TILING OF THE PLANE

By

Emnet Assefa

A PROJECT SUBMITTED TO
THE DEPARTMENT OF MATHMATICS

PRESENTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF MASTER OF SCIENCE

ADDIS ABABA UNIVERSITY
ADDIS ABABA, ETHIOPIA
JUNE 2013

ADDIS ABABA UNIVERSITY
DEPARTMENT OF
MATHEMATICS

The undersigned hereby certifies that they have read this manuscript and recommends to the School of Graduate Studies its acceptance. The title of the project is, "**Tiling of the plane**" by **Emnet Assefa** in partial fulfillment of the requirements for the degree of **Master of Science**.

Dated: June 2013

Advisor: ______________________
Dr. Yirgalem Tesgaye

Examining Committee: ______________________
Dr.

____
Dr.

Table of Contents

Abstract

This project discribes the concept and properties of tiling of the plane with regular polygones. The regular tilings that is a monohedral tilings whose tiles is regular polygones and semi regular tilings that uses a mix of regular polygons with different numbers of sides but in which all vertex figures are alike.

It also discuss K-uniform tilings by regular polygones that are tiling with k equivalence classes of vertices with respect to the symmetries of the tiling.

Acknowledgements

First of all I would like to thank the Almighty GOD for his endless grace in my life and especially during the years in Addis Ababa university. I am extremely thankful to my advisor Dr.Yirgalem Tesgaye for her invaluable advise, encouragement and motivation in my seminar paper work throughout the semester.

Also I would like to express my deep appreciation to the department of mathematics for the support in supplying the necessary materials for this seminar report and giving me this opportunity. Finally I would like to thank my parents and friends for their cooperation and motivation.

Addis Ababa, Ethiopia — Emnet Assefa
June 2013.

Introduction

Every known human society has made use of tilings or patterns in some form or another. The Sumerians (about 4000 B.C) in the Mesopotamian Valley built homes and temples decorated with mosaics in geometric patterns. Later, Persians showed that they were masters in tile decorations. Similarly, the Moors used congruent, multicolored tiles on the walls and floors of their buildings. Moslem and Islamic tile patterns with striking colors still survive. Roman buildings, floors, and pavements were decorated with tiles[1].

The major mathematical question about tilings is: given one or more shapes (of specific sizes) of tiles, can they tile the plane? And, if so, how?

Tilings are geometrical patterns that appear in many everyday situations. The act of tiling that is filling a plane, space etc. with units of a few defined shapes creates pavements, brick walls, mosaics, etc. Even when efficiency is more important than aesthetics, designers value clever tiling patterns. In manufacturing, for example, stamping the components from a sheet of metal is most economical if the shapes of the components fit together without gaps in other words, if the shapes form a tiling. In science also, tilings are important for the study of crystal structures (Conrad, Krumeich, Reich, and Harbrecht, 2000)[8], and provide inspiration for synthetic organic chemistry(Bunz, Rubin, and Tobe, 1999)[7].

The purpose of this project is to introduce the basic concepts on tiling the plane using regular polygons. Mainly, we will focus on tiling using equilateral triangles, squares and hexagones. This project report is divided into three parts.

The first part is concerned with preliminary concepts of Euclidean topology of R^2 such as open and closed, bounded, simply connected sets in R^2. Important properties of topological spaces, like connectedness. The ideal structure of tiles based on the idea of continuity and homeomorphism and some examples of tiles. We will discuss also regular polygons and thire angle measures.

The second part of this project is concerend with some examples of tiling follwed by the mathmatical definition of tiling and the introduction of terminologies used in tiling. As many important properties of tiling depend on the idea of symmetry, it also gives a basic understanding of properties like isometry, symmetry and transitivity.

The third part of this project discuss tilings made up of regular polygons. We will see tilings that are edge-to-edge, regular and uniform, Archimedean, and also illustrate eleven types of vertices that can be extended to form Archimedean tilings. We will also see K-unifom tilings with regular polygons.

Chapter 1

Preliminaries

1.1 Countable set

Definition 1.1.1. Let S_1, S_2 be sets. We say S_1 and S_2 have the same cardinality if there exists a bijection f: $S_1 \longrightarrow S_2$. In other words:

- If $f(a) = f(b)$ then $a = b$. This holds for all $a, b \in S_1$.
- For each $b \in S_2$, there is some a in S_1 such that $f(a) = b$.

and we will use the notation $S_1 \sim S_2$.

Definition 1.1.2. Let S be a non-empty set. S is finite, with cardinality n, if $S \sim I_n$. S is countably infinite if $S \sim N$. In either case, S is said to be countable. If S is not countable, we say S is uncountable.

1.2 Topology of the plane

Definition 1.2.1. A topology on a set X is a collection T of subsets of X having the following properties:

1. $\varnothing$ and X are in T.

2. the union of any subcollection of T is in T.

3. the intersection of any finite subcollection of T in T.

A set X for which a topology T has been specified is called a **topological space**.

Given $x = (x_1, x_2) \in R^2$ and a positive real number $r > 0$ we define the open disk of radius r centered at x to be the set

$$B(x, r) = \{y \in R^2 \mid d(x, y) < r\}$$

Definition 1.2.2. A set Q $\subseteq R^2$ is open if for every p in Q, there is a real number $r > 0$ such that the open disk $B(p, r)$ is completely contained in Q. The interior of S $\subseteq R^2$ is define as the union of all open sets contained in S.

Definition 1.2.3. Let A is subset of R^2. Then a boundary point of A is a point $x \in R^2$ not necessarily in A so that every open set containing x contains points in A and points in A complement. The set of all boundary points of a set A is called the boundary of A.

Definition 1.2.4. A subset K of R^2 is closed if it contains all its boundary points. Closure of a subset K is K plus all its boundary points.

Definition 1.2.5. A set $A \subset R^2$ is bounded if there is an open disk around some point $x \in R^2$ so that $A \subset B(x, r)$. Basically it means a set is not infinite in any dimension.

Definition 1.2.6. Let X be a topological space. A separation of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be connected if there does not exist a separation of X. A topological space is said to be disconnected if it is not connected.

Example 1.2.1. $\{(x, y) \in R^2 : y = x sin(1/x), x \neq 0\} \cup \{(0, 0)\}$

Definition 1.2.7. An object in R^2 is simply connected if it consists of one piece and does not have any "holes" that pass all the way through it or if any simple closed curve can be shrunk to a point continuously in the set.

1.3 Continuity

Definition 1.3.1. A function $f : X \longrightarrow Y$ is continuous if and only if the pre-image of any open set in Y is open in X.

Equivalentely we can also define continuity as follows

Definition 1.3.2. A function $f : X \longrightarrow Y$ is continuous if and only if the pre-image of any closed set in Y is closed in X.

1.4 Homeomorphisms

Definition 1.4.1. A function $f : X \longrightarrow Y$ between two topological spaces X and Y is called a homeomorphism if it has the following properties:

- f is a bijection
- f is continuous and
- The inverse function f^{-1} is continuous.

