ADDIS ABABA UNIVERSITY

COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCE

DEPARTMENT OF MATHEMATICS
Graduate Thesis Report on Solving PDEs Using Fourier Transform

project on

Solving second order PDEs using Fourier Transform

A Project submitted to the School of Graduate Studies of Addis Ababa University in Partial fulfillment of the requirements for the Degree of Master of Science in Mathematics

By: Kassa Mekuria
Advisor: Dr.Tesfa Biset
date: June/2019 G.C
Adiss Ababa;Ethiopia

Advisor: Dr.Tesfa Biset: Signature:____________________Date:______________

Examiner 1:______________________________Signature:__________Date:__________________

Examiner 2:______________________________Signature:__________Date:__________________

Solving second order PDEs using Fourier Transform

KASSA MEKURIA
ID GSE\7748\09

July 3, 2019

Abstract

This project is concerned with the Fourier transform methods for second order partial differential equations. In particular, solving wave, Laplace and heat equations using Fourier transform. We also introduce the theory of distributions and examine their relation to the Fourier transform, and we then use this methods to find solutions to linear partial differential equations. In particular, fundamental solutions to heat operators

Acknowledgement

Firstly, I would like to express my gratitude to my advisor D.r Tesfa Biset for the useful comments, remarks , gave invaluable advice on how to organize the ideas and provide motivation through out the learning process of this project. Secondly,I would like to thank my husband Wubet Terfe ,My family and friends, who have supported me throughout entire process.

Notations

$\Re^n$ .... real Euclidian space in n dimension
PDEs .... partial differential equations
Ω .... open set in $\Re^n$
$D(\Re^n)$ .... the space of test function over $\Re^n$)
$D^{'}(\Re^n)$ .... the space of distribution $\Re^n$)
$S(\Re^n)$ .... the space of Schwartz $\Re^n$)
$S^{'}(\Re^n)$ .... the space of tempered distribution $\Re^n$)

Contents

Chapter 1

INTRODUCTION

Fourier transform is the useful method to solve partial differential equations.In the second chapter it contains some preliminaries, definitions of some related concepts , definition of Fourier transform and some properties of it.In the third chapter we deal with how to solve wave,heat and laplace equation using Fourier transform. To solve this kinds of partial differential equation by using Fourier method we transform PDEs to ordinary differential equation. suppose that $u(x,t)$ is a function of two variables x and t where $(-\infty, \infty)$ and $t > 0$.Because of the presence of two variables,care is needed in identifying with respect to which the Fourier transform is computed. For example,for fixed t the function $U(x,t)$ becomes a function of the spatial variable x,in such a case we can take the Fourier transform with respect to the variable x we denote the transformation by $\hat{u}(\xi, t)$ .Thus

$$F(u(x,t))(\xi) = \hat{u}(\xi, t) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} u(x,t) e^{-i\xi x} dx$$

This transformation is called Fourier transformation in the variable x And the inverse Fourier transform is given by

$$u(x,t)) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} \hat{u}(\xi, t) e^{ix\xi} d\xi$$

In the Fourth and fifth chapter we discuss on the theory of Distribution,test function, schwartz and tempered distributions ,also the some operations,differentiation ,convolution and direct product of distribution. in this chapter we also discuss The Fourier transform in S and $S^{'}$. those operations and properties of Fourier transform of distribution used to solve partial differential equations,in particular how to find the fundamental solution of heat operator

Chapter 2

PRELIMINARIES

2.1 Definitions and Terminologies

DEFINITION 2.1.1. *A piecewise smooth function can be broken in to distinct pieces both the functions and their derivatives are continuous every where ,How ever the only discontinuities that are allowed are a finite number of jumps discontinuities.*

DEFINITION 2.1.2. *An absolutely integrable function is a function whose absolute value is integrable meaning that the integral of the absolute value over the whole domain is finite*

$$\int_{\Re} |f| < \infty$$

DEFINITION 2.1.3. *Let* f *be function on an interval* (a, b)*, the function* f *is said to be increasing function if for any two points* $X_1, X_2 \in (a, b)$ *such that* $X_1 < X_2$ *then* $f(X_1) < f(X_2)$

DEFINITION 2.1.4. *Let* f *be function on an* $\Re^n$ *is locally integrable function if it is integrable on every compact sub-set of* $\Re^n$

2.2 Fourier Transform

DEFINITION 2.2.1. *For any absolutely integrable function* f *defined on* $\Re$ *the Fourier transform of* f *is given by*

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} f(x) e^{-\xi i x} dx \tag{2.1}$$

