Weak Idempotent Nil-neat Rings

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This is to certify that the dissertation prepared by **Biadiglign Asmare Alamneh** entitled: "WEAK IDEMPOTENT NIL-NEAT RINGS" submitted in fulfillment of the requirements for the Degree of Doctor of Philosophy in Mathematics complies with the regulations of the University and meets the accepted standards concerning originality and quality.

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Abstract

We introduce the concept of a weak idempotent nil-clean ring which is a generalization of weakly nil-clean ring. We give certain characterizations for a weak idempotent nil-clean ring in terms of the Jacobson radical and nil-radical. In addition to this, we prove that $n \times n$ upper (lower) triangular matrix over a ring R is weak idempotent nil-clean if and only if so is R.

We introduce the concept of a strongly weak idempotent nil-clean ring which is a generalization of a strongly weakly nil clean ring. We characterize strongly weak idempotent nil-clean rings in terms of the set of nilpotent elements, homomorphic images, and Jacobson radicals. Moreover, we give necessary and sufficient conditions of a strongly weak idempotent nil-clean ring in relation to periodic rings, and also we give a characterization between strongly weak idempotent nil-clean rings and strongly π-regular rings and strongly clean rings elementwise. Furthermore, we prove that a strongly weak idempotent nil-clean ring R with $2 \in J(R)$ satisfies nil-involution property.

We define the concept of a weak idempotent nil-neat ring which is the generalization of a weakly nil-neat ring. We characterize reduced weak idempotent nil-clean rings. Also, we give a characterization of weak idempotent nil-neat rings in terms of semiprime ideals, maximal ideals and Jacobson radicals. Moreover, we prove that every nonzero prime ideal of a strongly weak idempotent nil-clean ring is maximal. Finally, we investigate the condition for which the group ring $R[G]$ becomes a weak idempotent nil-clean ring and a weak idempotent nil-neat ring.

List of Notations and Conventions

Notations

Notation	Meaning
R	an associative ring with unity.
$wi(R)$	A collection of all weak idempotent elements of a ring R.
$Nil(R)$	The collection of nilpotent elements of a ring R.
$J(R)$	The Jacobson radical of a ring R.
$Id(R)$	The collection of idempotent elements of a ring R.
$U(R)$	The set of all unit elements of R.
I	An ideal of R.
$C(R)$	The set containing all central elements of a ring R.
$Inv(R)$	The set containing involutions of a ring R.
$M_n(R)$	The ring of all square matrices of order n over a ring R.
$\mathbb{T}_n(R)$	The ring of all upper triangular matrices of order n over a ring R.
I_n	The identity element of a matrix ring $M_n(R)$.
C_n	The cyclic group of order n.
S_n	The symmetric group of degree n.
G	The multiplicative abelian group.
$R[G]$	The group ring of R over group G.
$\mathbb{F}_n$	A field with n elements.
$rk(A)$	Rank of a matrix A.
$det(A)$	Determinant of a square matrix A.
$char(R)$	Characteristic of a ring R.
$\mathbb{Z}_{(n)}$	The set of rational numbers of the form $\frac{r}{s}$, where n does not divide s.
$\mathbb{N}$	The set of natural numbers.
$\mathbb{Z}$	The set of integers.

Contents

Chapter 1

Introduction

In this dissertation, we propose and study weak idempotent nil-clean rings and commutative weak idempotent nil-neat rings in this dissertation as a special class of rings. Further, we assume that every ring is associative ring with unity. We denote the collection of all nilpotent elements, the collection of all idempotent elements , the collection of all weak idempotent elements, the group of unit elements and the Jacobson radicals of a ring R by $Nil(R)$, $Id(R)$, $wi(R)$, $U(R)$ and $J(R)$ respectively. It is commonly known that:

$$\text{strongly nil-clean} \Rightarrow \text{nil-clean} \Rightarrow \text{weakly nil-clean} \Rightarrow \text{clean.}$$

It is observed that each element can be expressed as the sum of a certain element and an idempotent element in all the above-said rings. It is quite natural to ask whether the representation can be generalized. In any ring R, if $a^4 = a^2$, then such a is called weak idempotent element. Clearly, every idempotent is weak idempotent but not conversely. For instance, consider the ring of integers modulo 4. Clearly, every element is a weak idempotent element but 2 is not idempotent. In view of these observations, is it possible to replace the idempotent element with a weak idempotent element in the above-said classes of rings? To some extent the answer is affirmative. We introduce here the concept of weak idempotent nil-clean rings, a wider class to the class of weakly nil-clean rings and a subclass of the class of clean rings. We prove that homomorphic image of a weak idempotnet nil-clean ring is weak idempotnet nil-clean ring, finite product of weak idempotnet nil-clean rings is a weak idempotnet nil-clean ring and every weak idempotnet nil-clean ring can be expressed as a direct product of nil- clean ring and weak idempotnet nil-clean ring with 3 belongs to its Jacobson radical. We will also consider the matrix extensions of weak idempotent nil-clean rings.
We recall the following definitions from ([4]), ([8]) and ([11]). A ring R is called

(1) strongly nil-clean if, for every $r \in R$, there exists a nilpotent n and an idempotent $e^2 = e \in R$ such that $r = n + e$ and $ne = en$.

(2) strongly weakly nil-clean if each element in R is the sum or difference of a nilpotent and an idempotent that commutes.

(3) strongly clean if each element in R can be expressed as the sum of a unit and an idempotent that commute.

(4) periodic if there are distinct $m, k \in \mathbb{N}$ such that $a^m = a^k$ for any $a \in R$.

(5) strongly π-regular if, for each element a in a ring R, there exists $r \in R$ and a positive integer k such that $a^k = a^{k+1}r$.

A ring satisfies nil-involution property if each element is the sum of a unipotent and an involution (i.e., an element whose square is 1). In other words, a ring R satisfies the nil-involution property if, for each $r \in R$, $r = u + v$, where $u \in Nil(R) + 1$(equivalently, $u \in Nil(R) - 1$) and $v^2 = 1$. The following hold:
strongly nil-clean rings $\implies$ strongly weakly nil-clean rings $\implies$ strongly clean rings.
In this dissertation, we present a new class of rings called strongly weak idempotent nil-clean rings (SWIN-clean rings) that is a super class of strongly weakly nil-clean rings and a subclass of strongly clean rings. This class of rings generalizes the idea of strongly weakly nil-clean rings . In respect to periodic rings, strongly π-regular rings, and strongly clean rings, we derive the necessary and sufficient conditions for strongly weak idempotnet nil-clean rings. We prove that finite direct product of strongly weak idempotnet nil-clean rings and homomorphic image of a strongly weak idempotnet nil-clean ring are strongly weak idempotnet nil-clean rings. We also prove that $Nil(R)$ of a strongly weak idempotnet nil-clean ring R forms an ideal and $R/Nil(R)$ is reduced weak idempotnet nil-clean ring with all of the elements of $R/Nil(R)$ are weak idempotent elements. Moreover, we prove that a ring R is strongly weak idempotent nil-clean if and only if R has no homomorphic image $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $a^k - a^{k+2}$ is nilpotent element in R for every $a \in R$ and some positive integer k. Further, we characterize strongly weak idempotnet nil-clean rings, strongly π-regular rings and strongly clean rings elementwise.
Consider the rings $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Then, they are not weakly nil-neat rings since they contain a proper homomorphic image $\mathbb{Z}_3 \times \mathbb{Z}_3$ which is not weakly nil-clean. Now $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ is weak idempotent nil-neat, but not weak idempotent nil-clean (see example 5.2). Also, it is not weakly nil-neat ring since it has a proper homomorphic image $\mathbb{Z}_3 \times \mathbb{Z}_3$, that is , not weakly nil-clean ring.

The following ring hierarchy hold: nil-neat $\implies$ weakly nil-neat $\implies$ neat.

We introduce a class of rings, called weak idempotent nil-neat rings which is a subclass of neat rings and a wider class of weakly nil-neat rings. The study of weak idempotent nil-neat rings is considerably to enlarge the results for weakly nil-neat rings obtained in ([20]) to this new point of view. We first characterize commutative weak idempotnet nil-clean rings and commutative reduced weak idempotnet nil-clean rings. Then we focus our attention to obtain the relationship between the class of commutative weak idempotent nil-clean rings and the class of weak idempotent nil-neat rings. Also, demonstrate a classification of weak idempotent nil-neat rings. We conclude the thesis by characterizing weak idempotent nil-neat group rings and win-clean group rings.

1.1 The Main Objectives of This PhD Dissertation

The main objectives of this PhD dissertation are to study the structures and properties of commutative weak idempotent nil-neat rings and weak idempotnet nil-clean rings, as well as to generalize these properties. The specific objectives of this dissertation are as follows:

(1) To identify in between which class of rings do the class of weak idempotnet nil-clean rings, the class of strongly weak idempotnet nil-clean rings, and the class of weak idempotent nil-neat rings lie.

(2) To prove under what condition our class of rings coincide with their supper class and their lower class.

(3) To investigate properties and structures of weak idempotnet nil-clean rings, the class of strongly weak idempotnet nil-clean rings and weak idempotent nil-neat rings.

(4) To investigate extension of weak idempotnet nil-clean rings to some extent.

1.2 Organization of The Dissertation

The remaining part of this PhD dissertation is organized as follows:
Chapter 2 contains *preliminaries and literature review*, here we give some basic concepts, definitions and theorems that play important roles in the discussions of the main results.
In Chapter 3, we introduce the concept of weak idempotnet nil-clean rings and furnish certain examples, following this we will see some of the properties of weak idempotent elements of a ring and reduced weak idempotnet nil-clean rings. Also, we obtain some basic results concerning weak idempotnet nil-clean rings. Next, we characterize weak idempotnet nil-clean rings in Proposition 3.4 and also we prove that every weak idempotnet nil-clean ring R is a direct product of weak idempotnet nil-clean rings R_1 and R_2, where $2 \in J(R_1)$ and $3 \in J(R_2)$. Finally, we characterize weak idempotent nil-clean matrix rings.
In Chapter 4, we introduce the concept of strongly weak idempotnet nil-clean rings and then we look at examples and basic properties of strongly weak idempotnet nil-clean rings. Next, we deal with some results associated to homomorphic images of strongly weak idempotnet nil-clean rings. Finally, we characterize strongly weak idempotnet nil-clean rings, strongly π-regular rings and strongly clean rings elementwise.

In Chapter 5, we deal with some of the properties of commutative weak idempotnet nil-clean rings. Then, we give a characterization of commutative weak idempotent nil-clean group rings. we also introduce the notion of weak idempotent nil-neat rings as generalization of the class of weakly nil-neat rings which is a subclass of neat rings. We first introduce the notion of weak idempotent nil-neat rings and give certain examples. Next, we will see some results associated to commutative weak idempotent nil-neat rings which are a base for the proof of main results of weak idempotent nil-neat rings (see Proposition 5.1) and then we focus our attention to obtain the relationship between commutative weak idempotent nil-clean and commutative weak idempotent nil-neat rings. We also characterize weak idempotent nil-neat group rings.
Finally, we conclude the thesis with a future plan by posing open questions related to this Ph.D. dissertation.

Chapter 2

Preliminaries and Literature Review

This chapter is devoted to some basic definitions and properties of nil-clean rings, weakly nil-clean rings, strongly weakly nil-clean rings, strongly π-regular and clean rings. First we look at the definitions of some basic terms.

2.1 Definitions of Basic Terms

In this section, we look at definitions of some basic terms we use in this dissertation as well as examples to illustrate the definitions.

Definition 2.1 ([33])**.** An element n of a ring R is said to be nilpotent if there exists a positive integer k such that $n^k = 0$.

Example 2.1. In $\mathbb{Z}_4$, 0 and 2 are the nilpotent elements.

Definition 2.2 ([24])**.** An element e of a ring R is called idempotent if $e = e^2$.

Example 2.2. In $\mathbb{Z}_6$, 0, 1, 3 and 4 are idempotent elements.

Remark 2.1 ([24])**.** In any ring, 0 and 1 are called trivial idempotents.

Definition 2.3 ([24])**.** A ring R is said to be Boolean if each element of R is idempotent.

Example 2.3. The ring $\mathbb{Z}_2$ is Boolean ring.

Definition 2.4 ([26])**.** A ring is called abelian if every idempotent is central, that is, every idempotent element commutes each element of the ring.

Definition 2.5 (26)**.** A ring R is said to be reduced if R has no non-trivial nilpotent element, that is, $Nil(R) = \{0\}$.

Example 2.4. The rings $\mathbb{Z}_6$ and $\mathbb{Z}_p$, where p is a prime number, are reduced rings.

Definition 2.6 ([17])**.** An element r of a ring R is said to be unipotent if there exists a nilpotent element n such that $r = n + 1$.

Example 2.5. In $\mathbb{Z}_{12}$, 1 and 7 are unipotent elements.

Definition 2.7 ([27]). Let R be a ring and $I \subseteq R$. Then I is said to be left (respectively, right) ideal of R if it satisfies

(1) $x - y \in I$ for all $x, y \in R$.

(2) $rx \in I$(respectively, $xr \in I$) for all $x \in I$ and $r \in R$.

Example 2.6. Consider the ring $M_2(\mathbb{R})$. Then $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$ is left ideal of $M_2(\mathbb{R})$.

Definition 2.8 ([5]). Let R be a ring and I be an ideal of R. Then, the idempotents can be lifted modulo I if for a given $a \in R$ with $a^2 - a \in I$, there exists $e = e^2 \in R$ such that $e - a \in I$.

Example 2.7. Consider the ring $\mathbb{Z}_6$ and its ideal $I = \{0, 2, 4\}$. Then $Id(\mathbb{Z}_6) = \{0, 1, 3, 4\}$ and
$1^2 - 1 = 0 \in I \implies \exists 1 \in Id(\mathbb{Z}_6)$ such that $1 - 1 = 0 \in I$.
$2^2 - 2 = 2 \in I \implies \exists 0 \in Id(\mathbb{Z}_6)$ such that $2 - 0 = 2 \in I$.
$3^2 - 3 = 0 \in I \implies \exists 1 \in Id(\mathbb{Z}_6)$ such that $3 - 1 = 2 \in I$.
$5^2 - 5 = 2 \in I \implies \exists 3 \in Id(\mathbb{Z}_6)$ such that $5 - 3 = 2 \in I$.
Hence, idempotents can be lifted modulo I, that is, I lifts the idempotents of the quotient $\mathbb{Z}_6/I$ to idempotents of the ring $\mathbb{Z}_6$.

Proposition 2.1 ([5]). Idempotents lift modulo nil ideals.

Definition 2.9 ([33]). An ideal I of a ring R is said to be a nilpotent ideal if there exists a natural number k such that $I^k = 0$, that is, the product of any k elements of I is 0.

Example 2.8. In the ring $\mathbb{Z}/p^n\mathbb{Z}$, where p is a prime number and $n \in \mathbb{N}$, every ideal except the ring itself is nilpotent.

Definition 2.10 ([5]). An ideal I of a ring R is said to be nil ideal if each element of I is nilpotent.

Example 2.9. In $\mathbb{Z}_{12}$, $Nil(\mathbb{Z}_{12}) = \{0, 6\}$ is nil ideal of $\mathbb{Z}_{12}$.

Definition 2.11 ([40]). A ring R is said to be Von Neumann regular (or regular) if, for every $a \in R$, there is $b \in R$ such that $a = aba$.

Example 2.10. Fields are regular rings.

Definition 2.12 ([21]). A ring R is said to be strongly regular if $a \in Ra^2 \cap a^2R$, that is, for every $a \in R$, there is $b \in R$ such that $a = a^2b$ (with $ab = ba$).

Definition 2.13 ([21]). A ring R is called strongly π-regular if, for each $a \in R$, there is $n \in \mathbb{N}$ depending on a and possessing the property $a^n \in Ra^{n+1} \cap a^{n+1}R$, i.e., $a^n = a^{n+1}r$ for some $r \in R$ (with ar=ra).

Lemma 2.1 ([22]). Let R be a ring, and I be a nilpotent ideal of R. An element $x \in R$ is strongly π-regular if and only if $\bar{x}$ is strongly π-regular in $\bar{R} = R/I$.

Definition 2.14 ([27]). A homomorphism $f : R \to S$ from a ring R into a ring S is called a zero morphism if $f(r) = 0_S$ for all $r \in R$.

Definition 2.15 (27)**.** If $f : R \to S$ is a homomorphism from a ring R into a ring S, then the set $f(R) = \{f(x) \mid x \in R\} \subseteq S$ is called f-homomorphic image of R.

Remark 2.2 ([27])**.** If $f : R \to S$ is an epimorphism from a ring R into a ring S, then S is a homomorphic image of R. For example, if R is a ring and I is an ideal of R, then R/I is the homomorphic image of R by a canonical epimorphism $\alpha : R \to R/I$ given by $\alpha(r) = r + I$.

2.2 Some Types of Rings

In this section, we look at definitions, examples and results on some types of rings that we use in our dissertation.

2.2.1 Nil-clean Rings

Definition 2.16 (21)**.** Let R be a ring. If there is an idempotent $e \in R$ and a nilpotent $n \in R$ such that $r = n + e$, then an element $r \in R$ is said to be nil-clean. The element r is further called strongly nil-clean if such an idempotent and nilpotent can be chosen such that $ne = en$. A ring is called nil-clean (or strongly nil-clean) if every element in it is nil-clean (or strongly nil-clean).

Example 2.11. The ring $\mathbb{Z}_4$ is nil-clean ring but the ring $\mathbb{Z}_3$ is not nil-clean because 2 can not be written as a sum of a nilpotent and an idempotent element in $\mathbb{Z}_3$.

Definition 2.17 ([21])**.** Let R be a ring and $r \in R$. Then writing r as a sum of a nilpotent and an idempotent elements is called nil-clean decomposition of r.

It was demonstrated by Diesl ([21]) that all strongly nil-clean elements are both strongly clean and strongly π-regular. Furthermore, each strongly nil-clean element of a ring is said to have a unique strongly nil-clean decomposition. It is also proved that a ring R is strongly nil-clean if and only if R is strongly π-regular and every unit of R is unipotent. We now list a few fundamental characteristics of nil-clean rings.

Proposition 2.2 ([21])**.**

(1) Any quotient of a nil-clean ring (respectively, a strongly nil-clean ring) is nil-clean (respectively, strongly nil-clean ring).

(2) Any finite direct product of nil-clean rings (respectively, strongly nil-clean rings) is nil-clean (respectively, strongly nil-clean rings).

Proposition 2.3 ([21])**.** Let R be a ring, and I be a nil-ideal of R. Then, R is nil-clean if and only if R/I is nil-clean.

Corollary 2.1 ([21])**.** Let R be a ring. Then, R is nil-clean if and only if $J(R)$ is nil and $R/J(R)$ is nil-clean.

Every nilpotent element of a nil-clean abelian ring R is contained in $J(R)$. A ring R is reduced and hence Boolean if it is a nil-clean abelian ring with $J(R) = 0$.

Proposition 2.4 ([21])**.** Let R be a ring. An element $a \in R$ is strongly π-regular if and only if there is an idempotent $e \in R$ and a unit $u \in R$ such that $a = u + e$, $ae = ea$ and eae is nilpotent. The element a is strongly regular if and only if there is an idempotent $e \in R$ and a unit $u \in R$ such that $a = u + e$, $ae = ea$ and eae is zero.

From Proposition 2.4, we can have the following definition:

Definition 2.18. Let R be a ring and $a \in R$. Then the clean decomposition of a given by $a = u + e$, where $u \in U(R)$ and $e \in Id(R)$, is called strongly π-regular decomposition of a if $ae = ea$ and eae is nilpotent element in R.

Proposition 2.5 ([21])**.** Let R be a ring and $a \in R$ is a strongly π-regular element with strongly π-regular decomposition $a = e + u$, where $u \in U(R)$ and $e \in Id(R)$. If $a = f + v$, where $f \in Id(R)$ and $v \in U(R)$ is another strongly π-regular decomposition of a, then $e = f$ and $u = v$.

The following corollary states the relationship between nil-clean ring and Boolean ring.

Corollary 2.2 ([21])**.** Let R be a commutative ring. Then, R is nil-clean if and only if $R/J(R)$ is Boolean and $J(R)$ is nil.

Kosan et al. ([31]) proved that every strongly nil-clean element of a ring is a strongly π-regular element. Also, a ring R is strongly nil-clean if and only if $R/J(R)$ is Boolean and $J(R)$ is nil.

Proposition 2.6 ([31])**.** An element $a \in R$ is strongly nil-clean if and only if a is strongly clean in R and $a - a^2$ is a nilpotent element in R.

Lemma 2.2 ([31])**.** Let a be a strongly nil-clean element of R. Then,

(1) a has a unique strongly nil-clean decomposition in R.

(2) a is a strongly π-regular element of R.

(3) a is a uniquely strongly clean element of R.

2.2.2 Weakly Nil-clean Rings

Let us begin by recalling the following definitions, examples and theorems from S. Breaz et al. ([4]).

Definition 2.19. A ring is said to be weakly nil-clean if every element can be written as either a sum or a difference of a nilpotent and an idempotent.

Example 2.12. The ring $\mathbb{Z}_{12}$ is weakly nil-clean ring but $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not weakly nil-clean ring since $(1, 2)$ can not be written as a sum or difference of a nilpotent and an idempotent element in $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Proposition 2.7. The homomorphic image of a weakly nil-clean ring is weakly nil-clean.

Theorem 2.1. Let R be a ring. The following are equivalent:

(1) R is weakly nil-clean.

(2) 6 is nilpotent and $R/6R$ is weakly nil-clean.

(3) $R/J(R)$ is weakly nil-clean and $J(R)$ is nil.

Corollary 2.3. A ring R satisfies the nil-involution property if and only if R is weakly nil-clean with $2 \in U(R)$.

Corollary 2.4. Every weakly nil-clean ring is clean ring.

Example 2.13. Let $n \geq 2$ be an integer. Then, the ring $\mathbb{Z}_n$ is weakly nil-clean if and only if $n = 2^\ell 3^k$, where $\ell, k \geq 0$ are integers.

Example 2.14. For any positive integer $n \geq 2$, $\mathbb{T}_n(R)$ is weakly nil-clean matrix ring if and only if R is a weakly nil-clean ring.

2.2.3 Commutative Weakly Nil-clean Rings

Danchev and McGovern ([18]) proved that any commutative weakly nil-clean ring is zero dimensional and also a clean ring. It also proved that a commutative reduced indecomposable ring is weakly nil-clean ring if and only if it is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. It has been proved that the class of weakly nil-clean rings is not closed under finite direct products. It also states the condition for which weakly nil-clean rings become nil-clean rings. The following theorems, propositions and corollaries are taken from Danchev and McGovern ([18]).

Theorem 2.2. Let R be a commutative reduced ring. The following statements are equivalent.

(i) $R = Id(R) \cup [-Id(R)]$.

(ii) R is either Boolean ring B, or $\mathbb{Z}_3$, or $B \times \mathbb{Z}_3$.

(iii) For all $x \in R$, either $x^2 = x$ or $x^2 = -x$.

(iv) R is weakly nil-clean.

Proposition 2.8. If R is a reduced weakly nil-clean ring, then $U(R)$ is a group of at most two elements.

Corollary 2.5. Let R be a commutative ring. If R is weakly nil-clean, then $U(R) = Nil(R) \pm 1$.

We can now characterize weakly nil-clean rings in general.

Theorem 2.3. Let R be a commutative ring. The following statements are equivalent.

(i) R is a weakly nil-clean ring.

(ii) R is zero dimensional and there is at most one maximal ideal of R, say M, such that $R/M \cong \mathbb{Z}_3$ while for all other maximal ideals N of R we have $R/N \cong \mathbb{Z}_2$.

(iii) $R/Nil(R)$ is isomorphic to either a Boolean ring, or $\mathbb{Z}_3$, or the product of two such rings.

(iv) $J(R)$ is nil and $R/J(R)$ is isomorphic to either a Boolean ring, or $\mathbb{Z}_3$, or the product of two such rings.

Corollary 2.6. A ring R is weakly nil-clean if and only if $R/Nil(R)$ is weakly nil-clean if and only if $R/J(R)$ is weakly nil-clean and $J(R)$ is nil.

2.2.4 Strongly Weakly Nil-clean Rings

The following definitions and theorems are found in ([8]).

Definition 2.20. A ring R is called strongly weakly nil-clean if every element in R is the sum or difference of a nilpotent and an idempotent that commute.

Example 2.15. The ring $T_2(\mathbb{Z}_2)$ is strongly weakly nil-clean ring.

The following theorem characterizes weakly nil-clean rings elementwise.

Theorem 2.4. Let R be a ring. Then, the following are equivalent:

(i) R is strongly weakly nil-clean.

(ii) For any $a \in R$, $a \pm a^2$ is nilpotent.

(iii) For any $a \in R$, there exists an idempotent $e \in \mathbb{Z}[a]$ such that $a \pm e$ is nilpotent.

Lemma 2.3 ([25])**.** Let R be a ring, and let $k \geq 2$ be a positive integer. Then, the following statements are equivalent:

(1) $Nil(R)$ forms an ideal whenever $a - a^k \in Nil(R)$ for all $a \in R$.

(2) $k \not\equiv 1(mod\ 3)$ and $k \not\equiv 1(mod\ 8)$.

Theorem 2.5. Let R be a ring. Then, the following statements are equivalent:

(1) R is strongly weakly nil-clean.

(2) R is weakly nil-clean and $Nil(R)$ forms an ideal of R.

(3) R is weakly nil-clean and $R/J(R)$ is commutative.

Definition 2.21 ([11])**.** A ring R is said to be periodic if, for any $a \in R$, there exist distinct $m, k \in \mathbb{N}$ such that $a^m = a^k$.

Example 2.16. Nil rings and direct sums of matrix rings over finite fields are periodic rings.

The following proposition is about a periodic ring and it can be found in ([11]).

Proposition 2.9. Let I be a nil ideal of a ring R. Then, R is periodic if and only if R/I is periodic. In particular, R is periodic if and only if $J(R)$ is nil and $R/J(R)$ is periodic.

Proposition 2.10 ([25])**.** Let R be a ring. For every $x \in R$, $x - x^2 \in Nil(R)$ if and only if $Nil(R)$ is an ideal and $R/Nil(R)$ is a Boolean ring.

Proposition 2.11 ([7])**.** If each element a of R is an algebraic co-integer, then R is a periodic ring.

Let P, N and N^* denote respectively the sets of potent elements, nilpotent elements, and elements whose square is zero, of a ring R respectively. The Proof of the following theorem found in ([25]).

Theorem 2.6 ([25])**.** Let R be a ring. The following statements are equivalent:

(i) R is $(P - N)$ representable, that is, for each $x \in R$, $x = p + n$, where $p \in P$ and $n \in N$.

(ii) R is $[P - N]$ representable, that is, $[p, n] = pn - np = 0$ for any $p \in P$ and $n \in N$, and $E - N^*$ orthogonal, that is, $en^* = 0 = n^*e$ for any $e \in E$ and $n^* \in N^*$.

(iii) $R = P \oplus N$.

2.2.5 Clean Rings

Definition 2.22 ([14])**.** An element $x \in R$ is said to be clean if $x = u + e$ for some $u \in U(R)$ and $e \in Id(R)$,and the ring R is called clean if each element of R is clean.

Example 2.17. Division rings, local rings and semiperfect rings are clean rings.

Definition 2.23 ([37])**.** An element $a \in R$ is said to be strongly clean if $a = u + e$ with $u \in U(R), e \in Id(R)$ and $ue = eu$, and the ring R is strongly clean if every element of R is strongly clean.

Example 2.18. Strongly π-regular rings, all commutative clean rings and local rings are strongly clean rings.

Definition 2.24 ([14])**.** A ring R is said to be weakly clean if, for every $r \in R$, there are a unit u and an idempotent e in R such that $r = u + e$ or $r = u - e$.

Example 2.19. $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$ weakly clean ring.

Proposition 2.12 ([12])**.** Suppose that R is a ring with $2 \in J(R)$. Then, R is weakly clean if and only if R is clean.

2.2.6 Weakly UU Rings

Definition 2.25 ([36])**.** A ring R is said to be exchange if, for each $a \in R$, there is an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$.

Example 2.20.

(1) Let $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. Then R is exchange ring.

(2) Clean rings are exchange rings.

Definition 2.26 ([6])**.** A ring is called unit unipotent (UU) if all units are unipotents, that is, $U(R) = Nil(R) + 1$, that is, each unit can be presented as $q+1$, where $q \in Nil(R)$. The set of unipotents of a ring R is denoted by $Uni(R)$.

Example 2.21. The rings $\mathbb{Z}_{2^n}$ and $T_n(\mathbb{Z}_2)$ are UU rings.

Definition 2.27 ([13])**.** A ring R is called weakly unit unipotent (WUU) if $U(R) = Nil(R) \pm 1$, that is, every unit can be presented as either $n+1$ or $n-1$, where $n \in Nil(R)$. The set of unipotents of a ring R of the form $n-1$ for some nilpotent element n is denoted by $-Uni(R)$.

