ADDIS ABABA UNIVERSITY
COLLEGE OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS

ANALYSIS OF BOUNDARY-DOMAIN INTEGRAL EQUATIONS FOR VARIABLE COEFFICIENT (THE CASE OF DIRICHLET BVP IN 2D)

A Thesis Submitted to the Department of Mathematics of Addis Ababa University in Partial Fulfillment of the Requirements of the Master of Science Degree in Mathematics

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We, the undersigned, hereby certify that we have read and examined this thesis on **ANALYSIS OF BOUNDARY-DOMAIN INTEGRAL EQUATIONS FOR VARIABLE COEFFICIENT (THE CASE OF DIRICHELET BVP IN 2D)**, which is done by SHITAYE ASCHALE in partial fulfillment of the requarements for the degree of master of science and recommend to the school of graduate studies for acceptance of a thesis.

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Abstract

Using an appropriate parametrix (Levi function), Dirichlet boundary value problem is reduced to some direct segregated systems of Boundary- Domain Integral Equations (BDIEs). Although the theory of BDIEs in $3D$ is well developed, the BDIEs in $2D$ need a special consideration due to their different equivalence properties. Consequently, we need to set conditions on the domain or the spaces to insure the invertibility of corresponding parametrix-based integral layer potentials and hence the unique solubility of BDIEs. The properties of corresponding potential operators are investigated. The equivalence of the original BVP and the obtained BDIEs are analysed and the invertibility of the BDIE operators is proved.

Notations

$\mathbb{R}^2$ Two-dimensional Euclidean space.

Ω Bounded open set in $\mathbb{R}^2$.

$\partial\Omega$ The boundary of Ω.

$\bar{\Omega}$ The closure of Ω.

$B(x,r)$ Ball of radius r about x in $\mathbb{R}^2$.

$\partial B(x,r)$ Boundary of ball of radius r about x in $\mathbb{R}^2$.

$C^\infty(\Omega)$ The set of all infinitely differentiable function on Ω.

$D(\Omega)$ The set of all infinitely differentiable function on Ω with compact support.

$D'(\Omega)$ The space of continuous linear functionals on $D(\Omega)$.

$H^s(\Omega)$ Sobolev spaces .

Δ The Laplace's operator.

∇ The gradient operator.

Introduction

Many applications in science and engineering can be modeled by boundary-value problems (BVPs) for partial differential equations with variable coefficients. Reduction of the BVPs with arbitrarily variable coefficients to explicit boundary integral equations is usually not possible, since the fundamental solution necessary for such reduction is generally not available in an analytical form (except for some special dependence of the coefficients on coordinates). Laplace's equation is one of the most important partial differential equations in applied mathematics, because it occurs in gravity, electrostatic, steady state heat conduction, compressible fluid flow and so on.

An Italian mathematician, Levi introduced method of the parametrix which is a way to construct fundamental solutions for elliptic PDE with variable coefficients.The Dirichlet boundary value problem for the Laplace equation with variable coefficient is reduced to boundary-domain integral equations (BDIEs) based on a specially constructed parametrix. The BDIEs contain potential-type integral operators defined on the domain under consideration and acting on the unknown solution as well as integral operators defined on the boundary and acting on the trace and/or co-normal derivative of the unknown solutions [5].

There are two approaches to derive BDIEs of BVPs for PDE with constant coefficients. The first integral formulation is often named as a direct method and the integral equations are derived through the application of the second Greens identity. The second integral formulation known as an indirect method is founded on single or double layer potentials. The method of boundary integral equations has always had two important applications in the theory of boundary value problems for partial differential equations: as a theoretical tool for proving the existence of solutions and as a practical tool for the construction of solutions [8]. The BDIEs are called segregated BDIEs when the unknown boundary functions are considered as formally unrelated to the unknown functions inside the domain whereas for the united BDIEs, the unknown boundary functions are related to the unknown functions inside the domain.

Let Ω be a domain in $\mathbb{R}^n$ bounded by simple closed infinitely differentiable curve $\partial\Omega$, the set of all infinitely differentiable function on Ω with compact support is denoted by $D(\Omega)$. The function space $D'(\Omega)$ consists of all continuous linear functionals over $D(\Omega)$. For any non-empty open set $\Omega \in \mathbb{R}^n$ we define $H^s(\Omega) = \{u \in D'(\Omega) : u = U|\Omega \ for\ some\ U \in H^s(\mathbb{R}^n)\}$. $H^{s,t}(\Omega; L_*)$ is the subspace of $H^s(\Omega)$ defined by $H^{s,t}(\Omega; L_*) := \{g : g \in H^s(\Omega), L_* g \in H^t(\Omega)\}$, for all $s, t \in \mathbb{R}$ and $H^{s,p}(\Omega) = H^s(\Omega) := \{u \in L_p : D^\alpha u \in L_p, |\alpha| \leq s,\ s \in \mathbb{N}_0,\ p \in [1, \infty)\}$.

The main part of this paper is divided into two chapters. In Chapter 1, we go through some preliminary results concerning Laplace's Equation, Mean Value Property and Potential Theory for constant coefficient in $\mathbb{R}^n$. These results are supposed to help to get into the subject and later on, also convince from the validity of some rather complicated proofs. In Chapter 2, we discuss the main result of this thesis, which is the Analysis of Boundary-Domain Integral Equations for variable coefficient (the case of Dirichlet boundary value problem in $2D$). First, we formulate the boundary value problem, construct prametrix and potential operators, then investigate the invertibility of these operators as well as analyse the corresponding BDIEs.

Chapter 1

Preliminaries

1.1 Laplace's Equation

The n-dimensional **Laplace's Equation**:

$$\Delta u = 0$$

and its inhomogeneous version, **Poisson's equation**

$$\Delta u = f.$$

A function u satisfying Laplace's equation is a harmonic functions.

1.1.1 The Fundamental Solution

Consider Laplaces equation in $\mathbb{R}^n$,

$$\Delta u = 0 \ , \ x \in \mathbb{R}^n$$

Given the symmetric nature of Laplace's equation, we look for a radial solution. That is, we look for a harmonic function u on $\mathbb{R}^n$ such that $u(x) = v(|x|)$. In addition, to being a natural choice due to the symmetry of Laplaces equation, radial solutions are natural to look for because they reduce a PDE to an ODE, which is generally easier to solve. Therefore, we look for a radial solution.

If $u(x) = v(|x|)$, then

$$\begin{aligned}
&u_{x_i} = \frac{x_i}{|x|} v'(|x|), \quad |x| \neq 0 \\
&\Rightarrow u_{x_i x_i} = \frac{1}{|x|} v'(|x|) - \frac{x_i^2}{|x|^3} v'(|x|) + \frac{x_i^2}{|x|^2} v''(|x|), \quad |x| \neq 0, \quad i = 1, 2, 3, ..., n \\
&\Rightarrow \Delta u = u_{x_1 x_1} + u_{x_2 x_2} + ... + u_{x_n x_n} \\
&= \frac{n-1}{|x|} v'(|x|) + v''(|x|).
\end{aligned}$$

Letting $r(x) = |x| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$,

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}(x_1^2 + x_2^2 + ... + x_n^2)^{-\frac{1}{2}}.2x_i = \frac{x_i}{r}, \quad x \neq 0, \quad i = 1, 2, 3, ..., n$$

we see that $u(x) = v(|x|)$ is a radial solution of Laplace's equation implies v satisfies

$$\frac{n-1}{r}v'(r) + v''(r) = 0.$$

Assuming $v(r) \neq 0$

$$\begin{aligned} \frac{v''}{v'} &= \frac{1-n}{r} \\ &\Rightarrow \ln v' = (1-n)\ln r + \tilde{C} \\ &\Rightarrow v' = \frac{C}{r^{n-1}}, \end{aligned}$$

$$v(r) = \begin{cases} c_1 \ln r + c_2 & n = 2 \\ \frac{c_1}{(2-n)r^{n-2}} + c_2 & n \geq 3 \end{cases}$$

From these calculations, we see that for any constants $c_1, \ c_2$ the function

$$u(x) \equiv \begin{cases} c_1 \ln |x| + c_2 & n = 2 \\ \frac{c_1}{(2-n)|x|^{n-2}} + c_2 & n \geq 3 \end{cases} \tag{1.1}$$

for $x \in \mathbb{R}^n, \ |x| \neq 0$ is a solution of Laplace's equation in $\mathbb{R}^n - \{0\}$. We notice that the function u defined in (1.1) satisfies $\Delta u(x) = 0$ for $x \neq 0$, but at $x = 0, \ \Delta u(0)$ is undefined. Define the function Φ as follows. For $|x| \neq 0$,

$$\Phi = \begin{cases} \frac{1}{2\pi} \ln |x| & n = 2 \\ -\frac{1}{n(n-2)\alpha(n)|x|^{n-2}} & n \geq 3 \end{cases}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$ and Φ satisfies Laplaces equation on $\mathbb{R}^n - \{0\}$.

Claim 1. Φ is a fundamental solution of Laplace's equation and satisfies

$$\Delta\Phi = \delta_0$$

in the sense of distributions. That is, for all $g \in D$,

$$\int_{R^n} \Phi(x)\Delta g(x)dx = g(0).$$

Proof. Let Φ be the fundamental solution defined:

$$(\Phi, g) = \int_{R^n} \Phi(x)g(x)dx$$

for all $g \in D$. Therefore, the distributional Laplacian of Φ is defined as

$$(\Delta\Phi, g) = (-1)^2(\Phi, \Delta g) = (\Phi, \Delta g)$$

for all $g \in D$. We will show that

$$(\Phi, \Delta g) = (\delta_0, g) = g(0),$$

which means $\Delta\Phi = \delta_0$ in the sense of distributions. Now, we would like to apply the divergence theorem, but Φ has a singularity at $x = 0$. We get around this, by breaking up the integral into two pieces: one piece consisting of the ball of radius δ about the origin, $B(0, \delta)$ and the other piece consisting of the complement of this ball in $\mathbb{R}^n$. Therefore, we have

$$\begin{aligned}(\Phi, \Delta g) &= \int_{\mathbb{R}^n} \Phi(x)\Delta g(x)dx \\ &= \int_{B(0,\delta)} \Phi(x)\Delta g(x)dx + \int_{\mathbb{R}^n - B(0,\delta)} \Phi(x)\Delta g(x)dx \\ &= I + J\end{aligned}$$

For $n = 2$, term I is bounded as follows,

$$\begin{aligned}\left|\int_{B(0,\delta)} \frac{1}{2\pi} \ln|x|\Delta g(x)dx\right| &\leq \tilde{C}|\Delta g(x)|_{L^\infty} \left|\int_{B(0,\delta)} \ln|x|dx\right| \\ &\leq C_1 \left|\int_0^{2\pi}\int_0^\delta \ln|r|rdrd\theta\right| \\ &\leq C_2 \left|\int_0^\delta \ln|r|rdr\right| \\ &\leq C\delta^2 \ln\delta.\end{aligned}$$

For $n \geq 3$, term I is bounded as follows,

$$\begin{aligned}\left|\int_{B(0,\delta)} -\frac{1}{n(n-2)\alpha(n)|x|^{n-2}}\Delta g(x)dx\right| &\leq C'|\Delta g(x)|_{L^\infty} \int_{B(0,\delta)} \frac{1}{|x|^{n-2}}dx \\ &\leq C\int_0^\delta \left(\int_{\partial B(0,r)} \frac{1}{|y|^{n-2}}dS(y)\right)dr \\ &= \int_0^\delta \frac{1}{|r|^{n-2}}\left(\int_{\partial B(0,r)} dS(y)\right)dr \\ &= \int_0^\delta \frac{1}{|r|^{n-2}}n\alpha(n)r^{n-1}dr \\ &= n\alpha(n)\int_0^\delta rdr \\ &= \frac{n\alpha(n)}{2}\delta^2.\end{aligned}$$

Therefore, as $\delta \to 0^+$, $|I| \to 0$.

Next, we look at term J. Applying the divergence theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^n-B(0,\delta)} \Phi(x)\Delta g(x)dx &= \int_{\mathbb{R}^n-B(0,\delta)} \Delta\Phi(x)g(x)dx - \int_{\partial(\mathbb{R}^n-B(0,\delta))} \frac{\partial\Phi}{\partial n}g(x)dS(x) \\ &+ \int_{\partial(\mathbb{R}^n-B(0,\delta))} \Phi(x)\frac{\partial g}{\partial n}dS(x) \\ &= -\int_{\partial(\mathbb{R}^n-B(0,\delta))} \frac{\partial\Phi}{\partial n}g(x)dS(x) + \int_{\partial(\mathbb{R}^n-B(0,\delta))} \Phi(x)\frac{\partial g}{\partial n}dS(x) \\ &= J_1 + J_2\end{aligned}$$

using the fact that $\Delta\Phi(x) = 0$ for $x \in \mathbb{R}^n - B(0,\delta)$.
We first look at term J_1. Now, by assumption, $g \in D$, and, therefore, g vanishes at ∞. Consequently, we only need to calculate the integral over $\partial B(0,\epsilon)$ where the normal derivative n is the outer normal to $\mathbb{R}^n - B(0,\delta)$. By a straightforward calculation, we see

$$\nabla\Phi(x) = \frac{x}{n\alpha(n)|x|^n}.$$

The outer unit normal to $\mathbb{R}^n - B(0,\delta)$ on $B(0,\delta)$ is given by

$$n(x) = \frac{x}{|x|}.$$

Therefore, the normal derivative of Φ on $B(0,\delta)$ is given by

$$\begin{aligned}\frac{\partial\Phi}{\partial n} &= \left(\frac{x}{n\alpha(n)|x|^n}\right)\cdot\left(\frac{x}{|x|}\right) \\ &= \frac{1}{n\alpha(n)|x|^{n-1}}.\end{aligned}$$

Therefore, J_1 can be written as

$$\begin{aligned}\int_{\partial B(0,\delta)} \frac{1}{n\alpha(n)|x|^{n-1}}g(x)dS(x) &= \frac{1}{n\alpha(n)|\delta|^{n-1}}\int_{\partial B(0,\delta)} g(x)dS(x) \\ &= \fint_{\partial B(0,\delta)} g(x)dS(x).\end{aligned}$$

Now if g is a continuous function, then

$$\fint_{\partial B(0,\delta)} g(x)dS(x) \to g(0) \ as \ \delta \to 0$$

Lastly, we look at term J_2. Now using the fact that g vanishes as $|x| \to +\infty$, we only need to integrate over $\partial B(0,\delta)$. Using the fact that $g \in D$, and, therefore, infinitely differentiable, we have

$$\left|\int_{\partial B(0,\delta)} \Phi(x)\frac{\partial g}{\partial n}dS(x)\right| \leq \left|\frac{\partial g}{\partial n}\right|_{L^\infty(\partial B(0,\delta))}\int_{\partial B(0,\delta)} |\Phi(x)|dS(x)$$
$$\leq C\int_{\partial B(0,\delta)} |\Phi(x)|dS(x).$$

Now first, for $n = 2$,

$$\begin{aligned}\int_{\partial B(0,\delta)} |\Phi(x)|dS(x) &= C\int_{\partial B(0,\delta)} |\ln|x||dS(x)\\ &\leq C|\ln\delta|\int_{\partial B(0,\delta)} dS(x)\\ &= C|\ln\delta|(2\pi\delta)\\ &\leq C'\delta|\ln\delta|.\end{aligned}$$

Next, for $n \geq 3$,

$$\begin{aligned}\int_{\partial B(0,\delta)} |\Phi(x)|dS(x) &= C\int_{\partial B(0,\delta)} \frac{1}{|x|^{n-2}}dS(x)\\ &\leq \frac{C}{\delta^{n-2}}\int_{\partial B(0,\delta)} dS(x)\\ &= \frac{C}{\delta^{n-2}}n\alpha(n)\delta^{n-2} \leq C_1\delta.\end{aligned}$$

Therefore, we conclude that term J_2 is bounded in absolute value by

$$\begin{array}{cc} C'\delta|\ln\delta| & n = 2 \\ C_1\delta & n \geq 3. \end{array}$$

Therefore, $|J_2| \to 0$ as $\delta \to 0^+$. Combining these estimates, we see that

$$\int_{\mathbb{R}^n} \Phi(x)\Delta g(x)dx = \lim_{\delta\to 0^+} I + J_1 + J_2 = g(0).$$

Therefore, $\Delta\Phi = \delta_0$. Thus, Φ is a fundamental solution of Laplace's equation. □

Theorem 1.1.1. ***(Solving Poisson's Equation)*** *Assume $f \in C^2(\mathbb{R}^n)$ and has compact support. Let*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy$$

where Φ is the fundamental solution of Laplace's equation. Then

1. *$u \in C^2(\mathbb{R}^n)$*
2. *$\Delta u = f$ in $\mathbb{R}^n$.*

Proof. 1. By a change of variables, we write

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy = \int_{\mathbb{R}^n} \Phi(y)f(x-y)dy.$$

Let $e_i = (0, ..., 0, 1, 0, ..., 0)$ be the unit vector in $\mathbb{R}^n$ with a 1 in the i^{th} slot. Then

$$\frac{u(x+he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y) \left[\frac{f(x+he_i - y) - f(x-y)}{h} \right] dy.$$

Now $f \in C^2$ implies

$$\frac{f(x+he_i-y) - f(x-y)}{h} \to \frac{\partial f}{\partial x_i}(x-y) \quad as \quad h \to 0$$

uniformly on $\mathbb{R}^n$. Therefore,

$$\frac{\partial u}{\partial x_i}(x) = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial f}{\partial x_i}(x-y)dy$$

Similarly,

$$\frac{\partial^2 u}{\partial x_i x_j}(x) = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial^2 f}{\partial x_i x_j}(x-y)dy.$$

This function is continuous because the right-hand side is continuous.

