right)^{\frac{1}{2}}$$

$$= (2p_0p_1 - 4q_1)^{\frac{1}{2}} \neq 0$$

$$\sigma = \frac{\frac{1}{4}F_0F_1 - \frac{1}{2}G_1}{\rho} = 0,$$

$$\lambda = \pm(2p_0p_1 - 4q_1)^{\frac{1}{2}}$$

$$\mu = 0 - \frac{1}{2}F_1 = \frac{1}{2} - p_1$$

Therefore, if $4q_1 \neq 2p_0p_1$,then (7) has an irregular singularity at infinity with an equal characteristic values $\pm(2p_1p_0 - 4q_1)^{\frac{1}{2}}$ and hence we can construct formal series expansions for w of the form (6) with x replaced by t.

Restoring the original variables in the case $4q_1 \neq 2p_0p_1$ we obtain series solutions of the form

$$y = e^{\left\{-\frac{1}{2}p_0x \pm (2p_1p_0-4q_1)^{\frac{1}{2}}x^{\frac{1}{2}}\right\}}x^{\frac{(1-2p_1)}{4}}\sum_{n=0}^{\infty}\widetilde{a_n}x^{\frac{-n}{2}}$$

Again, the coefficients $\widetilde{a_n}$ may be found by direct substitution in the original differential equation. Expansions of this kind, involving fractional powers of x, are called subnormal solutions.

3.2 Examples

Example 1. The modified Bessel equation:

Examine the asymptotic series solutions of

$$y'' + \frac{1}{x}y' - \left(1 + \frac{v^2}{x^2}\right)y = 0 \text{ as } x \to \infty.$$

In chapter two we have showed that this equation has an irregular singularity at infinity.

Now, in the notation above, we have

$$p(x) = 0 + \tfrac{1}{x} + \tfrac{0}{x^2} + \cdots,$$

and

$$q(x) = -1 + \frac{0}{x} - \frac{v^2}{x^2} + \cdots$$

Then

$$p_0 = 0, p_1 = 1, p_n = 0 \ \forall n \geq 2$$

$$q_0 = -1, q_1 = 0, q_2 = -v^2, q_n = 0 \ \forall n \geq 3.$$

Since ${p_0}^2 = 0 \neq 4q_0 = -4$, we can use the argument of theorem 2.

$$\rho = \left(\frac{1}{4}{p_0}^2 - q_0\right)^{\frac{1}{2}} = 1,$$

$$\sigma = \frac{\left(\frac{p_0 p_1}{4} - \frac{q_1}{2}\right)}{\rho} = 0.$$

and hence

$$\lambda = \pm\rho - \frac{1}{2}p_0 = \pm 1,$$

$$\mu = \pm\sigma - \tfrac{1}{2}p_1 = -\tfrac{1}{2}.$$

Thus the equation has asymptotic solutions of the form

$$y_i \sim e^{\pm x} x^{\frac{-1}{2}} \textstyle\sum_{n=0}^{\infty} a_{n,i} x^{-n} \ \text{ as } x \to \infty.\ i = 1,2$$

Using the recursion relation(6.03),

$$2\lambda_i n a_{n,i} = \left(\frac{(2n-1)^2-4v^2}{4}\right) a_{n,i-1}$$

$$\Rightarrow a_{n.i} = \left(\frac{(2n-1)^2-4v^2}{8\lambda_i n}\right) a_{n,i-1}$$

$$= (\pm 1)^n \frac{(1-4v^2)(3^2-4v^2)\dots\left((2n-3)^2-4v^2\right)\left((2n-1)^2-4v^2\right)}{8^n n!}$$

With $a_{0,1} = a_{0,2} = 1$.