A function with these three properties is sometimes called bicontinuous. If such a function exists, we say X and Y are homeomorphic.

1.5 Tiles

A closed topological disk is any plane set which is the image of a closed circular disk under some homeomorphism. Similarly, an open topological disk is the image of an open circular disk under some homeomorphism. Each topological disk, closed or open is:

- bonuded and connected set
- simply connected set

A **tiles** is a subset of R^2 that is a closed or open topological disk. Homeomorphism preserve topological properties, so tiles inherite topological properties from the closed cicular disk or open topological disk.

This definition of a tile is very general. Later, additional restriction will be added as needed. For example,we may restrict our attention to tiles that are polygons. Here are some examples of tiles:

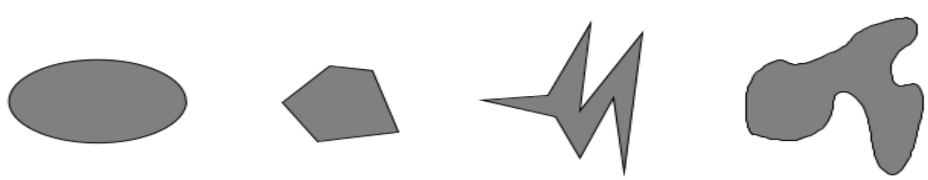

Figure 1.1: some examples of tiles

but these are not tiles

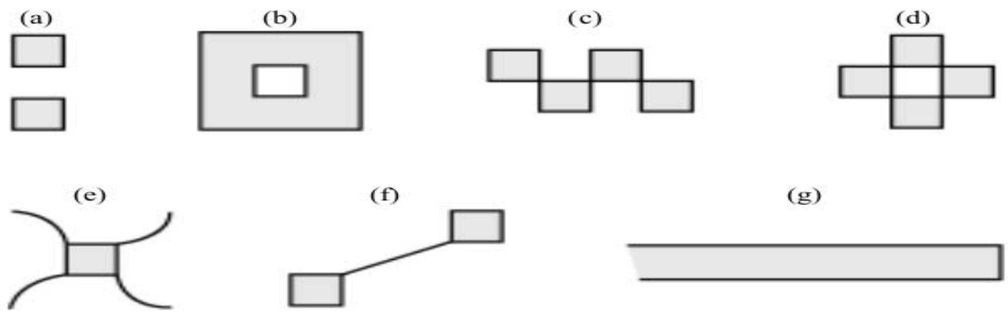

Figure 1.2: tiles which are not topological disks

From the above Fig , one can see that

(a) is not connected.

(b) is not simply connected.

(c) and (d) becomes disconnected.

(e) and (f) are made up partly of figures of zero area like line segments and arcs.

(g) is unbounded.

1.6 polygones

Definition 1.6.1. A **polygon** is a closed plane figure formed by three or more line segments. A polygon whose line segments are congruent and whose interior angles are all congruent is called a regular polygon. If a regular polygon consists of n sides then we will refer to it as regular n-gon see fig 1.3.

A vertex angle (**interior** angle) is formed by two consecutive sides. An **exterior** angle is formed by one side together with the extension of an adjacent side.

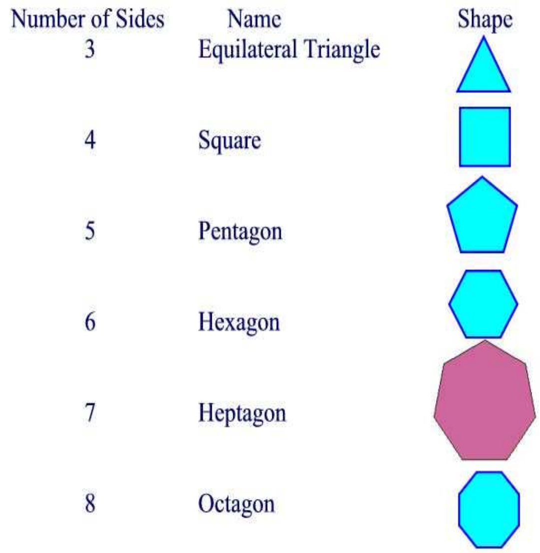

Figure 1.3: types of regular n-gons

1.6.1 Angles Measures in Regular Polygons

Lets first find the measure of a central angle in a regular n-gon. Connecting the center of the n-gon to the n vertices we create n congruent central angles. Since the sum of the measures of the n central angles is 360^0 then the measure of each central angle is $360^0/n$.

Next, we will find the measure of each interior angle of a regular n-gon. We will use the method of recognizing patterns for that purpose. Since the angles are congruent then the measure of each is the sum of the angles divided by n. Hence, we need to find the sum of the interior angles. This can be achieved by dividing the n-gon into triangles and using the fact that the sum of the three interior angles in a triangle is 180^0.

So, in general, the measure of an interior angle of a regular n-gon is

$$\frac{(n-2)*180^0}{n} = 180^0 - \frac{360^0}{n} \tag{1.6.1}$$

To measure the exterior angles in a regular n-gon, notice that the interior angle and the corresponding adjacent exterior angle are supplementary.

$$180^0 - \frac{(n-2)*180^0}{n} = \frac{360^0}{n} \tag{1.6.2}$$

Chapter 2

Tiling

A tiling refers to any pattern that covers a flat surface, like a painting on a convas, using non-overlapping repetion. In mathematical literature, the words tessellation, paving, mosaic and parquetting are used with similar meanings as tiling[12]. There are several ways to creat a tiling, here are some examples.

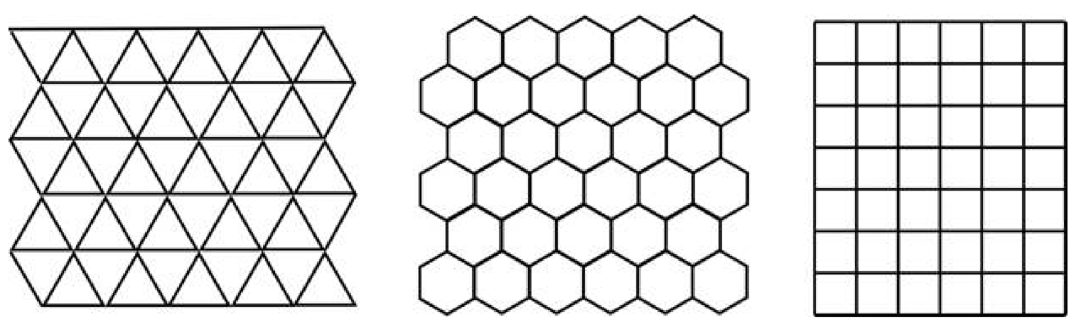

Figure 2.1: examples of tiling

2.1 Mathmatical Defintion of Tiling

Mathematically, a plane tiling T is a countable family of closed sets which covers the plane **without any gaps** or **without overlaps**. From this point on, we will use the word tiling to refer to a tiling of the plane.

$$T = \{T_1, T_2, T_3, \ldots\} \tag{2.1.1}$$

where T_1, T_2, T_3, ... are known as tiles of T.

