BASIC STRUCTURE OF H^P-SPACES ON THE UNIT DISK

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Declaration

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any degree, diploma, associate ship, fellowship or any other similar title to me.

Eshetu Belete

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INTRODUCTION

This paper begins with the classical representation theorems for certain class of harmonic functions in the unit disc, together with some basic results on boundary behavior and some discussion of subhamonic functions.

The **Hardy spaces** are certain space of holomorphic functions on the unit disk. They were introduced by Frigyes Riesz (Riesz 1923), who named then after G.H.Hardy, because of the paper (Hardy 1915)

The Hardy space H^P for $0 < P < \infty$ is the class of holomorphic functions f on the open unit disk U satisfying

$$\sup_{r\to 1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^P d\theta\right\}^{1/P} = \|f\|_P < \infty$$

and H^∞ is the class of bounded holomorphic functions on the unit disk .

The paper shows that H^P is a vector space and when $P \geq 1$, H^P is a Banach space with norm $\|.\|_P$.

Fatou's theorem showed that each harmonic function u on U such that

$$\sup_{0<r<1}\|u_r\|_P < \infty$$

for $1 < P \leq \infty$ has a non-tangential limits $f(e^{i\theta})$ almost everywhere on T, where T denotes the unit circle which we equip with normalized Lebesgue measure .Also u can be expressed as the Poisson integral of f.

After introducing Blaschke product and F.Riesz theorem this paper shown that every function f in H^P,for $0 < P \leq \infty$ has nontangential limits $f^* \in L^P(T)$ almost everywhere on T ,and t $f(z) = 0$ if its boundary function vanish on a set of positive measure: Moreover $\|f\|_P = \|f^*\|_P$.

Finally, for $0 < P \leq \infty$, every non-zero function f in H^P can be written as the product $f = MQ$, where M is an inner function and Q is an outer function.

Chapter 1

Preliminaries

As a preparation, we discuss some facts about harmonic function, Poisson integral formula, nontangential Maximal function and nontangential limits, subharmonic functions on a unit disc.

1.1. Harmonic function.

1.1.1. Definition. If G is an open subset of $\mathbb{C}$ then a function $u: G \to R$ is **harmonic** if u has continuous second order partial derivatives and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

A **complex function** f in the open set G is called **harmonic** if both its real part and its imaginary part are harmonic in G.

To prove Theorem 1.1.2 we use Theorem 2.29 in John B.Conway [2], page 42.

1.1.2 Theorem. Every holomorphic functions are harmonic.

Proof Suppose f is holomorphic in the open set G and

$u(x, y) = Ref(x + iy), v(x, y) = Imf(x + iy) \ \ for\ x + iy\ in\ G.$

Then $u\ and\ v$ satisfy the Cauchy-Riemann equation

$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$,

and $u\ and\ v$ have continuous second partial derivative. Differentiating the Cauchy-Riemann equations again we get

$$\frac{\partial^2 u}{\partial^2 x} = -\frac{\partial^2 v}{\partial x \partial y} \quad and \qquad \frac{\partial^2 u}{\partial^2 y} = \frac{\partial^2 v}{\partial y \partial x}$$

$$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = 0$$

Hence u is harmonic, similarly v is also harmonic, so is f. $\blacksquare$

1.2. Poisson integrals.

1.2.1. Definition. The function

$$P_r(\theta) = \sum_{-\infty}^{\infty} r^{|n|} e^{in\theta}$$

for $0 \leq r < 1$ and $-\infty < \theta < \infty$,is called the **Poisson Kernel**.

Let $z = re^{i\theta}, 0 \leq r < 1$; then

$$\frac{1 + re^{i\theta}}{1 - re^{i\theta}} = (1 + z)(1 + z + z^2 + \cdots) = 1 + 2\sum_{n=1}^{\infty} z^n$$

$$= 1 + 2\sum_{n=1}^{\infty} r^n e^{in\theta}$$

Hence,

$$Re\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right) = 1 + 2\sum_{n=1}^{\infty} r^n cosn\theta = 1 + \sum_{n=1}^{\infty} r^n \left(e^{in\theta} + e^{-in\theta}\right) = \sum_{-\infty}^{\infty} r^{|n|} e^{in\theta} = P_r(\theta)$$

Also $\frac{1+re^{i\theta}}{1-re^{i\theta}} = \frac{1+re^{i\theta}-re^{-i\theta}-r^2}{|1-re^{i\theta}|^2}$, so that

$$P_r(\theta) = \frac{1 - r^2}{1 - 2rcos\theta + r^2}$$

1.2.2 Theorem The Poisson Kernel satisfies the following

(a).$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r\,(\theta)d\theta = 1$;

(b). $P_r(\theta) > 0$ for all $\theta, P_r(-\theta) = P_r(\theta)$ $and\ P_r\ is$ periodic in θ with period 2π;

(c).$P_r(\theta) < P_r(\delta)\ if\ 0 < \delta < |\theta| \leq \pi$

(d).for each $\delta > 0$,$\lim_{r \to 1^-} P_r(\theta) = 0$ uniformly in θ for $\pi \geq |\theta| \geq \delta$.

1.2.3. Definition. Let T is the boundary of U and if $f \in L^1(T)$,

$$F(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r\,(\theta - t) f(e^{it})dt \qquad (1)$$

then the function F so defined in U is called **Poisson integral of $\boldsymbol{f}$**. We shall sometimes abbreviate the relation (1) to

$$F = P[f]$$

Note:-Theorem 1.2.5 will provides the solution of a boundary value problem(namely the Dirichlet problem).A continuous function f is given on T and it is reguired to find a harmonic function $F\ in\ U$ whose boundary values are f.The theorem exhibits a solution, by the means of the Poisson integral of f. To do this first we need the following theorems.

1.2.4 Theorem If $f \in L^1(T)$ then the Poisson integral $P[f]$ is a harmonic function in U.

Proof Without loss of generality assume f is real valued function.

If $0 \leq r < 1\ then$

$$P[f](re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} Re\left(\frac{1 + re^{i(\theta-t)}}{1 - re^{i(\theta-t)}}\right) f(e^{it})dt$$

$$= Re\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} f(e^{it})dt\right)$$

So defined $g: U \to R\ by$

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} f\left(e^{it}\right)dt$$

Since $P[f] = Re(g)$ we need only show that g is holomorphic on U. But this is a consequence of Leibniz's rule which is stated and proved in John B.Conway [2], page 73.$\blacksquare$

1.2.5 Theorem Suppose $f{:}\,T \to R$ is a continuous function then there is a continuous function $u{:}\,\overline{U} \to R$ such that

(a).$u(z) = f(z)$ for all $z \in T$

(b).u is harmonic in D.

Moreover u is unique and is defined by

$$u\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f\left(e^{it}\right)dt \qquad (1)$$

for $0 \leq r < 1, 0 \leq \theta \leq 2\pi$.

Proof Define $u{:}\,\overline{U} \to R$ by letting $u(re^{i\theta})$ be in (1) if $0 \leq r < 1$ and $u\left(e^{i\theta}\right) = f(e^{i\theta})$

Clearly u satisfies part (a) and from Theorem 1.2.4 u is harmonic on U; it remain to show that u is continuous on $\overline{U}$.This is also proved in John B.Conway [2], page 253.$\blacksquare$

1.2.6 Corollary If $u{:}\,\overline{U} \to R$ is a continous function that is harmonic in U then

$$u\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) u\left(e^{it}\right)dt$$

for $0 \leq r < 1$ and all θ.Moreover, u is the real part of the holomorphic function

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} u\left(e^{it}\right)dt$$

Proof From Theorem 1.2.5 the first part of the corollary is a direct consequence and the proof of Theorem 1.2.4 proved that $u(z) = Re(f(z))$ and f is holomorphic.$\blacksquare$

1.2.8 Boundary Behavior of Poisson integrals

Our next objective is to find analogues of Theorem 1.2.5 for Poisson integrals of L^P −functions and measures on T.Let us associate to any function u in U a family of functions u_r on T, defined by

$$u_r(e^{i\theta}) = u(re^{i\theta}) \qquad ,(0 \leq r < 1) \qquad (1)$$

Thus u_r is essentially the restriction of u to the circle with radius r and center 0, but we shift the domain of u_r to T

Using this terminology, Theorem 1.2.5 can be state in the following form:

If f is continoues on T and $F = P[f]$, then $F_r \to f$ uniformly on T,as $r \to 1$.In other words,

$$\lim_{r\to 1} ||F_r - f||_\infty = 0 \qquad (2)$$

This implies of course that

$$\lim_{r\to 1} F_r\left(e^{i\theta}\right) = f(e^{i\theta})$$

at every point of T .we call f is the radial limits of F.As regards (2), we shall now see (theorem 1.2.9) that corresponds norm-convergence result is just as easy in L^P.