Example 2.22. The rings $\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $T_n(\mathbb{Z}_2)$ are WUU rings.

2.2.7 Weakly Clean WUU Rings

Definition 2.28 ([14])**.** A ring R is called clean WUU if every element of R is clean and every unit of R weak unipotent, that is, every unit u is either $u = n+1$ or $u = n-1$ for some nilpotent n.

Example 2.23. The rings $\mathbb{Z}_{3^n}$ are clean WUU rings.

Definition 2.29 ([14])**.** A ring R is said to be weakly clean WUU if every element of R is weakly clean and every unit of R is weak unipotent.

Example 2.24. The rings $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $T_n(\mathbb{Z}_2)$ are weakly clean WUU rings because clean rings are weakly clean rings.

Definition 2.30 ([14])**.** A ring R is said to be exchange WUU if every element of R is exchange and every unit of R is weak unipotent.

Example 2.25. The rings $\mathbb{Z}_2$ and $\mathbb{Z}_{3^n}$ are exchange WUU rings since clean rings are exchange rings.

Theorem 2.7 ([14])**.** Let R be a ring. Then, the following three conditions are equivalent.

(i) R is a weakly clean WUU.

(ii) R is a clean WUU.

(iii) R is with nil $J(R)$ and either $R/J(R) \cong B$, or $R/J(R) \cong \mathbb{Z}_3$, or $R/J(R) \cong B \times \mathbb{Z}_3$, where B is a Boolean ring.

2.2.8 UNI Rings

Definition 2.31 ([8])**.** An element r in a ring R is said to be an involution element if $r^2 = 1$, and the set of all involution elements of R is denoted by $Inv(R)$.

Example 2.26. In $\mathbb{Z}_8$, $Inv(\mathbb{Z}_8) = \{1, 3, 5, 7\}$.

Definition 2.32 ([15])**.** A ring R is called UNI if for each $u \in U(R)$, there are $n \in Nil(R)$ and $i \in Inv(R) \cap C(R)$ such that $u = n + i$.

Example 2.27. The ring $T_n(\mathbb{Z}_2)$ is UNI ring.

Definition 2.33 ([16])**.** The unit group $U(R)$ of a ring R is strongly invo-fine if for every $u \in U(R)$, there are $v \in Inv(R)$ and $q \in Nil(R)$ such that $u = v + q$ with $vq = qv$.

Example 2.28. The unit group of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are strongly invo-fine.

Lemma 2.4 ([16])**.** In a weakly clean UNI ring, 6 or 30 is nilpotent.

Lemma 2.5 ([15])**.** Let R be a UNI ring. Then, the following statements are true:

(1) $2 \in U(R) \iff 3 \in Nil(R)$;

(2) $3 \in U(R) \iff 2 \in Nil(R)$.

Lemma 2.6 ([15])**.** A ring R is UNI for which $6 \in J(R)$ if and only if $R \cong R_1 \times R_2$, where R_1 is a UU ring and R_2 is either $\{0\}$ or a UNI ring with $3 \in J(R_2)$.

Theorem 2.8 ([15])**.** A ring R is clean UNI if and only if R can be decomposed as the direct product of a ring which is Boolean modulo its nil Jacobson radical and of a UNI clean ring whose nil Jacobson radical contains 3.
In addition, if the latter ring modulo its Jacobson radical is commutative, then this factor ring can be embedded in the direct product of the fields $\mathbb{Z}_3$.
In particular, a commutative ring R is UNI clean with $3 \in J(R)$ if and only if $J(R)$ is nil and $R/J(R) \subseteq \prod_\mu \mathbb{Z}_3$, where μ is an ordinal.

Corollary 2.7 ([15])**.** A commutative ring R is clean UNI if and only if $R \cong R_1 \times R_2$, where $J(R_1)$ is nil with $R_1/J(R_1) \subseteq \prod_\lambda \mathbb{Z}_2$ for some ordinal λ and $J(R_2)$ is nil with $R_2/J(R_2) \subseteq \prod_\mu \mathbb{Z}_3$ for some ordinal μ.

Theorem 2.9 ([16])**.** Let R be a commutative ring. Then, the following statements are equivalent.

(i) R is a weakly clean UNI ring.

(ii) R is a clean UNI ring.

(iii) $J(R)$ is nil with $R \cong L \times P$, where $L/J(L) \subseteq \prod_\lambda \mathbb{Z}_2$ and $P/J(P) \subseteq \prod_\mu \mathbb{Z}_3$ for some ordinals λ and μ.

Definition 2.34 ([16])**.** Let R be a ring. We say that $U(R)$ is invo-fine if the equality $U(R) = Inv(R) + Nil(R)$ holds.

Example 2.29. Consider the ring $T_n(\mathbb{Z}_3)$. Then $U(T_n(\mathbb{Z}_3))$ is invo-fine because $U(T_n(\mathbb{Z}_3)) = Inv(T_n(\mathbb{Z}_3)) + Nil(T_n(\mathbb{Z}_3))$.

Theorem 2.10 ([16])**.** A ring R is an exchange ring with strongly invo-fine $U(R)$ if and only if $J(R)$ is nil and $R \cong R_1 \times R_2$, where $R_1/J(R_1) \subseteq \prod_\lambda \mathbb{Z}_2$ and $R_2/J(R_2) \subseteq \prod_\mu \mathbb{Z}_3$ for some ordinals λ and μ.

2.2.9 Commutative Weakly Nil-neat Rings

Definition 2.35 ([20])**.** A ring R is called weakly nil-neat if each of its proper homomorphic image is weakly nil-clean.

Example 2.30. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ are weakly nil-neat rings.

Corollary 2.8 ([20])**.** Let R be a ring such that $J(R) \neq 0$ and let R be not a domain. Consider the following statements:

(1) R is a weakly nil-clean ring;

(2) R is a weakly nil-neat ring;

(3) R is a clean WUU ring.
Then, (1) $\implies$ (2) $\implies$ (3). Moreover, if $2 \in Nil(R)$, then the above three statements are equivalent and also equivalent to:

(4) R is a clean UU ring;

(5) R is a nil-clean ring;

(6) R is a uniquely nil-clean ring;

(7) R is a uniquely clean ring such that every prime ideal of R is maximal;

(8) $J(R)$ is a nil ideal, and $R/J(R)$ is a Boolean ring;

(9) R is an exchange UU ring; and

(10) R is a nil-neat ring.

2.2.10 Group Rings

Definition 2.36 ([35])**.** Let p be a prime number. A group G is said to be

(1) torsion if every element has finite order.

(2) locally finite if every finitely generated subgroup of G is finite.

(3) p-group if the order of each element of G is a power of p. We usually denoted p-group by G_p.

(4) p-torsion free if $G_p \subseteq G$ and G_p is trivial, that is, $G_p = \{1_G\}$. If G is abelian group, then G_p is subgroup of G.

Example 2.31.

(1) Finite groups are torsion groups.

(2) The cyclic group C_4 and the Klein four group V_4 are both 2-groups of order 4.

(3) The ring $\mathbb{Z}$ is a torsion-free abelian group.

Definition 2.37 ([10])**.** Let R be a ring, and let G be a multiplicative group. Then, the group ring of G with coefficients in R, denoted by $R[G]$ is the set of formal sums:

$$R[G] = \{\sum_{g\in G} r_g.g : r_g \in R\}$$

where the sum $\sum_{g\in G} r_g.g$ is finite, that is, has only finitely many nonzero coefficients r_g with addition and multiplication defined as follows:

$$(\sum a_g.g) + (\sum b_g.g) = (a_g + b_g).g$$

$$(\sum_{x\in G} a_x.x)(\sum_{y\in G} b_y.y) = \sum_{x,y\in G} (a_x b_y).(xy) = \sum_{z\in G} c_z.z$$

where $c_z = \sum_{xy=z} a_x b_y = \sum_{x\in G} a_x b_{x^{-1}z}$

Example 2.32. $\mathbb{F}_3C_3 = \{a.1+b.x+c.x^2 \mid a,b,c \in \mathbb{Z}_3\}$ and $\mathbb{F}_3C_2 = \{a.1+b.x \mid a,b \in \mathbb{F}_3\}$ are group rings.

Definition 2.38 ([35])**.**

(1) For $\alpha \in R[G]$ with $\alpha = \sum_{g\in G} r_g g$, the set $supp(\alpha) = \{g \in G : r_g \neq 0\} = \{1 - g : g \in G\}$ is called support of α. $supp(\alpha)$ is finite set.

(2) The map $\Psi : R[G] \to R$ defined by $\Psi(\sum_{g\in G} r_g g) = \sum_{g\in G} r_g$ is a surjective ring homomorphism called the trace map. The kernel $ker(\Psi)$ is called the augmentation ideal of the group ring $R[G]$ and is denoted by $\omega(R[G])$.

Remark 2.3 ([35])**.** $\omega(R[G])$ is an ideal of $R[G]$ generated by the set $\{1-g : g \in G\}$.

Lemma 2.7 ([35])**.** Let R be a commutative ring and G be an abelian group. Suppose $2 \in Nil(R)$ and $g \in G_2$. Then, the element $1 - g \in R[G]$ is nilpotent.

Corollary 2.9 ([23])**.** Let p be a prime number. If D is an integral domain of characteristic $p \neq 0$, then $D[G]$ is reduced if and only if G is p-torsion free.

Let $H \leq G$. Then, $\Psi_H : R[G] \to R[G/H]$ given by

$$\Psi_H(\sum_{g\in G} r_g g) = \sum_{g\in G} r_g(g + H)$$

maps the group ring $R[G]$ naturally onto $R[G/H]$. In other words, $R[G/H]$ is homomorphic image of $R[G]$, Glimer ([23]).

Proposition 2.13 ([34])**.** If $R[G]$ is a clean ring, then R is clean ring and G is a torsion group.

Corollary 2.10 ([35])**.** Let R be a ring. R is nil-clean if and only if $R/Nil(R)$ is Boolean. In particular, if R is nil-clean, then 2 is nilpotent.

Corollary 2.11 ([38])**.** The group ring $R[G]$ is reduced if and only if the following two conditions hold:

(i) R is reduced.

(ii) For all primes p for which there is an element $g \in G$ with order p, p is not a zero-divisor of R.

Theorem 2.11 ([18])**.** Let R be a ring and G be a group. The group ring $R[G]$ is weakly nil-clean if and only if exactly one of the following three conditions is satisfied:

(1) R is nil-clean and G is a non-trivial torsion 2-group;

(2) $R/Nil(R) \cong \mathbb{Z}_3$ and G is a non-trivial torsion 3-group;

(3) R is weakly nil-clean and G is trivial.

Theorem 2.12 ([23])**.** Let R be a ring and G be a group. The group ring $R[G]$ is quasi-local (that is, has a unique maximal ideal) if and only if R is quasi-local with maximal ideal M, $char(R/M) = p \neq 0$, where p is prime number, and G is a p-group.

Theorem 2.13 ([23])**.** Assume that R is a ring of prime power characteristic p^m and G is a group. The nil radical of $R[G]$ is the ideal $N(R)[G] + I$, where $N(R)$ is the nilradical of R and I is the kernel ideal of the congruence $\tilde{p}$.

Corollary 2.12 ([9])**.**

(1) G torsion: $R[G]$ is semisimple, that is, $J(R[G]) = 0$ if and only if R is semisimple and any prime that is the order of an element of G is not a zero divisor of R.

(2) G torsion free: $R[G]$ is semisimple if and only if R is reduced (i.e. R has trivial nil-radical) and any prime that is the order of an element of G is not a zero divisor of R.

Proposition 2.14 ([10])**.** Let R be a commutative ring, $I \lhd R$ and let G be a group. Then $I[G] = \{\sum_{g \in G} a_g g \mid a_g \in I\} \lhd R[G]$ and also $R[G]/I[G] \cong (R/I)[G]$.

Chapter 3

Weak Idempotent Nil-clean Rings

In our discussions, by a ring R we mean an associative ring with unity. We introduce the notion of weak idempotent nil-clean rings (here after we call it WIN-clean rings) and furnish certain examples. Further, we obtain some basic results concerning WIN-clean rings and we also characterize the WIN-clean rings (in Proposition 3.4). The main result of this chapter is that, every WIN-clean ring R is isomorphic to a direct product of WIN-clean rings R_1 and R_2, with the special property that $2 \in J(R_1)$ and $3 \in J(R_2)$.

3.1 Definition and Examples of WIN-clean rings

In this section, we give definition and examples of WIN-clean rings and we also describe properties of weak idempotent elements of a ring. Moreover, we characterize reduced WIN-clean rings.
We will start our discussion by defining weak idempotent elements and WIN-clean rings.

Definition 3.1. Let R be a ring. An element w in R is said to be weak idempotent if $w^2 = w^4$. An element $a \in R$ is called WIN-clean if $a = n + w$ for some nilpotent element n of R and some weak idempotent element w of R. A ring R is said to be WIN-clean if every element of R is WIN-clean.

In any ring R, nilpotents and weak idempotents are WIN-clean elements. Because if n is a nilpotent element of R, then we can write $n = 0 + n$. Again, if $w \in wi(R)$, then $2w - w^3 \in wi(R)$, $w^3 - w \in Nil(R)$ and $w = (2w - w^3) + (w^3 - w)$. Also, w can be expressed as $w = 0 + w$.
From the above discussion, we observe that weak idempotents are not uniquely expressed as a sum of a nilpotent and a weak idempotent element.

Example 3.1. Let $R = M_2(\mathbb{Z}_3)$. Then, there are 81 elements in R.

$$Nil(R) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \right\}$$

$$wi(R) = \Bigg\{ \begin{pmatrix}0&0\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}2&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&0\\0&2\end{pmatrix}, \begin{pmatrix}1&1\\0&0\end{pmatrix}, \begin{pmatrix}1&2\\0&0\end{pmatrix}, \begin{pmatrix}2&1\\0&0\end{pmatrix},$$
$$\begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&2\\0&0\end{pmatrix} \begin{pmatrix}2&2\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\0&2\end{pmatrix}, \begin{pmatrix}0&2\\0&1\end{pmatrix}, \begin{pmatrix}0&2\\0&2\end{pmatrix}, \begin{pmatrix}1&0\\1&0\end{pmatrix},$$
$$\begin{pmatrix}1&0\\2&0\end{pmatrix}, \begin{pmatrix}2&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\2&0\end{pmatrix} \begin{pmatrix}2&0\\2&0\end{pmatrix}, \begin{pmatrix}1&1\\1&1\end{pmatrix}, \begin{pmatrix}2&2\\2&2\end{pmatrix}, \begin{pmatrix}0&0\\1&1\end{pmatrix},$$
$$\begin{pmatrix}0&0\\2&1\end{pmatrix}, \begin{pmatrix}0&0\\1&2\end{pmatrix}, \begin{pmatrix}0&0\\2&2\end{pmatrix}, \begin{pmatrix}1&2\\2&1\end{pmatrix}, \begin{pmatrix}2&1\\1&2\end{pmatrix}, \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&2\end{pmatrix}, \begin{pmatrix}2&0\\0&1\end{pmatrix},$$
$$\begin{pmatrix}2&0\\0&2\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&2\\2&0\end{pmatrix}, \begin{pmatrix}2&0\\1&1\end{pmatrix}, \begin{pmatrix}1&0\\1&2\end{pmatrix}, \begin{pmatrix}2&1\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\0&2\end{pmatrix}, \begin{pmatrix}2&0\\2&1\end{pmatrix},$$
$$\begin{pmatrix}1&2\\0&2\end{pmatrix}, \begin{pmatrix}1&0\\2&2\end{pmatrix}, \begin{pmatrix}2&2\\0&1\end{pmatrix} \Bigg\}.$$

Next, we need to show that the remaining elements of R can be written as a sum of a nilpotent and a weak idempotent element in R.

$$\begin{pmatrix}1&2\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&2\\0&0\end{pmatrix}; \quad \begin{pmatrix}1&1\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&1\\0&0\end{pmatrix}; \begin{pmatrix}0&1\\2&0\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix} + \begin{pmatrix}0&0\\1&0\end{pmatrix}$$
$$\begin{pmatrix}1&0\\1&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&0\\1&0\end{pmatrix}; \quad \begin{pmatrix}2&2\\0&2\end{pmatrix} = \begin{pmatrix}2&0\\0&2\end{pmatrix} + \begin{pmatrix}0&2\\0&0\end{pmatrix}; \begin{pmatrix}2&0\\2&2\end{pmatrix} = \begin{pmatrix}2&0\\0&2\end{pmatrix} + \begin{pmatrix}0&0\\2&0\end{pmatrix}$$
$$\begin{pmatrix}0&2\\2&2\end{pmatrix} = \begin{pmatrix}0&2\\2&0\end{pmatrix} + \begin{pmatrix}0&2\\0&0\end{pmatrix}; \quad \begin{pmatrix}1&1\\1&2\end{pmatrix} = \begin{pmatrix}1&1\\2&2\end{pmatrix} + \begin{pmatrix}0&0\\2&0\end{pmatrix}; \begin{pmatrix}0&2\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix} + \begin{pmatrix}0&1\\0&0\end{pmatrix}$$
$$\begin{pmatrix}2&1\\1&1\end{pmatrix} = \begin{pmatrix}2&2\\1&1\end{pmatrix} + \begin{pmatrix}0&2\\0&0\end{pmatrix}; \quad \begin{pmatrix}1&1\\1&0\end{pmatrix} = \begin{pmatrix}1&1\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\1&0\end{pmatrix}; \begin{pmatrix}2&2\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&0\\1&0\end{pmatrix}$$
$$\begin{pmatrix}1&2\\1&1\end{pmatrix} = \begin{pmatrix}2&2\\1&1\end{pmatrix} + \begin{pmatrix}2&0\\0&0\end{pmatrix}; \quad \begin{pmatrix}1&1\\2&1\end{pmatrix} = \begin{pmatrix}1&1\\2&2\end{pmatrix} + \begin{pmatrix}0&0\\0&2\end{pmatrix} \begin{pmatrix}1&2\\2&2\end{pmatrix} = \begin{pmatrix}1&1\\2&2\end{pmatrix} + \begin{pmatrix}0&1\\0&0\end{pmatrix}$$
$$\begin{pmatrix}2&1\\2&2\end{pmatrix} = \begin{pmatrix}1&1\\2&2\end{pmatrix} + \begin{pmatrix}1&0\\0&0\end{pmatrix}; \quad \begin{pmatrix}2&2\\1&2\end{pmatrix} = \begin{pmatrix}2&2\\1&1\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix}; \begin{pmatrix}2&2\\2&0\end{pmatrix} = \begin{pmatrix}2&0\\2&0\end{pmatrix} + \begin{pmatrix}0&2\\0&0\end{pmatrix}$$
$$\begin{pmatrix}2&1\\0&2\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix} + \begin{pmatrix}2&0\\0&2\end{pmatrix}; \quad \begin{pmatrix}1&0\\2&1\end{pmatrix} = \begin{pmatrix}0&0\\2&0\end{pmatrix} + \begin{pmatrix}1&0\\0&1\end{pmatrix}; \begin{pmatrix}2&0\\1&2\end{pmatrix} = \begin{pmatrix}0&0\\1&0\end{pmatrix} + \begin{pmatrix}2&0\\0&2\end{pmatrix}$$
$$\begin{pmatrix}2&0\\2&2\end{pmatrix} = \begin{pmatrix}0&0\\2&0\end{pmatrix} + \begin{pmatrix}2&0\\0&2\end{pmatrix}; \quad \begin{pmatrix}0&1\\1&1\end{pmatrix} = \begin{pmatrix}0&1\\0&1\end{pmatrix} + \begin{pmatrix}0&0\\1&0\end{pmatrix} \begin{pmatrix}0&1\\2&1\end{pmatrix} = \begin{pmatrix}0&1\\0&1\end{pmatrix} + \begin{pmatrix}0&0\\2&0\end{pmatrix}$$
$$\begin{pmatrix}0&1\\1&2\end{pmatrix} = \begin{pmatrix}0&1\\0&2\end{pmatrix} + \begin{pmatrix}0&0\\1&0\end{pmatrix}; \quad \begin{pmatrix}0&2\\1&1\end{pmatrix} = \begin{pmatrix}0&2\\0&1\end{pmatrix} + \begin{pmatrix}0&0\\1&0\end{pmatrix}; \begin{pmatrix}0&2\\1&2\end{pmatrix} = \begin{pmatrix}0&0\\1&2\end{pmatrix} + \begin{pmatrix}0&2\\0&0\end{pmatrix}$$
$$\begin{pmatrix}0&2\\2&1\end{pmatrix} = \begin{pmatrix}0&2\\0&1\end{pmatrix} + \begin{pmatrix}0&0\\2&0\end{pmatrix}; \quad \begin{pmatrix}0&1\\2&2\end{pmatrix} = \begin{pmatrix}0&1\\0&2\end{pmatrix} + \begin{pmatrix}0&0\\2&0\end{pmatrix}; \begin{pmatrix}1&2\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\1&0\end{pmatrix} + \begin{pmatrix}0&2\\0&0\end{pmatrix}$$
$$\begin{pmatrix}2&1\\1&0\end{pmatrix} = \begin{pmatrix}2&0\\1&0\end{pmatrix} + \begin{pmatrix}0&1\\0&0\end{pmatrix}; \quad \begin{pmatrix}1&1\\2&0\end{pmatrix} = \begin{pmatrix}1&0\\2&0\end{pmatrix} + \begin{pmatrix}0&1\\0&0\end{pmatrix}; \begin{pmatrix}2&2\\1&0\end{pmatrix} = \begin{pmatrix}2&0\\1&0\end{pmatrix} + \begin{pmatrix}0&2\\0&0\end{pmatrix}$$
$$\begin{pmatrix}1&2\\2&0\end{pmatrix} = \begin{pmatrix}1&0\\2&0\end{pmatrix} + \begin{pmatrix}0&2\\0&0\end{pmatrix}; \quad \begin{pmatrix}2&1\\2&0\end{pmatrix} = \begin{pmatrix}2&0\\2&0\end{pmatrix} + \begin{pmatrix}0&1\\0&0\end{pmatrix}; \begin{pmatrix}1&2\\1&2\end{pmatrix} = \begin{pmatrix}1&0\\1&2\end{pmatrix} + \begin{pmatrix}0&2\\0&0\end{pmatrix}$$
$$\begin{pmatrix}2&1\\2&1\end{pmatrix} = \begin{pmatrix}0&0\\2&0\end{pmatrix} + \begin{pmatrix}2&1\\0&1\end{pmatrix}; \quad \begin{pmatrix}2&2\\2&1\end{pmatrix} = \begin{pmatrix}2&0\\2&1\end{pmatrix} + \begin{pmatrix}0&2\\0&0\end{pmatrix}$$

Hence, $M_2(\mathbb{Z}_3)$ is WIN-clean ring.

Example 3.2. Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. Then, $wi(R) = R$ since $wi(\mathbb{Z}_3) = \mathbb{Z}_3$. So, R is WIN-clean ring.

The following lemma states that any positive power of a weak idempotent element, w, is either w itself or w^2 or w^3, and it helps us to prove Theorem 3.1.

Lemma 3.1. Let R be a ring and w be a weak idempotent element of R. Then,

1. for $n \in \mathbb{N}$, $w^{2n} = w^2$ and $w^{2n+1} = w^3$;

2. $Id(R) \cup [-Id(R)] \subseteq wi(R)$.

Proof.

1. We proceed the proof by induction on n. Let $n = 1$. Then, $w^{2(1)} = w^2$ and $w^{2(1)+1} = w^3$. So, it is true for $n = 1$. Assume that it is true for $n > 1$ and $n = k - 1$, i.e., $w^{2(k-1)} = w^2$ and $w^{2(k-1)+1} = w^3$.
Now $w^{2k} = w^{2k-2}w^2 = w^{2(k-1)}w^2 = w^2w^2$ by induction assumption and hence $w^{2k} = w^4 = w^2$ since w is weak idempotent element. Again, $w^{2k+1} = w^{(2k-2)+1}w^2 = w^3w^2$ by induction assumption. Thus, $w^{2k+1} = w^5 = w^4w = w^2w = w^3$. Hence, it is true for $n = k$.

2. Since idempotents are weak idempotent elements, we have $Id(R) \subseteq wi(R)$. Let $-e \in -Id(R)$. Then $(-e)^2 = e^2 = e = (-e)^4$ implies that $-e \in wi(R)$. Thus, $-Id(R) \subseteq wi(R)$. Hence, $Id(R) \cup [-Id(R)] \subseteq wi(R)$.

□

Using Lemma 3.1 (2), we can easily verify that every weakly nil-clean ring is a WIN-clean ring, but the converse is not true. For instance, $\mathbb{Z}_3 \times \mathbb{Z}_3$ is WIN-clean ring, but it is not weakly nil-clean ring. This is because $(2, 1)$ cannot be expressed as a sum or a difference of a nilpotent and an idempotent element in $\mathbb{Z}_3 \times \mathbb{Z}_3$.
Now we see the properties of weak idempotent elements in a ring.

Theorem 3.1. Let R be a ring and $w \in R$ be a weak idempotent element. Then,

1. for any $k \in \mathbb{N}$, w^k is a weak idempotent element;

2. w^2 and $1 - w^2$ are idempotent elements;

3. $2w^2 - 1$ and $w - 1 + w^2$ are units;

4. for every $k \in \mathbb{N}$, $w^k - w^{k+2}$ is nilpotent;

5. $(1 - w^2)w^2 = 0$; and

6. w is clean.

Proof.

1. Let $k \in \mathbb{N}$. Then,

$$(w^k)^4 = w^{4k} = (w^4)^k = (w^2)^k = (w^k)^2.$$

Thus, w^k is a weak idempotent element.

2. $w^2 = w^4 = (w^2)^2$ and $(1-w^2)^2 = 1-2w^2+w^4 = 1-2w^2+w^2 = 1-w^2$. Hence, w^2 and $1-w^2$ are idempotent elements.

3. $(2w^2-1)^2 = 4w^4-4w^2+1 = 4w^2-4w^2+1 = 1$, and
   $[(-1-w+w^2)(-1+w+w^2)] = 1-2w^2$

$$\begin{aligned}[(-1-w+w^2)(-1+w+w^2)]^2 =&(1-2w^2)^2\\ =&(-1+w+w^2)(-1-w+w^2+3w^3)\\ =&[(-1-w+w^2)(-1+w+w^2)]^2\\ =&(1-2w^2)^2 = 1\end{aligned}$$

4. Let $k \in \mathbb{N}$. For $k=1$, we have $(w-w^3)^2 = w^2-2w^4+w^6 = 0$. Hence, it is true for $k=1$.
   Assume that it is for $k>1$, that is, $w^k - w^{k+2}$ is nilpotent. Then, we want to show that it is true for $k+1$, that is, $w^{k+1}-w^{k+3}$ is nilpotent.
   Now, $w^{k+1}-w^{k+3} = w(w^k-w^{k+2}) = wn$, where $n = w^k-w^{k+2}$ is nilpotent by induction assumption. So, wn is also nilpotent. Hence, $w^{k+1}-w^{k+3}$ is nilpotent.

5. $(1-w^2)w^2 = w^2-w^4 = w^2-w^2 = 0$.

6. $w = (w-1+w^2)+(1-w^2)$, where $w-1+w^2 \in U(R)$ and $1-w^2 \in Id(R)$. Hence, w is clean.

□

Remark 3.1. For any ring R, a weak idempotent element w in R can be

(1) an idempotent element;

(2) a unit with its own inverse, that is, $w^2 = 1$;

(3) a nilpotent with index 2, that is, $w^2 = 0$;

(4) $w^2 = e$, where e is non trivial idempotent.

In (1), (3) and (4), w is a zero divisor in R.

Example 3.3. Consider the ring $\mathbb{Z}_{12}$. Then $Id(\mathbb{Z}_{12}) = \{0,1,4,9\}$, $wi(\mathbb{Z}_{12}) = \mathbb{Z}_{12}$ and the zero divisors of $\mathbb{Z}_{12}$ are 2, 3, 4, 6, 8, 9 and 10. Then $2^2=4$, $3^2=9$, $6^2=0$, $8^2=4$, $10^2=4$ and $5^2=7^2=11^2=1$.

Recall that a ring is said to be abelian if all its idempotents are central. The following theorem shows the way we form idempotents, weak idempotents and units using a combination of idempotent and weak idempotent elements.

Theorem 3.2. Let R be an abelian ring, $w \in wi(R)$ and $e \in Id(R)$. Then, each of the following hold true.

1. ew^2 is an idempotent element.

2. $1-e-w^2$ is a weak idempotent element.

3. $1 - 2ew^2$ is a unit element.

4. $e - ew^2$ and $1 - e + ew^2$ are idempotent elements.

Proof.

1. $(ew^2)^2 = e^2w^4 = ew^2$. Hence, ew^2 is an idempotent element.