The terms in bold are the conditions required of a tiling as explained below:

1) Without gaps: The union of all the sets T_1, T_2, T_3, ... is to be the whole plane, i.e.

$$\{T_1 \cup T_2 \cup T_3 \cup \ldots\} = \text{whole plane.}$$

2) Without overlaps: The interior of the sets are to be pairwise disjoint, i.e.

$$\{\text{interior of } T_i \cap \text{interior of } T_j\} = \varnothing \text{ where } i \neq j \text{ for any } i \text{ and } j.$$

The countable condition excludes tilings in which every tile has zero area (such as points or line segments). This is because tiles with zero area are uncountable when condition one is satisfied[1].

In this project, we only consider tiles which are closed topological disks, Therefore, the condition that tiles are closed topological disks eliminates tiles which are not closed topological disks as shown on fig 1.2.

2.2 Terminologies Used in Tiling and Tiles

- **Tiling**
  - Edge:- It is a boundary which separates the two tiles.
  - Vertex:- It is an endpoint of an edge.

- **Tiles**
  - Side:- It is an edge of a tile.
  - Corner:- It is a point where its sides intersect.

To get a clearer picture, we consider the following example

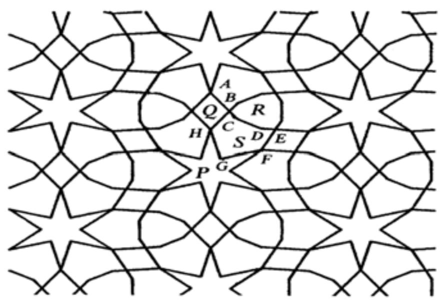

Figure 2.2: Tiling contains terminologies of tiling and tiles

BCDE is an edge of the tiling as it separates tile R and S, but it is not a side of any tile. BC, CD, DE, HG and GF are sides of tile S but are not edges of the tiling. G is a corner of tile S and tile P but it is not a vertex of the tiling[1].

In the subsequent chapters, for most of the part, we shall focus our attention to the special case of tilings in which each tile is a polygon.

We say that the tiling by polygons is edge-to edge if the corners and sides of the polygons coincide with the vertices and edges of the tiling, otherwise we say that the tiling by polygons is not edge-to-edge[1].

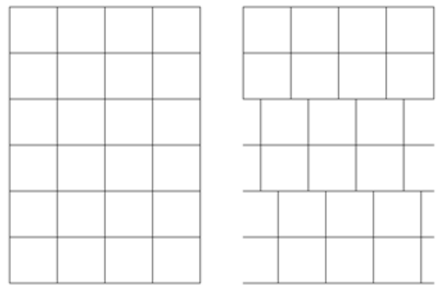

Figure 2.3: An edge-to-edge tiling by squares (left) and a tiling by squares that is not edge-to-edge.

Now, we shall see what we mean by saying two tilings are congruent. Two tilings T_1 and T_2 are congruent if T_1 may be made to coincide with T_2 by a rigid motion of the plane. Often, we are interested in equal tilings. Two tilings are said to be equal or the same if one of them can be changed in scale (magnified or contracted equally throughout the plane) so as to be congruent to the other. In other words, the tilings have to be the same size to be congruent while size does not matter for them to be equal[1]. Let's look the following example

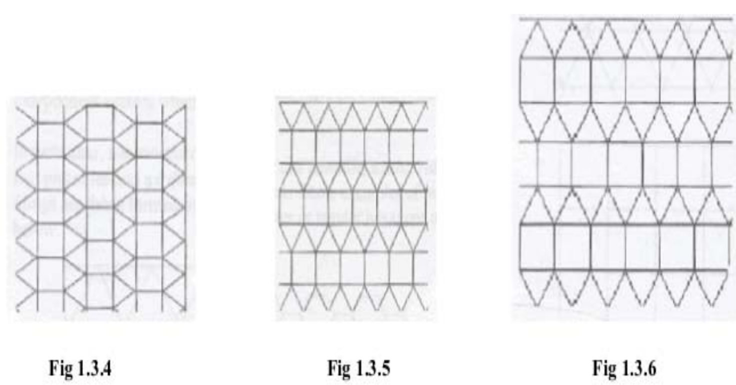

Figure 2.4: One can easily see that Fig 1.3.4 is congruent to Fig 1.3.5 while Fig 1.3.5 is equal to Fig 1.3.6.

2.3 Tilings with Tiles of a few shapes

We have already mentioned that we only consider tiles which are closed topological disks and as a further simplification, for the most part, we shall be concerned with monohedral tilings.

Monohedral: Every tile T_i in the tiling T is congruent to one fixed set T, meaning all the tiles are of the same shape and size. The set T is called the prototile of T. Figures (2.1) are Some examples of monohedral tilings formed by equilateral triangles, squares or regular hexagons[1].

In a similar way, we define dihedral, trihedral, 4-hedral, ... , n-hedral tilings in which all the tiles in the tiling are congruent to two, three, four, ..., n distinct prototiles of the set T respectively. Some examples of dihedral tilings are shown below.

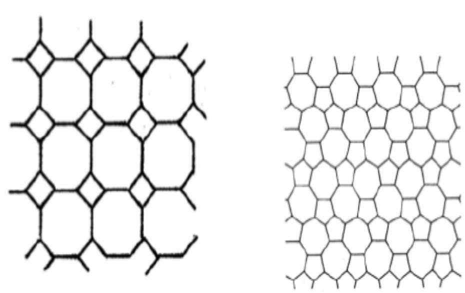

Figure 2.5: Some examples of dihedral

2.4 symmetry

Many important properties of tilings depend upon the idea of symmetry. Here, in this project, we explain what is meant by this term and give examples of tilings with various kinds of symmetry.

Some tilings have a certain kind of symmetry in themselves, namely that under a rigid motion of the plane, like a rotation or translation, the tiling remains the same. So that we can classify tilings according to symmetry properties. We start by defining the rigid motions we have in mind:

Definition 2.4.1. Isometry is any mapping of the Euclidean plane R^2 onto itself which preserves all distances. Let A, B be any two points, then the distance between A and B is equal to the distances between their images A^1 and B^1[1].

There are four types of isometric motions in the plane, called the plane isometries.

1. **Rotation** maps all points of a figure through a given angle about a given point O. The point O is called the center of rotation. Let θ be the angle of rotation about a point O that maps a figure onto itself. If $360^0/\theta = n$, which is a natural number, then the order of rotation of the figure is equal to n.

2. **Translation** shifts all points of a figure in a given direction through a given distance. We will use an arrow to represent a translation. The magnitude and direction of the arrow denote the distance and direction of translation, respectively. Translation can take place in any direction.

3. **A reflection** maps all points of a figure on one side of any straight line L, to the opposite side of L such that the perpendicular distance between any point

P, on one side of L, to the line L is equal to the perpendicular distance of P^1, the image of P on the opposite side of L, to the line L. The line L is called the mirror or the axis of reflection.

