On The Theory of Abstract R-Vector spaces over a commutative regular ring

By

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Declaration

I, Litegebe Wondie with student number $GSR/1247/05$, here by declare that this thesis is my own work and that it has not previously been submitted for assessment or completion of any post graduate qualification to another university or for another qualification.

____________________________ Date ________________

Litegebe Wondie

Certificate

I, here by certify that I have read this dissertation prepared by Litegebe Wondie under my supervision and recommended that, it is accepted as fulfilling the dissertation requirement.

____________________________ Date ________________

Kolluru Venkateswarlu

Publications

- Litegebe Wondie, Zelalem Teshome and K. Venkateswarlu, *Special Homomorphims in R-Vector spaces*, International Mathematical Forum, Vol.9, no.18, 849-856, 2014. http://dx.doi.org/10.12988/imf.2014.4450
- Litegebe Wondie, Zelalem Teshome and K. Venkateswarlu, *Functionals in R-Vector Space*, International Journal of Algebra, Vol.9, no.10, 487-494, 2015. http://dx.doi.org/10.12988/ija.2015.5849
- Litegebe Wondie, Zelalem Teshome and K.Venkateswarlu, *Fractions in R-Vector spaces*, Submitted to Kyungpook Mathematical Journal, South Korea.

Contents

Abstract

The notion of an Abstract Boolean Vector space (an Abstract B-Vector space) is introduced by Subrahmanyam N.V. and he studied intensively on this spaces. Later N.Raja Gopala Rao introduced the concept of an Abstract R-Vector spaces as a generalization of Abstract Boolean Vector space of Subrahmanyam N.V. He introduced the notion of linear endomorphisms and affine transformations in Abstract R-Vector spaces and studied its properties. Further, he made a study on the geometric aspect of these spaces. Later K.Venkateswarlu introduced the notion of direct sums in Abstract R-Vector spaces and established that every direct sum of Abstract R-Vector spaces has a basis provided each component has a basis.

This thesis is a further continuation on the theory of Abstract R-Vector spaces. It is investigated by introducing special homomorphisms, strong special homomorphisms, bilinear maps and fractions in Abstract R-Vector Spaces. It is observed that special homomorphism is a normed Abstract R-Vector Space with a suitable norm. Certain properties regarding dual spaces has been obtained like some necessary and sufficient condition for two Abstract R-

Vector Spaces to be dual and some interesting results have been proved on fractions in Abstract R-Vector Spaces.

Chapter 1

Introduction

The study of autometrized Boolean algebras initiated by David Ellis [6,7] has been followed by the study of some special types of 'Boolean metric spaces' by several authors Blumenthal [4,5], Zemmer [31], Melter [20], Penning [24]. Blumental has studied the geometry of autometrized Boolean algebras by establishing theorems about betwenness and linearity and obtaining properties of segments. Zemmer has studied the geometry of p-spaces(= the Boolean metric space obtained from a p-ring (with unity)).

Zemmer has proved that a p-space has the property of free mobility if and only if the corresponding Boolean algebra of the p-ring is complete. He has also proved that every two congruent, finite subsets of a p-space are superposable. Penning has considered the Boolean metric spaces obtained from associate rings. He has extended results obtained by Blumenthal in [4,5] to these spaces. Melter has studied the results which continue the programme started in the paper [31] of Zemmer. He has obtained necessary and sufficient conditions for a subset of a p-space forms a metric basis and further he has exhibited an infinite minimal metric basis for a p-space. Further he has determined the congruence order of a p-space with respect to the class of all Boolean metric spaces. He has also determined the group of motions of a p-space and further he has shown that every rotation can be expressed as a product of a finite number of involutions.

Thus while the study of abstract Boolean metric space might be interesting in itself, the cases studied so far have been Boolean metric spaces with internal algebraic structures. It has been observed by Subrahmanyam that the several cases studied by Ellis, Blumenthal, Zemmer, Penning and Melter can be unified under the broad theory of what he called 'Abstract (normed) Boolean Vector spaces' [26,27,28]

Subrahmanyam has defined a Boolean Vector space in a way similar to that of the ordinary vector space over a field of real numbers and made an intensive study of these spaces in his papers [26,27]. He also noticed that any bounded Boolean extension V of an abelian group $G(=$ B-extension of G, where B is a Boolean algebra) can be made in to a normed Boolean Vector space by introducing a suitable multiplication of the elements of V by the elements of the corresponding Boolean algebra. In his paper [26,27], he has introduced a notion of a basis for a normed Boolean Vector space and proved that if G^* is a basis for a normed Boolean Vector space V, then $G = G^*$ together with the zero (of V) is a subgroup of V and further V is isomorphic with the bounded Boolean extension of G. Conversely, he also proved that any bounded Boolean extension of a group G is a normed Boolean Vector space with a basis isomorphic with G.

This result of Subrahmanyam, in particular, implies that a given abelian group G is isomorphic to the B-extension of a given abelian group $G^{'}$ if and only if it is possible to introduce a multiplication of elements of G by elements of B such that G together with this multiplication forms a normed Boolean Vector space with basis $G^{'}$.

Since a Boolean Vector space has been defined in a way similar to that of an ordinary Vector space over a field of real numbers, it is natural to ask whether it is possible to introduce the concept of a Vector space over a generalized Boolean ring and extend the results obtained in the case of a Boolean Vector space by Subrahmanyam to these generalised systems. Raja Gopala Rao [21] has investigated an answer to this problem by introducing the concept of an 'R-Vector space', where R is a commutative regular ring (of Von Neumann [30]) with 1. To this end he has imitated the idea of B-extension of an abelian group of Subrahmanyam and introduced the concept of "R-extension of a group G (definition 2.4.3) and this concept served as the generalization to the notion of B-extension of G. Also he observed that the R-extension V of a group G is again a group and just as subrahmanyam did in the case of B-extension of a group, it is possible to introduce a scalar multiplication of elements of V by elements of R and this scalar multiplication is found to have certain peculiar properties with respect to the addition in V, which will finally serve us as axioms to define a more general system called an 'Abstract R-Vector space'.

This notion of an 'Abstract R-Vector space' has become a generalisation to that of an 'Abstract Boolean Vector space' and stand in relation to a Boolean Vector space the same way as does the unitary space in relation to the real Vector space.

Thus, in this thesis we further investigate the study of R-Vector spaces by considering special homomorphisms, strong special homomorphisms, bilinear functionals (special class of special homomorphisms) and fractions of R-Vector spaces.

The rest of the thesis is divided into four chapters. In the second chapter, we collect certain definitions, examples and results concerning regular rings and Abstract R-Vector spaces from N.Raja gopala Rao [21].

In chapter three, we introduce the notion of special homomorphisms and furnish example that this notion is different from the notion of linear homomorphism of R-Vector spaces defined by Raja Gopala Rao [21]. Also we have proved that the set of all special homomorphisms, denoted by $SHom(V, W)$ where V and W are R-Vector spaces, forms an R-Vector space with suitable scalar multiplication. Further we proved that this $SHom(V, W)$ is a normed R-Vector space under suitable norm. We have also introduced the strong special homomorphism, which is denoted by $SSHom(V, W)$, and proved that the set of all strong speical homomorphisms forms a subspace of an R-Vector space $SHom(V, W)$. Finally we have obtained results concerning basis of R-Vector spaces and special homomorphisms.

In chapter four, we have introduced the notions of functionals, bilinear maps between two R-Vector space. We obtained some interesting properties regarding these notions. Also we have introduced the notion of dual spaces, inner product spaces of R-Vector spaces.

In chapter five, we have introduced the notion of fractions of R-Vector spaces and obtained that the fractions of R-Vector spaces is again an R-Vector space and also it is normed under suitable norm. Special class of subspaces were introduced and studied their properties. Finally we could obtain isomorphism theorems for the class of R-Vector spaces under strong special homomorphisms.

Chapter 2

Preliminaries

2.1 Introduction

In this chapter, we collect certain important definitions, examples and results from the existing literature on regular rings (of Von Neumann, [30]) and R-Vector spaces of Rajo Gopala Rao[21]. Here throughout this chapter B stands for Boolean Algebra of idempotents of a regular ring R.

2.2 Regular rings

Definition 2.2.1. *A ring R is called a regular ring if and only if to each $a \in R$ there is an element $x \in R$ such that $axa = a$.*

Definition 2.2.2. *An element a in a ring R is said to be nilpotent if and only if there is a positive integer n such that $a^n = 0$.*

Remark 2.2.3. *If R is a ring without non-zero nilpotent elements, then all the idempotents of R are in its centre.*

Definition 2.2.4. *A non empty subset S of a commutative regular ring R with 1 is said to be multiplicative if $1 \in S$ and $a, b \in S$ implies $ab \in S$.*

Example 2.2.5. *Every field F is a regular ring (for every $a \in F$, take $x = a^{-1}$).*

Example 2.2.6. *The ring $M_{2\times 2}$ of all 2×2 matrices whose entries are elements of Z_2 is a regular ring.*

Example 2.2.7. *Every Boolean ring is a regular ring.*

Note. The converse is not true. For instance, the regular ring given in example 2.2.5 is not a Boolean ring when the field $F \neq z_2$.

For a regular ring R we have the following

Lemma 2.2.8.
1. *If $a, x \in R$ and $axa = a$, then ax and xa are idempotents.*
2. *If all the idempotents of R are in its centre, then R has no non-zero nilpotent elements.*

Theorem 2.2.9. *On any regular ring R without non-zero nilpotent elements, there is a unique involution $a \to a^*$ with the property $aa^*a = a$*

If $a \in R$ and $axa = a$, put $a^* = xax$, then

Lemma 2.2.10.
1. *$aa^* = a^*a = ax = xa$;*
2. *If $a \in B$, then $a^* = a$ and $a^2 = a$;*
3. *$(-a)^* = -a^*$;*
4. *$a^*aa^* = a^*$;*
5. *$a^2a^* = a = a^*a^2$;*

6. $a(a^*)^2 = a^* = (a^*)^2 a$.

Corollary 2.2.11. *If* $a_i (i = 1, 2, ..., n) \in R$ *such that* $a_i a_j = 0$ *for all* $i \neq j$ *and* $\sum_{j=1}^{n} a_j = 0$*, then* $a_i = 0$ *for all* i*.*

Lemma 2.2.12. *1.* $a^{**} = a$

2. $(ab)^* = b^* a^*$

Corollary 2.2.13. *If* $ab = 0$*, then*

1. $ab^* = a^* b = a^* b^* = 0$

2. $b^* a = ba^* = b^* a^* = 0$

Lemma 2.2.14. *If* $ab = 0$*, then* $(a + b)^* = a^* + b^*$

Theorem 2.2.15. *Let* R *be any ring with* 1 *and* $*$ *be an involution on* R *satisfying for all* $a, b \in R$*, the properties (i)* $aa^*a = a$*. (ii)* $(ab)^* = b^* a^*$ *and (iii)* $(a + b)^* = a^* + b^*$ *if* $ab = 0$*. Then* R *has no non zero nilpotent elements and (hence by theorem 2.2.9) such* $'*'$ *is unique.*

Lemma 2.2.16. *1.* $0^* = 0$ *and* $1^* = 1$

2. If $b = b^2$*, then* $(1 - b)^* = 1 - b^*$

3. $(aa^*)^* = aa^*$ *and* $(a^*a)^* = a^*a$

4. $(1 - aa^*)^* = 1 - aa^*$ *and* $(1 - a^*a)^* = 1 - a^*a$

5. $(a + 1 - aa^*)^* = a^* + 1 - aa^*$

6. If $a^2 = 0$*, then* $a^* + 1 = (a + 1)^*$

7. If $a^2 = 0$*, then* $a = 0$

2.3 'Norm' of an element of a regular ring (with out non zero nilpotent elements) and its properties

Let R be a regular ring without non-zero nilpotent elements. For any $a \in R$, define norm of a, which will be denoted by $|a|$, by putting $|a| = aa^*$. It is clear that $|a| = a^*a$ and is well-defined. Following are the consequences of the definition of norm.

Lemma 2.3.1. *1. $|a| \in B$*

2. $|a| = a$ if and only if $a \in B$

3. $|a| = |a^*|$

4. If $ab = 0$, then $a|b|(= |b|a) = b|a|(= |a|b) = |a||b| = 0$

5. $|ab| = |a||b|(= |ba|)$

6. $|a + b| = |a| + |b|$ if $ab = 0$ (or equivalently if $ba = 0$)

For any $a, b \in B$, define $a < b$ if and only if $ab = a$(or equivalently $ba = a$). Then one can easily obtain '$<$' is a partial ordering on B.

Corollary 2.3.2. *1. $|a|$ is the 'minimal idempotent duplicator' of a in the sense that $|a|a = a$ and if $x \in B$ and $xa = a$, then $|a| < x$*

2. $|a|a^* = a^*$

Lemma 2.3.3. *1. $|a| = 0$ if and only if $a = 0$*

2. $|a| = |-a|$

3. $|a| + |b| - |a||b| \in B$ and $|a + b| < |a| + |b| - |a||b|$

Lemma 2.3.4. *1.* $|a| + |1-a| - |a||1-a| = 1$

2. $|a|(a+1-|a|) = a$

Let U denote the set of all invertible elements of R. Then the following

Lemma 2.3.5. *1.* $|a| = 1$ *if and only if* $a \in U$ *and further if* $a \in U$*, then* $a^* = a^{-1}$

2. $a \in U$ *if and only if* $a^* \in U$

3. $|a+1-|a|| = 1$

4. $(a+1-|a|)^* = a^* + 1 - |a|$

5. $a+1-|a| \in U$ *with* $a^* + 1 - |a|$ *as its inverse*

6. $(a+1-|a|)(b+1-|b|) \in U$ *and is the form* $c+1-|c|$*, where* $c = ab + a(1-|b|) + (1-|a|)b$

Lemma 2.3.6. *1.* $|1+a^*| = |1+a|$

2. $|1-a^*| = |1-a|$

3. To each $a \in R$*, we write* $\alpha_a = |a-a^2| = |a||1-a|$*. Then* $1-\alpha_a \in B$ *and* $(1-\alpha_a)a$ *also belongs to* B

2.4 Abstract R-Vector space

In this section we recall the following definitions from Subrahmanyam [26].

Definition 2.4.1. *Let* G *be a group and* $B = (B, \vee, \wedge, ')$ *(in what follows, for any* $a, b \in B$*, we write* ab *instead of* $a \wedge b$*,* $a+b+ab$ *instead of* $a \vee b$*,* $1+a$ *instead of* a'*) is a Boolean algebra with greatest* *element* 1. *The Bounded Boolean extension (or B-extension) of the group* G *means the set* $\tilde{G}$ *of all mappings* $f : G \to B$ *with the properties*

- *P1:* $f(x) = 0$ *except for finite number of elements* $x \in G$
- *P2:* $f(x)f(y) = 0$ *for all* $x, y \in G$ *with* $x \neq y$
- *P3 :* $\bigvee\limits_{x \in G} f(x) = 1$

Remark 2.4.2. *If for any* $f, g \in \tilde{G}$, *define* $f + g$ *by*
$(f + g)(x) = \bigvee_{\alpha+\beta=x} f(\alpha)g(\beta)$ *for any* $x \in G$, *then* $(\tilde{G}, +)$ *is a group.*

Definition 2.4.3. *Let* R *be a commutative regular ring without non zero nilpotent elements and with* 1. *By an R-extension of a group* G, *means the set* V *of all mappings*
$f : G \to R$ *with the properties*

- *P1:* $f(x) = 0$ *except for finite number of elements* $x \in G$
- *P2:* $f(x)f(y) = 0$ *for all* $x, y \in G$ *with* $x \neq y$
- *P3 :* $\sum\limits_{x \in G} |f(x)| = 1$

Definition 2.4.4. *Let* $\tilde{R}$ *be an R-extension of a group G. Then addition and multiplication in* $\tilde{R}$ *are defined by*

1 : $(f + g)(x) = \sum\limits_{\alpha+\beta=x} f(\alpha)g(\beta)$

2 : $(fg)(x) = \sum\limits_{\alpha\beta=x} f(\alpha)g(\beta)$

for all $f, g \in \tilde{R}$ *and* $x \in G$

Let $V = (V, +)$ be an abelian group and $B = (B, +, ., ')$ be a Boolean algebra. Let 0 be the 'null-element' of B and 1 be the 'universal element' of B.

Definition 2.4.5. *V is said to be a 'Boolean Vector space over B' (or simply a B-Vector space) if and only if there exists a mapping $\cdot : B \times V \to V$ (the image of (a, x) will be denoted by ax) such that, for all $x, y \in V$ and $a, b \in B$,*

1. *$a(x + y) = ax + ay$;*
2. *$(ab)(x) = a(bx)$;*
3. *$1x = x$;*
4. *if $ab = 0$, then $(a + b)x = ax + bx$.*

Remark 2.4.6. *Even if V is an arbitrary group(i.e., not necessarily abelian), we can extend the definition of a Boolean Vector space by retaining the same postulates in definition 2.4.5 above.*

Example 2.4.7. *Let V be a group (not necessarily abelian) and B be the (trivial) Boolean algebra of two elements 0 and 1. Define for any $x \in V$, $0x = 0$ and $1x = x$. Then obviously all the axioms in definition 2.4.5 are satisfied and hence, V is a 'Boolean Vector space' over B.*

Definition 2.4.8. *If V is normed, then*

1. *$[V] = \{|x| : x \in V\}$*
2. *$V_a = \{x : |x| < a\}$ for each $x \in V$ and $a \in B$.*

Let $V = (V, +)$ be any arbitrary group and $R = (R, +, .)$ be a commutative regular ring with unity element 1.

Definition 2.4.9. *V is said to be a Vector space over R*

(or 'an Abstract R-Vector space') if and only if there exist a mapping :$R \times V \to V$ *(the image of any* $(a, x) \in R \times V$ *will be denoted by* ax *) such that for all* $x, y \in V$ *and* $a, b \in R$*, all the following axioms hold.*

1. $a^2(x + y) = ax + ay$;
2. $a(bx) = (ab)x$ *if* $a^2 = a$;
3. $1x = x$;
4. $(a + b)x = ax + bx$ *if* $ab = 0$;
5. $r(sx) = (rs)x$ *if* r *and* s *are invertible elements of* R *and*
6. $a0 = 0$ *imply* $a^2 = a$

All the elements of V will be referred to as 'Vectors' and those elements of R are 'scalars' and the multiplication 'ax' of the elements of V by the elements of R will be referred to as scalar multiplication. Throughout the remaining part of this chapter, when we say V is a vector space over R, we mean V is an abstract R-Vector space in the sense of the definition 2.4.9.

Example 2.4.10. *Let* G *be any group and* R *be a commutative regular ring with* 1*. Suppose* V *is the* R*-extension of* G*. Then* V *satisfies all the axioms of the definition 2.4.9. Thus* V *is a vector space over* R*.*

Example 2.4.11. *Let* R *be the ring* $\{0, 1, 2\}$ *of residues modulo* 3*, which is a commutative regular ring with* 1 *and let* V *be the group* $\{0, 1, 2, 3\}$ *of addition modulo* 4*. Define the scalar multiplication by putting* $0x = 0,\ 1x = x,\ 2x = x + 2$ *for all* $x \in V$*. Then it can be* *verified that V is a Vector space over R.*

Remark 2.4.12. *(1). The above example is that of an R-Vector space which can not reduce to any R-extension of a group.*
(2). Any Boolean Vector space is an R-Vector space.

Example 2.4.13. *Let F be a field (which is obviously a commutative regular ring with 1) and F^* be its multiplicative group. Define for any $0 \neq a \in F$ and $x \in F^*$, the scalar product $ax =$ the product of a and x in F and $0x = 1$. Then obviously F^* is an F-Vector space.*

Example 2.4.14. *Let R be a commutative regular ring with 1 and U be its multiplicative group of invertible elements. Define for any $a \in R$ and $x \in U$, the scalar multiplication of a and x, which will be denoted by $a \otimes x$, by putting $a \otimes x = ax + 1 - |a|$. Then the mapping is well-defined (since $|ax+1-|a|| = |ax|+|1-|a|| = |a||x|+1-|a| = 1$, $ax + 1 - |a| \in U$). Then U is a vector space over R with respect to this scalar multiplication.*

Remark 2.4.15. *Raja Gopala Rao [21] remarked that any R-Vector space can be treated as a B-Vector space, where B is the Boolean algebra of idempotents of R with respect to the same scalar multiplication as in the R-Vector space.*

Let V be a vector space over a regular ring R. Then in the following Lemma we collect some known results as in ([21]) on abstract R-Vector spaces.