2. By using the above calculations and Using Claim1, we see that

$$\begin{aligned} \Delta_x u(x) &= \int_{\mathbb{R}^n} \Phi(y)\Delta_x f(x-y)dy \\ &= \int_{\mathbb{R}^n} \Phi(y)\Delta_y f(x-y)dy \\ &= f(x). \end{aligned}$$

□

1.1.2 Green's Identities and Green's Function

First state the Divergence theorem as the following:

Theorem 1.1.2. *(Divergence Theorem) Let Ω be a bounded solid region with a C^1 boundary curve $\partial\Omega$. Let n be the unit outward normal vector on $\partial\Omega$. Let u be any C^1 vector field on $\bar{\Omega} = \Omega \cup \partial\Omega$. Then*

$$\int_{\Omega} \nabla .u dV = \int_{\partial\Omega} u.n dS$$

where dV is the volume element in Ω and dS is the surface element on $\partial\Omega$.

Green's Identities

Green's Identities form an important tool in the analysis of Laplace equation. Let $u, w \in C^2(\bar{\Omega})$. Then we have

1. First Green's Identity:

$$\int_\Omega u\Delta w dx = -\int_\Omega \nabla u.\nabla w dx + \int_{\partial\Omega} u\frac{\partial w}{\partial n}dS$$

2. Second Green's Identity:

$$\int_\Omega (u\Delta w - w\Delta u)\, dx = \int_{\partial\Omega} \left(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\right) dS$$

Proof. Prove first Green's identity,

$$\begin{aligned}\nabla(u\nabla w) &= \nabla((uw_{x_1}, ..., uw_{x_n})) \\ &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(u\frac{\partial w}{\partial x_i}\right) \\ &= \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}.\frac{\partial w}{\partial x_i} + \sum_{i=1}^{n} u\frac{\partial^2 w}{\partial x_i^2} \\ &= \nabla u\nabla w + u\Delta w\end{aligned}$$

Integrating with respect to dx on Ω,

$$\begin{aligned}\int_\Omega \nabla u.\nabla w dx + \int_\Omega u\Delta w dx &= \int_\Omega \nabla(u\nabla w)dx \\ &= \int_{\partial\Omega} u\nabla w.n dS \\ &= \int_{\partial\Omega} u\frac{\partial w}{\partial n}dS\end{aligned}$$

Therefore,

$$\int_\Omega u\Delta w dx = -\int_\Omega \nabla u.\nabla w dx + \int_{\partial\Omega} u\frac{\partial w}{\partial n}dS \quad (1.2)$$

Prove second Green's identity:
First consider the following equation,

$$\int_\Omega w\Delta u dx = -\int_\Omega \nabla u.\nabla w dx + \int_{\partial\Omega} w\frac{\partial u}{\partial n}dS \quad (1.3)$$

Subtract equation (1.3) from equation (1.2), we get second Green's identity

$$\int_\Omega (u\Delta w - w\Delta u)dx = \int_{\partial\Omega} \left(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\right) dS.$$

□

Representation formula

Let $K(x, x_0) \equiv -1/(4\pi|x - x_0|)$. Any harmonic function u in an open solid region Ω can be expressed as an integral over the boundary $\partial\Omega$ as

$$u(x_0) = \int_{\partial\Omega} \left[u(x) \frac{\partial}{\partial n} K(x, x_0) - K(x, x_0) \frac{\partial}{\partial n} u(x) \right] dS$$

where $x_0 \in \Omega$ and x is on $\partial\Omega$.

Green's Function

Definition 1.1.1. *Let x_0 be an interior point of Ω. The Green's function $G(x, x_0)$ for the operator Δ and the domain Ω is a function defined for $x \in \Omega$ such that:*

(i) Let $K(x, x_0) = -1/(4\pi|x - x_0|)$. The function $H(x) = G(x, x_0) - K(x, x_0)$ has continuous second derivatives and is harmonic in Ω (including the point x_0).

(ii) $G(x, x_0) = 0$ for $x \in \partial\Omega$.

Theorem 1.1.3. *If $G(x, x_0)$ is the Green's function, then the solution of the Dirichlet problem is given by the formula*

$$u(x_0) = \int_{\partial\Omega} u(x) \frac{\partial G(x, x_0)}{\partial n} dS.$$

Proof. Recall that the representation formula is

$$u(x_0) = \int_{\partial\Omega} \left[u(x) \frac{\partial}{\partial n} K(x, x_0) - K(x, x_0) \frac{\partial}{\partial n} u(x) \right] dS$$

The result of applying Greens second identity to the pair of harmonic functions u and H is

$$\int_{\partial\Omega} \left[u(x) \frac{\partial}{\partial n} H(x) - H(x) \frac{\partial}{\partial n} u(x) \right] dS = 0.$$

Adding the two equations, the result becomes

$$\int_{\partial\Omega} \left[u(x) \frac{\partial}{\partial n} G(x, x_0) - G(x, x_0) \frac{\partial}{\partial n} u(x) \right] dS = \int_{\partial\Omega} u(x) \frac{\partial}{\partial n} G(x, x_0) dS.$$

Therefore,

$$u(x_0) = \int_{\partial\Omega} u(x) \frac{\partial}{\partial n} G(x, x_0) dS.$$

□

1.2 Mean Value Property

For a function u defined on $B(x,r)$, the average of u on $B(x,r)$ is given by

$$\fint_{B(x,r)} u(y)dy = \frac{1}{\alpha(n)r^n}\int_{B(x,r)} u(y)dy.$$

For a function u defined on $\partial B(x,r)$, the average of u on $\partial B(x,r)$ is given by

$$\fint_{\partial B(x,r)} u(y)dy = \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(x,r)} u(y)dS(y).$$

Theorem 1.2.1. *(Mean-Value Theorem) Let $\Omega \subset \mathbb{R}^n$. If $u \in C^2(\Omega)$ is harmonic, then*

$$u(x) = \fint_{\partial B(x,r)} u(y)dS(y) = \fint_{B(x,r)} u(y)dy$$

for every ball $B(x,r) \subset \Omega$.

Proof. Assume $u \in C^2(\Omega)$ is harmonic. For $r > 0$, define

$$\phi(r) = \fint_{\partial B(x,r)} u(y)dS(y).$$

For $r = 0$, define $\phi(r) = u(x)$. Notice that if u is a smooth function, then $\lim_{r\to 0^+}\phi(r) = u(x)$, and, therefore, ϕ is a continuous function. Therefore, if we can show that $\phi'(r) = 0$, then we can conclude that ϕ is a constant function, and, therefore,

$$u(x) = \fint_{\partial B(x,r)} u(y)dS(y).$$

We prove $\phi'(r) = 0$ as follows. First, making a change of variables, we have

$$\begin{aligned}\phi(x) &= \fint_{\partial B(x,r)} u(y)dS(y)\\ &= \fint_{\partial B(0,1)} u(x+rz)dS(y).\end{aligned}$$

Therefore,

$$\begin{aligned}
\phi'(r) &= \fint_{\partial B(0,1)} \nabla u(x+rz).z dS(z) \\
&= \fint_{\partial B(x,r)} \nabla u(y). \frac{y-x}{r} dS(y) \\
&= \fint_{\partial B(x,r)} \frac{\partial u}{\partial n}(y) dS(y) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial n}(y) dS(y) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \nabla.(\nabla u) dy \quad (\textit{ by the Divergence Theorem}) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y) dy = 0,
\end{aligned}$$

using the fact that u is harmonic. Therefore, we have proven the first part of the theorem. It remains to prove that

$$u(x) = \fint_{B(x,r)} u(y) dy.$$

We do so as follows, using the first result,

$$\begin{aligned}
\int_{B(x,r)} u(y) dy &= \int_0^r \left(\int_{\partial B(x,s)} u(y) dS(y) \right) ds \\
&= \int_0^r \left(n\alpha(n) s^{n-1} \fint_{\partial B(x,s)} u(y) dS(y) \right) ds \\
&= \int_0^r n\alpha(n) s^{n-1} u(x) ds \\
&= n\alpha(n) u(x) \int_0^r s^{n-1} ds \\
&= \alpha(n) u(x) s^n |_{s=0}^{s=r} \\
&= \alpha(n) u(x) r^n.
\end{aligned}$$

Therefore,

$$\int_{B(x,r)} u(y) dy = \alpha(n) u(x) r^n,$$

which implies

$$u(x) = \frac{1}{\alpha(n) r^n} \int_{B(x,r)} u(y) dy = \fint_{B(x,r)} u(y) dy$$

as claimed □

1.3 Potential Theory

1.3.1 Representation Formula

Consider Ω an open, bounded subset of R^n with C^2 boundary and $\Omega^c = R^n - \overline{\Omega}$ (the open complement of Ω). Two problems are:
(a) Interior Dirichlet Problem.

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ u = g & x \in \partial\Omega \end{cases}$$

(b) Exterior Dirichlet Problem.

$$\begin{cases} \Delta u = 0 & x \in \Omega^c \\ u = g & x \in \partial\Omega^c \end{cases}$$

Previously, we have used Green's representation, to show that if u is a C^2 solution of the Interior Dirichlet Problem, then u is given by

$$u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial n_y}dS(y),$$

where $G(x, y)$ is the Green's function for Ω. However, in general, it is difficult to calculate an explicit formula for the Green's function. Here, we use a different approach to look for solutions to the Interior/ Exterior Dirichlet Problem.

1.3.2 Double and Single Layer Potentials

It is known that

$$\Phi(x-y) = \begin{cases} \frac{1}{2\pi} ln|x-y| & n = 2 \\ \frac{-1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x-y|^{n-2}} & n \geq 3 \end{cases}$$

is the fundamental solution of Laplace equation $\Delta u = 0$ in $\mathbb{R}^n$, where $|x-y|$ is the distance between two points $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ in $\mathbb{R}^n$, $\alpha(n)$ is the volume of unit sphere in $\mathbb{R}^n$.
Let h be a continuous function on $\partial\Omega$. The **single layer potential** with moment h is defined as

$$V_\Delta(x) = -\int_{\partial\Omega} h(y)\Phi(x-y)dS(y). \tag{1.4}$$

The **double layer potential** with moment h is defined as

$$W_\Delta(x) = -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial n_y}(x-y)dS(y). \tag{1.5}$$

We plan to use these layer potentials to construct solutions of the Dirichlet problem. Notice that Green's function gives us a solution to the Interior Dirichlet Problem which is similar to a double layer potential. We will see that for an appropriate choice of h, we can write solutions of the Dirichlet problems $(a), (b)$ as double layer potentials.

Theorem 1.3.1. *For h a continuous function on $\partial\Omega$,*

1. *$V_\Delta(x)$ and $W_\Delta(x)$ are defined for all $x \in \mathbb{R}^n$.*

2. *$\Delta V_\Delta(x) = \Delta W_\Delta(x) = 0$ for all $x \notin \partial\Omega$.*

Proof.

We prove that $W_\Delta(x)$ is defined for all $x \in \mathbb{R}^n$. A similar proof works for $V_\Delta(x)$. First, suppose $x \notin \partial\Omega$. Therefore, $\frac{\partial\Phi}{\partial n_y}(x-y)$ is defined for all $y \in \partial\Omega$. Consequently, for all $x \notin \partial\Omega$; we have

$$|W_\Delta(x)| \leq |h(y)|_{L_\infty}(\partial\Omega) \int_{\partial\Omega} \left| \frac{\partial\Phi}{\partial n_y}(x-y) \right| dS_y \leq C$$

Next, consider the case when x is in $\partial\Omega$. In this case, the term $\frac{\partial\Phi}{\partial n_y}(x-y)$ in the integrand is undefined at $x = y$. We prove $W_\Delta(x)$ is defined at this point x by showing that the integral in (1.5) still converges. We need to look for a bound on $W_\Delta(x)$.

$$\Phi_{y_i}(x-y) = \frac{y_i - x_i}{n\alpha(n)|x-y|^n},$$

$$\begin{aligned} \frac{\partial\Phi}{\partial n_y}(x-y) &= \nabla_y \Phi(y-x).n(y) \\ &= \frac{(x-y).n(y)}{n\alpha(n)|x-y|^n}, \end{aligned}$$

where $n(y)$ is the unit normal to $\partial\Omega$ at y.
Claim: Fix $x \in \partial\Omega$. For all $y \in \partial\Omega$, there exists a constant $C > 0$ such that

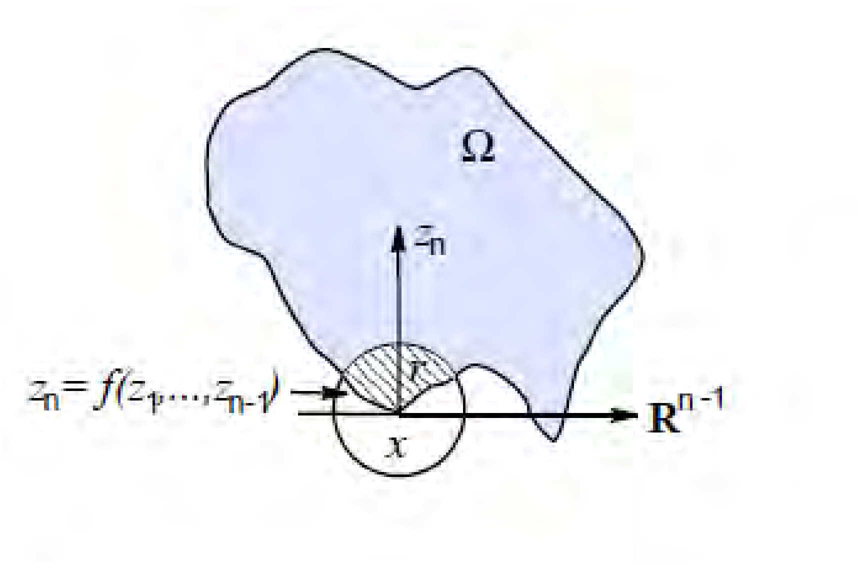
$$|(y-x).n(y)| \leq C|y-x|^2.$$

Proof of Claim. By assumption, $\partial\Omega$ is C^2. This means at each point $x \in \partial\Omega$ there exists an $r > 0$ and a C^2 function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that - upon relabelling and reorienting if necessary - we have

$$\Omega \cap B(x,r) = \{z \in B(x,r) | z_n > f(z_1, z_2, ..., z_{n-1})\}$$

Without loss of generality (by reorienting if necessary), we may assume $x = 0$ and $n(x) = (0,0,0,...,0,1)$. Using the fact that our boundary is C^2, we know there exists an $r > 0$ and a C^2 function $f : B(0,r) \subset \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\partial\Omega$ is given by the graph of the function f near x.
First, consider $x \in \partial\Omega$ such that $|y-x| \geq r$. In this case,

$$|(y-x).n(y)| \leq |y-x| \leq \frac{1}{r}|y-x|^2 = C(r)|y-x|^2.$$



Second, consider $y \in \partial\Omega$ such that $|x-y| \leq r$. In this case, we use the fact that

$$\begin{aligned}|(x-y).n(y)| = |(x-y).(n(x)+n(y)-n(x))| &\leq \frac{1}{r}|x-y|^2 \\ &\leq |(x-y).n(x)| + |(y-x)(n(y)-n(x))| \\ &= |y_n| + |(y-x).(n(y)-n(x))|\end{aligned}$$