1.2.9 Theorem If $1 \leq P \leq \infty, f \in L^P(T)$ and $U = P[f]$, then

$$||u_r||_P \leq \left||f|\right|_P \qquad 0 \leq r < 1 \qquad (1)$$

If $1 \leq P < \infty$, then

$$\lim_{r\to 1} ||u_r - f||_P = 0 \qquad (2)$$

Proof now

$$u_r\left(e^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, f\left(e^{it}\right)dt \qquad (3)$$

If we apply Jensen inequality (or Holder’s) to (3) we obtain

$$\left|u_r\left(e^{i\theta}\right)\right|^P \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\left|f\left(e^{it}\right)\right|^P dt$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|u_r\left(e^{i\theta}\right)\right|^P d\theta \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f\left(e^{it}\right)\right|^P P_r(\theta - t)dt\right]d\theta$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r\,(\theta - t)d\theta\right]|f(e^{it})|^P dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f\left(e^{it}\right)\right|^p dt \; = \int_T |f|^P d\sigma = \left||f|\right|_P^P$$

$$\Rightarrow ||u_r||_P \leq ||f||_P$$

To prove (2), since $C(T)$(a family of continues function) is dense in $L^P(\mu), 1 \leq P < \infty$

Suppose $f \in L^P(\mu)$,$1 \leq P < \infty$,then for each $\varepsilon > 0$, there is $g \in C(T)$ suchthat

$$||g - f||_P < \varepsilon$$

Let $v = P[g]. Then$

$$u_r - f = (u_r - v_r) + (v_r - g) + (g - f)$$

By (1),$||u_r - v_r||_P = ||(u - v)_r||_P \leq ||f - g||_P < \varepsilon$

Thus, $||u_r - f||_P \leq \|u_r - v_r\|_P + \|v_r - g\|_P + \|g - f\|_P$

$$\leq 2\varepsilon + \|v_r - g\|_\infty, for\ all\ r < 1.$$

Take limit $r \to 1, we\ get$

$$\lim_{r\to 1} ||u_r - f||_P \leq 2\varepsilon$$

Since $\lim_{r\to 1}\|v_r - g\|_\infty \; = 0$,by formula 1.2.8(2)

Hence, $\lim_{r\to 1} ||u_r - f||_P = 0.$ ■

1.2.10. Poisson integrals of Measure

If μ is a complex Measure on T, and if we want to replace integrals over T by integrals over intervals of length $2\pi\ in\ R^1$, these intervals have to be taken half open, because of the possible presence of point masses in μ.To avoid this problem, we shall keep integration on the circle in that follows, and will write the Poisson integral $u = P[d\mu]$ of μ in the form

$$u(z) = \int_T P(z, e^{it})\, d\mu(e^{it}) \qquad , z \in U$$

Where $P(z, e^{it}) = \frac{1-|z|^2}{|e^{it}-z|^2}$,

The proof of Lemma 1.2.4 applies without change to Poisson integrals of measures. Thus u is harmonic in U.

Setting $||\mu|| = |\mu|(T)$,the analogue of the first half of Theorem 1.2.9 is

$$||u_r||_1 = \frac{1}{2\pi}\int_T |u(re^{i\theta})|\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\left|\int_T P(z, e^{it})\, d\mu(e^{it})\right|\right] d\theta$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\int_T |P(re^{i\theta}, e^{it})||d\mu(e^{it})|\right] d\theta$$

$$= \int_T \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} P(re^{i\theta}, e^{it})\, d\theta\right] |d\mu(e^{it})|, \qquad by\ Fubini's\ theorem.$$

$$= \int_T |d\mu|\,(e^{it}) = |\mu|(T) = ||\mu||.$$

1.3. Nontangential Maximal Functions and Nontangential limits

For $0 < \alpha < 1$, we define $\Omega_\alpha e^{it}$ to be the union of the disc $D(0{:}\,\alpha)$ and the line segments from $z = e^{it}$ to points of $D(0{:}\,\alpha)$.

1.3.1 Definition If $0 < \alpha < 1$ and u is harmonic function on U,its **nontangential Maximal function** $N_\alpha U$ is defined on T by

$$(N_\alpha u)(e^{it}) = \sup\{|u(z)| : z \in e^{it}\Omega_\alpha\}$$

Similarly, the **radial maximal function** of u is

$(M_{rad}u)(e^{it}) = \sup\{|u(re^{it})|: 0 \leq r < 1\}$.

The set $\{e^{it}: (N_\alpha u)(e^{it}) < \lambda\} = \{e^{it}: |u(z)| \leq \lambda, z \epsilon \Omega_\alpha e^{it}\}$ is closed subset of T for each positive λ. so is $\{e^{it}: (M_{rad}u)(e^{it}) \leq \lambda\}$.

Consequently, ($N_\alpha u$ and $M_{rad}u$ are lower semicontinous on T) In particular, they are measurable.

Clearly, $M_{rad}u \leq N_\alpha u$, and the latter increases with α.

Let us replace ordinary Lebesgue Measure m on T by $\sigma = {}^{m}/_{2\pi}$ .Then σ is a rotation-invariant Positive Borel Measure on T,so normalized that $\sigma(T) = 1$.

Accordingly, M_N is now defined by

$$M_N(e^{i\theta}) = \lim \frac{N(I)}{\sigma(I)}$$

as the open arcs $I \subset T$ Shrink to their center $e^{i\theta}$.

1.3.2 Theorem Assume $0 < \alpha < 1$.Then there is a constant $c_\alpha > 0$ with the following Property: if μ is a positive finite Borel measure on T and $u = P[d\mu]$ is its Poisson integral, then the inequalities

$$c_\alpha (N_\alpha u)(e^{i\theta}) \leq \left(M_{rad}u(e^{i\theta})\right) \leq M_N(e^{i\theta})$$

hold at every point $e^{i\theta} \in T$.

1.3.3 Definition A function F, defined in U is said to have **non-tangential limit** λ at $e^{i\theta} \epsilon T$ if for each $\alpha < 1$,

$$\lim_{j \to \infty} F(z_j) = \lambda$$

for every sequences $\{z_j\}$ that converge to $e^{i\theta}$ and that lies in $e^{i\theta}\Omega_\alpha$.

The next theorem state and prove in rudin real and complex analysis [1] page 243.

1.3.4 Theorem if μ is a positive Borel measure on T and $(D\mu)\left(e^{i\theta}\right) = 0$ for some θ, then its Poisson integral $u = P[d\mu]$ has nontangential limits 0 at $e^{i\theta}$.

1.3.5 Theorem If $f \in L^1(T)$, then $P[f]$ has **nontangential limit** $f(e^{i\theta})$ at every Lebesgue point $e^{i\theta}$ of f.

Proof Suppose $e^{i\theta}$ is a lebesgue point of f.By substituting constant from f.We may assume without loss of generality, that $f\left(e^{i\theta}\right) = 0$.Then

$$\lim \frac{1}{\sigma(I)} \int_I |f| d\sigma = 0 \qquad (1)$$

as the open arcs $I \subset T$ shrinks to their center$e^{i\theta}$.