∴ The asymptotic expansions of the solutions are

$$y_1 \sim e^x x^{\frac{-1}{2}} \sum_{n=0}^{\infty} \left\{\frac{(1-4v^2)(3^2-4v^2)\dots\left((2n-3)^2-4v^2\right)\left((2n-1)^2-4v^2\right)}{8^n n!}\right\} x^{-n}$$

,and

$$y_2(x) \sim e^{-x} x^{\frac{-1}{2}} \sum_{n=0}^{\infty} (-1)^n \left\{\frac{(1-4v^2)(3^2-4v^2)\dots\left((2n-3)^2-4v^2\right)\left((2n-1)^2-4v^2\right)}{8^n n!}\right\} x^{-n}$$

Example 2. Consider the differential equation

$$y'' - \frac{1}{x} y = 0 \dots\dots\dots\dots (2.1)$$

We want to construct asymptotic series expansions of the solutions of (2.1) in a neighborhood of infinity.

First we need to classify the point at infinity.

Set: $t = \frac{1}{x}$,then equation (2.1) is transformed to

$$\ddot{y} + \frac{2}{t}\dot{y} - \frac{1}{t^3} y = 0 \dots\dots\dots\dots..(2.2)$$

Clearly equation (2.2) has an irregular singular point at $t = 0$. Hence equation (2.1) has irregular singular point at infinity.

Now,

$$p(x) \equiv 0 \ ,$$

$$q(x) = 0 - \frac{1}{x} + \frac{o}{x^2} + \cdots$$

Since ${p_0}^2 = 4q_0$, we use the transformation of Fabry,

$$y = e^{-p_0\frac{x}{2}}w = w \ , t = x^{\frac{1}{2}}$$

then equation (2.1) becomes

$$\frac{d^2y}{dt^2} - \frac{1}{t}\frac{dy}{dt} - 4y = 0 \ldots\ldots\ldots\ldots (2.3)$$

Thus

$$F(t) = 0 - \frac{1}{t} + \frac{0}{t^2} + \cdots,$$

$$G(t) = -4 + \frac{0}{t} + \frac{0}{t^2} + ..$$

Since $4q_1 = -4 \neq 2p_0p_1 = 0$, equation (2.3) has an irregular singular point at infinity.

Now,

$$\rho = (\tfrac{1}{4}F_0 - G_0)^{\frac{1}{2}} = \sqrt{4} = 2,$$

$$\sigma = \frac{\frac{1}{4}F_0 - \frac{1}{2}G_1}{\rho} = 0$$

Hence

$$\lambda = \pm\rho - \frac{1}{2}F_0 = \pm 2$$

$$\mu = \pm\sigma - \frac{1}{2}F_1 = \frac{1}{2}$$

Thus equation (2.3) has asymptotic solutions of the form

$$y(t) = e^{\pm 2t} t^{\frac{1}{2}} \sum_{n=0}^{\infty} a_{n,i} t^{-n}$$

Using the recurrence relation

$(F_0 + 2\lambda)na_n = (n-\mu)(n-1-\mu)a_{n-1} + \{\lambda F_2 + G_2 - (n-1-\mu)F_1\}a_{n-1} + \{\lambda F_3 + G_3 - (n-2-\mu)F_2\}a_{n-2} + \cdots \{\lambda F_{n+1} + G_{n+1} + \mu F_n\}a_0$, we have

$$2\lambda n a_n = \left(\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) + (n-\frac{3}{2})\right)a_{n-1}$$

$$= \left(n-\tfrac{3}{2}\right)(n+\tfrac{1}{2})a_{n-1}$$

$$\Longrightarrow a_n = (\pm 1)^n \frac{\left(n-\frac{3}{2}\right)\left(n+\frac{1}{2}\right)}{2\lambda n} a_{n-1}$$

$$= \frac{\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\ldots(n+\frac{1}{2})}{2^{2n} n!} \quad for\ n > 1$$

$$, a_0 = 1.$$

That is

$$a_{n,1} = \frac{\left(n-\frac{3}{2}\right)\left(n+\frac{1}{2}\right)}{2\lambda_1 n} a_{n,1-1}$$

$$= \frac{\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\ldots(n+\frac{1}{2})}{2^{2n} n!} \quad for\ n > 1, a_{0,1} = 1.$$

and

$$a_{n,2} = \frac{\left(n-\frac{3}{2}\right)\left(n+\frac{1}{2}\right)}{2\lambda_2 n} a_{n,2-1}$$