Now,

$$y_n = f(y_1, y_2, ..., y_{n-1})$$

where $f \in C^2, f(0) = 0$ and $\nabla f(0) = 0$ (*see.* [10] *page* 7). Therefore, by Taylor Theorem, we have

$$\begin{aligned}|y_n| &= |f(y_1, y_2, ..., y_{n-1})| \\ &\leq C|(y_1, y_2, ..., y_{n-1})|^2 \\ &\leq C|y|^2 \\ &= C|x-y|^2\end{aligned}$$

where the constant C depends only on the bound on the second partial derivatives of $f(y_1, y_2, ..., y_{n-1})$ for $|(y_1, y_2, ..., y_{n-1})| \leq r$, but this is bounded because by assumption $f \in C^2(B(0,r))$.
Next we look at $|(x-y).(n(y)-n(x))|$. By assumption, $\partial\Omega$ is C^2 and consequently, n is a C^1 function and therefore, there exists a constant $C > 0$ such that

$$|n(y) - n(x)| \leq C|x-y|.$$

Therefore,

$$|(x-y).(n(y)-n(x))| \leq C|x-y|^2.$$

Consequently, our claim is proven. We remark that the constant C will depend on

r, but once x is chosen r is fixed.
Therefore, we conclude that for all $x \in \partial\Omega$, all $y \in \partial\Omega$

$$\begin{aligned}\left|\frac{\partial \Phi}{\partial n_y}(x-y)\right| &= \left|\frac{(x-y).n_y}{n\alpha(n)|x-y|^n}\right| \\ &\leq C\frac{|x-y|^2}{|x-y|^n} \\ &= \frac{C}{|x-y|^{n-2}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left|-\int_{\partial\Omega} h(y)\frac{\partial \Phi}{\partial n_y}(x-y)dS(y)\right| &\leq |h(y)|_{L^\infty(\partial\Omega)}\int_{\partial\Omega}\left|\frac{\partial \Phi}{\partial n_y}(x-y)\right|dS(y) \\ &\leq C'\int_{\partial\Omega}\frac{1}{|x-y|^{n-2}}dS(y) \leq C\end{aligned}$$

using the fact that $\partial\Omega$ is of dimension $n-1$. Therefore, we conclude that u is defined for all $x \in \partial\Omega$ and consequently for all $x \in \mathbb{R}^n$ as claimed.
Next, we will prove $W_\Delta(x)$ is harmonic for all $x \in \Omega$. Similar proof work for $V_\Delta(x)$. Fix $x \in \Omega$. $\frac{\partial \Phi}{\partial n_y}(x-y)$ is smooth function for all $y \in \partial\Omega$ and $\Phi(x-y)$ is harmonic for all $x \neq y$ implies that $\Delta_x\frac{\partial \Phi}{\partial n_y}(x-y) = 0$ for all $y \in \partial\Omega$. Therefore, using the fact that our integral is finite and $\frac{\partial \Phi}{\partial n_y}(x-y)$ is smooth, we conclude that

$$\begin{aligned}\Delta_x W_\Delta(x) &= -\Delta_x\int_{\partial\Omega} h(y)\frac{\partial \Phi}{\partial n_y}(x-y)dS(y) \\ &= -\int_{\partial\Omega} h(y)\Delta_x\frac{\partial \Phi}{\partial n_y}(x-y)dS(y) \\ &= 0\end{aligned}$$

□

For a moment, consider the interior Dirichlet problem (a). As proven above, for h a continuous function on $\partial\Omega$, W_Δ defined in (1.5) is harmonic. Now, if we can choose h appropriately, such that for all $x_0 \in \partial\Omega$,

$$\lim_{x\in\Omega\to\ x_0} W_\Delta(x) = g(x_0),$$

then we will have found a solution of the interior Dirichlet problem. Consequently, we are interested in studying the limits of W_Δ as we approach the boundary of Ω. In order to study this, we must first prove the following lemma.

Lemma 1.3.1. *(Gauss's Lemma) Consider the double layer potential,*

$$W_\Delta(x) = -\int_{\partial\Omega}\frac{\partial \Phi}{\partial n_y}(x-y)dS(y).$$

Then,

$$W_\Delta(x) = \begin{cases} 0 & x \in \Omega^c \\ 1 & x \in \Omega \\ 1/2 & x \in \partial\Omega. \end{cases}$$

1. *Proof.* First, for $x \in \Omega^c$,

$$\begin{aligned} W_\Delta(x) &= -\int_{\partial\Omega} \frac{\partial\Phi}{\partial n_y}(x-y)dS(y) \\ &= -\int_{\Omega} \Delta_y\Phi(x-y)dS(y) \\ &= 0 \end{aligned}$$

using the Divergence Theorem and the fact that $\Phi(x-y)$ is smooth for $y \in \Omega$, $x \in \Omega^c$.

Now, for $x \in \Omega$, $\Phi(x-y)$ is not smooth for all $y \in \Omega$. In order to overcome this problem, we fix $\epsilon > 0$ sufficiently small such that $B(x,\epsilon)$ is contained within Ω. Then on the region $\Omega - B(x,\epsilon)$, $\Phi(x-y)$ is smooth, and, consequently, we can say

$$\begin{aligned} 0 &= \int_{\Omega - B(x,\epsilon)} \Delta_y\Phi(x-y)dS(y) \\ &= \int_{\partial(\Omega - B(x,\epsilon))} \frac{\partial\Phi}{\partial n_y}(x-y)dS(y) \\ &= \int_{\partial\Omega} \frac{\partial\Phi}{\partial n_y}(x-y)dS(y) + \int_{\partial B(x,\epsilon)} \frac{\partial\Phi}{\partial n_y}(x-y)dS(y) \end{aligned}$$

where n is outer unit normal to $\Omega - B(x,\epsilon)$.
As mentioned above,

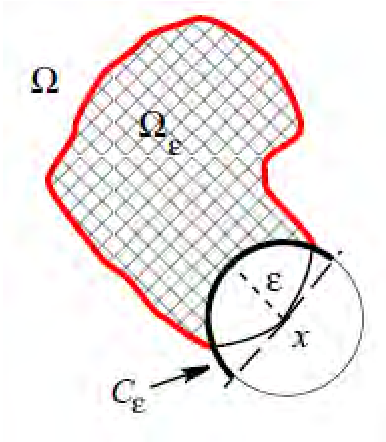
$$\Phi_{y_i}(x-y) = \frac{y_i - x_i}{n\alpha(n)|y-x|^n}.$$

For $y \in \partial B(x,\epsilon)$, the outer unit normal to $\Omega - B(x,\epsilon)$ is given by

$$n(y) = \frac{y-x}{|y-x|}.$$

Therefore, for $y \in \partial B(x,\epsilon)$,

$$\begin{aligned} \frac{\partial\Phi}{\partial n_y}(x-y) &= \nabla_y\Phi(y-x).n(y) \\ &= \frac{y-x}{n\alpha(n)|y-x|^n}.\frac{y-x}{|y-x|} \\ &= \frac{|y-x|^2}{n\alpha(n)|y-x|^{n+1}} \\ &= \frac{1}{n\alpha(n)|y-x|^{n-1}} \end{aligned}$$



Therefore,

$$\int_{\partial B(x,\epsilon)} \frac{\partial \phi}{\partial n_y}(x-y)dS(y) = \int_{\partial B(x,\epsilon)} \frac{1}{n\alpha(n)|y-x|^{n-1}} dS(y)$$
$$= \frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B(x,\epsilon)} dS(y)$$
$$= 1$$

Therefore, we conclude that

$$0 = \int_{\partial\Omega} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) + \int_{\partial B(x,\epsilon)} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y)$$
$$= \int_{\partial\Omega} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) + 1$$

which implies

$$-\int_{\partial\Omega} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) = 1.$$

Last, we consider the case $x \in \partial\Omega$. In this case, $\frac{\partial\Phi}{\partial n_y}(x-y)$ is not defined at y=x. Fix $x \in \partial\Omega$. Let $B(x,\epsilon)$ be the ball of radius ϵ about x. Let

$$\Omega_\epsilon \equiv \Omega - \Omega \cap B(x,\epsilon), \quad C_\epsilon \equiv \{y \in \partial B(x,\epsilon) : n(x).y < 0\}, \quad \tilde{C}_\epsilon \equiv \partial\Omega_\epsilon \cap C_\epsilon.$$

First, we note that

$$0 = \int_{\Omega_\epsilon} \Delta_y \Phi(x-y)dy \tag{1.6}$$
$$= \int_{\partial\Omega_\epsilon} \frac{\partial \phi}{\partial n_y}(x-y)dS(y)$$
$$= \int_{\partial\Omega-\tilde{C}_\epsilon} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) + \int_{\tilde{C}_\epsilon} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y)$$

where n_y is the outer unit normal to Ω_ϵ.
Now, first we recall that

$$\nabla_y \Phi(x-y) = \frac{x-y}{n\alpha(n)|x-y|^n}$$

For all $y \in \tilde{C}_\epsilon$, the outer unit normal is given by

$$n(y) = \frac{y-x}{|y-x|}.$$

Therefore,

$$\begin{aligned}\int_{\tilde{C}_\epsilon} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) &= \int_{\tilde{C}_\epsilon} \frac{1}{n\alpha(n)|y-x|^{n-1}} dS(y) \\ &= \frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\tilde{C}_\epsilon} dS(y)\end{aligned}$$

Next, we use the fact that

$$\int_{\tilde{C}_\epsilon} dS(y) \approx \int_{C_\epsilon} dS(y).$$

In fact, as we will show below,

$$\int_{\tilde{C}_\epsilon} dS(y) = \int_{C_\epsilon} dS(y) + O(\epsilon^n). \tag{1.7}$$

We omit the proof of (1.7) for now and will return to it below. Assuming this fact for now, we have

$$\int_{\tilde{C}_\epsilon} dS(y) = \frac{1}{2} n\alpha(n)\epsilon^{n-1} + O(\epsilon^n)$$

which implies

$$\begin{aligned}\int_{\tilde{C}_\epsilon} \frac{\partial \Phi}{\partial \nu_y}(x-y)dS(y) &= \frac{1}{n\alpha(n)\epsilon^{n-1}}[\frac{1}{2} n\alpha(n)\epsilon^{n-1} + O(\epsilon^n)] \\ &= \frac{1}{2} + \frac{1}{n\alpha(n)} O(\epsilon)\end{aligned} \tag{1.8}$$

Combing (1.6) and (1.8), we have

$$0 = \int_{\partial\Omega - \tilde{C}_\epsilon} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) + \frac{1}{2} + \frac{1}{n\alpha(n)} O(\epsilon)$$

which implies

$$\int_{\partial\Omega - \tilde{C}_\epsilon} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) = -\frac{1}{2} - \frac{1}{n\alpha(n)} O(\epsilon).$$

Taking the limit as $\epsilon \to 0^+$, we have

$$-\int_{\partial\Omega} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) = \frac{1}{2}.$$

Now we will prove (1.7).

Claim: For $\tilde{C}_\epsilon$ and C_ϵ as defined above, we have

$$\int_{\tilde{C}_\epsilon} dS(y) = \int_{C_\epsilon} dS(y) + O(\epsilon^n).$$

Proof. We just need to show that the surface area of $C_\epsilon - \tilde{C}'_\epsilon$ is $O(\epsilon^n)$. The surface area is approximately the surfaces area of the base times height. Now the surface area of the base is $O(\epsilon^{n-2})$. Therefore, we just need to show that the height is $O(\epsilon^2)$.
Without loss of generality, we let $x = 0$. Now, by assumption, $\partial\Omega$ is C^2. Therefore, $\partial\Omega$ can be written as the graph of a C^2 function $f : R^{n-1} \to R$ such that $f(0) = 0$ and $\nabla f(0) = 0$. Therefore, if $y \in C_\epsilon - \tilde{C}_\epsilon$, then

$$|y_n| \leq |f(y_1, y_2, ..., y_{n-1}) \leq C|(y_1, y_2, ..., y_{n-1})|^2 \leq C|y|^2 \leq C\epsilon^2,$$

using Taylor's theorem. Therefore, the height is $O(\epsilon^2)$ and the claim follows. □

Theorem 1.3.2. *Let h be a continuous function on $\partial\Omega$ and $x_0 \in \partial\Omega$. Then*

$$\lim_{x\in\Omega\to x_0} V_\Delta(x) = V_\Delta(x_0), \tag{1.9}$$

$$\lim_{x\in\Omega\to x_0} \frac{\partial V_\Delta(x)}{\partial n_x} = \frac{1}{2}h(x_0) + \frac{\partial V_\Delta(x_0)}{\partial n_x}, \quad \lim_{x\in\Omega^c\to x_0} \frac{\partial V_\Delta(x)}{\partial n_x} = -\frac{1}{2}h(x_0) + \frac{\partial V_\Delta(x_0)}{\partial n_x}, \tag{1.10}$$

$$\lim_{x\in\Omega\to x_0} W_\Delta(x) = \frac{1}{2}h(x_0) + W_\Delta(x_0), \quad \lim_{x\in\Omega^c\to x_0} W_\Delta(x) = -\frac{1}{2}h(x_0) + W_\Delta(x_0). \tag{1.11}$$

Proof. 1. Proof of equation (1.9).
Let $x \in \Omega,\ x_0 \in \partial\Omega$. We have

$$V_\Delta(x) = -\int_{\partial\Omega} \Phi(x,y)h(y)dS(y) \quad \textit{and}$$
$$V_\Delta(x_0) = -\int_{\partial\Omega} \Phi(x_0,y)h(y)dS(y)$$

We need to show that

$$\lim_{x\in\Omega\to x_0} V_\Delta(x) = V_\Delta(x_0).$$

That is for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|V_\Delta(x) - V_\Delta(x_0)| < \epsilon$ for $|x - x_0| < \delta$. Now,

$$V_\Delta(x) - V_\Delta(x_0) = -\int_{\partial\Omega} h(y)\left[\Phi(x,y) - \Phi(x_0,y)\right] dS(y)$$

By assumption, h is continuous, and as we know $\Phi(x,y)$ is smooth for $y \neq x$. Therefore, to get a bound on $|V_\Delta(x) - V_\Delta(x_0)|$, we divide $\partial\Omega$ into two pieces:
(1) $B(x_0,\gamma) \cap \partial\Omega$

(2) $\partial\Omega - B(x_0,\gamma)\cap\partial\Omega$. We look at these two pieces below. First for (1),

$$|V_\Delta(x) - V_\Delta(x_0)| \le |h(y)|_{L^\infty}(B(x_0,\gamma)\cap\partial\Omega)\int_{B(x_0,\gamma)\cap\partial\Omega}|\Phi(x,y)-\Phi(x_0,y)|dS(y)$$