Define a Borel measure μ on T by

$$\mu(E) = \int_E |f| d\sigma$$

Then (1) says that $(D\mu)\left(e^{i\theta}\right) = 0$;hence $P[d\mu]$ has nontangential limit 0 at $e^{i\theta}$, by Theorem 1.3.4.The same is true of $P[f]$, because

$$|P[f]| \leq P[|f|] = P[d\mu]\,. \qquad \blacksquare$$

The last two Theorems can be combined as follows.

1.3.6 Theorem. If $d\mu = f d\sigma + dN_s$ is the Lebesgue decomposition of a complex Borel measure μ on T, where $f \in L^1(T)$, $N_s \perp \sigma$,then $P[d\mu]$ has nontangential limit $f(e^{i\theta})$ at almost all points of T.

Proof:-Apply theorem 1.3.4 to the positive and negative Variations of the real and imaginary parts of N_s ,and apply Theorem 1.3.5 to f.$\blacksquare$

Here is another consequence of Theorem 1.3.2.

1.3.7 Theorem For $0 < \alpha < 1$ and $1 \leq P \leq \infty$, there are constants $A(\alpha, P)$ with the following properties:

(a).If μ is a complex Borel measure on T, and $u = P[d\mu]$,then

$$\sigma(N_\alpha u > \lambda) \leq \frac{A(\alpha,1)}{\lambda} \|\mu\|, \quad (0 < \lambda < \infty)$$

(b).If $1 < P \leq \infty$,$f \in L^P(T)$,and $u = P[f]$,then

$$||N_\alpha u||_P \leq A(\alpha, P)||f||_P$$

Define $h^P = \{u: u\ is\ harmonic\ in\ U\ and\ \sup_{0<r<1} ||u_r||_P = M < \infty\}$

1.3.8 Theorem (fatou's) Suppose $u \in h^P$, $1 \leq P \leq \infty$, and

(a).If $P = 1$,it follows that there is a unique complex Borel Measure μ on T so that $u = P[d\mu]$.

(b).If $P > 1$, it follows that there is a unique $f \in L^P(T)$ so that $u = P[f]$.

(c).Every positive harmonic function in U is the poisson integral of a unique positive Borel Measure on T.

The general proof is contained in [1], page 247.

1.3.9 Corollary If $u \in h^P$, then u has nontangential limit a.e on T.

Proof From the above theorem there is an $f \in L^P(T)$ so that $u = P[f]$. Then by Theorem 1.3.5 u has nontangential limit f a.e on T

1.4 Subharmonic functions

1.4.1. Definition A function u defined in an open set Ω in the plane is said to be **subharmonic** if it has the following four properties.

(a).$-\infty \leq u(z) < \infty$ for all $z \in \Omega$

(b).u is upper semi continous in Ω.

(c).whenever, $\overline{D}(a,r) \subset \Omega$, then

$$u(a) \le \frac{1}{2\pi}\int_{-\pi}^{\pi} u(a + re^{i\theta})d\theta$$

(d).None of the integrals in (c) is $-\infty$.

Note that the integral in (c) always exist and are not $+\infty$ since (a) and (b) imply that u is bounded above on every compact $K \subset \Omega$[proof: If K_n is the set of all $z \in K$ at which $u(z) \ge n$,then $K \supset K_1 \supset K_2 \supset \cdots$ so either $K_n = \emptyset$ for some n, or $\bigcap K_n \neq 0$, in which case $u(z) = \infty$ for some $z \in K$.].hence (d) says that the integrand in (c) belong to $L^1(T)$.

Every real harmonic function is obviously subharmonic.

1.4.2 Example If Ω is a region and f is holomorphic on Ω, and f is not identically zero, then log $|f|$ is subharmonic.

Solution Let $u(z) = \log|f(z)|\ and\ u(z) = -\infty$ if $f(z) = 0$.

First to show u is upper semi continous.

Let $A = \{z \in \Omega: u(z) < \alpha\} = \{z \in \Omega: \log|f(z)| < \alpha\} = \{z\epsilon\Omega: |f(z)| < e^{\alpha}\}$

Since f is holomorphic, hence A is open.

Therefore, u is upper semicontinuous

The next lemma and two theorems show that u is subharmonic.

1.4.3 Lemma $\frac{1}{2\pi}\int_0^{2\pi} \log\ |1 - e^{i\theta}|\,d\theta = 0$.

1.4.4 Lemma suppose $\Omega = D(0.R), f \in H(\Omega), f(0) \not\equiv 0, 0 < r < R, and\ \alpha_1, \alpha_{2,\ldots}\alpha_N$ are the zeros of f in $\overline{D}(0,r), listed\ accordingly\ to\ thier\ multiplicities$.Then

$$|f(0)|\prod_{n=1}^{N}\frac{r}{|\alpha_n|} = exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left|f\left(re^{i\theta}\right)\right|d\theta\right\}.$$

This is known as Jensen’s formula.

1.4.5 Theorem If f is a holomorphic bounded function on u, f is not identically 0, define

$$N_r(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|f\left(re^{i\theta}\right)\right| d\theta \qquad (0 < r < 1) \tag{1}$$

and

$$N^*(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|f^*\left(e^{i\theta}\right)\right| d\theta \tag{2}$$

Where f^* is the radial limit function of f.Then

$N_r(f) \leq N_s(f)$ if $0 < r < s < 1$

$N_r(f) \to \log|f(0)| \qquad as\ r \to 0$

Note the following consequence: One can choose r so that $f(z) \neq 0$ if $|z| = r; then$

$N_r(f)$ is finite, and so is $N^*(f)$, by (5).thus $\log|f^*| \in L^1(T), and\ f^*(e^{i\theta}) \neq 0$ at almost every point of T.

Proof There is an integer $m \geq 0$ such that $f(z) = Z^m g(z), g \in H^\infty$, and $g(0) \neq 0$.Apply Jensen's formula theorem 1.4.4 for g in place of f .Its left side obviously cannot decrease if r increases.

Thus $N_r(g) \leq N_s(g)$ if $r < s$.

Since

$$N_r(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\ |r|^m\ |g(re^{i\theta})| d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} mlogr + \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\ |g(re^{i\theta})| d\theta$$

$$= mlogr + N_r(g) \leq mlogr + N_s(g) = N_s(f)$$

Hence $N_r(f) \leq N_s(f)$, we have provided (3).

Let us now assume without loss of generality that $|f| \leq 1$, write $f_r(e^{i\theta})$ in place of $f(re^{i\theta})$.Then $f_r \to f(0)\ as\ r \to 0$ a.e as $r \to 1$.

Since $\log\left({}^{1}/_{|f_r|}\right) \geq 0$,then by fatous lemma

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(\frac{1}{|f(0)|}\right) d\theta \leq \varliminf_{r\to 0} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left({}^{1}/_{|f_r(e^{i\theta})|}\right) d\theta$$

Then, $\lim_{r\to 0} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|f\left(re^{i\theta}\right)\right| d\theta \leq \log\ |f(0)|$

Hence, $N_r(f) \to \log\ |f(0)|$ as $r \to 0$

and,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\,({}^{1}/_{|f^*(re^{i\theta})|}) d\theta \leq \varliminf_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\,({}^{1}/_{|f_r(e^{i\theta})|}) d\theta$$

$$\Rightarrow \varliminf_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\ |f_r(e^{i\theta}| d\theta \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\ |f^*(e^{i\theta}| d\theta$$

$$\Rightarrow \varliminf_{r\to 1} N_r(f) \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\ |f^*(e^{i\theta}| d\theta$$

Hence, $N_r(f) \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\ |f^*(e^{i\theta}| d\theta$, for $0 < r < 1$.