Lemma 2.4.16. ([21]) *Let V be an abstract R-Vector space. Then*

1. $0x = 0$

2. *if* $a \in B$*, then* $a0 = 0$

3. *if* $a \in B$*, then* $a(-x) = -ax$*,* $a(x + y) = ax + ay$
   and (hence) $a(x - y) = ax - ay$

4. *if* $a \in B$*, then* $(1 - a)x = x - ax = -ax + x$

5. *if* $a, b \in B$*, then* $(a \vee b)x = ax + bx - abx = ax - abx + bx$*.*

Lemma 2.4.17. ([21])

1. $a^2 0 = a0 + ax = ax + a0$ *and*
   (hence, in particular) $a^2 0 = a0 + a0$

2. $-ax = -a^2 0 + a(-x) = a(-x) - a^2 0$

3. $-a(-x) = -a^2 0 + ax = ax - a^2 0$

Lemma 2.4.18. ([21])

1. $ax - ay = a(x - y) - a0 = -a0 + a(x - y)$

2. $-ay + ax = -a0 + a(-y + x) = a(-y + x) - a0$

Lemma 2.4.19. ([21]) *If* $a, b \in U$*, then*

1. $ax - a0 = x = -a0 + ax$

2. $ax - ay = x - y$

3. $-ay + ax = -y + x$

4. $a(x + y) = x + ay = ax + y$

5. $ax + by = (ab)(x + y)$

Lemma 2.4.20. ([21]) *If* $|a| = |b|$*, then* $ax + by = (ab)(x + y)$

Corollary 2.4.21. ([21]) *If* $|a| = |b|$*, then*

1. $ax - by = (ab^*)(x - y)$

2. $-by + ax = (ab^*)(-y + x)$

Lemma 2.4.22. ([21]) $ax - |a|x = a0 = -|a|x + ax$

Lemma 2.4.23. ([21]) *If* $ab = 0$*, then* $ay + bx = bx + ay$

Corollary 2.4.24. *Any 'n' vectors of the form* $\pm a_1x_1, \pm a_2x_2, ..., \pm a_nx_n$*, where* $a_ia_j = 0$ *for* $i \neq j$*, will commute with each other(and hence write the sum of* $a_1x_1, a_2x_2, ..., a_nx_n$ *in any order).*

Theorem 2.4.25. V *is an R-module if and only if R is a Boolean ring and* $x + x = 0$ *for all* $x \in V$*.*

Definition 2.4.26. *Let* V*,* W *be R-Vector spaces. A mapping* $T : V \to W$ *is a linear homomorphism of* V *to* W *provided* $T(ax+by) = aTx+bTy$ *if* $ab = 0, \forall x, y \in V$ *and* $a, b \in R$*. The set of linear homomorphisms from* V *to* W *will be denoted by* $Hom(V, W)$*.*

Definition 2.4.27. *Let* V *be R-Vector space. A mapping* $T : V \to V$ *is called a linear endomorphism of* V *if and only if* $x, y \in V$*,* $a, b \in R$ *and* $ab = 0$ *imply* $T(ax + by) = aTx + bTy$ *. The set of such linear endomorphisms will be denoted by* $Hom(V, V)$*.*

Definition 2.4.28. *Let* W *and* U *be R-Vector spaces. Then the mapping* $T : W \to U$ *is a strongly linear homomorphism if* $T(ax + by) = aTx + bTy$ *for all* $a, b \in R$ *and* $x, y \in W$*.*

Definition 2.4.29. *Let* $V_1, V_2, ..., V_n$ *be vector spaces over regular ring R. The direct sum,* $\sum_{i=1}^{n} V_i$*, of the spaces* $V_1, V_2, ..., V_n$ *is defined to be the Cartesian product set* $\{(x_1, x_2, ..., x_n) : x_i \in V_i\}$ *on which addition and scalar multiplication are defined as follows:*

1. $(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$

2. $a(x_1, x_2, ..., x_n) = (ax_1, ax_2, ..., ax_n)$

Theorem 2.4.30. *If $V_1, V_2, ..., V_n$ are vector spaces over regular ring R, then $\sum_{i=1}^{n} V_i$ is a vector space over R.*

Theorem 2.4.31. *If $V_1, V_2, ..., V_n$ are vector spaces over the same regular ring R having basis $G_1^*, G_2^*, ..., G_n^*$ respectively, then $(\sum_{i=1}^{n} G_i)^*$ is a basis for $\sum_{i=1}^{n} V_i$, where $\sum_{i=1}^{n} G_i$ is considered as the direct sum of additive groups.*

2.5 Normed R-Vector spaces

In this section we recall the definition of 'normed R-Vector space', examples of normed R-Vector space and its properties.

Definition 2.5.1. *An R-Vector space V is said to be B-normed (or simply normed) if and only if there exist a mapping $|.| : V \to B$ (called norm) satisfying the following properties:*

1. *$|x| = 0$ if and only if $x = 0$ and*
2. *$|ax| = a|x|$ for all $x \in V$ and $a \in B$*

Lemma 2.5.2. *If V is a normed R-Vector space, then $|-x| = |x|$, for all $x \in V$.*

Example 2.5.3. *Let V be the R-extension of any group G. Then as we have seen in example 2.4.10, V is an R-Vector space. Now for any $f \in V$, put $|f| = |1 - f(0)|$, then V is normed.*

Example 2.5.4. *Let V be the R-Vector space as in the example 2.4.11. For any $x \in V$, $|x| = 1,\ x \neq 0$ and $|x| = 0,\ x = 0$. Then V is normed.*

Example 2.5.5. *The R-Vector space U as in example 2.4.14 is* *normed by putting* $||x|| = |1 - x|$ *for any* $x \in U$.

Remark 2.5.6. *Here any* $x \in U$ *is a vector as well as a scalar and hence to avoid confusion, the symbol* $||x||$ *to denote the norm of the vector* x *and* $|x|$ *for the norm of the scalar* x.

Theorem 2.5.7. *In any R-Vector space* V*, the following statements are equivalent*

1. V *is B-normed*
2. *To each* $x \in V$*, there exist a 'minimal idempotent duplicator'* a_x*, i.e., there is an element* $a_x \in B$ *such that* (i) $a_x x = x$*, and* (ii) *if* $b \in B$ *and* $bx = x$*, then* $a_x < b$*. (Such* a_x *, given* x*, is clearly unique)*

The following is an observation: if (1) holds, then $|x|$ is the a_x required in (i) of (2) and conversely if (i) and (ii) holds, then a_x is actually $|x|$ for all $x \in V$. Therefore the above theorem can be restated as follows

Theorem 2.5.8. *If* V *is B-normed, then for any* $x \in V$*,* $|x|$ *is the 'minimal idempotent duplicator' of* x*, i.e.,* $|x|x = x$ *and if* $b \in B$ *and* $bx = x$*, then* $|x| < b$*, and conversely, if for each* $x \in V$*, there is an element* $a_x \in B$ *satisfying the properties* (i) *and* (ii) *of* (2) *of theorem 2.5.7, then* V *is B-normed and* $a_x = |x|$ *for all* $x \in V$.

Corollary 2.5.9. *If* V *is a normed R-Vector space, then* $|x + y| < |x| + |y| - |x||y|$.

Corollary 2.5.10. *If* V *is a normed R-Vector space, then* $d(x, y) = |x - y|$ *defines a Boolean metric on* V*, i.e.,* $d(x, y) = 0$ *if and only if* $x = y$*,* $d(x, y) = d(y, x)$ *and* $d(x, z) < d(x, y) + d(y, z) -$

$d(x, y)d(y, z)$.

Theorem 2.5.11. *If in a normed R-Vector space V, $ax = bx$ and $|a| = |b|$, then $a = b$.*

Lemma 2.5.12. *If V is normed, and $a_1, a_2, ..., a_n \in R$ such that $a_i a_j = 0$ if $i \neq j$ and $x = \sum_{i=1}^{n} a_i x_i$, then $|x| = \sum_{i=1}^{n} |a_i x_i|$.*

Corollary 2.5.13. *If $a \in U$, then $|a0| = |1 - a|$.*

Corollary 2.5.14. *If $a \in U$ and $a0 = 0$, then $a = 1$.*

Definition 2.5.15. *If G^* is a basis of V and $g \in G^*$, then*
(i). $|g| = 1$ and (ii). $-g \in G^*$.

Definition 2.5.16. *A finite subset of non zero elements $x_1, ..., x_n$ of an R-Vector space V is called (linearly) independent over R if and only if $a_1x_1 + ... + a_nx_n = 0$ and $a_1...a_n \neq 0$ imply $x_1 + ... + x_n = 0$ and a subset S (of non zero elements)of V is said to be an independent subset of V if and only if every finite subset of S is linearly independent over R.*

Definition 2.5.17. *If S is a subset of V, we say S spans V if and only if each $x \in V$ can be written as $x = \sum_{g \in S \cup \{0\}} a_g g$, where $a_g a_h = 0$ if $g, h \in S \cup \{0\}$ and $g \neq h$, $a_g = 0$ for almost all $g \in S \cup \{0\}$ and $\sum_{g \in S \cup \{0\}} |a_g| = 1$*

Definition 2.5.18. *Let V be an R-Vector space. An independent subset of V which spans V is called basis for V.*

Remark 2.5.19. *If G is a basic group for V, then $a_{x,g}$ represent the uniquely determined coefficient of any $g \in G$ in the representation of any vector x in terms of G and $x = \sum_{g \in G} a_{x,g} g$, where $a_{x,g} a_{x,h} = 0$ for $g \neq h$.*

Theorem 2.5.20. *Any R-extension of a group is an R-Vector space with a basis and conversely any R-Vector space with a basis is isomorphic to the R-extension of a suitable group.*

Lemma 2.5.21. *If* $c \in R$, *then* $a_{cx,g} = ca_{x,g}$, $\forall g \in G^*$ *and* $a_{cx,0} = 1 - |x| + ca_{x,0}$.

Lemma 2.5.22. *Let* S *be any subset of* V *and* $x \in V$. *Suppose* $x = \sum_{g \in S} a_g g$, *where* $a_g a_h = 0$ *for* $g, h \in S$ *with* $g \neq h$, $a_g = 0$ *for almost all* $g \in S$ *and* $\sum_{g \in S} |a_g| = 1$. *Then* $cx = \sum_{g \in S} (ca_g) g$ *for all* $c \in R$.

Corollary 2.5.23. *If* $c \in U$, *then* $a_{cx,g} = ca_{x,g}$, $\forall g \in G$.

Chapter 3

Special Homomorphisms in R-Vector space

3.1 Introduction

N.Raja Gopala Rao [21,22] introduced the concept of Vector spaces over regular rings (simply R-Vector spaces) as a generalization of Boolean Vector space of Subrahmanyam N.V [26]. He introduced the notion of linear endomorphisms and affine transformations in R-Vector spaces and studied its properties. Further he made a study on the geometric aspect of these spaces. Later K.Venkateswarlu [16] introduced the notion of direct sums in R-Vector spaces and established that every direct sum of R-Vector spaces has a basis provided each component has a basis. The set of all linear endomorphisms denoted by $Hom(V, V)$, where V is an R-Vector space in [17], is not closed under usual addition of linear endomorphisms. Keeping this in view, we introduce the notion of special homomorphisms in R-Vector spaces and studied its properties. The rest of this chapter is divided in two sections. In the second section, we introduce the notion of special homomorphisms in R-Vector spaces and study its certain properties. We give some examples of special homomorphisms which do not subsume the notion of linear endomorphism of Raja Gopala Rao [21,22] and B-Vector space of Subrahmanyam [26] (see example 3.2.8). We prove that the set of all special homomorphisms in R-Vector space forms an R-Vector space (see theorem 3.2.13). We also introduce the notion of strong special homomorphism and in theorem 3.2.18, we prove that the set of all strong special homomorphisms form a subspace. In section three, we characterize bases in R-Vector spaces and further we prove that $SHom(V, W)$ is isomorphic to the finite direct sum of W's (see theorem 3.3.6). Even though the condition 6 of definition 2.4.9 is independent of the other conditions 1 through 5, which was substantiated by an example in [21,22] given by Raja Gopala Rao, it can be relaxed without affecting the validity of the results obtained in [21,22]. We restate the definition of R-Vector space below by relaxing the condition 6 and adopt the same nomenclature given by Raja Gopala Rao [21,22].

3.2 Definition and Examples of Special Homomorphism

In this section we restate the definition of R-Vector spaces by relaxing the condition (6) of definition 2.4.9, keeping the nomenclature given by Raja Gopala Rao [21].

Definition 3.2.1. *Let* $V = (V, +)$ *be any group and* $R = (R, +, .)$ *be a commutative regular ring with unity element* 1. *Then* V *is said to be a Vector space over* R *(or simply* R*-Vector space) if and only if there exists a mapping* $\cdot : R \times V \to V$ *(the image of any* $(a, x) \in R \times V$ *will be denoted by* ax*) such that for all* $x, y \in V$ *and* $a, b \in R$*, all the following properties hold:*

1. $a^2(x + y) = ax + ay$
2. $a(bx) = (ab)x$ *if* $a^2 = a$
3. $1x = x$
4. $(a + b)x = ax + bx$ *if* $ab = 0$
5. $r(sx) = (rs)x$ *if* r *and* s *are invertible elements of* R

Raja Gopala Rao remarked as follows

Remark 3.2.2. *It may appear that the axiom 6 follows from the remaining axioms of 1 to 5 of definition 2.4.9. But in reality it is not true. The axiom 6 is independent of the axioms of 1 to 5.*

Example 3.2.3. *Let* R *be any commutative regular ring with 1, which is not a Boolean ring and* $V = (V, +)$ *be the additive group of* R*. For any* $a \in R$*,* $x \in V$*, define the scalar multiplication of* a *and* x*, which will be denoted by* $a \otimes x$*, by putting* $a \otimes x = |a|x$*, the ring product of* $|a|$ *and* x*. Then it is easy to verify that all the axioms* 1 - 5 *of definition 2.4.9 are satisfied. But* 6 *is not true for this system.*

Throughout the remaining chapters of this thesis, when we say V is a vector space over R, we mean V is an abstract R-vector space in the sense of the definition 3.2.1.

Remark 3.2.4. *The set of all linear endomorphisms defined by Raja Gopala Rao [21] in definition 2.4.27 is not closed under the binary operation* $'+'$.

Consider the following example

Example 3.2.5. *Let* $V = (Z_6, +)$ *be the group of addition modulo* 6 *and* $R = (Z_3, +, .)$ *be the ring of residues modulo* 3*, which is a commutative regular ring with* 1*. Define scalar multiplication as* $\cdot : R \times V \to V$*, by* $0x = 0$*,* $1x = x$*,* $2x = x + 3$*, for all* $x \in V$*. Define* $T, S : V \to V$ *by* $T(0) = 0$*,* $T(1) = 5$*,* $T(2) = 1$*,* $T(3) = 3$*,* $T(4) = 2$*,* $T(5) = 4$ *and* $S(0) = 0$*,* $S(1) = 4$*,* $S(2) = 2$*,* $S(3) = 3$*,* $S(4) = 1$*,* $S(5) = 5$*. Letting* $a = 0$*,* $b = 2$ *and* $y = 1$ *in definition 2.4.27,* $(T + S)(0x + 2y) = 3 \neq 0 = 0(T + S)x + 2(T + S)y$*. Thus* $(T + S) \notin Hom(V, V)$.

Remark 3.2.6. *In view of the above example 3.2.5, we introduce the notion of special homomorphism in R-Vector spaces which in turn gives the set of all special homomorphisms closed under the binary operation* $'+'$ *(see theorem 3.2.12).*

Definition 3.2.7. *Let* V *and* W *be R-Vector spaces. A mapping* $T : V \to W$ *is a special homomorphism of* V *to* W *provided* $T(ax + by) = \mid a \mid Tx + \mid b \mid Ty$ *if* $ab = 0$*,* $\forall x, y \in V$ *and* $a, b \in R$.

Notation. *The set of all special homomorphisms from* V *to* W *will be denoted by* $SHom(V, W)$.

Example 3.2.8. *Let* V *be an R-Vector space as in example 3.2.5 above and* $T : V \to V$ *be a map defined by* $T(0) = 0 = T(3)$*,*

$T(1) = 2 = T(4)$, $T(2) = 3 = T(5)$. *Then by easy verification of definition 3.2.7, T is a special homomorphism.*

Remark 3.2.9. *It seems to be that special homomorphism defined above coincides with the linear homomorphism in B-Vector space of Subrhamanyam, N.V.[26]. But this is not the case. For instance in any B-Vector space the property $(a + b)x = ax + bx - abx$ holds. However this is not true in the case of special homomorphism of R-Vector spaces. It is justified in the following example.*

Example 3.2.10. *Let V be an R-Vector space as in example 3.2.5 and $T : V \to V$ be a map defined by $T(0) = 0 = T(3)$, $T(1) = 2 = T(4)$, $T(2) = 3 = T(5)$. Let $a = 1, b = 2$ in Z_3 and $x = 1$ in Z_6. Clearly by the definition of scalar multiplication in example 3.2.5 and properties of Z_3, Z_6, we will have*

$$(1 + 2)T1 = 3T1 = 0T1 = 0$$

but

$$1T1 + 2T1 - (1.2)T1 = 1T1 + 2T1 - (1)(2)T1 = 2$$

We have the following

Theorem 3.2.11. *Let V and W be R-Vector spaces and T be a mapping from V to W. $T \in SHom(V, W)$ if and only if $T(ax) =\mid a \mid Tx$, for all $x \in V$ and $a \in R$.*

Proof. ($\Rightarrow$) Suppose $T \in SHom(V, W)$. Put $b = 0$ in definition 3.2.7, we have $T(ax) =\mid a \mid Tx$ for all $x \in V$ and $a \in R$.

($\Leftarrow$) Suppose $T(ax) =\mid a \mid Tx$ for all $x \in V$ and $a \in R$ such that $ab = 0$.

Then

$$
\begin{aligned}
T(ax+by) &= (|a|+1-|a|)T(ax+by) \quad (\text{since } |a|+1-|a|=1)\\
&= |a|T(ax+by)+(1-|a|)T(ax+by) \quad (\text{by 4 of definition 2.4.5})\\
&= ||a||T(ax+by)+|1-|a||T(ax+by) \quad (\text{by 2 of lemma 2.3.1})\\
&= T(|a|(ax+by))+T((1-|a|)(ax+by)) \quad (\text{by hypothesis})\\
&= T[(|a|a)x+(|a|b)y)]+T[(1-|a|)ax+(1-|a|)by]\\
&= T(ax)+T(by) \quad (\text{since } ab=0 \text{ and } |a|a=a)\\
&= |a|Tx+|b|Ty \quad (\text{ by hypothesis})
\end{aligned}
$$

Hence T is a special homomorphism from V to W. □

Theorem 3.2.12. *Let V and W be R-Vector spaces. Define '+' on $SHom(V,W)$ by $(T+S)(x) = Tx+Sx$ for all $x \in V$ and $T,S \in SHom(V,W)$. Then $(SHom(V,W),+)$ is an abelian group.*

Proof. 1. Let $T,S \in SHom(V,W)$. Then for $a,b \in R$ such that $ab=0$,

$$
\begin{aligned}
(T+S)(ax+by) &= T(ax+by)+S(ax+by) \quad (\text{by hypothesis})\\
&= (|a|Tx+|b|Ty)+(|a|Sx+|b|Sy) \quad (\text{by hypothesis})\\
&= |a|(Tx+Sx)+|b|(Ty+Sy) \quad (\text{since } |a|,|b| \in B)\\
&= |a|(T+S)x+|b|(T+S)y \quad (\text{by hypothesis})
\end{aligned}
$$

Hence $T+S \in SHom(V,W)$.

2. For $T,S,K \in SHom(V,W)$, it is routine verification of $(T+S)+K = T+(S+K)$ and $T+S = S+T$.

3. Let $\overline{0} : V \to W$ be a map defined by $\overline{0}x = 0$ for all $x \in V$. Since $\overline{0}(ax) = 0 = |a|\overline{0}x$ for all $x \in V$ and $a \in R$, $\overline{0} \in SHom(V,W)$

(by theorem 3.2.11). For each $T \in SHom(V,W)$, we have

$$\begin{aligned}(T+\overline{0})(ax+by) &= T(ax+by)+\overline{0}(ax+by)\\ &= |a|Tx+|b|Ty+|a|\overline{0}x+|b|\overline{0}y\\ &= |a|Tx+|b|Ty\\ &= T(ax+by)\end{aligned}$$

Hence, $\overline{0}$ is the additive identity in $SHom(V,W)$.

4. For each $T \in SHom(V,W)$, let $-T : V \to W$ be a map defined by $(-T)x = -(Tx)$ for each $x \in V$. If $T \in SHom(V,W)$, $x \in V$ and $a \in R$, then $(-T)(ax) = -(T(ax))$.
   By theorem 3.2.11,

$$\begin{aligned}-(T(ax)) &= -(|a|Tx)\\ &= |a|(-(Tx)) \quad \text{(by 3 of lemma 2.4.16).}\end{aligned}$$

   Hence, $(-T)(ax) = |a|(-(Tx)) = |a|((-T)x)$. Then $-T \in SHom(V,W)$ (by theorem 3.2.11).

$$\begin{aligned}\textit{Since } (T+(-T))(ax+by) &= T(ax+by)+(-T)(ax+by)\\ &= |a|Tx+|b|Ty-|a|Tx-|b|Ty = 0,\end{aligned}$$

   T has an additive inverse in $SHom(V,W)$.

Hence the theorem holds. □

Theorem 3.2.13. *Let V and W be R-Vector spaces. Then $SHom(V,W)$ is an R-Vector space if the scalar multiplication is defined by $(aT)(x) = |a|Tx$, for all $x \in V$, $T \in SHom(V,W)$ and $a \in R$.*

Proof. 1. Let $S,\ T \in SHom(V,W),\ a \in R\ and\ x \in V$. Then

$$\begin{aligned}(a^2(T+S))(x) &= |a^2|(T+S)x \\ &= |a|^2(Tx+Sx) \\ &= |a|Tx+|a|Sx \\ &= (aT)(x)+(aS)(x)\end{aligned}$$

2. For $T \in SHom(V,W),\ a,b \in R\ and\ x \in V$,

$$\begin{aligned}(a(bT))(x) &= |a|(bT)x \\ &= |a|(|b|Tx) \\ &= (|a||b|)Tx \\ &= |ab|Tx \\ &= ((ab)T)(x)\end{aligned}$$

3. $(1T)(x) = |1|Tx = Tx$

4. Let $a,b \in R$ with $ab = 0$, $x \in V$ and $T \in SHom(V,W)$. Then

$$\begin{aligned}((a+b)T)(x) &= |a+b|Tx \\ &= |a|Tx+|b|Tx \\ &= (aT)(x)+(bT)(x) \\ &= (aT+bT)(x)\end{aligned}$$

5. let $r,s \in R$ be invertible elements. Then

$$\begin{aligned}(r(sT))(x) &= |r|(sT)x \\ &= |r|(|s|Tx) \\ &= (|r||s|)Tx \\ &= |rs|Tx \\ &= ((rs)T)x.\end{aligned}$$

Hence $SHom(V, W)$ is an R-Vector space.