By assumption, h is continuous. Therefore, for all $\tilde{\epsilon}>0$ there exists a $\gamma>0$ such that $|h(y)|<\tilde{\epsilon}$ for $y<\gamma$. In addition that

$$\int_{B(x_0,\gamma)\cap\partial\Omega}|\Phi(x,y)-\Phi(x_0,y)|dS(y)\le C$$

using the fact that V_Δ is defined for all $x\in\mathbb{R}$. Therefore, we conclude that for any $\tilde{\epsilon}>0$, $|(1)|\le C_1\tilde{\epsilon}$ for γ chosen appropriately small.
Next, for (2), we use the fact that $\Phi(x,y)$ is continuous in x for x away from y. Consequently, we have

$$|V_\Delta(x)-V_\Delta(x_0)|\le$$
$$|h(y)|_{L^\infty}(\partial\Omega-B(x_0,\gamma)\cap\partial\Omega)|\Phi(x,y)-\Phi(x_0,y)|L^\infty(\partial\Omega-B(x_0,\gamma)\cap\partial\Omega)\left|\int_{\partial\Omega-(B(x_0,\gamma)\cap\partial\Omega)}dS(y)\right|$$

Now, first h is bounded on $\partial\Omega$. Therefore, $|h(y)|\le C$. Next, $\left|\int dS(y)\right|\le C$. Lastly, using the fact that $\Phi(x-y)$ is continuous in x uniformly for y, we conclude that there exists a δ such that

$$|\Phi(x,y)-\Phi(x_0,y)|L^\infty(\partial\Omega-B(x_0,\gamma)\cap\partial\Omega)\le\tilde{\epsilon},$$

for $|x-x_0|<\delta$. Therefore, $|(2)|\le C_2\tilde{\epsilon}$ if $|x-x_0|<\gamma$ where δ is chosen appropriately small. Consequently, for $\epsilon>0$ choose $\tilde{\epsilon}>0$ such that $C_1\tilde{\epsilon}+C_2\tilde{\epsilon}<\epsilon$. Then choosing $\gamma>0$ sufficiently small such that $|(1)|\le C_1\tilde{\epsilon}$ and $\delta>0$ sufficiently small such that $|(2)|\le C_2\tilde{\epsilon}$ when $|x-x_0|<\delta$, we conclude that

$$|V_\Delta(x)-V_\Delta(x_0)|\le C_1\tilde{\epsilon}+C_2\tilde{\epsilon}<\epsilon \quad for \quad |x-x_0|<\delta.$$

2. Proof of equation (1.10) is similar to the following Proof of equation (1.11).
We will prove only the first case, when $x\in\Omega$. The second case works similarly. Let $x\in\Omega$, $x_0\in\partial\Omega$. We have

$$\begin{aligned}W_\Delta(x) &= -\int_{\partial\Omega}h(y)\frac{\partial\Phi}{\partial n_y}(x-y)dS(y)\\ &= -\int_{\partial\Omega}h(y)\frac{\partial\Phi}{\partial n_y}(x-y)dS(y)+h(x_0)\int_{\partial\Omega}\frac{\partial\Phi}{\partial n_y}(x-y)dS(y)-h(x_0)\int_{\partial\Omega}\frac{\partial\Phi}{\partial n_y}(x-y)dS(y)\\ &= -\int_{\partial\Omega}[h(y)-h(x_0)]\frac{\partial\Phi}{\partial n_y}(x-y)dS(y)+h(x_0)\\ &\equiv I(x)+h(x_0)\end{aligned}$$

using the fact that

$$-\int_{\partial\Omega} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) = 1$$

for $x \in \Omega$, proven in Gauss' lemma. Similarly,

$$\begin{aligned}
W_\Delta(x_0) &= -\int_{\partial\Omega} h(y)\frac{\partial \Phi}{\partial n_y}(x_0-y)dS(y) \\
&= -\int_{\partial\Omega} [h(y)-h(x_0)]\frac{\partial \Phi}{\partial n_y}(x_0-y)dS(y) - h(x_0)\int_{\partial\Omega} \frac{\partial \Phi}{\partial n_y}(x_0-y)dS(y) \\
&= -\int_{\partial\Omega} [h(y)-h(x_0)]\frac{\partial \Phi}{\partial n_y}(x_0-y)dS(y) + \frac{1}{2}h(x_0) \\
&\equiv I(x_0) + \frac{1}{2}h(x_0)
\end{aligned}$$

again using Gauss' lemma. Therefore,

$$W_\Delta(x) - W_\Delta(x_0) = I(x) + h(x_0) - I(x_0) - \frac{1}{2}h(x_0),$$

which implies

$$W_\Delta(x) = I(x) - I(x_0) + \frac{1}{2}h(x_0) + W_\Delta(x_0).$$

Therefore, to prove our theorem, we need only show that

$$\lim_{x\in\Omega\to x_0}[I(x) - I(x_0)] = 0,$$

where

$$I(x) \equiv -\int_{\partial\Omega} [h(y)-h(x_0)]\frac{\partial \Phi}{\partial n_y}(x-y)dS(y).$$

Now,

$$I(x) - I(x_0) = -\int_{\partial\Omega} [h(y)-h(x_0)]\left[\frac{\partial \Phi}{\partial n_y}(x-y) - \frac{\partial \Phi}{\partial n_y}(x_0-y)\right]dS(y).$$

We need to show that for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|I(x) - I(x0)| < \epsilon$ for $|x - x_0| < \delta$.

By assumption, h is continuous, and as we know $\Phi(x-y)$ is smooth for $y \neq x$. Therefore, to get a bound on $|I(x) - I(x_0)|$, we divide $\partial\Omega$ into two pieces:

(1) $B(x_0,\gamma) \cap \partial\Omega$

(2) $\partial\Omega - B(x_0,\gamma) \cap \partial\Omega$. We look at these two pieces below. First for (1),

$$\begin{aligned}
&\left|-\int_{\partial\Omega} [h(y)-h(x_0)]\left[\frac{\partial \Phi}{\partial n_y}(x-y) - \frac{\partial \Phi}{\partial n_y}(x_0-y)\right]dS(y)\right| \\
&\leq |h(y)-h(x_0)|_{L^\infty(B(x_0,\gamma)\cap\partial\Omega)}\int_{B(x_0,\gamma)\cap\partial\Omega}\left|\frac{\partial \Phi}{\partial n_y}(x-y) - \frac{\partial \Phi}{\partial n_y}(x_0-y)\right|dS(y)
\end{aligned}$$

By assumption, h is continuous. Therefore, for all $\tilde{\epsilon} > 0$ there exists a $\gamma > 0$ such that if $|h(y)-h(x_0)|<\tilde{\epsilon}$ if $|y-x_0|<\gamma$. In addition,

$$\int_{B(x_0,\gamma)\cap\partial\Omega}\left|\frac{\partial\Phi}{\partial n_y}(x-y)-\frac{\partial\Phi}{\partial n_y}(x_0-y)\right|dS(y)\leq C$$

using the fact that $W_\Delta(x)$ is defined for all $x\in\mathbb{R}$. Therefore, we conclude that for any $\tilde{\epsilon}>0$,

$$|(1)|\leq C1\tilde{\epsilon}$$

for γ chosen appropriately small. Next, for (2), we use the fact that $\frac{\partial\Phi}{\partial n_y}(x-y)$ is continuous in x for x away from y. Consequently, we have

$$\begin{aligned}&\left|-\int_{\partial\Omega-B(x_0,\gamma)\cap\partial\Omega}[h(y)-h(x_0)][\frac{\partial\Phi}{\partial n_y}(x-y)-\frac{\partial\Phi}{\partial n_y}(x_0-y)]dS(y)\right|\\ &\leq|h(y)-h(x_0)|_{L^\infty}\left|\frac{\partial\Phi}{\partial n_y}(x-y)-\frac{\partial\Phi}{\partial n_y}(x_0-y)\right|_{L^\infty(\partial\Omega-B(x_0,\gamma)\cap\partial\Omega)}\left|\int dS(y)\right|.\end{aligned}$$

Now, first h is bounded on $\partial\Omega$. Therefore, $|h(y)-h(x_0)|\leq C$. Next, $|\int dS(y)|\leq C$. Lastly, using the fact that $\frac{\partial\Phi}{\partial n_y}(x-y)$ is continuous in x uniformly for y, we conclude that there exists a $\delta>0$ such that

$$\left|\frac{\partial\Phi}{\partial n_y}(x-y)-\frac{\partial\Phi}{\partial n_y}(x_0-y)\right|_{L^\infty(\partial\Omega-B(x_0,\gamma)\cap\partial\Omega)}\leq\tilde{\epsilon},$$

for $|x-x_0|<\delta$. Therefore,

$$|(2)|\leq C_2\tilde{\epsilon}$$

if $|x-x_0|<\delta$ where δ is chosen appropriately small. Consequently, for $\epsilon>0$ choose $\tilde{\epsilon}>0$ such that

$$C_1\tilde{\epsilon}+C_2\tilde{\epsilon}<\epsilon$$

Then choosing $\gamma>0$ sufficiently small such that

$$|(1)|\leq C_1\tilde{\epsilon}$$

and $\delta>0$ sufficiently small such that

$$|(2)|\leq C_2\tilde{\epsilon}$$

when $|x-x_0|<\delta$, we conclude that

$$|I(x)-I(x_0)|\leq C_1\tilde{\epsilon}+C_2\tilde{\epsilon}\leq\epsilon,$$

for $|x-x_0|<\delta$, implies that

$$|I(x)-I(x_0)|\leq\epsilon$$

Therefore, we have shown that

$$\lim_{x \in \Omega \to x_0} [I(x) - I(x_0)] = 0.$$

Consequently,

$$\begin{aligned}\lim_{x \in \Omega \to x_0} W_\Delta(x) &= \lim_{x \in \Omega \to x_0} \left([I(x) - I(x_0)] + \frac{1}{2} h(x_0) + W_\Delta(x_0) \right) \\ &= \frac{1}{2} h(x_0) + W_\Delta(x_0)\end{aligned} \tag{1.12}$$

as claimed.

□

1.3.3 Solution of Laplace's Equation

We begin by considering the interior Dirichlet problem,

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ u = g & x \in \partial\Omega \end{cases}$$

For a given function h, define the double-layer potential $W_\Delta(x)$ associated with h as

$$W_\Delta(x) = -\int_{\partial\Omega} h(y) \frac{\partial \Phi}{\partial n_y}(x - y) dS(y).$$

In the previous section, we proved that $W_\Delta(x)$ is a harmonic function in Ω. In addition, we proved that for $x_0 \in \partial\Omega$,

$$\lim_{x \in \Omega \to x_0} W_\Delta(x) = \frac{1}{2} h(x_0) + W_\Delta(x_0).$$

Therefore, if we can find a continuous function h such that for all $x_0 \in \partial\Omega$,

$$g(x_0) = \frac{1}{2} h(x_0) - \int_{\partial\Omega} h(y) \frac{\partial \Phi}{\partial n_y}(x_0 - y) dS(y)$$

and we define

$$W_\Delta(x) = -\int_{\partial\Omega} h(y) \frac{\partial \Phi}{\partial n_y}(x - y) dS(y),$$

for that choice of h, then $W_\Delta(x)$ will give us a solution of our interior Dirichlet problem. Next, consider the exterior Dirichlet problem,

$$\begin{cases} \Delta u = 0 & x \in \Omega^c \\ u = g & x \in \partial\Omega^c \end{cases}.$$

As proven in the previous section, for any continuous function h,

$$W_\Delta(x) = -\int_{\partial\Omega} h(y) \frac{\partial \Phi}{\partial n_y}(x - y) dS(y),$$

is harmonic in Ω^c and satisfies

$$\lim_{x\in\Omega^c\to x_0} W_\Delta(x) = -\frac{1}{2}h(x_0) + W_\Delta(x)(x_0).$$

Therefore, if we can find a continuous function h such that for all $x_0 \in \partial\Omega^c$,

$$g(x_0) = -\frac{1}{2}h(x_0) - \int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial n_y}(x_0 - y)dS(y),$$

then defining

$$W_\Delta(x) \equiv -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial n_y}(x - y)dS(y),$$

for that choice of h, $W_\Delta(x)$ will give us a solution of our exterior Dirichlet problem.

Chapter 2

Analysis of Boundary-Domain Integral Equations for Variable-Coefficient (The case of Dirichlet BVP in 2D)

2.1 Formulation of Boundary Value Problem

Let Ω be a bounded open domain of $\mathbb{R}^2$. For simplicity, we assume that the boundary $\partial\Omega$ is a simply connected, closed, infinitely smooth curve. Let us denote $\partial x_j = \frac{\partial}{\partial x_j}(j = 1, 2), \partial x = (\partial x_1, \partial x_2)$.

For a linear operator L_* (either L_a or Δ), we introduce the subspace of $H^1(\Omega)$ $\Big(H^1(\Omega) := \{u \in L_2(\Omega) : \frac{\partial}{\partial x_i} u(x) \in L_2(\Omega)\}\Big)$

$$H^{1,0}(\Omega; L_*) := \{g : g \in H^1(\Omega), L_* g \in L_2(\Omega)\},$$

endowed with the norm

$$||g||^2_{H^{1,0}(\Omega;L_*)} := ||g||^2_{H^1(\Omega)} + ||L_* g||^2_{L_2(\Omega)}.$$

Definition 2.1.1. *For any $u \in C^\infty(\Omega)$, define the trace operator $\gamma^+|_{\partial\Omega}$ by $\gamma^+|_{\partial\Omega} u(x) = u(x),\ x \in \partial\Omega$.*

Theorem 2.1.1. ***(Trace Theorem)*** *If $s > \frac{1}{2}$, then γ^+ has a unique extension to a bounded linear operator*

$$\gamma^+ : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$$

and we have

$$||\gamma^+ u||_{H^{s-\frac{1}{2}}(\partial\Omega)} \leq C||u||_{H^s(\Omega)},$$

$$H^{s-\frac{1}{2}}(\partial\Omega) = \{\gamma^+ u : u \in H^s(\Omega)\}.$$

Note that the trace operator γ^+ has a bounded right inverse

$$\gamma^+_{-1} : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega),$$

i.e. $\gamma^+\gamma^+_{-1}w = w, \; w \in H^{s-\frac{1}{2}}(\partial\Omega)$ and

$$||\gamma^+_{-1}w||_{H^s(\Omega)} \leq C||w||_{H^{s-\frac{1}{2}}(\partial\Omega)}.$$

From the trace theorem for $u \in H^1(\Omega)$, it follows that $u^+ := \gamma^+_{\partial\Omega}u \in H^{\frac{1}{2}}(\partial\Omega)$, where $\gamma^+_{\partial\Omega}$ is the trace operator on $\partial\Omega$ from Ω.
We consider the following PDE with a scalar variable coefficient $a \in C^\infty(\mathbb{R}^2), \; a(x) > 0$,

$$\begin{aligned} L_a u(x) &:= L_a(x, \partial_x)u(x) \\ &:= \sum_{i=1}^{2} \frac{\partial}{\partial_{x_i}} \left(a(x) \frac{\partial u(x)}{\partial x_i} \right) \\ &= f(x) \; , \quad x \in \Omega \end{aligned} \tag{2.1}$$

where u is an unknown function and f is a given function in Ω.