2. Consider $1 - e - w^2$.

$$\begin{aligned}(1-e-w^2)^2 =& [(1-e)-w^2]^2\\ =&(1-e)^2 - 2(1-e)w^2 + w^4\\ =&1 - 2e + e - 2w^2 + 2ew^2 + w^2\\ =&1 - e - w^2 + 2ew^2\\ (1-e-w^2)^4 =&(1-e-w^2+2ew^2)^2\\ =&(1-e-w^2)^2 - 2(1-e-w^2)2ew^2 + (2ew^2)^2\\ =&1 - e - ew^2 - 4ew^2 + 4ew^2 + 4ew^2 + 4ew^4\\ =&1 - e - w^2 + 2ew^2\end{aligned}$$

This implies $(1 - e - w^2)^2 = (1 - e - w^2)^4$ and hence $1 - e - w^2$ is weak idempotent element.

3. Since $(1 - 2ew^2)^2 = 1$, we have that $1 - 2ew^2$ is unit.

4. $(e - ew^2)^2 = e - 2ew^2 + ew^2 = e - ew^2$ and

$$(1 - e + ew^2)^2 = (1 - e)^2 + 2(1 - e)ew^2 + e^2w^4 = 1 - e + ew^2.$$

Hence, both $e - ew^2$ and $1 - e + ew^2$ are idempotent elements.

□

Proposition 3.1. All elements of a reduced WIN-clean ring are weak idempotent elements.

Proof. Let R be a reduced WIN-clean ring. Then, every element of R is a sum of a nilpotent and a weak idempotent element, and $Nil(R) = \{0\}$. So, for any $x \in R$, we have $x = 0 + x$. Hence, x is weak idempotent element. □

Remark 3.2. A subring of a WIN-clean ring may not be WIN-clean ring. For instance, consider the WIN-clean ring $M_2(\mathbb{Z}_3)$. Let
$T = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \right\}$.
Then, T is subring of a WIN-clean ring $M_2(\mathbb{Z}_3)$, $Nil(T) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ and $wi(T) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\}$. This implies that, T is reduced ring, but $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is not weak idempotent element. Hence, T is not WIN-clean ring.

In reduced nil-clean rings, 1 is the only unit element where as 1 and -1 are the only unit elements in a reduced weakly nil-clean ring. In the next discussions, we characterize unit elements in a reduced WIN-clean ring.

Proposition 3.2. In a reduced WIN-clean ring, every unit element is its own multiplicative inverse.

Proof. Let R be a reduced WIN-clean ring and $x \in R$ be a unit. Then, x is weak idempotent by Proposition 3.1. So, $x^2 = x^4$ implies that $x^2(x^2-1) = 0$ which in turn implies that $x^2 = 1$, i.e, $x = x^{-1}$. Hence, every unit element is its own multiplicative inverse. $\square$

Remark 3.3. A reduced WIN-clean ring may not be Boolean ring. For example, consider the reduced WIN-clean subring $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ of $T_2(\mathbb{Z}_3)$. Then R is not Boolean ring because $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ is not idempotent element.

From the remark 3.3, we conclude that reduced commutative WIN-clean ring may not be Boolean ring.

Proposition 3.3. Every subring of a reduced WIN-clean ring is a reduced WIN-clean ring.

Proof. Let R be a reduced WIN-clean ring and S be a subring of R. Then, $R = wi(R)$. Let $s \in S$. Then, $s \in R$ implies that $s^2 = s^4$, that is, $s \in wi(S)$. Thus, $S = wi(S)$. Hence, S is reduced WIN-clean ring. $\square$

Proposition 3.4. If R is a reduced WIN-clean ring, then $U(R) = Inv(R)$.

Proof. Let R be a reduced WIN-clean ring and $u \in U(R)$. Then, u is weak idempotent element. Thus, u^2 is idempotent. So, $u^2 = 1$ since 1 is the only unit and idempotent element. Thus, $u \in Inv(R)$ and hence $U(R) \subseteq Inv(R)$. By definition, $Inv(R) \subseteq U(R)$. Hence, $U(R) = Inv(R)$. $\square$

Remark 3.4. The converse of Proposition 3.4 is not true. For instance, $\mathbb{Z}_4$ is not reduced WIN-clean ring, but $U(\mathbb{Z}_4) = Inv(\mathbb{Z}_4) = \{1, 3\}$.

Corollary 3.1. If R is a reduced WIN-clean domain, then $U(R)$ has at most two elements.

Proof. Let R be a reduced WIN-clean domain, that is, R is reduced WIN-clean ring and also an integral domain, and let $u \in U(R)$. Then, u is a weak idempotent element. Thus, $u^4 - u^2 = 0$ implies that $u^2(u^2 - 1) = 0$. So, $u^2 = 1$ by Proposition 3.4 and hence $(u-1)(u+1) = 0$. Thus, either $u - 1 = 0$ or $u + 1 = 0$ since R is an integral domain. This implies that either $u = 1$ or $u = -1$. Hence, $U(R)$ has at most two elements. $\square$

Definition 3.2. A weak idempotent element in a ring is said to be central if it commutes with every element of the ring.

Example 3.4. The central weak idempotent elements in $T_2(\mathbb{Z}_3)$ are $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Definition 3.3. An ideal I of a ring is said to be WIN-clean ideal if each element of I is WIN-clean.

Example 3.5. $\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}2&0\\0&2\end{pmatrix}\right\}$ is WIN-clean ideal of $T_2(\mathbb{Z}_3)$.

The following proposition characterizes ideals of a WIN-clean ring.

Proposition 3.5. Every ideal I in a WIN-clean ring R with central weak idempotent elements w having the form either $w^2 = 0$ or $w^2 = 1$ is WIN-clean ring.

Proof. Let R be a WIN-clean ring and I be an ideal of R. Let $x \in I$. Then, $x = n + w$ for some $n \in Nil(R)$ and $w \in wi(R)$. If $w^2 = 0$, then w is nilpotent and hence x is nilpotent. Assume that $w^2 \neq 0$. Then, $n^k = 0$ for some positive integer k which implies that $n^k = (x - w)^k = 0$. Using binomial expansion, we have

$$(x-w)^k = x^k - \binom{k}{1}x^{k-1}w + \binom{k}{2}x^{k-2}w^2 + \cdots + \binom{k}{k-1}(-1)^{k-1}xw^{k-1} + (-w)^k = 0$$

Thus, $-(-w)^k = x^k - \binom{k}{1}x^{k-1}w + \binom{k}{2}x^{k-2}w^2 + \cdots + \binom{k}{k-1}(-1)^{k-1}xw^{k-1} \in I$.

(i) By hypothesis, $(-w)^k = w^2 = 1$ if k is even. In this case, $I = R$.

(ii) If k is odd, then $(-w)^k = -w^3 = -w$ and also $w \in I$.

So, $n = x - w \in I$ and hence I is WIN-clean ring. □

Ibrahim and Fadil ([28]) proved that for a weakly nil-clean ring R and $a \in R$ such that aR contains no non-zero idempotent, a is a sum of a nilpotent element and a right unit element. We adapt this result to our case as follows.

Proposition 3.6. Let R be a WIN-clean ring with central weak idempotent elements and $a \in R$. If aR contains no non-zero idempotent element, then a is the sum of two nilpotent elements.

Proof. Let R be a WIN-clean ring with central weak idempotent element. Suppose aR contains no nonzero idempotent element. Let $w \in wi(R)$ and $n \in Nil(R)$ such that $a = n + w$. Then, $aw^3 = nw^3 + w^4 = nw^3 + w^2 = (nw + 1)w^2$. So, $aw^3(nw+1)^{-1} = (nw+1)w^2(nw+1)^{-1}$ in aR. As nw is nilpotent, $nw+1$ is a unit and w^2 is idempotent. Thus, $(nw + 1)w^2(nw + 1)^{-1}$ is idempotent. Since aR does not contain a nonzero idempotent element, we have $(nw+1)w^2(nw+1)^{-1} = 0$ which implies $w^2 = 0$ and hence w is nilpotent. Therefore, a is the sum of two nilpotent elements. □

The following proposition gives a condition in which a division ring becomes a WIN-clean ring.

Proposition 3.7. Every division ring with all units have their own inverses is WIN-clean ring.

Proof. Since division rings are reduced rings, the proof follows from Proposition 3.2. □

A ring R is said to be indecomposable if $Id(R) = \{0, 1\}$. The following lemma describes the properties of weak idempotent elements in indecomposable WIN-clean rings.

Lemma 3.2. The weak idempotents of indecomposable WIN-clean ring are either trivial idempotents or nilpotents with index 2 or units with their own inverses.

Proof. Suppose R is indecomposable WIN-clean ring. Let $w \in wi(R)$. Then, $w^2 = 0$ or $w^2 = 1$ since $Id(R) = \{0, 1\}$. Thus, $w = 0$ or $w = 1$ or w is nilpotent with nilpotent index 2 or w is unit with its own inverse. Hence, w is either trivial idempotent or a nilpotent with index 2 or a unit with its own inverse. □

3.2 Some Properties of WIN-clean rings

In this section, we prove that homomorphic image of WIN-clean rings is WIN-clean ring, and finite product of WIN-clean rings is WIN-clean ring. We will also prove that every WIN-clean ring is isomorphic to a product of two WIN-clean rings R_1 and R_2 with $2 \in J(R_1)$ and $3 \in J(R_2)$. In addition to all these, WIN-clean rings are clean rings. We will start this section by defining lifting weak idempotent elements of a ring.

Definition 3.4. Let R be a ring and I be an ideal of R. Then, the weak idempotents can be lifted modulo I if for a given $a \in R$ with $a^4 - a^2 \in I$, there exists $w \in wi(R)$ such that $w - a \in I$.

Calugareanu ([5]) proved that nil-ideals lift idempotents, nilpotents and units. Then, the following proposition guarants us that nil-deals lift weak idempotents.

Proposition 3.8. Let I be a nil ideal of a ring R. If $\bar{w} = w + I$ is a weak idempotent element in R/I, then w can be lifted to a weak idempotent element in R.

Proof. Let $\bar{w} \in R/I$ be a weak idempotent element and w be a pre-image for $\bar{w}$. Then, $\bar{w}^2 = \bar{w}^4$ implies that $w^2 - w^4 \in I$ or $w^2 \equiv w^4 (modI)$, where w^2 and w^4 are pre-images of $\bar{w}^2$ and $\bar{w}^4$, in R respectively. Let $z = 1 - w^2$. Then, we have the following conditions.

(a) $w^2 z = z w^2$

(b) $w^2 + z \equiv 1 (modI)$

Now $w^2 z = w^2 - w^4 \in I$. Then, $0 = (w^2 z)^k = w^{2k} z^k$ for some positive integer k and w^{2k} is also a pre-image of $\bar{w}$, since $w^{2k} \equiv w^2 (modI)$. Conditions (a) and (b) are preserved when w and z are replaced by w^{2k} and z^k. Moreover, condition

(c) $w^2 z = z w^2 = 0$

is also preserved.
From condition (b), we have $x = 1 - w^2 - z \in I$. Then, $(1 - w^2 - z)^m = 0$ for some positive integer m. Thus, $1 = 1 - x^m = (1 - x)(1 + x + \cdots + x^{m-1})$ and it follows that $1 - x$ has an inverse $u = 1 + x + \cdots + x^{m-1}$. The element u commutes with w and z as x commutes with w and z.
Since $x \in I$, $u \equiv 1 (modI)$. We can replace w and z with uw^2 and uz, in this case w is again a pre-image for $\bar{w}$ and also conditions (a), (b), and (c) hold true. Further, it is true that $w^2 + z = 1$. By condition (c), we have $w^2 z = 0$, so, it gives that $w^2 = w^2(w^2 + z) = w^4 + w^2 z = w^4$. Therefore, $\bar{w}$ lifted to the weak idempotent w in R. □

Lemma 3.3. The homomorphic images of nilpotent and weak idempotent elements are nilpotent and weak idempotent elements respectively.

Proof. Let $f : R \to S$ be a ring homomorphism and $n \in Nil(R)$. Then, $n^k = 0$ for some positive integer k. So, $f(n)^k = f(n^k) = 0$ since $f(0) = 0$ implies that $f(n) \in Nil(S)$. Similarly, if $w \in wi(R)$, then $f(w) \in wi(S)$. □

Breaz et al. ([3]) proved that the homomorphic image of a weakly nil-clean ring is weakly nil-clean, and Diesl ([21]) proved that the homomorphic image of a nil-clean ring is nil-clean. We have the same result for WIN-clean rings, which is stated as follows:

Proposition 3.9. The homomorphic image of a WIN-clean ring is WIN-clean ring.

Proof. Let $f : R \to S$ be a ring homomorphism from ring R into a ring S. Suppose R is WIN-clean ring. Then we show that $f(R)$ is WIN-clean ring. Let $b \in f(R)$. Then there exists $a \in R$ such that $f(a) = b$. By hypothesis, $a = n + w$ for some $n \in Nil(R)$ and $w \in wi(R)$. This implies that $b = f(a) = f(n) + f(w)$. By Lemma 3.3, $f(n) \in Nil(f(R))$ and $f(w) \in wi(f(R))$. So b is WIN-clean element in $f(R)$. Hence, $f(R)$ is WIN-clean ring. □

Remark 3.5. The converse of Theorem 3.9 is not true. For example, consider the canonical epimorphism $\alpha : \mathbb{Z} \to \mathbb{Z}/(3)$ given by $\alpha(n) = n + (3)$. Then, $\mathbb{Z}_3 \cong \mathbb{Z}/(3)$ is a WIN-clean ring, but $\alpha^{-1}(\mathbb{Z}/(3)) = \mathbb{Z}$ is not a WIN-clean ring.

The next lemma describes that homomorphic image of an indecomposable WIN-clean ring is an indecomposable WIN-clean ring.

Proposition 3.10. Every homomorphic image of an indecomposable WIN-clean ring is an indecomposable WIN-clean ring.

Proof. Let R be an indecomposable WIN-clean ring. Then, $Id(R) = \{0, 1\}$ and for any $r \in R$, either $r = n$,or $r = n \pm 1$ or $r = n + w$, where $w^2 = 0$ or $w^2 = 1$, for some nilpotent element n of R.
Let $\alpha : R \to S$ be epimorphism of rings. Then, $\alpha(n)$ is nilpotent element in S, $\alpha(n \pm 1) = \alpha(n) \pm 1$ and $\alpha(n + w) = \alpha(n) + \alpha(w)$, where $\alpha(w)^2 = 0$ or $\alpha(w)^2 = 1$. By Lemma 3.2, $Id(S) = \{0, 1\}$. Moreover, by Proposition 3.9, S is WIN-clean ring. Hence, S is indecomposable WIN-clean ring. □

Let R be a ring and M a left R-module. Consider the idealization of R and M given by $R(M) = R \oplus M$. For $(r, m), (s, t) \in R(M)$, product and sum are defined as follows:

$$(r, m)(s, t) = (rs, rt + sm); (r, m) + (s, t) = (r + s, m + t).$$

Then, one can easily prove that $R(M)$ is a ring.
The following proposition gives the relationship between a WIN-clean ring R and an idealization of R and an R-module M.

Proposition 3.11. Let R be a ring and M be a left R-module. Then, R is WIN-clean if and only if $R(M)$ is WIN-clean.

Proof. Let R be a WIN-clean ring and $(r,m) \in R(M)$, where $r \in R$ and $m \in M$. Then, $r = n+w$ for some $n \in Nil(R)$ and $w \in wi(R)$. Thus, $n^k = 0$ for some $k \in \mathbb{N}$. So, $(n,m)^{k+1} = (n^{k+1}, (k+1)n^k m) = (0,0)$ which implies that

$$(r,m) = (n+w,m) = (n,m) + (w,0)$$

is WIN-clean expression of (r,m) of $R(M)$. Hence, $R(M)$ is WIN-clean. Conversely, $R \cong R(M)/(0 \oplus M)$ is homomorphic image of $R(M)$. So, by Theorem 3.9, R is WIN-clean ring. □

The following proposition is useful in proving that the Jacobson radical of a WIN-clean ring is nil ideal.

Proposition 3.12. Let R be a ring. Then, weak idempotent elements in $J(R)$ are nilpotents.

Proof. Let R be a ring and $w \in J(R)$ be a weak idempotent element. Then, $w^2 \in J(R)$ and also $1 - w^2$ is an idempotent element. Again, $w^2 \in J(R)$ implies that $1 - w^2 \in U(R)$. So, $1 - w^2$ is both idempotent and unit. Thus, $1 - w^2 = 1$, since 1 is the only unit and idempotent element. This implies that $w^2 = 0$. Hence, w is nilpotent element. □

Lemma 3.4. Let R be a ring. Then, $J(R)$ does not contain non-zero idempotent and unit elements of R.

Proof. Let $e \in J(R)$ and $e^2 = e$. Then, $1 - e \in U(R)$. Since $1 - e$ is an idempotent and a unit, $1 - e = 1$. So, $e = 0$. □

The following proposition is useful to prove Proposition 3.14 and the detail can be found in ([41]).

Proposition 3.13 ([41])**.** Let R be a ring with 1 and $a, b \in R$ such that $ab \neq ba$. Then,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} D_k b^{n-k}$$

where $d_a(x) = ax - xa$ and $D_k = D_k(b,a) = (a+d_b)^k 1 - a^k$, $D_0(b,a) = 0$, $D_{n+1}(b,a) = d_b a^n + (A + d_b) D_n(b,a)$.

Proposition 3.14. Let R be a WIN-clean ring. Then, $J(R) \subseteq Nil(R)$.

Proof. Let R be a WIN-clean ring and $a \in J(R)$. Then, $a = n + w$ for some nilpotent n and weak idempotent w. So, $(a-w)^k = 0$ for some $k \in \mathbb{N}$ and hence $(w-a)^k \in J(R)$. Now

$$(w-a)^k = \sum_{k=0}^{n} \binom{n}{k} w^k a^{n-k} + \sum_{k=0}^{n} \binom{n}{k} D_k a^{n-k}$$

implies that $(w-a)^k - [\sum_{k=0}^{n-1} \binom{n}{k} w^k a^{n-k} + \sum_{k=0}^{n} \binom{n}{k} D_k a^{n-k}] = w^k \in J(R) \cap wi(R)$. Since $J(R)$ does not contain units and nonzero idempotents, w must be nilpotent. Now $a - w, w \in Nil(R)$ which in turn implies that $a \in Nil(R)$.
Hence, $J(R) \subseteq Nil(R)$. □

Proposition 3.15. Every ideal of a local ring with central weak idempotents is WIN-clean ideal.

Proof. Let R be a local ring. Then, $J(R) = Nil(R)$, since R has a unique left (or right) maximal ideal. First, we show that R is a WIN-clean ring. Let $x \in R$. Then, either $x \in Nil(R)$ or $x \in U(R)$. If $x \in Nil(R)$, then we are done. Assume that $x \in U(R)$. Then, $1 - x \in Nil(R)$. So, $1 - x = n$ for some $n \in Nil(R)$. Thus, $x = -n + 1$. Hence, R is a WIN-clean ring. Next, we show that any ideal of R is WIN-clean. Let $w \in wi(R)$. Then, either $w = 0$ or w^2 is idempotent. So, $w^2 = 0$ or $w^2 = 1$ since $Id(R) = \{0, 1\}$. Hence, the proof follows from Proposition 3.5. □

Theorem 3.3. Let I be a nil ideal of a ring R. Then, R is WIN-clean if and only if R/I is WIN-clean and weak idempotents of R can be lifted modulo I.

Proof. Let I be a nil ideal of a ring R.
($\Longrightarrow$) Suppose R is a WIN-clean ring. Then, R/I is the homomorphic image of R of the canonical epimorphism. So, it is a WIN-clean ring by Proposition 3.10, and I lifts weak idempotents by Proposition 3.8.
($\Longleftarrow$) Suppose R/I is a WIN-clean ring and the weak idempotents can be lifted modulo I. Let $r \in R$. Then, $\bar{r} = r + I \in R/I$. We can write $\bar{r} = \bar{n} + \bar{w}$, where $\bar{n} \in Nil(R/I)$ and $\bar{w} \in wi(R/I)$ implies that $r + I = (n + w) + I$. The nilpotent $\bar{n}$ in R/I lifted to a nilpotent n in R. To see this, $\bar{n}^k = 0$ for some positive integer k in R/I implies that $n^k \in I$. Since I is nil, $(n^k)^m = 0$ for some positive integer m. So, $n^{km} = 0$. We know that weak idempotents lift modulo any nil ideal (Proposition 3.8) and this allows us to assume that w is a weak idempotent in R.
Moreover, $r - n - w \in I$ and it follows that $r - w = n + d$, where $d \in I$. Since $n^m = 0$ for some $m \in \mathbb{N}$, we have $(n + d)^m \in I$ because I is an ideal of R. Thus, $(n + d)^{mk} = 0$ for some $k \in \mathbb{N}$ as I is a nil ideal. So, $n + d$ is a nilpotent element. Hence, R is WIN-clean, as desired. □

Remark 3.6. In Theorem 3.3, the assumption that I is a nil ideal is necessary. For example, consider the ring $\mathbb{Z}_{10}$. Then, $I = \{0, 2, 4, 6, 8\}$ is an ideal of $\mathbb{Z}_{10}$ but not nil and $\mathbb{Z}_{10}/I = \{I, 1 + I\}$ is a WIN-clean ring. Now $wi(\mathbb{Z}_{10}) = \{0, 1, 4, 5, 6, 9\}$. To see that I lifts weak idempotents, consider the following expressions:

$$\begin{aligned} 8^4 - 8^2 = 2 \in I &\Longrightarrow \exists\, 6 \in wi(\mathbb{Z}_{10}) \text{ such that } 8 - 6 \in I \\ 7^4 - 7^2 = 2 \in I &\Longrightarrow \exists\, 1 \in wi(\mathbb{Z}_{10}) \text{ such that } 7 - 1 \in I \\ 3^4 - 3^2 = 2 \in I &\Longrightarrow \exists\, 1 \in wi(\mathbb{Z}_{10}) \text{ such that } 3 - 1 \in I \\ 2^4 - 2^2 = 2 \in I &\Longrightarrow \exists\, 0 \in wi(\mathbb{Z}_{10}) \text{ such that } 2 - 0 \in I. \end{aligned}$$

This implies that weak idempotents can be lifted modulo I, but $\mathbb{Z}_{10}$ is not a WIN-clean ring.

Corollary 3.2. A ring R is WIN-clean if and only if $R/J(R)$ is WIN-clean ring and $J(R)$ is nil.

Proof. Since $J(R)$ is nil, the proof follows from Theorem 3.3. □

Remark 3.7. The converse of Proposition 3.14 is not true. Consider Example 1.2 Lee et al. ([32]). If we take a simple domain $F = \mathbb{Z}_5$, then $A = M_2(\mathbb{Z}_5)$ is a ring of 2×2 matrices over integer modulo 5, and $B = D_2(\mathbb{Z}_5)$ is a ring of 2×2 diagonal

matrices over integer modulo 5 such that $Nil(B) = \begin{pmatrix} 0 & \mathbb{Z}_5 \\ 0 & 0 \end{pmatrix}$.
Define $R = B + A[[x]]x$, where $A[[x]]$ denotes the formal power series ring with an indeterminate x over a ring R. Then, $Nil(R) \subsetneq J(R) = Nil(B) + A[[x]]x$ and $R/J(R) \cong \mathbb{Z}_5$. But $\mathbb{Z}_5$ is not WIN-clean and hence $R/J(R)$ is not WIN-clean. Therefore, by Corollary 3.2, R is not WIN-clean ring.

Corollary 3.3. Let R be a commutative ring. Then, R is WIN-clean if and only if $R/Nil(R)$ is WIN-clean ring such that all of its elements are weak idempotents.

Proof. Since R is a commutative ring, $Nil(R)$ is a nil ideal of R. So, the proof follows from Theorem 3.3. □

Definition 3.5. Let R be a WIN-clean ring. An element $r \in R$ is said to be center of R if $rs = sr$ for all $s \in R$.

Example 3.6. Centers of the ring $T_2(\mathbb{Z}_3)$ are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

The following corollary states the condition for which the centre of a WIN-clean ring becomes a WIN-clean ring.

Corollary 3.4. Let R be a WIN-clean ring such that the weak idempotents are central. Then, the centre of R is a WIN-clean ring.

Proof. Let $x \in C(R)$. Then, $x = n + w$, where $n \in Nil(R)$ and $w \in wi(R)$. But $w \in C(R)$. Thus, $n = x - w \in C(R)$. Hence, R is WIN-clean ring. □

Remark 3.8. There are WIN-clean rings with non-central weak idempotents whose centre is WIN-clean. For example, consider the WIN-clean ring $M_2(\mathbb{Z}_3)$ and it has non-central weak idempotent element. Then,

$$C(M_2(\mathbb{Z}_3)) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\}$$

is WIN-clean.

Remark 3.9. Given a ring R, it is clear that if $x \in R$ is a nonzero central nilpotent, then $1 - xr \in U(R)$ for all $r \in R$. Hence, $x \in J(R)$, that is, the nonzero central nilpotents are contained in Jacobson radical, $J(R)$.

Deisl ([21]) proved that 2 is the least central nilpotent element in nil-clean rings where as in Breaz et al. ([3]) proved that 6 is the least central nilpotent element in weakly nil-clean rings.
Next, we investigate central nilpotent element of a WIN-clean ring.

Theorem 3.4. The following statements are equivalent for a ring R:

(1) R is WIN-clean.

(2) 12 is a nilpotent element and $R/12R$ is a WIN-clean ring.

(3) $R/J(R)$ is a WIN-clean ring and $J(R)$ is nil.

Proof. Let R be a ring.
(1) $\implies$ (2). Suppose R is a WIN-clean ring. If $12 = 0$, then we are done. Assume that $12 \neq 0$. As R is a ring with 1, $1+1 = 2 \in R$ is the least non-unit central element of R. Then, there exist a weak idempotent w and a nilpotent n such that $2 = n+w$. Thus, $(2-n)^2 = (2-n)^4 \implies 2^2 - 4n + n^2 = 2^4 - 32n + 24n^2 - 8n^3 + n^4$. So, $n(-n^3 + 8n^2 - 23n + 28) = 12$. Thus, 12 is nilpotent element. Since R is WIN-clean ring, $R/12R$ is WIN-clean ring by Theorem 3.3.
(2) $\implies$ (1) follows from Theorem 3.3 and the equivalence (1) $\iff$ (3) follows immediately from Corollary 3.2. □

The following proposition establishes the relationship between local rings and WIN-clean rings.

Proposition 3.16. Let R be a ring with only trivial idempotents. Then, R is WIN-clean if and only if R is a local ring with $J(R)$ nil and $R/J(R) \cong \mathbb{F}_2$ or $R/J(R) \cong \mathbb{F}_3$.

Proof. Let R be a WIN-clean ring, $Id(R) = \{0, 1\}$ and $r \in R$. Then, either r or $r-1$ is a nilpotent. This implies that either $1-r$ or r is a unit and hence R is a local ring. By Corollary 3.2, $J(R)$ is nil. Thus, $R/J(R)$ is WIN-clean division ring. Hence, $R/J(R)$ must be isomorphic to either $\mathbb{F}_2$ or $\mathbb{F}_3$. Conversely, assume that R is local ring with $J(R)$ nil. Then, $R = J(R) \cup (1 - J(R))$. Let $r \in R$. Then, $r \in J(R)$ or $r \in 1 - J(R)$. If $r \in J(R)$, then it is nilpotent element and hence $r = 0 + r$. If $r \in 1 - J(R)$, then $r - 1$ is nilpotent element. So, $r = 1 + (r-1)$ which is WIN-clean element. □

The following propositions establish the conditions for which WIN-clean rings become nil-clean and weakly nil-clean rings.

Proposition 3.17. A ring R is nil-clean if and only if R is WIN-clean and $2 \in J(R)$.

Proof. Let R be a ring.
($\implies$) Suppose R is a nil-clean ring and $r \in R$. Then, $r = n + e$, where $n \in Nil(R)$ and $e \in Id(R)$. Thus, $e \in wi(R)$. So, r is WIN-clean element and hence R is WIN-clean ring. Also, $2 = n + e$ implies that $1 - e = n - 1$ is both a unit and idempotent. So, $n - 1 = 1$ implies $n = 2$. Thus, 2 is central nilpotent element. This implies that $2 \in J(R)$.
($\impliedby$) Suppose R is a WIN-clean ring and $2 \in J(R)$. Then, $J(R)$ is nil and $2 + J(R) = 0 + J(R)$. We know that a nilpotent modulo nil ideal lifted to nilpotent in R. So, we have $2 = 0$, that is, $char(R/J(R)) = 2$. Thus, for all $r \in R$, we have $2\bar{r} = \bar{0}$ and $1 - 2\bar{r} = \bar{1}$. So, $R/J(R)$ is boolean and hence $R/J(R)$ is nil-clean. Hence, by Proposition 2.3, R is nil-clean ring. □

Corollary 3.5. Let R be a commutative ring. Then, R is WIN-clean such that 2 is nilpotent if and only if $R/J(R)$ is Boolean.