$$= (-1)^n \frac{\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\ldots\left(n+\frac{1}{2}\right)}{2^{2n} n!}, n > 1$$

$$a_{0,2} = 1.$$

$$\therefore y_1(t) \sim e^{2t} t^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\ldots\left(n+\frac{1}{2}\right)}{2^{2n} n!\, t^n}$$

$$y_2(t) \sim e^{-2t} t^{\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n \frac{\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\ldots(n+\frac{1}{2})}{2^{2n} n!\, t^n}$$

are the asymptotic series solutions of $(2.3)\ as\ t \to \infty$.

Substituting $t = x^{\frac{1}{2}}$,we obtain

$$y_1(x) \sim e^{2\sqrt{x}} x^{\frac{1}{4}} \sum_{n=0}^{\infty} \frac{\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\ldots\left(n+\frac{1}{2}\right)}{2^{2n} n!\, x^{\frac{n}{2}}}$$

and

$$y_2(x) \sim e^{-2\sqrt{x}} x^{\frac{1}{4}} \sum_{n=0}^{\infty} (-1)^n \frac{\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\ldots\left(n+\frac{1}{2}\right)}{2^{2n} n!\, x^{\frac{n}{2}}}$$

,which are the asymptotic series expansions of solutions of (2.1)

3.3/ Differential Equations with Irregular Singularity of Arbitrary Finite Rank

In the previous section we have derived the asymptotic expansions to the solutions of the differential equation

$$y'' + p(x)y' + q(x)y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

with an irregular singularity at infinity of unit rank. In this section we show that how to find the asymptotic expansions to solutions of (1) with an irregular singularity at infinity of arbitrary finite rank.

First we find the asymptotic approximation of (1) in a neighborhood of infinity by the method we develop in chapter two, say $y_0(x)$.

That is $y(x) \sim y_0(x)\ as\ x \to \infty$. Meaning, we first factored off the singular behavior for y since the asymptotic approximation is the leading term in an asymptotic expansion. Then we might expect what is left over is well behaved enough to expand in Taylor series about infinity.

Hence assuming that we can expand the solution for y in the form

$$y(x) \sim y_0(x) \sum_{n=0}^{\infty} a_n x^{-n} \qquad as\ x \to \infty.$$

Substituting this series in (1) and equating coefficients we determine $a_0, a_1, a_2,$ ...This is demonstrated through the following example.

3.4. Example: The Parabolic Cylinder Equation

Examine the behavior of the solutions of the parabolic cylinder equation

$$y'' + \left(v + \frac{1}{2} - \frac{1}{4}x^2\right) y = 0\ as\ x \to \infty.$$

One can show that this equation has an irregular singular point at infinity of rank two. First we have to find the asymptotic approximation of the solutions.

Here, $f(x) = -\left(v + \frac{1}{2} - \frac{1}{4}x^2\right)$.

$f(x) > 0$ in a neighborhood of ∞ , and $\left|\frac{4ff'' - 5f'^2}{16f^3}\right| \ll 1 \quad as\ x \to \infty$.

Hence the asymptotic solutions becomes

$$y(x) \sim f^{\frac{-1}{4}} e^{\pm \int f^{\frac{1}{2}}(x)dx} \text{ as } x \to \infty.$$

To simplify the result

$$\left(v + \frac{1}{2} - \frac{1}{4}x^2\right)^{\frac{1}{2}} = \frac{1}{2}x - \left(v + \frac{1}{2}\right)\frac{1}{x} + O(\,x^{-3})$$

and hence

$$\int f^{\frac{1}{2}} dx = \frac{1}{4}x^2 - \left(v + \frac{1}{2}\right)\log x + O(x^{-2}) \ \ as\ x \to \infty.$$

$$f^{\frac{-1}{4}} \sim \frac{\sqrt{2}}{\sqrt{x}} + O\left(x^{\frac{-5}{2}}\right) \quad as\ x \to \infty.$$

$$\therefore\ y_1(x) \sim \frac{1}{\sqrt{x}} e^{\frac{1}{4}x^2 - \left(v + \frac{1}{2}\right)\log x} = x^{-1-v} e^{\frac{x^2}{4}}$$

and

$$y_2(x) \sim \frac{1}{\sqrt{x}} e^{-\frac{1}{4}x^2 + \left(v + \frac{1}{2}\right)\log x} = x^v e^{-\frac{1}{4}x^2} \qquad x \to \infty.$$