Since N_r is increasing of r.$\blacksquare$

1.4.6. Theorem. If u is subharmonic in Ω, and if φ is a monotonically increasing convex function on R^1,then $\varphi o u$ is subharmonic.

Note. To prove this we use Jenson inequality which is stated and proved in Walter rudin [1], page[62] we will state as again.

Let μ be a positive Measure on a σ-algebra in a set Ω, so that $\mu(\Omega) = 1$.If f is a real function in $L^1(\mu)$,if $a < f(x) < b$ for all $x \epsilon \Omega$, and if φ is convex on (a, b), then

$$\varphi\left(\int_{\Omega} (\varphi o f) d\mu\right) \leq \int_{\Omega} (\varphi o f) d\mu$$

Proof To show $\varphi o u$ is upper semicontinuous.

Let A=$\{x: \varphi o u(x) < \alpha\} = \left\{x: \varphi\left(u(x)\right) < \alpha\right\} = \{x: u(x) < \beta, for\ \beta\ such\ that\ \alpha = \varphi(\beta)\}$

Since U *is* harmonic.Then A is open, for each α.

Hence $\varphi o u$ is upper semicontinuous.

Next, if $\overline{B(a,r)} \subset \Omega$, we have

$u(a) \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(a + re^{i\theta}\right)d\theta$ (1)

$$\varphi\big(u(a)\big) \leq \varphi\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(a + re^{i\theta}\right)d\theta\right)$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi o u(a + re^{i\theta})\, d\theta \qquad (2)$$

The first inequality (2) holds since φ is increasing and u is subharmonic and the second follow from the Jenison's inequalities.$\blacksquare$

1.4.7 Example If Ω is a region, $f \in H(\Omega)$,and f is not identically 0,then

$log^{+}|f| =$ max $\{log|f|, 0\}$ and $|f|^{P}(0 < P < \infty)$ are subharmonic.

Solution Let us define

$\varphi(t) =$ max $\{t, 0\}$ and $\psi(t) = e^{Pt}$

Clearly, φ *and* ψ are increasing convex function on R^{1} and let $u(z) = \log|f(z)|$,then u is harmonic in Ω.Then by above theorem

$\varphi o u = \varphi(\log|f|) = \max\{\log|f|\,, 0\} = log^{+}|f|$ and

$\varphi o u = \varphi(\log|f|) = e^{\log\,|f|^{P}} = |f|^{P}$ are subharmonic in Ω. $\blacksquare$

1.4.8 Theorem Suppose u is a continous subharmonic function in Ω, K is a compact subset of Ω, h is a continuous real function on K which is harmonic in the interior V of K, and $u(z) \leq h(z)$ for all $z \epsilon K$.

Proof Put $u_1 = u - h$, and assume to get a contradiction, that $u_1(z) > 0$ for some $z \in V$.

Since u_1 is continuous on K,u_1 attains its maximum M on K; and since $u_1 \leq 0$ on the boundary of K, the set $E = \{z \in K\colon u_1(z) = M\}$ is a compact subset of V.

Let z_0 be a boundary point of E.Then for some $r > 0$, we have $\bar{B}(z_0; r) \subset V$,but some subarc of the boundary of $\bar{B}(z_0; r)$ lies in the compliment of E. Hence

$$u_1(z) = M > \frac{1}{2\pi}\int_{-\pi}^{\pi} u_1\left(z_0 + re^{i\theta}\right)d\theta,$$

and this means that u_1 is not subharmonic in V. But if u is subharmonic, so is $u - h$ and we have our contradiction. $\blacksquare$

1.4.9 Theorem Suppose u is a continuous subharmonic functions in U, and

$$M(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{i\theta})\, d\theta \qquad\qquad (0 \le r < 1)$$

If $r_1 < r_2$ then $m(r_1) < m(r_2)$.

Proof Let h be the continuous function on $\bar{B}(0; r_2)$ which coincides with u on the boundary of $\bar{B}(0; r_2)$ and which is harmonic in $B(0; r_2)$.By above theorem,

$u \le h\ in\ \bar{B}(0; r_2)$.Hence

$$m(r_1) \le \frac{1}{2\pi}\int_{-\pi}^{\uparrow\pi} h(r_1 e^{i\theta}) = h(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(r_2 e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(r_2 e^{i\theta}) = m(r_2). \blacksquare$$

CHAPTER 2

BASIC STRUCTURE OF H^P −SPACES ON THE UNIT DISC

2.1. Introduction of $\mathrm{H^P}$ space

Hardy spaces (or Hardy classes) H^P are certain spaces of holomorphic functions on the unit disc or on the upper half plane, but this paper concentrated on the unit disc.

Throughout this section we shall use the notation $U = \{z: |z| < 1\}$, $T = \{z: |z| = 1\}$
$H(U) =$ the collection of all holomorphic function defined on U.

If f is any continous function with domain U and we let σ lebesgue measure on T, so normalized that $\sigma(T) = 1$ we define $f_r(0 \leq r < 1): T \to ₵$ such that

$$f_r(e^{i\theta}) = f(re^{i\theta})$$

Clearly for each $r, 0 \leq r < 1$, f_r is measurable on T.

Accordingly, L_P norms will refer to $L^P(\sigma)$.In particular,

$$||f_r||_P = \left\{\int_T |f_r|^P d\sigma\right\}^{1/P} \qquad (0 < P < \infty)$$

$$||f_r||_\infty = sup_\theta\, |f_r(e^{i\theta})|$$

and we also introduce

$$||f_r||_0 = exp\left\{\int_T log^+|f_r|\, d\sigma\right\}$$

, where $log^+|f_r| = max\{0, log|f_r|\}$

2.1.1 Definitions If $0 < P \leq \infty$,define

$$H^P = \left\{f: U \to ₵: f \in H(U) and \sup_{0\leq r<1} \|f_r\|_P < \infty\right\}$$

and

$$N = \left\{f: U \to ₵: f \in H(U) and \sup_{0\leq r<1} \big||f_r|\big|_0 < \infty\right\}$$

If $f \in H(U)$ and $0 \leq P \leq \infty$, we put

$$\|f\|_P = sup\,\{||f_r||_P : 0 \leq r < 1\}$$

Remarks:

(a).If $0 < S < K < \infty$, then $H^\infty \subset H^K \subset H^S \subset N$.To see this let $P = \frac{K}{S} > 1$ and $\frac{1}{q} = 1 - \frac{1}{p}$

If $f \in H^K$ then by Holider's inequality

$$||f_r||_S^S = \int_T |f_r|^S \, d\sigma \leq \left\{\int_T (|f_r|^S)^P\right\}^{1/P} . \left\{\int_T |1|^q\right\}^{1/q} = ||f_r||_K^S$$

$$\Longrightarrow \|f\|_S \leq \|f\|_K$$

hence $f \in H^S$.