Proof. Let R be a commutative ring.
($\impliedby$) Suppose $R/J(R)$ is a Boolean ring. Then, $Nil(R) = J(R)$ and hence R is nil-clean by ([21], Corollary 3.20) with $char(R/J(R)) = 2$, that is, $2 + J(R) = J(R)$ implies that $2 \in J(R)$.
($\implies$) Suppose R is a commutative WIN-clean ring. Then, $R/J(R)$ is WIN-clean ring with all elements are weak idempotents. Since 2 is nilpotent, $2 \in J(R)$ and hence $R/J(R)$ is Boolean. □

Proposition 3.18. A ring R is weakly nil-clean if and only if R is WIN-clean and $2 \in J(R)$ or $3 \in J(R)$.

Proof. Let R be a ring.
$(\implies)$ Suppose R is a weakly nil-clean ring. Then, R is WIN-clean. Assume that $2 \notin J(R)$. Then, $2 \in U(R)$ and $6^n = 0$ for some positive integer n as 6 is nilpotent element in R by Theorem 2.1. Thus, $2^n 3^n = 0$ implies that $3^n = 0$. Hence, $3 \in J(R)$.
$(\impliedby)$ Assume that R is a WIN-clean ring and $2 \in J(R)$ or $3 \in J(R)$. Then, $R/J(R)$ is WIN-clean ring and $J(R)$ is nil by Corollary 3.2. If $2 \in J(R)$, then by Proposition 3.17, R is nil-clean and hence R is weakly nil-clean. Again, if $3 \in J(R)$, then $3 + J(R) = 0 + J(R)$ and also 2 is invertible in R. We can assume $3 = 0$ so that $char(R/J(R)) = 3$. So, $\bar{2}$ is unit in $R/J(R)$. Moreover, $3\bar{r} = \bar{0}$, $\bar{1} - 3\bar{r} = \bar{1}$ and $\bar{2} - 3\bar{r} = \bar{2}$ for all $r \in R$. Thus, $R/J(R) \cong \mathbb{Z}_3$ and also $R/J(R)$ is weakly nil-clean ring. Hence, R is weakly nil-clean ring. □

Proposition 3.19. Let $R_1, R_2, \cdots, R_m$ be rings for some natural number m. Then, direct product $R = R_1 \times R_2 \times \cdots \times R_m = \prod_{i=1}^{m} R_i$ is a WIN-clean ring if and only if each R_i is WIN-clean ring for $i = 1, 2, \cdots, m$.

Proof. Let $R_1, R_2, \cdots, R_m$ be rings.
$(\implies)$ Suppose $R = \prod_{i=1}^{m} R_i$ is a WIN-clean ring. Then, $R/(\prod_{i=1, i\neq k}^{m} R_i) \cong R_k$ is homomorphic image of R. Hence, R_k is WIN-clean ring for each k.
$(\impliedby)$ Suppose each R_i is a WIN-clean ring for each $i = 1, 2, \cdots, m$ and $x \in R$. Then, $x = (x_1, x_2, \cdots, x_m)$, where $x_i \in R_i$ and hence $x_i = n_i + w_i$, where $n_i \in Nil(R_i)$ and $w_i \in wi(R_i)$. Thus, $n = (n_1, n_2, \cdots, n_m) \in Nil(R)$ and $w = (w_1, w_2, \cdots, w_m) \in wi(R)$ such that $n + w = (n_1, n_2, \cdots, n_m) + (w_1, w_2, \cdots, w_m) = (x_1, x_2, \cdots, x_m) = x$. Hence, R is WIN-clean ring. □

Remark 3.10. An infinite direct product of WIN-clean rings may not be a WIN-clean ring. For example, consider $R = \prod_{m\in\mathbb{N}} \mathbb{Z}_{2^m}$. Then, $0, 2 \in \mathbb{Z}_{2^m}$ for $m \in \mathbb{N}$ are nilpotent elements, but the infinite sequence $r = (0, 2, 2, 2, \cdots) \in R$ is not nilpotent since there is no positive integer k such that $r^k = 0$ as well as it can not be written as a sum of a nilpotent element and a weak idempotent element in R. Thus, r is not WIN-clean element. Hence, R is not WIN-clean ring.

Now we give the definition of comaximal ideals.

Definition 3.6. Let R be a ring. Then, ideals I and J are comaximal ideals of R if $I + J = R$.

Next we state the famous Chinese Remainder Theorem for rings and its proof found in ([27], P. 131).

Proposition 3.20 (Chinese Remainder Theorem for Rings)**.** Let R be a ring and $I_1, I_2, \cdots, I_m$ be ideals of R. Then, the map $\phi : R \to (R/I_1) \times (R/I_2) \times \cdots \times (R/I_m)$ defined by $\phi(r) = (r + I_1, r + I_2, \cdots, r + I_m)$ is a ring homomorphism with kernel $I_1 \cap I_2 \cap \cdots \cap I_m$. If all of the ideals $I_1, I_2, \cdots, I_m$ are pairwise comaximal, then ϕ is surjective and $I_1 \cap I_2 \cap \cdots \cap I_n = I_1 I_2 \cdots I_m$, and hence $R/(I_1 I_2 \cdots I_m) \cong (R/I_1) \times (R/I_2) \times \cdots, (R/I_m)$.

Breaz et al. ([3]) proved that every weakly nil-clean ring is isomorphic to a nil-clean ring or an indecomposable weakly nil-clean ring with 3 contained in its Jacobson radical or a product of nil-clean ring and an indecomposable weakly nil-clean ring with 3 contained in its Jacobson radical. Now the following proposition describes that every WIN-clean ring is isomorphic to the product of a nil-clean ring and a WIN-clean ring with 3 belong to its Jacobson radical.

Theorem 3.5. Let R be a ring. Then, R is WIN-clean ring if and only if $R \cong R_1 \times R_2$, where R_1 is WIN-clean ring with $2 \in J(R_1)$ and R_2 is 0 or a WIN-clean ring with $3 \in J(R_2)$.

Proof. Let R be a ring.
($\implies$) Suppose R is a WIN-clean ring. Then, 12 is nilpotent element in R so that $(12)^m = 0$ for some positive integer m. Then, $4^m R \cap 3^m R = 0$ and $4^m R + 3^m R = R$. Thus, $R \cong (R/2^{2m}R) \times (R/3^m R)$ by Chinese Remainder Theorem. By Theorem 3.3, $R_1 = R/2^{2m}R$ and $R_2 = R/3^m R$ are WIN-clean rings. Thus, 2 is central nilpotent in R_1 and hence $2 \in J(R_1)$. Assume that $R_2 \neq 0$. Then, 3 is central nilpotent element in R_2 and hence $3 \in J(R_2)$.
($\Longleftarrow$) Let $R \cong R_1 \times R_2$, where R_1 is WIN-clean with $2 \in J(R_1)$ and R_2 is 0 or a WIN-clean ring with $3 \in J(R_2)$. Since R_1 and R_2 are WIN-clean rings, $R_1 \times R_2$ is WIN-clean ring by Proposition 3.19. So, $R \cong R_1 \times R_2$ implies that R is WIN-clean ring. □

The following examples explain the above proposition. In these examples, we learn that indecomposability is not mandatory as in weakly nil-clean rings.

Example 3.7. Consider the matrix ring $R = M_2(\mathbb{Z}_2)$. Then,
$Nil(R) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ and

$$wi(R) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

Thus, R is WIN-clean ring and $J(R) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$. Also $2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in J(R)$ and $3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus, $R_1 = R/2R = R$, $R_2 = R/3R = 0$, $R_1 \cap R_2 = \{0\}$ and $R_1 + R_2 = R$. Hence, $R \cong R_1 \times R_2$, where $R_1 = R/2R$ with $2 \in J(R_1)$ and $R_2 = 0$.

Example 3.8. Consider the non-commutative ring with unity,

$$R = T_2(\mathbb{Z}_3) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_3 \right\}.$$

Then, $Id(R) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$,

$Nil(R) = \left\{\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 2\\0 & 0\end{pmatrix}\right\}$ and also we have

$$wi(R) = \left\{\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}2 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}2 & 2\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 1\end{pmatrix},\right.$$
$$\begin{pmatrix}0 & 2\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 2\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 2\end{pmatrix}, \begin{pmatrix}0 & 2\\0 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\0 & 2\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix},$$
$$\left.\begin{pmatrix}2 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}1 & 2\\0 & 0\end{pmatrix}, \begin{pmatrix}2 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}2 & 2\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 2\\0 & 0\end{pmatrix}\right\}.$$

Thus, R is WIN-clean ring and $J(R) = \left\{\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 2\\0 & 0\end{pmatrix}\right\} = Nil(R)$. So, $2 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} + \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} \notin J(R)$ and hence $R_1 = R/2R = 0$, where $2R = R$ but $3 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} + \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} + \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \in J(R)$. So,

$$R/J(R) = \left\{J(R), \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} + J(R), \begin{pmatrix}2 & 0\\0 & 1\end{pmatrix} + J(R), \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} + J(R), \begin{pmatrix}0 & 0\\0 & 2\end{pmatrix} + J(R),\right.$$
$$\left.\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix} + J(R), \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} + J(R), \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + J(R), \begin{pmatrix}2 & 0\\0 & 0\end{pmatrix} + J(R)\right\}.$$

Thus, $R_2 = R/J(R)$ is reduced commutative WIN-clean ring with $char(R/J(R)) = 3$, but not indecomposable since

$$Id(R_2) = \left\{J(R), \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} + J(R), \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + J(R), \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} + J(R)\right\}$$

is non-trivial. Then, $R_1 \cap R_2 = 0$, $R_1 + R_2 = R$ and hence $R \cong R_1 \times R_2$.

Corollary 3.6. The following statements are equivalent for a ring R.

(1) R is a WIN-clean ring with central weak idempotent elements.

(2) $R \cong R_1 \times R_2$, where R_1 is WIN-clean with central weak idempotent elements and $J(R_1)$ nil such that $R_1/J(R_1)$ is Boolean, and R_2 is 0 or $R_2/J(R_2)$ is isomorphic to a direct product of copies of $\mathbb{Z}_3$ with $J(R_2)$ nil.

(3) R is a ring with central weak idempotent elements, $J(R)$ is nil, and $R/J(R)$ is isomorphic to either a Boolean ring, or to $\mathbb{Z}_3$, or to the direct product of two such rings.

Proof. Let R be a ring.
(1) $\implies$ (2) Using Theorem 3.5, we can write $R \cong R_1 \times R_2$, where R_1 is WIN-clean ring with central weak idempotents and $3 - 1 = 2 \in J(R_1)$; and R_2 is 0 or a WIN-clean ring with central weak idempotents and $3 \in J(R_2)$. Thus, $char(R_1/J(R_1)) = 2$ which in turn implies that $\bar{x} = -\bar{x}$ for all $\bar{x} \in R_1/J(R_1)$ and hence $R_1/J(R_1)$ is Boolean. Assume $R_2 \neq 0$. Since R_2 is WIN-clean and $3 \in J(R_2)$, $R_2/J(R_2)$ is WIN-clean and $char(R_2/J(R_2)) = 3$. Moreover, 2 is unit in R_2, since $2 \notin J(R_2)$. From

this, we conclude that $R_2/J(R_2) = \{3R_2, 1-3R_2, 2-3R_2\}$ so that every element of $R_2/J(R_2)$ is nilpotent or invertible which implies that $R_2/J(R_2) \cong \mathbb{Z}_3$. Hence, by Corollary 3.2, $J(R_1)$ and $J(R_2)$ are nil ideals.
The proof of the implication (2) $\implies$ (3) follows from Theorem 3.5 and that of (3) $\implies$ (1) follows from Theorem 3.4. □

Definition 3.7. Let R be a ring. Then, an element x in R is called a square root of idempotent element if there exists an idempotent element e in R such that $x^2 = e$.

Proposition 3.21. Let R be a WIN-clean ring with $2 \in U(R)$. Then, every element of R can be written as a sum of a nilpotent and a square root of idempotent element.

Proof. Let $a \in R$. Then, $a = n + w$ for some $n \in Nil(R)$ and $w \in wi(R)$. Let $v = 2w^2 - 1$. Then, $v^2 = (2w^2-1)^2 = 4w^4 - 4w^2 + 1 = 4w^2 - 4w^2 + 1 = 1$. Thus, $vv^{-1} = (2w^2-1)(2w^2-1)^{-1} = 1$. Now $v = 2w^2 - 1$ implies $w^2 = (v+1)/2$ and $[(v+1)/2]^2 = (v+1)/2$. Hence, w is a square root of idempotent. □

Next, we prove that a WIN-clean ring is a subclass of clean rings.

Theorem 3.6. Every WIN-clean ring is clean ring.

Proof. Let R be a WIN-clean ring and $a \in R$. Then, $a = n+w$ for some nilpotent n and weak idempotent w. So, $a = n+w = (n+w-1+w^2)+(1-w^2)$. By Theorem 3.1, $w - 1 + w^2$ is unit and $1 - w^2 \in Id(R)$. To see $n + w - 1 + w^2$ is unit, let $u = w - 1 + w^2$. Then, $n + w - 1 + w^2 = n + u$. Since n and $(u^{-1}n)$ are nilpotents, we have $n^m = 0$ and $(u^{-1}n)^m = 0$ for some positive integer m. Now one can easily show that

$$\begin{aligned}(n+u)^{-1} =&[u(1+\frac{n}{u})]^{-1} = [1 - \frac{n}{u} + (\frac{n}{u})^2 - (\frac{n}{u})^3 + \cdots + (-\frac{n}{u})^{m-1}]u^{-1} \\ =&[1 - u^{-1}n + (u^{-1}n)^2 - \cdots + (-1)^{m-1}(u^{-1}n)^{m-1}]u^{-1}.\end{aligned}$$

Thus, a is clean element and hence, R is clean ring. □

In the proof of Theorem 3.6, we observed that WIN-clean element is clean. In general, the converse of Theorem 3.6 does not hold true. For example, integer modulo 5, $\mathbb{Z}_5$, is clean ring, but not WIN-clean ring.

Lemma 3.5. If w is a weak idempotent element in a WIN-clean ring R and $2 \in J(R)$, then $w \pm w^2$ is nilpotent element.

Proof. Let R be a WIN-clean ring and $w \in R$ be a weak idempotent element. Suppose $2 \in J(R)$. Then, we have $(w \pm w^2)^2 = 2(w^2 \pm w^3) \in J(R)$. As $J(R)$ is nil, there exists some positive integer m such that $2^m = 0$ and also $(w \pm w^2)^{2m} = 0$. Hence, $w \pm w^2$ is nilpotent element. □

The following proposition sets a condition for which a clean element becomes WIN-clean element.

Proposition 3.22. Let R be a ring with central weak idempotents, $2 \in J(R)$ and x be a clean element in R with clean decomposition $x = u + e$, where $u \in U(R)$ and $e \in Id(R)$. Then, x is WIN-clean element if and only if $2e - 1 + u$ is nilpotent element.

Proof. Let R be a ring with central weak idempotents, $2 \in J(R)$ and x be a clean element in R with clean decomposition $x = u + e$, where $u \in U(R)$ and $e \in Id(R)$.
($\implies$) Suppose x is a WIN-clean element. Then, $x = n + f$ for some $n \in Nil(R)$ and $f \in wi(R)$. Now $x = n + f = (n - 1 + f + f^2) + (1 - f^2)$. Since $2 \in J(R)$, $f + f^2$ is nilpotent by Lemma 3.5. Then, take $u = n - 1 + f + f^2$ and $e = 1 - f^2$. So, $2e - 1 + u = 2(1 - f^2) - 1 + (n - 1 + f + f^2) = n + f - f^2$. But $n + f - f^2$ is nilpotent. Thus, $2e - 1 + u$ is nilpotent element.
($\impliedby$) Suppose $2e - 1 + u$ is nilpotent element. We can rewrite $x = u + e$ as $x = (u + 2e - 1) + (1 - e)$. Then, x is nil-clean element. Hence, it is WIN-clean element. □

Proposition 3.23. Let R be a ring. Then, the following statements are equivalent.

(1) R is WIN-clean ring such that $2 \in Nil(R)$ or $3 \in Nil(R)$;

(2) R is a clean WUU ring;

(3) R is a weakly nil-clean ring having the strong property;

(4) $J(R)$ is a nil ideal, and either $R/J(R) \cong B$, or $R/J(R) \cong \mathbb{Z}_3$, or $R/J(R) \cong B \times \mathbb{Z}_3$, where B is a Boolean ring;

(5) R is an exchange WUU ring;

(6) R is a weakly nil-neat ring;

(7) R is weakly clean WUU.

Proof. (1) $\iff$ (3) follows from Proposition 3.18.
The equivalence (2) $\iff$ (3) $\iff$ (4) $\iff$ (5) proved in ([13], Corollary 2.15).
The equivalence (2) $\iff$ (7) proved in ([14], Theorem 2.7). □

3.3 Extensions of WIN-clean rings

In this section, we will see the condition for which the matrix ring $M_n(R)$, where R is WIN-clean ring and n is a positive integer, will be WIN-clean ring. Diesl ([21]) proved that R is nil-clean ring if and only if the upper triangular matrix over R is nil-clean. Breaz et al. ([3]) sets a condition for which the matrix ring becomes weakly nil-clean ring for division rings. Next we define WIN-clean matrix ring as follows.

Definition 3.8. Let R be a ring. Then, $A \in M_n(R)$ is said to be WIN-clean if there exists $W \in wi(M_n(R))$ and $N \in Nil(M_n(R))$ such that $A = N + W$. The matrix ring $M_n(R)$ is said to be WIN-clean if every element in $M_n(R)$ is WIN-clean.

Example 3.9. The matrix $M_2(\mathbb{Z}_2)$ and $M_2(\mathbb{Z}_3)$ are both WIN-clean rings.

Remark 3.11. If matrix A is weakly nil-clean, then it is WIN-clean. The converse is not true. For instance, the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(\mathbb{Z})$ is not weakly nil-clean, but it is WIN-clean since $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^4$.

Bisht ([2]) described Morita context as follows: Let R_1 and R_2 be rings with unity, M be an R_1-R_2-bimodule and N be an R_2-R_1-bimodule. The algebraic structure $(R_1, R_2, M, N, \phi, \psi)$ is said to be Morita context, where

$$\phi : M \otimes_{R_2} N \to R_1 \; and \; \psi : N \otimes_{R_1} M \to R_2$$

are bimodule homomorphisms satisfying $b\phi(m \otimes n) = \psi(b \otimes m)n$ for all $b \in N$ and $a\psi(n \otimes m) = \phi(a \otimes n)m$ for all $a \in M$. The bimodule homomorphisms ϕ and ψ are called pairings. The collection $T = \left\{ \begin{pmatrix} a & m \\ n & b \end{pmatrix} : a \in R_1, b \in R_2, m \in M, n \in N \right\}$ forms a ring under usual addition and multiplication defined by the following:

$$\begin{pmatrix} a & m \\ n & b \end{pmatrix} \begin{pmatrix} a' & m' \\ n' & b' \end{pmatrix} = \begin{pmatrix} aa' + \phi(m \otimes n') & am' + mb' \\ na' + bn' & \psi(n \otimes m') + bb' \end{pmatrix}.$$

This ring is called the ring of Morita context. If pairings are zero-morphisms, then T is called Morita context with zero pairings. It is interesting to note that if $R_1 = R_2 = R$ and $N = 0$, then the ring of Morita context T becomes the idealization R(M) of the ring R and the R- module M.
The following lemma describes the set of nilpotent elements of the ring of Morita context T and it is used to prove Theorem 3.6.

Lemma 3.6 ([2])**.** Let T be a ring of Morita context with zero pairings. Then,

$$Nil(T) = \left\{ \begin{pmatrix} x_1 & m \\ n & x_2 \end{pmatrix} : x_1 \in Nil(R_1), x_2 \in Nil(R_2), m \in M, n \in N \right\}.$$

Proposition 3.24. Let $T = \begin{pmatrix} R_1 & M \\ N & R_2 \end{pmatrix}$ be a ring of Morita context with zero pairings. Then, T is WIN-clean if and only if R_1 and R_2 are WIN-clean rings.

Proof. Let T be a WIN-clean ring and $a \in R_1$. Then, $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in T$. It can be written as $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_1 & m_1 \\ n_1 & x_2 \end{pmatrix} + \begin{pmatrix} w_1 & m_1' \\ n_1' & w_2 \end{pmatrix}$, where $\begin{pmatrix} x_1 & m_1 \\ n_1 & x_2 \end{pmatrix} \in Nil(T)$ and $\begin{pmatrix} w_1 & m_1' \\ n_1' & w_2 \end{pmatrix} \in wi(R)$. This gives $a = x_1 + w_1$, where $x_1 \in Nil(R_1)$ and $w_1 \in wi(R_1)$. Hence, R_1 is WIN-clean ring. Similarly, it can be proved that R_2 is WIN-clean ring. Conversely, let R_1 and R_2 be WIN-clean rings and $\begin{pmatrix} a & m \\ n & b \end{pmatrix} \in T$ be any element. Then, we can write $\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix} + \begin{pmatrix} x_1 & m \\ n & x_2 \end{pmatrix}$, where $x_1 \in Nil(R_1), x_2 \in Nil(R_2), w_1 \in wi(R_1), w_2 \in wi(R_2)$. By Lemma 3.6, $\begin{pmatrix} x_1 & m \\ n & x_2 \end{pmatrix} \in Nil(T)$ and also $\begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix} \in wi(T)$. Hence, T is WIN-clean ring. □

Remark 3.12. For a ring R and $w \in wi(R)$, wRw and $(1-w)R(1-w)$ are rings without unity. But w^2Rw^2 and $(1-w^2)R(1-w^2)$ are rings with unity w^2 and $1-w^2$ respectively.

P. N. Ánh et al. ([1]) stated that an idempotent $e = e^2$ in a ring R not necessarily with unity induces the (two sided) Peirce decomposition

$$R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$$

or more transparently, e induces on R the generalized matrix ring structure

$$R = \begin{bmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{bmatrix},$$

with the obvious matrix addition and multiplication.

Corollary 3.7.

(i) Let R_1 and R_2 be rings and M be an R_1-R_2 bimodule. Then, $T' = \begin{pmatrix} R_1 & M \\ 0 & R_2 \end{pmatrix}$ is WIN-clean if and only if R_1 and R_2 are WIN-clean.

(ii) The idealization $R(M) = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} | r, s \in R, m \in M \right\}$ of a ring R and an R-module M is a WIN-clean ring if and only if R is WIN-clean ring.

(iii) Let R be an abelian ring and $w \in wi(R)$. Then, R is WIN-clean ring if and only if w^2Rw^2 and $(1-w^2)R(1-w^2)$ are WIN-clean rings.

(iv) Let R be an abelian ring and $w_1, w_2, \cdots, w_n \in wi(R)$ such that $w_i^2 w_j^2 = 0$ for all $i \neq j$ and $w_1^2 + w_2^2 + \cdots + w_n^2 = 1$. Then, R is WIN-clean ring if and only if for each $i = 1, 2, \cdots, n$, $w_i^2 R w_i^2$ is a WIN-clean ring.

(v) The $n \times n$ upper (lower) triangular matrix over a ring R is WIN-clean if and only if so is R.

Proof.

(i) We observe that T' is a ring of Morita context with zero pairings and $N = 0$. In other words, T' is obtained from T by replacing $N = 0$ in Theorem 3.24. Hence, the result follows from Theorem 3.24.

(ii) Its proof follows from (i).

(iii) Since R is abelian, $w^2R(1-w^2) = 0$ and $(1-w^2)Rw^2 = 0$. Then, by Pierce decomposition, we have $R = \begin{pmatrix} w^2Rw^2 & 0 \\ 0 & (1-w^2)R(1-w^2) \end{pmatrix}$. So, the proof follows from (i).

(iv) For each i, $1 - w_i^2 = w_1^2 + \cdots + w_{i-1}^2 + w_{i+1}^2 + \cdots + w_n^2$ and $w_i^2 w_j^2 = 0$ for all $i \neq j$ which implies that $w_i^2 R(1-w_i^2) = 0$ and $(1-w_i^2)Rw_i^2 = 0$. So, $R = w_i^2 R w_i^2 \oplus (1-w_i^2)R(1-w_i^2)$. Thus, by (iii), R is WIN-clean if and only if $w_i^2 R w_i^2$ and $(1-w_i^2)R(1-w_i^2)$ are WIN-clean.

(v) Let R be a ring.
($\Longleftarrow$) Let R be a WIN-clean ring. Then, consider $T_n(R)$ the ring of all $n \times n$ upper triangular matrix over R. We prove by induction on n. For $n = 2$, $T_2(R) = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ is WIN-clean by Proposition 3.24. So, it is true for $n = 2$. Assume that it is true for $n - 1$. Then,

$$T_{n-1}(R) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n-1)} \\ 0 & a_{22} & \cdots & a_{2(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a_{(n-1)(n-1)} \end{pmatrix}$$

is WIN-clean. Now $T_n(R) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$. Taking $M = \{(a_{12}, \cdots, a_{1n}) : a_{1i} \in R, i = 2, 3, \cdots, n\}$, we can rewrite $T_n(R)$ as follows:

$$T_n(R) = \begin{pmatrix} R & M \\ 0 & T_{n-1}(R) \end{pmatrix}.$$

Clearly, M is (R,$T_n(R)$)-module. So, $T_n(R)$ is WIN-clean ring by by Proposition 3.24. Similarly, one can prove for lower triangular matrix over R.
($\Longrightarrow$) Let the $n \times n$ upper (lower) triangular matrix over a ring R be a WIN-clean ring and I be the ideal of $\mathbb{T}_n(R)$ which consists of all matrices with zeroes along the main diagonal. Then, I is nilpotent ideal and that $\mathbb{T}_n(R)/I$ is isomorphic to the direct product of n copies of R. By Theorem 3.3, $\mathbb{T}_n(R)/I$ is WIN-clean ring and by Proposition 3.19, the direct product of n copies of R is WIN-clean ring. So, R is WIN-clean ring by Proposition 3.19.

□

Corollary 3.8. Let R be a ring and $a \in R$. Then, a is WIN-clean element if and only if $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_2(R)$ is WIN-clean element.

The proof follows from Corollary 3.7 (v).
We know that every nil-clean ring is WIN-clean ring. Now we adapt Theorem 3 from ([30]) to our case as follows.

Theorem 3.7. Suppose that K is a field. The following statements are equivalent.

(1) $K \cong \mathbb{F}_2$.

(2) For every positive integer n, the matrix ring $M_n(K)$ is WIN-clean with $2 \in J(M_n(K))$.

(3) There exists a positive integer n such that the matrix ring $M_n(K)$ is WIN-clean with $2 \in J(M_n(K))$.

Proof. (1) $\implies$ (2) Suppose $K \cong \mathbb{F}_2$. Then $M_n(K)$ is nil-clean matrix ring by ([30], Theorem 3). By Proposition 3.17, $M_n(K)$ is WIN-clean ring with $2 \in J(M_n(K))$.
(2) $\implies$ (3) It is obvious.
(3) $\implies$ (1) Suppose there exists a positive integer n such that the matrix ring $M_n(K)$ is WIN-clean with $2 \in J(M_n(K))$. Then $M_n(K)$ is nil-clean by Proposition 3.17. Again, by ([30], Theorem 3), $K \cong \mathbb{F}_2$. □

Proposition 3.25. Let D be a division ring and $n \geq 1$. If $M_n(D)$ is WIN-clean, then $|D| \leq 3$.

Proof. For $n = 1$, we observe that D is WIN-clean if and only if $|D| \leq 3$. So, we can assume for $n \geq 2$. Let $A = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M_n(D)$, where $a \in D \backslash \{-1, 0, 1\}$. Then, $a, a+1, 1-a$ are not nilpotents of D implies that $a \in D$ is not a sum a nilpotent element and a weak idempotent element in D which in turn implies that $M_2(D)$ is not WIN-clean by Corollary 3.8. Hence, the claim holds true for $n = 2$. By adapting the proof of ([3], Theorem 3), we conclude that A is not WIN-clean in $M_n(D)$. Hence, $M_n(D)$ is WIN-clean implies $|D| \leq 3$. □

Theorem 3.8. Let D be a division ring and $n \geq 1$. Then, $M_n(D)$ is WIN-clean ring if and only if either $D \cong \mathbb{Z}_2$ or $D \cong \mathbb{Z}_3$.