Le us examine the behavior of the bounded solution. The asymptotic approximation (the leading order behavior) of this solution is

$$y_2(x) \sim x^v e^{-\frac{1}{4}x^2} \quad as\ \ x \to \infty.$$

Since we factored off the singular behavior for y, assume that we can expand the solution for y in the form

$$y(x) \sim x^v e^{-\frac{1}{4}x^2}\psi(x) = \ x^v e^{-\frac{1}{4}x^2}\sum_{n=0}^{\infty} a_n x^{-n} \quad\quad as \ \ x \to \infty.$$

Differentiating $y = x^v e^{-\frac{1}{4}x^2}\psi(x)$,

$$y^{'} = \left[vx^{v-1} - \frac{1}{2}x^{v+1}\right]e^{-\frac{1}{4}x^2}\psi(x) + x^v e^{-\frac{1}{4}x^2}\psi^{'}(x)$$

$$\begin{aligned} y^{''} &= \left[v(v-1)x^{v-2} - \frac{1}{2}vx^v - \frac{1}{2}(v+1)x^v + \frac{1}{4}x^{v+2}\right]e^{-\frac{1}{4}x^2}\psi(x) \\ &\quad + 2\left[vx^{v-1} - \frac{1}{2}x^{v+1}\right]e^{-\frac{1}{4}x^2}\psi^{'(x)} + x^v e^{-\frac{1}{4}x^2}\psi^{''(x)}. \end{aligned}$$

Substituting this in to the equation for y,

$$\begin{aligned} &\left[v(v-1)x^{-2} - \left(v + \frac{1}{2}\right) + \frac{1}{4}x^2\right]\psi(x) + 2\left[vx^{-1} - \frac{1}{2}x\right] + \psi^{''}(x) \\ &\quad + \left[v + \frac{1}{2} - \frac{1}{4}x^2\right]\psi(x) = 0 \end{aligned}$$

$$\Longrightarrow x^2\psi^{''}(x) + (2vx - x^3)\psi^{'}(x) + v(v-1)x^{-2}\psi(x) = 0 \ldots\ldots\ldots..(*)$$

Differentiating $\psi(x) = \sum_{n=0}^{\infty} a_n\, x^{-n}$,

$$\psi^{'}(x) = \sum_{n=1}^{\infty} -na_n x^{-n}$$

$$= \sum_{n=-1}^{\infty} -(n+2)a_{n+2}x^{-n-3}$$

$$\psi^{''}(x) = \sum_{n=1}^{\infty} n(n+1)a_n x^{-n-2}$$

Substituting this in to the equation$(*)$, we obtain

$$\sum_{n=1}^{\infty} n(n+1)a_n x^{-n-2} + 2v\sum_{n=1}^{\infty} -na_n x^{-n} - \sum_{n=-1}^{\infty} -(n+2)a_{n+2}x^{-n-3}$$

$$+ v(v+1)\sum_{n=0}^{\infty} a_n x^{-n} = 0.$$

Equating the coefficient of x^1 to zero yields

$$a_1 x = 0 \Longrightarrow a_1 = 0.$$

Equating the coefficient of x^0 to zero yields

$$2a_2 + v(v+1)a_0 = 0 \Longrightarrow a_2 = -\frac{1}{2}v(v+1)$$

And from the coefficient of x^{-n} $for\ n > 0$,

$$n(n+1)a_n - 2vna_n + (n+2)a_{n+2} + v(v-1)a_n = 0$$

$$\Longrightarrow (n+2)a_{n+2} = -[n(n+1) - 2vn + v(v-1)]a_n$$

$$= -[n^2 + n - 2vn + v(v-1)]a_n$$

$$= -(n-v)(n-v+1)a_n$$

Thus the recursion formula for the a_n is

$$a_{n+2} = -\frac{(n-v)(n-v+1)}{n+2}a_n \, , a_0 = 1, a_1 = 0.$$

$$\therefore \psi(x) = 1 - \frac{v(v-1)}{2\,!}x^{-2} + \frac{v(v-1)(v-2)(v-3)}{2^2 2!}x^{-4} - \cdots$$