(b).When $P < 1$, Example 1.4.7 and Theorem1.4.9 show that $||f_r||_P$ is an increasing function of r, for every $f \in H(U)$;When $P = \infty$, the same follows from the maximum modulus theorem. Hence

$$\|f\|_P = \lim_{r \to 1} \|f_r\|_P$$

2.1.2 Theorem If $f, g \in H^P (0 < P \leq \infty)$, then $f + g \in H^P$

Proof: If f and g are in H^P then

$$\|(f+g)_r\|_P = \int_T |(f+g)_r|^P d\sigma$$

$$\leq \int_T (|f_r| + |g_r|)^P \, d\sigma$$

$$\leq \int_T (2max\,(|f_r| + |g_r|))^P \, d\sigma$$

$$\leq 2^P \int_T |f_r|^P \, d\sigma + 2^P \int_T |g_r|^P \, d\sigma$$

$$\leq 2^P ||f_r||_P^P + 2^P ||g_r||_P^P$$

If we take limit $r \to 1$, the right side converges to $2^P||f||_P^P + 2^P||g||_P^P$ it is finite, so

$f + g \in H^P$. $\blacksquare$

2.1.3 Theorem If $1 \leq P \leq \infty$ $\|.\|_P$ is a norm in H^P.

Proof formula 1 show one condition of a norm.

i. $\|f\|_P = sup_{r\to 1}\left\{\int_T |f_r|^P d\sigma\right\}^{1/P} = 0\ iff\ \int_T |f_r|^P d\sigma = 0, for\ 0 \leq r < 1$

$$iff\ \ |f_r|^P = 0, for\ 0 \leq r < 1$$
$$iff\ f = 0$$

ii. If $\alpha\epsilon$₵ $and\ f\epsilon H^P$, then

$$\int_T |(\alpha f)_r|^P\, d\sigma = |\alpha|^P \int_T |f_r|^P\, d\sigma$$

$$\Longrightarrow \|\alpha f\|_P = |\alpha|\|f\|_P < \infty$$

iii. if $0 < r < 1$,

$$\|(f+g)_r\|_P = \|f_r + g_r\|_P \leq \|f_r\|_P + \|g_r\|_P$$

as $r \to 1$,we obtain

$$\|f+g\|_P = \|f\|_P + \|g\|_P \blacksquare$$

Actually H^P is a vector space for $0 < P \leq \infty$.

2.1.4 Theorem For $1 \leq P \leq \infty$, H^P is a Banach space.

Proof: We have seen from the above theorem that H^P is a normed space, we suffice to show that it is complete.

Suppose $\{f_n\}$ is a cauch sequence in H^P, let $|z| \leq R < 1$, and apply the Cauchy formula to

$f_n - f_m$, integrating around the circle of radius R with center 0.

Let $\gamma(t) = Re^{i\theta}, -\pi \leq \theta < \pi$

$$f_n(z) - f_m(z) = \frac{1}{2\pi i}\int_\gamma \frac{f_n(w) - f_m(w)}{w - z}\,dw \;= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{f_n\left(Re^{i\theta}\right) - f_m\left(Re^{i\theta}\right)}{Re^{i\theta} - z}\,d(Re^{i\theta})$$

$$= \frac{\mathrm{Ri}e^{i\theta}}{2\pi}\int_{-\pi}^{\pi} \frac{(f_n - f_m)(Re^{i\theta})}{Re^{i\theta} - z}\,d\theta$$

for all $|z| < R$

Then,

$$|f_n(z) - f_m(z)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left|(f_n - f_m)\left(Re^{i\theta}\right)\right|}{R - r}\,d\theta$$

$$= \frac{1}{2\pi(R - r)}\int_{-\pi}^{\pi} \left|(f_n - f_m)\left(Re^{i\theta}\right)\right| d\theta$$

$$= \frac{1}{R - r}\int_T |(f_n - f_m)_R|\,d\sigma$$

$$= \frac{1}{R - r}\|(f_n - f_m)_R\|_1$$

$$\leq \frac{1}{R - r}\|(f_n - f_m)_R\|_P$$

$$\leq \frac{1}{R - r}||f_n - f_m\|_P$$

from which we conclude that $\{f_n\}$ converges $f \in H(U)$.

Given $\varepsilon > 0$, there is an N such that $||f_n - f_m||_P < \varepsilon$

Now fix $n > N$ *and,* letting $m \to \infty$ and, then for every $r < 1$,

$$||(f_n - f)_{\mathrm{r}}\|_P = \|\lim_{\mathrm{m}\to\infty}(f_n - f_m)_r\|_{\mathrm{P}} = lim_{m\to\infty}\|(f_n - f_m)_r\|_P \leq \varepsilon$$

$$\Longrightarrow \quad ||f_n - f\|_P \leq \varepsilon$$

Hence $f_n - f \in H^P$, so is f and $||f - f_n||_P \to 0$ as $n \to \infty$.$\blacksquare$

2.2 Boundary behavior of Poisson integrals and some properties of H^P- Spaces

2.2.1 Theorem If $f \in H^P(1 < P \leq \infty)$ then the nontangential limit exist a.e on T.

Proof Since holomorphic functions are harmonic, by fatou's theorem f has nontangential limit $f^* \in L^P(T)$ almost every point of T. $\blacksquare$

We have seen a harmonic function in h^P have a radial limit in almost every direction. The above theorem also shows that the same is true for H^P functions $(1 < P \leq \infty)$.Surprsingly, this section will show that it is true for $0 < P \leq 1$.

First let us introduce the definition of Blaschke product and some theorems corresponding to Blaschke product. Blaschke product plays an expanding role in this chapter

A finite Blaschke product is a function of the form

$$B(z) = e^{i\varphi} \prod_{j=1}^{n} \frac{z - z_j}{1 - \overline{z}_j z}$$

, Where $\varphi \varepsilon R$ and $|z_j|$<1 for $j = 1,2,3, \ldots, n$.It is easy to verify that B has the following properties.

(a).$B\ is\ analytic\ in\ U\ and\ continous\ in\ \overline{U}$.

(b).$|B| = 1\ on\ the\ boundary\ of\ U$

(c).$B\ has\ zeros\ at\ z_{1,}z_{2,} \ldots, z_n\ only$.

This section will show that the zeros of any $f \in H^P$ can be divided out without increasing the norm of f.First let as see some lemmas taken from Walter rudin [1].

We shall see Szegö lemma that the zero of any $f \in N$ satisfies the Blaschke condition

$$\sum_{n=1}^{\infty} (1 - |\alpha_n|) < \infty.$$

Hence the same is true in H^P $becouse\ H^P \subset N$.

2.2.2 Lemma$(\boldsymbol{G.Szeg\ddot{o}})$. $Suppose\ f \in N, f$ is not identically 0 in U, and $\alpha_1, \alpha_{2,\ldots,}$are the zeros of f, listed according to their multiplicities, then

$$\sum_{n=1}^{\infty} (1-|\alpha_n|) < \infty$$

2.2.3 Lemma. Let $\{\alpha_n\}$ *is a seguence in* U *such that* $\alpha_n \neq 0$ *and*

$$\sum_{n=1}^{\infty}(1-|\alpha_n|) < \infty \qquad (1)$$

Let k *is a nonnegative integer, and if*

$$B(z) = z^k \prod_{n=1}^{\infty} \frac{z-\alpha_n}{1-\overline{\alpha_n}z}\frac{|\alpha_n|}{\alpha_n}, \quad (z \in U)$$

, then $|B(z)| \leq 1, \forall z \in U$, $|B| = 1$ on T, *and* B *has no zeros exept at the points*

$\alpha_n \in U$ (and at the origin, if $k > 0$).We call this function B an infinite Blaschke product. Note that some of the α_n may be repeated, in which case B has multiple zeros at those points.

The term Blaschke product" will also be used if there are only finitly many

factors, and even if there are none, in which B(z)=1.

2.2.4 Lemma. If B *is a Blascke product, then* $\left|B^*\left(e^{i\theta}\right)\right| = 1$ *a.e* and

$$\lim_{r\to 1}\frac{1}{2\pi}\int_{-\pi}^{\pi} log\left|B\left(re^{i\theta}\right)\right| \, d\theta = 0$$

2.2.5 Theorem (F.Riesz) *Suppose* $f \in N$, f is not identically zero, and B is the Blaschke product formed with the zeros of f . *Put* $g = {}^{f}/_{B}$. *Then* $g \in N$ *and*

$\|g\|_0 = \|f\|_0$ *Moreover, if* $f \in H^P$, *then* $g \in \mathrm{H}^{\mathrm{P}}$ *and*

$\|g\|_P = \|f\|_P (0 < P \leq \infty)$

Proof .Note first that

$$|g(z)| \geq |f(z)| \qquad (z \in U) \qquad (1)$$

In fact, strict inequality holds for every $z \in U$, unless f has no zero in $U \; and \; g = f$.