Proof. Let D be a division ring.
($\implies$) Let $M_n(D)$ be a WIN-clean ring. Then, by Proposition 3.25, $|D| \leq 3$. Hence, either $|D| = 1$ or $|D| = 2$ or $|D| = 3$. It is trivial for $|D| = 1$. For $|D| = 2$, we have $M_n(D)$ is nil-clean if and only if $D = \mathbb{F}_2$ by Theorem 3.7. For $|D| = 3$, assume that D is not isomorphic to $\mathbb{Z}_3$. To get a contradiction, take $a \in D \setminus \{-1, 0, 1\}$. Then, $a, a+1, 1-a$ are not nilpotents of D implies that $a \in D$ is not a sum a nilpotent element and a weak idempotent element. Hence, the claim holds true for $n = 1$. Let us assume that $n \geq 2$. By hypothesis,

$$A := \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = W + N$$

where $W \in wi(M_n(D))$ and $N \in Nil(M_n(D))$. By adapting the proof of ([3], Theorem 3), we conclude that A is not WIN-clean element in $M_n(D)$. Thus, $D \cong \mathbb{Z}_3$.
($\impliedby$) Let $D \cong \mathbb{Z}_2$ or $D \cong \mathbb{Z}_3$. Then $M_n(\mathbb{Z}_2)$ is WIN-clean ring by Theorem 3.7. Next we show that $M_2(\mathbb{Z}_3)$ is WIN-clean ring. For $n = 2$, $M_2(\mathbb{Z}_3) = \begin{pmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ \mathbb{Z}_3 & \mathbb{Z}_3 \end{pmatrix}$ is WIN-clean ring by Example 3.1. For $n > 2$, assume that it is true for $n-1$, that is, $M_{n-1}(\mathbb{Z}_3)$ is WIN-clean ring. Then, consider $M_n(\mathbb{Z}_3) = \begin{pmatrix} M_{n-1}(\mathbb{Z}_3) & M_{(n-1)\times 1}(\mathbb{Z}_3) \\ M_{1\times(n-1)}(\mathbb{Z}_3) & \mathbb{Z}_3 \end{pmatrix}$. Since $\mathbb{Z}_3$ is WIN-clean ring, $M_{(n-1)\times 1}(\mathbb{Z}_3)$ and $M_{1\times(n-1)}(\mathbb{Z}_3)$ are WIN-clean rings by Proposition 3.19, and also $M_{n-1}(\mathbb{Z}_3)$ is WIN-clean ring by induction assumption. So, $M_n(\mathbb{Z}_3)$ is WIN-clean ring because 2×2 matrix over $\mathbb{Z}_3$ is WIN-clean ring. □

3.4 Commutative WIN-clean Rings

In this section, R stands for a commutative ring with unity. Commutative WIN-clean rings is a special class of WIN-clean rings. We characterize reduced commutative WIN-clean rings and also we look at some of the results of commutative WIN-clean rings.

Proposition 3.26. Let $R, R_1, R_2, \cdots, R_m$ be rings. Then,

(1) R is WIN-clean if and only if $R/Nil(R)$ is a reduced WIN-clean ring with all elements of $R/Nil(R)$ are weak idempotents.

(2) $R = \prod_{i=1}^{m} R_i$ is WIN-clean ring if and only if each R_i is WIN-clean.

(3) Let I be a nil ideal of a ring R. Then, R is a WIN-clean ring if and only if R/I is WIN-clean.

(4) The homomorphic image of a WIN-clean ring is WIN-clean.

Proof. Let R be a ring.

(1) Assume that R is a WIN-clean ring. Let $\bar{x} = x + Nil(R) \in R/Nil(R)$ for some $x \in R$. Then, $x = n + w$ for some $n \in Nil(R)$ and $w \in wi(R)$ and hence $\bar{x} = (n + w) + Nil(R) = (n + Nil(R)) + (w + Nil(R)) = w + Nil(R)$. Thus, $w + Nil(R) \in wi(R/Nil(R))$, which implies that $\bar{x}$ is weak idempotent element in $R/Nil(R)$. Since $\bar{x}$ is arbitrary, $R/Nil(R)$ is reduced WIN-clean ring. Conversely, suppose $R/Nil(R)$ is a reduced WIN-clean ring and $r \in R$. Then, $r + Nil(R) = w + Nil(R)$ for some $w + Nil(R) \in wi(R/Nil(R))$. Thus, by Proposition 3.8, the weak idempotent $w + Nil(R)$ can be lifted to a weak idempotent $w \in wi(R)$ such that $r - w = n$ for some $n \in Nil(R)$, that is, $r = n + w$. This shows that r is WIN-clean element. Hence, R is WIN-clean ring.

(2) follows from Proposition 3.19.

(3) follows from Theorem 3.3.

(4) follows from Proposition 3.9.

□

Theorem 3.9. Let R be a reduced ring. Then, the following statements are equivalent.

(i) $R = wi(R)$.

(ii) R is isomorphic to either a Boolean ring B, or $\mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $B \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$.

(iii) For all $x \in R$, $x^4 = x^2$.

(iv) R is a WIN-clean ring.

Proof. For a reduced ring R, $(i) \iff (iii) \iff (iv)$.
Thus, it remains to show the equivalence of (i) and (ii).
$(i) \implies (ii)$: Suppose $R = wi(R)$. If $y \in R$, y^2 is an idempotent. If R is indecomposable, then either $y^2 = 0$ or $y^2 = 1$ for any $y \in R$. This implies that $y = 0$ or $y^2 = 1$ for all $y \in R$. Thus, each nonzero element of R is a unit and hence R is a field. Hence, R is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$.
Next, assume that R is not indecomposable. Let $R = S \times T$ and $s \in S$, where S and T are coprime ideals of R, that is, $S + T = R$. Then, $(s, 0)$ is not a unit implies that either $(s, 0) = (0, 0)$, or $(s, 0)^2 = (0, 0)$, or $(s, 0)^2 = (s, 0)$, or $(s, 0)^2 = (s, 0)^4$ and $(s, 0)^2 \neq (1, 0)$. If $(s, 0) = (0, 0)$ or $(s, 0)^2 = (0, 0)$, then $(s, 0) = (0, 0)$ since S is reduced. In this case, S is a field. So, S is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. If $(s, 0)^2 = (s, 0)$, then $s \in Id(S) \cup [-Id(S)]$ and hence $S = Id(S) \cup [-Id(S)]$. By Theorem 2.2, S is isomorphic to either a Boolean ring, or $\mathbb{Z}_3$, or $B \times \mathbb{Z}_3$, where B is a Boolean ring. The same holds for T. As a direct product of two Boolean rings is a Boolean ring we get R is isomorphic to a Boolean ring B, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times B$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$. If $(s, 0)^2 = (s, 0)^4$ and $(s, 0)^2 \neq (1, 0)$, then let $y = (s, 0)$. Now $R = R(y^2) \oplus R(1 - y^2)$ is the decomposition of R.
Assume that $R(y^2)$ is not a Boolean ring. Then, we show that $R(1 - y^2)$ is Boolean. Suppose ry^2 is not idempotent. Then, for any $s \in R$, $ry^2 + (-s)(1 - y^2)$ is not idempotent. Thus, $-(ry^2 + (-s)(1 - y^2)) = -ry^2 + s(1 - y^2)$ is idempotent. So, each $s(1 - y^2)$ is idempotent. Thus, $R(1 - y^2)$ is Boolean and also $2R(1 - y^2) = 0$. Hence, for each $y \in R$, either $2y^2 = 0$ or $2(1 - y^2) = 0$.
If $(0 : 2) = \{y \in R \mid 2y^2 = 0\} = R$, then $char(R) = 2$. Hence, $R = wi(R) = Id(R)$ and so R is Boolean. Now assume $(0 : 2) \neq R$. Then, we claim that $(0 : 2)$ is a maximal ideal of R. Suppose there is a maximal ideal M such that $(0 : 2) \subseteq M$. Let $y^2 \in M - (0 : 2)$. Then, $y^2 \in wi(R) = R$ and $y^2 \notin (0 : 2)$. Thus, $2y^2 \neq 0$ and hence $2(1 - y^2) = 0$. So, $1 - y^2 \in (0 : 2) \subseteq M$, a contradiction. Hence, $(0 : 2)$ is a maximal ideal. So, $\bar{R} = R/(0 : 2)$ is an indecomposable ring with $\bar{R} = wi(\bar{R})$. By the idea in the first part of this proof, we have that $\bar{R}$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.
Next we show that $2R \cap (0 : 2) = 0$. Assume that $y \in 2R \cap (0 : 2)$. Then, $y = 2s$ and $2y^2 = 0$. But then $y^2 = y^4 = (2s)^4 = 2(2s)^2(2s)^2 = 2y^2 = 0$. If $2R = 0$, then R is Boolean.
Now assume that $2R \neq 0$. If $2R = R$, then $(0 : 2) = 0$ is a maximal ideal of R. Thus, R is a field and hence by the first paragraph of this proof, it is isomorphic to $\mathbb{Z}_3$. If $2R \neq R$, then $R = 2R \oplus (0 : 2)$, where $(0 : 2)$ is a Boolean ring and $2R \cong R/(0 : 2)$ is isomorphic to $\mathbb{Z}_3$ since $2R \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$ by the first paragraph of this proof and $2R \nsubseteq (0 : 2)$. Therefore, R is isomorphic to either a Boolean ring, or $\mathbb{Z}_3$, or $B \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$, where B is a Boolean ring.
$(ii) \implies (i)$: It is obvious. □

Remark 3.13. For a positive integer n, if $\mathbb{Z}_n$ is a reduced WIN-clean ring, then $wi(\mathbb{Z}_n) = \mathbb{Z}_n$. But the converse is not true. For instance, consider the ring $\mathbb{Z}_4$. Then, $wi(\mathbb{Z}_4) = \mathbb{Z}_4$, but $\mathbb{Z}_4$ is not reduced.

The following proposition gives a condition for which a given ring is a union of the set of its nilpotent elements and the set of its weak idempotent elements.

Proposition 3.27. Let R be a ring. Then, $R = Nil(R) \cup wi(R)$ if and only if $R = wi(R)$ or $R = \mathbb{Z}_8$.

Proof. Let R be a ring.
($\Longleftarrow$) If $R = wi(R)$, then $Nil(R) = 0$ and hence $R = 0 \cup wi(R)$.
As $Nil(\mathbb{Z}_8) = \{0, 2, 4, 6\}$ and $wi(\mathbb{Z}_8) = \{0, 1, 3, 4, 5, 7\}$, if $R = \mathbb{Z}_8$, then it is clear that $\mathbb{Z}_8 = Nil(\mathbb{Z}_8) \cup wi(\mathbb{Z}_8)$.
($\Longrightarrow$) Suppose R is a ring with $R = Nil(R) \cup wi(R)$.
If $Nil(R) \subseteq wi(R)$, then $R = Nil(R) \cup wi(R) = wi(R)$.
Suppose $Nil(R) \nsubseteq wi(R)$. Then, there is a nilpotent $0 \neq n \in Nil(R)$ and $n \notin wi(R)$. So, $n^2 \neq 0$. The unit $1-n \in Nil(R) \cup wi(R)$ and a unit element cannot be nilpotent element. If $1 - n$ is weak idempotent, then $1 - n = 1$ or $1 - n = -1$. This implies that $n = 0$ or $n = 2$.
Since n was assumed to be any nonzero nilpotent element with $n^2 \neq 0$, we conclude that $Nil(R) = \{0, 2, 4\}$ and $2^2 \neq 0$. Since $2^3 = 8 \in Nil(R)$ we get that either $8 = 0$ or $8 = 4$. The only possible option is that $8 = 0$. Moreover, we conclude that the characteristic of R is 8.
Next, let $u \in U(R)$. Then, $u \notin Nil(R)$ and so u is weak idempotent, that is , $u^2 = 1$ or $u^{-1} = u$. It follows that $U(R) = \{1, -1, u, -u\}$. Since $char(R) = 8$ we infer that $u \neq -u$. So, we have produced at least 8 elements in R. If $x \in R - \{0, 1, 2, 3, 4, 5, 6, 7\}$, then $x \in wi(R)$. Since $\{0, 1, 2, 3, 4, 5, 6, 7\}$ is closed under negation, it follows that if R has more than 8 elements, then there is weak idempotent $w \in wi(R)$ such that $w \neq 0, 1$.
Consider $1+w$ for any weak idempotent $w \neq 0, 1$. If $1+w = 0$, then $w = -1$ is weak idempotent which has already been established not to be the case. If $1 + w = 2$, then $w = 1$, which we are assuming is not the case. If $-(1 + w) \in wi(R)$, then $[-(1+w)]^2 = [-(1+w)]^4$ implies $2w = 4w^3 + 6w^2 + 4w$ and hence $4 = 0$ for $w = 2$. The same is true for other values of w. Thus, for any weak idempotent w different than 0 and 1, one must have $1+w \in wi(R)$. Also, notice that in this case $1+w \neq 0, 1$. So, we can make the same argument to conclude that $1 + (1 + w) = 2 + w$ is a weak idempotent. This last statement implies that $(2 + w)^2 = (2 + w)^4$ and hence $4 = 0$, which gives the desired contradiction. Consequently, $R = \{0, 1, 2, 3, 4, 5, 6, 7\} = \mathbb{Z}_8$. □

Recall that a ring is said to be zero dimensional if every prime ideal is maximal ideal. The following corollary discusses about WIN-clean reduced indecomposable rings and also WIN-clean rings are clean rings.

Corollary 3.9. Let R be a ring. The following statements are true.

(i) A reduced indecomposable ring is WIN-clean if and only if it is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. In particular, any WIN-clean domain is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

(ii) A WIN-clean ring is zero-dimensional and hence it is a clean ring.

Proof. Let R be a ring.

(i) Suppose R is a reduced indecomposable WIN-clean ring. Then, 0 is the only nilpotent element, and its idempotents are only 0 and 1. Let $w \in wi(R)$. Then, $w^2 \in Id(R)$ implies that $w^2 = 0$ or $w^2 = 1$. If $w^2 = 0$, then w is both weak idempotent and nilpotent. So, $w = 0$. If $w^2 = 1$, then w is a unit and a weak idempotent element. Now we have $R = \{0, 1, w\}$ and hence $w + 1 \in R$ which implies that $w + 1 = 0$ or $w + 1 = 1$, or $w + 1 = w$, because R is closed

under addition. If $w+1=0$, then $w=-1$. In this case, $R=\{0,1,-1\}$ which is isomorphic to $\mathbb{Z}_3$. If $w+1=1$ or $w+1=w$, then $w=0$ as $0\neq 1$.
Hence, $R=\{0,1\}$ which is isomorphic to $\mathbb{Z}_2$ and the converse is clear.

(ii) Let R be a WIN-clean ring and P be a nonzero prime ideal of R. Then, R/P is a WIN-clean domain. So, by (i), the quotient R/P is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. This implies that R/P is a field. Hence, P is a maximal ideal.

□

The following proposition states that for a WIN-clean ring R, $R/Nil(R)$ is isomorphic to either a Boolean ring, or integer modulo 3 or, the direct product of Boolean ring and integer modulo 3, or direct product of two copies of integer modulo 3, or the direct product of two copies of integer modulo 3 and Boolean ring.

Theorem 3.10. Let R be a ring. The following statements are equivalent:

(i) R is a WIN-clean ring.

(ii) R is zero-dimensional and $R/M\cong\mathbb{Z}_3$ for at least one maximal ideal M or $R/N\cong\mathbb{Z}_2$ for at least one other maximal ideal N.

(iii) $R/Nil(R)$ is isomorphic to either a Boolean ring B, or $\mathbb{Z}_3$, or $B\times\mathbb{Z}_3$, or $\mathbb{Z}_3\times\mathbb{Z}_3$, or $\mathbb{Z}_3\times\mathbb{Z}_3\times B$.

(iv) $J(R)$ is nil and $R/J(R)$ is isomorphic to either a Boolean ring B, or $\mathbb{Z}_3$, or $\mathbb{Z}_3\times\mathbb{Z}_3$, or $B\times\mathbb{Z}_3$, or $\mathbb{Z}_3\times\mathbb{Z}_3\times B$.

Proof. Let R be a ring. From Proposition 3.9 (ii) follows that $(i)\iff(iii)$ and also $(iii)\iff(iv)$, because $Nil(R)=J(R)$.
$(i)\implies(ii)$: Let R be a WIN-clean ring. By Corollary 3.9(ii), R is zero-dimensional. For any maximal ideal M, R/M is reduced WIN-clean domain so that either $R/M\cong\mathbb{Z}_2$ or $R/M\cong\mathbb{Z}_3$. By Chinese Remainder Theorem, for any two maximal ideals M and N of R, we have $R/(M\times N)\cong(R/M)\times(R/N)$ which is WIN-clean ring. By Proposition 3.19, R/M and R/N are WIN-clean rings. We can apply Theorem 3.9 to finish the proof.
$(ii)\implies(i)$: Assume that R is zero-dimensional and also there is at least one maximal ideal of R, say M, which satisfies $R/M\cong\mathbb{Z}_3$ or there is at least one maximal ideal N of R such that $R/N\cong\mathbb{Z}_2$. It follows that $R/Nil(R)=R/J(R)$ is embeddable inside of $\prod_{M\in Max(R)}(R/M)$, where $Max(R)=\{M: M \text{ is maximal ideal of } R\}$, which is isomorphic to either a product of copies of $\mathbb{Z}_2$ or a product of copies of $\mathbb{Z}_2$ and copies of $\mathbb{Z}_3$ or a product of copies of $\mathbb{Z}_3$. In all cases we have that $R/Nil(R)$ is a subring of a reduced WIN-clean ring and hence reduced WIN-clean ring. Hence, R is WIN-clean ring by Proposition 3.26 (1). □

Corollary 3.10. A ring R is WIN-clean if and only if

(1) R is a zero-dimensional ring; and

(2) either $R/M\cong\mathbb{Z}_3$ for at least one maximal ideal M containing a nonzero semiprime ideal I of R or $R/N\cong\mathbb{Z}_2$ for at least one other maximal ideal N containing a nonzero semiprime ideal I of R.

Proof. It is an immediate consequence of Theorem 3.10. □

The following lemma describes the relationship between indecomposable WIN-clean and WUU rings.

Lemma 3.7. A ring R is indecomposable WIN-clean if and only if for every element r in R, either $r \in Nil(R)$ or $r \in Uni(R)$ or $r \in -Uni(R)$.

Proof. Let R be a ring.
($\implies$) Suppose R is an indecomposable WIN-clean ring and $r \in R$. Then, $Id(R) = \{0, 1\}$ and $r = n + w$ for some nilpotent n and weak idempotent w. For any weak idempotent w, either $w^2 = 0$ or $w^2 = 1$ which implies that w is either nilpotent or unit with its own inverse. If w is nilpotent, then $r \in Nil(R)$. If w is unit, then $w + 1, w - 1 \in J(R) = Nil(R)$. Thus, $r = n + w = (n + w - 1) + 1$ or $r = (n + w + 1) - 1$. Hence, $r \in Uni(R)$ or $r \in -Uni(R)$.
($\impliedby$) Assume that $r = n$ or $r = n + 1$ or $r = n - 1$ for some nilpotent n and for every r in R. Then, R is weakly nil-clean ring and also $Id(R) = \{0, 1\}$. So, R is indecomposable weakly nil-clean which implies that R is indecomposable WIN-clean ring. □

The following proposition gives the equivalence among a local WIN-clean ring, an indecomposable WIN-clean ring and a weakly unit unipotent rings (denoted by WUU).

Theorem 3.11. The following statements are equivalent for a ring R:

(1) R is a local WIN-clean ring.

(2) R is an indecomposable WIN-clean ring.

(3) For all $x \in R$, either $x \in Nil(R)$, or $x \in Uni(R)$, or $x \in -Uni(R)$.

(4) R is a WUU ring and R has exactly one prime ideal.

Proof. (2) $\iff$ (3). It follows from Lemma 3.7.
(1) $\implies$ (3). Suppose R is a local WIN-clean ring and $r \in R$. Then, 0 and 1 are the only idempotent elements. So, R is indecomposable. By Lemma 3.7, we have either $x \in Nil(R)$, or $x \in Uni(R)$, or $x \in -Uni(R)$.
(3) $\implies$ (1): For each $x \in R$, we have either $x \in Nil(R)$ or $x \in Uni(R)$, or $x \in Uni(R)$. Then, either $x \in J(R)$ or $x \in U(R)$. Hence, R is local ring.
(2) $\implies$ (4). Suppose that R is an indecomposable WIN-clean ring. Then, every weak idempotent is either a nilpotent or a unit. So for every $x \in U(R)$, we have $x = n+w$ for some nilpotent n and weak idempotent w. In this case, w must be unit otherwise $n + w$ will be nilpotent which is impossible. So, $w \in U(R)$ implies that $w \pm 1 \in Nil(R)$ which in turn implies $x = (n+w\pm 1) \mp 1$, where $n+w\pm 1 \in Nil(R)$. Thus, x is a WUU element and hence R is a WUU ring. Now we show that R has exactly one prime ideal. Assume that P_1 and P_2 are nonzero prime ideals of R. Then, R/P_1 and R/P_2 are indecomposable WIN-clean domains. By Proposition 3.26, R/P_1 and R/P_2 are isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Moreover, $R/P_1P_2 \cong R/P_1 \times R/P_2$ by Chinese Remainder Theorem. Thus, $R/P_1P_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $R/P_1P_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ or $R/P_1P_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. So, R/P_1P_2 is not indecomposable since $(\bar{0}, \bar{1}) \in Id(R/P_1P_2)$. Hence, by contrapositive of Proposition 3.10, R is not indecomposable which is a

contradiction.
(4) $\implies$ (2) Assume that R is a WUU ring and P is the only prime ideal of R. Then, all elements of P are nilpotents as $rad(R) = \bigcap P = P$ and also for any $r \in R$, we have either $r = n$ or $r = n' \pm 1$, where $n, n' \in Nil(R)$. Thus, $R/P \cong \mathbb{Z}_3$. So, $Id(R/P) = \{\bar{0}, \bar{1}\}$ implies that $Id(R) = \{0, 1\}$. Hence, R is an indecomposable WIN-clean ring. □

3.5 Commutative WIN-clean Group Rings

In this section, we will see definition of WIN-clean group rings, examples of WIN-clean group rings and characterization of WIN-clean group rings. Throughout this section, R and G stand for a commutative ring with unity and multiplicative abelian group respectively.

Definition 3.9. Let R be a ring and G be a multiplicative abelian group. Then, the group ring $R[G]$ is said to be WIN-clean if each element of $R[G]$ is WIN-clean.

Example 3.10. Consider the ring $\mathbb{Z}_2$ and the cyclic group $C_2 = \{1, x\}$, where $x^2 = 1$. Then, $\mathbb{Z}_2[C_2] = \{a.1 + b.x \mid a, b \in \mathbb{Z}_2\} = \{0, 1, x, 1 + x\}$, $Nil(\mathbb{Z}_2[C_2]) = \{0, 1 + x\}$ and $wi(\mathbb{Z}_2[C_2]) = \mathbb{Z}_2[C_2]$. Hence, $\mathbb{Z}_2[C_2]$ is WIN-clean group ring.

Remark 3.14. Both nil-clean group rings and weakly nil-clean group rings are WIN-clean group rings. But the converse is not true. For instance, Consider the group ring $\mathbb{Z}_3[C_2] = \{a.1+b.x \mid a, b \in \mathbb{Z}_3\} = \{0, 1, 2, x, 2x, 1+x, 2+x, 1+2x, 2+2x\}$. Then, $Nil(\mathbb{Z}_3[C_2]) = \{0\}$ and $wi(\mathbb{Z}_3[C_2]) = \{0, 1, 2, x, x^2, 2x, 1 + x, 2 + x, 2 + 2x\}$. Thus, $\mathbb{Z}_3[C_2]$ is WIN-clean ring. Now $Id(\mathbb{Z}_3[C_2]) = \{0, 1, 2 + x, 2 + 2x\}$ and $\mathbb{Z}_3[C_2]$ is reduced ring, but x can not be written as a sum or a difference of a nilpotent element and an idempotent element of $\mathbb{Z}_3[C_2]$. So, $\mathbb{Z}_3[C_2]$ is neither nil-clean nor weakly nil-clean ring.

Proposition 3.28. Let R be a ring and G be a group. If $R[G]$ is WIN-clean, then R is WIN-clean.

Proof. It follows from the fact that the homomorphic image of a WIN-clean ring is a WIN-clean ring and R is homomorphic image of $R[G]$ through the trace map. □

Remark 3.15. The converse of Proposition 3.28 is not true. For instance, Consider the group ring

$$\mathbb{Z}_2[C_3] = \{a.1+b.x+c.x^2 \mid a, b, c \in \mathbb{Z}_2\} = \{0, 1, x, x^2, 1+x, 1+x^2, x+x^2, 1+x+x^2\},$$

where $C_3 = \{1, x, x^2\}, x^3 = 1$ is the cyclic group.
Then, $\mathbb{Z}_2[C_3]$ is not WIN-clean ring with $Nil(\mathbb{Z}_2[C_3]) = \{0\}$ and $wi(\mathbb{Z}_2[C_3]) = \{0, 1, x + x^2, 1 + x + x^2\}$ but $\mathbb{Z}_2$ is WIN-clean ring.

The following proposition gives the characterization of WIN-clean group rings.

Theorem 3.12. Let R be a ring and G be a group. The group ring $R[G]$ is WIN-clean if and only if one of the following three conditions is satisfied:

(1) G is trivial and R is WIN-clean.

(2) G is non-trivial torsion 2-group and R is nil-clean such that $R/Nil(R)$ is Boolean.

(3) G is non-trivial torsion 3-group and R is WIN-clean with $3 \in Nil(R)$ such that $R/Nil(R)$ is isomorphic to $\mathbb{Z}_3$, or with $(3,3) \in Nil(R)$ such that $R/Nil(R)$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.

(4) $G = C_2$ and $R \cong \mathbb{Z}_3$.

Proof. Let R be a ring and G be a group.
($\Longrightarrow$) Suppose $R[G]$ is a WIN-clean ring. Then, $R[G]$ is clean by Corollary 3.9 (ii) and also G is torsion group by Proposition 2.13. Moreover, R is a WIN-clean ring by Proposition 3.28. So, $R/Nil(R)$ is isomorphic to either a Boolean ring, or $\mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times B$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$ for some Boolean ring B by Theorem 3.10. Now we will see the proof in four cases.
Case 1. Suppose $R/Nil(R)$ is Boolean. Then, R is nil-clean by Corollary 2.10. So, there is a maximal ideal M such that $R/M \cong \mathbb{Z}_2$ by Proposition 3.26. The map $\alpha_{G_2} : R[G] \to R[G/G_2]$ given by $\alpha_{G_2}(\sum_{g\in G} r_g g) = \sum_{g\in G} r_g(g + G_2)$ is natural epimorphism. As $R[G]$ is WIN-clean group ring, so is $\mathbb{Z}_2[G/G_2]$ by Proposition 3.9 and also it is reduced by Corollary 2.11, because $\mathbb{Z}_2$ is reduced and has no prime zero divisor which is the order of an element of G/G_2. Thus, G/G_2 is 2-torsion free by Corollary 2.9. Hence, $G = G_2$, that is, G is a torsion 2-group.
Case 2. Suppose $R/Nil(R) \cong \mathbb{Z}_3$. Then, $3 \in Nil(R)$ and $\mathbb{Z}_3[G]$ is a WIN-clean, because $R[G]$ is a WIN-clean ring. So, $\mathbb{Z}_3[G/G_3]$ is reduced WIN-clean ring as a homomorphic image of $\mathbb{Z}_3[G]$ by Proposition 3.9 and Corollary 2.11. So, G/G_3 is 3-torsion free by Corollary 2.9. Hence, $G = G_3$, that is, G is a torsion 3-group.
Case 3. Assume that $R/Nil(R) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Then, $(3,3) \in Nil(R)$. As $R[G]$ is WIN-clean group ring, $(\mathbb{Z}_3 \times \mathbb{Z}_3)[G]$ is WIN-clean and also $(\mathbb{Z}_3 \times \mathbb{Z}_3)[G/G_3] = \mathbb{Z}_3[G/G_3] \times \mathbb{Z}_3[G/G_3]$ is a WIN-clean ring as a homomorphic image of $(\mathbb{Z}_3 \times \mathbb{Z}_3)[G]$. Thus, by Theorem 3.10, $\mathbb{Z}_3[G/G_3]$ is reduced WIN-clean ring. So, G/G_3 is 3-torsion free by case 2 above and hence $G = G_3$ is the torsion 3-group.
Case 4. Suppose $R/Nil(R) \cong B \times \mathbb{Z}_3$ for some non-trivial Boolean ring B. Then, $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are proper homomorphic images of R. Since $R[G]$ is WIN-clean ring and $(B \times \mathbb{Z}_3)[G] \cong B[G] \times \mathbb{Z}_3[G]$, we have WIN-clean rings $B[G]$ and $\mathbb{Z}_3[G]$. So, by Proposition 3.9, $\mathbb{Z}_2[G/G_2]$ and $\mathbb{Z}_3[G/G_3]$ are WIN-clean rings. From case 1 and case 2 above, we get that $G = G_2 = G_3$. Hence, G is trivial. Following the same procedure for $R/Nil(R) \cong B \times \mathbb{Z}_3 \times \mathbb{Z}_3$, we arrive at the same result.
If $G = C_2$ and $R \cong \mathbb{Z}_3$, then $R[G] \cong \mathbb{Z}_3[C_2] \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ is WIN-clean ring by Theorem 3.10.
($\Longleftarrow$) The first case is obvious. Suppose the second case holds true. Then, $R[G]$ is nil-clean group ring by Theorem 2.11 and hence it is WIN-clean ring. Assume that the third case holds true. If $R/Nil(R) \cong \mathbb{Z}_3$, then $R[G]$ is weakly nil-clean group ring by Theorem 2.11 and so it is WIN-clean group ring. If $R/Nil(R) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and G is torsion 3-group, then $char(R/Nil(R)) = 3$ and $(R/Nil(R))[G] \cong R[G]/(Nil(R)[G]) = \mathbb{Z}_3[G] \times \mathbb{Z}_3[G]$ by Proposition 2.14. This implies that $R[G]/(Nil(R)[G])$ is reduced WIN-clean ring, because $\mathbb{Z}_3[G] \times \mathbb{Z}_3[G]$ is reduced WIN-clean ring. Now $Nil(R[G]) = Nil(R)[G] + I$ is nil ideal of $R[G]$ by Proposition 2.13, where I is the kernel ideal of the congruence $\tilde{p}$ and $Nil(R)$ is nil ideal of R. Thus $Nil(R)[G]$ is nil ideal of $R[G]$ by Proposition 2.14, since $Nil(R)[G] \subseteq Nil(R[G]) = Nil(R)[G] + I$. Hence, the group ring $R[G]$ is WIN-clean by Proposition 3.26 (3). $\square$

The following corollary characterizes the group ring $\mathbb{Z}_n[C_m]$ for positive integer n and m, where $C_m = \{1, x, x^2, \cdots, x^{m-1}\}$, $x^m = 1$.