□

Definition 3.2.14. *Let V, W be R-Vector spaces. A mapping $T : V \to W$ is a strong special homomorphism of V to W provided $T(ax + by) = |a|Tx + |b|Ty$, for all $x, y \in V$ and $a, b \in R$.*

Notation. *The set of all strong special homomorphisms from V to W will be denoted by $SSHom(V, W)$.*

We characterize strong special homomorphism in the following

Theorem 3.2.15. *Let V, W be R-Vector spaces and $T \in SHom(V, W)$. Then $T \in SSHom(V, W)$ if and only if $T(x + y) = Tx + Ty$, for all $x, y \in V$.*

Proof. Letting $a = b = 1$ in definition 3.2.14, we have

$$T(x + y) = Tx + Ty$$

Conversely, let $T(x + y) = Tx + Ty$, for all $x, y \in V$. Then for all $a, b \in R$,

$$\begin{aligned}T(ax + by) &= T(ax) + T(by) \quad \text{(by hypothesis)} \\ &= |a|Tx + |b|Ty \quad \text{(by theorem 3.2.11)}\end{aligned}$$

Hence, T is strong special homomorphism. □

Definition 3.2.16. *Let V be an R-Vector space. A non empty subset W of V is called a sub R-Vector space of V if*

1. *For $x, y \in W$, $x - y \in W$.*
2. *For $a \in R$ and $x \in W$, $|a|x \in W$.*

Remark 3.2.17. *If V is an R-Vector space and W is sub R-Vector space of V, then W is itself an R-Vector space.*

Theorem 3.2.18. *Let V and W be R-Vector spaces. Then $SSHom(V, W)$ is a subspace of $SHom(V, W)$.*

Proof.

$$Since\ \tilde{0}(ax + by) = 0$$
$$= | a | 0 + | b | 0$$
$$= | a | \tilde{0}x + | b | \tilde{0}y \quad (\text{ for all } x, y \in V \text{ and } a, b \in R).$$

Thus, $\tilde{0} \in SSHom(V, W)$. Hence, $SSHom(V, W) \neq \emptyset$.

Now, let $T, S \in SSHom(V, W)$ and $x, y \in V$.

$$(T - S)(x + y) = (T + (-S))(x + y)$$
$$= T(x + y) + (-S)(x + y) \quad (\text{ by theorem 3.2.12})$$
$$= Tx + Ty + (-S)x + (-S)y \quad (\text{ by theorem 3.2.15})$$
$$= (T - S)x + (T - S)y \quad (\text{ by theorem 3.2.12}).$$

Thus $T - S \in SSHom(V, W)$.

Finally, let $a \in R$, $x, y \in V$ and $T \in SSHom(V, W)$. Then we have

$$\begin{aligned}(aT)(x+y) &= |a|T(x+y) \text{ (by theorem 3.2.13)}\\ &= |a|(Tx+Ty) \text{ (by theorem 3.2.15)}\\ &= |a|Tx+|a|Ty \text{ (since } |a|\in B)\\ &= (aT)x+(aT)y \text{ (by theorem 3.2.13).}\end{aligned}$$

Thus $(aT) \in SSHom(V, W)$. Hence, $SSHom(V, W)$ is a subspace of $SHom(V, W)$.

□

Remark 3.2.19. *Observe that* $T(ax) = |a|Tx$ *and* $(aT)(x) = |a|Tx$. *Hence* $T(ax) = (aT)(x)$.

3.3 Basis and Special Homomorphisms

In this section we extend W-extension of a group to an element of $SHom(V, W)$, where W is an R-Vector space.

Theorem 3.3.1. *Let* V *and* W *be R-Vector spaces with* V *having basis* G^*. *Let* $L : G \to W$ *be a map such that* $L0 = 0$, *where* $G = G^* \cup \{0\}$ *is a group. Then* L *can be uniquely extended to an element of* $SHom(V, W)$.

Proof. Let $L : G \to W$ be a mapping such that $L0 = 0$. If $x \in V$, then $x = \sum_{g\in G^*} a_{x,g}g,\ a_{x,g} \in R$.

Define $T : V \to W$ by $T(x) = \sum_{g\in G^*} \mid a_{x,g} \mid Lg,\ x \in V$. Now, let

$c \in R$ and $x \in V$, then

$$\begin{aligned}
T(cx) &= \sum_{g \in G^*} \mid a_{cx,g} \mid Lg \text{ (by definition)} \\
&= \sum_{g \in G^*} \mid ca_{x,g} \mid Lg \quad \text{(by lemma 2.5.21)} \\
&= \sum_{g \in G^*} \mid c \mid\mid a_{x,g} \mid Lg \\
&= \mid c \mid \sum_{g \in G^*} \mid a_{x,g} \mid Lg \text{ (since } |c| \in B) \\
&= \mid c \mid T(x).
\end{aligned}$$

Thus, $T \in SHom(V, W)$. We claim that $Tg = Lg$ for all $g \in G$.
For $x = 0$,

$$\begin{aligned}
T0 = & \sum_{g \in G^*} \mid a_{0,g} \mid Lg \\
= & \mid a_{0,0} \mid L0 + \sum_{g_i \neq 0} \mid a_{0,g_i} \mid Lg_i \\
= & \ L0.
\end{aligned}$$

For $x = g$,

$$\begin{aligned}
Tg = & \sum_{g \in G^*} \mid a_{x,g} \mid Lg \\
= & \mid a_{g,g} \mid Lg \\
= & \ 1.Lg \\
= & \ Lg
\end{aligned}$$

Thus, $Tg = Lg$, for all $g \in G$. Therefore, T is an extension of L to an element of $SHom(V, W)$.
Suppose S is an extension of L such that $S \in SHom(V, W)$.

Since S is an extension of L, $Sg = Lg$, for all $g \in G$.
Let $x \in V$, then $x = \sum_{g \in G^*} a_{x,g} g$.

$$\begin{aligned} Sx &= S(x) \\ &= S(\sum_{g \in G^*} a_{x,g} g) \\ &= \sum_{g \in G^*} |a_{x,g}| Sg \\ &= \sum_{g \in G^*} |a_{x,g}| Lg \\ &= Tx. \end{aligned}$$

Thus, $Sx = Tx$, for all $x \in V$. Hence, $S = T$. □

Theorem 3.3.2. *Let V, W be normed R-Vector spaces and V has basis a G^*. If B is complete up to cardinality of G^*, then*

$$\| T \| = \sum_{g \in G^*} | Tg |$$

defines a norm on $SHom(V, W)$ and hence $SHom(V, W)$ is a normed R-Vector space.

Proof. It is clear that B is complete upto the cardinality of G^* guarantees the existence of $\sum_{g \in G^*} | Tg |$.
Let $\tilde{0} \in SHom(V, W)$. Then $\tilde{0}(x) = 0$, for all $x \in V$ and

$$\begin{aligned} \| \tilde{0} \| &= \sum_{g \in G^*} | \tilde{0} g | \\ &= \sum_{g \in G^*} | 0 | \\ &= 0. \end{aligned}$$

Let $T \in SHom(V,W)$ and $\| T \| = 0$. Then $\sum_{g \in G^*} | Tg | = 0$. Hence $| Tg | = 0$, for all $g \in G^*$ and it follows that $Tg = 0$ for all $g \in G^*$. Also $T0 = 0$. Thus, $Tg = 0 = \tilde{0}g$, for all $g \in G$. By theorem 3.3.1, $Tx = 0 = \tilde{0}x$, for all $x \in V$. Therefore, $||T|| = 0 \Leftrightarrow T = \tilde{0}$.

Let $a \in B$ and $T \in SHom(V,W)$. Now,

$$\begin{aligned}
\| aT \| &= \sum_{g \in G^*} | (aT)g | \\
&= \sum_{g \in G^*} || a \mid Tg | \quad \text{(by theorem 3.2.13)} \\
&= | a | \sum_{g \in G^*} | Tg | \\
&= a \| T \| .
\end{aligned}$$

□

Theorem 3.3.3. *Let V, W be R-Vector spaces and G^* be basis of V. If H is a subgroup of G and $S \in SHom(< H >, W)$, then S can be extended to an element of $SHom(V,W)$.*

Proof. Define $L : G \to W$ by $Lg = 0$ for $g \notin H$ and $Lg = Sg$ for $g \in H$. Since $S \in SHom(< H >, W)$, $S0 = 0$. Hence $L0 = S0 = 0$. Thus, $L : G \to W$ such that $L0 = 0$. By theorem 3.3.1, L can be extended to $T \in SHom(V,W)$. Let H^* is a basis of $< H >$ and $x \in < H >$. Then $x = a_1g_1 + ... + a_ng_n$, $a_i \in R$, where $a_ia_j = 0$ for $i \neq j$.

Now

$$
\begin{aligned}
Tx &= T(\sum_{g_i \in H^*} a_i g_i) \\
&= \sum_{g_i \in H^*} |a_i| T g_i \quad \text{(by definition 3.2.7)} \\
&= \sum_{g_i \in H^*} |a_i| L g_i \quad \text{(by theorem 3.3.1)} \\
&= \sum_{g_i \in H^*} |a_i| S g_i \quad \text{(by hypothesis)} \\
&= S(\sum_{g_i \in H^*} a_i g_i) \quad \text{(by definition 3.2.7 and remark 3.2.19)} \\
&= Sx, where\ 1 \leq i \leq n.
\end{aligned}
$$

Hence, T is an extension of S to an element of $SHom(V, W)$. □

Remark 3.3.4. *In the above theorem, the extension of S to an element of $SHom(V, W)$ is not necessarily unique. It can be shown in the following*

Example 3.3.5. *Let V be an R-Vector space such that V has a basis G^*. Let H be any proper subgroup of G and I denote the identity mapping on V. Clearly, $I \in SHom(V, V)$. Let S denote the identity mapping on $< H >$. Then $S \in SHom(< H >, V)$. The extension T of S given in the proof of the above theorem is such that $Tg = 0$ for $g \in G$ and $g \notin H$. Since H is a proper subgroup of G, there exists an element g of G such that $g \notin H$. Thus, $Tg = 0$. Since $Ig = g$, $T \neq I$. For each $g \in H$, $Tg = Sg = g = Ig$.*
Consequently, $Tx = Sx = Ix$ for each $x \in < H >$. Therefore, T and I are distinct extensions of S to elements of $SHom(V, V)$.

We conclude this chapter by the following

Theorem 3.3.6. *Let V and W be R-Vector spaces. If V has a finite basis $G^* = \{x_1, ..., x_n\}$, then $SHom(V, W)$ is isomorphic to $\sum_{i=1}^{n} W_i$, $W_i = W$, for $i = 1, 2, ..., n$.*

Proof. Let $y \in \sum_{i=1}^{n} W_i$. Then $y = (y_1, ..., y_n)$, $y_i \in W$ for $i = 1, 2, ..., n$. Define a mapping $\theta : SHom(V, W) \to \sum_{i=1}^{n} W_i$ by $\theta(T) = (Tx_1, ..., Tx_n)$ for each $T \in SHom(V, W)$.

Let $T, S \in SHom(V, W)$ and $a \in R$.

Now

$$\begin{aligned}\theta(S + T) &= ((S + T)x_1, ..., (S + T)x_n) \\ &= (Sx_1 + Tx_1, ..., Sx_n + Tx_n) \quad \text{(by theorem 3.2.12)} \\ &= (Sx_1, ..., Sx_n) + (Tx_1, ..., Tx_n) \\ &= \theta(S) + \theta(T).\end{aligned}$$

Simiarily,

$$\begin{aligned}\theta(aT) &= ((aT)x_1, ..., (aT)x_n) \\ &= (|a|Tx_1, ..., |a|Tx_n) \quad \text{(by remark 3.2.19)} \\ &= |a|(Tx_1, ..., Tx_n) \\ &= |a|\theta(T).\end{aligned}$$

Hence, θ is a homomorphism.

Let $T, S \in SHom(V, W)$ such that

$$\theta(T) = \theta(S) \Leftrightarrow (Tx_1, ..., Tx_n)$$
$$= (Sx_1, ..., Sx_n) \Leftrightarrow Tx_i$$
$$= Sx_i \ for \ i = 1, 2, ..., n.$$

Also we have $S0 = 0 = T0$. So $Tx = Sx$, for each $x \in V$. Thus $T = S$ (theorem 3.3.1). Hence, θ is one to one.

Let $y \in \sum_{i=1}^{n} W_i$. Then $y = (y_1, ..., y_n)$ for some $y_i \in W$.

Let $\alpha : G \to W$ be the mapping defined by $\alpha 0 = 0$ and $\alpha x_i = y_i$, $i = 1, 2, 3, ..., n$. Then by theorem 3.3.1, there is a unique extension $T : SHom(V, W) \to \sum_{i=1}^{n} W_i$ of α.

Consequently, $\theta(T) = (Tx_1, ..., Tx_n) = (\alpha x_1, ..., \alpha x_n) = (y_1, ..., y_n)$. Thus, θ is on to. Hence, θ is an isomorphism of $SHom(V, W)$ on to $\sum_{i=1}^{n} W_i$. □

Chapter 4

Functionals in R-Vector spaces

4.1 Introduction

In this chapter we introduce the notion of functionals in R-Vector space which generalizes the notion of functionals of Boolean Vector space. Similarly we introduce the notion of bilinear maps in R-Vector spaces. Further we study certain properties regarding these notions in section two. Section three is meant for the study of dual spaces and obtain the necessary and sufficient condition for two R-Vector spaces to be dual in theorem 4.3.11. Finally, in section four, we introduce the notion of inner product and prove that V is a dual space to itself if V is normed in theorem 4.4.5. Here in this chapter B stands for the set of all idempotents of a commutative regular ring R.

4.2 Functionals

Now we begin with the following

Theorem 4.2.1. *Let* $(R, +, .)$ *be a commutative regular ring with* 1 *which is not a Boolean ring and* $W = (R, +)$*. Define* $\otimes : R \times W \to W$ *by* $a \otimes x = \mid a \mid x$*, for all* $a \in R$*,* $x \in W$*, which is the ring product of* $\mid a \mid$ *and* x*. Then* W *is an R-Vector space.*

Proof. We verify the axioms 1 to 5 of definition 3.2.1.

1. Let $a \in R$ and $x, y \in W$. Then

$$\begin{aligned} a^2 \otimes (x+y) &= |a^2|(x+y) \quad \text{(by definition)} \\ &= |a|^2(x+y) \\ &= |a|(x+y) \\ &= |a|x + |a|y \quad (\text{ since } |a| \in B) \\ &= a \otimes x + a \otimes y \text{ for all } a \in R \text{ and } x, y \in W. \end{aligned}$$

2. $1 \otimes x = |1|x = 1.x = x$ for all $x \in W$ *and* $1 \in R$

3. Let $a, b \in R$ and $x \in W$. Then

$$\begin{aligned} a \otimes (b \otimes x) &= |a|(b \otimes x) \\ &= |a|(|b|x) \\ &= (|a||b|)x \\ &= (a \otimes |b|)x \\ &= |a \otimes b|x \\ &= (a \otimes b) \otimes x \quad \text{for all } a, b \in R \text{ and } x \in W. \end{aligned}$$

4. Let $a, b \in R$ with $ab = 0$ and $x \in W$

$$\begin{aligned}(a+b) \otimes x &= |a+b|x \\ &= (|a| + |b|)x \text{ (by 5 of lemma 2.3.1)} \\ &= |a|x + |b|x \\ &= a \otimes x + b \otimes x\end{aligned}$$

5. For all $x \in W$ and $r, s \in R$ be units,

$$\begin{aligned}r \otimes (s \otimes x) &= |r|(s \otimes x) \\ &= |r|(|s|x) \\ &= (|r||s|)x \\ &= (r \otimes |s|)x \\ &= |r \otimes s|x \\ &= (r \otimes s) \otimes x \qquad \text{(by 1 of lemma 2.3.5)}\end{aligned}$$

□

Corollary 4.2.2. *If $R = (R, +, .)$ is a regular ring and $W = (R, +)$ is an R-Vector space, then*

1. *$\{1\}$ is a basis of W*
2. *$\mid x \mid = x$ for each $x \in B$.*

Proof. 1. Since $x = x.1$ for each $x \in W$, $\{1\}$ spans W. If $a \in R$ and $a.1 = 0$, then $a = 0$. Thus $\{1\}$ is a linearly independent set. Hence $\{1\}$ is a basis for W over R.

2. Clearly for each $x \in B$, and by definition 2.5.15, we have

$$\begin{aligned} |x| &= |x.1| \\ &= x|1| \\ &= x.1 \\ &= x \end{aligned}$$

□

Definition 4.2.3. *Let V be an R-Vector space and $W = (R, +)$. A mapping T from V to W is called a linear functional on V provided $T(ax + by) = a \otimes Tx + b \otimes Ty$, if $ab = 0$, $x, y \in V$ and $a, b \in R$.*

Notation. The set of all linear functionals on V, denoted by $\overline{V}$, is nothing but $SHom(V, W)$, where $W = (R, +)$.

Definition 4.2.4. *Let V be an R-Vector space and $W = (R, +)$. A mapping T from V to W is called a strong linear functional on V provided $T(ax+by) = a \otimes Tx + b \otimes Ty$, for all $x, y \in V$ and $a, b \in R$.*

Theorem 4.2.5. *Let $\overline{V} = \{T | T : V \to W \text{ is a Special Homomorphism}\}$ where V is an R-Vector space and $W = (R, +)$. Then $(\overline{V}, +)$ is an abelian group if we define $(T + S)(x) = Tx + Sx$ for all $x \in V$ and $T, S \in \overline{V}$.*

Proof. Similar to the lines of the proof of theorem 3.2.12 □

Theorem 4.2.6. *$\overline{V}$ is an R-Vector space if we define the scalar multiplication $\otimes : R \times \overline{V} \to \overline{V}$ by $(a, T)(x) = (a \otimes T)(x) = | a | (Tx) = a \otimes Tx$.*

Proof. Similar to the lines of the proof of theorem 3.2.13 □

Theorem 4.2.7. *If V is an R-Vector space of finite dimension $'n'$ and $W_i = (R, +)$ for $i = 1, 2, ..., n$, then $\overline{V}$ is isomorphic to $\sum_{i=1}^{n} W_i$, where $(R, +)$ is considered as an R-Vector space.*

Proof. Since $SHom(V, W)$ is isomorphic to $\sum_{i=1}^{n} W_i$ (by theorem 3.3.6) where V and W are any R-Vector spaces, this theorem is a special case. □

Lemma 4.2.8. *If V is a normed R-Vector space and $Nx = |x|$ for some $x \in V$. Then $N \in \overline{V}$.*

Proof. For $a \in R$ and $x \in V$, we have

$$\begin{aligned} N(ax) &= |ax| \quad \text{(by hypothesis)} \\ &= |a||x| \\ &= |a|Nx \quad \text{(by hypothesis)} \\ &= a \otimes Nx \quad \text{(by theorem 4.2.1).} \end{aligned}$$

Since N may be considered as a mapping of $V \to W$, $W = (R, +)$, it follows from theorem 3.2.11 that $N \in SHom(V, W)$, $W = (R, +)$. Hence, $N \in \overline{V}$. □

Remark 4.2.9. *Let V be a normed R-Vector space. For each $a \in R$, $\overline{a}$ be a mapping of V to B defined by $\overline{a}(x) =| a || x |$ for each $x \in V$.*

Lemma 4.2.10. *If V is a normed R-Vector space and $a \in R$, then $\overline{a} \in \overline{V}$.*

Proof. For each $x \in V$ and $a, b \in R$, we have

$$\begin{aligned}
\overline{a}(bx) &= |a||bx| \quad \text{(by remark 4.2.9)} \\
&= |a|(|b||x|) \\
&= (|a||b|)|x| \quad \text{(by 2 of definition 2.4.5)} \\
&= (|b||a|)|x| \\
&= |b|(|a||x|) \\
&= |b|\overline{a}(x) \\
&= b \otimes \overline{a}(x).
\end{aligned}$$

Thus the result holds by theorem 3.2.11 and lemma 4.2.8. □

Lemma 4.2.11. *If V is a normed R-Vector space and $a, b \in B$, then*

1. $\overline{a+b} = \overline{a} + \overline{b}$
2. $a \otimes \overline{b} = \overline{ab}$
3. $\overline{ab} = \overline{a} \otimes \overline{b}$

Proof. From the above lemma 4.2.10, $\overline{a}$, $\overline{b}$, $\overline{a+b}$, $\overline{ab} \in \overline{V}$. The expressions $\overline{a} + \overline{b}$, $a \otimes \overline{b}$ and $\overline{a} \otimes \overline{b}$ are well defined according to $(T+S)x = Tx + Sx$, $(a \otimes T)x = |a|Tx$ and $(T \otimes S)x = |Tx|(Sx)$ respectively.

1. $(\overline{a+b})(x) = |a+b||x| = (a+b)|x| = a|x| + b|x| = |a||x| + |b||x| = \overline{a}(x) + \overline{b}(x) = (\overline{a} + \overline{b})(x)$.
2. $(a \otimes \overline{b})(x) = |a|(\overline{b}(x)) = |a|(|b||x|) = (|a||b|)|x| = |ab||x| = \overline{ab}(x)$.

3. $\overline{ab}(x) = |ab||x| = (|a||b|)|x| = (|a||x|)(|b||x|) = \overline{a}(x) \otimes \overline{b}(x) = |\overline{a}(x)|\overline{b}(x) = (\overline{a} \otimes \overline{b})(x)$.

$\square$

Lemma 4.2.12. *If V is a normed R-Vector space and $[V] = B$, then $\overline{a} = \overline{b} \Leftrightarrow a = b$.*

Proof. Suppose $\overline{a} = \overline{b}$ for some a, b $\in B$. Since $[V] = B$, there exist $x, y \in V$ such that $|x| = |a| = a$ and $|y| = |b| = b$. Then by the remark 4.2.9, we have

$$\begin{aligned} \overline{a}(x) &= |a||x| \\ &= aa \\ &= a \end{aligned}$$

and

$$\begin{aligned} \overline{b}(x) &= |b||x| \\ &= |b||a| \\ &= ba. \end{aligned}$$

since $\overline{a} = \overline{b}$, we have $a = ba$ and similarly, $b = ab$. Hence, $a = b$. Suppose $a = b \in B = [V]$. Then there exists $x \in V$ such that $a = |x| = b$. Thus, $\overline{a}(x) = |a||x| = a|x| = b|x| = |b||x| = \overline{b}(x)$. $\square$

Theorem 4.2.13. *If V is a normed R-Vector spaces and $[V] = B$, then*

1. *The R-Vector spaces $W = (R, +)$ is isomorphically contained in the R-Vector space $\overline{V}, +, .)$.*

2. *The regular ring* $(R, +, .)$ *is isomorphically contained in the regular ring* $(\overline{V}, +, .)$.