4. **Glide reflection**, as the name suggests, is a combination of the two isometries mentioned earlier, namely, translation and reflection.

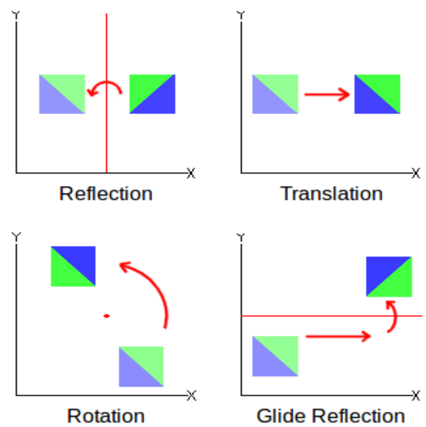

Figure 2.6: The four types of isometries

A symmetry of a figure is an isometry that maps the figure onto itself. Besides the four isometries mentioned, there is a trivial isometry called the identity isometry. The identity isometry maps every point of a figure onto itself and it is a symmetry of every figure. The rotation of a figure through 360^0 is an identity isometry of the figure[1].

A symmetry group of a set S is the set consisting of all the symmetries of S. The number of symmetries in a symmetry group is called the order of the symmetry group. We will denote the symmetry group of a single tile T by S(T) and the symmetry group of a tiling T by S(T).

2.4.1 Symmetry groups of tiles

The symmetry groups of tiles are classified into two types: cyclic groups and dihedral groups.

1. **cyclic group** The first type of symmetry groups is called cyclic groups. A cyclic group consists of rotations through angle $360j/n(j = 0, 1, ..., n-1)$ about a fixed point. Therefore, order of a cyclic group is n because the cyclic group consists of only n rotations. The notation for a cyclic group of order n is $C_n(n \geq 1)$. C_1 is the group consisting of only the identity symmetry[1]. For example:

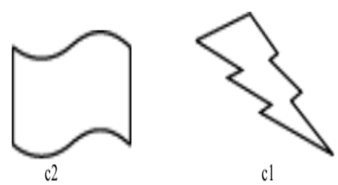

Figure 2.7: Examples of cyclic group

2. **dihedral group** The second type of symmetry groups is called dihedral groups. A dihedral group includes all the isometries in C_n and reflections in n lines. The n lines of reflection must be equally inclined to one another. This means that the angle between any two lines of reflection is the same. The order of a dihedral group is $2n$. This is because a dihedral group consists of n rotations and n reflections. Thus, the total number of symmetries in a dihedral group is $2n$. The notation for a dihedral group of order $2n$ is $d_n(n \geq 1)$. d_1 is the group consisting of the identity symmetry and reflection about a line[1]. For example:

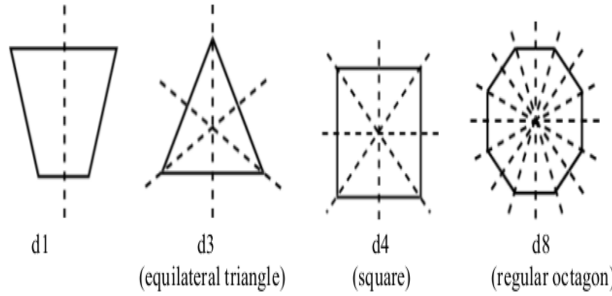

Figure 2.8: Examples of dihedral group

Note:-

- The dotted lines denote the line of reflection.
- The order of a cyclic group is n because the cyclic group consists of only n rotations including the identity symmetry (i.e. rotation about 360^0 about a fixed point).
- The order of a dihedral group is 2n because a dihedral group consists of n reflections and n rotations.

We exatend the difinition of symmetry of tilings as follow: A symmetry of a tiling is an isometry that maps the tiling onto itself. We will consider an example shown below fig 2.9. We can see that some of the symmetries of this tiling are:

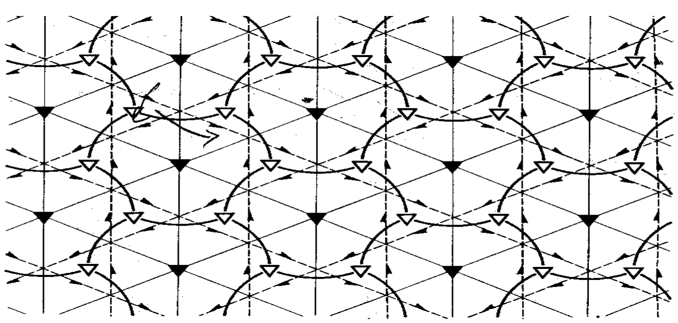

Figure 2.9: Examples of symmetry of tiling

- Rotations through 120^0, 240^0 about each point in the figure marked by a small triangle (solid or open).
- Translations, which take any of the solid black triangles onto one another.
- Reflections in each of the solid lines.
- Glide reflections consisting of reflection in the dashed lines followed by translation along them. This translation is through half the distance between the solid black triangles and is marked in the diagram by half arrowheads.

Now, we will compare symmetry of a tiling with that of its tiles. Let T be a tile of any tiling $\mathcal{T}$. Then every symmetry of $\mathcal{T}$ which maps T onto itself is clearly a symmetry of T. But the converse is not true in general. In other words, every symmetry of T may not be a symmetry of $\mathcal{T}$. For example in Fig 2.10 shown below, the only symmetry of $\mathcal{T}$ which maps a tile onto itself is the identity symmetry. However, the tile itself, being a square has seven other symmetries[1].

Hence, we must carefully distinguish between S(T), the group of symmetries of the

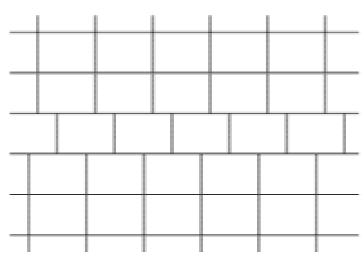

Figure 2.10: symmetry of tilings and tiles

tile T, and S($\mathcal{T}$), the group of symmetries of T which also are symmetries of the tiling $\mathcal{T}$.

2.5 Transitivity classes

Two tiles T_1, T_2 of a tiling T are said to be equivalent if the symmetry group S(T) contains a transformation that maps T_1 onto T_2. The collection of all tiles of T that are equivalent to T_1 is called the transitivity class of T_1. If all the tiles of T form one transitivity class, T is said to be isohedral. If T is a tiling with k transitivity classes, T is called k-isohedral. If a tiling T admits only the identity symmetry, then every tile in T is a transitivity class on its own. Figure 2.11 (a) and (b) shows an isohedral tiling. Figure 2.11 (c) and (d) show a 2-isohedral tiling. All the tiles in these tilings are either equivalent to A or B as denoted in the diagram, meaning these are tilings with two transitivity classes[1].