The Fourier of f in "transform space" can be recover via an inversion formula that defines the inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{\xi i x} d\xi \tag{2.2}$$

2.2.1 some properties of fourier transform

1. Linearity: For two absolutely integrable functions $f(x)$ and $g(x)$ and for any $a, b \in \Re$

   $$F[af(x) + bg(x)] = aF[f(x)] + bF[g(x)] = a\hat{f(\xi)} + b\hat{g(\xi)} \tag{2.3}$$

   PROOF 2.2.1. *For two absolutely integrable functions $f(x)$ and $g(x)$ and for any $a, b \in \Re$ by definition 2.1*

   $$\begin{aligned} F[af + bg] &= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} [af + bg](x) e^{-\xi i x} dx \\ &= \frac{a}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} f(x) e^{-\xi i x} dx + \frac{b}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} g(x) e^{-\xi i x} dx \\ &= aF[f] + bF[g] \end{aligned}$$

2. Shifting properties:

   (a)

   $$F[f(x - c)] = e^{i\xi c} \hat{f}(\xi) \tag{2.4}$$

   PROOF 2.2.2. *For any $c \in \Re$ and $c > 0$ and $f(x)$ is absolutely integrable functions,$F[f]$ be the Fourier transform*
   claim

   $$\begin{aligned} F[f(x - c)] &= e^{i\xi c} F[f(x)] \\ F[f(x - c)] &= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} f(x - c) e^{-\xi i x} dx \\ &= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} f(\omega) e^{-\xi i (\omega +) c} d\omega \end{aligned}$$

where $\omega = x - c$,$x = \omega + c$

$$\begin{aligned} &= \frac{1}{\sqrt{2\Pi}} e^{i\xi c} \int_{-\infty}^{\infty} f(\omega) e^{-i\xi\omega} d\omega \\ &= e^{i\xi c} \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} f(\omega) e^{-i\xi\omega} d\omega \\ &= e^{i\xi c} F[f(x)] \end{aligned}$$

(b)

$$F[e^{icx} f(x)] = F[\xi + c] \tag{2.5}$$

PROOF 2.2.3. *For any* $c \in \Re$ *and* $c > 0$ *and* $f(x)$ *is absolutely integrable functions/*$F[f]$ *be the Fourier transform/*

$$\begin{aligned} F[e^{icx} f(x)] &= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} [e^{icx} f(x)] e^{-i\xi x} dx \\ &= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} f(x) e^{icx} e^{-i\xi x} dx \\ &= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} f(x) e^{-ix(\xi + c)} dx \\ &= F[\xi + c] \end{aligned}$$

3. Derivatives:

   i. $F[\frac{df}{dx}] = i\xi F[f(x)] = -i\xi f\hat{(}\xi)$

   ii. $F[\frac{d^2f}{d^2x}] = i\xi^2 F[f(x)] = \xi^2 \hat{f}(\xi)$, and

   $$F[\frac{d^n f}{d^n x}] = (-i\xi)^n F[f(x)] \tag{2.6}$$

   iii.

   $$F[\frac{\partial f}{\partial t}] = \frac{\partial}{\partial t} F[f(x,t)] \tag{2.7}$$

   Let f be continuous and piecewise smooth in$(-\infty, \infty)$ let f approach zero as $|x| \to \infty$ if f $f^{'}$ are absolutely integrable ,then

   $$F[f^{'}(x)] = i\xi F[f]$$

PROOF 2.2.4.

$$F[f^{'}(x)] = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty} f^{'}(x)e^{-i\xi x}dx$$

using integration by parts

$$\begin{aligned}
\frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty} f^{'}(x)e^{-i\xi x}dx &= \frac{1}{\sqrt{2\Pi}}[f(x)e^{i\xi x}|_{-\infty}^{\infty} - i\xi\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx] \\
&= -i\xi\frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty} f(x)e^{-\xi ix}dx \\
&= -i\xi F[f]
\end{aligned}$$

The result can be extended $F[f^n(x)] = (-i\xi)^n F[f]$ *for n=0,1,2,...n*

DEFINITION 2.2.2. *For two absolutely integrable functions* f *and* g *the convolution of* f *and* g *over the interval* $(-\infty,\infty)$*is defined as*