If s and t are nonnegative real numbers, the inequality

$$log^+(st) \leq log^+ s + log^+ t \qquad (2)$$

$holds \; since \; the \; left \; side \; is$ 0 $if \; st < 1 \; and \; is \; logs + logt$ if $st \geq 1$.

$Since \; |g| = {}^{|f|}\!/_{|B|}, \quad (2) \; gives$

$$log^+|g| \leq log^+|f| + log^+\left|\frac{1}{B}\right|$$

$$\Longrightarrow log^+|g| \leq log^+|f| - log\left|B\left(re^{i\theta}\right)\right|$$

$$\Longrightarrow log^+\left|g\left(re^{i\theta}\right)\right| \leq log^+\left|f\left(re^{i\theta}\right)\right| - log\left|B\left(re^{i\theta}\right)\right|, 0 < r < 1.$$

$Now \; integrate \; both \; side \; on \; T \; and \; take \; limit \; on \; r, we \; get$

$$\lim_{r \to 1} \int_T log^+\left|g_r\left(e^{i\theta}\right)\right| d\sigma \leq \lim_{r \to 1} \int_T log^+|f_r| d\sigma - \lim_{r \to 1} \int_T log\left|B\left(re^{i\theta}\right)\right| d\sigma$$

$$= \lim_{r \to 1} \int_T log^+|f_r| d\sigma \text{ , } by \; Theorem \; 2.2.4.$$

$Hence, ||g||_0 \leq ||f||_0, and \;\; from \; (1), ||g||_0 \geq ||f||_0$

Therefore, $||f||_0 = \left||g|\right|_0 \; and \; g \in N$.

Now suppose $f \; in \; H^P \; for \; some \; P. let \; B_n$ be the finite Blaschke product formed with the first n zeros of f, (we arrange these zeros in some sequence, taking multiplicities into account).

$Put \; g_n = {}^{f}\!/_{B_n}$, for each n, $\left|B_n\left(re^{i\theta}\right)\right| \rightarrow 1$ uniformly, as $r \rightarrow 1$.

Hence $||g_n||_P = ||f||_P$ $as\ n \to \infty, |g_n|$ increases to $|g|$,so that

$$||g_r||_P = \lim_{n\to\infty} ||(g_n)_r||_P \ \ (0 < r < 1) \qquad (3)$$

by the monotonic convergence theorem.

Then right side of $(3) is\ at\ most\ ||f||_P, for\ all\ r < 1$.If we let $r \to 1$, we obtain

$$||g||_p \leq ||f||_P$$

However, $|g(z)| \geq |f(z)|, for\ all\ z \in U$, so we have equality.$\blacksquare$

2.2.6 Theorem. Suppose $0 < P < \infty, f \in \mathrm{H}^{\mathrm{P}}, f$ is not identically zero, and B is a Blaschke product formed with the zeros of f.Then there is a zero free function $h \in \mathrm{H}^2$ such that

$$f = B.h^{2/P} \qquad (1).$$

In particular, every $f \in H^1$ is a product

$$f = gh \qquad (2)$$

in which both factors are in H^2.

Proof $Suppose\ f \in H^P (0 < P < \infty)$.Then by theorem 2.2.5

$$f/B \in H^P \ \ and\ ||f/B||_P = ||f||_P$$

Since f/B has no zero in U, and U is simply connected, there exists $\varphi \epsilon H(U)$ so that

$\exp(\varphi) = f/B$ (This is taken from Walter rudin [1], theorem 13.11).Put $h = exp({}^{P\varphi}/_{2})$.

Then, $h \in H^2$, and

$f = B.h^{2/P}. For\ P = 1, then\ f = Bh^2 = (Bh).h$

$Put\ g = Bh$.Then, $g \in H(U)$ and

$$||g||_2 = ||Bh||_2 \leq ||h||_2 < \infty.$$

Hence $g \in H^2$, and we obtain (2).$\blacksquare$

We can now easily prove some of the most important properties of the H^P_spaces.

2.2.7 Theorem If $0 < P \leq \infty and\ f \in H^P$, then

(a). the nontangential Maximal function $N_\alpha f$ are in $L^P(T)$, for all $\alpha < 1$

(b). the nontangential limits f^* exists a.e on T, and $f^* \in L^P(T)$;

(c). $\lim_{r\to 1}\|f_r - f^*\|_P = 0$, and

(d). $\|f^*\|_P = \|f\|_{P.}$

Moreover if $f \in H^1$ then f is the Cauchy integral as well as the Poisson integral of f^*.

Proof We begin by proving (a) and (b) for the case $1 < P \leq \infty$. Since every holomorphic function are harmonic, then by fatou's theorem there is a function $f^* \in L^P(T)$ such that $f = P[f^*]$ and by theorem 1.3.7 there is a constant $A(\alpha, P)$ such that

$$||N_\alpha f||_P \leq A(\alpha, P)\ \|f^*\|_P < \infty$$

Hence $N_\alpha f$ are in $L^P(T)$(for all $\alpha < 1$), and $f^*(e^{i\theta})$ is the nontangential limit of f at almost every $e^{i\theta} \epsilon\, T$, by Theorem 1.3.5.

If $0 < P \leq 1$ and $f \in H^P$ then there is a zero free function $h \in H^2$ such that

$$f = B.h^{2/p} \qquad (1)$$

where B is a Blaschke product formed with the zeros of f, by Theorem 2.2.6.

Since $|f| \leq |h|^{2/P}$ in U,it follow that

$$(N_\alpha f)^P \leq (N_\alpha h)^2$$

So that $N_\alpha f \in L^P(T)$, because $N_\alpha h \in L^2(T)$.
Similarly, the existence of $B^*\ and\ h^*$ a.e on T implies that the nontangential limits of f (call them f^*) exist a.e on T and

$\left|f^*(e^{i\theta})\right| = |\lim_{r\to 1} f(re^{i\theta})| = \lim_{r\to 1} |f(re^{i\theta})| \le N_\alpha f(e^{i\theta})$, for each α, $0 < \alpha < 1$.

$$\Longrightarrow \int_T |f^*|^P d\sigma \le \int_T (N_\alpha f)^P d\sigma < \infty$$

Hence, $f^* \in L^P(T)$. This proves (a) and (b) for $0 < P \le \infty$.

To prove (c), since $f_r \to f^*\ a.e, as\ r \to 1\ then\ |f_r - f^*|^P \to 0\ as\ r \to 1\ and$

$|f_r - f^*|^P \le 2^P (N_\alpha f)^P$.

Then by dominated convergence theorem

$$0 = \int_T |f_r - f^*|^P d\sigma = ||f^* - f_r||_P^P$$

Hence $\lim_{r\to 1} ||f^* - f_r||_P = 0$

To prove (d), If $P \ge 1$, then from (c)

$$\lim_{r\to 1} ||f^* - f_r||_P = 0$$

$$\Longrightarrow |\lim_{r\to 1} ||f^*||_P - ||f_r||_P = 0| \Longrightarrow \lim_{r\to 1} ||f_r||_P = \left||f^*|\right|_P$$

$$\Longrightarrow ||f||_P = ||f^*||_P$$

If $P < 1$,then

$$\left|\int_T |f_r|^P - \int_T |f^*|^P d\sigma\right| = \left|\int_T (|f_r|^P - |f^*|^P) d\sigma\right| \le \int_T ||f_r|^P - |f^*|^P)| \le \int_T |f_r - f^*|^P d\sigma$$

$$= \|f_r - f^*\|_P^P$$

Then by letting $r \to 1$,we obtain

$$\lim_{r\to 1} \int_T |f_r|^p d\sigma = \int_T |f^*|^P\, d\sigma \Longrightarrow \|f\|_P = \|f^*\|_P$$