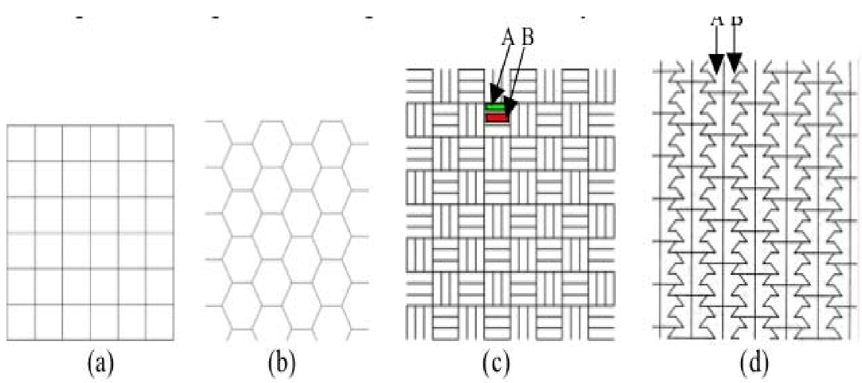

Figure 2.11: isohedral tilings and 2-isohedral tilings

Compare between monohedral and isohedral:

The definition of monohedral is explained in section 2.3. Both consist of tiles which have same shape and size. In isohedral tiling there is a symmetry that makes all the tiles equivalent. However in monohedral tilings, there might not exist a symmetry which maps one tile onto another.

In Fig 2.11(a) and (b), the monohedral tilings are isohedral. However, the monohedral tilings in Fig 2.11(c) and (d) are not isohedral as there is no symmetry of the tiling that maps tile A onto tile B. In short, isohedral implies monohedral but not the other way round. A brief explanation is given below.

For Fig 2.11(c), some of the ways to map tile A onto tile B is by translation by 1 unit down or reflection. However, you can easily see that these are not symmetries of the tiling. For Fig 2.11(d), you may try reflection to map tile A onto tile B at first thought but you can easily see that reflection is not a symmetry of the tiling.

Next, we go on to look at vertices. We will introduce two definitions, monogonal and isogonal. A monogonal tiling is one in which every vertex, together with its incident edges, forms a figure congruent to that of any other vertex and its incident edges. It simply means that vertices with its incident edges form one congruent class. An isogonal tiling is one in which all its vertices form one transitivity class.

Thus far, we have looked at tiles and vertices. Now, we will look at the last element of the tilings, that is, edges. As usual, we introduce two more definitions: isotoxal and monotoxal. Isotoxal tilings are tilings in which every edge can be mapped onto any other edge by a symmetry of the tiling. In short, it simply means that edges of the tiling form one transitivity class. From the definition of monohedral and monogonal, we try to deduce the meaning of monotoxal. We come up with the following two definitions.

1. analogy with monohedral: all edges are congruent, meaning all edges have the same length.
2. analogy with monogonal: all edge figures are congruent. This means that every edge, together with its incident vertices and edges form one congruent class. Note that an edge has two vertices.

The example (Fig 2.12) below shows an isotoxal tiling and a monotoxal tiling. It is monotoxal because all edges have the same length or it can be easily seen that all edge figures are congruent.

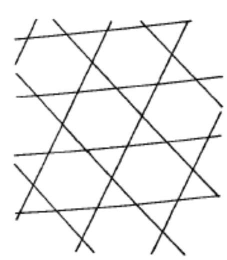

Figure 2.12: an isotoxal tiling and a monotoxal tiling

To summarize, hedral refers to tiles, gonal refers to vertices and toxal refers to edges. Note that if a tiling has k transitivity classes and m congruent classes, $m \leqslant k$. In other words, transitivity classes are contained in congruent classes, or one congruent class can be split into many transitivity classes[1].

Chapter 3

Tiling With Regular Polygon

3.1 Regular and Uniform Tilings

In this chapter, we are going to study tilings by regular polygons. We will also consider tilings that are edge-to-edge. It is important to study them as they form the basis for pattern construction in many traditions of sacred and decorative art. They can be found in Celtic and Islamic patterns, and in the natural world, they appear as crystal and cellular structures. Many people used them widely for their wallpaper and fabric designs[1].

Two tiles are called edge neighbors (vertex neighbors) if their intersection is an edge (vertex, respectively) of the polygons. Intersections of vertex neighbors are called the vertices of the tiling. Clearly the vertices of the polygons are then exactly the same points as the vertices of the tiling[2].

For the remainder of this project, all our tilings are assumed to be edge-to-edge tilings of regular polygons.

The surroundings of a vertex can be described by its vertex figure (see Figure 3.1), a term that is defined slightly differently by different authors. Krotenheerdt (1969)[11] defines vertex figure (or actually Eckgebilde) as the union of a vertex and its surrounding tiles, but Chavey (1984a)[9] defines it as the union of a vertex and its surrounding edges. As long as we are dealing with tilings by regular polygons, this distinction is of little practical importance. Vertex figures give rise to the related concepts of vertex species and vertex types (see Figure 3.1). The species of a vertex is the number and shape of the polygons in the corresponding vertex figure. The type of a vertex also takes the order of polygons into account. To describe vertex types and the tilings

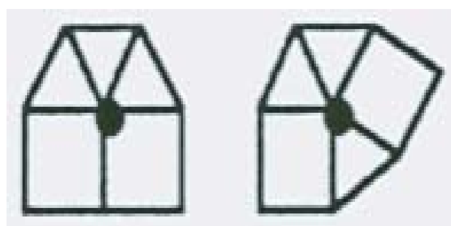

Figure 3.1: The vertices have the same species, but different types

they form, we need some notation. Following Grunbaum and Shephard (1987)[1], I will denote vertex types by listing the number of sides of each incident polygon, in cyclic order, separated by full stops (for example 3.7.42). The starting point is chosen to give the lexicographically smallest listing (for example 3.4.6.4 rather than 4.6.4.3). If several say k congruent polygons with say n sides are adjacent to each other, the notation is abbreviated by replacing the k ns by n^k, for example 4^4 rather than 4.4.4.4. A tiling formed by a specific vertex type is simply denoted by putting the symbol of the vertex type in parentheses[2].