Corollary 3.11. Consider the ring $\mathbb{Z}_n$ and the cyclic group C_m. Then, the group rings $\mathbb{Z}_2[G_2]$, $\mathbb{Z}_3[G_3]$, $\mathbb{Z}_2[C_{2^k}]$ and $\mathbb{Z}_3[C_{3^k}]$, where k is a positive integer, are WIN-clean.

Proof. $\mathbb{Z}_2[G_2]$ and $\mathbb{Z}_3[G_3]$ are WIN-clean group rings by Theorem 3.12 (2) and (3) respectively, because $\mathbb{Z}_2$ is nil-clean ring with $2 \in Nil(\mathbb{Z}_2)$ and $\mathbb{Z}_3$ is WIN-clean ring with $3 \in Nil(\mathbb{Z}_3)$. Again, the group rings $\mathbb{Z}_2[C_{2^k}]$ and $\mathbb{Z}_3[C_{3^k}]$ are WIN-clean, the reason is that C_{2^k} and C_{3^k} are special types of the groups G_2 and G_3 respectively. □

Remark 3.16.

(i) $\mathbb{Z}_2[G_3]$ is not WIN-clean group ring. For instance, $\mathbb{Z}_2[C_3] = \{a.1 + b.x + c.x^2 \mid a, b, c \in \mathbb{Z}_2\} = \{0, 1, x, x^2, 1 + x, 1 + x^2, x + x^2, 1 + x + x^2\}$ is not WIN-clean ring with $Nil(\mathbb{Z}_2[C_3]) = \{0\}$ and $wi(\mathbb{Z}_2[C_3]) = \{0, 1, x + x^2, 1 + x + x^2\}$.

(ii) $\mathbb{Z}_3[C_2]$ is WIN-clean group ring, but $\mathbb{Z}_3[G_2]$ is not WIN-clean group ring in general. For example, consider the torsion 2-group

$$G = \{1,\ a\ ,\ b\ ,\ c,\ d\ ,\ e\ ,\ f\ ,\ g \mid a^2 = 1,\ b^4 = 1,\ c^2 = 1,\ d^4 = 1,\ e^2 = 1,\ f^2 = 1,\ g^2 = 1\}.$$

Then, the group ring $\mathbb{Z}_3[G_2]$ is reduced by Corollary 2.11, because $\mathbb{Z}_3$ is reduced ring and prime orders of each element of G is not a zero divisor in $\mathbb{Z}_3$. However, $b + d$ is not weak idempotent element in $\mathbb{Z}_3[G_2]$. Hence, $\mathbb{Z}_3[G_2]$ is not a WIN-clean group ring.

Chapter 4

Strongly Weak Idempotent Nil-clean Rings

By a ring R, we mean that R is an associative ring with unity. This chapter deals with strongly weak idempotent nil-clean rings (afterwards we will call them SWIN-clean rings) which generalizes the notion of strongly weakly nil-clean rings. It is a super class of strongly weakly nil-clean rings and a subclass of strongly clean rings. We obtain the necessary and sufficient conditions for strongly weak idempotent nil-clean rings in relation to periodic rings, strongly π-regular rings and strongly clean rings.

4.1 Examples and Properties of SWIN-clean Rings

In this section, we look at the definition, examples and basic properties of SWIN-clean rings. We begin with the definition of SWIN-clean rings.

Definition 4.1. Let R be a ring. Then, an element $a \in R$ is called strongly weak idempotent nil-clean if there exists a nilpotent element n and a weak idempotent element w such that $a = n + w$ and $nw = wn$. A ring R is said to be strongly weak idempotent nil-clean if each element of the ring is strongly weak idempotent nil-clean.

Remark 4.1. In a ring, every weak idempotent element and nilpotent element are SWIN-clean elements.

Example 4.1. Consider the ring $\mathbb{Z}_3$. Then, $\mathbb{Z}_3$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ are SWIN-clean because $wi(\mathbb{Z}_3) = \mathbb{Z}_3$ and $wi(\mathbb{Z}_3 \oplus \mathbb{Z}_3) = \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

Example 4.2. Consider the upper triangular matrix ring,
$T_2(\mathbb{Z}_3) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_3 \right\}$. Then, $Nil(T_2(\mathbb{Z}_3)) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \right\}$

$$wi(T_2(\mathbb{Z}_3)) = \Bigg\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \Bigg\}.$$

Those elements of $T_2(\mathbb{Z}_3)$, which are neither nilpotent nor weak idempotent elements, can be written as a sum of commuting nilpotents and weak idempotents as follows:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \qquad \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix};$$
$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}; \qquad \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Hence, $T_2(\mathbb{Z}_3)$ is SWIN-clean ring.

Remark 4.2. Every SWIN-clean ring is WIN-clean ring, but the converse is not true. For example, $M_2(\mathbb{Z}_2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}_2 \right\}$ is WIN-clean ring, but not SWIN-clean because $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ cannot be written as a sum of commuting nilpotent and weak idempotent elements in $M_2(\mathbb{Z}_2)$.

Remark 4.3. Observe that every strongly weakly nil-clean ring is SWIN-clean ring, but the converse is not true. For example, consider the ring
$M = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \right\}$
which is a subring of the matrix ring $T_2(\mathbb{Z}_3)$.
Then, $Id(M) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, $Nil(M) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ and $wi(M) = M$. This implies M is SWIN-clean ring.
As $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ can not be written as a sum or difference of a nilpotent element and an idempotent element of M, M is not weakly nil-clean.

Remark 4.4. Subring of a SWIN-clean ring may not be SWIN-clean ring. For example, consider the SWIN-clean ring $T_2(\mathbb{Z}_3)$. Then

$$S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} \right\}$$

is a reduced subring of $T_2(\mathbb{Z}_3)$. But not WIN-clean ring. Hence, S is not SWIN-clean ring.

The following proposition states that any positive integral power of a SWIN-clean element of a ring is SWIN-clean element.

Proposition 4.1. Let R be a ring. If x is a SWIN-clean element in a ring R, so is x^m for any positive integer m.

Proof. Let x be a SWIN-clean element. Then, $x = n + w$ for some $n \in Nil(R)$, $w \in wi(R)$ and $nw = wn$. Now we prove by induction on m.
For $m = 2$, $x^2 = (n + w)^2 = w^2 + 2nw + n^2 = w^2 + n(2w + n)$, where $w^2 \in wi(R)$ and $(2w + n)n \in Nil(R)$.
Assume that it is true for $m-1$. Then, $x^{m-1} = (n+w)^{m-1} = \sum_{k=0}^{m-1} \binom{m-1}{k} n^k w^{m-1-k} = \binom{m-1}{0} w^{m-1} + \binom{m-1}{1} nw^{m-2} + \binom{m-1}{2} n^2 w^{m-3} + \cdots + \binom{m-1}{m-1} n^{m-1}$ and

$$\begin{aligned}
x^m =&(n + w)^m = \sum_{k=0}^{m} \binom{m}{k} n^k w^{m-k} = w^m + \binom{m}{1} nw^{m-1} + \cdots + \binom{m}{m-1} n^{m-1} w + n^m \\
=&w^m + \left[\binom{m-1}{0} + \binom{m-1}{1}\right] nw^{m-1} + \cdots + \left[\binom{m-1}{m-2} + \binom{m-1}{m-1}\right] n^{m-1} w + n^m \\
=&\left[\binom{m-1}{0} nw^{m-1} + \cdots + n^m\right] + \left[w^m + \binom{m-1}{1} nw^{m-1} + \cdots + \binom{m-1}{m-1} n^{m-1} w\right] \\
=&\left[w^{m-1} + \binom{m-1}{1} nw^{m-2} + \cdots + n^{m-1}\right] n + \left[w^{m-1} + \cdots + \binom{m-1}{m-1} n^{m-1}\right] w \\
=&x^{m-1} n + x^{m-1} w \text{ by induction assumption.} \\
=&(n' + w')n + (n' + w')w \text{ , where } x^{m-1} = n' + w', \\
&n' = \binom{m-1}{1} nw^{m-2} + \binom{m-1}{2} n^2 w^{m-3} + \cdots + \binom{m-1}{m-1} n^{m-1}, and\ w' = w^{m-1} \\
=&(n'n + w'n + n'w) + w'w \\
=&n'' + w''
\end{aligned}$$

As $n'' \in Nil(R)$ and $w'' \in wi(R)$, x^m is SWIN-clean element. □

In the following proposition, we will see that the Jacobson radical of any SWIN-clean ring is always nil ideal.

Proposition 4.2. Let R be a SWIN-clean ring, Then $J(R) \subseteq Nil(R)$. In particular, if R is SWIN-clean ring, then $J(R)$ is nil.

Proof. Let $a \in J(R)$. Then, $a = n + w$ and $nw = wn$ for some $n \in Nil(R)$ and $w \in wi(R)$. Thus, $(a - w)^k = 0$ for some $k \in \mathbb{N}$. So, $(w - a)^k \in J(R)$. Next we show that $w \in J(R)$.

$$\begin{aligned}
(w - a)^k =&w^k - \binom{k}{1} w^{k-1} a + \binom{k}{2} w^{k-2} a^2 - \binom{k}{3} w^{k-3} a^3 + \cdots + \binom{k}{k} (-1)^k a^k \\
=&w^k - a\left[\binom{k}{1} w^{k-1} - \binom{k}{2} w^{k-2} a + \binom{k}{3} w^{k-3} a^2 - \cdots + \binom{k}{k} (-1)^{k-1} a^{k-1}\right] \\
=&w^k - a\left[\sum_{i=1}^{k} \binom{k}{i} (-1)^{i-1} w^{k-i} a^{i-1}\right] \\
=&w^k - as \text{ , where } s = \sum_{i=1}^{k} \binom{k}{i} (-1)^{i-1} w^{k-i} a^{i-1}
\end{aligned}$$

Now $w^k - as \in J(R)$. Thus, $w^k = (w^k - as) + as \in J(R)$ and hence $w^k \in J(R) \cap wi(R)$. Since $J(R)$ does not contain units and nonzero idempotents, w must be nilpotent. Now $a - w, w \in Nil(R)$ which in turn implies that $a \in Nil(R)$. Hence, $J(R) \subseteq Nil(R)$. □

In the next proposition, we will prove that each element of a SWIN-clean ring induces nilpotent element.

Proposition 4.3. If R is a SWIN-clean ring, then $a - a^3$ is nilpotent element for any $a \in R$.

Proof. Suppose R is a SWIN-clean ring and $a \in R$. Then, $a = n + w$ and $nw = wn$ for some $n \in Nil(R)$ and $w \in wi(R)$. Then, $a^3 = (n+w)^3 = n^3 + 3n^2w + 3nw^2 + w^3$ and hence $a - a^3 = n(1 - n^2 - 3nw - 3w^2) + (w - w^3)$ is nilpotent element because $n(1 - n^2 - 3nw - 3w^2)$ and $(w - w^3)$ are nilpotent elements. □

Chacron ([7]) proved that the direct sum of strongly nil-clean rings and strongly weakly nil-clean rings is strongly weakly nil-clean rings. Using this result as a base, next we prove that the direct sum of strongly weakly nil-clean rings and SWIN-clean rings is SWIN-clean rings.

Proposition 4.4. Let R and S be rings. If R is a SWIN-clean ring and S is a strongly weakly nil-clean ring, then $R \oplus S$ is a SWIN-clean ring.

Proof. Let $A = R \oplus S$ and $(a, b) \in A$. Then, $a = n + w$ for some $n \in Nil(R)$ and $w \in wi(R)$, and $b = n' + e$ or $b = n' - e$ for some $n' \in Nil(S)$ and $e \in Id(S)$. Thus, $(a, b) = (n, n') + (w, e)$ or $(a, b) = (n, n') + (w, -e)$, where $(n, n') \in Nil(A)$ and $(w, \pm e) \in wi(A)$. We also have that $(n, n')(w, \pm e) = (w, \pm e)(n, n')$ because $nw = wn$ and $n'e = en'$. Hence, $R \oplus S$ is SWIN-clean ring. □

Now we give the definition of a quasipolar ring and then we will prove that every WIN-clean ring is quasipolar. For $a \in R$, the commutant and the double commutant of a, denoted by $comm_R(a)$ and $comm^2_R(a)$, are defined by

$$comm_R(a) = \{x \in R : ax = xa\} \text{ and } comm^2_R(a) = \{x \in R : xy = yx \; \forall y \in comm_R(a)\},$$

respectively.
Notation: Let R be a ring. We denote $R^{qnil} = \{a : 1 + xa \in U(R) \; \forall x \in comm_R(a)\}$. We consider the following definition quasinilpotent element and quasipolar element of a ring from ([39]).

Definition 4.2. Let R be a ring.

(1) An element $a \in R$ is said to be quasinilpotent if $a \in R^{qnil}$; and

(2) An element $a \in R$ is said to be quasipolar if there exists $p^2 = p \in comm^2_R(a)$ such that $a + p \in U(R)$ and $ap \in R^{qnil}$.

(3) A ring R is said to be quasipolar if each element of R is quasipolar.

Proposition 4.5. Every SWIN-clean ring is quasipolar.

Proof. Let R be a SWIN-clean ring and $a \in R$. Then, $a = n+w$ for some $n \in Nil(R)$, $w \in wi(R)$ and $nw = wn$. So, $a = n + w = (n + w - 1 + w^2) + (1 - w^2)$, where $1-w^2 \in comm^2(a)$, $n+w-1+w^2 \in U(R)$ and $(1-w^2)a = (1-w^2)n+(w-w^3) \in R^{qnil}$. Hence, R is quasipolar. □

4.2 Homomorphic Images of SWIN-clean rings

This section deals with the various homomorphic images of SWIN-clean rings. Then, we will prove that a ring R is SWIN-clean if and only if $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ is not a homomorphic image of R and $a^k - a^{k+2} \in Nil(R)$ for every $a \in R$ and for some positive integer k. We will also prove that for a SWIN-clean ring R, $Nil(R)$ forms an ideal and $R/Nil(R)$ is a commutative reduced WIN-clean ring. Moreover, we will prove that 6 is central nilpotent element in any SWIN-clean ring.

Proposition 4.6. Let R be a ring and $\{R_i\}$ be rings, where $i = 1, 2, \cdots, n$. Then, the following statements are true.

(1) If R is a SWIN-clean ring, then every homomorphic image of R is SWIN-clean ring.

(2) Let I be a nil ideal of R. Then, R is SWIN-clean ring if and only if R/I is SWIN-clean ring.

(3) The direct product $\prod_{i=1}^{m} R_i$ is SWIN-clean ring if and only if each R_i is SWIN-clean ring for $i = 1, 2, \cdots, m$.

Proof.

(1) It follows from Proposition 3.9 and form the fact that homomorphic images of two commuting elements is commuting elements.

(2) Let R be a ring and I be a nil ideal of R.
($\implies$) Suppose R is a SWIN-clean ring. Then, R/I is SWIN-clean ring as a homomorphic image of a SWIN-clean ring R.
($\Longleftarrow$) Let $a \in R$. Then, $\bar{a} \in \bar{R} = R/I$. By hypothesis, $\bar{a} = \bar{n} + \bar{w}$, where $\bar{n} \in Nil(\bar{R})$, $\bar{w} \in wi(\bar{R})$ and $\bar{n}\bar{w} = \bar{w}\bar{n}$. Since nilpotents and weak idempotents lift modulo nil ideal, $\bar{n}$ and $\bar{w}$ lifted to $n \in Nil(R)$ and $w \in wi(R)$ respectively. So, $a = n + w$. Hence, R is SWIN-clean.

(3) Let R_i be rings for $i = 1, 2, \cdots, m$.
($\implies$) Suppose $R = \prod_{i=1}^{m} R_i$ is SWIN-clean ring. Then, for some k, $R/(\prod_{i=1, i\neq k}^{m} R_i) \cong R_k$ is homomorphic image of R. Hence, R_i is SWIN-clean ring for each i.
($\Longleftarrow$) Assume that each R_i is SWIN-clean ring. Let $x \in R$. Then, $x = (x_1, x_2, \cdots, x_m)$, where $x_i \in R_i$. So, $x_i = n_i + w_i$, where $n_i w_i = w_i n_i$, $n_i \in Nil(R_i)$ and $w_i \in wi(R_i)$. Thus, $n = (n_1, n_2, \cdots, n_m) \in Nil(R)$ and $w = (w_1, w_2, \cdots, w_m) \in wi(R)$ such that

$$n + w = (n_1, n_2, \cdots, n_m) + (w_1, w_2, \cdots, w_m) = (x_1, x_2, \cdots, x_m) = x, \quad \text{and}$$

$$(n_1, n_2, \cdots, n_m)(w_1, w_2, \cdots, w_m) = (w_1, w_2, \cdots, w_m)(n_1, n_2, \cdots, n_m).$$

Hence, R is SWIN-clean ring.

□

The next proposition states that the set of nilpotent elements, $Nil(R)$, of a SWIN-clean ring R form an ideal of R and its quotient ring, $R/Nil(R)$, is reduced strongly WIN-clean ring.

Theorem 4.1. The following statements are equivalent for a ring R.

(1) R is SWIN-clean.

(2) 6 is nilpotent element in R and $R/6R$ is SWIN-clean.

(3) $Nil(R)$ forms an ideal of R and $R/Nil(R)$ is reduced SWIN-clean ring.

Proof. Let R be a ring.
(1) $\Longrightarrow$ (2) Suppose R is a SWIN-clean ring. Then, $2^3-2 \in Nil(R)$ by Proposition 4.3. Thus, $6 \in Nil(R)$ and hence $6R$ is nil ideal of R. So, $R/6R$ is SWIN-clean ring as a homomorphic image of R.
(2) $\Longrightarrow$ (1) Suppose 6 is nilpotent element in R and $R/6R$ is SWIN-clean. Then, $6R$ is nil ideal and $R/6R$ is SWIN-clean ring implies R is SWIN-clean ring by Proposition 4.6 (2).
(1) $\Longrightarrow$ (3) Suppose R is SWIN-clean and $a \in R$. Then, $a - a^3 \in Nil(R)$. By Lemma 2.3, $Nil(R)$ forms an ideal of R and hence $R/Nil(R)$ is reduced SWIN-clean ring.
(3) $\Longrightarrow$ (1) Suppose $Nil(R)$ forms an ideal of R and $R/Nil(R)$ is reduced SWIN-clean ring. This implies that R is SWIN-clean ring by Proposition 4.6 (2). □

Corollary 4.1. The following statements are equivalent for a ring R.

(1) R is SWIN-clean ring.

(2) $R/Nil(R)$ is SWIN-clean ring and $Nil(R)$ forms an ideal of R.

(3) $R/J(R)$ is SWIN-clean ring and $J(R)$ is nil.

Proof. Let R be a ring.
(1) $\Longleftrightarrow$ (2) follows from Theorem 4.1.
(2) $\Longrightarrow$ (3) Assume that $R/Nil(R)$ is SWIN-clean ring and $Nil(R)$ forms an ideal of R. Then, R is SWIN-clean ring by Proposition 4.6(2) and hence $J(R)$ is nil by Proposition 4.2. Therefore, $R/J(R)$ is SWIN-clean ring by Proposition 4.6(2).
(3) $\Longrightarrow$ (1) Let $a \in R$. Then, $\bar{a} \in \bar{R} = R/J(R)$ and hence $\bar{a} - \bar{a}^3 \in \bar{R}$. So, $a - a^3 \in J(R)$. Thus, $J(R)$ is nil. Hence, R is SWIN-clean ring by Proposition 4.6(2). □

Theorem 4.2. Let R be a ring. Then, R is SWIN-clean if and only if $J(R)$ is nil and $R/J(R)$ is isomorphic to a Boolean ring or $\mathbb{Z}_3$ or the direct product of two such rings.

Proof. Suppose R is SWIN-clean ring. Then, 6 is nilpotent by Theorem 4.1. So, $6^k = 0$ for some positive integer k. Thus, $2^kR \cap 3^kR = 0$ and $2^kR + 3^kR = R$. Hence, $R \cong R/2^kR \oplus R/3^kR$ by Chinese Remainder Theorem. By Proposition 4.6, $R_1 = R/2^kR$ and $R_2 = R/3^kR$ are SWIN-clean rings. Then, R_1 is Boolean since $2 \in J(R_1) = 2^kR$ and $J(R_1)$ is nil by Proposition 4.2. If $R_2 \neq 0$, then $3 \in J(R_2) = 3^kR$ and also $2 = 3-1$ is unit in R_2. Thus, R_2 is commutative division ring such that $char(R_2) = 3$. Hence, $R_2 \cong \mathbb{Z}_3$ and $J(R_2)$ is nil by Proposition 4.2. Therefore, we finish the proof by Proposition 4.6(3). The converse is obvious. □

Corollary 4.2. A ring R is a SWIN-clean if and only if $R \cong R_1$, $R \cong R_2$ or $R \cong R_1 \oplus R_2$, where R_1 is strongly nil-clean ring , R_2 is SWIN-clean ring with $3 \in J(R_2)$, $R_1/J(R_1)$ is Boolean with $J(R_1)$ is nil and $R_2/J(R_2) \cong \mathbb{Z}_3$ with $J(R_2)$ is nil.

Proof. It follows from Theorem 4.2. □

The following proposition gives the relationship between SWIN-clean ring and periodic ring.

Proposition 4.7. Let R be a ring. Then, R is SWIN-clean if and only if R is periodic and $R/J(R)$ is SWIN-clean.

Proof. Let R be a ring.
($\Longrightarrow$) Suppose R is a SWIN-clean ring and $a \in R$. Then, $a - a^3 \in Nil(R)$ by Proposition 4.3. Thus, $(a-a^3)^k = 0$ for some positive integer k. Now $a^k = a^{k+1}f(a)$, where $f(a) = \sum_{i=1}^{k} \binom{k}{i}(-1)^{i-1}a^{2i-1}$. Hence, R is periodic by Proposition 2.11.
($\Longleftarrow$) Suppose R is periodic and $R/J(R)$ is SWIN-clean ring. By Proposition 2.9, $J(R)$ is nil. Then, R is SWIN-clean by Corollary 4.1. □

The next proposition states that no SWIN-clean ring has homomorphic image $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ and each element of it induces nilpotent element.

Theorem 4.3. Let R be a ring. Then, R is SWIN-clean if and only if

(1) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ is not a homomorphic image of R; and

(2) For any $a \in R$, there exists k(depend on a) such that $a^k - a^{k+2} \in Nil(R)$.

Proof. Let R be a ring.
($\Longrightarrow$) Suppose R a SWIN-clean. Then,

(1) Assume that R has homomorphic image $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ and $f : R \to \mathbb{Z}_3 \oplus \mathbb{Z}_3$ is an epimorphism. Then, $R \cong S$, where S is a subring of $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. But the subrings of $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ are 0, $\mathbb{Z}_3 \oplus 0$, $0 \oplus \mathbb{Z}_3$, and $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. If $R \cong 0$ or $\mathbb{Z}_3 \oplus 0$ or $0 \oplus \mathbb{Z}_3$, then we are done. Assume that $R \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$. But this is impossible by Corollary 4.2. Hence, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ is not a homomorphic image of R.

(2) Let $a \in R$. Then, $a - a^3 \in Nil(R)$ by Proposition 4.3. Choose $k = 1$, required.

($\Longleftarrow$) Let $a \in R$. Then, $a^k - a^{k+2} \in Nil(R)$ for any positive integer k. Thus, $a^k(1-a^2) \in Nil(R)$. Now $a^{2k}(1-a^2)^k(1-a^2)a^2 \in Nil(R)$ implies that $(a^2-a^4)^{k+1} \in Nil(R)$. So, $a^2 - a^4 \in Nil(R)$. Thus, $2^2 - 2^4 = -2^2 \oplus 3 \in Nil(R)$ since $6 \in Nil(R)$. By Proposition 4.2, we have $R \cong R_1 \oplus R_2$, where $2 \in Nil(R_1)$, $3 \in Nil(R_2)$, $J(R_1)$ and $J(R_2)$ are nil.
For $x \in R_1$, since $\overline{R_1} = R_1/J(R_1)$ is Boolean, $\bar{x} - \bar{x}^2 \in Nil(\overline{R_1})$ by Proposition 2.10. Then, $x - x^2 \in Nil(R_1)$ and hence $x(x-x^2) = x - x^3 \in Nil(R_1)$. By Proposition 2.4, R_1 is strongly weakly nil-clean with $2 \in Nil(R_1)$. Hence, R_1 is strongly nil-clean.
Let $y \in R_2$. Then, $y^k - y^{k+2} \in Nil(R_2)$. Thus, $y^k(1 - y^2) \in Nil(R_2)$ and hence $y^k(1 - y^2)^k(y - y^3) \in Nil(R_2)$ which implies that $y - y^3 \in Nil(R_2)$. By Proposition 2.3, $Nil(R_2)$ forms an ideal of R_2. Then, $J(R_2) = Nil(R_2)$ is nil. Thus, $\bar{y} = \bar{y}^3$ in $R_2/J(R_2)$. Let $M \in Max(R_2)$. Then, $R_2/M \cong (R_2/J(R_2))/(M/J(R_2))$. So, for any $\bar{d} \in R_2/M$, we have $\bar{d} = \bar{d}^3$. Hence, R_2/M is a commutative simple ring by

Theorem 2.6 and also it is a field with 2 invertible. Hence, $R_2/M \cong \mathbb{Z}_3$. Construct a ring morphism $\phi : R_2/J(R_2) \to \prod_{M\in Max(R_2)} R_2/M, x + J(R_2) \mapsto (x + M_i)$. Then, ϕ is injective. Since $\mathbb{Z}_3$ is simple, it follows that $R_2/J(R_2) \cong \prod_{M_i\in Max(R_2), i\in I} R_2/M_i$. By hypothesis, R has no homomorphic image $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, and hence $R_2/J(R_2)$ is not isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. So, $|I| = 1$. Therefore, $R_2/J(R_2) \cong \mathbb{Z}_3$. By Corollary 4.2, R is SWIN-clean. □

Corollary 4.3. Let R be a ring. Then, R is SWIN-clean if and only if

(1) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ is not a homomorphic image of R; and

(2) for any $a \in R$, $a^2 - a^4 \in Nil(R)$.

Proof. Let R be a ring.
($\Longrightarrow$) Suppose R is SWIN-clean. Then,
(1) is obvious by Theorem 4.3.
(2) Let $a \in R$. Then, $a - a^3 \in Nil(R)$. Thus, $a^2 - a^4 = a(a - a^3) \in Nil(R)$, since $Nil(R)$ forms an ideal of R.
($\Longleftarrow$) The result follows from Theorem 4.3 by putting $k = 2$. □

The following corollary states that each element of a SWIN-clean ring can be written as a sum of a weak idempotent element and two nilpotent elements that commute.

Corollary 4.4. Let R be a ring. Then, R is SWIN-clean if and only if

(1) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ is not a homomorphic image of R; and

(2) Every element of R is the sum of a weak idempotent element and two nilpotent elements that commute.

Proof. Let R be a ring.
($\Longrightarrow$) Suppose R is SWIN-clean. Then,

(1) follows from Theorem 4.3.

(2) Let $a \in R$. Then, there exist a weak idempotent $w \in wi(R)$ and a nilpotent $n \in Nil(R)$ such that $a - 6 = n + w$. Hence, $a = 6 + n + w$, as desired.