Definition 2.1.2. *For $u \in H^2(\Omega)$, the co-normal derivative is defined as*

$$\begin{aligned} T^+(x, n^+(x), \partial_x)u(x) &:= \sum_{i=1}^{2} a(x) n_i^+(x) \left[\frac{\partial u(x)}{\partial x_i} \right]^+ \\ &= a(x) \left[\frac{\partial u(x)}{\partial n^+(x)} \right]^+ \end{aligned}$$

where $n^+(x)$ is the exterior (to Ω) unit normal vectors at the point $x \in \partial\Omega$.
In general the boundary differential operator T^+ is continuous mapping from $H^s(\Omega)$ *to* $H^{s-\frac{3}{2}}(\partial\Omega)$. That is

$$T^+ : H^s(\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega) \; , \quad s > \frac{3}{2}.$$

For $u \in H^1(\Omega)$ the co-normal derivative operators on $\partial\Omega$ do not generally exist in the trace sense. However if $u \in H^{1,0}(\Omega; \Delta)$ one can correctly define the generalized (canonical) co-normal derivative $T^+u \in H^{-1/2}(\partial\Omega)$ with the help of the first Green's identity as

$$\left\langle T^+u, w \right\rangle_{\partial\Omega} := \int_\Omega \left((\gamma^+_{-1}w) L_a u(x) + E_a(u, \gamma^+_{-1}w) \right) dx \; for \; all \; w \in H^{1/2}(\partial\Omega),$$

where $\gamma^+_{-1} : H^{-1/2}(\partial\Omega) \to H^1(\Omega)$ is a right inverse to the trace operator γ^+ and $\langle .,. \rangle_{\partial\Omega}$ denotes the duality brackets between the spaces $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. Then for $u \in H^{1,0}(\Omega; \Delta)$, $v \in H^1(\Omega)$ the first Green identity holds:

$$\left\langle T^+u, \gamma^+v \right\rangle_{\partial\Omega} := \int_\Omega (v(x) L_a u(x) + E_a(u, v)) dx \tag{2.2}$$

where

$$E_a(u,v) := \sum_{i=1}^{2} a(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial_{xi}} := a(x)\nabla u(x)\nabla v(x) \tag{2.3}$$

We will investigate the following Dirichlet boundary value problem.
Find a function $u \in H^1(\Omega)$ satisfying the conditions

$$L_a u = f \quad in \quad \Omega, \tag{2.4}$$

$$u^+ = \varphi_0 \quad on \quad \partial\Omega, \tag{2.5}$$

where $\varphi_0 \in H^{1/2}(\partial\Omega)$ and $f \in L_2(\Omega)$. Equation (2.4) is understood in the distributional sense and condition (2.5) in the trace sense.

Recall that the operator $A : H \to H'$ induces a bilinear form $\mathcal{E}(u,v) := \langle Au, v\rangle$ for all $u,\ v\ \in H$ with the mapping property $\mathcal{E}(.,.) : H \times H \to \mathbb{R}$.

Theorem 2.1.2. ***(Lax-Milgram Theorem)*** *Let H be a Hilbert space and $\mathcal{E} : H \times H \to \mathbb{R}$ a bilinear form satisfying the following conditions.*

1. $|\mathcal{E}(u,v)| \leq C_1||u||_H||v||_H$ *for all* $u,\ v \in H,\ C_1 > 0$
2. $\mathcal{E}(u,u) \geq C_2||u||_H^2$ *for all* $u \in H,\ C_2 > 0.$

Theorem 2.1.3. ***(Uniqueness Theorem)***
BVP $(2.4) - (2.5)$ *with* $\varphi_0 \in H^{1/2}(\partial\Omega)$ *and* $f \in L_2(\Omega)$ *has at most one solution in* $H^1(\Omega)$*.*

Proof. Applying the first Green identity (2.2) and (2.3) with $v = u$ as a solution of the homogeneous Dirichlet problem, i.e., with $f = 0,\ \varphi_0 = 0.$

$$\langle T^+u, \gamma^+u\rangle_{\partial\Omega} = \int_\Omega (u(x)L_a u(x) + E_a(u,u))dx$$
$$\Rightarrow 0 = \int_\Omega E_a(u,u)dx = \mathcal{E}(u,u).$$

By Lax-Milgram theorem, the bilinear form $\mathcal{E}(u,u) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is bounded and is $H^1(\Omega)$-elliptic. Hence

$$0 = \mathcal{E}(u,u) \geq c||u||^2_{H^1(\Omega)}$$

is true if and only if $u = 0$ in $H^1(\Omega)$. Therefore, BVP $(2.4) - (2.5)$ with $\varphi_0 \in H^{1/2}(\partial\Omega)$ and $f \in L_2(\Omega)$ has at most one solution in $H^1(\Omega)$. □

2.2 Parametrix and Potential-type Operators

The function

$$P_a(x,y) = \frac{1}{2\pi a(y)} \ln|x-y|, \quad x,y \in \mathbb{R}^2 \tag{2.6}$$

is the parametrix (Levi -function) for the operator $L(x, \partial_x)$ if

$$L_a(x,\partial_x)P_a(x,y) = \delta(x-y) + R_a(x,y), \quad x,y \in \mathbb{R}^2 \tag{2.7}$$

where δ is Dirac distribution, while $R_a(x, y)$ is a remainder possessing at most a weak singularity at $x = y$. We can show that the corresponding remainder is given by

$$R_a(x, y) = \sum_{i=1}^{2} \frac{x_i - y_i}{2\pi a(y)|x - y|^2} \frac{\partial a(x)}{\partial x_i}, \quad x, y \in \mathbb{R}^2. \tag{2.8}$$

Indeed,

$$\frac{\partial P_a(x, y)}{\partial x_i} = \frac{x_i - y_i}{2\pi a(y)|x - y|^2}, \; i = 1, 2$$

and

$$\frac{\partial}{\partial x_i}\left[a(x)\frac{\partial P_a(x, y)}{\partial x_i}\right] = \frac{a(x)}{a(y)}\frac{\partial^2}{\partial {x_i}^2}\left[\frac{\ln|x - y|}{2\pi}\right] + \frac{x_i - y_i}{2\pi a(y)|x - y|^2}\frac{\partial a(x)}{\partial x_i}$$

So,

$$\begin{aligned} L_a(x, \partial_x)P_a(x, y) &= \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial P_a(x, y)}{\partial x_i}\right] \\ &= \frac{a(x)}{a(y)}\sum_{i=1}^{2} \frac{\partial^2}{\partial {x_i}^2}\left[\frac{\ln|x - y|}{2\pi}\right] + \sum_{i=1}^{2} \frac{x_i - y_i}{2\pi a(y)|x - y|^2}\frac{\partial a(x)}{\partial x_i} \\ &= \frac{a(x)}{a(y)}\delta(x - y) + R_a(x, y) \\ &= \delta(x - y) + R_a(x, y) \end{aligned}$$

Here, we use $\sum_{i=1}^{2} \frac{\partial^2}{\partial {x_i}^2}\left[\frac{\ln|x - y|}{2\pi}\right] = \Delta_x\left[\frac{\ln|x - y|}{2\pi}\right] = \delta(x - y)$ that followed from the definition of the fundamental solution of the Laplace operator. Also, the multiplication of a smooth function $a(x)$ by δ, gives $a(x)\delta(x - y) = a(y)\delta(x - y)$. Observe that, $L_a(x, \partial_x)P_a(x, y) = \delta(x-y)+\nabla a(x).\nabla P_a(x, y)$ and we can write the remainder as $R_a(x, y) = \nabla a(x).\nabla P_a(x, y)$. And for constant coefficient $a(x) = 1$, then becomes the Laplace operator, Δ, and the parametrix $P_a(x, y)$ becomes its fundamental solution, and $R_a(x, y) \equiv 0$. Evidently, the parametrix $P_a(x, y)$ given by (2.6) is a fundamental solution to the operator $L_a(y, \partial_x) := a(y)\Delta(\partial_x)$ with Frozen coefficient $a(x) = a(y)$, that is,

$$L_a(y, \partial_x)P_a(x, y) = \delta(x - y).$$

Proof. Show that $P_a(x,y)$ is fundamental solution to the operator $L_a(y,\partial_x) := a(y)\Delta(\partial_x)$

$$\begin{aligned}
L_a(y,\partial_x)P_a(x,y) &= a(y)\Delta(\partial_x)P_a(x,y)\\
&= a(y)\sum_{i=1}^{2}\frac{\partial^2}{\partial x_i^2}P_a(x,y)\\
&= a(y)\sum_{i=1}^{2}\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{2\pi a(y)}\ln|x-y|\right)\\
&= \frac{a(y)}{a(y)}\sum_{i=1}^{2}\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{2\pi}\ln|x-y|\right)\\
&= \delta(x-y)
\end{aligned}$$

(since $\frac{1}{2\pi}\ln|x-y|$ is a fundamental solution of Laplace's operator). □

For some scalar function g, let

$$V_a g(y) := -\int_{\partial\Omega} P_a(x,y)g(x)dS_x, \quad y \notin \partial\Omega, \tag{2.9}$$

$$W_a g(y) := -\int_{\partial\Omega} [T(x,n(x),\partial_x)P_a(x,y)]g(x)dS_x, \quad y \notin \partial\Omega, \tag{2.10}$$

be the single and the double layer surface potential operators.
The corresponding boundary integral operators of direct surface values of the simple layer potential $\mathcal{V}_a$ and of the double layer potential $\mathcal{W}_a$, and the co-normal derivatives of the simple layer potential $\mathcal{W}'_a$ and of the double layer potential $\mathcal{L}^+_a$ are

$$\mathcal{V}_a g(y) := -\int_{\partial\Omega} P_a(x,y)g(x)dS_x, \quad y \in \partial\Omega, \tag{2.11}$$

$$\mathcal{W}_a g(y) := -\int_{\partial\Omega} [T(x,n(x),\partial_x)P_a(x,y)]g(x)dS_x, \quad y \in \partial\Omega, \tag{2.12}$$

$$\mathcal{W}'_a g(y) := -\int_{\partial\Omega} [T(y,n(y),\partial_y)P_a(x,y)]g(x)dS_x, \quad y \in \partial\Omega \tag{2.13}$$

$$\mathcal{L}^+_a g(y) := [T(y,n(y),\partial_y)W_a g(y)]^+, \quad y \in \partial\Omega \tag{2.14}$$

The parametrix-based logarithmic and the remainder potential operators, corresponding to parametrix (2.6) and to remainder (2.8) are

$$\mathcal{P}_a g(y) := \int_{\Omega} P_a(x,y)g(x)dx, \quad y \in \mathbb{R}^2 \tag{2.15}$$

$$\mathcal{R}_a g(y) := \int_{\Omega} R_a(x,y)g(x)dx, \quad y \in \mathbb{R}^2. \tag{2.16}$$

Let $\mathcal{P}_\Delta$, V_Δ, W_Δ, $\mathcal{V}_\Delta$, $\mathcal{W}_\Delta$, $\mathcal{L}^+_\Delta$ denote the potential corresponding to the operator $L_a = \Delta$.

Then the equations (2.9) − (2.16) have the following relations.

$$\mathcal{P}_a g = \frac{1}{a}\mathcal{P}_\Delta g \ , \ \mathcal{R}_a g = -\frac{1}{a}\sum_{j=1}^{2} \partial_j \mathcal{P}_\Delta \left[g(\partial_j a)\right] \tag{2.17}$$

$$V_a g = \frac{1}{a} V_\Delta g \ , \ W_a g = \frac{1}{a} W_\Delta (ag) \tag{2.18}$$

$$\mathcal{V}_a g = \frac{1}{a}\mathcal{V}_\Delta g \ , \ \mathcal{W}_a g = \frac{1}{a}\mathcal{W}_\Delta (ag) \tag{2.19}$$

$$\mathcal{W}'_a g = \mathcal{W}'_\Delta g + \left[a\frac{\partial}{\partial_n}\left(\frac{1}{a}\right)\right]\mathcal{V}_\Delta g \tag{2.20}$$

$$\mathcal{L}^+_a g = \mathcal{L}^+_\Delta (ag) + \left[a\frac{\partial}{\partial_n}\left(\frac{1}{a}\right)\right]\gamma^+ W_\Delta (ag) \tag{2.21}$$

Proof. Prove equation (2.17),

$$\begin{aligned}
\mathcal{P}_a g &= \int_\Omega P_a(x,y) g(x) dx \\
&= \int_\Omega \frac{1}{2\pi a(y)} \ln|x-y| g(x) dx \\
&= \frac{1}{a(y)} \int_\Omega \frac{1}{2\pi} \ln|x-y| g(x) dx \\
&= \frac{1}{a(y)} \int_\Omega P_\Delta g(x) dx \\
&= \frac{1}{a}\mathcal{P}_\Delta g \\
\mathcal{R}_a g &= \int_\Omega R_a(x,y) g(x) dx \\
&= \int_\Omega \sum_{i=1}^{2} \frac{x_i - y_i}{2\pi a(y)|x-y|^2} \frac{\partial a(x)}{\partial x_i} g(x) dx \\
&= \sum_{i=1}^{2} \frac{-1}{a(y)} \int_\Omega \frac{\partial}{\partial y_i} P_\Delta \left[g(x)\frac{\partial a(x)}{\partial x_i}\right] dx \\
&= -\frac{1}{a(y)} \int_\Omega \sum_{i=1}^{2} \partial_i P_\Delta \left[g(x)\partial_i a(x)\right] dx \\
&= \frac{-1}{a} \sum_{i=1}^{2} \partial_i \mathcal{P}_\Delta \left[g \partial_i a\right]
\end{aligned}$$

where $P_\Delta = \frac{1}{2\pi}\ln|x-y|$ is a fundamental solution of Laplace operator Δ.

Prove Equation (2.18),

$$
\begin{aligned}
V_a g(y) &= -\int_{\partial\Omega} P_a(x,y) g(x) dS_x \\
&= -\int_{\partial\Omega} \frac{1}{2\pi a(y)} \ln|x-y| g(x) dS_x \\
&= -\frac{1}{a(y)} \int_{\partial\Omega} \frac{1}{2\pi} \ln|x-y| g(x) dS_x \\
&= -\frac{1}{a(y)} \int_{\partial\Omega} P_\Delta(x,y) g(x) dS_x \\
&= \frac{1}{a(y)} V_\Delta(x,y) g(x) \\
W_a g(y) &= -\int_{\partial\Omega} [T(x,n(x),\partial_x) P_a(x,y)] g(x) dS_x \\
&= -\int_{\partial\Omega} \left[\sum_{i=1}^{2} a(x) n_i(x) \frac{\partial}{\partial x_i} P_a(x,y)\right] g(x) dS_x \\
&= -\int_{\partial\Omega} \left[\sum_{i=1}^{2} n_i(x) \frac{\partial}{\partial x_i} \frac{P_\Delta(x,y)}{a(y)}\right] (ag)(x) dS_x \\
&= -\frac{1}{a(y)} \int_{\partial\Omega} \left[\sum_{i=1}^{2} n_i(x) \frac{\partial}{\partial x_i} P_\Delta(x,y)\right] (ag)(x) dS_x \\
&= \frac{1}{a(y)} W_\Delta(ag)(y)
\end{aligned}
$$

Prove Equation (2.19),

$$
\begin{aligned}
\mathcal{V}_a g(y) &= -\int_{\partial\Omega} P_a(x,y) g(x) dS_x \\
&= -\int_{\partial\Omega} \frac{1}{2\pi a(y)} \ln|x-y| g(x) dS_x \\
&= -\frac{1}{a(y)} \int_{\partial\Omega} \frac{1}{2\pi} \ln|x-y| g(x) dS_x \\
&= -\frac{1}{a(y)} \int_{\partial\Omega} P_\Delta(x,y) g(x) dS_x \\
&= \frac{1}{a(y)} \mathcal{V}_\Delta g(y)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W}_a g(y) &= -\int_{\partial\Omega}[T(x,n(x),\partial_x)P_a(x,y)]g(x)dS_x \\
&= -\int_{\partial\Omega}\left[\sum_{i=1}^{2} a(x)n_i(x)\frac{\partial}{\partial x_i}P_a(x,y)\right]g(x)dS_x \\
&= -\int_{\partial\Omega}\left[\sum_{i=1}^{2} n_i(x)\frac{\partial}{\partial x_i}\frac{P_\Delta(x,y)}{a(y)}\right](ag)(x)dS_x \\
&= \frac{1}{a(y)}\int_{\partial\Omega}\left[\sum_{i=1}^{2} n_i(x)\frac{\partial}{\partial x_i}P_\Delta(x,y)\right](ag)(x)dS_x \\
&= \frac{1}{a(y)}\mathcal{W}_\Delta(ag)(y)
\end{aligned}
$$