Theorem 3.1.1. *The only edge-to-edge monohedral tilings by regular polygons are* (3^6)*,*(4^4) *and* (6^3)*. They consist of equilateral triangles, squares and regular hexagons respectively[1].*

Proof. Consider a surface to be covered with regular polygons, leaving no spaces between the meeting points of their vertices. If n denotes the number of sides each regular polygon will have, then the interior angles at each vertex of each polygon will be $(n-2) * 180^0/n$, lets denote by a. Now remember that our regular polygon will make a regular tiling only when the angle, a, divides exactly into 360^0. In other words, using our formula.

$$a = 180^0 - \frac{360^0}{n} \tag{3.1.1}$$

should satisfy

$$\frac{360^0}{180^0 - \frac{360^0}{n}} \tag{3.1.2}$$

Simplifying things we get

$$\frac{360^0}{180^0 - \frac{360^0}{n}} = \frac{1}{\frac{1}{2} - \frac{1}{n}} = \frac{2n}{n-2} = 1, 2, 3, ... \tag{3.1.3}$$

In other words we have turn our geometric problem into an algebra problem: Find the numbers n such that $2n/(n-2)$ is a whole number. Substituting $n = 8$, 9,... will not work because, we can write

$$\frac{2n}{n-2} = 2 + \frac{4}{n-2} \tag{3.1.4}$$

and notice that if n is bigger than 8 then

$$\frac{4}{n-2} < \frac{4}{8-2} = \frac{2}{3} < 1 \tag{3.1.5}$$

this means that

$$2 < \frac{2n}{n} = 2 + \frac{4}{n-2} < 3 \qquad \textit{when} \;\; n > 8 \tag{3.1.6}$$

There are no whole numbers between 2 and 3, so no regular polygon with more than eight sides can form a regular tiling. For this to be a whole number for $n > 2$, n must have values equal to 3, 4 or 6. Therefore, this corresponds to tilings which consist of regular hexagons, squares and equilateral triangles respectively[12]. □

We have seen that there are only 3 shapes that make a regular tiling. What happens if we use more than one shape. For now lets keep the rule that we want the vertexes to match up and that all shapes keeps the same sidelength.

When we consider vertex which is surrounded by more than one type of polygon, noted that there cannot be four or more different types of polygons around a vertex. This is because if we add up the four different polygons with the smallest angles, that is triangle with angle of 60^0, square with angle of 90^0, pentagon with angle of 108^0 and hexagon with angle of 120^0, we get a total of 378^0 which is greater than 360^0.

In order to figure this out we use the same principle as above that at each vertex the sum of the angles meeting has to add up to 360^0. Suppose that we have k regular polygons with number of sides $n_1, ..., n_k$ meeting at a vertex. Our basic principle means that

$$180^0 - \frac{360^0}{n_1} + 180^0 - \frac{360^0}{n_2} + ... + 180^0 - \frac{360^0}{n_k} = 360^0 \tag{3.1.7}$$

or doing a little rearranging

$$\frac{1}{n_1} + \frac{1}{n_2} + ... + \frac{1}{n_k} = \frac{k-2}{2} \tag{3.1.8}$$

Now all we need to do is solve this to figure out how many shapes meet at each vertex.

This is no easy task, there are too many unknowns.

We can, however, get bounds on k. For instance we cannot have $k = 1$ or 2. So $k \geqslant 3$ for a lower bound. To get an upper bound notice we must have n_1 , n_2,...,n_k all bigger than 3 because each regular polygon has at least three sides. This makes

$$\frac{k-2}{2} = \frac{1}{n_1} + \frac{1}{n_2} + ... + \frac{1}{n_k} \leqslant \frac{1}{3} + \frac{1}{3} + ... + \frac{1}{3} = \frac{k}{3} \qquad (3.1.9)$$

Cross multiply to get an upper bound on k, the number of regular polygons that meet at a vertex that is $3 \leqslant k \leqslant 6$.

We have made things more manageable for our selves by showing that in order to have a mixed regular tiling, we need at least three and at most six regular polygons to meet at each vertex. However we still have to find all solutions to the equations.

let us consider case by case

Case 1: 3 polygons

The sum of the interior angles of the 3 polygons around a vertex add up to 360^0,

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = \frac{1}{2} \qquad (3.1.10)$$

We find that the solutions which satisfy the above equation are as follows:

(1) 3.7.42

(2) 3.8.24

(3) 3.9.18

(4) 3.10.15

(5) 3.12.12

(6) 4.5.20

(7) 4.6.12

(8) 4.8.8

(9) 5.5.10

(10) 6.6.6

Case 2: 4 polygons

Following the same procedure as above, we have

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} = 1 \tag{3.1.11}$$

Thus, the solutions which satisfy the above equation are as follows:

(11) (i) 3.3.4.12 (ii) 3.4.3.12

(12) (i) 3.3.6.6 (ii) 3.6.3.6

(13) (i) 3.4.4.6 (ii) 3.4.6.4

(14) 4.4.4.4

Case 3: 5 polygons

Following the same procedure as above, we have

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} + \frac{1}{n_5} = \frac{3}{2} \tag{3.1.12}$$

Thus, the solutions which satisfy the above equation are as follows:

(15) 3.3.3.3.6

(16) (i) 3.3.3.4.4 (ii) 3.3.4.3.4

Case 4: 6 polygons

Following the same procedure as above, we have

$$\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}+\frac{1}{n_4}+\frac{1}{n_5}+\frac{1}{n_6}=2 \tag{3.1.13}$$

Thus, the only solution which satisfy the above equation are as follows:

(17) 3.3.3.3.3.3

It turns out there are 17 solutions to the above equations. The first equation has 10 solutions, the second equation has 4 solutions, the third equation has two solutions, and the last equation has only one solution.

It can be seen that in 4 of the species, there are 2 distinct ways in which the polygons may be arranged, meaning the position of the polygons is different, resulting in different look and shape. Take the vertices of types (3.3.6.6) and (3.6.3.6) shown in Fig 3.2 for example. Both belong to the same species but they are different. For the type (3.3.6.6), the triangles are adjacent to each other, same for the hexagons. But for type (3.6.3.6), the triangles are opposite to each other, same for the hexagons. Note that for the case with 3 polygons, all have only 1 distinct type by virtue of the fact that no matter how you permutate the 3 polygons around a vertex, the look and shape is still the same. Therefore, there are altogether 21 possible types of vertices.

Fig. 3.2 shows the 21 types of vertices possible with regular polygonal tiles.

We shall restrict our study to those tilings whose vertices are of the same type. As

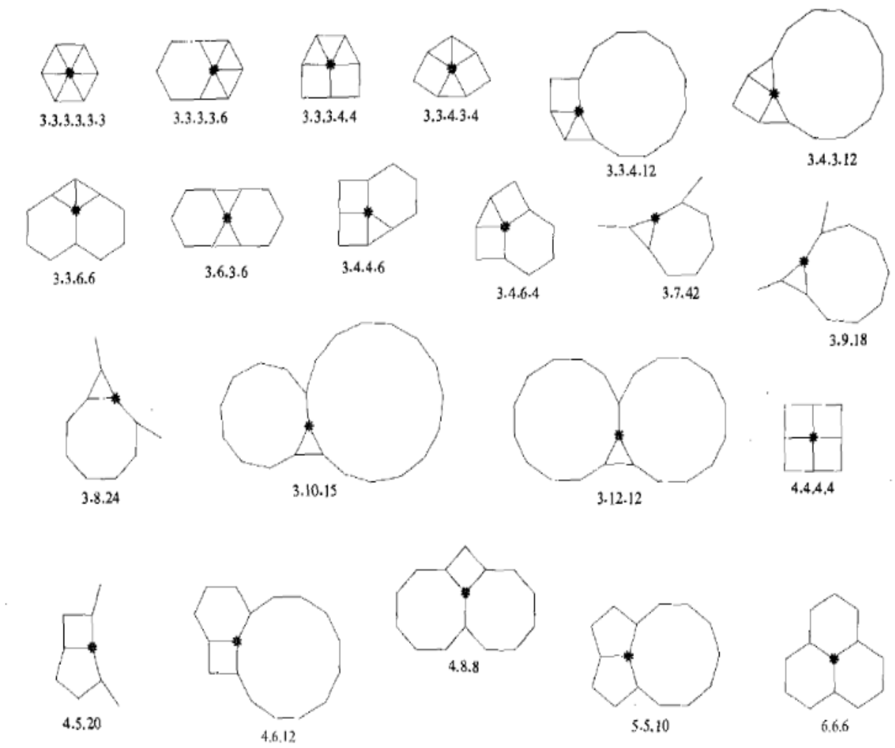

Figure 3.2:

shown above, we have found 21 ways to fit regular polygons around a vertex. Unfortunately, not all of these extend to tilings of the plane. We have only found possible tilings as we know only that these vertex combinations add up to 360^0 at each vertex. Now, our goal is to find out what is the total number of edge-to-edge tilings which use polygons as tiles and whose vertices are of the same type[1].

1. Consider a vertex of type $3.x.y$, $x \leq y$. At vertex A, we have the vertex

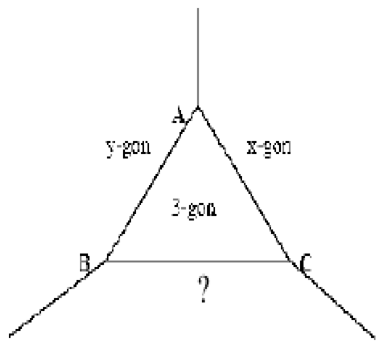

Figure 3.3:

configuration $3.x.y$. A triangle and a y-sided polygon meet at B, so the polygon marked ? ought to have x sides. But at vertex C, there are already a triangle and a x-sided polygon, so the polygon marked ? should have y sides. Thus, we cannot have any vertex configuration of the form $3.x.y$ if $x \neq y$. Hence, if $3.x.y$ is possible, $x = y$. Therefore, the vertex of type (1), (2), (3), (4) are not possible.

2. Similarly, consider a vertex of type $x.5.y$, $x \leq 5$ Consider a regular 5-gon

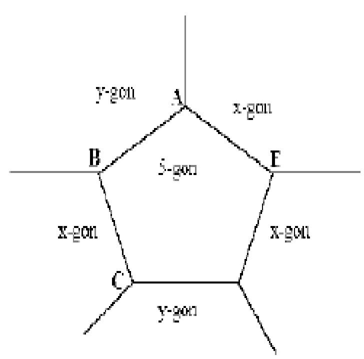

Figure 3.4:

ABCDE. All the vertices are of the type $x.5.y$, except for vertex E, which is forced to be of the type $x.5.x$. Therefore, if $x.5.y$ is possible, $x = y$. Thus, the vertex of type (6) and (9) are not possible.

3. Lastly, consider a vertex of type $3.x.y.z$.

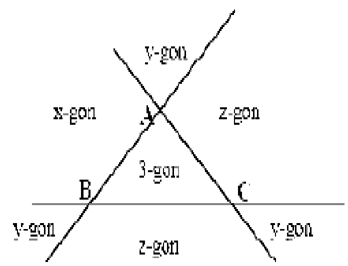

Figure 3.5:

Using the same reasoning as above, we can show that if $3.x.y.z$ is possible, $x = z$. Thus,the vertex of type (11)(i) and (ii), (12)(i) and (13)(i) are not possible.

Therefore, from the above solutions 11 types of vertices remain. It can be shown that each of them can be extended to form tilings See Fig 3.6. They are usually called **Archimedean tilings** (some authors call them homogeneous or semiregular), and they clearly include the three regular tilings. We call them "Archimedean" as it simply means tilings with only one type of vertex. In general, there can be more than one type of vertices in the tiling. If there are k types of vertices, we call it k-Archimedean[1].

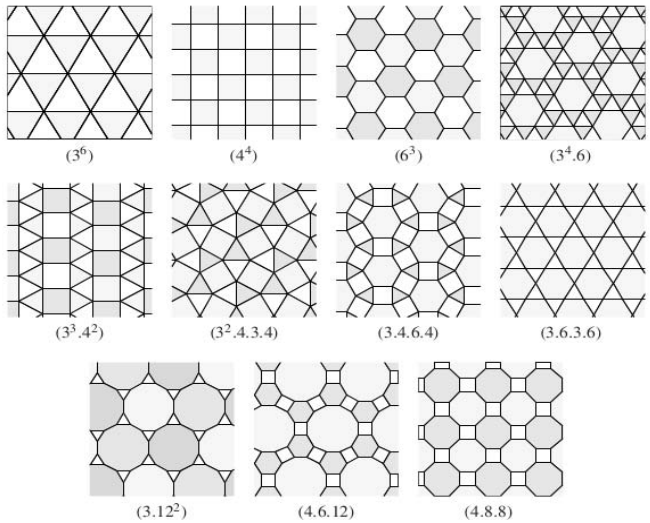

Figure 3.6:

Let us discuss some of the special properties of Archimedean tilings.

Firstly, we observe that the vertex of $(3^4.6)$ occurs in two forms which are mirror images of one another. We recall from Section 2.2 that two tilings are equal if one may be made to coincide with the other by a rigid motion of the plane (possibly including reflection) followed by a change of scale. In fact, except for the tiling $(3^4.6)$, reflections are never required to establish the equality of two tilings of the same type. But in the case of $(3^4.6)$, reflections may be required and we describe this situation by saying that $(3^4.6)$ occurs in two enantiomorphic (mirror image) forms[1].

Next, one accidental but very special and important feature of the Archimedean tilings is that each is isogonal. Therefore, we call Archimedean tilings also uniform. The distinction between the meanings of the two words is that Archimedean refers only to the fact that the tiling is monogonal, that is, the immediate neighbourhood of any two vertices look the same, while the term uniform implies the much stronger property of isogonality. We will discuss the meaning of k-uniform in the next section.

3.2 K-Uniform Tiling

For a tiling with tiles of regular polygons, we call it k-uniform if and only if it is k-isogonal, we recall in Section 2.5 that a tiling is k-isogonal if its vertices form precisely k transitivity classes with respect to the group of symmetries of the tiling. In this terminology, uniform tilings are 1-uniform. An example is shown below to illustrate this definition.
Lets take a look at Fig below.