($\Longleftarrow$) Suppose (1) and (2) hold true and $a \in R$. Then, by hypothesis, $a = n+b+w$ for some $n, b \in Nil(R)$ and $w \in wi(R)$. Then, $a^2 = (n+b+w)^2 = (n+b)^2+2w(n+b)+w^2$, $a^4 = (n+b+w)^4 = (n+b)^4 + 4(n+b)^3w + 6(n+b)^2w^2 + 4(n+b)w^3 + w^4$ and hence $a^2 - a^4 = (b+n)[(n+b) + 2w - (n+b)^3 - 4(b+n)^2w - 6(b+n)w^2 - 4w^3] \in Nil(R)$. Hence, R is SWIN-clean by Corollary 4.3. □

The next corollary states that $a^2 \in R$ is strongly nil-clean for every a in a SWIN-clean ring R.

Corollary 4.5. Let R be a ring. Then, R is SWIN-clean if and only if

(1) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ is not a homomorphic image of R; and

(2) for any $a \in R$, $a^2 \in R$ is strongly nil-clean.

Proof. Let R be a ring.
($\Longrightarrow$) Suppose R is a SWIN-clean ring. Then,

(1) follows from Theorem 4.3.

(2) Let $a \in R$. Then, there exist a weak idempotent $w \in R$ and a nilpotent $n \in Nil(R)$ such that $a = n + w$ and $nw = wn$. Thus, $a^2 = (w+n)^2 = w^2 + 2wn + n^2 = w^2 + n(2w+n)$, as required.

($\Longleftarrow$) Suppose (1) and (2) are true. By hypothesis, $a^2 = e + n$ and $ne = en$ for some $e \in Id(R)$ and $n \in Nil(R)$. Now $a^4 = (n+e)^2 = n^2 + 2ne + e$. So, $a^2 - a^4 = (e+n) - (e+2ne+n^2) = n(1-2e-n) \in Nil(R)$. Hence, R is SWIN-clean by Corollary 4.3. □

4.3 Strongly π-regular and Strongly Clean Rings

In this section, we investigate the various rings such as strongly π-regular rings, strongly clean rings and rings with nil-involution property. The following proposition gives a relationship between SWIN-clean elements and strongly clean elements of a ring.

Proposition 4.8. Let R be a ring. An element $a \in R$ is a SWIN-clean if and only if a is strongly clean in R and $a - a^3$ is nilpotent.

Proof. Let R be a ring.
($\Longrightarrow$) Suppose an element a in R is SWIN-clean and $a = n + w$ is a SWIN-clean decomposition in R, where $n \in Nil(R)$ and $w \in wi(R)$. Then, $a = (n + w - 1 + w^2) + (1 - w^2)$ is a strongly clean decomposition in R. Moreover,

$$a^3 = (n+w)^3 = n^3 + 3n^2w + 3nw^2 + w^3, \text{ and so}$$

$$a - a^3 = (n+w) - (n^3 + 3n^2w + 3nw^2 + w^3) = n(1 - n^2 - 3nw - 3w^2) + (w - w^3).$$

Hence, $a - a^3$ is nilpotent element.
($\Longleftarrow$) Suppose $a \in R$ is a strongly clean element and $a - a^3$ is nilpotent element. Let $a = u + e$ be a strongly clean decomposition in R and $a - a^3$ be a nilpotent element. Then,

$$a^3 = (u+e)^3 = u^3 + 3u^2e + 3ue + e, \text{ and hence}$$

$$a - a^3 = (u+e) - (u^3 + 3u^2e + 3ue + e) = u(1 - u^2 - 3ue - 3e).$$

It follows that $1 - u^2 - 3ue - 3e$ is nilpotent element. Take $u^2 + u + 4e + 3ue - 1$ as weak idempotent element. Now we claim that these satisfy a clean decomposition of a.

$$\begin{aligned} n + w - 1 + w^2 =& (1 - 3e - 3ue - u^2) + (u^2 + u + 4e + 3ue - 1) - 1 + \\ & (u^2 + u + 4e + 3ue - 1)^2 \\ =& (u + e - 1) + u^4 + 2u^3 - u^2 - 2u + 1 + 6u^3e + 23u^2e + 26ue + 8e \\ =& u^4 + 2u^3 - u^2 - u + 6u^3e + 23u^2e + 26ue + 9e \\ 1 - w^2 =& 1 - [u^2 + u + 4e + 3ue - 1]^2 \\ =& - u^4 - 2u^3 + u^2 + 2u - 1 - 6u^3e - 23u^2e - 26ue - 8e \end{aligned}$$

Thus, $n+w-1+w^2$ is a unit element of R, $1-w^2$ is an idempotent element of R and

$$\begin{aligned}(n+w-1+w^2)+(1-w^2) =&(u^4+2u^3-u^2-u+6u^3e+23u^2e+26ue+9e)+\\ &(-u^4-2u^3+u^2+2u-1-6u^3e-23u^2e-26ue-8e)\\ =&u+e=a\end{aligned}$$

Hence, a is SWIN-clean element in R. □

The next proposition gives elementwise characterization of SWIN-clean rings.

Proposition 4.9. Any SWIN-clean element is strongly π-regular.

Proof. It can be proved elementwise as follows. If n is nilpotent and w is a weak idempotent such that $nw = wn$ and $a = n+w$, then $a = (n+w-1+w^2)+(1-w^2)$ is a strongly π-regular decomposition of a. □

The following proposition states that SWIN-clean elements have unique SWIN-clean decomposition.

Proposition 4.10. If an element of a ring is SWIN-clean, then it has precisely one SWIN-clean decomposition.

Proof. By Proposition 4.9, any SWIN-clean decomposition is automatically strongly π-regular decomposition. So, the result holds by Proposition 2.5. □

The following proposition gives the relationship between a SWIN-clean element and its nilpotent summand of strongly π-regular element from its strongly π-regular decomposition.

Theorem 4.4. Let R be a ring and $a \in R$. Suppose that a is strongly π-regular with strongly π-regular decomposition $a = u + e$, where $u \in U(R)$ and $e \in Id(R)$. Then, a is SWIN-clean element of R if and only if $1 - u^2 - 3ue - 3e$ is nilpotent element of R.

Proof. Let R be a ring. Suppose a is a SWIN-clean element of R. Then, $a = n + w$ and $nw = wn$, where $n \in Nil(R)$ and $w \in wi(R)$. Thus, $a = (n+w-1+w^2)+(1-w^2)$ is strongly π-regular decomposition. By Proposition 2.5, $u = n + w - 1 + w^2$ and $e = 1 - w^2$. Then,

$$\begin{aligned}1-u^2-3ue-3e =&1-(n+w-1+w^2)^2-3(n+w-1+w^2)(1-w^2)-3(1-w^2)\\ =&1-n^2+2n-2nw-2nw^2+2w-1-2w^3-3n+3nw^2-3w+3\\ &+3w^3-3w^2-3+3w^2\\ =&n(w^2-n-1-2w)+(w^3-w).\end{aligned}$$

Thus, $n(w^2-n-1-2w)$ and w^3-w are nilpotent elements and their sum is also a nilpotent element, because n commutes with w. The converse follows directly from the proof of Proposition 4.8. □

The following corollary follows from Theorem 4.4 and it states that a unit element of a ring is SWIN-clean if and only if its square is a nilpotent element plus 1, which is unipotent.

Corollary 4.6. Let R be a ring. A unit $u \in R$ is SWIN-clean if and only if it is a square root of unipotent.

Proof. Suppose $u \in U(R)$ is SWIN-clean. Then, $u = 0 + u$ is strongly π-regular decomposition of u. By Theorem 4.4, $1 - u^2$ is nilpotent element. So, $1 - u^2 = n$, where n is nilpotent element. Hence, $u^2 = -n+1$. Conversely, assume that $a \in U(R)$ and $a^2 = n + 1$ is unipotent. Then, $1 - a^2 = -n$. By Theorem 4.4, we have $e = 0$ and hence $a = 0 + a$ is strongly π-regular decomposition of a. Hence, a is SWIN-clean. □

The following proposition generalizes Proposition 4.8 and Proposition 4.9.

Theorem 4.5. Every SWIN-clean ring R is a strongly π-regular and also strongly clean.

Proof. Its proof follows from Proposition 4.8 and Proposition 4.9. □

The following corollary generalizes Proposition 4.9 and Corollary 4.6.

Corollary 4.7. Let R be a ring. A unit element u of R is SWIN-clean if and only if R is strongly π-regular and every unit of R is a square root of unipotent.

Proof. The forward direction holds by Proposition 4.9 and Corollary 4.6. For the reverse direction, let $a \in R$. By hypothesis, we have a strongly π-regular decomposition $a = u + e$. We claim that $-u^2 - 3ue - 3e$ is a square root of a unit, that is, $-u^2 - 3ue - 3e = a^2$, where a is a unit.
We consider the Peirce decomposition with respect to e with which $-u^2 - 3ue - 3e$ commutes. Then, $(-u^2 - 3ue - 3e)(1 - e) = -u^2(1 - e) - 3ue(1 - e) = -u^2(1 - e)$ is a unit in $(1 - e)R(1 - e)$ and hence $-u^2 - 3ue - 3e$ is a square of a unit u in R. Moreover,

$$\begin{aligned} e(-u^2 - 3ue - 3e) =& e[(-u^2 - 3ue) - 3e] = e[-u(u + 3e) - 3e] \\ =& e[-u(a + 2e) - 3e] = -eu(a + 2e) - 3e \\ =& - u(ea) - 2ue - 3e = -u(ea) + e(-2u - 3) \\ =& - u(ea) + e(-2(u + 1) - 1) \end{aligned}$$

Thus, $-u(ea) + e(-2(u + 1) - 1)$ is unipotent in eRe. Since $-u^2 - 3ue - 3e$ is unit, $1 - u^2 - 3ue - 3e$ must be nilpotent. Hence, by Theorem 4.4, a is SWIN-clean. □

The next Corollary follows from Propositions 4.8 and 4.6.

Corollary 4.8. Let R be a ring and $a \in R$. The following statements are equivalent.

(1) a is SWIN-clean.

(2) a is strongly π-regular and $a - a^3$ is a nilpotent.

(3) a is uniquely strongly clean and $a - a^3$ is a nilpotent.

Proof. Let R be a ring.
(1) $\implies$ (2) Assume that a is SWIN-clean. Then, a is strongly π-regular by Theorem 4.5 and $a - a^3$ is nilpotent by Proposition 4.3.
(2) $\implies$ (3) Suppose a is strongly π-regular and $a - a^3$ is a nilpotent. Then, by

Proposition 2.5 and 4.10, a is uniquely strongly clean.
(3) $\implies$ (1) Let $a = u + e$ and $ue = eu$, where $u \in U(R)$ and $e \in Id(R)$. Then, $a - a^3 = (u + e) - (u^3 + 3u^2e + 3ue + e) = u(1 - u^2 - 3ue - 3e)$ is nilpotent. So, $1 - u^2 - 3ue - 3e$ is nilpotent. Hence, by Theorem 4.4, a is SWIN-clean. □

Corollary 4.9. A SWIN-clean ring is uniquely strongly clean, that is, every element is uniquely strongly clean.

The following proposition generalizes Proposition 4.10, Theorem 4.5 and Corollary 4.8.

Theorem 4.6. Let a be a SWIN-clean element of R. Then,

(1) a has a unique SWIN-clean decomposition in R;

(2) a is a strongly π-regular element of R; and

(3) a is a uniquely strongly clean element of R.

Proof.

(1) Let $a = n_1 + w_1$ and $a = n_2 + w_2$ be two SWIN-clean decompositions in R. Then, $w_1 = (n_2 - n_1) + w_2$, but this is impossible because $n_2 - n_1$ not necessarily nilpotent element.

(2) Suppose a is SWIN-clean element of a ring R. Then, $a = n + w$ and $nw = wn$, where $n \in Nil(R)$ and $w \in wi(R)$. Now $a = n+w = (n-1+w+w^2)+(1-w^2)$. Let $u = n - 1 + w + w^2$ and $e = 1 - w^2$. Then,

$$ae = (n + w)(1 - w^2) = n(1 - w^2) + w(1 - w^2) = n - nw^2 + w - w^3$$

$$ea = (1 - w^2)(n + w) = (1 - w^2)n + (1 - w^2)w = n - nw^2 + w - w^3$$

Thus, $ae = ea$. So, $eae = ea = (1 - w^2)(n + w) = (1 - w^2)n + (w - w^3)$ is nilpotent. By Proposition 2.4, a is strongly π-regular element and also a is strongly clean. Hence, R is strongly π-regular and strongly clean.

(3) By Proposition 4.8, a is strongly clean element and $a - a^3$ is nilpotent element in R. By proof of Proposition 4.8, two different idempotents which give strongly clean decompositions of a must yield two different idempotents which give SWIN-clean decompositions of a. But this is impossible. Thus, a is uniquely strongly clean.

□

The following proposition states the relationship between SWIN-clean rings and the set of unipotent elements.

Proposition 4.11. A ring R is SWIN-clean and $2 \in J(R)$ if and only if R is strongly π-regular ring and $U(R) = -1 + Nil(R)$.

Proof. Let R be a ring.
$(\implies)$ Suppose R is SWIN-clean and $a \in R$. Then, its WIN-clean decomposition is $a = n + w$, where $n \in Nil(R)$ and $w \in wi(R)$. So, its strongly π-regular decomposition is $a = (n + w - 1 + w^2) + (1 - w^2)$ by Proposition 4.9.
Next we show that $n + w + w^2$ is nilpotent. Now $(w + w^2)^2 = 2(w^2 + w^3) \in J(R)$ since $2 \in J(R)$. As $J(R)$ is nil, $[(w + w^2)^2]^k = 0$ for some positive integer k. So, $(w + w^2)^{2k} = 0$. This implies that $w + w^2$ is nilpotent in R. As R is SWIN-clean, $n_1 = n + (w + w^2)$ is nilpotent. Hence, $(n + w + w^2) - 1 = n_1 - 1 = u$ for some unit u in R.
$(\impliedby)$ Suppose R is strongly π-regular and $U(R) = -1 + Nil(R)$. Let $r \in R$. Then, its strongly π-regular decomposition is $r = u + e$, $ue = eu$, $re = er$ and er is nilpotent. By assumption, $u = n - 1$ for some $n \in Nil(R)$ so that $r = n - 1 + e = n + (e - 1)$. Also, $(e - 1)^2 = 1 - e = (e - 1)^4$ and hence $e - 1 \in wi(R)$. Moreover, $(e - 1)n = en - n = ne - n = n(e - 1)$ since $eu = ue$ and $u = n - 1$. Thus, r is SWIN-clean element. Hence, R is SWIN-clean ring.
Take $1 \in R$. Then, $1 = u + e$ implies $1 - u = e$ and hence $1 - u = e = e^2 = (1 - u)^2$. Thus, $u^2 = u$ implies $u = 1$ and hence $e = 0$. So, $1 = u + e = n - 1$, that is, $n = 2$. Hence, $2 \in J(R)$. □

The following proposition states that the inverse image of a SWIN-clean element is SWIN-clean.

Proposition 4.12. Let R be a ring and I be a nilpotent ideal of R. Let $\bar{R} = R/I$ and $a \in R$. If $\bar{a}$ is SWIN-clean in $\bar{R}$, then a is SWIN-clean in R.

Proof. Let R be a ring and I be a nilpotent ideal of R. Suppose $\bar{a}$ is SWIN-clean. Then, we may write $\bar{a} = \bar{n} + \bar{w}$ for some nilpotent $\bar{n}$ and weak idempotent $\bar{w}$ which commute. By Proposition 2.1, $\bar{a} = \overline{n + w - 1 + w^2} + \overline{1 - w^2}$ is a strongly π-regular decomposition of $\bar{a}$. By Lemma 4.9, there exists an idempotent f, lifting $1 - w^2$, and a unit u such that $a = f + u$ is a strongly π-regular decomposition in R by Lemma 2.1. By Theorem 4.4, we need only show that $1 - u^2 - 3uf - 3f$ is nilpotent. Since $\bar{f} = \overline{1 - w^2}$ and $\bar{u} = \overline{n + w - 1 + w^2}$, we can calculate that
$\overline{1 - u^2 - 3uf - 3f} = \overline{1 - (n + w - 1 + w^2) - 3(n + w - 1 + w^2)(1 - w^2) - 3(1 - w^2)}$
$= \overline{n(w^2 - n - 1 - 2w) + (w^3 - w)} = \overline{n_1}$. Since n_1 is nilpotent modulo I, $1 - u^2 - 3uf - 3f$ is nilpotent. □

Corollary 4.10. Suppose that R is a ring with a nilpotent ideal I. Then, R is SWIN-clean if and only if R/I is SWIN-clean.

Proof. Let R be a ring with a nilpotent ideal I.
$(\implies)$ It follows from Proposition 4.6(2).
$(\impliedby)$ It is an immediate consequence of Proposition 4.12. □

Chapter 5

Commutative Weak Idempotent Nil-neat Rings

Throughout this chapter, R stands for a commutative ring with unity. The notion of weakly nil-neat rings was introduced by Danchev ([20]), as a subclass of the class of neat rings and we can observe that every weakly nil-neat ring is a neat ring. In this chapter, we will initiate the study of weak idempotent nil-neat rings as a generalization of the class of weakly nil-neat rings, which is a subclass of neat rings. Weak idempotent nil-neat rings are a further generalization of weak idempotent nil-clean rings. So, the fifth chapter is a generalization of Chapters 3 and 4. We have extended many of the properties of weakly nil-neat rings to the class of weak idempotent nil-neat rings.

5.1 Commutative Weak Idempotent Nil-neat Rings

In this section, we introduce the notion of weak idempotent nil-neat rings (hereafter we call them WIN-neat rings) and furnish certain examples. We will characterize WIN-neat rings in terms of direct product of Boolean rings and $\mathbb{Z}_3$. Also, we establish the relationship between weak idempotent nil-clean and WIN-neat rings. Moreover, we obtain a complete classification of the WIN-neat rings. By a proper homomorphic image of a ring R we mean R/I where I is a nonzero ideal of R.

Definition 5.1. A ring R is called a WIN-neat if every proper homomorphic image of R is weak idempotent nil-clean.

Example 5.1.

(1) $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ are WIN-neat rings.

(2) $\mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_{10} \times \mathbb{Z}_2 \times \mathbb{Z}_3$ are not WIN-neat rings, because they contain homomorphic images $\mathbb{Z}_5 \times \mathbb{Z}_3$ and $\mathbb{Z}_{10} \times \mathbb{Z}_2$, respectively, which are not WIN-clean rings.

WIN-neat rings are neat rings, because WIN-clean rings are clean rings. However, the converse is not true. For example, consider the ring $\mathbb{Z}_{(3)} \times \mathbb{Z}_3$. Then, it is neat ring, because $(\mathbb{Z}_{(3)} \times \mathbb{Z}_3)/\mathbb{Z}_3$ and $(\mathbb{Z}_{(3)} \times \mathbb{Z}_3)/\mathbb{Z}_{(3)}$ are clean rings. But it is not WIN-neat ring, the reason is that $(\mathbb{Z}_{(3)} \times \mathbb{Z}_3)/\mathbb{Z}_3$ is not WIN-clean ring.

Proposition 5.1. A homomorphic image of a WIN-neat ring is again WIN-neat ring.

Proof. The proof follows from Definition 5.1 and Proposition 3.9. □

Proposition 5.2. Let R be a ring. If R is a WIN-neat ring which is not WIN-clean, then R is reduced.

Proof. Let R be a ring. Assume, on the contrary, that R is a WIN-neat ring which is not a WIN-clean ring and $Nil(R) \neq 0$. Thus, by Definition 5.1, $R/Nil(R)$ is WIN-clean ring and by Proposition 3.26 (1), R is WIN-clean, but this is a contradiction. Hence, R is reduced ring. □

The following proposition states the condition for which a ring R is WIN-neat ring and WIN-clean ring.

Proposition 5.3. Let R be a decomposable ring. Then, R is a WIN-neat ring if and only if R is WIN-clean.

Proof. Suppose R is a decomposable ring. Then, there are nonzero ideals I and J such that $R = I \times J$. If R is a WIN-neat, then $I \cong R/J$ and $J \cong R/I$ are WIN-clean rings. Thus, R is a direct product of WIN-clean rings. Hence, by Proposition 3.26 (2), R is WIN-clean. Conversely, assume that R is a WIN-clean ring and I is the nonzero ideal of R. Then, R/I is WIN-clean by Proposition 3.26 (3). Therefore, R is the WIN-neat ring. □

The following lemma induces an equivalence among a WIN-neat ring R, R/aR for every nonzero a in R, and R/I for every nonzero semiprime ideal I of R.

Lemma 5.1. Let R be a ring. Then, the following statements are equivalent.

(1) R is WIN-neat ring.

(2) R/aR is WIN-clean ring for every nonzero $a \in R$.

(3) For any collection of nonzero prime ideals $\{P_j\}_{j\in J}$ of R with $I = \cap_{j\in J} P_j \neq 0$, the factor ring R/I is WIN-clean.

(4) R/aR is WIN-neat for every $a \in R$.

(5) R/I is WIN-clean for every nonzero semiprime ideal I.

(6) $R/I = wi(R/I)$ for every nonzero semiprime ideal I.

(7) R/I is isomorphic to either Boolean, or $\mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times B$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$ for some Boolean ring B and for every nonzero semiprime ideal I.

(8) For every nonzero semiprime ideal I of R, the factor ring R/I is zero dimensional and $R/P \cong \mathbb{Z}_3$ for at least one maximal ideal P containing I, or $R/Q \cong \mathbb{Z}_2$ for at least one other maximal ideal Q containing I.

(9) For every nonzero semiprime ideal I of R it must be that $J(R/I) = 0$ and R/I is isomorphic to either a Boolean ring B, or $\mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times B$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$.

Proof. (1) $\iff$ (2) follows from Definition 5.1 and also from the fact that homomorphic image of a WIN-clean ring is WIN-clean ring and any nontrivial ideal contains a principal nontrivial ideal.
(1) $\implies$ (4) For a nonzero element a in R, R/aR is the proper homomorphic image of R. So, R/aR is WIN-clean ring.
(4) $\implies$ (1) is clear by choosing $a = 0$.
(3) $\iff$ (5) is instant since intersection of any family of prime ideals is semiprime ideal.
(1) $\implies$ (5) is obvious.
(5) $\implies$ (1) Suppose I is a nonzero ideal of R. Then, $\sqrt{I}$ is semiprime ideal of R. By assumption, $R/\sqrt{I}$ is WIN-clean ring and hence $R/\sqrt{I} \cong (R/I)/(\sqrt{I}/I)$ is WIN-clean. Since $I \subseteq \sqrt{I}$ and $\sqrt{I} = \{a \in R : a+I \text{ is nilpotent in } R/I\}$, $\sqrt{I}/I = \sqrt{I}$ and $\sqrt{I}/I$ is nil ideal of R/I. By Proposition 3.26 (3), R/I is WIN-clean ring. Hence, R is WIN-neat ring.
(5) $\implies$ (6). Suppose R/I is WIN-clean for every nonzero semiprime ideal I. Let $a + I$ be a nilpotent element of R/I. Thus, $(a + I)^k = I$ for some $k \in \mathbb{Z}$. So, $a^k + I = I$ implies that $a^k \in I$ which in turn implies that $a \in \sqrt{I}$. As $I = \sqrt{I}$, $a \in I$, that is, $a + I = I$. Hence, $R/I = wi(R/I)$. The converse is obvious.
(6) $\iff$ (7) follows from Theorem 3.9.
(1) $\iff$ (8) $\iff$ (9) is clear using Theorem 3.9. □

Remark 5.1. Every WIN-clean ring is WIN-neat, but the converse is not true. It is illustrated by the following example.

Example 5.2. Consider the ring of the localization of integers at the prime ideal (3), $R = \mathbb{Z}_{(3)}$. Then, $0_{(3)} = \{0\}$, $2_{(3)}$ and $3_{(3)}$ are the only prime ideals of $\mathbb{Z}_{(3)}$. Now $\mathbb{Z}$ is an ideal of R such that $0_{(3)} \subset \mathbb{Z} \subset R$. So, $0_{(3)}$ is not maximal ideal of R and hence R is not zero dimensional. By Corollary 3.9 (ii), R is not a WIN-clean ring. Define a homomorphism $\alpha : \mathbb{Z}_{(3)} \to \mathbb{Z}_3$ by $\alpha(\frac{m}{n}) = \bar{0}$ if $m \equiv 0(mod\ 3)$; $\alpha(\frac{m}{n}) = \bar{1}$ if $m \equiv 1(mod\ 3)$ and $\alpha(\frac{m}{n}) = \bar{2}$ if $m \equiv 2(mod\ 3)$. Then, it can be verified that α is an epimorphism. Thus, $ker\alpha = 3_{(3)}$ and $\mathbb{Z}_{(3)}/3_{(3)} \cong \mathbb{Z}_3$ by first isomorphism theorem. Since every prime ideal is semiprime ideal, $3_{(3)}$ is semiprime. Also, $\mathbb{Z}_3$ is WIN-clean ring. Hence, by Lemma 5.1, $\mathbb{Z}_{(3)}$ is WIN-neat ring.

Corollary 5.1. A ring R is WIN-neat if and only if

(i) Every nonzero prime ideal of R is maximal.

(ii) For any nonzero semiprime ideal I of R, either $R/M \cong \mathbb{Z}_3$ for at least one maximal ideal M containing I or $R/N \cong \mathbb{Z}_2$ for at least one other maximal ideal N containing I.

Proof. ($\implies$). Suppose R is WIN-neat ring.
(*i*) Let P be a nonzero prime ideal of R. Then, R/P is the WIN-clean domain. By Corollary 3.9 (i), either $R/P \cong \mathbb{Z}_2$ or $R/P \cong \mathbb{Z}_3$. This implies that R/P is a field. Hence, P is a maximal ideal.
(*ii*) Assume that I is a nonzero semiprime ideal of R. Then, R/I is a WIN-clean ring by Lemma 5.1(5). Hence, the result obtained from Lemma 5.1(8).
($\Longleftarrow$). It is obvious by using Lemma 5.1 ((8) $\implies$ (1)) and Theorem 3.10. □

Proposition 5.4. If R is a reduced WIN-clean ring, then either $|U(R)| = 1$ or $|U(R)| = 2^{|\lambda|}$ for some ordinal λ.

Proof. Suppose R is reduced WIN-clean ring. Then, by proposition 5.1, we have R is isomorphic to either a Boolean ring B, or $\mathbb{Z}_3$, or $B\times\mathbb{Z}_3$, or $\mathbb{Z}_3\times\mathbb{Z}_3$, or $\mathbb{Z}_3\times\mathbb{Z}_3\times B$. From this, we observe that $|U(R)|=1$, or $|U(R)|=2^{|\lambda|}$. Hence, this is the required one. □

The following proposition characterizes WIN-neat rings.

Theorem 5.1. A ring R is WIN-neat if and only if exactly one of the following is true.

(1) R is a field.

(2) $J(R)\neq 0$ and $R/J(R)$ is isomorphic to either a Boolean ring (i.e, to a subring of a direct product of copies of $\mathbb{Z}_2$), or $\mathbb{Z}_3$, or $\mathbb{Z}_3\times\mathbb{Z}_3$, or $\mathbb{Z}_3\times B$, or $\mathbb{Z}_3\times\mathbb{Z}_3\times B$ for some Boolean ring B.

(3) $J(R)=0$, R is not a field, and R is isomorphic to either a Boolean ring B (i.e., to a subring of a direct product of copies of $\mathbb{Z}_2$), or $B\times\prod_\mu\mathbb{Z}_3$, or $\prod_\lambda\mathbb{Z}_3$ for some ordinals μ and λ. Moreover, in all cases every nonzero prime ideal of R is maximal.