Prove Equation (2.20),

$$
\begin{aligned}
\mathcal{W}'_a g(y) &= -\int_{\partial\Omega}[T(y,n(y),\partial_y)P_a(x,y)]g(x)dS_x \\
&= -\int_{\partial\Omega}\left[\sum_{i=1}^{2} a(y)n_i(y)\frac{\partial}{\partial y_i}P_a(x,y)\right]g(x)dS_x, \\
&= -\int_{\partial\Omega}\left[\sum_{i=1}^{2} n_i(y)\frac{\partial}{\partial y_i}\left(\frac{P_\Delta(x,y)}{a(y)}\right)\right]g(x)dS_x \\
&= -\int_{\partial\Omega}\left[\sum_{i=1}^{2} a(y)n_i(y)\frac{1}{[a(y)]^2}\left(a(y)\frac{\partial}{\partial y_i}P_\Delta(x,y) - P_\Delta(x,y)\frac{\partial}{\partial y_i}a(y)\right)\right]g(x)dS_x \\
&= -\int_{\partial\Omega}\left[\sum_{i=1}^{2} \frac{1}{[a(y)]^2}\left([a(y)]^2 n_i(y)\frac{\partial}{\partial y_i}P_\Delta(x,y) - a(y)n_i(y)P_\Delta(x,y)\frac{\partial}{\partial y_i}a(y)\right)\right]g(x)dS_x \\
&= -\int_{\partial\Omega}\left[\sum_{i=1}^{2}\left(n_i(y)\frac{\partial}{\partial y_i}P_\Delta(x,y)\right)\right]g(x)dS_x \\
&+ \int_{\partial\Omega}\left[\sum_{i=1}^{2}\frac{1}{a(y)}\left(n_i(y)P_\Delta(x,y)\frac{\partial}{\partial y_i}a(y)\right)\right]g(x)dS_x \\
&= \mathcal{W}'_\Delta g(y) + \sum_{i=1}^{2}\frac{n_i(y)}{a(y)}\frac{\partial a(y)}{\partial y_i}\int_{\partial\Omega}P_\Delta(x,y)g(x)dS_x \\
&= \mathcal{W}'_\Delta g(y) + \left[a(y)\frac{\partial}{\partial_{n(y)}}\left(\frac{1}{a(y)}\right)\right]\mathcal{V}_\Delta g(y)
\end{aligned}
$$

Prove Equation (2.21),

$$\begin{aligned}
\mathcal{L}_a^+ g(y) &= -\left[T^+(y, n(y), \partial_y)\right] W_a g(y) \\
&= \sum_{i=1}^{2} a(y) n_i(y) \gamma^+ \left[\frac{\partial}{\partial y_i} W_a g(y)\right] \\
&= a(y)\gamma^+ \left[\frac{\partial}{\partial n(y)} W_a g(y)\right] \\
&= a(y)\gamma^+ \left[\frac{\partial}{\partial n(y)} \frac{W_\Delta(ag)(y)}{a(y)}\right] \\
&= a(y)\gamma^+ \left[\frac{1}{a(y)} \frac{\partial}{\partial n(y)} W_\Delta(ag)(y) + W_\Delta(ag)(y) \frac{\partial}{\partial_{n(y)}} \left(\frac{1}{a(y)}\right)\right] \\
&= \gamma^+ \left[\frac{\partial}{\partial_{n(y)}} W_\Delta(ag)(y)\right] + \left[a(y) \frac{\partial}{\partial n(y)} \left(\frac{1}{a(y)}\right)\right] \gamma^+ W_\Delta(ag)(y) \\
&= \mathcal{L}^+(ag)(y) + \left[a(y) \frac{\partial}{\partial n(y)} \left(\frac{1}{a(y)}\right)\right] \gamma^+ W_\Delta(ag)(y)
\end{aligned}$$

□

Theorem 2.2.1. *For $s \in \mathbb{R}$, the following operators are continuous.*

$$V_a : H^s(\partial\Omega) \to H^{s+3/2}(\Omega),$$

$$W_a : H^s(\partial\Omega) \to H^{s+1/2}(\Omega),$$

$$\mathcal{V}_a, \mathcal{W}_a, \mathcal{W}'_a : H^s(\partial\Omega) \to H^{s+1}(\partial\Omega),$$

$$\mathcal{L}_a^+ : H^s(\partial\Omega) \to H^{s-1}(\partial\Omega).$$

Proof. We have the corresponding mapping for the corresponding constant coefficient operators [*see*, [1] *C.Constanda*, [4] *W.McLean*, [9] *O.Steinbach*]. Then (2.18) – (2.21) imply the theorem claim. □

Theorem 2.2.2. *For $g_1 \in H^{-1/2}(\partial\Omega)$ and $g_2 \in H^{1/2}(\partial\Omega)$, there hold the jump relations on $\partial\Omega$*

$$[V_a g_1(y)]^+ = \mathcal{V}_a g_1(y)$$

$$[W_a g_2(y)]^+ = -\frac{1}{2} g_2(y) + \mathcal{W}_a g_2(y),$$

$$[T(y, n(y), \partial_y) V_a g_1(y)]^+ = \frac{1}{2} g_1(y) + \mathcal{W}'_a g_1(y),$$

Proof. For the constant coefficient case, this theorem is proved in Chapter 1 Theorem 1.4.2. Then taking into account the relations (2.17) – (2.21), we can prove the theorem for the variable positive coefficient $a \in C^\infty(\mathbb{R}^2)$ as well.

$$\begin{aligned}
[V_a g_1(y)]^+ &= \lim_{y_0 \to y} (V_a g_1(y_0)) \\
&= \lim_{y_0 \to y} \left(\frac{1}{a(y_0)} V_\Delta g_1(y_0) \right) \\
&= \frac{1}{a(y)} \mathcal{V}_\Delta g_1(y) \\
&= \mathcal{V}_a g_1(y) \quad for \quad y_0 \in \Omega \ , \ y \in \partial\Omega. \\
[W_a g_2(y)]^+ &= \lim_{y_0 \to y} (W_a g_2(y_0)) \\
&= \lim_{y_0 \to y} \left(\frac{1}{a(y_0)} W_\Delta (a(y_0) g_2(y_0)) \right) \\
&= \frac{1}{a(y)} \left(\frac{-1}{2} a(y) g_2(y) + \mathcal{W}_\Delta (a(y) g_2(y) \right) \\
&= -\frac{1}{2} g_2(y) + \frac{1}{a(y)} \mathcal{W}_\Delta (a(y) g_2(y)) \\
&= -\frac{1}{2} g_2(y) + \mathcal{W}_a g_2(y) \quad for \quad y_0 \in \Omega^c \ , \ y \in \partial\Omega. \\
[T(y, n(y), \partial_y) V_a g_1(y)]^+ &= \left[\frac{\partial V_a(y)}{\partial n_y} \right]^+ \\
&= \lim_{y_0 \to y} \left(\frac{\partial V_a(y_0)}{\partial n_{y_0}}) \right) \\
&= \lim_{y_0 \to y} \left(\frac{\partial V_\Delta(y_0)}{\partial n_{y_0}} + \left[a(y_0) \frac{\partial}{\partial n(y_0)} \left(\frac{1}{a(y_0)} \right) \right] \mathcal{V}_\Delta g_1(y_0) \right) \\
&= \frac{1}{2} g_1(y) + \frac{\partial V_\Delta(y)}{\partial n_y} + \left[a(y) \frac{\partial}{\partial_{n(y)}} \left(\frac{1}{a(y)} \right) \right] \mathcal{V}_\Delta g_1(y) \\
&= \frac{1}{2} g_1(y) + \mathcal{W}'_\Delta g_1(y) + (\mathcal{W}'_a g_1(y) - \mathcal{W}'_\Delta g_1(y)) \\
&= \frac{1}{2} g_1(y) + \mathcal{W}'_a g_1(y) \quad for \quad y_0 \in \Omega \ , \ y \in \partial\Omega.
\end{aligned}$$

□

Recall that a linear operator $K : X \to Y$ is said to be compact if only if every bounded sequence $\{x_n\}$ in X has a subsequence $\{x_{n_i}\}$ such that $\{Kx_{n_i}\}$ converges in Y. By the Rellich compact embedding theorem, the inclusion $H^t(\Omega) \subset H^s(\Omega)$ is compact for $s < t$ ([4], theorem 3.27).

Corollary 2.1. *The following operators are compact,*

$$\mathcal{V}_a, \mathcal{W}_a, \mathcal{W}'_a : H^s(\partial\Omega) \to H^s(\partial\Omega), \quad s \in \mathbb{R} \tag{2.22}$$

$$\mathcal{R}_a : H^s(\Omega) \to H^s(\Omega), \quad s > \frac{1}{2}, \tag{2.23}$$

$$\gamma^+ \mathcal{R}_a : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad s > \frac{1}{2}, \tag{2.24}$$

$$T^+ \mathcal{R}_a : H^s(\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega), \quad s > \frac{1}{2}. \tag{2.25}$$

Proof. The embedding, $H^{s+1}(\Omega) \subset H^s(\Omega)$ is compact. Let $\{g_n\}$ be a bounded sequence in

$H^s(\Omega)$, then the continuity $\mathcal{R}_a : H^s(\Omega) \to H^{s+1}(\Omega), \quad s > -\frac{1}{2}$ implies

$$||\mathcal{R}_a g_n||_{H^{s+1}(\Omega)} \leq C||g_n||_{H^s(\Omega)}$$

Thus the sequence $\{\mathcal{R}_a g_n\}$ is bounded in $H^{s+1}(\Omega)$. Hence it has a convergent subsequence $\{\mathcal{R}_a g_{n_i}\}$ in $H^s(\Omega)$,i.e., there exists subsequence $\{g_{n_i}\}$ such that $\{\mathcal{R}_a g_{n_i}\}$ converges in $H^s(\Omega)$, which implies the compactness of the operator $\mathcal{R}_a$.
To prove equation (2.22),
since the embedding $H^{s+1}(\partial\Omega) \subset H^s(\partial\Omega)$ is compact, and similar to the first proof. □

2.3 Invertibility of the Single Layer Potential Operator

The boundary integral operator $\mathcal{V}_\Delta : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is Fredholm operator of index zero ([4], *theorem* 7.6). Thus the relation (2.19), leads to the same result for single layer potential $\mathcal{V}_a$. For the three dimensional case, the following holds. For $\psi^* \in H^{-1/2}(\partial\Omega)$, if $V_a\psi^*(y) = 0,\ y \in \Omega$, then $\psi^* = 0$, which implies the invertibilty of single layer potential operator mapping from $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$. But it is not true the two dimensional case. It is well know (*see*, [1] *Remark* 1.42(*ii*) , [9], *proof of thm* 6.22) for some $2D$ domains the kernel of the operator $\mathcal{V}_\Delta$ is non-zero, which by (2.19) also implies that $ker\mathcal{V}_a \neq \{0\}$ for the same domains. The following example illustrates this fact.

Example 2.3.1. *Take the density function $\phi \equiv 1$ and $\Omega = B(0,R)$ to be a disc of radius R centered at the origin and $\partial\Omega = \partial B(0,R)$ be the circular boundary of the disc. We can show that* $a(y)V_a\phi(y) = V_\Delta\phi(y) = \begin{cases} R\ln|y|, & for\ |y| > R, \\ R\ln R, & for\ |y| \leq R. \end{cases}$

Proof. Let $\phi \equiv 1$. Then

$$V_\Delta\phi(y) = \frac{1}{2\pi}\int_{|x|=R} \ln|y-x|dS_x.$$

If $|y| > R$, then the function $g(x) = \ln|y-x|$ is harmonic in the disk $B(0,R)$. Then $g(x)$ has the mean value property,

$$\ln|y| = g(0) = \frac{1}{2\pi R}\int_{|x|=R} g(x)dS_x.$$

Therefore,

$$\frac{1}{2\pi}\int_{|x|=R} \ln|y-x|dS_x = R\ln|y|, \quad for\ |y| > R. \tag{2.26}$$

For $|y| \leq R$, in particular take $y = 0$,

$$(V_\Delta \phi)(0) = \frac{1}{2\pi} \int_{|x|=R} \ln |x| dS_x = R \ln R.$$

The relation (2.26) implies that, the limit of the value of the potential when $|y|$ approach the boundary from exterior is given by

$$\lim_{|y| \to R^+} (V_\Delta \phi)(y) = R \ln R \; for \; |y| = R.$$

Furthermore, since the single layer potential is continuous on $\mathbb{R}^2$ we have

$$(V_\Delta \phi)(y) = R \ln R \quad for \; |y| = R.$$

To determine the value of the potential inside the disc for $y \neq 0$, we use the maximum/minimum principle. Since the single layer potential is harmonic on Ω it has neither maximum nor minimum in the disc. Let

$$C_0 = (V_\Delta \phi)(y_0) \quad for \;\; 0 < |y_0| < R.$$

If we assume $C_0 \neq R \ln R$, i.e., C_0 is different from the value of potential on the boundary, we will arrive contradiction of the maximum principle. Thus $(V_\Delta \phi)(y)$ is constant on $\bar{\Omega}$. Therefore, $(V_\Delta \phi)(y) = R \ln R, \quad for \;\; |y| \leq R.$ □

Remark 2.1. *In the above example, if we take the value of $R = 1$, and since $a(y) \neq 0$, then $(V_a \phi)(y) = 0$ in $\bar{\Omega}$.*

Example 2.3.1 show that, the kernel of the operator $\mathcal{V}_a : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ contains non zero element for a unit ball, i.e., $ker\mathcal{V}_a \neq \{0\}$ for $\Omega = B(0,1)$, which means, the operators $\mathcal{V}_a$ is not one to one for this particular domain. In order to have invertibility for the single layer potential operator in $2D$, we define the following subspace of the space $H^{-\frac{1}{2}}(\partial\Omega)$,

$$H_{**}^{-\frac{1}{2}}(\partial\Omega) := \{\phi \in H^{-\frac{1}{2}}(\partial\Omega) : \langle \phi, 1 \rangle_{\partial\Omega} = 0\},$$

where the norm in $H_{**}^{-\frac{1}{2}}(\partial\Omega)$ is the induced by the norm in $H^{-\frac{1}{2}}(\partial\Omega)$.

Theorem 2.3.1. *If $\psi \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ satisfies $\mathcal{V}_a \psi = 0$ on $\partial\Omega$, then $\psi = 0$.*

Proof. The theorem holds for the operator $\mathcal{V}_\Delta$ $(see, [4] \; corollary \; 8.11(ii))$,

$$\begin{aligned} &\mathcal{V}_a \psi = 0 \\ &\Rightarrow \frac{1}{a(y)} \mathcal{V}_\Delta \psi = 0 \\ &\Rightarrow \psi = 0, \quad (since \;\; a(y) \neq 0, \Rightarrow \mathcal{V}_\Delta \neq 0). \end{aligned}$$

□

In the two dimensional case $n = 2$, the logarithmic capacity is defined by $cap_{\partial\Omega} := e^{-2\pi\mathcal{V}_\Delta\psi_{eq}}$, $\psi_{eq} \in H^{-1/2}(\partial\Omega)$ is natural density, so that $\frac{1}{2\pi}\ln\frac{1}{cap_{\partial\Omega}} = \mathcal{V}_\Delta\psi_{eq}$.
Note that $\mathcal{V}_\Delta\psi_{eq} = 0$ if and only if $1 = cap_{\partial\Omega}$.

Theorem 2.3.2. *The single layer potential operator* $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$*, is* $H^{-\frac{1}{2}}(\partial\Omega)$*-elliptic, i.e.,*

$$\langle\mathcal{V}_\Delta\psi, \psi\rangle_{\partial\Omega} \geq C||\psi||^2_{H^{-\frac{1}{2}}(\partial\Omega)} \quad for \quad all \quad \psi \in H^{-\frac{1}{2}}(\partial\Omega)$$

if and only if $cap_{\partial\Omega} < 1$.

