ADDIS ABABA UNIVERSITY

COLLEGE OF NATURAL SCIENCE

DEPARTMENT OF MATHEMATICS

A PROJECT REPORT

ON

HAUSDORFF MEASURE

BY: TSEGAYE ABEBE

ADVISOR: MENGISTU GOA (PhD)

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Permission

This is to certify that this project is a record of the Research done by TsegayeAbebe in the department of Mathematics, Addis Ababa University, under my supervision. I also confirm that the project can be submitted for evaluation by examiners and eventual defense.

Signature _____

Abstract

Measure Theory is the rigorous mathematical study of the field commonlyknown as Hausdorff measure.Hausdorff measures were introduced as certain lower dimensional measures on $\mathfrak{R}^N$ *which allow us to measure "small" subsets in* $\mathfrak{R}^N$*. The Hausdorff measure and the associated Hausdorff dimension of the set provide a more delicate sense of the size of a set in* $\mathfrak{R}^N$ *than the Lebesguemeasure provides.*

In this work we study means of constructingHausdorff measures, via the so-called "outer measure" and "Carathéodory measure",which isolate certain small-scale features of complicated sets in a metric space. The construction is explicit andcovered in detail, after which specific instances of constructed measures areinvestigated in depth.

ACKNOWLEDGEMENT

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LIST OF MATHEMATICAL TERMINOLOGIES

$\Re$	The set of real numbers
A^c	The complement of A
A\B	A without B
⊆	Is subset of
∈	Is an element of
Ø	Empty set
∪	Union
∩	Intersection
$\mu\|_A$	The restriction of μ on A
Σ_μ	The collection of all μ-measurable sets
$\boldsymbol{\mathcal{B}}$(X)	The Borel $\boldsymbol{\sigma}$-field of X
δ (A), diam (A)	The diameter of A
λ^N	N-dimensional lebesgue outer measure
λ^*	Lebesgue outer measure on $\Re$
sup (f)	Supremum of f
inf (f)	Infimum of f
d(x , y)	The distance from x to y
$d_x(A, B)$	$d_x(A, B) = \inf_{\substack{a\in A \\ b\in B}} d_x(a, b) > 0$
$\mathcal{T}_\varepsilon$ (C)	The family of all δ-covers of the set C
$\mu^{(s)}$	Hausdorff s-dimensional outer measure

INTRODUCTION

This paper introduces Hausdorff measures, which provide the means to estimate the “size" or “dimension" of “thin" or “highly irregular” sets.

It contains two chapters. In the first chapter, we present preliminary concepts that are some basic definitions and facts from measure theory like algebra and σ-algebra, outer measure, measurable set,Borel set, metric space, compact set, Hausdorff space, Borel measure, regular measure etc. with relevant definitions, examples and theorems which will be needed in what follows.