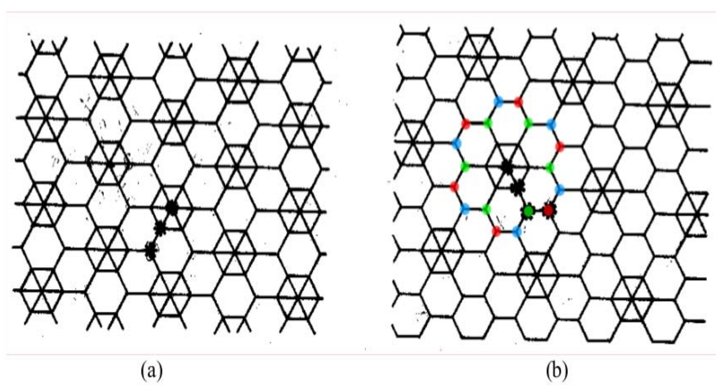

Figure 3.7: uniform tilings

In both Fig 3.7(a) and (b), the tilings consist of 3 types of vertices,(3^6),(6^3) and $(3^2.6^2)$, thus we know that these 2 tilings must be at least 3-uniform.

For Fig 3.7(a), vertices of type (3^6) can be mapped onto one another by translations; therefore they form 1 transitivity class. Vertices of type (6^3) can be mapped onto one another by 60^0 rotation followed by translations, therefore they form 1 transitivity class. Similar to vertices of type (6^3), vertices of type $(3^2.6^2)$ also form 1 transitivity class. As there are a total of 3 transitivity classes, we can conclude that Fig 3.7(a) is 3-uniform.

Similarly, for Fig 3.7(b), vertices of type (3^6) and vertices of type $(3^2.6^2)$ form 1 transitivity class each. However, vertices of type (6^3) form 3 transitivity classes, as the vertices indicated by red dots, green dots and blue dots can be mapped onto one another by 60^0 rotation and they can all be mapped to other corresponding (6^3) vertices by translations. As Fig 3.7(b) has altogether 5 transitivity classes, it is 5-uniform.

There exist 20 distinct types of 2-uniform edge-to edge tilings by regular polygons shown in Fig 3.8 These result was obtained by krotenheerdt. He consider those k-uniform tilings in which the k transitivity classes of vertices consist of k distinct types of vertices. If we denoted by K(k) the number of distinct krotenheerdt tilings his result are:

$K(1) = 11$

$K(2) = 20$

$K(3) = 39$

$K(4) = 33$

$K(5) = 15$

$K(6) = 10$

$K(7) = 7$ and $K(k) = 0$ for $k \geqslant 8$

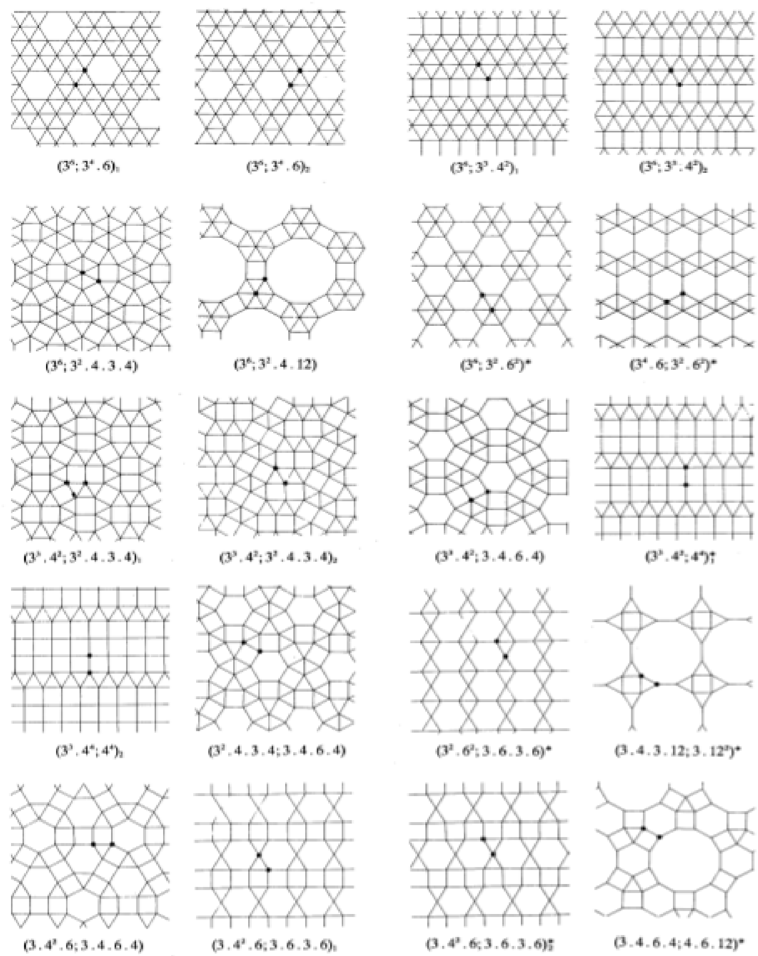

Figure 3.8: 2-uniform tilings

However, if we do not impose Krotenheerdt condition,even for small k such as $k = 4$, it is not known how many k-uniform tilings exist, nor is any kind of asymptotic estimate available for the number of k-uniform tilings with large k. We just state here that for k=3, $Chavey[1984b]$ determined that there are 61 tilings of this kind.

Reference

[1] B. Grunbaum and G.C. Shephard, "Tilings and Patterns", W. H. Freeman and Company, New Yoke,1987.

[2] Nils Lenngren, "K−Uniform tilings by regular polygons", Uppsala University, 2009.

[3] Renzos Math 490, "Introduction to topology", Winter, 2007.

[4] George E.Martin, "Transformation geometry: an introduction to symmetry", Springer-Verlag, New York,1982.

[5] Rich Schwartz, "Countable and Uncountable sets", November 12, 2007.

[6] James R. Munkres, "Topology", Second edition, Prentice-Hall Inc., Englewood Cliffs., N.J., 2000.

[7] Bunz, U. H. F., Rubin, Y., and Tobe, Y. (1999). Polyethynylated cyclic −systems: scaffoldings for novel two and three-dimensional carbon networks. Chemical Society reviews, 28, 107119.

[8] Conrad, M., Krumeich, F., Reich, C., and Harbrecht, B. (2000). Hexagonal approximants of a dodecagonal tantalum telluride the crystal structure of $Ta_{21}Te_{13}$. Materials science and engineering, 294296, 3740.

[9] Chavey, D. (1984a). Periodic tilings and tilings by regular polygons. Unpublished doctoral dissertation. University of WisconsinMadison.

[10] Chavey, D. (1984b). Periodic tilings and tilings by regular polygons. I: Bounds on the number of orbits of vertices, edges and tiles. Mitteilungen aus dem Math− ematischen Seminar Giessen, 164, 3750.

[11] Krotenheerdt, O. (1969). Die homogenen Mosaike $n-$ter Ordnung in der euko lidischen Ebene I. Wissenschaftliche Zeitschrift der Martin−Luther−Universitat HalleWittenberg, 18, 273290.

[12] http://www.dehn.wustl.edu/∼blake/circles/talks/2011–Mar06–kabe_Moen–Tilings.pdf.

[13] http://www.users.utu.fi/jkari/tilings/part2.pdf.

[14] http://www.faculty.atu.edu/mfinan/2043/section40.pdf.