Therefore, our asymptotic expansion for y is

$$y(x) \sim x^{v} e^{-\frac{x^2}{4}} \left[1 - \frac{v(v-1)}{2\,!} x^{-2} + \frac{v(v-1)(v-2)(v-3)}{2^2 2!} x^{-4} - \cdots \right] \ , x \to \infty.$$

If we check the radius of convergence of this series,

$$\lim_{n\to\infty} \left| \frac{a_{n+2} x^{-n-2}}{a_n x^{-n}} \right| < 1$$

$$\Rightarrow \lim_{n\longrightarrow\infty} \left| -\frac{(n-v)(n-v+1)}{n+2} x^{-2} \right| < 1$$

$$\Rightarrow \frac{1}{x} = 0$$

⟹The radius of convergence is zero.

Thus, if $v \neq 0,1,2$ ...this asymptotic expansion diverges for all x. However, this solution is still very useful. If we only use a finite number of terms, we will get a very good numerical approximation for large x.

3.5 Inhomogeneous Equations

We have already derived the asymptotic expansions of the solutions $y_1(x)$ and $y_2(x)$ of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

In this section we consider the construction of the asymptotic expansion of the particular solution of the inhomogeneous equation

$$y'' + p(x)y' + q(x)y = x^{\alpha} e^{\beta x} r(x) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . (1.1),$$

in which α and β are real constants, and $p(x), q(x)\ and\ r(x)$ are analytic functions having convergent series expansions

$$p(x) = \sum_{n=0}^{\infty} p_n x^{-n} \quad ,$$

$$q(x) = \sum_{n=0}^{\infty} q_n x^{-n} \; ,$$

$$r(x) = \sum_{n=0}^{\infty} r_n x^{-n}$$

for $|x| > a, a > 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . (1.2)$

The general solution of (1.1) has the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

where $c_1 \;\; and \; c_2$ are arbitrary constants, $y_1(x) \; and \; y_2(x)$ are independent solutions of the corresponding homogeneous equation, and $y_p(x)$ is a particular solution of (1.1).

First if we let $y = e^{\beta x} v$

then using the chain rule, we have

$$y^{'} = (\beta v + v')e^{\beta x}$$

and

$$y^{''} = (\beta^2 v + 2\beta v^{'} + v'')e^{\beta x}$$

Substituting these values in (1.1), we have

$$\beta^2 v + 2\beta v^{'} + v^{''} + \beta p v + p v^{'} + q v = x^{\alpha} r$$

$$\Rightarrow v^{''} + (2\beta + p(x))v^{'} + (\beta^2 + \beta p(x) + q(x))v = x^{\alpha} r(x) \ldots \ldots \ldots \ldots \ldots \ldots . (1.3),$$

which has the form as (1.1) but without an exponential factor in the inhomogeneous term.

Hence without loss of generality, we consider the equation

$$y^{''} + p(x)y^{'} + q(x)y = x^{\alpha} r(x) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . (1.4)$$

in which $p(x), q(x)\ and\ r(x)$ have the expansions (1.2)

As a possible formal series solutions of (1.4) we try the series

$$y = x^{\alpha} \sum_{n=0}^{\infty} a_n x^{-n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.5)$$