Proof. Consider the mapping $\gamma : B \to \overline{V}$ defined by $\gamma(a) = \overline{a}$ for each $a \in B$. By lemma 4.2.12, γ is one to one.

1. let $x, y \in W = (R, +)$ and $a \in B$. Then by lemma 4.2.11, we have $\gamma(x + y) = \overline{x + y} = \overline{x} + \overline{y} = \gamma(x) + \gamma(y)$ and $\gamma(a \otimes x) = \gamma(|a|x) = |a|\gamma(x) = a \otimes \gamma(x)$. Hence, γ is an isomorphism.
2. let $a, b \in B$, then $\gamma(ab) = \overline{ab} = \overline{a} \otimes \overline{b} = \gamma(a) \otimes \gamma(b)$. Similarly $\gamma(a + b) = \overline{a} + \overline{b} = \gamma(a) + \gamma(b)$. Hence, γ is an isomorphism of the regular ring $(R, +, .)$ in to the regular ring $\overline{V}$.

□

4.3 Dual spaces

In this section we introduce the notion of dual space and study its properties.
We begin with the following

Definition 4.3.1. *Let* V *and* W *be R-Vector spaces and* $V \times W = \{(x, z) : x \in V, z \in W\}$. *A mapping* $U : V \times W \to R$ *is called a bilinear function on* $V \times W$ *provided:*

1. $U(ax + by, z) = a \otimes U(x, z) + b \otimes U(y, z)$
2. $U(x, aw + bz) = a \otimes U(x, w) + b \otimes U(x, z)$, *for all* $x, y \in V$ *and* $w, z \in W$ *whenever* $a, b \in R$, $ab = 0$.

Definition 4.3.2. *If* V *and* W *are R-Vector spaces, then a bilinear* *function U on $V \times W$ is called non-degenerate provided:*

1. $U(x, z) = 0$ *for each* $z \in W \Rightarrow x = 0$
2. $U(x, z) = 0$ *for each* $x \in V \Rightarrow z = 0$

Definition 4.3.3. *If V and W are R-Vector spaces and U is a non-degenerate bilinear function on $V \times W$, then V and W are said to be dual spaces with respect to U.*

Remark 4.3.4. *Two R-Vector spaces will be called dual spaces if they are dual with respect to at least one bilinear function.*

Remark 4.3.5. *Let V and W be R-Vector spaces.*
If $U : V \times W \to R$ is a bilinear function on $V \times W$, then
$U' : W \times V \to R$ is also a bilinear function on $W \times V$ when
$U(x, z) = U'(z, x)$.

Corollary 4.3.6. *If U is non-degenerate, then U' is also non-degenerate.*

Proof. Let the bilinear function U on $V \times W$ be non degenerate. Then by definition 4.3.2, $U(x, z) = 0$ for each $z \in W$. Then $x = 0$ and $U(x, z) = 0$ for each $x \in V$. Hence $z = 0$. Thus, by remark 4.3.5, $U'(z, x) = 0$ for each $z \in W$. This implies $x = 0$ and $U'(z, x) = 0$ for each $x \in V$. Thus $z = 0$. Hence, U' is non degenerate. □

Lemma 4.3.7. *Let V and W be R-Vector spaces. If U is a bilinear function on $V \times W$, then*

1. $U(ax, z) = a \otimes U(x, z)$
2. $U(x, bz) = b \otimes U(x, z)$, *for all* $x \in V$, $z \in W$ *and* $a, b \in R$.

Proof. Letting $b = 0$ in (1) of definition 4.3.1 and $a = 0$ in (2) of definition 4.3.1 respectively yields the desired conclusions. □

Theorem 4.3.8. *If V is a normed R-Vector space and $\overline{V}$ is a space of linear functionals on V, then V and $\overline{V}$ are dual spaces.*

Proof. Let $U : V \times \overline{V} \to R$ be a map defined by $U(x,T) = Tx$ for all $x \in V$ and $T \in \overline{V}$. Suppose $x, y \in V$, $T, S \in \overline{V}$ and $a, b \in R$ with $ab = 0$. Then

$$\begin{aligned} U(ax+by,T) &= T(ax+by) \quad \text{(by hypothesis)} \\ &= a \otimes Tx + b \otimes Ty \quad \text{(by definition 4.2.3)} \\ &= a \otimes U(x,T) + b \otimes U(y,T) \quad \text{(by hypothesis)} \end{aligned}$$

Similarly,

$$\begin{aligned} U(x, a\otimes T + b \otimes S) &= (a \otimes T + b \otimes S)x \quad \text{(by hypothesis)} \\ &= (a \otimes T)x + (b \otimes S)x \quad \text{(by theorem 4.2.5)} \\ &= a \otimes Tx + b \otimes Sx \quad \text{(by theorem 4.2.6)} \\ &= a \otimes U(x,T) + b \otimes U(x,S) \quad \text{(by hypothesis)} \end{aligned}$$

Thus, by definition 4.3.1, U is a bilinear functional on $V \times \overline{V}$.
Suppose $x \in V^*$ and $T \in \overline{V^*}$. To show U is non degenerate, it will suffice to establish the existence of elements $N \in \overline{V}$ and $y \in V$ such that $U(x,N) \neq 0$ and $U(y,T) \neq 0$. Since $T \in \overline{V^*}$, T is not the zero mapping. Hence, there exists at least one element $y \in V$ for which $Ty \neq 0$. Let N denote the norm mapping on V. Then by lemma 4.2.8, it is clear that $N \in \overline{V}$. Since $x \neq 0$, $Nx = |x| \neq 0$. Then we have $U(x,N) = Nx \neq 0$ and $U(y,T) = Ty \neq 0$. Hence by definition 4.3.2, U is a non degenerate bilinear functional on $V \times \overline{V}$. Thus, V and $\overline{V}$ are dual spaces. □

Definition 4.3.9. *Let V be an R-Vector space. A subset K of $\overline{V}$ is called a total subset of $\overline{V}$ provided that for each $x \in V^*$ there exists an element T of K such that $Tx \neq 0$.*

Lemma 4.3.10. *If V is a normed R-Vector space, then $\overline{V}$ is a total set.*

Proof. Let $Nx = |x|$ for each $x \in V$. Then $N \in \overline{V}$ by lemma 4.2.8. If $x \neq 0$, then $Nx = |x| \neq 0$. Thus, $\overline{V}$ is a total set. □

Now we close this section by the following

Theorem 4.3.11. *Let V and M be R-Vector spaces. A necessary and sufficient condition for V and M to be dual spaces is the existence of a special homomorphism T of V in to $\overline{M}$ such that $T(V)$ is a total subset of $\overline{M}$ and $T^{-1}\{0\} = \{0\}$.*

Proof. Necessity:

Suppose V and M are dual spaces with respect to the non degenerate bilinear function U on $V \times M$. For each $x \in V$, let $\overline{x}$ denote the mapping of M in to R defined as follows:

$\overline{x}(z) = U(x, z)$ for each $z \in M$. Now, for $w, z \in M$ and $a, b \in R$ with $ab = 0$,

$$\begin{aligned}
\overline{x}(aw + bz) &= U(x, aw + bz) \text{ (by definition)} \\
&= a \otimes U(x, w) + b \otimes U(x, z) \text{ (by 2 of definition 4.3.1)} \\
&= a \otimes \overline{x}(w) + b \otimes \overline{x}(z).
\end{aligned}$$

Then $\overline{x} \in \overline{M}$ for each $x \in V$. Now consider the map $T : V \to \overline{M}$ defined by $Tx = \overline{x}$ for each $x \in V$.

Let $a \in R, x \in V$ and $z \in M$. By definition of $\overline{ax}$, we have that

$$\begin{aligned}(\overline{ax})(z) &= U(ax, z) \\ &= a \otimes U(x, z) \text{ (by 1 of lemma 4.3.7)} \\ &= a \otimes \overline{x}(z)(= |a|\overline{x}(z)) \\ &= (a \otimes \overline{x})(z)\end{aligned}$$

Thus $T(ax) = \overline{ax} = a \otimes \overline{x} = a \otimes Tx = |a|Tx$. Then T is a special homomorphism of V in to $\overline{M}$.

Let $z \in M^*$. Since U is non degenerate, there exists $x \in V$ such that $U(x, z) \neq 0$. From the definitions of T and $\overline{x}$, we have $(Tx)(z) = \overline{x}(z) = U(x, z) \neq 0$. Thus, $T(V)$ is a total subset of $\overline{M}$, since for each $z \in M^*$ there exists Tx in $T(V)$ such that $(Tx)(z) \neq 0$. Suppose $Tx = 0$ (zero mapping in $\overline{M}$) for some $x \in V$. Then $(Tx)(z) = 0$ for each $z \in M$. Thus, $0 = (Tx)(z) = \overline{x}(z) = U(x, z)$ for each $z \in M$. Then $x = 0$ (by 1 of definition 4.3.2). Hence $T^{-1}\{0\} = \{0\}$.

Sufficiency:

Suppose T is a special homomorphism of V in to $\overline{M}$ such that $T(V)$ is a total subset of $\overline{M}$ and $T^{-1}\{0\} = \{0\}$.

Let $U : V \times M \to R$ be a map defined by $U(x, z) = (Tx)(z)$ for all $x \in V$ and $z \in M$. Let $x, y \in V, z, w \in M$ and $a, b \in R$ with $ab = 0$. Since $T \in SHom(V, \overline{M})$ and $Tx \in \overline{M}$, it follows from definition 4.2.3, that $T(ax + by) = a \otimes Tx + b \otimes Ty$ and

$(Tx)(aw + bz) = a \otimes [(Tx)(w)] + b \otimes [(Tx)(z)]$. Thus,

$$\begin{aligned} U(ax + by, z) &= [T(ax + by)](z) \quad \text{(by hypothesis)} \\ &= (a \otimes Tx + b \otimes Ty)(z) \text{ (by definition 4.2.3)} \\ &= (a \otimes Tx)(z) + (b \otimes Ty)(z) \\ &= a \otimes [(Tx)(z)] + b \otimes [(Ty)(z)] \\ &= a \otimes U(x, z) + b \otimes U(y, z). \end{aligned}$$

Also by similar procedure, we have

$$\begin{aligned} U(x, aw + bz) &= (Tx)(aw + bz) \\ &= a \otimes [(Tx)(w)] + b \otimes [(Tx)(z)] \\ &= a \otimes U(x, w) + b \otimes U(x, z) \end{aligned}$$

Hence, U is a bilinear function on $V \times M$.
Suppose $x \in V^*$ and $z \in M^*$. Since $T^{-1}\{0\} = \{0\}$ and $x \neq 0$, it follows that Tx is not the zero mapping in $\overline{M}$. Hence there exists $w \in M$ such that $(Tx)(w) \neq 0$. Consequently, $U(x, w) = (Tx)(w) \neq 0$. Since $T(V)$ is a total subset of $\overline{M}$ and $z \in M^*$, there exists $y \in V$ such that $(Ty)(z) \neq 0$. Thus, $U(y, z) = (Ty)(z) \neq 0$. Hence U is non degenerate. Since U is non degenerate bilinear function on $V \times M$, we have that V and M are dual spaces(relative to U). $\square$

4.4 Inner Product

Subrahmanyam established that any normed Boolean Vector space V admits a unique "inner product" mapping, $< >$, of $V \times V$ in to B. In this section we introduce the notion of inner product on R-Vector spaces and study its properties.

Definition 4.4.1. *Let V be a normed R-Vector space. The mapping, $< \quad >: V \times V \rightarrow B$, is an inner product mapping with the following properties*

1. $< x, y > |x - y| = 0$
2. $< x, y > +|x - y| = |x| + |y| - |x||y|$
3. $< x, y >=< y, x >$
4. *If* $ab = 0$, *then* $< ax + bz, y >= a\otimes < x, y > +b\otimes < z, y >$, *for all* $x, y \in V$ *and* $a, b \in R$.

Remark 4.4.2. *Let V be a normed R-Vector space. For each $x \in V$, let $\bar{x}$ denote the mapping of V in to B defined by $\bar{x}(y) =< x, y >$ for each $y \in V$.*

Lemma 4.4.3. *If V is a normed R-Vector space, then*
$\{\bar{x} : x \in V\}$ is a total subset of $\bar{V}$.

Proof. Let $x, y, z \in V$ and $a, b \in R$ with $ab = 0$. Then

$$\begin{aligned}
\overline{x}(ay + bz) &=< x, ay + bz > && \text{(by remark 4.4.2)} \\
&=< ay + bz, x > && \text{(by 3 of definition 4.4.1)} \\
&= a\otimes < y, x > +b\otimes < z, x > && \text{(by 4 of definition 4.4.1)} \\
&= a\otimes < x, y > +b\otimes < x, z > && \text{(by 3 of definition 4.4.1)} \\
&= a \otimes \overline{x}(y) + b \otimes \overline{x}(z) && \text{(by remark 4.4.2)}
\end{aligned}$$

Thus $\overline{x} \in \overline{V}$. Suppose $x \in V^*$. By 2 of definition 4.4.1, $< x, x >= |x| + |x| - |x||x| = |x|$. Since $x \neq 0$, it is clear that $|x| \neq 0$. Thus,

$\overline{x}(x) =< x, x >\neq 0$. Hence, the lemma holds. □

Remark 4.4.4. *Let V be a normed R-Vector space. If $< x, y >=< z, y >$ for each $y \in V$ then $x = z$.*

Theorem 4.4.5. *If V is a normed R-Vector space, then V is a dual space to itself.*

Proof. Consider the mapping $T : V \to \overline{V}$ defined by $Tx = \overline{x}$ for each $x \in V$, where $\overline{x}(y) =< x, y >$ for each $y \in V$. By lemma 4.4.3, $T(V)$ is a total subset of $\overline{V}$. Now let $x, y, z \in V$ and $a, b \in R$ with $ab = 0$. Then

$$\begin{aligned}\overline{ax + by}(z) &=< ax + by, z > \text{ (by remark 4.4.2)}\\ &= a\otimes < x, z > +b\otimes < y, z > \text{ (by 4 of definition 4.4.1)}\\ &= a \otimes \overline{x}(z) + b \otimes \overline{y}(z)\\ &= (a \otimes \overline{x} + b \otimes \overline{y})(z).\end{aligned}$$

Hence,

$$\begin{aligned}T(ax + by) &= \overline{ax + by}\\ &= a \otimes \overline{x} + b \otimes \overline{y}\\ &= a \otimes Tx + b \otimes Ty.\end{aligned}$$

Thus, $T \in SHom(V, \overline{V})$. Finally, suppose $Tx = \overline{x}$ is the zero mapping in $\overline{V}$. Then $\overline{x}(y) =< x, y >= 0$ for each $x \in V$. Taking $a = b = 0$ in 4 of definition 4.4.1 yields $< 0, y >= 0$ for each $y \in V$. Hence by remark 4.4.4, $x = 0$. Thus $T^{-1}\{0\} = \{0\}$. Hence, V is a dual space to itself (definition 4.4.1 and theorem 4.3.11). □

Chapter 5

Fractions in R-Vector spaces

5.1 Introduction

The concept of R-Vector spaces has been studied by many authors in different ways. But N.Raja Gopala Rao[21,22] has intensively studied on it. Imitating the line of thought of N.Raja Gopala Rao, we introduce the notion of Fraction of R-Vector spaces, its norm, its sub $S^{-1}R$-Vector spaces and their properties. Here throughout this chapter, B denotes the set of all idempotents of a commutative regular ring R and $B_{S^{-1}R}$ denotes the set of all idempotents of fractions of a commutative regular ring R. The rest of this chapter consists of five sections. In section two, we collect certain definitions and examples in fraction of commutative regular rings. In section three, we introduce the notion of fractions of R-Vector spaces and we also see that fractions of R-Vector space is a vector space over fraction of regular rings. In section four, we establish the norm of fraction of R-Vector spaces and study its properties. In section five, we introduce the sub vector space of a vector space over fraction of commutative regular rings, and establish that $\frac{S^{-1}V}{S^{-1}U}$ is isomorphic to $S^{-1}(\frac{V}{U})$ where U is a sub R-Vector space of an R-Vector space V. Finally, in section six, we investigate the isomorphism theorems in R-Vector spaces.

5.2 Fractions of commutative regular ring

In this section we recall certain definitions and results of fractions of commutative rings.

Theorem 5.2.1. *Let R be a commutative regular ring with 1. Let S be a multiplicatively closed subset in R and V be an R-Vector space. Define a relation $\sim$ on $R \times S$ by $(r_1, s_1) \sim (r_2, s_2) \Leftrightarrow u(s_2 r_1) = u(s_1 r_2)$ for some $u \in S$, for all $r_1, r_2 \in R$ and $s_1, s_2 \in S$. Then $\sim$ is an equivalence relation.*

Remark 5.2.2. *The equivalence class containing $(r, s) \in R \times S$ is denoted by $\frac{r}{s}$. The set of all equivalence classes in $R \times S$ is denoted by $S^{-1}R = \{\frac{r}{s} : r \in R, s \in S\}$.*

Lemma 5.2.3. *Let R be a commutative regular ring with 1 and S be a multiplicatively closed subset of R. Then*

1. *For $r, p \in R$ and $s, t \in S$, $\frac{r}{s} = \frac{p}{t} \Leftrightarrow u(tr) = u(sp)$ for some $u \in S$.*
2. *$\frac{r}{s} = \frac{rt}{st} = \frac{tr}{st} = \frac{tr}{ts}$, for all $r \in R$ and $t, s \in S$*
3. *$\frac{rs_1}{s_1} = \frac{rs_2}{s_2}$, for all $r \in R$ and $s_1, s_2 \in S$*
4. *$\frac{s_1}{s_1} = \frac{s_2}{s_2}$, for all $s_1, s_2 \in S$*

5. $\frac{0}{s} = \overline{0} \in S^{-1}R$ *for* $0 \in R$

6. $\frac{s}{s} = \frac{1}{1} = \overline{1} \in S^{-1}R$ *for* $1 \in R$

7. *For* $r_1, r_2 \in R$ *and* $s \in S$, $\frac{r_1}{s} + \frac{r_2}{s} = \frac{r_1+r_2}{s}$ *and* $\frac{r_1}{s} \cdot \frac{r_2}{s} = \frac{r_1 r_2}{ss}$.

Theorem 5.2.4. *Let* S *be a multiplicatively closed subset of a commutative regular ring* R. *Define the binary operations* $+$ *and* $\cdot$ *on* $S^{-1}R$ *as* $\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_2 r_1 + s_1 r_2}{s_1 s_2}$ *and* $\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}$, *for all* $r_1, r_2 \in R$ *and* $s_1, s_2 \in S$. *Then* $S^{-1}R$ *is a commutative regular ring.*

Remark 5.2.5. *The usual partial ordering,* $<$, *on* $B_{S^{-1}R}$ *is defined as* $|\frac{r}{s}| < |\frac{q}{t}| \Leftrightarrow |\frac{r}{s}||\frac{q}{t}| = |\frac{r}{s}|$.

Remark 5.2.6. *For any* $\frac{a}{s}, \frac{b}{t} \in B_{S^{-1}R}$,

1. $\frac{a}{s}\frac{a}{s} = \frac{a}{s}$

2. $\frac{a}{s}\frac{b}{t} < \frac{a}{s}$, $\frac{a}{s}\frac{b}{t} < \frac{b}{t}$

Remark 5.2.7. $|a|$ *is an idempotent element of* R.

5.3 Construction of Fractions in R-Vector spaces

In this section we construct fractions in R-Vector spaces and study certain properties.

Remark 5.3.1. *Let* B *be the set of idempotents of* R, *then* B *is a multiplicatively closed set.*

Theorem 5.3.2. *Let* V *be an R-Vector space over a commutative regular ring* R, $S = B$ *be the set of idempotents of* R. *Now define a relation* $\sim$ *on* $V \times S$ *as* $(x, s) \sim (y, t) \Leftrightarrow u(tx) = u(sy)$ *for some* $u \in S$. *Then* $\sim$ *is an equivalence relation.*

Proof. 1. For any $u \in S$,

$$u(sx) = u(sx).$$
$$\Rightarrow (x, s) \sim (x, s).$$

2. For $x_1, x_2 \in V$ and $s_1, s_2 \in S$,

$$\begin{aligned}
&\textit{Let } (x_1, s_1) \sim (x_2, s_2).\\
&\Rightarrow u(s_2x_1) = u(s_1x_2) \quad \text{for some } u \in S.\\
&\Rightarrow u(s_1x_2) = u(s_2x_1)\\
&\Leftrightarrow (x_2, s_2) \sim (x_1, s_1).
\end{aligned}$$

3. Let $(x_1, s_1) \sim (x_2, s_2)$ and $(x_2, s_2) \sim (x_3, s_3)$.
$\Rightarrow u(s_2x_1) = u(s_1x_2)$ and $v(s_3x_2) = v(s_2x_3)$ for some $u, v \in S$.
Now, for $uvs_2 \in S$, by definition 2.4.5, we have

$$\begin{aligned}
(uvs_2)(s_3x_1) &= (uvs_2s_3)x_1\\
&= (vs_3us_2)x_1\\
&= vs_3(us_2x_1)\\
&= vs_3(us_1x_2)\\
&= us_1(vs_3x_2)\\
&= us_1(vs_2x_3)\\
&= (uvs_2)(s_1x_3)
\end{aligned}$$
$$\Rightarrow (x_1, s_1) \sim (x_3, s_3).$$

Hence '$\sim$' is an equivalence relation. □

Remark 5.3.3. *The equivalence class containing $(x, s) \in V \times S$ is* *denoted by $\frac{x}{s}$. The set of all equivalence classes in $V \times S$ is denoted by $S^{-1}V = \{\frac{x}{s} : x \in V, s \in S\}$.*

Lemma 5.3.4. *Let V be an R-Vector space over a commutative regular ring R, $S = B$ be the set of idempotents of R. Then*

1. $\frac{x_1}{s_1} = \frac{s_2x_1}{s_2s_1} = \frac{s_2x_1}{s_1s_2}$ *for* $s_1, s_2 \in S$ *and* $x_1 \in V$.
2. $\frac{s_1x_1}{s_1} = \frac{s_2x_1}{s_2} = \frac{x_1}{1}$ *for* $s_1, s_2 \in S$ *and* $x_1 \in V$.
3. $\frac{0}{s} = \frac{0}{t} = \overline{0}$ *for any* $t, s \in S$.
4. $\frac{x}{s} = \frac{0}{t} \Leftrightarrow ux = 0$ *for some* $u \in S$, $s, t \in S$ *and* $x \in V$

Proof. 1. For $u, s_1 \in S$ and $x_1 \in V$, $u(s_1x_1) = u(s_1x_1)$. Then frequently using 2 of definition 2.4.5, we have

$\Rightarrow s_2(u(s_1x_1)) = s_2(u(s_1x_1))$

$\Rightarrow u(s_2s_1x_1)) = u(s_2s_1x_1)$

$\Rightarrow u((s_2s_1)x_1) = u(s_1(s_2x_1))$

$\Leftrightarrow \frac{x_1}{s_1} = \frac{s_2x_1}{s_2s_1}$.