Proof. Put $\lambda = \mathcal{V}_\Delta\psi_{eq} = \frac{1}{2\pi}\ln(\frac{1}{cap_{\partial\Omega}})$. For an arbitrary $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$, let $(1, \psi)_{\partial\Omega} = \alpha \in \mathbb{R}$ and define $\psi = \psi_0 + \alpha\psi_{eq}$ for $\psi_0 \in H_*^{-\frac{1}{2}}(\partial\Omega)$ and $\mathcal{V}_\Delta\psi = \mathcal{V}_\Delta\psi_0 + \lambda\alpha$. Since $\langle\mathcal{V}_\Delta\psi_0, \psi_{eq}\rangle_{\partial\Omega} = \left\langle\psi_0, \mathcal{V}_\Delta\psi_{eq}\right\rangle_{\partial\Omega} = 0$, we have

$$\langle\mathcal{V}_\Delta\psi, \psi\rangle_{\partial\Omega} = \langle\mathcal{V}_\Delta\psi_0, \psi_0\rangle_{\partial\Omega} + \lambda\alpha^2. \tag{2.27}$$

If $cap_{\partial\Omega} \geq 1$, then $\langle\mathcal{V}_\Delta\psi_{eq}, \psi_{eq}\rangle_{\partial\Omega} = \lambda \leq 0$. Suppose that $cap_{\partial\Omega} < 1$, then $\lambda > 0$. It is known that $\langle\mathcal{V}_\Delta\psi_0, \psi_0\rangle_{\partial\Omega} > 0$ ([9] *thm* 8.12). Hence both terms the right hand side of (2.27) are non-negative, and by theorem 2.3.1, the first is zero if and only if $\psi_0 = 0$. Thus, $\langle\mathcal{V}_\Delta\psi, \psi\rangle_{\partial\Omega} \geq 0$, with equality if and only if ψ_0 and $\alpha = 0$, i.e., if and only if $\psi = 0$. Hence, $\mathcal{V}_\Delta$ is strictly positive-definite, on the whole of $H^{-1/2}(\partial\Omega)$. Since, the boundary integral operator $\mathcal{V}_\Delta : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is Fredholm operator of index zero, and $\mathcal{V}_\Delta$ is strictly positive-definite, $ker\mathcal{V}_\Delta = \{0\}$, we conclude that $\mathcal{V}_\Delta$ is $H^{-1/2}(\partial\Omega)$ -elliptic. □

There is a connection between the logarithmic capacity and the euclidean diameter of Ω. In the particular, $cap_{\partial\Omega} \leq diam(\Omega)$. Therefore, to ensure $cap_{\partial\Omega} < 1$ a sufficient criteria is to assume $diam(\Omega) < 1$.

Theorem 2.3.3. *Let* $\Omega \subset \mathbb{R}^2$ *have the diameter* $diam(\Omega) < 1$. *Then the single layer potential* $\mathcal{V}_a : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ *is invertible.*

Proof. By theorem 2.3.2 for $diam(\Omega) < 1$ the operator $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is $H^{-\frac{1}{2}}(\partial\Omega)$-elliptic i.e

$$||\mathcal{V}_\Delta\psi||_{H^{\frac{1}{2}}(\partial\Omega)} \geq C||\psi||_{H^{-\frac{1}{2}}(\partial\Omega)} \quad for \ \psi \in H^{-\frac{1}{2}}(\partial\Omega)$$

and since it is also bounded by theorem 2.2.1 for $s = -1/2$, Lax-Milgram theorem ([9], *thm* 3.4) implies its invertibility. Then by the first relation in (2.19) invertibility of the single layer potential $\mathcal{V}_a : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ also follows. That is

$$\mathcal{V}_a^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$$

is bounded and satisfying $||\mathcal{V}_a^{-1}\psi||_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C||\psi||_{H^{\frac{1}{2}}(\partial\Omega)}$ for $\psi \in H^{\frac{1}{2}}(\partial\Omega)$.
Hence, $\mathcal{V}_a$ is invertible. [[9], *thm* 6.23] □

2.4 Formation of Boundary Domain Integral Equations

Let $u \in H^{1,0}(\Omega; \Delta)$, $v \in H^{1,0}(\Omega; \Delta)$ be some functions. Then, subtracting (2.2) from its counterpart with exchanged roles of u and v i.e.,

$$(T^+v, \gamma^+u)_{\partial\Omega} := \int_\Omega (u(x)L_a v(x) + E_a(v,u))dx,$$

we obtain the second Green identity for the operator $L_a(x, \partial_x)$,

$$\int_\Omega (v(x)L_a u(x) - u(x)L_a v(x))dx = \langle T^+u, \gamma^+v\rangle_{\partial\Omega} - \langle T^+v, \gamma^+u\rangle_{\partial\Omega} \tag{2.28}$$

Applying the second Green identity (2.28) and $v(x) = P_a(x,y)$. We get the third Green identity.
That is

$$\int_\Omega (v(x)L_a u(x) - u(x)L_a v(x))dx = (T^+u, \gamma^+v)_{\partial\Omega} - (T^+v, \gamma^+u)_{\partial\Omega}$$
$$\Rightarrow \int_\Omega v(x)L_a u(x)dx - \int_\Omega u(x)L_a v(x)dx = \int_{\partial\Omega} T^+u\gamma^+v dS_x - \int_{\partial\Omega} T^+v\gamma^+u dS_x$$
$$\Rightarrow \int_\Omega P_a(x,y)L_a u(x)dx - \int_\Omega u(x)L_a P_a(x,y)dx = \int_{\partial\Omega} T^+u\gamma^+P_a(x,y)dS_x - \int_{\partial\Omega} T^+P_a(x,y)\gamma^+u dS_x$$
$$\Rightarrow \int_\Omega P_a(x,y)L_a u(x)dx - \int_\Omega u(x)[\delta(x-y) + R_a(x,y)]dx$$
$$= \int_{\partial\Omega} T^+u(x)\gamma^+P_a(x,y)dS_x - \int_{\partial\Omega} T^+P_a(x,y)\gamma^+u dS_x$$
$$\Rightarrow \int_\Omega u(x)\delta(x-y)dx + \int_\Omega u(x)R_a(x,y)dx + \int_{\partial\Omega} P_a(x,y)T^+u(x)dS_x - \int_{\partial\Omega} T^+P_a(x,y)\gamma^+u dS_x$$
$$= \int_\Omega P_a(x,y)L_a u(x)dx$$
$$\Rightarrow u(y) + \int_\Omega u(x)R_a(x,y)dx + \int_{\partial\Omega} P_a(x,y)T^+u(x)dS_x - \int_{\partial\Omega} T^+P_a(x,y)\gamma^+u dS_x$$
$$= \int_\Omega P_a(x,y)L_a u(x)dx$$

Using equations (2.9), (2.10), (2.15) and (2.16) one operator third Green identity can be written as:

$$u(y) + \mathcal{R}_a(x,y)u(y) - V_a T^+u(y) + W_a\gamma^+u(y) = \mathcal{P}_a L_a u(y)\ , \quad y \in \Omega \tag{2.29}$$

Taking trace and co-normal derivative of the third Green identity (2.29),
Take the trace of equation (2.29) on $\partial\Omega$ and using jump relations, we obtain

$$\gamma^+u(y) + \gamma^+\mathcal{R}_a(x,y)u(y) - \gamma^+V_aT^+u(y) + \gamma^+W_a\gamma^+u(y) = \gamma^+\mathcal{P}_aL_au(y)$$
$$\Rightarrow \gamma^+u(y) + \gamma^+\mathcal{R}_a(x,y)u(y) - \mathcal{V}_aT^+u(y) - \frac{1}{2}\gamma^+u(y) + \mathcal{W}_a\gamma^+u(y) = \gamma^+\mathcal{P}_aL_au(y)$$
$$\Rightarrow \frac{1}{2}\gamma^+u(y) + \gamma^+\mathcal{R}_a(x,y)u(y) - \mathcal{V}_aT^+u(y) + \mathcal{W}_a\gamma^+u(y) = \gamma^+\mathcal{P}_aL_au(y)\ , \quad y \in \partial\Omega.$$

Again by taking the co-normal derivative of (2.29) on $\partial\Omega$ and using jump relations, we have

$$T^+u(y) + T^+\mathcal{R}_a(x,y)u(y) - T^+V_aT^+u(y) + T^+W_a\gamma^+u(y) = T^+\mathcal{P}_aL_au(y)$$
$$\Rightarrow T^+u(y) + T^+\mathcal{R}_a(x,y)u(y) - \frac{1}{2}T^+u(y) - \mathcal{W}'_aT^+u(y) + \mathcal{L}_a^+\gamma^+u(y) = T^+\mathcal{P}_aL_au(y)$$
$$\Rightarrow \frac{1}{2}T^+u(y) + T^+\mathcal{R}_a(x,y)u(y) - \mathcal{W}'_aT^+u(y) + \mathcal{L}_a^+\gamma^+u(y) = T^+\mathcal{P}_aL_au(y)$$

Therefore,

$$u(y) + \mathcal{R}_a(x,y)u(y) - V_aT^+u(y) + W_a\gamma^+u(y) = \mathcal{P}_aL_au(y)\ ,\ y \in \Omega \tag{2.30}$$

$$\frac{1}{2}\gamma^+u(y) + \gamma^+\mathcal{R}_a(x,y)u(y) - \mathcal{V}_aT^+u(y) + \mathcal{W}_a\gamma^+u(y) = \gamma^+\mathcal{P}_aL_au(y)\ , \quad y \in \partial\Omega \tag{2.31}$$

$$\frac{1}{2}T^+u(y) + T^+\mathcal{R}_a(x,y)u(y) - \mathcal{W}'_aT^+u(y) + \mathcal{L}_a^+\gamma^+u(y) = T^+\mathcal{P}_aL_au(y)\ ,\ y \in \partial\Omega \tag{2.32}$$

For arbitrary functions f, Ψ, Φ, let us consider a more general indirect integral relation, associated with (2.30), namely,

$$u(y) + \mathcal{R}_a(x,y)u(y) - V_a\Psi(y) + W_a\Phi(y) = \mathcal{P}_af\ ,\ y \in \Omega \tag{2.33}$$

Lemma 2.4.1. *Let $\Psi \in H^{-1/2}(\partial\Omega), \Phi \in H^{1/2}(\partial\Omega)$, and $f \in L_2(\Omega)$. Suppose a function $u \in H^1(\Omega)$ satisfies (2.33). Then $u \in H^{1,0}(\Omega;\Delta)$, it is a solution of $L_au = f$ in Ω, and $V_a(\Psi - T^+u)(y) - W_a(\Phi - \gamma^+u)(y) = 0$, $y \in \Omega$.*

Proof. Since $u \in H^1(\Omega)$, we have to show that $L_au \in L_2(\Omega)$. First observe that,

$$\begin{aligned}
\Delta(au) &= \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial_{x_i}}(au)\right) \\
&= \sum_{i=1}^{2} \frac{\partial}{\partial_{x_i}}\left[u(x)\frac{\partial}{\partial x_i}a(x) + a(x)\frac{\partial}{\partial x_i}u(x)\right] \\
&= \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[u(x)\frac{\partial}{\partial x_i}a(x)\right] + \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial}{\partial x_i}u(x)\right] \\
&= \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[u(x)\frac{\partial a(x)}{\partial x_i}\right] + L_au
\end{aligned}$$

Therefore we have,

$$L_a u = \Delta(au) - \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[u(x)\frac{\partial a(x)}{\partial x_i}\right]$$

Also note that $u \in H^1(\Omega)$ implies $\frac{\partial u}{\partial x_i} \in H^0(\Omega) = L_2(\Omega)$, hence for $a(x) \in C^\infty(\mathbb{R}^2)$ we have $u(x)\frac{\partial a(x)}{\partial x_i} \in H^1(\Omega)$, implies the last term is belongs to $L_2(\Omega)$. Now we need to show that $\Delta(au) \in L_2(\Omega)$.
Rewriting (2.33) as a function of u

$$u(y) = \mathcal{P}_a f - \mathcal{R}_a(x,y)u(y) + V_a\Psi(y) - W_a\Phi(y).$$

Then multiply by $a(y)$

$$\begin{aligned} a(y)u(y) &= a(y)\mathcal{P}_a f - a(y)\mathcal{R}_a(x,y)u(y) + a(y)V_a\Psi(y) - a(y)W_a\Phi(y) \\ &= P_\Delta f - a(y)\mathcal{R}_a(x,y)u(y) + V_\Delta\Psi(y) - W_\Delta(a(y)\Phi(y)). \end{aligned}$$

Then take Laplace's operator of the last equation

$$\Delta(a(y)u(y)) = \Delta(P_\Delta f) - \Delta(a(y)\mathcal{R}_a(x,y)u(y)) + \Delta(V_\Delta\Psi(y)) - \Delta(W_\Delta(a(y)\Phi(y)))$$

The last two terms $\Delta(V_\Delta\Psi(y))$ and $\Delta(W_\Delta(a(y)\Phi(y))$ are harmonic for $y \in \Omega/\partial\Omega$ i.e., $\Delta(V_\Delta\Psi(y)) = \Delta(W_\Delta(a(y)\Phi(y)) = 0$.

$$\begin{aligned} \Delta(P_\Delta f) &= \Delta(P_\Delta * \tilde{f}) \\ &= \Delta P_\Delta * \tilde{f} \\ &= \delta * \tilde{f} = \tilde{f} \in L_2(\mathbb{R}^2), \end{aligned}$$

where $\tilde{f} \in L_2(\mathbb{R}^2)$ is the extension of f by zero outside Ω. If $u \in H^1(\Omega)$, then by the mapping property of $\mathcal{R}_a$ $(\mathcal{R}_a : H^s(\Omega) \to H^{s+1}(\Omega), \quad s > -\frac{1}{2})$, we have $\mathcal{R}_a(x,y)u(y) \in H^2(\Omega)$ and hence $a(y)\mathcal{R}_a(x,y)u(y) \in H^2(\Omega)$. By the definition of the space $H^2(\Omega)$, we have $\Delta(a(y)\mathcal{R}_a(x,y)u(y)) \in L_2(\Omega)$. Therefore, $u \in H^{1,0}(\Omega; L_a)$ $(i.e., H^{1,0}(\Omega; L_a) = \{u \in H^1(\Omega) : L_a u \in L_2(\Omega)\})$. Subtracting (2.29) from (2.33)

$$-V_a(T^+u - \Psi) + W_a(\gamma^+u - \Phi) = P_a(L_a u - f) \;\; in \;\; \Omega$$

$$-V_a\Psi^* + W_a\Phi^* = P_a(L_a u - f) \;\; in \;\; \Omega \tag{2.34}$$

where $\Psi^* := T^+u - \Psi, \Phi^* := \gamma^+u - \Phi$.
Multiplying equation (2.34) by $a(y)$, we get

$$\begin{aligned} &- a(y)V_a\Psi_a^*(y) + W_a\Phi^*(y) = a(y)P_a(L_a u - f) \\ &\Rightarrow -V_\Delta\Psi^* + W_\Delta\Phi^* = P_\Delta(L_a u - f) \quad in \;\; \Omega. \end{aligned}$$

Applying Laplace's operator to the last equation, we get $L_a u - f = 0$. This show that u solves PDE, $L_a u = f$ in Ω. Now substituting $L_a u = f$ into $-V_a(T^+u-\Psi)+W_a(\gamma^+u-\Phi) = P_a(L_a u - f)$, we obtain $V_a(T^+u - \Psi) - W_a(\gamma^+u - \Phi) = 0, \quad y \in \Omega$. □

Lemma 2.4.2. *(i) Let either $\Psi^* \in H^{-1/2}(\partial\Omega)$ and $diam(\Omega) < 1$ or $\Psi^* \in H_{**}^{-1/2}(\partial\Omega)$. If $V_a\Psi^*(y) = 0$, $y \in \Omega$, then $\Psi^* = 0$ on $\partial\Omega$.*
(ii) If $\Phi^* \in H^{1/2}(\partial\Omega)$, and $W_a\Phi^*(y) = 0$ in Ω, then $\Phi^* = 0$ on $\partial\Omega$.