Differentiating $y = \sum_{n=0}^{\infty} a_n x^{\alpha-n}$

$$y^{'} = \sum_{n=0}^{\infty} (\alpha - n) a_n x^{\alpha-n-1}$$

$$= \sum_{n=1}^{\infty} (\alpha - n + 1) a_{n-1} x^{\alpha-n}$$

$$y^{''} = y = \sum_{n=0}^{\infty} (\alpha - n)(\alpha - n - 1) a_n x^{\alpha-n-2}$$

$$= \sum_{n=2}^{\infty} (\alpha - n + 1)(\alpha - n + 2) a_{n-2} x^{\alpha-n}$$

Substituting $y, y^{'}, y''$ and the expansions (1.2) in (1.4),

$$\sum_{n=2}^{\infty} (\alpha - n + 1)(\alpha - n + 2) a_{n-2} x^{\alpha-n}$$
$$+ \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} a_{n-i} p_{i-1} (\alpha - n + i) \right] x^{\alpha-n} + \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} a_{n-i} q_i + q_0 a_n \right] x^{\alpha-n}$$
$$= \sum_{n=0}^{\infty} r_n x^{\alpha-n}$$

Equating the coefficient of $x^{\alpha-n}$ to zero, we derive

$$q_0 a_n + \sum_{i=1}^{n}\{q_i + p_{i-1}(\alpha - n + i)\}a_{n-i} + (\alpha - n + 2)(\alpha - n + 1)a_{n-2} = r_n \ldots\ldots\ldots\ldots\ldots .(1.6) \text{ for } n = 0,1,2\ldots$$

Equation (1.6) determines the a_n recursively provided that $q_0 \neq 0$. For simplicity we assume this to be the case.

In particular,

$$a_0 = {q_0}^{-1} r_0$$

$$a_1 = {q_0}^{-1} r_1 - {q_0}^{-2} r_0 (q_1 + \alpha p_0)$$

The structure of the recurrence relation (1.6) indicates that in general the series (1.5) diverges for all finite values of x.

Example: Consider the inhomogeneous form of modified Bessel equation

$$y'' + \frac{1}{x} y' - \left(1 + \frac{v^2}{x^2}\right) y = \left(\frac{1}{2}x\right)^{v-1}$$

We want to find the particular solution of this equation.

Now

$$p(x) = \frac{1}{x} \ ,$$

$$q(x) = -1 - \frac{v^2}{x^2} \, ,$$

$$r(x) = \frac{1}{2^{v-1}}$$

Thus

$$p_0 = 0, p_1 = 1$$

$$q_0 = -1, q_1 = 0, \qquad q_2 = -v^2$$

$$r_0 = \frac{1}{2^{v-1}}, r_n = 0, \forall n \geq 1$$

Then

$$a_0 q_0 = r_0 \Longrightarrow a_0 = -\frac{1}{2^{v-1}}$$

$$a_1 = {q_0}^{-1} r_1 - {q_0}^{-2} r_0 (q_1 + \alpha p_0) \Rightarrow a_1 = 0.$$

$$q_0 a_2 = -\left\{\sum_{i=1}^{2} [q_i + p_{i-1}(v - 3 + i)] a_{2-i} + (v-1)(v-2) a_0\right\}$$

$$\Rightarrow a_2 = (1 - 2v) a_0$$

and

$$q_0 a_4 = -\left\{\sum_{i=1}^{4} [q_i + p_{i-1}(v - 5 + i)] a_{4-i} + (v-3)(v-4) a_2\right\}$$

$$\Rightarrow a_4 = -(6v - 9) a_2 = (3^2 - 2(3v)) a_2$$

$$\therefore a_{2n} = \{(2n-1)^2 - 2(2n-1)v\} a_{2n-2} \text{ for } n \geq 1$$

$$a_0 = -\tfrac{1}{2^{v-1}}, \text{ and } a_{2n+1} = 0.$$

Therefore the particular solution of this equation has asymptotic expansion

$$y_p(x) \sim x^{v-1}\left[-\tfrac{1}{2^{v-1}} + \textstyle\sum_{n=1}^{\infty} a_{2n} x^{-2n}\right] \ \ as\ x \to \infty.$$

The general solution of the given differential equation has asymptotic expansion

$$y(x) \sim c_1 y_1(x) + c_2 y_2(x) + y_p(x) \text{ as } x \to \infty,$$

where $y_1(x)\ and\ y_2(x)$ are the solutions we found in example 1 of section 3.2.