The rest will be proved in similar fashion.

2. For $u, s_2 \in S$ and $x_1 \in V$, $u(s_2x_1) = u(s_2x_1) \Leftrightarrow u(1.(s_2x_1)) = u(s_2x_1) \Leftrightarrow \frac{s_2x_1}{s_2} = \frac{x_1}{1}$. The rest are in the same line of proof.
3. For every $u \in S$, $u(t0) = u(s0)$ for some $s, t \in S$. Hence, $\frac{0}{s} = \frac{0}{t}$.
4. $\frac{x}{s} = \frac{0}{t} \Leftrightarrow$ there exist some $u \in S$ such that $u(tx) = u(s0) = 0$. $\Leftrightarrow (ut)x = 0 \Leftrightarrow (tu)x = 0 \Leftrightarrow t(ux) = 0 \Leftrightarrow ux = 0$.

□

Definition 5.3.5. *For $\frac{x}{s}, \frac{y}{t} \in S^{-1}V$ and $\frac{r}{u} \in S^{-1}R$, we define the binary operations addition and scalar multiplication on $S^{-1}V$ as* *follows:*

1. $\frac{x}{s} + \frac{y}{t} = \frac{tx+sy}{st}$
2. $\frac{r}{u} \odot \frac{x}{t} = \frac{|r|x}{ut}$

Theorem 5.3.6. *With the above definition of addition and scalar multiplication, $S^{-1}V$ is an $S^{-1}R$-Vector space.*

Proof. Since $\frac{0}{s} = \overline{0} \in S^{-1}V$, $S^{-1}V \neq \emptyset$.
Let $\frac{x_1}{s_1}, \frac{x_2}{s_2}, \frac{x_3}{s_3}, \frac{x_4}{s_4} \in S^{-1}V$ such that $\frac{x_1}{s_1} = \frac{x_2}{s_2}, \frac{x_3}{s_3} = \frac{x_4}{s_4} \Leftrightarrow \exists u, v \in S$ such that $u(s_2x_1) = u(s_1x_2)$ and $v(s_4x_3) = v(s_3x_4)$.
Now, by applying 2 of definition 2.4.5 frequently, we have

$$\begin{aligned}(uv)[(s_2s_4)(s_3x_1 + s_1x_3)] &= (vs_4s_3)u(s_2x_1) + u(s_2s_1)(vs_4x_3)\\ &= (vs_4s_3)u(s_1x_2) + u(s_2s_1)(vs_3x_4)\\ &= (uv)[(s_1s_3)(s_4x_2 + s_2x_4)]\end{aligned}$$

$\Rightarrow \frac{s_3x_1+s_1x_3}{s_1s_3} = \frac{s_4x_2+s_2x_4}{s_2s_4}$ (by theorem 5.3.2).
Thus " + " is well defined.
Similarly, let $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in S^{-1}R$ and $\frac{x}{t_1}, \frac{y}{t_2} \in S^{-1}V$ such that $\frac{r_1}{s_1} = \frac{r_2}{s_2}$ and $\frac{x}{t_1} = \frac{y}{t_2}$, then there exist $u, v \in S$ such that $u(s_2r_1) = u(s_1r_2)$ and $v(t_2x) = v(t_1y)$ respectively. Now, by applying 2 of definition 2.4.5 repeatedly, we have

$$\begin{aligned}(uv)[(s_2t_2)(|r_1|x)] &= (u(s_2|r_1|))(v(t_2x))\\ &= (u(s_1|r_2|))(v(t_1y))\\ &= (uv)[(s_1t_1)(|r_2|y)].\end{aligned}$$

$\Rightarrow \frac{|r_1|x}{s_1t_1} = \frac{|r_2|y}{s_2t_2}$
$\Rightarrow \frac{r_1}{s_1} \odot \frac{x}{t_1} = \frac{r_2}{s_2} \odot \frac{y}{t_2}$ (definition 5.3.5). Thus, " $\odot$" is also well defined.

Now, let $\frac{x_1}{s_1}, \frac{x_2}{s_2}, \frac{x_3}{s_3} \in S^{-1}V$.

1.

$$\begin{aligned}
\frac{x_1}{s_1} + (\frac{x_2}{s_2} + \frac{x_3}{s_3}) &= \frac{x_1}{s_1} + (\frac{s_3x_2 + s_2x_3}{s_2s_3}) \quad \text{(by definition 5.3.5)} \\
&= \frac{x_1}{s_1} + \frac{s_3x_2}{s_2s_3} + \frac{s_2x_3}{s_2s_3} \\
&= \frac{x_1}{s_1} + \frac{s_1(s_3x_2)}{s_1(s_2s_3)} + \frac{s_1(s_2x_3)}{s_1(s_2s_3)}) \quad (by\ 1\ of\ lemma\ 5.3.4) \\
&= \frac{x_1}{s_1} + \frac{s_1(s_3x_2) + s_1(s_2x_3)}{s_1(s_2s_3)} \quad \text{(by definition 5.3.5)} \\
&= \frac{(s_2s_3)x_1 + s_1(s_3x_2) + s_1(s_2x_3)}{s_1(s_2s_3)} \\
&= \frac{(s_2s_3)x_1}{s_1(s_2s_3)} + \frac{s_1(s_3x_2)}{s_1(s_2s_3)} + \frac{s_1(s_2x_3)}{s_1(s_2s_3)} \\
&= \frac{(s_2s_3)x_1}{s_1(s_2s_3)} + \frac{s_1(s_3x_2)}{s_1(s_2s_3)} + \frac{s_1(s_2x_3)}{s_1(s_2s_3)} \\
&= \frac{s_2x_1}{s_1s_2} + \frac{s_1x_2}{s_1s_2} + \frac{x_3}{s_3} \quad \text{(by 1 of lemma 5.3.4)} \\
&= \frac{s_2x_1 + s_1x_2}{s_1s_2} + \frac{x_3}{s_3} \quad \text{(by definition 5.3.5)} \\
&= (\frac{x_1}{s_1} + \frac{x_2}{s_2}) + \frac{x_3}{s_3}.
\end{aligned}$$

2.

$$\begin{aligned}
\frac{x_1}{s_1} + \frac{x_2}{s_2} &= \frac{s_2x_1 + s_1x_2}{s_1s_2} \quad \text{(by definition 5.3.5)} \\
&= \frac{s_1x_2 + s_2x_1}{s_1s_2} \\
&= \frac{x_2}{s_2} + \frac{x_1}{s_1} \quad \text{(by 1 of lemma 5.3.4).}
\end{aligned}$$

Thus " + " is commutative.

3. For $\frac{x}{s} \in S^{-1}V$, consider $\frac{0}{s} = \overline{0} \in S^{-1}V$. Then

$$\begin{aligned}\frac{x}{s} + \frac{0}{s} &= \frac{sx + s0}{ss} \\ &= \frac{sx}{ss} \\ &= \frac{x}{s} \\ &= \frac{0 + x}{s} \\ &= \frac{0}{s} + \frac{x}{s}.\end{aligned}$$

4. For any $\frac{x}{s} \in S^{-1}V$, consider $\frac{-x}{s} \in S^{-1}V$. Then

$$\begin{aligned}\frac{x}{s} + \frac{-x}{s} &= \frac{sx + -sx}{ss} \\ &= \frac{0}{ss} \\ &= \overline{0}.\end{aligned}$$

Thus, $\frac{-x}{s}$ is the inverse of $\frac{x}{s}$ in $S^{-1}V$ so that $(S^{-1}V, +)$ is an abelian group. Now, we can easily verify $S^{-1}V$ is an $S^{-1}R$-Vector space.

1. Let $\frac{r}{t} \in S^{-1}R$ *and* $\frac{x}{s}, \frac{y}{u} \in S^{-1}V$

$$\begin{aligned}(\frac{r}{t})^2 \odot (\frac{x}{s} + \frac{y}{u}) &= (\frac{r}{t})^2 \odot (\frac{ux + sy}{su}) \quad \text{(by definition 5.3.5)} \\ &= \frac{|r|(ux + sy)}{t(su)} \\ &= \frac{|r|(ux) + |r|(sy)}{t(su)} \\ &= \frac{|r|(ux)}{t(su)} + \frac{|r|(sy)}{t(su)} \\ &= \frac{|r|x}{ts} + \frac{|r|y}{tu} \quad \text{(by 1 of lemma 5.3.4)} \\ &= \frac{r}{t} \odot \frac{x}{s} + \frac{r}{t} \odot \frac{y}{u} \quad \text{(by definition 5.3.5)}\end{aligned}$$

2. Let $\frac{q}{t}, \frac{r}{s} \in S^{-1}R$ *and* $\frac{x}{t}, \frac{y}{u} \in S^{-1}V$

$$
\begin{aligned}
(\frac{q}{t}\frac{r}{s}) \odot \frac{x}{u} &= (\frac{qr}{tu}) \odot \frac{x}{u} \\
&= \frac{(|qr|)x}{(ts)u} \\
&= \frac{|q|(|r|x)}{t(su)} \quad \text{(by 2 of definition 2.4.5)} \\
&= \frac{q}{t} \odot (\frac{|r|x}{su}) \\
&= \frac{q}{t} \odot (\frac{r}{s} \odot \frac{x}{u}).
\end{aligned}
$$

3. Let $\overline{1} \in S^{-1}R$ and $\frac{x}{s} \in S^{-1}V$. Then

$$
\begin{aligned}
\overline{1}\frac{x}{s} &= \frac{1}{1}\frac{x}{s} \\
&= \frac{1.x}{1.s} \\
&= \frac{x}{s}.
\end{aligned}
$$

4. Suppose $\frac{q}{t}\frac{r}{s} = \overline{0}$. Then

$$
\begin{aligned}
(\frac{q}{t} + \frac{r}{s}) \odot \frac{x}{u} &= (\frac{sq + tr}{ts}) \odot \frac{x}{u} \quad \text{(by definition 5.3.5)} \\
&= \frac{|(sq + tr)|x}{(ts)u} \quad \text{(by definition 5.3.5)} \\
&= \frac{|sq|x + |tr|x}{(ts)u} \quad \text{(since qr=0)} \\
&= \frac{|sq|x}{(ts)u} + \frac{|tr|x}{(ts)u} \\
&= \frac{|q|x}{tu} + \frac{|r|x}{su} \quad \text{(by 1 of lemma 5.3.4)} \\
&= \frac{q}{t} \odot \frac{x}{u} + \frac{r}{s} \odot \frac{x}{u} \quad \text{(by definition 5.3.5).}
\end{aligned}
$$

5. Let $\frac{q}{t}, \frac{r}{s} \in S^{-1}R,$ *where* $q, r \in R$ *are units and* $\frac{x}{t}, \frac{y}{u} \in S^{-1}V$

$$\begin{aligned}(\frac{q}{t}\frac{r}{s}) \odot \frac{x}{u} &= (\frac{qr}{tu}) \odot \frac{x}{u}\\ &= \frac{(|qr|)x}{(ts)u}\\ &= \frac{|q|(|r|x)}{t(su)} \text{ (by 2 of definition 2.4.5)}\\ &= \frac{q}{t} \odot (\frac{|r|x}{su})\\ &= \frac{q}{t} \odot (\frac{r}{s} \odot \frac{x}{u}) \text{ (by definition 5.3.5).}\end{aligned}$$

□

Remark 5.3.7. $S^{-1}V$ *is an R-Vector space with scalar multiplication* $r \odot \frac{x}{s} = \frac{|r|x}{s}$ *for all* $r \in R, x \in V, s \in S$.

5.4 Normed $S^{-1}R$-Vector Spaces

Here we introduce norm on fraction of R-Vector spaces (normed $S^{-1}R$-Vector spaces) and observe certain properties on it.

Definition 5.4.1. *Let* V *be a normed R-Vector space and* $S = B$. *A vector space* $S^{-1}V$ *over* $S^{-1}R$ *is said to be normed provided there exists a mapping* $\|.\| : S^{-1}V \to B_{S^{-1}R}$ *defined by* $\|\frac{x}{s}\| = \frac{|x|}{|s|}$, *satisfying:*

1. $\|\frac{x}{s}\| = \frac{0}{s} = \overline{0} \Leftrightarrow \frac{x}{s} = \frac{0}{s} = \overline{0}$
2. $\|\frac{r}{t}\frac{x}{s}\| = \frac{r}{t}\|\frac{x}{s}\|$, *for all* $\frac{r}{t} \in B_{S^{-1}R}$ *and* $\frac{x}{s} \in S^{-1}V$.

Remark 5.4.2. *If* $\frac{a}{s} \in B_{S^{-1}R}$ *and* $\frac{x}{t} \in S^{-1}V$, *then* $\frac{a}{s} \odot \frac{x}{t} = \frac{a}{s}.\frac{x}{t} = \frac{ax}{st}$

Lemma 5.4.3. *If* $S^{-1}V$ *is a normed* $S^{-1}R$*-Vector space, then* $\|\frac{x}{s}\|\frac{x}{s} = \frac{x}{s}$ *for each* $\frac{x}{s} \in S^{-1}V$.

Proof. For $\frac{x}{s} \in S^{-1}V$,

$$\begin{aligned}\|\frac{x}{s}\|\frac{x}{s} &= \frac{|x|}{|s|}\frac{x}{s} \quad \text{(by definition 5.4.1)}\\ &= \frac{|x|x}{ss} \quad \text{(by 1 of lemma 2.3.1)}\\ &= \frac{x}{s} \quad \text{(by theorem 2.5.8 and lemma 2.2.10).}\end{aligned}$$

□

Lemma 5.4.4. *If $S^{-1}V$ is a normed vector space over $S^{-1}R$, then $\|\frac{x}{s} + \frac{y}{t}\| < \|\frac{x}{s}\| + \|\frac{y}{t}\| - \|\frac{x}{s}\|\|\frac{y}{t}\|$ for all $\frac{x}{s}, \frac{y}{t} \in S^{-1}V$.*

Proof. Let $\frac{x}{s}, \frac{y}{t} \in S^{-1}V$. Then

$$\begin{aligned}\|\frac{x}{s} + \frac{y}{t}\| &= \|\frac{tx+sy}{st}\| \quad \text{(by definition 5.3.5)}\\ &= \frac{|tx+sy|}{|st|} \quad \text{(by definition 5.4.1)}\\ &< \frac{|tx|+|sy|-|tx||sy|}{st} \quad \text{(by corollary 2.5.9)}\\ &= \frac{t|x|}{st} + \frac{s|y|}{st} - \frac{t|x|s|y|}{stst} \quad \text{(by definition 2.5.1)}\\ &= \frac{t|x|}{st} + \frac{s|y|}{st} - \frac{t|x|}{st}\frac{s|y|}{st}\\ &= \frac{|x|}{s} + \frac{|y|}{t} - \frac{|x|}{s}\frac{|y|}{t} \quad \text{(by lemma 5.3.4)}\\ &= \|\frac{x}{s}\| + \|\frac{y}{t}\| - \|\frac{x}{s}\|\|\frac{y}{t}\| \quad \text{(by definition 5.4.1)}\end{aligned}$$

□

Definition 5.4.5. *If $S^{-1}V$ is a normed vector space over $S^{-1}R$, then*

1. $[S^{-1}V] = \{\|\frac{x}{s}\| : \frac{x}{s} \in S^{-1}V\}$
2. $S^{-1}V_{\frac{r}{s}} = \{\frac{x}{t} : \|\frac{x}{t}\| < \frac{|r|}{s}\}$ *for each* $\frac{x}{t} \in S^{-1}V$ *and* $\frac{r}{s} \in S^{-1}R$.

Lemma 5.4.6. *If $S^{-1}V$ is a normed vector space over $S^{-1}R$, then $\|\frac{-x}{s}\| = \|\frac{x}{s}\|$ for each $\frac{x}{s} \in S^{-1}V$.*

Proof. For any $\frac{x}{s} \in S^{-1}V$, we have

$$\begin{aligned}\|\frac{-x}{s}\| &= \frac{|-x|}{|s|} \quad \text{(by definition 5.4.1)}\\ &= \frac{|x|}{s}\\ &= \|\frac{x}{s}\| \quad \text{(by definition 5.4.1).}\end{aligned}$$

□

Lemma 5.4.7. *If $S^{-1}V$ is a normed vector space over $S^{-1}R$ and $\|\frac{x}{s}\|\|\frac{y}{t}\| = \frac{0}{s} = \overline{0}$, then for all $\frac{x}{s}, \frac{y}{t} \in S^{-1}V$*

1. $\|\frac{x}{s}\|\frac{y}{t} = \frac{0}{s} = \overline{0}$
2. $\|\frac{x}{s} + \frac{y}{t}\| = \|\frac{x}{s}\| + \|\frac{y}{t}\|$
3. $[S^{-1}V] = \{\|\frac{x}{s}\| : \frac{x}{s} \in S^{-1}V\}$ *is an ideal of* $B_{S^{-1}R}$.

Proof. Suppose $\frac{x}{s}, \frac{y}{t} \in S^{-1}V$ and $\|\frac{x}{s}\|\|\frac{y}{t}\| = \frac{0}{s}$.

1.

$$\begin{aligned}Since\ \|\|\frac{x}{s}\|\frac{y}{t}\| &= \|\frac{x}{s}\|\|\frac{y}{t}\| \quad \text{(by definition 5.4.1)}\\ &= \frac{0}{s} \quad \text{(by hypothesis)}\\ &\Leftrightarrow \|\frac{x}{s}\|\frac{y}{t} = \frac{0}{s} \quad \text{(by definition 5.4.1)}\end{aligned}$$

2. For $\frac{x}{s}, \frac{y}{t} \in S^{-1}V$,

$$(\|\frac{x}{s}\| + \|\frac{y}{t}\|)\|\frac{x}{s} + \frac{y}{t}\| = \|\frac{x}{s}\|\|\frac{x}{s} + \frac{y}{t}\| + \|\frac{y}{t}\|\|\frac{x}{s} + \frac{y}{t}\|$$
$$= \|\|\frac{x}{s}\|\frac{x}{s} + \|\frac{x}{s}\|\frac{y}{t}\| + \|\|\frac{y}{t}\|\frac{x}{s} + \|\frac{y}{t}\|\frac{y}{t}\|$$

(by definition 5.4.1)

$$= \|\frac{x}{s}\| + \|\frac{y}{t}\| \quad \text{(by case 1 above).}$$
$$\Rightarrow \|\frac{x}{s}\| + \|\frac{y}{t}\| < \|\frac{x}{s} + \frac{y}{t}\| \quad \text{(by remark 5.2.5)}$$

On the other hand,

$$\|\frac{x}{s} + \frac{y}{t}\| < \|\frac{x}{s}\| + \|\frac{y}{t}\| - \|\frac{x}{s}\|\|\frac{y}{t}\| \quad \text{(by lemma 5.4.4)}$$
$$= \|\frac{x}{s}\| + \|\frac{y}{t}\| \quad \text{(by hypothesis).}$$

Hence the result holds.

3. Let $a, b \in [S^{-1}V]$ such that $a = \|\frac{x}{s}\|$ and $b = \|\frac{y}{t}\|$. Then,

$$\|(1-b)\frac{x}{s}\|\|(1-a)\frac{y}{t}\| = (1-b)\|\frac{x}{s}\|(1-a)\|\frac{y}{t}\| \quad \text{(by definition 5.4.1)}$$
$$= (1-b)a(1-a)b$$
$$= (1-b)(1-a)ab$$
$$= \overline{0} \quad (\text{since } ab = \overline{0})$$

$$\Rightarrow \|(1-b)\frac{x}{s} + (1-a)\frac{y}{t}\| = \|(1-b)\frac{x}{s}\| + \|(1-a)\frac{y}{t}\| \text{(by 2 above)}$$
$$= (1-b)\|\frac{x}{s}\| + (1-a)\|\frac{y}{t}\| \quad \text{(by definition 5.4.1)}$$
$$= (1-b)a + (1-a)b$$
$$= a - ab + b - ab$$
$$= a - b \in [S^{-1}V] \quad \text{(by hypothesis).}$$

Let $a \in [S^{-1}V]$ and $\frac{r}{u} \in B_{S^{-1}R}$, then $a = \|\frac{x}{t}\|,\ for\ some\ \frac{x}{t} \in S^{-1}V$. Now

$$\begin{aligned}\frac{r}{u}a &= \frac{r}{u}\|\frac{x}{s}\| \\ &= \|\frac{r}{u} \odot \frac{x}{s}\| \quad \text{(by definition 5.4.1)} \\ &= \|\frac{|r|x}{us}\| \in [S^{-1}V] \quad \text{(by definition 5.3.5).}\end{aligned}$$

Thus the result holds.