Finally, if $f \in H^1, r < 1$, and $f_r(z) = f(rz), then\ f_r \in H\left(B\left(0, {}^1\!/_r\right)\right)$, and therefore f_r can be represented in U by the Cauchy formula

$$f_r(z) = \frac{1}{2\pi i}\int_T \frac{f_r(w)}{w-z}dw = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{f_r\left(e^{it}\right)}{e^{it}-z} i\, e^{it}dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f_r(e^{it})}{1-e^{-it}z}dt \qquad (2)$$

and by the Poisson formula

$$f_r(z) = P\left[f_r\left(e^{it}\right)\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r\,(\theta - t) f_r\left(e^{it}\right)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} P\left(z, e^{it}\right) f_r\left(e^{it}\right)dt \qquad (3)$$

for each $z \in U, |1-e^{-it}z|$and $P(z, e^{it})$ are bounded function on T

The case $P = 1$ of (c) $(i.e \lim_{r\to 1}\|f^* - f_r\|_1 = 0 \implies \lim_{r\to 1}\int_T f_r d\sigma = \int_T f^* d\sigma)$

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f^*(e^{it})}{1-e^{-it}z}dt$$

and

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P\left(z, e^{it}\right) f^*\left(e^{it}\right)dt \;. \blacksquare$$

2.2.8 Theorem. For $0 < P \leq \infty$, H^P and $L^P(T)$ are isometricaly isomorphic.

Proof .Define $F: H^P \to L^P(T)$ by

$$F(f) = f^*$$

Clearly F is a bounded linear operator and if $f \in H^P$

$\|F(f)\|_P = \|f^*\|_P = \|f\|_P \blacksquare$

2.2.9 Theorem If $0 < P \leq \infty$,$f \in H^P$, and $f^*\left(e^{i\theta}\right) = 0$ a.e on T, then $f(z) = 0$ for every $z \in U$.

Proof $\|f\|_P = \|f^*\|_P = 0$, hence $\|f\|_P = 0$, hence $f = 0$.$\blacksquare$

The space H^2 has a particular simple characterization in terms of power series coefficients:

2.2.9 Theorem Suppose $f \in H(U) and$

$$f(z)=\sum_{n=0}^{\infty}a_n\,z^n$$

Then $f\in H^2$ iff $\sum_{n=0}^{\infty}|a_n|^2<\infty$

Proof for $r<1$,

$$\|f_r\|_2^2=\int_T|f_r|^2\,d\sigma=\int_T f_r\overline{f_r}\,d\sigma=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=0}^{\infty}a_nr^ne^{in\theta}\,.\sum_{n=0}^{\infty}\overline{a_nr^ne^{\imath\theta}}\,d\theta$$

$$=\sum_{n,m=0}^{\infty}a_n\overline{a_m}r^nr^M\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(n-m)\theta}d\theta$$

$$=\sum_{n=0}^{\infty}|a_n|^2r^{2n}$$

$\Longrightarrow \|f\|_2^2=\sum_{n=0}^{\infty}|a_n|^2$ ∎

2.2.10 Theorem H^2 is a Hilbert space with inner product

$$(f,g)=\int_T f^*\left(e^{i\theta}\right)\overline{g^*(e^{\imath\theta})}dm(\theta)=\sum_{n=0}^{\infty}a_n\overline{b_n}$$

where $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}b_nz^n$

Proof i. $(f,f)=\sum_{n=0}^{\infty}|a_n|^2\geq 0$ and

$$(f,f)=0\ iff\ \sum_{n=0}^{\infty}|a_n|^2=0\ iff\ a_n=0, for\ n=0,1,2,\ldots,$$

$$iff\ f=0$$

(ii). Let $\alpha\epsilon ₵$,

$$(\alpha f,g)=\sum_{n=0}^{\infty}\alpha a_n\overline{b_n}=\alpha\sum_{n=0}^{\infty}a_n\overline{b_n}=\alpha(f,g)$$

(iii).$(g,f)=\sum_{n=0}^{\infty} b_n\overline{a_n}=\overline{\sum_{n=0}^{\infty}\overline{b_n\overline{a_n}}}=\overline{\sum_{n=0}^{\infty} a_n\overline{b_n}}=\overline{(f,g)}$

(iv).Let $h(z)\in H^2$ with $h(z)=\sum_{n=0}^{\infty} c_n$

$$(f+g,h)=\sum_{n=0}^{\infty}(a_n+b_n)\overline{c_n}=\sum_{n=0}^{\infty} a_n\overline{c_n}+\sum_{n=0}^{\infty} b_n\overline{c_n}=(f,h)+(g,h)$$

$\blacksquare$

Chapter 3

Factorization Theorems

In this section we are concerned with the multiplicative structure of the Hardy spaces, in that we want to factorize a general Hardy class functions into an inner factor and outer factor. Here are their definitions, and theorems.

Definition3.1 An **inner function** is a function $M \in H^{\infty}$ for which $|M^*| = 1$ a.e on T(As usual M^* denotes the radial limits of M)

If φ is a positive Measurable functions on T such that $log\varphi\epsilon L^1(T)$ and if

$$Q(z) = cexp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log\varphi(e^{it})dt\right\}$$

for $z \in U$, then Q is called an **outer function**. Here c is a constant, $|c| = 1$.

Exercise 3.2 Every Blaschke product is an inner function.

This follows from Theorem 2.2.3.

But there are also some inner functions without zeros.

Example 3.3 Suppose $f \in H^{\infty}$and$1/_f \in H^1$, then f is an outer function.

Solution:-There exist $\varphi\epsilon H(U)$ such that

$$f = e^{\varphi}$$

$$\Longrightarrow |f| = e^{Re(\varphi)}$$

$$\Longrightarrow \log|f| = Re(\varphi)$$

Since $f \in H^{\infty}$, then there exist $f^* \in L^{\infty}(T)$ such that

$$f(re^{i\theta}) \to f^*(e^{i\theta})$$

as $r \to 1$.This implies log $|f(re^{i\theta})| \to$ log $|f^*(e^{i\theta})|$

Then, log $|f^*| \in L^1(T)$ and

$$\log|f(z)| = P[\log|f^*|] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Re\left(\frac{e^{it}+z}{e^{it}-z}\right)\log\left|f^*\left(e^{it}\right)\right| dt$$

$$\Rightarrow |f(z)| = exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} Re\left(\frac{e^{it}+z}{e^{it}-z}\right)\log\left|f^*\left(e^{it}\right)\right| dt\right\}$$

Hence f is an outer function.■

Theorem 3.4 Suppose c is a constant,$|c| = 1$,B is a blaschke product,μ is a finite positive Borel measure on T which is singular with respect to lebesgue measure, and

$$M(z) = cB(z)exp\left\{-\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}d\mu(t)\right\} \qquad (z \in U) \qquad (1)$$

Then M is an inner function, and every inner function is of this form.

Proof Suppose (1) holds and $g = M/B$, then

$$\Rightarrow \quad \log|g| = \int_{-\pi}^{\pi} Re\left(\frac{e^{it}+z}{e^{it}-z}\right)\left(-d\mu(t)\right)$$

$$= P[-d\mu] \leq 0$$

Then, $|g| \leq 1$.Hence $g \in H^\infty$, and the same is true of M.

Also $D\mu = 0$ $a.e$, since μ is singular (see $[1]$, $Theorem$ 7.13), and the radial limit of log $|g|$ are 0 a.e (Theorem1.3.4).since $|B^*| = 1$ $a.e$ and $|g^*| = 1$, we see that M is an inner function.

Conversely, let B be the Blaschke product formed with the zeros of a given inner function M and put $g = M/B$.Then log $|g|$ is harmonic in U.since M is inner function, then

$M \in H^\infty$ and $|M^*| = 1$ $a.e$ on T and

$$|g(z)| \leq ||g||_\infty = ||M||_\infty = ||M^*||_\infty = 1\,, for\ all\ z \in U$$

$and \qquad |g^*| = \left|\frac{M^*}{B^*}\right| = 1 \; a.e \; on \; T$

Thus $\log|g| \leq 0$.Then $-\log|g| \geq 0$

Then, by fatou theorem

$\log|g| = P[-d\mu]$, for some positive measure μ on T.