APPENDIX

ASYMPTOTIC RELATIONS AND EXPANSIONS

A/ Asymptotic Relations

Let us recall a few definitions used in asymptotic relations.

Let $f, g: D \subseteq \mathbb{R} \to \mathbb{R}, x_0 \in \overline{D}$, the asymptotic behavior of f for $x \to x_0$ is given by the value of $f(x)$ for $x \to x_0$. We shall use the following notations to describe the behavior of a function $f(x)$ in the limit as $x \to x_0$.

i) $f \sim g$ as $x \to x_0$,is read as f is asymptotic to g or g is an asymptotic approximation to $f\ as\ x \to x_0$. This means $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$.

ii) $f = o(g)$ as $x \to x_0$, is read as f is of order less than g as $x \to x_0$. This means $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$. Also we write $f \ll g\ as\ x \to x_0$.

iii) $f = O(g)$ as $x \to x_0$, is read as f is of order not exceeding g as $x \to x_0$. This means $\frac{f}{g}$ is bounded in a neighborhood of x_0.

Note:

a) $f = o(1)$ as $x \to x_0$, meaning f vanishes as $x \to x_0$

b) $f = O(1)$ as $x \to x_0$,meaning that $|f|$ is bounded as $x \to x_0$.

Examples:

a) $(x+1)^2 \sim x^2$ as $x \to \infty$.

b) $\sin x \sim x$ as $x \to 0$.

c) $\frac{1}{x} \ll 1$ as $x \to \infty$.

d) $\frac{1}{x^2} = o\left(\frac{1}{x}\right)$ as $x \to \infty$.

The ordinary arithmetic operations (addition, multiplication, and division) are valid on asymptotic relations.

As a rule, asymptotic relations may be integrated subject to obvious restrictions on the convergence of the integrals involved. However, differentiation of asymptotic relation is not always valid.

Example: Let $f(x) = x + \cos x$. Then $f(x) \sim x \; as \; x \to \infty$. But it is not true that $f'(x) \to 1 \; as \; x \to \infty$.

B) Asymptotic Expansions

The set $\{f_i\}^{\infty}{}_{i=0}$ of functions $f_i : D \to \mathbb{R}, f_{i+1} \ll f_i \; \forall i$ as $x \to x_0 \in \overline{D}$ is called an asymptotic sequence.

An asymptotic series is a sum

$$\sum_{i=0}^{\infty} a_i f_i(x) \text{ , where } f_{i+1} \ll f_i \; \forall i \; as \; x \to x_0 \in \overline{D}.$$

An asymptotic expansion of f with respect to the asymptotic sequence $\{f_i\}$ is asymptotic series

$$\sum_{i=0}^{\infty} a_i f_i(x)$$

such that for each non negative integer n

$$f(x) = \sum_{i=0}^{n} a_i f_i(x) + O(f_{n+1}(x)) \qquad (x \to x_0 \in \overline{D})$$

In this case we write

$$f(x) \sim \sum_{i=0}^{\infty} a_i f_i(x) \qquad (x \to x_0 \in \overline{D})$$

Hence

$$f(x) - \sum_{i=0}^{n} a_i f_i(x) = o(f_n(x))$$

and

$$\lim_{x \to x_0} \frac{f(x) - \sum_{i=0}^{k} a_i f_i(x)}{f_k(x)} = a_k < \infty, k = 0,1,2,3 \dots$$

The first term $a_0 f_0$ is called the leading term or the asymptotic representation of f.

Note that an asymptotic series may be convergent or divergent. If the series is convergent, then we have that

$$f(x) - \sum_{i=0}^{n} a_i f_i(x) \to 0 \ as \ n \to \infty \text{ for fixed } x.$$

As far as the operations on asymptotic expansions concerned, the arithmetic operation addition is valid. Multiplication and division of asymptotic expansions is not always valid because of the fact that it is not always possible to arrange the double infinite array $f_n(x) f_m(x)$ as a single asymptotic series. Integration of an asymptotic expansion is valid where as differentiation may be invalid.

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