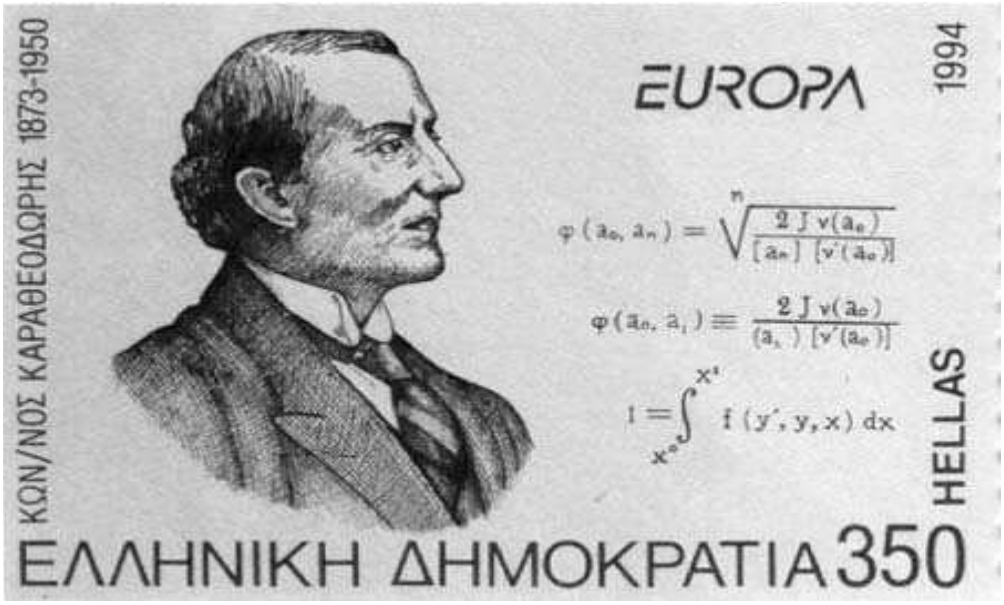
In chapter two we introduce and study metric outer measure/Carathéodoryouter measure, Hausdorff measures and the Hausdorff dimension of sets. Among other things we calculate the Hausdorff measure and Hausdorff dimension of some sets in $\mathfrak{R}$like Cantor sets. From these calculations, the reader will realize that the Hausdorff measure and the Hausdorff dimension of sets maybe interesting.

Felix Hausdorff was born on 8th November 1868 in Breslau, Germany (which isnow Wroclaw, Poland) into a wealthy family. His Father was a textile merchant.In fact, Felix grew up in Leipzig after his parents moved there when he was achild. He studied Mathematics at Leipzig University, completing his PhD there in1891. He was subsequently a Privatdozent, and then an Extraordinary Professorin Leipzig. However, Hausdorffreally wanted to be a writer and actually publishedbooks on philosophy and poetry under a pseudonym. In 1904 he even published afarce which, when eventually produced, turned out to be very successful. Followingthis literary phase, he concentrated again on mathematics, and during the nextdozen years he made major contributions to both topology and set theory. In 1910he moved to Bonn, and then

in 1913 he moved again to take up an ordinary professorship in Greifswalf before finally, in 1921, hereturned again to Bonn. In 1919 heintroduced the notion of Hausdorff dimension in a seminal paper on analysis. Thiswas essentially a generalization of an idea introduced earlier by Carathodory, butHausdorffrealized that the construction actually allows a definition of \fractionaldimensions". In particular, Hausdorff's paper includes a proof of the famous resultthat the dimension of the middle-third Cantor set is $\log 2 / \log 3$.Unfortunately,the final years of Hausdorff's life were tragic. He had come from a Jewish family,and in 1935 he was forced to retire by the Nazi regime in power in Germany. In1941 he was scheduled to be sent to an internment camp, but managed to avoidbeing sent through the intervention of the University. However, this was merelya postponement, and on 26th Januray 1942 Hausdorff, his wife and sister-in-lawcommitted suicide when internment seemed inevitable.

Felix Hausdorff (1868-1942)

ConstantinCarathéodory was born on 13th September 1873, in Berlin. He wasof Greek extraction, being the son of a secretary in the Greek embassy in Berlin.As a stundent, he studied as a military engineer at the ÈcoleMilitaire de Belgique.Subsequently, he joined the British colonial service and worked on the construction of the Assiut dam in Egypt in 1900. He then went on to study for his PhDin Berlin, and then Gottingen, before becoming a Provatdozent in Bonn in 1908.The following year he married - his own aunt! In the following years Carathéodorywent on to hold chairs at Universities in Hanover, Breslau, Gottingen and Bonn. However, in 1919 the Greek Government asked him to help establish a new university in Smyrna. However, this was not a happy experience since the project wasthwarted by a Turkish attack. Eventually, following this interlude he was appointedto a chair in Munich, which he held until his retirement in 1938. He died there on2nd February 1950.



ConstantinCarathéodory (1873-1950)