Since $\log|g^*| = 0$ $a.e$ on T, we have $D\mu = 0$ $a.e$ on T, so μ is singular.

Finally,

$$\log|g| = -\int_{-\pi}^{\pi} Re\left(\frac{e^{it}+z}{e^{it}-z}\right) d\mu(t)$$

$$\Rightarrow |g| = exp\left\{-\int_{-\pi}^{\pi} Re\left(\frac{e^{it}+z}{e^{it}-z}\right) d\mu(t)\right\}$$

$$\Rightarrow |g| = |exp\left\{-\int_{-\pi}^{\pi} \left(\frac{e^{it}+z}{e^{it}-z}\right) d\mu(t)\right\}|$$

$$\Rightarrow g = cexp\left\{-\int_{-\pi}^{\pi} \left(\frac{e^{it}+z}{e^{it}-z}\right) d\mu(t)\right\}$$

for some constant c,$|c| = 1$.Then, M is of the form (1).$\blacksquare$

Example 3.5 Take $c = 1$ and $B = 1$, and let μ be the unit mass at t=0.Then $M(z)exp\left\{\frac{z+1}{z-1}\right\}$ which tends to 0 very rapidly along the radius which ends at z=1.

Theorem 3.6 Suppose Q is the outer function related to φ as in Definition 3.1. Then

(a) $\log|Q|$ is the poisson integral of $log\varphi$.

(b) $\lim_{r\to 1} |Q(re^{i\theta})| = \varphi\left(e^{i\theta}\right)$ $a.e$ on T.

(c) $Q \in H^P$ if and $only$ if $\varphi \epsilon L^P(T)$.

Proof

(a). since Q is outer function,

$$Q(z) = cexp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log\varphi\left(e^{it}\right)dt\right\}$$

$$\Longrightarrow |Q(Z)| = exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}Re\left(\frac{e^{it}+z}{e^{it}-z}\right)log\varphi(e^{it})dt\right\}$$

$$\Longrightarrow \log|Q(z)| = \frac{1}{2\pi}\int_{-\pi}^{\pi}P_r\left(z,e^{it}\right)log\varphi\left(e^{it}\right)dt = P[log\varphi]$$

Hence, $\log|Q| = P[log\varphi]$.

(b).Since $log\varphi \epsilon L^1(T), then\ P[log\varphi]$ has nontangential limit $log\varphi\left(e^{i\theta}\right)a.e\ on\ T$

This implies that $\lim_{r\to 1}\log\left|Q\left(re^{i\theta}\right)\right| = log\ \varphi\left(e^{i\theta}\right)\ a.e\ on\ T$

(c).Suppose $Q \in H^p$.Then from Fatou's lemma, we get

$$||Q^*||_P \leq ||Q||_P$$

Since$||Q^*||_P = ||\varphi||_P$, hence $||\varphi||_P \leq ||Q||_P < \infty$.therfore, $\varphi\epsilon L^P(T)$.

Conversely, if $\varphi\epsilon L^P(T)$,then

$$\left|Q\left(re^{i\theta}\right)\right|^P = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)\log\varphi^P\left(e^{it}\right)dt\right\}$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}P_r\left(\theta-t\right)\varphi^P\left(e^{it}\right)dt$$

$$\Longrightarrow \int_T\left|Q\left(re^{it}\right)\right|^P d\sigma \leq \int_T\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}P(\theta-t)_r\varphi^P\left(e^{i\theta}\right)dt\right)d\sigma$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_T P_r(\theta-t)d\sigma\right)\varphi^P\left(e^{it}\right)dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)\,d\theta\right)\varphi^P\left(e^{it}\right)dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi^P(e^{it})\,dt = \int_T \varphi^P dt = ||\varphi||_P^P$$

$$\Longrightarrow ||Q_r||_P \le ||\varphi||_P < \infty, for\ P < \infty$$

$For\ P = \infty$ is trivial.therfore,$Q \in H^P$.$\blacksquare$

Theorem 3.7 Suppose $0 < P \le \infty, f \in H^P$, and $f \not\equiv 0$.Then log $|f^*| \in L^1(T)$, the outer function

$$Q_f(z) = exp\left\{\frac{1}{2\pi}\int_T \frac{e^{it}+z}{e^{it}-z}\log\ |f^*|dt\right\} \tag{1}$$

is in H^P, and there is an inner function M_f such that

$$f = M_f Q_f \tag{2}$$

Furthermore,

$$\log|f(0)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\ |f^*(e^{it})|\,dt \tag{3}$$

Equality holds in (3) if and only if M_f is constant.

The functions $M_f\ and\ Q_f$ are called the inner and outer factors of f, respectively.

Proof We assume first that $f \in H^1$.If B is a Blaschke product formed with the zeros of f and if $g = f/B$

Then, by F.Riesz lemma,$g \in H^P$ and $||g||_1$ and $||g||_1$=$||f||_1$.

Since $|g^*| = |f^*|\ a.e\ on\ T$, it suffices to prove the theorem with g implice of f.

So Let us assume that f has no zeros in U and that $f(0) = 1$

Then log$|f|$ is harmonic on U, log $|f(0| = 0$,and since

$$log = log^+ - log^-$$

Then

$$0 = \log|f(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\ |f(re^{i\theta})|d\theta$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} log^{-}\left|f\left(re^{i\theta}\right)\right|d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} log^{+}|f(re^{i\theta})|\, d\theta \leq ||f||_0 \leq ||f||_1 \leq \infty \qquad (4)$$

for all $0 < r < 1$.

It now follows from Fatou's Lemma that both $\log^{+}|f^{*}|\, and log^{-}|f^{*}|$ are in $L^1(T)$, hence so is $\log\ |f^{*}|$.

This shows that the definition (1) makes sense. By Theorem 3.6 (c),$Q_f \in H^1$.Also, $|{Q_f}^{*}| = |f^{*}| \neq 0\ a.e.$since $\log|f^{*}| \in L^1(T)$.

If we can prove that

$$|f(z)| \leq \left|Q_f(z)\right| \qquad (z \in U) \qquad\qquad (z \in U) \qquad (5)$$

Thenf/Q_f will be inner function, and we obtain the factorization (2) by putting $M = {}^{f}\!/_{Q_f}$.

Since $\log\left|Q_f\right| = P[\log\ |f^{*}|]$, (5) is equivalent to the inequality

$$\log|f| \leq P[\log|f^{*}|], \qquad (6)$$

which we shall now prove.

For $|z| \leq 1$ and $0 < R < 1, put\ f_R(z) = f(Rz)$

Fix $z \in U$.Then $\log|f_r(z)| = P[log^{+}|f_R|](z) - P[log^{-}|f_r|](z)$.

Since $|log^{+}u - log^{-}v| \leq |u - v|$ for all real numbers u and v, and since

$||f_R - f^{*}||_1 \rightarrow 0\ as\ R \rightarrow 1$(Theorem 2.2.7), the first Poisson integral in (7) converges to $P[log^{+}|f^{*}|]$, as $R \rightarrow 1$.Hence Fatou's lemma gives

$$P[log^{-}|f^{*}|] \leq \lim_{R\rightarrow 1} inf\ P[log^{-}|f_R|] = P[log^{+}|f^{*}|] - \log|f|, \qquad (8)$$

which is the same as (6).

We have now established the factorization (2).If we put z=0 in (5) we obtain (3);equality holds in (3) if and only if $|f(0)| = |Q_f(0)|$ if and only if

$|M_f(0)| = 1$, this happens only when M_f is a constant.

This completes the proof for the case $P = 1$.

If $1 < P \leq \infty, then\ H^P \subset H^1$, hence all that remains to be proved is that $Q_f \in H^P$,

by Theorem 3.6(c).

Theorem 2.2.6 reduces the case $P < 1$ to the case $P = 2$. $\blacksquare$

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