□

5.5 Sub Vector Spaces in Fraction of R-Vector Spaces

In this section we introduce the concept of sub vector spaces in fractions of R-Vector spaces and study certain properties. Now before introducing sub vector spaces in fractions of R-Vector spaces we look on certain properties of sub vector spaces of R-Vector spaces given in definition 3.2.16.

Lemma 5.5.1. *Let V be an R-Vector space and W, U be sub R-vector spaces of V. Then $W \cap U$ is a sub vector space of V over R.*

Proof. Let $x, y \in W \cap U$.

$$\begin{aligned}&\Rightarrow x, y \in W, U \\ &\Rightarrow x - y \in W, U \quad \text{(by (1) of definition 3.2.16)} \\ &\Rightarrow x - y \in W \cap U.\end{aligned}$$

For $a \in R$ and $x \in W, U$,

$$\Rightarrow |a|x \in W, U \quad \text{(by (2) of definition 3.2.16).}$$

$$\Rightarrow |a|x \in W \cap U.$$

Hence the lemma holds. □

Definition 5.5.2. *Let $S^{-1}V$ be a vector space over $S^{-1}R$. A non empty subset $S^{-1}W$ of $S^{-1}V$ is called a sub $S^{-1}R$-Vector space of $S^{-1}V$ provided:*

1. $\frac{x}{s} - \frac{y}{t} \in S^{-1}W$ *for* $\frac{x}{s}, \frac{y}{t} \in S^{-1}W$.
2. $\frac{r}{s} \odot \frac{x}{t} = \frac{|r|x}{st} \in S^{-1}W$ *for* $\frac{r}{s} \in S^{-1}R$ *and* $\frac{x}{t} \in S^{-1}W$.

Lemma 5.5.3. *Let W be a sub vector space of a vector space V over R. Then*

1. $S^{-1}W$ *is a sub vector space of* $S^{-1}V$ *over* $S^{-1}R$
2. $s^{-1}x \in S^{-1}W \Leftrightarrow tx \in W$ *for some* $t \in S$

Proof. 1. Let $\frac{x}{s}, \frac{y}{t} \in S^{-1}W$ and $\frac{r}{u} \in S^{-1}R$. Then we can easily see that

$$\frac{r}{u} \odot \frac{x}{s} = \frac{|r|x}{us} \in S^{-1}W \quad \text{(by definition 3.2.16).}$$

$$\frac{x}{s} - \frac{y}{t} = \frac{tx - sy}{st} \in S^{-1}W \quad \text{(by definition 3.2.16).}$$

2. Suppose $\frac{x}{s} \in S^{-1}W$.

$$\Rightarrow \frac{tx}{ts} = \frac{x}{s} \in S^{-1}W for\ some\ t \in S \quad \text{(by (1) of lemma 5.3.4).}$$

$$\Rightarrow tx \in W,\ ts \in S\ for\ some\ t \in S \quad \text{(by remark 5.3.3).}$$

Conversely, suppose $tx \in W$ for some $t \in S$. $\Rightarrow \frac{x}{s} = \frac{tx}{ts} \in S^{-1}W$ for some $t, s, st \in S$ (by lemma 5.3.4 and remark 5.3.3). □

Theorem 5.5.4. *If $S^{-1}V$ is a normed vector space over $S^{-1}R$ and $\frac{r}{s} \in S^{-1}R$, then $S^{-1}V_{\frac{r}{s}}$ is a sub vector space of $S^{-1}V$ over $S^{-1}R$.*

Proof. Since $\frac{0}{s} \in (S^{-1}V)_{\frac{r}{s}}$, $(S^{-1}V)_{\frac{r}{s}}$ is non empty.
Let $\frac{x}{s}, \frac{y}{t} \in (S^{-1}V)_{\frac{r}{s}}$, then $\|\frac{x}{s}\| < \frac{|r|}{s}$ and $\|\frac{y}{t}\| < \frac{|r|}{s}$.
Now

$$\begin{aligned}
\|\frac{x}{s} - \frac{y}{t}\| &= \|\frac{x}{s} + (-\frac{y}{t})\| \\
&< \|\frac{x}{s}\| + \| - \frac{y}{t}\| - \|\frac{x}{s}\|\|\frac{-y}{t}\| && \text{(by lemma 5.4.4)} \\
&= \|\frac{x}{s}\| + \|\frac{y}{t}\| - \|\frac{x}{s}\|\|\frac{y}{t}\| && \text{(by lemma 5.4.6)} \\
&< \frac{|r|}{s} + \frac{|r|}{s} - \frac{|r|}{s}\frac{|r|}{s} && \text{(by definition 5.4.5)} \\
&= \frac{|r|}{s} + \frac{|r|}{s} - \frac{|r|}{s} && \text{(by lemma 2.2.10)} \\
&= \frac{|r|}{s}.
\end{aligned}$$

Then $\frac{x}{s} - \frac{y}{t} \in S^{-1}V_{\frac{r}{s}}$.
Again, let $\frac{x}{s} \in (S^{-1}V)_{\frac{r}{s}}$ and $\frac{q}{u} \in S^{-1}R$.

$$\begin{aligned}
\|\frac{q}{u} \odot \frac{x}{s}\| &= \|\frac{|q|x}{us}\| \quad \text{(by definition 5.3.5)} \\
&= \frac{||q|x|}{|us|} \quad \text{(by definition 5.4.1)} \\
&= \frac{|q||x|}{us} \\
&= \frac{|q|}{u}\frac{|x|}{s} \\
&= \frac{|q|}{u}\|\frac{x}{s}\| \quad \text{(by definition 5.4.1)} \\
&< \frac{|q|}{u}\frac{|r|}{s} \quad \text{(by hypothesis)} \\
&< \frac{|r|}{s}.
\end{aligned}$$

Thus $\frac{q}{u} \odot \frac{x}{s} \in (S^{-1}V)_{\frac{r}{s}}$. Hence the result holds. □

Lemma 5.5.5. *Let V be an R-Vector space and W, U be sub R-Vector spaces of V. Then*

1. $S^{-1}(W \cap U) = S^{-1}W \cap S^{-1}U$.
2. $S^{-1}(W + U) = S^{-1}W + S^{-1}U$.

Proof. 1. Since W and U are sub R-Vector spaces of V, $W \cap U$ is a sub R-Vector space of V (by lemma 5.5.1).

$$\begin{aligned}
Let\ \frac{x}{s} \in S^{-1}(W \cap U) &\Leftrightarrow tx \in W \cap U, for\ some\ t \in S \\
\text{(by 2 of lemma 5.5.3)} &\Leftrightarrow tx \in U, W for\ some\ t \in S \\
&\Leftrightarrow \frac{x}{s} \in S^{-1}W, S^{-1}U
\end{aligned}$$

(by 2 of lemma 5.5.3)

$$\Leftrightarrow \frac{x}{s} \in S^{-1}W \cap S^{-1}U.$$

2. Let $\frac{w}{s} \in S^{-1}W$, $\frac{u}{t} \in S^{-1}U$. $\Rightarrow \frac{w}{s} \in S^{-1}(W+U)$ and $\frac{u}{t} \in S^{-1}(W+U)$. Since $S^{-1}(W+U)$ is a sub $S^{-1}R$-Vector space of $S^{-1}V$, $\frac{w}{s} + \frac{u}{s} = \frac{tw+su}{st} \in S^{-1}(W+U)$. Let $\alpha \in S^{-1}(W+U)$. $\Rightarrow \alpha = \frac{w+u}{s} = \frac{w}{s} + \frac{u}{s} \in S^{-1}W + S^{-1}U$.

Thus the result holds. □

Theorem 5.5.6. *Let $S^{-1}W$ and $S^{-1}U$ be an $S^{-1}R$-Vector spaces. A mapping $S^{-1}T$ from $S^{-1}W$ in to $S^{-1}U$ is an element of $Hom(S^{-1}W, S^{-1}U)$ $\Leftrightarrow (S^{-1}T)(\frac{a}{s} \odot \frac{x}{t}) = \frac{a}{s} \odot (S^{-1}T)(\frac{x}{t})$ for $\frac{a}{s} \in S^{-1}R$ and $\frac{x}{t} \in S^{-1}W$.*

Proof. Let $S^{-1}T : S^{-1}W \to S^{-1}U, \frac{x}{t} \longmapsto \frac{Tx}{t}$, where $T \in Hom(V, W)$. Suppose $S^{-1}T \in Hom(S^{-1}W, S^{-1}U)$. If $\frac{x}{t} \in S^{-1}W$ and $\frac{a}{s} \in S^{-1}R$, then

$$
\begin{aligned}
(S^{-1}T)(\frac{a}{s} \odot \frac{x}{t}) &= (S^{-1}T)(\frac{|a|x}{st}) \quad \text{(by definition 5.3.5)} \\
&= \frac{T(|a|x)}{st} \quad \text{(by hypothesis)} \\
&= \frac{|a|Tx}{st} \quad \text{(by definition 2.4.26)} \\
&= \frac{a}{s} \odot \frac{Tx}{t} \quad \text{(by definition 5.3.5)} \\
&= \frac{a}{s} \odot (S^{-1}T)(\frac{x}{t}) \quad \text{(by hypothesis).}
\end{aligned}
$$

Suppose $(S^{-1}T)(\frac{a}{s} \odot \frac{x}{t}) = \frac{a}{s} \odot (S^{-1}T)(\frac{x}{t})$ and let $a, b \in R$ such that

$ab = 0$. Now

$$(S^{-1}T)(\frac{a}{s_1} \odot \frac{x}{t_1} + \frac{b}{s_2} \odot \frac{y}{t_2}) = (S^{-1}T)(\frac{|a|x}{s_1t_1} + \frac{|b|y}{s_2t_2}) \text{ (by definition 5.3.5)}$$
$$= (S^{-1}T)(\frac{(s_2t_2)(|a|x) + (s_1t_1)(|b|y)}{(s_1t_1)(s_2t_2)})$$

(by definition 5.3.5)

$$= \frac{T[(s_2t_2)(|a|x) + (s_1t_1)(|b|y)]}{(s_1t_1)(s_2t_2)} \text{ (by hypothesis)}$$
$$= \frac{T[((s_2t_2)|a|)x + ((s_1t_1)|b|)y]}{(s_1t_1)(s_2t_2)}$$
$$= \frac{((s_2t_2)|a|)Tx + ((s_1t_1)|b|)Ty}{(s_1t_1)(s_2t_2)} \text{ (by definiton 2.4.26)}$$
$$= \frac{|a|Tx}{s_1t_1} + \frac{|b|Ty}{s_2t_2} \text{ (by 1 of lemma 5.3.4)}$$
$$= \frac{a}{s_1} \odot \frac{Tx}{t_1} + \frac{b}{s_2} \odot \frac{Ty}{t_2} \text{ (by definition 5.3.5)}$$
$$= \frac{a}{s_1} \odot (S^{-1}T)(\frac{x}{t_1}) + \frac{b}{s_2} \odot (S^{-1}T)(\frac{y}{t_2}) \text{ (by hypothesis)}$$

□

Theorem 5.5.7. *Let $S^{-1}W$ and $S^{-1}U$ be an $S^{-1}R$-Vector spaces. $S^{-1}T \in Hom(S^{-1}W, S^{-1}U)$ is strongly linear $\Leftrightarrow (S^{-1}T)(\frac{x}{s} + \frac{y}{t}) = (S^{-1}T)(\frac{x}{s}) + (S^{-1}T)(\frac{y}{t})$ for $\frac{x}{s}, \frac{y}{t} \in S^{-1}W$.*

Proof. Suppose $S^{-1}T$ is a strongly linear homomorphism. Let $\frac{a}{s_1}, \frac{b}{s_2} \in S^{-1}R$ and $\frac{x}{t_1}, \frac{y}{t_2} \in S^{-1}W$. If $\frac{a}{s_1} = \overline{1} = \frac{b}{s_2}$ then $(S^{-1}T)(\frac{x}{t_1} + \frac{y}{t_2}) = (S^{-1}T)(\frac{x}{t_1}) + (S^{-1}T)(\frac{y}{t_2})$.

Suppose $S^{-1}T \in Hom(S^{-1}W, S^{-1}U)$ and $(S^{-1}T)(\frac{x}{s}+\frac{y}{t}) = (S^{-1}T)(\frac{x}{s})+ (S^{-1}T)(\frac{y}{t})$.

Now for any $\frac{x}{t_1}, \frac{y}{t_2} \in S^{-1}W$ and $\frac{a}{s_1}, \frac{b}{s_2} \in S^{-1}R$, we have

$$\begin{aligned}(S^{-1}T)(\frac{a}{s_1} \odot \frac{x}{t_1} + \frac{b}{s_2} \odot \frac{y}{t_2}) &= (S^{-1}T)(\frac{a}{s_1} \odot \frac{x}{t_1}) + (S^{-1}T)(\frac{b}{s_2} \odot \frac{y}{t_2}) \text{ (by hypothesis)} \\ &= \frac{a}{s_1} \odot (S^{-1}T)(\frac{x}{t_1}) + \frac{b}{s_2} \odot (S^{-1}T)(\frac{y}{t_2}) \text{ (by hypothesis)}\end{aligned}$$

Hence, $S^{-1}T$ is a strongly linear homomorphism. □

Theorem 5.5.8. *Let W and U be R-Vector spaces and $T : W \to U$ be a strongly linear homomorphism. Then $S^{-1}T : S^{-1}W \to S^{-1}U$, $\frac{w}{s} \longmapsto \frac{T(w)}{s}, w \in W, s \in S$ is also a strongly linear homomorphism.*

Proof. Let $\frac{x}{t_1}, \frac{y}{t_2} \in S^{-1}W$ and $\frac{a}{s_1}, \frac{b}{s_2} \in S^{-1}R$. Now

$$\begin{aligned}(S^{-1}T)(\frac{a}{s_1} \odot \frac{x}{t_1} + \frac{b}{s_2} \odot \frac{y}{t_2}) &= (S^{-1}T)(\frac{|a|x}{t_1s_1} + \frac{|b|y}{t_2s_2}) \text{ (by definition 5.3.5)} \\ &= (S^{-1}T)(\frac{(s_2t_2)(|a|x) + (s_1t_1)(|b|y)}{(s_1t_1)(s_2t_2)})\end{aligned}$$

(by definition 5.3.5)

$$\begin{aligned}&= \frac{T[(s_2t_2)(|a|x) + (s_1t_1)(|b|y)]}{(s_1t_1)(s_2t_2)} \text{ (by definition 5.3.5)} \\ &= \frac{(s_2t_2)T(|a|x) + (s_1t_1)T(|b|y)}{(s_1t_1)(s_2t_2)} \text{ (by definition 2.4.28)} \\ &= \frac{|a|Tx}{s_1t_1} + \frac{|b|Ty}{s_2t_2} \text{ (by 1 of lemma 5.3.4)} \\ &= \frac{a}{s_1} \odot \frac{Tx}{t_1} + \frac{b}{s_2} \odot \frac{Ty}{t_2} \text{ (by definition 5.3.5)} \\ &= \frac{a}{s_1} \odot (S^{-1}T)(\frac{x}{t_1}) + \frac{b}{s_2} \odot (S^{-1}T)(\frac{y}{t_2}).\end{aligned}$$

□

Corollary 5.5.9. *Let $S^{-1}W$ and $S^{-1}U$ be an $S^{-1}R$-Vector spaces. If $S^{-1}T : S^{-1}W \to S^{-1}U$, $\frac{w}{s} \longmapsto \frac{T(w)}{s}$ and T is a strongly linear* *homomorphism, then kernel of $S^{-1}T$ is a sub $S^{-1}R$-Vector space of $S^{-1}W$.*

Proof. Let kernel of $S^{-1}T = \{\frac{x}{s} \in S^{-1}W : (S^{-1}T)(\frac{x}{s}) = \frac{0}{s}\}$.
Now for $\frac{x}{s}, \frac{y}{t} \in ker(S^{-1}T)$, we have

$$\begin{aligned}(S^{-1}T)(\frac{x}{s} - \frac{y}{t}) &= (S^{-1}T)(\frac{tx - sy}{st}) \quad \text{(by definition 5.3.5)}\\ &= \frac{T(tx - sy)}{st} \quad \text{(by hypothesis)}\\ &= \frac{tTx - sTy}{st} \quad \text{(by definition 2.4.28)}\\ &= \frac{Tx}{s} - \frac{Ty}{t} \quad \text{(by 1 of lemma 5.3.4)}\\ &= (S^{-1}T)(\frac{x}{s}) - (S^{-1}T)(\frac{y}{t})\\ &= \overline{0}.\end{aligned}$$

Let $\frac{a}{s} \in S^{-1}R$ and $\frac{x}{t} \in ker(S^{-1}T)$. Then we the following

$$\begin{aligned}(S^{-1}T)(\frac{a}{s} \odot \frac{x}{t}) &= (S^{-1}T)(\frac{|a|x}{st}) \quad \text{(by definition 5.3.5)}\\ &= \frac{T(|a|x)}{st} \quad \text{(by hypothesis)}\\ &= \frac{|a|Tx}{st} \quad \text{(by definition 2.4.28)}\\ &= \frac{a}{s} \odot \frac{Tx}{t} \quad \text{(by definition 5.3.5)}\\ &= \frac{a}{s} \odot (S^{-1}T)(\frac{x}{t}) \quad \text{(by hypothesis)}\\ &= \overline{0}.\end{aligned}$$

Hence kernel of $S^{-1}T$ is a sub $S^{-1}R$-Vector space of $S^{-1}W$. □

Theorem 5.5.10. *Let V be an R-Vector space and U be a sub R-Vector space of V. Then $S^{-1}(V/U) \cong S^{-1}V/S^{-1}U$.*

Proof. Let $T : \frac{S^{-1}V}{S^{-1}U} \to S^{-1}(\frac{V}{U})$ is a map defined by
$T(\frac{x}{s} + S^{-1}U) = \frac{x+U}{s}$, for all $x \in V$ and $s \in S$.
Let $T(\frac{x}{s} + S^{-1}U) = T(\frac{y}{s} + S^{-1}U)$ for $x, y \in V$ and $s \in S$
$\Rightarrow \frac{x+U}{s} = \frac{y+U}{s} \Leftrightarrow \exists v \in S$ such that

$$\begin{aligned} v(s(x+U)) &= v(s(y+U)) \\ &\Rightarrow \beta x + U = \beta y + U, \beta = vs \in S \\ &\Rightarrow \beta(x-y) \in U \\ &\Rightarrow \frac{\beta s(x-y)}{ss} \in S^{-1}U \quad \text{(by 1 of lemma 5.3.4 and remark 5.3.3)} \\ &\Rightarrow \beta(\frac{x}{s} - \frac{y}{s}) \in S^{-1}U \quad \text{(by definition 5.3.5)} \\ &\Rightarrow \frac{x}{s} + S^{-1}U = \frac{y}{s} + S^{-1}U \end{aligned}$$

$\Rightarrow T$ is one to one. Clearly T is on to.
Let $\frac{x}{s} + S^{-1}U$, $\frac{y}{t} + S^{-1}U \in \frac{S^{-1}V}{S^{-1}U}$. Now

$$\begin{aligned} T(\frac{x}{s} + S^{-1}U + \frac{y}{t} + S^{-1}U) &= T(\frac{x}{s} + \frac{y}{t} + S^{-1}U) \\ &= T(\frac{tx+sy}{st} + S^{-1}U) \\ \text{(by definition 5.3.5)} \\ &= \frac{(tx+sy)+U}{st} \quad \text{(by hypothesis)} \\ &= \frac{t(x+U)}{ts} + \frac{s(y+U)}{st} \\ &= T(\frac{x}{s} + S^{-1}U) + T(\frac{y}{t} + S^{-1}U) \end{aligned}$$

(by 1 of lemma 5.3.4 and hypothesis)

.

Let $\frac{a}{s} \in S^{-1}R$ and $\frac{x}{t} + S^{-1}U \in \frac{S^{-1}V}{S^{-1}U}$. Then

$$\begin{aligned}
T(\frac{a}{s} \odot (\frac{x}{t} + S^{-1}U)) &= T(\frac{|a|x}{st} + S^{-1}U) \quad \text{(by definition 5.3.5)} \\
&= \frac{|a|x + U}{st} \\
&= \frac{|a|(x + U)}{st} \\
&= \frac{a}{s} \odot (\frac{x + U}{t}) \\
&= \frac{a}{s} \odot T(\frac{x}{t} + S^{-1}U).
\end{aligned}$$

Hence the result holds. □

5.6 Isomorphism theorems in R-Vector spaces

In this section we investigate the isomorphism theorems in R-Vector spaces.

Theorem 5.6.1. *Let U and W be R-Vector spaces. If $T : U \to W$ is a strong linear homomorphism, then*

1. *$ker(T)$ is a sub vector space of U.*
2. *$U/ker(T)$ is isomorphic to $Im(T)$*

Proof. 1. By definition $ker(T) = \{x \in U : T(x) = 0\}$.

$$\begin{aligned}
\text{Let } x, y \in ker(T) &\Rightarrow T(x) = 0 = T(y) \\
&\Rightarrow T(x) - T(y) = 0 \\
&\Rightarrow T(x - y) = 0 \quad \text{(by definition 2.4.28)} \\
&\Rightarrow x - y \in ker(T)
\end{aligned}$$

Let $a \in R$ and $x \in ker(T)$. Now we have the following
$T(ax) = |a|T(x) = |a|0 = 0$. Thus, $ax \in ker(T)$.
Hence, $ker(T)$ is a sub vector space of U.

2. Define $\Theta : U/ker(T) \to Im(T)$ by $\Theta(x + ker(T)) = T(x)$ for $x \in U$.

$$\begin{aligned}
let\ x + ker(T) = y + ker(T) \\
&\Rightarrow x - y \in ker(T) \\
&\Rightarrow T(x - y) = 0 \\
&\Rightarrow T(x) - T(y) = 0 \quad \text{(by definition 2.4.28)} \\
&\Rightarrow T(x) = T(y) \\
&\Rightarrow \Theta(x + ker(T)) = \Theta(y + ker(T))
\end{aligned}$$

Thus, Θ is well defined.
let $\Theta(x + ker(T)) = \Theta(y + ker(T))$, for all $x, y \in U$.

$$\begin{aligned}
&\Rightarrow T(x) = T(y) \\
&\Rightarrow T(x) - T(y) = 0 \\
&\Rightarrow T(x - y) = 0 \quad \text{(by definition 2.4.28)} \\
&\Rightarrow x - y \in ker(T) \\
&\Rightarrow x + ker(T) = y + ker(T)
\end{aligned}$$

Hence, Θ is one to one.
let $y \in Im(T)$. Then there exists $x \in U$ such that $T(x) = y$.
$\Rightarrow \Theta(x + ker(T)) = T(x) = y$. Thus Θ is on to.