Proof. (i) Taking the trace of $V\Psi^*(y) = 0$ on $\partial\Omega$, by the jump relation we have

$$\gamma^+V_a\Psi^*(y) = \mathcal{V}_a\Psi^*(y) = 0 \quad in \quad \partial\Omega$$

If $\Psi^* \in H^{-1/2}(\partial\Omega)$ and $diam(\Omega) < 1$, the result follows from invertibility of the single layer potential given by theorem 2.3.3. On the other hand if $\Psi^* \in H_{**}^{-1/2}(\partial\Omega)$, then the result is implied by theorem 2.3.1.
(ii) Taking the trace of $W_a\Phi^*(y) = 0$, and use the jump relation to obtain

$$\gamma^+W_a\Phi^*(y) = -\frac{1}{2}\Phi^*(y) + \mathcal{W}_a\Phi^*(y) = 0 \quad in \quad \partial\Omega$$

Multiply by $a(y)$, denoting $\tilde{\Phi}^* = a\Phi^*$ and using the relation $\mathcal{W}_a = \frac{1}{a}\mathcal{W}_\Delta(ag)$, we obtain equation

$$\tilde{\Phi}^*(y) - (\frac{1}{2}I + \mathcal{W}_\Delta)\tilde{\Phi}^* = 0 \quad in \quad \partial\Omega$$

It is well known that this equation has only trivial solution. It is particularly due to the contraction property of the operator $\frac{1}{2}I + \mathcal{W}_\Delta$. Therefore, $\Phi^*(y) = 0$ since $a(y) \neq 0$. □

To reduce the variable coefficient Dirichlet BVP $(2.4)-(2.5)$ to a segregated boundary-domain integral equation system, let us denote the unknown function of co-normal derivative as $\psi := T^+u \in H^{-1/2}(\partial\Omega)$ and the known function of trace as $\varphi_0 := \gamma^+u$. Then substituted this function and $L_a u = f$ into the third Green identity (2.30) and either into its trace (2.31) or into its co-normal derivative (2.32) on $\partial\Omega$. we can reduce the BVP $(2.4)-(2.5)$ to the following two different systems of Boundary Domain-Integral Equations.

Integral Equation System (D1)

Boundary-domain Integral Equation System $(D1)$ obtained from equations (2.30) and (2.31) is:

$$u(y) + \mathcal{R}_a u(y) - V_a\psi(y) = F_0(y) \quad , \ y \in \Omega \tag{2.35}$$

$$\gamma^+\mathcal{R}_a u(y) - \mathcal{V}_a\psi(y) = \gamma^+F_0(y) - \varphi_0(y) \quad , \ y \in \partial\Omega \tag{2.36}$$

where

$$F_0(y) = \mathcal{P}_a f(y) - W_a\varphi_0(y). \tag{2.37}$$

Remark 2.2. *$F_0 = 0$ if and only if $(f, \varphi_0) = 0$.*

Proof. Let $(f, \varphi_0) = 0$. Then $F_0 = 0$. Inversely, let $F_0 = 0$. Keeping in mind equation (2.37), Lemma 2.4.1 with $F_0 = 0$ for u implies $f = 0$ and $W_a\varphi_0 = 0$ in Ω. Lemma 2.4.2 then gives $\varphi_0 = 0$ on $\partial\Omega$. □

The system can be written in matrix form as $\mathcal{A}^1\mathcal{U} = \mathcal{F}^1$,
where $\mathcal{U} := (u, \psi)^T \in H^{1,0}(\Omega; L_a) \times H^{-1/2}(\partial\Omega)$ and

$$\mathcal{A}^1 = \begin{bmatrix} I + \mathcal{R}_a & -V_a \\ \gamma^+\mathcal{R}_a & -\mathcal{V}_a \end{bmatrix} \ , \ \mathcal{F}^1 = \begin{bmatrix} F_0 \\ \gamma^+F_0 - \varphi_0 \end{bmatrix}.$$

From the mapping properties of W_a in theorem (2.2.1) and $\mathcal{P}_a$ ($\mathcal{P}_a : H^s(\Omega) \to H^{s+2}(\Omega),\ s > -\frac{1}{2}$), we get the inclusion $F_0 \in H^{1,0}(\Omega; L_a)$, and the trace theorem implies $\gamma^+F_0 \in H^{1/2}(\partial\Omega)$. Therefore, $\mathcal{F}^1 \in H^1(\Omega) \times H^{1/2}(\partial\Omega)$. Due to the mapping properties of the operators involved in $\mathcal{A}^1$, the operator $\mathcal{A}^1 : H^{1,0}(\Omega; L_a) \times H^{1/2}(\partial\Omega) \to H^1(\Omega) \times H^{1/2}(\partial\Omega)$ is bounded.

Remark 2.3. $\mathcal{F}^1 = 0$ *if and only if* $(f, \varphi_0) = 0$.

Proof. If $(f, \varphi_0) = 0$, then $\mathcal{F}^1 = 0$. Inversely, if $\mathcal{F}^1 = 0$, then $F_0 = 0$ and $\gamma^+F_0 - \varphi_0 = 0$. And $F_0 = 0$ implies $\varphi_0 = 0$ and then $\mathcal{P}_af = 0$ in Ω. Multiplying this by a and applying Laplace operator, we get $f = 0$. □

Integral Equation System (D2)

Boundary-domain Integral Equation System ($D2$) obtained from equations (2.30) and (2.32) is:

$$u(y) + \mathcal{R}_au(y) - V_a\psi(y) = F_0(y)\ ,\ y\ \in \Omega \tag{2.38}$$

$$\frac{1}{2}\psi(y) + T^+\mathcal{R}_au(y) - \mathcal{W}'_a\psi(y) = T^+F_0(y)\ \ ,\ y\ \in \partial\Omega \tag{2.39}$$

where F_0 is given by (2.37). The system can be written in matrix form as $\mathcal{A}^2\mathcal{U} = \mathcal{F}^2$,
where $\mathcal{U} := (u, \psi)^T \in H^{1,0}(\Omega, L_a) \times H^{-1/2}(\partial\Omega)$ and

$$\mathcal{A}^2 = \begin{bmatrix} I + \mathcal{R}_a & -V_a \\ T^+\mathcal{R}_a & \frac{1}{2}I - \mathcal{W}'_a \end{bmatrix} \ , \ \mathcal{F}^2 = \begin{bmatrix} F_0 \\ T^+F_0 \end{bmatrix}.$$

Note that the operator $\mathcal{A}^2 : H^{1,0}(\Omega; L_a) \times H^{-1/2}(\partial\Omega) \to H^1(\Omega) \times H^{-1/2}(\partial\Omega)$ is bounded.

Remark 2.4. $\mathcal{F}^2 = 0$ *if and only if* $(f, \varphi_0) = 0$.

2.5 Analysis of Boundary-Domain Integral Equations

Theorem 2.5.1. ***(Equivalence Theorem)***
Let $f \in L_2(\Omega)$ and $\varphi_0 \in H^{1/2}(\partial\Omega)$

1. *If some $u \in H^1(\Omega)$ solve BVP (2.4) – (2.5), then the pair (u, ψ), where*

$$\psi = T^+u \in H^{-1/2}(\partial\Omega); \tag{2.40}$$

   solves BDIE system ($D1$) and ($D2$).

2. *If pair $(u, \psi) \in H^1(\Omega) \times H^{-1/2}(\partial\Omega)$ solves BDIE system ($D1$), and diam$(\Omega) < 1$, then u solves BDIE ($D2$), and BVP (2.4) – (2.5), this solution is unique,and ψ satisfies (2.40).*

3. *If pair* $(u, \psi) \in H^1(\Omega) \times H^{-1/2}(\partial\Omega)$ *solves BDIE system* $(D2)$ *then* u *solves BDIE* $(D1)$*, and BVP* $(2.4)-(2.5)$*, this solution is unique, and* ψ *satisfies* (2.40).

Proof. 1. let $u \in H^1(\Omega)$ be solution of the BVP $(2.4)-(2.5)$. Since $f \in L_2(\Omega)$, we have that $u \in H^{1,0}(\Omega; L_a)$. Setting ψ by (2.40) and recalling how BDIE systems $(D1)$ and $(D2)$ were constructed, we obtain that (u, ψ) solve them.
Let now a pair $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve systems $(D1)$ *or* $(D2)$. Due to the first equations in the BDIE systems, the hypotheses of Lemma (2.4.1) are satisfying implying that u belongs to $H^{1,0}(\Omega; L_a)$ and solves PDE (2.4) in Ω, while the following equation also holds,

$$V_a(\psi - T^+u)(y) - W(\varphi_0 - \gamma^+u)(y) = 0, \quad y \in \Omega. \tag{2.41}$$

2. Let $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve system $(D1)$. Taking the trace of the first equation in $(D1)$ and subtracting the second equation from it, we get $\gamma^+u = \varphi_0$ on $\partial\Omega$. Thus the Dirichlet boundary condition is satisfied, and using it in (2.41), we have $V_a(\psi - T^+u)(y) = 0, \quad y \in \Omega$. Lemma 2.4.2$(i)$ then implies $\psi = T^+u$.

3. Let now $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve system $(D2)$. Taking the co-normal derivative of the first equation in $(D2)$ and subtracting the second equation from it, we get $\psi = T^+u$ on $\partial\Omega$. Then inserting this in (2.41) gives $W(\varphi_0 - \gamma^+u)(y) = 0, \quad y \in \Omega$ and Lemma 2.4.2(ii) implies $\varphi_0 = \gamma^+u$ on $\partial\Omega$.
The uniqueness of the BDIE system solutions follows form the fact that the corresponding homogeneous BDIE systems can be associated with the homogeneous Dirichlet problem, which has only the trivial solution. Then paragraphs (2) and (3) above imply that the homogeneous BDIE systems also have the trivial solutions.

□

Theorem 2.5.2. *If* $diam(\Omega) < 1$*, then the following operators are invertible.*

$$\mathcal{A}^1 : H^1(\Omega) \times H^{-1/2}(\partial\Omega) \to H^1(\Omega) \times H^{1/2}(\partial\Omega) \tag{2.42}$$

$$\mathcal{A}^1 : H^{1,0}(\Omega; \Delta) \times H^{-1/2}(\partial\Omega) \to H^{1,0}(\Omega; \Delta) \times H^{1/2}(\partial\Omega) \tag{2.43}$$

Proof. Theorem 2.5.1(2) implies that operators (2.42) and (2.43) are injective. To prove the invertibility, let us denote

$$\mathcal{A}_0^1 = \begin{bmatrix} I & -V_a \\ 0 & -\mathcal{V}_a \end{bmatrix}.$$

Then $\mathcal{A}_0^1 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is bounded. It is invertible due to its triangular structure and invertibility of its diagonal operators $I : H^1(\Omega) \to H^1(\Omega)$ and $-\mathcal{V}_a : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$. By compactness of $\mathcal{R}_a$ and $\mathcal{R}_a^+$, the operator

$$\mathcal{A}^1 - \mathcal{A}_0^1 = \begin{bmatrix} \mathcal{R}_a & 0 \\ \mathcal{R}_a^+ & 0 \end{bmatrix} : H^1(\Omega) \times H^{-1/2}(\partial\Omega) \to H^1(\Omega) \times H^{1/2}(\partial\Omega)$$

is compact, implying that operator (2.42) is Fredholm operator with zero index. Then the injectivity of operator (2.42) implies its invertibility. To prove invertibility of operator (2.43), for any $\mathcal{F}^1 \in H^{1,0}(\Omega; L_a) \times H^{1/2}(\partial\Omega)$, a solution of the equation $\mathcal{A}^1\mathcal{U} = \mathcal{F}^1$ can be written as $\mathcal{U} = (\mathcal{A}^1)^{-1}\mathcal{F}^1$, where $(\mathcal{A}^1)^{-1} : H^1(\Omega) \times H^{1/2}(\partial\Omega) \to H^1(\Omega) \times H^{-1/2}(\partial\Omega)$ is continuous inverse to operator (2.42). But due to Lemma 2.4.1 the first equation of system $(D1)$ implies that $\mathcal{U} = (\mathcal{A}^1)^{-1}\mathcal{F}^1 \in H^{1,0}(\Omega; L_a) \times H^{1/2}(\partial\Omega)$ and moreover, the operator $(\mathcal{A}^1)^{-1} : H^{1,0}(\Omega; L_a) \times H^{1/2}(\partial\Omega) \to H^{1,0}(\Omega; L_a) \times H^{-1/2}(\partial\Omega)$ is continuous, which implies invertibility of the operator (2.43). □

Theorem 2.5.3. *The following operators are invertible.*

$$\mathcal{A}^2 : H^1(\Omega) \times H^{-1/2}(\partial\Omega) \to H^1(\Omega) \times H^{-1/2}(\partial\Omega) \tag{2.44}$$

$$\mathcal{A}^2 : H^{1,0}(\Omega; L_a) \times H^{-1/2}(\partial\Omega) \to H^{1,0}(\Omega; L_a) \times H^{-1/2}(\partial\Omega) \tag{2.45}$$

Proof. Theorem 2.5.1 implies that operators (2.44) and (2.45) are injective. To prove the invertibility, let us consider the operator

$$\mathcal{A}_0^2 = \begin{bmatrix} I & -V_a \\ 0 & \frac{1}{2}I \end{bmatrix}.$$

Then $\mathcal{A}_0^2 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is bounded.It is invertible due to its triangular structure and invertibility of its diagonal operators $I : H^1(\Omega) \to H^1(\Omega)$ and $I : H^{-\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$. By compactness of $\mathcal{R}_a$ and $\mathcal{W}'_a$, the operator

$$\mathcal{A}^2 - \mathcal{A}_0^2 = \begin{bmatrix} \mathcal{R}_a & 0 \\ T^+\mathcal{R}_a & -\mathcal{W}'_a \end{bmatrix} : H^1(\Omega) \times H^{-1/2}(\partial\Omega) \to H^1(\Omega) \times H^{-1/2}(\partial\Omega)$$

is compact. This implies that operator (2.44) is Fredholm operator with zero index. Then the injectivity of operator (2.44) implies its invertibility. To prove invertibility of operator (2.45), for any $\mathcal{F}^2 \in H^{1,0}(\Omega; L_a) \times H^{-1/2}(\partial\Omega)$, a solution of the equation $\mathcal{A}^2\mathcal{U} = \mathcal{F}^2$ can be written as $\mathcal{U} = (\mathcal{A}^2)^{-1}\mathcal{F}^2$, where $(\mathcal{A}^2)^{-1} : H^1(\Omega) \times H^{-1/2}(\partial\Omega) \to H^1(\Omega) \times H^{-1/2}(\partial\Omega)$ is continuous inverse to operator (2.44). But due to Lemma 2.3.1 the first equation of system $(D2)$ implies that $\mathcal{U} = (\mathcal{A}^2)^{-1}\mathcal{F}^2 \in H^{1,0}(\Omega; L_a) \times H^{-1/2}(\partial\Omega)$ and moreover,the operator $(\mathcal{A}^2)^{-1} : H^{1,0}(\Omega; L_a) \times H^{-1/2}(\partial\Omega) \to H^{1,0}(\Omega; L_a) \times H^{-1/2}(\partial\Omega)$ is continuous, which implies invertibility of the operator (3.45). □

Conclusion and Future Work

In this paper, we have considered the interior Dirichlet problem for variable coefficient PDE in a two-dimensional domain, where the right hand side function is from $L_2(\Omega)$ and the Dirichlet data from the space $H^{1/2}(\partial\Omega)$. The BVP was reduced to two systems of Boundary Domain Integral Equations and their equivalence to the original BVP was shown. The invertibility of the associated operators in the corresponding Sobolev spaces was also proved.

An interesting feature is that the Dirichlet BVP for variable coefficient PDE can be equivalently reduce to four different systems of boundary domain integral equations in $2D$ and $3D$. Two of them segregated BDIEs, which is discussed in this paper and other two systems united boundary domain integro-differential equations, these systems to be done in the future study.

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