Proof. Let R be a ring.
($\Longrightarrow$) Suppose R is a WIN-neat ring. If R is a field, then we are done. Now assume that R is a WIN-neat ring which is not a field.
Let $J(R)\neq 0$. Then, $rad(J(R))=J(R)$ implies that $J(R)$ is semiprime ideal of R. By Lemma 5.1 (5) and (7), $R/J(R)$ is isomorphic to either a Boolean ring, or $\mathbb{Z}_3\times\mathbb{Z}_3$, or $\mathbb{Z}_3\times B$, or $\mathbb{Z}_3\times\mathbb{Z}_3\times B$ for some Boolean ring B, as required.
Suppose $J(R)=0$ and $Max(R)=\{M_i\}_{i\in I}$ for some index set I. Since R is not a field, $M_i\neq 0$. This implies that R has at least two maximal ideals. If $I=\{0,1\}$, then by Lemma 5.1 (8), we have either $R/M_i\cong\mathbb{Z}_2$ or $R/M_i\cong\mathbb{Z}_3$ for $i\in I$. So, R is isomorphic to a subring of either $\mathbb{Z}_2\times\mathbb{Z}_2$ or $\mathbb{Z}_2\times\mathbb{Z}_3$ or $\mathbb{Z}_3\times\mathbb{Z}_3$. Assume that $|I|>2$. Then, $i>2$ and set $I_k=\cap_{i\neq k}M_i$. Again by Lemma 5.1(8), either $R/M_i\cong\mathbb{Z}_2$ or $R/M_i\cong\mathbb{Z}_3$. If $R/M_k\cong\mathbb{Z}_3$, then $R/Max(R)\cong\prod_i\mathbb{Z}_2\times\mathbb{Z}_3$, or $R/Max(R)\cong\prod\mathbb{Z}_2\times\prod\mathbb{Z}_3$, or $R/Max(R)\cong\prod\mathbb{Z}_3$ is WIN-clean ring by Proposition 3.26 (2). If $R/M_k\cong\mathbb{Z}_2$, then $R/Max(R)\cong\prod\mathbb{Z}_2$, or $R/Max(R)\cong\prod\mathbb{Z}_2\times\prod\mathbb{Z}_3$, or $R/Max(R)\cong\mathbb{Z}_2\times\prod_i\mathbb{Z}_3$ is WIN-clean ring by Proposition 3.26 (2). Hence, we conclude that R is isomorphic to either a subring of $\prod_\mu\mathbb{Z}_2\times\prod_\lambda\mathbb{Z}_3$, or a subring of $\prod_\mu\mathbb{Z}_2$, or a subring of $\prod_\lambda\mathbb{Z}_3$.
($\Longleftarrow$) Assume that one of the statements (1), (2) and (3) holds true. By Corollary 5.1, we have that every nonzero prime ideal of R is maximal. If R is a field, then R is a WIN-neat ring since R has no proper ideal. Now assume that R is not a field. If $J(R)\neq 0$ and I is a nonzero semiprime ideal of R, then by assumption $R/J(R)$ is isomorphic to a Boolean ring, or $\mathbb{Z}_3\times\mathbb{Z}_3$, or $\mathbb{Z}_3\times B$, or $\mathbb{Z}_3\times\mathbb{Z}_3\times B$ for some Boolean ring B. Now $J(R)\subseteq I$ since for any $x\in J(R)$, we have $x^k=0\in I$ for some $n\in\mathbb{N}$ which implies that $x\in I$. So R/I is isomorphic to a Boolean ring, or $\mathbb{Z}_3\times\mathbb{Z}_3$, or $\mathbb{Z}_3\times B$, or $\mathbb{Z}_3\times\mathbb{Z}_3\times B$ for some Boolean ring B by Lemma 5.1. If $J(R)=0$, then two cases arise.
Case 1. Assume that R is isomorphic to a subring of a direct product of copies of $\mathbb{Z}_2$ and a direct product of copies of $\mathbb{Z}_3$. So $\phi:R\to\prod_\mu\mathbb{Z}_2\times\prod_\lambda\mathbb{Z}_3$ is monomorphism. We know that the order of the element 1_R divides the order of $1_{\phi(R)}$. This implies

that $O(1_R)$ is either 2 or 3 or 6 since $\prod_\mu \mathbb{Z}_2 \times \prod_\lambda \mathbb{Z}_3$ has characteristic exactly 6. Let I be a nonzero semiprime ideal of R and M_i be a maximal ideal of R containing I. Consider the epimorphism $\pi_i : R \to R/M_i$. Then, $\pi_i(1_R) = 1_{R/M_i}$ as π_i is ring homomorphism. Since R/M_i is a field, 2 or 3 divides the order of the element $1_{R/M_i}$. Thus, R/M_i is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Hence, R is a WIN-neat ring by Corollary 5.1.
Case 2. Suppose R is isomorphic to $\prod_\lambda \mathbb{Z}_3$. Then, $\alpha : R \to \prod_\lambda \mathbb{Z}_3$ is monomorphism. So, R embeds into a ring of order $3^{|\lambda|}$ and hence R has either 1, or 3, or $\cdots$, or $3^{|\lambda|}$ elements. If R embeds in a one element ring, then it is trivial. Assume that $|R| = 3$. Then, it is an integral domain with no nontrivial ideal. So, it must be isomorphic to $\mathbb{Z}_3$. For the case where $|R| = 3^{|\lambda|}$, we have $R \cong \prod_\lambda \mathbb{Z}_3$ which is weak idempotent nil-clean ring by Proposition 3.26 (2). □

The following corollary gives a condition for which WIN-neat rings and clean UNI rings are equivalent.

Corollary 5.2. Let R be a ring. Then, the following are equivalent.

(1) R is WIN-neat ring such that R is not a field;

(2) R is clean UNI ring;

(3) R is weakly clean UNI ring;

(4) R is exchange with strongly invo-fine $U(R)$;

(5) $J(R)$ is nil with $R/J(R)$ is isomorphic to either $\prod_\lambda \mathbb{Z}_2$, or $\prod_\mu \mathbb{Z}_3$, or $\prod_\lambda \mathbb{Z}_2 \times \prod_\mu \mathbb{Z}_3$ for some ordinals λ and μ.

Proof. Let R be a ring.
(1) $\iff$ (2) follows from Theorem 5.1 and Theorem 2.9.
(2) $\implies$ (3) It is obvious.
(3) $\implies$ (2) Suppose R is weakly clean UNI. Then, $6 \in J(R)$ or $30 \in J(R)$ by Lemma 2.4.
Case 1. If $6 \in J(R)$, then R can be decomposed as $R \cong R_1 \times R_2$, where R_1 is a UU ring and R_2 is either $\{0\}$ or a UNI ring with $3 \in J(R_2)$ by Lemma 2.6. Since homomorphic images of weakly clean rings is weakly clean, R_1 and R_2 are weakly clean rings. As $2 \in R_1$ is nilpotent, R_1 is clean by Proposition 2.12. Thus, R_1 is clean UNI. Also, R_2 is clean UNI by Theorem 2.8. Hence, R is clean because the direct product of clean UNI rings is clean UNI.
Case 2. If $30 \in J(R)$, then $(30)^n = 0$ for some natural number n. Thus, either $(2^n, 3^n, 5^n) = 1$, that is, there exist integers u, v and w such that either $2^n u + 3^n v + 5^n w = 1$. So, this allows us to write that either $R = 2^n R + 3^n R + 5^n R$ and also $2^n R \cap 3^n R \cap 5^n R = \{0\}$. Thus, $R = 2^n R \oplus 3^n R \oplus 5^n R$ and hence $R \cong (R/2^n R) \times (R/3^n R) \times (R/5^n R) = R_1 \times R_2 \times R_3$ with $R_1 = R/2^n R \cong (R/3^n R) \times (R/5^n R)$, $R_2 = R/3^n R \cong (R/2^n R) \times (R/5^n R)$ and $R_3 = R/5^n R \cong (R/2^n R) \times (R/3^n R)$. For the case $R_1 \cong R_2 \times R_3$, we have $3 \in J(R_2)$ and $5 \in J(R_3)$. By Theorem 2.8, R_2 is clean UNI ring. Next, we claim that R_3 is trivial ring. Now $6 = 1 + 5 \in 1 + J(R_3)$ is a unit in R_3. So, 2 and 3 are units in R_3. But this contradicts Lemma 2.5. Hence, the claim. In this case, $R_1 \cong R_2$ is a clean UNI ring.
Therefore, R is clean UNI.

(1) $\iff$ (2) $\iff$ (5) follows from Corollary 5.3.
(4) $\iff$ (5) obtained from Theorem 2.10. $\square$

The next corollary gives a hierarchy among WIN-clean rings, WIN-neat rings and clean UNI rings.

Corollary 5.3. Let R be a ring. Consider the following:

(1) R is a WIN-clean ring;

(2) R is a clean UNI ring;

(3) R is a WIN-neat ring.
Then, (1) $\implies$ (2) $\implies$ (3).

Proof. Let R be a ring.
(1) $\implies$ (2). Suppose R is a WIN-clean ring. Then, $R/J(R)$ is isomorphic to either Boolean, or $\mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times B$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$ for some Boolean ring B by Theorem 3.10. Hence, R is clean UNI ring by Corollary 2.7.
(2) $\implies$ (3). Assume that R is clean UNI ring. Then, by Proposition 2.7, $R \cong R_1 \times R_2$, where $J(R_1)$ is nil with $R_1/J(R_1) \subseteq \prod_\lambda \mathbb{Z}_2$ for some ordinal λ and $J(R_2)$ is nil with $R_2/J(R_2) \subseteq \prod_\mu \mathbb{Z}_3$ for some ordinal μ. Thus, R is WIN-clean ring by Theorem 5.1. $\square$

The following proposition gives an equivalence on a local ring with unique maximal ideal which is not a field.

Theorem 5.2. Let (R, M) be a local ring which is not a field. The following statements are equivalent.

(1) R is a clean UNI ring.

(2) R is a WIN-clean ring.

(3) R is a WIN-neat ring.

(4) R is a UNI ring.

Proof. Suppose (R, M) is a local ring with the nonzero maximal ideal M. Then, $Nil(R) = J(R) = M$ is nil ideal, $Id(R) = \{0, 1\}$ and $U(R) = Nil(R) \pm 1$. By Corollary 5.3, we have (1) $\iff$ (2) $\iff$ (3).
(4) $\implies$ (1) is obvious.
(1) $\implies$ (4). Let $r \in R$. Then, $r = u + e$, where $u \in U(R)$ and $e \in Id(R)$. Thus, $r = u$ or $r = u + 1 \in Nil(R)$ but $u = n \pm 1$ for some $n \in Nil(R)$. Hence, R is the UNI ring. $\square$

5.2 Commutative WIN-neat Group Rings

In this section, we will see the conditions for which the group ring over a WIN-neat ring is WIN-neat ring. First, we will state the following theorem that will help us to prove the next proposition.

Theorem 5.3 ([10])**.** Let R be a ring and let G and H be groups. Then,

$$R[G \times H] \cong (R[G])[H].$$

The following proposition characterizes WIN-neat group rings.

Theorem 5.4. Let R be a ring and G be a group. The group ring $R[G]$ is WIN-neat if and only if exactly one of the following four conditions is satisfied.

(1) G is trivial and R is WIN-neat.

(2) G is non-trivial such that G is a torsion 2-group and R is nil-clean such that $R/Nil(R)$ is Boolean.

(3) G is non-trivial such that G is a torsion 3-group and R is WIN-clean ring with $3 \in Nil(R)$ such that $R/Nil(R)$ is isomorphic to $\mathbb{Z}_3$ or $(3,3) \in Nil(R)$ such that $R/Nil(R)$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.

(4) $G = \prod_{i=1}^{k} C_2$, where k is a positive integer and $R \cong \mathbb{Z}_3$ or $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Proof. Let R be a ring and G be a group.
($\Longrightarrow$) Assume that $R[G]$ is a WIN-neat ring. Then, $R[G] \cong R$ if G is trivial. This implies that R is a WIN-neat. Now let us assume that G is non-trivial. Then, using the standard augmentation map $R[G] \to R$ we get that R is a proper homomorphic image of $R[G]$ and hence it is a WIN-clean ring.
If R is not reduced, then $J(R) \neq \{0\}$ and $R/J(R)$ is isomorphic to either a Boolean ring B, or $\mathbb{Z}_3$, or $B \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$ by Theorem 3.10(iv). So, R has a proper homomorphic image either B, or $\mathbb{Z}_3$, or $B \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$ implies that either $B[G]$, or $\mathbb{Z}_3[G]$, or $(B \times \mathbb{Z}_3)[G]$, or $(\mathbb{Z}_3 \times \mathbb{Z}_3)[G]$, or $(\mathbb{Z}_3 \times \mathbb{Z}_3 \times B)[G]$ is a proper homomorphic image of $R[G]$. Now $B \times \mathbb{Z}_3$ is WIN-clean but not nil-clean as well as $3 \notin Nil(B \times \mathbb{Z}_3) = \{0\}$, that is, $3 \neq 0$ if B is a non-trivial direct factor. Using Theorem 3.12 and the fact that every proper homomorphic image of a WIN-neat ring is WIN-clean, we deduce that either $B[G]$ or $\mathbb{Z}_3[G]$ is a WIN-clean ring. Again using Theorem 3.12, we have G as either a non-trivial torsion 2-group or a non-trivial torsion 3-group, respectively. Regarding to R, we have either R is nil-clean ,that is, $R/Nil(R) \cong B$ or R is WIN-clean with $3 \in Nil(R)$, that is, $R/Nil(R) \cong \mathbb{Z}_3$ or R is WIN-clean with $(3,3) \in Nil(R)$, that is, $R/Nil(R) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ respectively.
If R is reduced, then it is isomorphic to either Boolean ring, or $\mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $B \times \mathbb{Z}_3$, or $B \times \mathbb{Z}_3 \times \mathbb{Z}_3$ by Theorem 3.9, where B is Boolean.
Case 1: $R \cong B$
(1.1) If G_2 is non-trivial, then $B[G/G_2]$ is WIN-clean ring as a homomorphic image of $B[G]$. Thus, by Theorem 3.12(1) we have $G/G_2 = \{1\}$ and hence $G = G_2$.
(1.2) If G_2 is trivial, we can get that G is either torsion-free, that is, $G_t := \prod_{\forall p} G_p = \{1\}$ or that G is q-torsion for some single prime $q \neq 2$. If there are two primes $p, q \neq 2$ such that G_p, G_q are non-identity, then both $B[G/G_p]$ and $B[G/G_q]$ will be WIN-clean. From (1) and (2) of Theorem 3.12, we deduce that G/G_p and G/G_q are either non-trivial 2-groups or they are simultaneously trivial. In the first case, we get that $G = G_p = G_q$ as both G_p and G_q are 2-divisible which is wrong contradicting the non-triviality of both quotients. In the other case, we again will have that $G = G_p = G_q$ which, in turn, will yield that $G = \{1\}$ as $G_p \cap G_q = \{1\}$, which

strongly contradict to the assumption that $G \neq \{1\}$ stated at the beginning of our proof. That is the reason why either G is torsion-free or is a q-group.
So, one follows now by the explicit formula

$$J(R[G]) = J(R)[G] + < r(g_p - 1) \mid r \in R, pr \in J(R), g_p \in G_p >$$

from ([29], Theorem) that $J(B[G]) = 0$, because $J(B) = 0$ and since $(q, 2) = 1$, the integer q will invert in B as B is of characteristic 2, so that $qr = 0$ for any $r \in B$ will imply that $r = 0$.
Moreover, using Theorem 5.1, $B[G]$ is a Boolean ring, because $char(B[G]) = 2$. So, for any $g \in G \subseteq B[G]$, the equality $g^2 = g$ holds and hence $g = 1$,because of the assumption on the non-triviality of G. We also observe that $B[G]$ cannot be a field by direction computation.
Case 2. $R \cong \mathbb{Z}_3$.
(2.1) Suppose that G_3 is non-trivial. Then, $\mathbb{Z}_3[G/G_3]$ is a WIN-clean ring and so either G/G_3 is a non-trivial torsion 3-group by Theorem 3.12(3), which is meaning less because it will lead to $G = G_3$ and hence $G/G_3 \cong \{1\}$ which is a contradiction, or G/G_3 is trivial by Theorem 3.12(1), which lead us to the desired equality $G = G_3$.
(2.2) Let G_3 be trivial. We shall show as above in case (1.2) that G is either torsion-free or a q-group for some single prime $q \neq 3$. On the contrary, suppose that $G_q \neq \{1\}$, for any prime $q \neq 3$. Consequently, $\mathbb{Z}_3[G/G_q]$ is a WIN-clean ring as a proper homomorphic image of $\mathbb{Z}_3[G]$. Therefore, Theorem 3.12(3) applies to get that G/G_q is a non-trivial 3-group and so we deduce that $G = G_q$, because each q-torsion component is 3-divisible, that is, $(G_q)^3 = G_q$. This equality for G contradicts, certainly, the non-triviality of the factor group G/G_q. That is why, Theorem 3.12(3) is now applicable to get that G/G_q has to be the identity group which, indeed, assures that $G = G_q$. But the validity of the equality $G = G_p = G_q$ for two distinct primes $p, q \neq 3$ means that $G = G_p \cap G_q = \{1\}$, so that we find that $G = \{1\}$, which is manifestly untrue. Thus, one infers that either $G_t := \cup_{\forall p} G_p = \{1\}$ or $G = G_q$ is a q-torsion for some integer q.
Therefore, based on the formula mentioned in ([29], Theorem), it can be concluded that $J(\mathbb{Z}_3[G]) = 0$. Considering that $J(\mathbb{Z}_3) = 0$ and $qr = 0$ for all $r \in \mathbb{Z}_3$ insures $r = 0$ since q is a unit in $\mathbb{Z}_3$.
Furthermore, Theorem 5.1 implies that $\mathbb{Z}_3[G]$, with characteristics 3, could be isomorphic to $\prod \mathbb{Z}_3$; it is important to note that being isomorphic to a subring of a direct product(=a subdirect product) can be interpreted as a direct isomorphism. A simple verification ensures that the elements of these rings meet the condition $x^3 = x$. Therefore, in these situations, we reach the conclusion that, for every $g \in G \subseteq \mathbb{Z}_3[G]$, the equation $g^3 = g$ holds true. It equivalently forces that $g^2 = 1$.
If $\mathbb{Z}_3[G] \cong \mathbb{Z}_3$, we may conclude that G is trivial, which once again contradicts our former assumption. We can directly check that $\mathbb{Z}_3[G]$ is not a field provided $G \neq \{1\}$.
If the isomorphism $\mathbb{Z}_3[G] \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ is fulfilled, we can say that $G =< g >= \{1, g \mid g^2 = 1\}$. In fact, G is necessarily finite (as for otherwise, $\mathbb{Z}_3[G]$ will be infinite which is impossible). Since $|\mathbb{Z}_3 G| = |\mathbb{Z}_3|^{|G|} = |\mathbb{Z}_3 \times \mathbb{Z}_3|$, one extracts that $3^{|G|} = 9$ yielding that $|G| = 2$, as expected.
If $\mathbb{Z}_3[G] \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, then $|\mathbb{Z}_3|^{|G|} = |\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3| = 3^3$ implies that $|G| = 3$, but $\mathbb{Z}_3[C_3] \cong \mathbb{Z}_3 \times \mathbb{Z}_{3^2} \ncong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Hence, such isomorphism does not hold.
If $\mathbb{Z}_3[G] \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, then $|\mathbb{Z}_3|^{|G|} = |\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3| = 3^4$. So, $|G| = 4$.

Thus, either $\mathbb{Z}_3[C_4] \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_3[C_2 \times C_2] \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. But $\mathbb{Z}_3[C_4] \cong \mathbb{Z}_3 \times \mathbb{Z}_{3^3} \ncong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Now $\mathbb{Z}_3[C_2 \times C_2] \cong (\mathbb{Z}_3[C_2])[C_2] \cong (\mathbb{Z}_3 \times \mathbb{Z}_3)[C_2] \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ by Theorem 5.3. Hence, $\mathbb{Z}_3[C_2 \times C_2] \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Therefore, by induction, we have $G = \prod_{i=1}^{k} C_i$ provided $R \cong \mathbb{Z}_3$. We get the same result in the case of $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.
Case 3. $R \cong B \times \mathbb{Z}_3$.
(3.1) Let $G_3 \neq \{1\}$. Then, the group ring $(B \times \mathbb{Z}_3)[G/G_3]$ is WIN-clean as being a proper homomorphic image of $(B \times \mathbb{Z}_3)[G]$ and so, G/G_3 has to be trivial in virtue of Theorem 3.12(1). Thus, $G = G_3$, as required. Consequently, $(B \times \mathbb{Z}_3)[G_3] \cong B[G_3] \times \mathbb{Z}_3[G_3]$ being a WIN-neat ring ensures that both $B[G_3]$ and $\mathbb{Z}_3[G_3]$ are WIN-clean rings as proper homomorphic images of whole WIN-neat ring. However, according to Theorem 3.12(2), $B[G_3]$ cannot be such a ring, so that this case is unavailable.
It is worthwhile noticing that the same conclusion may be drawn provided $G_2 \neq \{1\}$, as in that case $\mathbb{Z}_3[G_2]$ need not be WIN-clean owing to Theorem 3.12(3).
Another approach might be that we definitely will have that $G = G_2 = G_3 \neq \{1\}$, which is a contradiction since $G_2 \cap G_3 = \{1\}$.
(3.2) Assume now that $G_2 = G_3 = \{1\}$. Again adapting the idea of the previous point quoted above, we shall detect that $J((B \times \mathbb{Z}_3)[G]) = \{0\}$. Indeed, to show that, one needs to get that either G is torsion-free or is a q-group for some single prime $q \neq 2, 3$. For otherwise, assume $G_q \neq \{1\}$. Thus, $(B \times \mathbb{Z}_3)[G/G_q]$ is WIN-clean as being a proper homomorphic image of $(B \times \mathbb{Z}_3)[G]$. It follows from Theorem 3.12(1) that G/G_q has to be trivial, that is, $G = G_q$. Thus, if there exist two such different primes p and q that $G = G_q = G_p$, the group G must be trivial, because $G_p \cap G_q = \{1\}$. This is, of course, a contradiction with our frontier's assumption that $G \neq \{1\}$. Thereby, the claim about G sustained, as pursued.
Furthermore, with the given above pivotal formula from ([29]) at hand, we may repeat the same trick already demonstrated above to get that the group ring $(B \times \mathbb{Z}_3)[G]$ is semi-primitive(=semi-simple in the sense of Jacobson) as $J(B \times \mathbb{Z}_3) \cong J(B) \times J(\mathbb{Z}_3) = \{0\}$ and $qr = 0$ for any $r \in B \times \prod \mathbb{Z}_3$ enables us that $r = 0$, because the characteristic of $B \times \prod \mathbb{Z}_3$ is exactly 6 since $B \neq \{0\}$, as required.
After bearing that in mind, we may write by Theorem 5.1 $(B \times \mathbb{Z}_3)[G]$ is isomorphic to a subring of one of direct products $B' \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$, where B' is a nonzero Boolean ring. Nevertheless, as observed above, $(B \times \mathbb{Z}_3)[G] = B[G] \times \mathbb{Z}_3[G]$ hence both direct factors $B[G]$ and $\mathbb{Z}_3[G]$ have to be WIN-clean rings as being proper homomorphic images of the former WIN-neat ring. However, Theorem 3.12 informs us that this cannot be happen when $G \neq \{1\}$, so this case is unrealistic,too.
As another argumentation, we may appeal to the fact that $B[G]$ and $\mathbb{Z}_3[G]$ are WIN-neat, which fact by referring to Theorem 5.1 will lead to a new promised contradiction, because both components G_2 and G_3 are trivial. We get the same result in the case of $B \times \mathbb{Z}_3 \times \mathbb{Z}_3$.
($\Longleftarrow$) (1) Suppose G is trivial and R is WIN-neat. Then, $G = \{1\}$ and hence $R[G] \cong R$. So, $R[G]$ is WIN-neat ring.
If the cases (2) and (3) hold true, then Theorem 3.12 helps us to show that the group ring $R[G]$ is WIN-clean and so it is WIN-neat ring.
(4) Suppose G is the cyclic 2-group and R is either the 3-elements field or the 9-elements ring. If $R \cong \mathbb{Z}_3$, then $R[G] \cong \mathbb{Z}_3[G] \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, which is WIN-neat ring using Theorem 5.1 (1). If $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then $R[G] \cong (\mathbb{Z}_3 \times \mathbb{Z}_3)[G] \cong \mathbb{Z}_3[G] \times \mathbb{Z}_3[G] \cong$

$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ is also WIN-neat ring by Theorem 5.1 (3). $\square$

Conclusion and Future plans

Breaz ([3]) proved that the finite direct product is a weakly nil-clean ring if and only if one is weakly nil-clean and the others are nil-clean rings and also proved that 6 is the central nilpotent element in a weakly nil-clean ring. Moreover, it is proved that every weakly nil-clean ring is isomorphic to a direct product of nil-clean ring and an indecomposable weakly nil-clean ring with 3 belongs to its Jacobson radical. Moreover, it is proved that every abelian weakly nil-clean ring is strongly π-regular. Finally, it is proved that for a division ring D, $M_n(D)$ is weakly nil-clean if and only if either $D \cong \mathbb{Z}_2$ or $D \cong \mathbb{Z}_3$ and $n = 1$.
Chen ([8]) proved that $a \pm a^2$ is nilpotent, 6 is central nilpotent, $Nil(R)$ is an ideal and $R/Nil(R)$ is weakly Boolean in a strongly weakly nil-clean ring R and for any $a \in R$. It is also proved that a ring R is strongly weakly nil-clean ring if and only if it is periodic and $R/J(R)$ is weakly Boolean if and only if R has no homomorphic image $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ and $a^k - a^{k+2}$ for any $a \in R$ and some positive integer k(depending on a).
Danchev et al. ([20]) proved that a ring R is weakly nil-neat if and only if one of the following holds: R is a field(in particular, R could be isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$); $J(R) \neq 0$ and $R/J(R)$ is isomorphic to either a Boolean ring or $\mathbb{Z}_3$ or direct product of two such rings; $J(R) = 0$, R is not a field, and R is isomorphic to a Boolean ring B or $B \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$.
Danchev ([19]) proved that the group rings $R[G]$ is weakly nil-neat if and only if one of the following satisfied: R is weakly nil-neat and G is trivial; R is nil-clean and G is torsion 2-group; R is weakly nil-clean with $3 \in Nil(R)$ and G is torsion 3-group; G is cyclic of order 2 and $R \cong \mathbb{Z}_2$. In Danchev ([18]) proved that the group ring $R[G]$ is weakly nil-clean ring if and only if one of the following conditions satisfied: R is weakly nil-clean and G is trivial; R is nil-clean and G is torsion 2-group; $R/Nil(R) \cong \mathbb{Z}_3$ and G is torsion 3-group.
We defined and gave examples of weak idempotent nil-clean rings, strongly weak idempotent nil-clean rings and weak idempotent nil-neat rings. We saw several results on weak idempotent nil-clean rings. Some of them stated in Theorem 3.4, Proposition 3.18, Proposition 3.19, Theorem 3.5, Theorem 3.6, Corollary 3.7 and Theorem 3.8. The result on central nilpotent, the set of nilpotent elements, the relationship between periodic rings and strongly weak idempotent nil-clean rings, and also among Strongly clean rings, strongly π-regular rings and strongly weak idempotent nil-clean rings stated in Proposition 4.3, Theorem 4.1, Theorem 4.2, Proposition 4.7, Theorem 4.3, Theorem 4.5, Theorem 4.6 and Theorem **??**. Moreover, results on commutative (reduced) weak idempotent nil-clean rings and local weak idempotent nil-clean rings stated and proved in Theorem 3.9, Theorem 3.10 and Theorem 3.11. In addition to this, we characterized weak idempotent nil-neat rings in Theorem 5.1 and also its immediate consequence Corollary 5.2 connects weakly clean UNI, clean

UNI and weak idempotent nil-neat rings. Finally, we characterized weak idempotent nil-clean group rings in Theorem 3.12 and weak idempotent nil-neat group rings in Theorem 5.4. This PhD thesis can serve as a base for researchers to do further study.
The current research aimed at to identify structure and properties of WIN-clean rings, SWIN-clean rings and WIN-neat rings. The central questions for this research were as follows:

1. In between which class of rings does the class of WIN-clean rings, class of SWIN-clean rings and class of WIN-neat rings lie?

2. Properties and structure of WIN-clean rings, SWIN-clean rings and WIN-neat rings what looks like?

3. Under what condition WIN-clean rings, SWIN-clean rings and WIN-neat rings overlap with their super class and lower class of rings?

4. The extension of WIN-clean rings, SWIN-clean rings and WIN-neat rings what look like?

The research questions we want to address are not all answered. Question number (1) and (3) are addressed in sections 3.1, 3.2, 4.1, 4.3 and 5.1. Question number (2) is also addressed in sections 3.2, 4.3 and 5.1. Question number (4) answered only for commutative rings and abelian groups in section 3.3,3.5 and 5.1. In this dissertation, we look at WIN-neat rings and weak idempotent nil-clean (or nil-neat) group rings for commutative cases. So, it is natural to ask what property and structure of weak idempotent nil-clean (or nil-neat) group rings look like for non-commutative case? Moreover, we investigate matrix extension of weak idempotent nil-clean rings for commutative rings. This brings a question that what matrix extension of weak idempotent nil-clean rings look like for non-commutative rings? In the future, we plan to address those questions not discovered in this dissertation.

List of publications

(1) B. Asmare, T. Abebaw, K. Venkateswarlu, *Weak idempotent nil-clean rings*, Journal of Algebraic Systems, DOI:10.22044/JAS.2023.13177.1725.

(2) B. Asmare, T. Abebaw, K. Venkateswarlu, *Strongly weak idempotent nil-clean rings*, Communicated.

(3) B. Asmare, W. Dereje, K. Venkateswarlu, *Commutative weak idempotent nil-neat rings*, Accepted for publication in Palestine Journal of Mathematics.

(4) B. Asmare, T. Abebaw, K. Venkateswarlu, *Commutative weak idempotent nil-neat group rings*, Communicated.

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