Let $r \in R$ and $x + ker(T) \in U/ker(T)$.

$$\begin{aligned}\Theta(r(x+ker(T))) &= \Theta(|r|x + ker(T))\\ &= T(|r|x)\\ &= |r|T(x)\\ &= |r|\Theta(x+ker(T))\\ &= r.\Theta(x+ker(T))\end{aligned}$$

Thus Θ is a strong linear homomorphism.

□

Theorem 5.6.2. *Let V, U and W be R-Vector spaces. If $W \subseteq U \subseteq V$ then $(V/W)/(U/W) \cong (V/U)$.*

Proof. Define $T : V/W \to V/U$ by $T(x+W) = x+U$, for all $x \in V$.
Let $x+W = y+W$, for all $x, y \in V$.

$$\begin{aligned}x+W &= y+W\\ &\Rightarrow x-y \in W\\ &\Rightarrow x-y \in U \qquad \text{(since } W \subseteq U\text{)}\\ &\Rightarrow x+U = y+U\\ &\Rightarrow T(x+W) = T(y+W).\end{aligned}$$

Hence T is well defined. Here for $x+U \in V/U$ there exist $x+W \in V/W$ such that $T(x+W) = x+U$.
Thus T is on to.

Let $r, r' \in R$ and $x + W, y + W \in V \diagup W$.

$$\begin{aligned}
T(r(x+W) + r'(y+W)) &= T(|r|x + W + |r'|y + W) \\
&= T(|r|x + |r'|y + W) \\
&= |r|x + |r'|y + U \\
&= (|r|x + U) + (|r'|y + U) \\
&= r(x+U) + r'(y+U) \\
&= rT(x+W) + r'T(y+W)
\end{aligned}$$

Thus, T is a strongly linear homomorphism.

Now we have to identify $ker(T)$ as follows:

$$\begin{aligned}
ker(T) &= \{x + W \in V \diagup W : T(x+W) = U\} \\
&= \{x + W \in V \diagup W : x + U = U\} \\
&= \{x + W \in V \diagup W : x \in U\} \\
&= U \diagup W
\end{aligned}$$

Hence, by the above theorem 5.6.1, $(V \diagup W) \diagup (U \diagup W) \cong V \diagup U$. □

Theorem 5.6.3. *Let V be an R-Vector space and U, W be sub vector spaces of V, then $(U + W) \diagup U \cong W \diagup U \cap W$.*

Proof. Define $T : (U + W) \to W \diagup U \cap W$

by $T(x + y) = y + U \cap W$, for all $x \in U$ and for all $y \in W$.

Suppose $x_1 + y_1 = x_2 + y_2$, for $x_1, x_2 \in U$ and $y_1, y_2 \in W$.

$$\Rightarrow x_1 - x_2 = y_2 - y_1 \in U, W$$

$$\Rightarrow x_1 - x_2 = y_2 - y_1 \in U \cap W$$

$$\Rightarrow y_1 + U \cap W = y_2 + U \cap W$$

$$\Rightarrow T(x_1 + y_1) = T(x_2 + y_2).$$

Thus T is well defined.

Now, let $r, r' \in R$, $x_1, x_2 \in U$ and $y_1, y_2 \in W$.

$$\begin{aligned} T(r(x_1 + y_1) + r'(x_2 + y_2)) &= T((|r|x_1 + |r|y_1) + (|r'|x_2 + |r'|y_2)) \\ &= T((|r|x_1 + |r'|x_2) + (|r|y_1 + |r'|y_2)) \\ &= (|r|y_1 + |r'|y_2) + U \cap W \\ &= (|r|y_1 + U \cap W) + (|r'|y_2 + U \cap W) \\ &= r(y_1 + U \cap W) + r'(y_2 + U \cap W) \\ &= rT(x_1 + y_1) + r'T(x_2 + y_2). \end{aligned}$$

Hence, T is a strong linear homomorphism.

Now we can identify the $ker(T)$ as follows:

$$\begin{aligned} ker(T) &= \{x + y : T(x + y) = U \cap W\} \\ &= \{x + y : y + U \cap W = U \cap W\} \\ &= \{x + y : y \in U \cap W\} \\ &= U. \end{aligned}$$

Hence by theorem 5.6.1, $(U + W) \diagup U \cong W \diagup U \cap W$. $\square$

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Annex - A :

Special Homomorphisms
in R-Vector spaces

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Special Homomorphisms in R-Vector Spaces

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Abstract

In this paper we introduce the notion of special homomorphism in R-vector spaces and study its properties.We show that the set of all special homomorpisms form an R-vector space with suitable addition and scalar multiplication.Also we prove that SHom(V,W),where V and W are R-Vector spaces over the same regular ring R,is a normed R-vector space.Further we prove certain results concerning R-Vector spaces.

Mathematics Subject Classification: 15A03, 15A04, 16G30

Keywords: R-vector space,Normed R-vector space,Special and Strong Special Homomorphsim

Introduction

N.Raja Gopala Rao[4]introduced the concept of vector spaces over regular rings(simply R-vector spaces) as a generalization of Boolean vector space of Subrahmanyam N.V[6].He introduced the notion of linear endomorphisms and affine transformations in R-vector spaces and studied its properties.Further he made a study on the geometric aspect of these spaces.Later K.Venkateswarlu [2]introduced the notion of direct sums in R-vector spaces and established that every direct sum of R-vector spaces has a basis provided each component has a basis. The set of all linear endomormpisms denoted by Hom(V,V),where V is an R-vector space in[5],is not closed under usual addition of linear endomorphisms.Keeping this in view,we introduce the notion of special homomorphisms in R-vector spaces and studied its properties. This paper is divided in 2 sections. The first section is devoted for certain definitions and results concerning regular rings, R-Vector spaces[4]and Boolean Vector spaces of Subrhmanyam [6].In section 2,we introduce the notion of special homomorphisms in R-vector spaces.We give some examples of Special homomorphims which do not subsume the notion of linear endomorphsim of Raja Gopala Rao[4]and B-vector space of Subrahmanyam[6](see example 2.8).We prove that the set of all special homomorpisms in R-vector space forms an R-vector space(see theorem 2.11).We introduce the notion of strong special homomorphism and in theorem 2.14,we prove that the set of all strong special homomorphism form a subspace.Further we prove that SHom(V,W)is isomorphic to the finite direct sum of W's.(see theorem 2.19).

1 Preliminaries

In this section we collect certain definitions and results concerning R-Vector spaces of Raja Gopala Rao[4]and freely make use of the results in [2,4,6].

Definition 1.1. *A ring R is called a regular ring if and only if to each $a \in R$ there is an element $x \in R$ such that $axa = a$. If $a \in R$ and $axa = a$ then we put $a^\star = xax$.*

In this B stands for Boolean algebra of idempotents of R-vector spaces , U is the set of units of R-vector spaces and $|a| = aa^\star$

Lemma 1.2. *Let R be a regular ring and $a, b \in R$.Then*

1. $|a| = a$ *if and only if* $a \in B$
2. $a|a| = a = |a|a$
3. *If* $ab = 0$ *then* $|a|b = |b|a = |a||b| = 0$
4. $|a|(a + 1 - |a|) = a$

Definition 1.3. *Let $V = (V, +)$ be any group and $R = (R, +, .)$ be a commutative regular ring with unity element 1.Then V is said to be a Vector space over R(or simply R-vector space)if and only if there exists a mapping : $R \times V \to V$ (the image of any $(a, x) \in R \times V$ will be denoted by ax) such that for all $x, y \in V$ and $a, b \in R$ the following properties hold:*

1. $a^2(x + y) = ax + ay$
2. $a(bx) = (ab)x$ *if* $a^2 = a$
3. $1x = x$

4. $(a+b)x = ax + bx$ *if* $ab = 0$

5. $r(sx) = (rs)x$ *if* r *and* s *are invertible elements of* R

6. $a0 = 0$ *imply* $a^2 = a$

The zero element of V *and as well as that of* R *will be denoted by the same symbol* $'0'$.

Example 1.4. *Let* R *be any commutative regular ring with 1,which is not a Boolean ring and* $V = (V, +)$ *be the additive group of* R.*For any* $a \in R$, $x \in V$, *define the scalar multiplication of* a *and* x,*which will be denoted by* $a \otimes x$,*by putting* $a \otimes x = |a|x$, *the ring product of* $|a|$ *and* x.*Then it is easy to verify that all the properties* 1 - 5 *of definition* 1.3 *are satisfied.But* 6 *is not true for this system since* R *is not a Boolean ring.*

Remark 1.5. *Raja Gopala Rao[4] remarked that every R-vector space can be treated as B-vector space over the Boolean algebra of idempotents of regular ring R with the same scalar multiplication as in the R-vector space.*

Definition 1.6. *An R-vector space* V *is said to be normed if and only if there exists a mapping norm* $\|\|$:$V \to B$ *satisfying the following properties.*
(1). $\| x \| = 0 \Leftrightarrow x = 0$ *and (2).* $\| ax \| = a \| x \|$ *for all* $x \in V, a \in B$

Definition 1.7. *A finite subset of non zero elements* $x_1, ..., x_n$ *of an R-vector space* V *is called (linearly) independent over* R *if and only if* $a_1x_1 + ... + a_nx_n = 0$ *and* $a_1...a_n \neq 0$ *imply* $x_1 + ... + x_n = 0$ *and a subset* S *(of non zero elements)of* V *is said to be an independent subset of* V *if and only if every finite subset of* S *is linearly independent over* R.

Definition 1.8. *Let* V *be a vector space over a regular ring* R.*An independent subset of* V *which spans* V *is called basis for* V.

Remark 1.9. *If* G *is a basis group for* V,*then* $a_{x,g}$ *represents the uniquely determined coefficient of any* $g \in G$ *in the representation of* x *in terms of* G *and* $x = \sum_{g \in G} a_{x,g} g$.

Definition 1.10. *Let* V,W *be R-vector spaces over a regular ring* R. *A mapping* $T : V \to W$ *is a homomorphism of* V *to* W *provided*
(1.10) $T(ax + by) = aTx + bTy$ *if* $ab = 0$, $\forall x, y \in V$ *and* $a, b \in R$.

The set of homomorphisms from V *to* W *will be denoted by* $Hom(V, W)$.

2 Special Homomorphisms

Even though the condition 6 of definition 1.3 is independent of the other conditions 1 through 5,which was substantiated by an example in [4]given by Raja Gopala Rao,it can be relaxed without affecting the validity of the results obtained in [4,5].We restate the definition of R-vector space below by relaxing the condition 6 and adopt the same nomenclature given by Raja Gopala Rao[4].

Definition 2.1. *Let $V = (V, +)$ be any group and $R = (R, +, .)$ be a commutative regular ring with unity element 1.Then V is said to be a Vector space over R (or simply R-vector space)if and only if there exists a mapping :$R \times V \to V$ (the image of any $(a, x) \in R \times V$ will be denoted by ax) such that for all $x, y \in V$ and $a, b \in R$, all the following properties hold:*

1. $a^2(x + y) = ax + ay$
2. $a(bx) = (ab)x$ *if* $a^2 = a$
3. $1x = x$
4. $(a + b)x = ax + bx$ *if* $ab = 0$
5. $r(sx) = (rs)x$ *if r and s are invertible elements of R*

Remark 2.2. *The set of all linear endomorpisms defined by Raja Gopala Rao[4]in definition* 1.10 *is not closed under the binary operation.Consider the following*

Example 2.3. *Let $V = (Z_6, +)$ be the group of addition modulo* 6 *and $R = (Z_3, +, .)$ be the ring of residues modulo* 3*, which is a commutative regular ring with* 1*. Define scalar multiplication as* $\cdot: R \times V \to V$*,by* $0x = 0, 1x = x, 2x = x + 3, \forall x \in V$*.Define* $T, S : V \to V$ *by* $T(0) = 0, T(1) = 5, T(2) = 1, T(3) = 3, T(4) = 2, T(5) = 4$ *and* $S(0) = 0, S(1) = 4, S(2) = 1, S(3) = 3, S(4) = 2, S(5) = 5$*.Letting* $a = 0, b = 2$ *and* $y = 1$ *in (1.10),*$(T + S)(0x + 2y) = 4 \neq 0 = 0(T + S)y$*.Thus* $(T + S) \notin Hom(V, V)$*.*

Remark 2.4. *In view of the above example,we introduce the notion of special homomorphism in R-vector spaces which in turn gives the set of all special homomorphisms closed under the binary operation '+'.*

Definition 2.5. *Let V,W be R-vector spaces over a regular ring R.A mapping $T : V \to W$ is a special homomorphism of V to W provided*
(2.5) $T(ax + by) =\mid a \mid Tx+ \mid b \mid Ty$ *if* $ab = 0$*,* $\forall x, y \in V$ *and* $a, b \in R$*.*

The set of special homomorphisms from V to W will be denoted by $SHom(V, W)$.

Example 2.6. *Let V be an R-vector space as in example* 2.3 *above and* $T: V \to V$ *be a map defined by* $T(0) = 0 = T(3)$, $T(1) = 2 = T(4)$, $T(2) = 3 = T(5)$. *Then by easy verification of (2.5) T is a special homomorphism.*

Remark 2.7. *It seems to be that special homomorphism defined above coincides with the linear homomorphism in B-Vector space of N.V.Subrhamanyam [6]. But this is not the case.For instance in any B-vector space the property* $(a + b)x = ax + bx - abx$ *holds.However this is not true in the case of special homomorphism of R-vector spaces.It is justified in the following example given under*

Example 2.8. *Let V be an R-vector space as in example* 2.3 *and* $T : V \to V$ *be a map defined by* $T(0) = 0 = T(3)$, $T(1) = 2 = T(4)$, $T(2) = 3 = T(5)$.*Let* $a = 1, b = 2$ *in* Z_3 *and* $x = 1$ *in* Z_6. *Clearly* $(1 + 2)Tx = 3T1 = 0T1 = 0 \neq 2 = 1T1 + 2T1 - (1.2)T1 = 1Tx + 2Tx - (1)(2)Tx$.

We have the following

Lemma 2.9. *Let V and W be R-vector spaces over R and T be a mapping from V to W.* $T \in SHom(V, W) \Leftrightarrow T(ax) =\mid a \mid Tx$, $\forall x \in V$ *and* $a \in R$.

Proof. ($\Rightarrow$) Suppose $T \in SHom(V, W)$.Letting $b = 0$ in (2.5),$T(ax) =\mid a \mid Tx$ Conversely, let $T(ax) =\mid a \mid Tx$ for all $a \in R$ and $ab = 0$. Then $T(ax + by) = (|a| + 1 - |a|)T(ax + by) = |a|T(ax + by) + (1 - |a|)T(ax + by) = T(|a|(ax + by)) + T((1 - |a|)(ax + by)) = T[(|a|a)x + (|a|b)y)] + T[(1 - |a|)ax + (1 - |a|)by] = T(ax) + T(by)$(by lemma 1.2(2),theorem 1.2(3))= $|a|Tx + |b|Ty$. □

Theorem 2.10. *Let V and W be R-Vector spaces. Define + on* $SHom(V, W)$ *by* $(T + S)(x) = Tx + Sx$ *for all* $x \in V$ *and* $T, S \in SHom(V, W)$.*Then* $(SHom(V, W), +)$ *is an abelian group.*

Proof. Let $T, S \in SHom(V, W)$.Then$(T + S)(ax + by) = T(ax + by) + S(ax + by) = (|a|Tx + |b|Ty) + (|a|Sx + |b|Sy) = |a|(Tx + Sx) + |b|(Ty + Sy) = |a|(T + S)x + |b|(T + S)y$. Hence $T + S \in SHom(V, W)$.It is routine to verify the remaining axioms of the abelian group. □

Theorem 2.11. *Let V and W be R-Vector spaces.Then* $SHom(V, W)$ *is an R-vector space if the scalar multiplication is defined by* $(aT)(x) = |a|Tx$,$\forall x \in V$, $T \in SHom(V, W)$ *and* $a \in R$.

Proof. Routine verification of axioms of definition 2.1. □

Definition 2.12. *Let V, W be R-vector spaces over a regular ring R. A mapping* $T : V \to W$ *is a strong special homomorphism of V to W provided*
(2.12) $T(ax + by) = |a|Tx + |b|Ty$, $\forall x, y \in V$ *and* $a, b \in R$.

The set of strong special homomorphism from V to W will be denoted by $SSHom(V, W)$.

We characterize $SSHom(V,W)$ in the following

Theorem 2.13. *Let V,W be R-vector spaces and $T \in SHom(V,W)$.*
$T \in SSHom(V,W) \Leftrightarrow T(x+y) = Tx + Ty, \forall x, y \in V$.

Proof. Letting $a = b = 1$ in(2.12),then we have $T(x+y) = Tx+Ty$.Conversely,let $T(x+y) = Tx+Ty$, $\forall x, y \in V$.Then $T(ax+by) = T(ax)+T(by) = |a|Tx+|b|Ty$ (by lemma 2.9). □

Theorem 2.14. *Let V and W be R-vector spaces.Then $SSHom(V,W)$ is a subspace of $SHom(V,W)$.*

Proof. Since $\tilde{0}(ax + by) = 0 =| a | \tilde{0}x+ | b | \tilde{0}y =| a | 0+ | b | 0$, it is clear that $\tilde{0} \in SSHom(V,W)$. Let $T, S \in SSHom(V,W)$ and $x, y \in V$. $(T - S)(x + y) = (T + (-S))(x + y) = T(x + y) + (-S)(x + y) = Tx + Ty + (-S)x + (-S)y = (T - S)x + (T - S)y$. Thus $T - S \in SSHom(V,W)$. Finally,$(aT)(x + y) = |a|T(x + y) = |a|(Tx + Ty) = |a|Tx + |a|Ty = (aT)x + (aT)y$. Thus $(aT) \in SSHom(V,W)$. Hence, $SSHom(V,W)$ is a subspace of $SHom(V,W)$. □

Remark 2.15. *Observe that $T(ax) = |a|Tx$ and $(aT)(x) = |a|Tx$.Hence $T(ax) = (aT)(x)$.*

Theorem 2.16. *let V and W be R-vector spaces with V having basis $G^\star$. Let $L : G \to W$ be a map such that $L0 = 0$, where G is a group. Then L can be uniquely extended to an element of $SHom(V,W)$.*

Proof. If $x \in V$, then $x = \sum_{g\in G^\star} a_{x,g}g, a_{x,g} \in R$. Define $T : V \to W$ by $T(x) = \sum_{g\in G^\star} | a_{x,g} | Lg$, $x \in V$. Now let $c \in R$,then $T(cx) = \sum_{g\in G^\star} | a_{cx,g} | Lg = \sum_{g\in G^\star} | ca_{x,g} | Lg = \sum_{g\in G^\star} | c || a_{x,g} | Lg =| c | \sum_{g\in G^\star} | a_{x,g} | Lg =| c | T(x)$.Thus, $T \in SHom(V,W)$.We claim that $Tg = Lg$ for all $g \in G^\star$.
For $x = 0$, $T0 = \sum_{g\in G^\star} | a_{0,g} | Lg =| a_{0,0} | L0 + \sum_{gi\neq 0} | a_{0,gi} | Lgi = L0$.
For $x = g$, $Tg = \sum_{g\in G^\star} | a_{x,g} | Lg =| a_{g,g} | Lg = 1.Lg = Lg$.Thus, $Tg = Lg, \forall g \in G$.Therefore, T is an extension of L to an element of $SHom(V,W)$.Suppose S is an extension of L such that $S \in SHom(V,W)$.Since S is an extension of L $Sg = Lg$,$\forall g \in G$.Let $x \in V$, then $x = \sum_{g\in G^\star} a_{x,g}g$. $S(x) = Sx = S(\sum_{g\in G^\star} a_{x,g}g) = \sum_{g\in G^\star} |a_{x,g}|Sg = \sum_{g\in G^\star} |a_{x,g}|Lg = Tx$.Thus $Sx = Tx$, $\forall x \in V$.Hence the theorem.

□

Theorem 2.17. *Let V,W be normed R-vector spaces and V has basis $G^\star$.If B is complete up to cardinality of $G^\star$, then $\| T \| = \sum_{g\in G^\star} | Tg |$ defines a norm on $SHom(V,W)$ and hence $SHom(V,W)$ is a normed R-vector space.*

Proof. Let $\tilde{0} \in SHom(V,W)$.Then $\tilde{0}(x) = 0$, $\forall x \in V$ and $\| \tilde{0} \| = \sum_{g\in G^\star} | \tilde{0}g | = \sum_{g\in G^\star} | 0 | = 0$.Let $T \in SHom(V,W)$ and $\| T \| = 0$.Then $\sum_{g\in G^\star} | Tg | = 0$.Hence $| Tg | = 0$, $\forall g \in G^\star$ and $T0 = 0$.Thus $Tg = 0 = \tilde{0}g$, $\forall g \in G$.Hence $Tx = 0 = \tilde{0}x, \forall x \in V$.Thus $T = \tilde{0}$.Let $a \in B$ and $T \in SHom(V,W)$.
Now, $\| aT \| = \sum_{g\in G^\star} | (aT)g | = \sum_{g\in G^\star} \| a \mid Tg | = | a | \sum_{g\in G^\star} | Tg | = a \| T \|$. □

Theorem 2.18. *Let V,W be R-vector spaces and $G^\star$ be basis of V. If H is a subgroup of G and $S \in SHom(< H >,W)$, then S can be extended to an element of $SHom(V,W)$.*

Proof. Define $R : G \to W$ by $Rg = 0$ for $g \notin H$ and $Rg = Sg$ for $g \in H$.Since $S \in SHom(< H >,W)$, $S0 = 0$. Hence $R0 = S0 = 0$. Thus $R : G \to W$ such that $R0 = 0$. By theorem 2.16, R can be extended to $T \in SHom(V,W)$. Let $H^\star$ is a basis of $< H >$ and $x \in < H >$. Then $x = a_1g_1 + ... + a_ng_n$.Now $Tx = T(\sum_{g_i\in H^\star} a_ig_i) = \sum_{g_i\in H^\star} |a_i|Tg_i = \sum_{g_i\in H^\star} |a_i|Rg_i = \sum_{g_i\in H^\star} |a_i|Sg_i = S(\sum_{g_i\in H^\star} a_ig_i) = Sx$.Where $1 \leq i \leq n$.Hence, T is an extension of S to an element of $SHom(V,W)$. □

We conclude this paper by the following

Theorem 2.19. *Let V and W be R-vector spaces over R.If V has a finite basis $G^\star = \{x_1, ..., x_n\}$, then $SHom(V,W)$ is isomorphic to $\sum_{i=1}^{n} W_i$, $W_i = W$, $\forall 1 \leq i \leq n$.*

Proof. Let $y \in \sum_{i=1}^{n} W_i$. Then $y = (y_1, ..., y_n)$, $y_i \in W$, $\forall 1 \leq i \leq n$.Define a mapping $\theta : SHom(V,W) \to \sum_{i=1}^{n} W_i$ by $\theta(T) = (Tx_1, ..., Tx_n)$ for each $T \in SHom(V,W)$.Let $T,S \in SHom(V,W)$ and $a \in R$. Now $\theta(S + T) = ((S + T)x_1, ..., (S + T)x_n) = (Sx_1 + Tx_1, ..., Sx_n + Tx_n) = (Sx_1, ..., Sx_n) + (Tx_1, ..., Tx_n) = \theta(S)+\theta(T)$.Simiarily,$\theta(aT) = ((aT)x_1, ..., (aT)x_n) = (|a|Tx_1, ..., |a|Tx_n) = |a|(Tx_1, ..., Tx_n) = |a|\theta(T)$.Hence, θ is a homomorphism.

Let $T, S \in SHom(V,W)$ such that $\theta(T) = \theta(S) \Leftrightarrow (Tx_1, ..., Tx_n) = (Sx_1, ..., Sx_n) \Leftrightarrow Tx_i = Sx_i, \forall 1 \leq i \leq n$.Also we have $S0 = 0 = T0$. So $Tx = Sx$, $\forall x$. Thus $T = S$.

Let $y \in \sum_{i=1}^{n} W_i$. Then $y = (y_1, ..., y_n)$ for some $y_i \in W$.Let $\alpha : G \to W$ be the mapping defined by $\alpha 0 = 0$ and $\alpha x_i = y_i$, $i = 1, 2, 3, ..., n$.Then by theorem 2.16, there is a unique extension $T : SHom(V,W) \to \sum_{i=1}^{n} W_i$ of α.

Consequently, $\theta(T) = (Tx_1, ..., Tx_n) = (\alpha x_1, ..., \alpha x_n) = (y_1, ..., y_n)$. Hence,$\theta$ is an isomorphism of $SHom(V,W)$ on to $\sum_{i=1}^{n} W_i$. □

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Annex - B :

Functionals in R-Vector spaces

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Functionals in R-vector Spaces

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Abstract

In this paper we introduce the notion of functionals on R-vector spaces and obtain various properties.We also introduce the concept of dual spaces and Inner product in R-vector spaces and study their properties.

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Keywords: R-vector space, Special homomorphism, Functional, Dual Space, Inner Product

Introduction

The notion of R-vector spaces was introduced by Raja Gopala Rao[4] as a generalization to the concept of Vector spaces over Boolean algebras (or simply Boolean vector spaces)of Subrahmanyam [6]. F.O.Stroup [1] has made a study on functionals , dual spaces, inner product spaces in Boolean vector spaces. In this paper we extend these notions to R-vector spaces and obtain certain properties.In [3] the authors have studied on special homomorphisms of R-vector spaces.This paper is a further continuation on special homomorphisms and we introduce the notion of functionals using special homomorphisms. This paper consists of four sections.In section one,we will collect certain basic definitions and results concerning R-vector spaces and special homomorphisms.In section two,we introduce the notion of functionals on R-vector spaces and obtain some basic results about functionals. Section three is meant for the study of dual spaces. we prove the necessary and sufficient condition for two R-vector spaces to be dual in theorem 3.11.Finally, In section four, we introduce the notion of inner product and prove that V is a dual space to itself if V is normed in theorem 4.4.

1 Preliminaries

We collect some definitions and results concerning R-vector spaces from [2,4,5] and special homomorphisms from [3].

We recall the following

Let $(R, +, .)$ be a commutative regular ring with 1. Let $a \in R$,then $a^* = yay$ for some y in R which is again a regular element and $\mid a \mid= aa^*$ .Clearly $\mid a \mid$ is an idempotent element of R and R_B denotes the set of all Boolean elements of R.

Definition 1.1. *Let $V = (V, +)$ be any group and $R = (R, +, .)$ be a commutative regular ring with unity element 1.Then V is said to be a Vector space over R (or simply R-vector space)if and only if there exists a mapping :$R \times V \to V$ (the image of any $(a, x) \in R \times V$ will be denoted by ax) such that for all $x, y \in V$ and $a, b \in R$, all the following properties hold.*

1. $a^2(x + y) = ax + ay$
2. $a(bx) = (ab)x$ *if* $a^2 = a$
3. $1x = x$
4. $(a + b)x = ax + bx$ *if* $ab = 0$
5. $r(sx) = (rs)x$ *if* r *and* s *are invertible elements of* R

Definition 1.2. *An R-vector space V is said to be normed if and only if there exists a mapping norm $||$:$V \to B$ satisfying the following properties.*
(1). $|x| = 0 \Leftrightarrow x = 0$ and (2). $|ax| = a|x|$ for all $x \in V, a \in B$

Definition 1.3. *If $G^\star$ is a basis of V and $g \epsilon G^\star$,then $|g| = 1$.*

Definition 1.4. *A finite subset of non zero elements $x_1, ..., x_n$ of an R-vector space V is called (linearly) independent over R if and only if $a_1x_1+...+a_nx_n = 0$ and $a_1...a_n \neq 0$ imply $x_1 + ... + x_n = 0$ and a subset S (of non zero elements)of V is said to be an independent subset of V if and only if every finite subset of S is linearly independent over R.*

Definition 1.5. *Let V,W be R-vector spaces over a regular ring R.A mapping $T : V \to W$ is a special homomorphism of V to W provided*
$T(ax + by) =| a | Tx+ | b | Ty$ *if* $ab = 0$, $\forall x, y \in V$ *and* $a, b \in R$.

The set of special homomorphisms from V to W will be denoted by $SHom(V, W)$.

Lemma 1.6. *Let V and W be R-vector spaces and T be a mapping from V to W. $T \in SHom(V, W) \Leftrightarrow T(ax) =| a | Tx$, $\forall x \in V$ and $a \in R$.*

Theorem 1.7. *Let V and W be R-Vector spaces.Then $SHom(V, W)$ is an R-vector space if the scalar multiplication is defined by $(aT)(x) = |a|Tx$,$\forall x \in V$, $T \in SHom(V, W)$ and $a \in R$.*

Theorem 1.8. *Let V and W be R-vector spaces.If V has a finite basis $G^\star = \{x_1, ..., x_n\}$, then $SHom(V, W)$ is isomorphic to $\sum_{i=1}^{n} W_i$, $W_i = W$, $\forall 1 \leq i \leq n$.*

2 Functionals

In this section we introduce functionals on R-vector spaces and obtain certain properties.

Let $(R, +, .)$ be a commutative regular ring with 1 which is not a Boolean ring and $W = (R, +)$ be any additive abelian group of R. Define $\otimes : R \times W \to W$ by $a \otimes x =| a | x$,$\forall a \in R$, $x \in W$, which is the ring product of $| a |$ and x.

We have the following simple

Remark 2.1. *W is an R-vector space.*

Lemma 2.2. *If $R = (R, +, .)$ is a regular ring and $W = (R, +)$ be an R-vector space.Then*
(i) 1 is a basis of $W = (R, +)$
(ii) $| x |= x$ for each $x \in R_B$.

Proof. Trivial □

Definition 2.3. *Let V and $W = (R, +)$ be R-vector spaces.A mapping T from V to W is called a linear functional on V provided:*
$T(a \otimes x + b \otimes y) = a \otimes Tx + b \otimes Ty$,$ab = 0$,$x, y \in V$ *and* $a, b \in R$.

Remark 2.4. *The set of all linear functionals on V will be denoted by*
$\overline{V} := SHom(V, W)$,*where* $W = (R, +)$.

Definition 2.5. *Let V and $W = (R, +)$ be R-vector spaces.A mapping T from V to W is called a strong linear functional on V provided:*
$T(a \otimes x + b \otimes y) = a \otimes Tx + b \otimes Ty, \forall x, y \in V$ *and* $a, b \in R$.

Remark 2.6. *Let $\overline{V}$=$\{T|T : V \to W$ is a SHom $\}$ where V, W are R-vector spaces.Clearly $(\overline{V}, +)$ is an abelian group.If we define $\otimes : R \times \overline{V} \to \overline{V}$ by $(a, T)(x) = (a \otimes T)(x) =| a | (Tx)$,where $(T+S)(x) = Tx+Sx$ and $(a \otimes T)(x) = | a | Tx$ for all ,$a \in R$,$x \in V$,$T, S \in \overline{V}$,$\overline{V}$ is an R-vector space.*

We furnish the following as a special case of theorem 1.8

Theorem 2.7. *If V is an R-vector space of finite dimension n and $Wi = (R, +)$ for $i = 1, 2, ..., n$,then $\overline{V}$ is isomorphic to $\sum_{i=1}^{n} W_i$,where $(R, +)$ is considered as an R-vector space.*

Lemma 2.8. *If V is a normed R-vector space and $Nx = |x|$ for some $x \in V$. Then $N \in \overline{V}$.*

Proof. $N(a \otimes x) = N(|a|x) = |a||x| = |a|Nx = a \otimes Nx$,$x \in V$ and $a \in R$ □

Remark 2.9. *Let V be a normed R-vector space.For each $a \in R$, $\overline{a}$ be a mapping of V to B defined by $\overline{a}(x) =| a || x |= a \otimes | x |$ for each $x \in V$.*

Lemma 2.10. *If V is a normed R-vector space and $a \in R$, then $\overline{a} \in \overline{V} = SHom(V, W)$.*

Proof. $\overline{a}(b \otimes x) = a \otimes | b \otimes x |= |a|||b|x| = |b|(a \otimes |x|) = |b|\overline{a}(x) = b \otimes \overline{a}(x)$.Thus the result holds by lemma 1.6.

□

Lemma 2.11. *If V is a normed R-vector space and $a, b \in R_B$,then*
(i). $\overline{a + b} = \overline{a} + \overline{b}$
(ii). $a\overline{b} = \overline{ab}$
(iii. $\overline{ab} = \overline{a}\overline{b}$

Proof. From the above lemma,$\overline{a}$, $\overline{b}$, $\overline{a + b}$, $\overline{ab} \epsilon \overline{V}$. The expressions $\overline{a}+\overline{b}$,$a\overline{b}$ and $\overline{a}\overline{b}$ are well defined according to $(T + S)x = Tx + Sx$,$(a \otimes T)x = |a|Tx$ and $(TS)x = |Tx|(Sx)$ respectively.
(i). $(\overline{a + b})(x) = |a + b||x| = (a + b)|x| = a|x| + b|x| = \overline{a}(x) + \overline{b}(x) = (\overline{a} + \overline{b})(x)$.
(ii). $(a\overline{b})(x) = a(\overline{b}(x)) = a(|b||x|) = a(b|x|) = (ab)|x| = |ab||x| = \overline{ab}(x)$.
(iii). $\overline{ab}(x) = |ab||x| = (|a||b|)|x| = (|a||x|)(|b||x|) = \overline{a}(x)\overline{b}(x) = |\overline{a}(x)|\overline{b}(x) = (\overline{a}\overline{b})(x)$. □

Lemma 2.12. *If V is a normed R-vector space and $[V] = R_B$, then $\overline{a} = \overline{b} \Rightarrow a = b$.*

Proof. Suppose $\overline{a} = \overline{b}$ for some a,b ϵR_B. Since $[V] = R_B$,there exist $x, y \epsilon V$ such that $|x| = |a| = a$ and $|y| = |b| = b$.
By the remark 2.9, $\overline{a}(x) = |a||x| = aa = a$ and $\overline{b}(x) = |b||x| = |b||a| = ba$.
since $\overline{a} = \overline{b}$,we have $a = ba$ and $b = ab$. □

Theorem 2.13. *If V is a normed R-vector spaces and $[V] = R_B$,then*
(i) The R-vector spaces $W = (R, +)$ is isomorphically contained in the R-vector space $\overline{V}$.
(ii) The regular ring $(R, +, .)$ is isomorphically contained in the regular ring $(\overline{V}, +, .)$.

Proof. Consider the mapping $\gamma : R_B \to \overline{V}$ defined by $\gamma(a) = \overline{a}$ for each $a \epsilon R_B$.By lemma 2.12, γ is one to one.To prove (i), let $x, y \epsilon W = (R, +)$ and $a \epsilon R_B$. Then by lemma 2,11, we have $\gamma(x + y) = \overline{x + y} = \overline{x} + \overline{y} = \gamma(x) + \gamma(x)$ and $\gamma(a \otimes x) = a \otimes \gamma(x)$.Hence, γ is an isomorphism. For proving (ii),let $a, b \epsilon R_B$, then $\gamma(ab) = \overline{ab} = \overline{a}\overline{b} = \gamma(a)\gamma(b)$. Similarly $\gamma(a + b) = \overline{a} + \overline{b} = \gamma(a) + \gamma(b)$.Hence, γ is an isomorphism of the regular ring $(R, +, .)$ in to the regular ring $(\overline{V}, +, .)$.

□

3 Dual spaces

In this section we introduce dual spaces and study certain properties.

Definition 3.1. *Let V and W be R-vector spaces and $V \times W = \{(x, z) : x \in V, z \in W\}$.A mapping $U : V \times W \to R$ is called a bilinear function on $V \times W$ provided:*
(3.1.1) $U(a \otimes x + b \otimes y, z) = a \otimes U(x, z) + b \otimes U(y, z)$
(3.1.2) $U(x, a \otimes w + b \otimes z) = a \otimes U(x, w) + b \otimes U(x, z)$, $\forall x, y \in V$ and $w, z \in W$ whenever $a, b \in R$,$ab = 0$.

Definition 3.2. *If V and W are R-vector spaces,then a bilinear function U on $V \times W$ is called non-degenerate provided:*
(3.2.1) $U(x, z) = 0$ for each $z \in W \Rightarrow x = 0$
(3.2.2) $U(x, z) = 0$ for each $x \in V \Rightarrow z = 0$

Definition 3.3. *If V and W are R-vector spaces and U is a non-degenerate bilinear function on $V \times W$,then V and W are said to be dual spaces with respect to U.*

Remark 3.4. *In general, two R-vector spaces will be called dual spaces if they are dual with respect to at least one bilinear function.*

Remark 3.5. *Let V and W be R-vector spaces.If $U : V \times W \to R$ is a bilinear function on $V \times W$, then $U' : W \times V \to R$ is also a bilinear function on $W \times V$ when $U(x, z) = U'(z, x)$.*

Corollary 3.6. *If U is non-degenerate,then U' is also non-degenerate.*

Proof. follows immediately from the definition. □

Lemma 3.7. *Let V and W be R-vector spaces. If U is a bilinear function on $V \times W$, then*
(i). $U(a \otimes x, z) = a \otimes U(x, z)$
(ii).$U(x, b \otimes z) = b \otimes U(x, z), \forall x \in V, z \in W$ and $a, b \in R$.

Proof. Letting $b = 0$ in (3.1.1) and $a = 0$ in (3.1.2) respectively yields the desired conclusions. □

Theorem 3.8. *If V is a normed R-vector space and $\overline{V}$ is a space of linear functionals on V, then V and $\overline{V}$ are dual spaces.*

Proof. Let $U : V \times \overline{V} \to \underline{(R, +)}$ defined by $U(x, T) = Tx$ for all $x \epsilon V$ and $T \epsilon \overline{V}$.Suppose $x, y \epsilon V$, $T, S\ \epsilon \overline{V}$ and $a, b \epsilon R$ with $ab = 0$.By definition 2.3, we have $U(a \otimes x + b \otimes y, T) = a \otimes U(x, T) + b \otimes U(y, T)$ and by remark 2.6,$U(x, a \otimes T + b \otimes S) = a \otimes U(x, T) + b \otimes U(x, S)$.Thus,by definition 3.1, U is a bilinear functional on $V \times \overline{V}$.Suppose $x \epsilon V^\star$ and $T \epsilon \overline{V^\star}$.To show U is non degenerate,it will suffice to establish the existence of elements $N \epsilon \overline{V}$ and $y \epsilon V$ such that $U(x, N) \neq 0$ and $U(y, T) \neq 0$.Since $T \epsilon \overline{V^\star}$, T is not the zero mapping.Hence, there exists at least one element $y \epsilon V$ for which $Ty \neq 0$.Let N denote the norm mapping on V.Then by lemma 2.8,it is clear that $N \epsilon \overline{V}$. Since $x \neq 0$, $Nx = |x| \neq 0$. we have $\Rightarrow U(x, N) = Nx \neq 0$ and $U(y, T) = Ty \neq 0$. Hence by definition 3.2, U is a non degenerate bilinear functional on $V \times \overline{V}$. Thus,V and $\overline{V}$ are dual spaces. □

Definition 3.9. *Let V be an R-vector space.A subset M of $\bar{V}$ is called a total subset of $\bar{V}$ provided that for each $x \epsilon V^\star$ there exists an element T of M such that $Tx \neq 0$.*

Lemma 3.10. *If V is a normed R-vector space, then $\bar{V}$ is a total set.*

Proof. Let $Nx = |x|$ for each $x \epsilon V$. Then $N \epsilon \overline{V}$ by lemma 2.8.If $x \neq 0$,then $Nx = |x| \neq 0$. Thus,$\overline{V}$ is a total set. □

Theorem 3.11. *Let V and W be R-vector spaces. A necessary and sufficient condition for V and W to be dual spaces is the existence of a special homomorphism T of V in to $\bar{W}$ such that $T(V)$ is a total subset of $\bar{W}$ and $T^{-1}\{0\} = \{0\}$.*

4 Inner Product

Subrahmanyam established that any normed boolean vector space V admits a unique "inner product" mapping,$[x, y]$,of $V \times V$ in to B.
In this section we introduce the notion of inner product on R-vector spaces and study its properties.

Definition 4.1. *Let V be a normed R-Vector space, $[\quad] : V \times V \to R_B$, is an inner product mapping such that*
(i). $[x, y]|x - y| = 0$
(ii). $[x, y] + |x - y| = |x| + |y|$
(iii).$[x, y] = [y, x]$
(iv). $[a \otimes x + b \otimes z, y] = a \otimes [x, y] + b \otimes [z, y]$, $ab = 0$

Let V be a normed R-vector space.For each $x \epsilon V$, let $\bar{x}$ denote the mapping of V in to R_B defined by $\bar{x}(y) = [x, y]$ for each $y \epsilon V$.

Lemma 4.2. *If V is a normed R-vector space, then $\{\bar{x} : x \epsilon V\}$ is a total subset of $\bar{V}$.*

Proof. Let $x, y, z \epsilon V$ and $a, b \epsilon R_B$ with $ab = 0$. Then $\overline{x}(a \otimes y + b \otimes z)$ $= [x, a \otimes y + b \otimes z] = [a \otimes y + b \otimes z, x] = a \otimes [y, x] + b \otimes [z, x] = a \otimes [x, y] + b \otimes [x, z]$ $= a \otimes \overline{x}(y) + b \otimes \overline{x}(z)$.Thus $\overline{x} \epsilon V$. Suppose $x \epsilon V^{\star}$.By 4.1 (ii), $[x, x] = |x| + |x| = |x|$.Since $x \neq 0$,it is clear that $|x| \neq 0$.Hence $\overline{x}(x) = [x, x] \neq 0$. Thus,the result holds. □

Remark 4.3. *Let V be a normed R-vector space.If $[x, y] = [z, y]$ for each $y \epsilon V$ then $x = z$.*

Theorem 4.4. *If V is a normed R-vector space, then V is a dual space to itself.*

Proof. Consider the mapping $T : V \to \overline{V}$ defined by $Tx = \overline{x}$ for each $x \epsilon V$,where $\overline{x}(y) = [x, y]$ for each $y \epsilon V$.By lemma 4.2,$T(V)$ is a total subset of $\overline{V}$.Now let $x, y, z \epsilon V$ and $a, b \epsilon R_B$ with $ab = 0$. Then $\overline{a \otimes x + b \otimes y}(z) = [a \otimes x + b \otimes y, z] = a \otimes [x, z] + b \otimes [y, z] = a \otimes [x, y] + b \otimes [x, z] = a \otimes \overline{x}(z) + b \otimes \overline{y}(z) = (a \otimes \overline{x} + b \otimes \overline{y})(z)$. Hence, $T(a \otimes x + b \otimes y) = \overline{a \otimes x + b \otimes y} = a \otimes \overline{x} + b \otimes \overline{y} = a \otimes Tx + b \otimes Ty$. Thus, $T \epsilon SHom(V, \overline{V})$.Finally suppose $Tx = \overline{x}$ is the zero mapping in $\overline{V}$. Then $\overline{x}(y) = [x, y] = 0$ for each $x \epsilon V$.Taking $a = b = 0$ in (iv) of definition 4.1 yields $[0, y] = 0$ for each $y \epsilon V$.Hence by remark 4.3, $x = 0$.Thus $T^{-1}\{0\} = \{0\}$. Hence the theorem follows from definition 4.1 and Theorem 3.11. □

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