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt \tag{2.8}$$

THEOREM 2.2.1. *If* $F(\xi)andG(\xi)$ *are the Fourier transform of* $f(x)$ *and* $g(x)$ *respectively, then the Fourier transform of the convolution* $f*g$ *is the product* $F(\xi)G(\xi)$ *that is*

$$(f*g) = F(f)F(g) \tag{2.9}$$

Proof. By the definition 2.2.1 and an interchange of the order of integration we have

$$\begin{aligned}
F(f*g) &= \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)g(x-t)e^{-i\xi x}dtdx \\
&= \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)g(x-t)e^{-i\xi x}dtdx
\end{aligned}$$

Now we make the substitution $x-t=v$,so that $x=t+v$

$$\begin{aligned}
F(f*g) &= \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)g(v)e^{-i\xi(t+v)}dvdt \\
&= \frac{1}{\sqrt{2\Pi}}[\int_{-\infty}^{\infty} f(t)e^{-i\xi t}dt\int_{-\infty}^{\infty} g(v)e^{-i\xi v}dv] \\
&= F(f)F(g))
\end{aligned}$$

□

2.3 Gaussian's Function and its integral

DEFINITION 2.3.1. *Let* $x \in \Re^n$ *the following function is called gaussian function:*

$$f(x) = e^{-bx^2} \tag{2.10}$$

for some $b > 0$

Let $f(x) = ae^{-bx^2}$ with $a > 0$,$b > 0$ be a Gaussian function, note that $f(x)$ is positive every where the integral I of $f(x)$ over $\Re$,in particular a and b Let

$$I = \int_{-\infty}^{\infty} f(x)dx$$

To solve this one dimensional integrals we will start by completing square. By separability property of exponential function it follows that we will get a two dimensional integral over a two dimensional gaussian .If we compute that the integral gives by positive square root of its integral

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} f(x)dx \int_{-\infty}^{\infty} f(y)dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)f(y)dydx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ae^{-bx^2}ae^{-by^2}dydx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a^2e^{-b(x^2+y^2)}dydx \end{aligned}$$

Now we will make a change of variable from (x, y) to polar coordinates (α, r)

then, let $u = e^{-br^2}$ $du = -2brdr$

$$\begin{aligned}
I^2 &= a^2 \int_0^{2\Pi} \int_0^{\infty} re^{-br^2} drd\alpha \\
&= a^2 \int_0^{2\Pi} \frac{1}{-2b} \int_0^{\infty} e^u dud\alpha \\
&= \frac{a^2}{-2b} \int_0^{2\Pi} [e^{-br^2}|_0^{\infty}] d\alpha \\
&= \frac{a^2}{-2b} \int_0^{2\Pi} -1 d\alpha \\
&= \frac{-2\Pi a^2}{-2b} \\
&= \frac{\Pi a^2}{b}
\end{aligned}$$

Take the positive square root gives

$$I = a\sqrt{\frac{\Pi}{b}} \tag{2.11}$$

Chapter 3

THE FOURIER TRANSFORM METHOD

3.1 Solving Some Second order PDEs Using Fourier Transform Method

we summarize the Fourier transform method as follows
step

1. Find the Fourier transform of the given boundary value problem in $u(x,t)$ and get the ordinary differential equation in $\hat{u}(\xi,t)$ in the variable t
2. solve the ordinary differential equation and find $\hat{u}(\xi,t)$.
3. inverse the Fourier transform $\hat{u}(\xi,t)$ to get $u(x,t)$

EXAMPLE 3.1.1. *consider the initial value problem for the wave equation*

$$u_{tt} = c^2 u_{xx}, -\infty < x < \infty, t > 0, c > 0 \tag{3.1}$$

u and u_x finite as $|x| \to \infty$,$t > 0$ $u(x,0) = f_1(x)$,$-\infty < x < \infty$,
$u_t(x,0) = f_2(x)$,$-\infty < x < \infty$
where the functions f_1 and f_2 are piecewise smooth and absolutely integrable in$(-\infty,\infty)$
To find the solution of this problem by 2.1 and 2.2

we also need the Fourier representation of the solution $u(x,t)$,

$$u(x,t) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} \hat{u}(\xi,t) e^{i\xi x} d\xi$$

where $\hat{u}(\xi,t)$ is unknown function ,which will be now determine .for this we substitute in to the differential equation 3.1 to obtain

$$0 = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} [\frac{\partial^2 \hat{u}(\xi,t)}{\partial t^2} + c^2\xi^2 \hat{u}(\xi,t)] e^{i\xi x} d\xi$$

This $\hat{u}$ must be a solution of the ordinary differential equation

$$\frac{\partial^2 \hat{u}(\xi,t)}{\partial t^2} + c^2\xi^2 \hat{u}(x,t) = 0$$

whose solution can be written as

$$\hat{u}(\xi,t) = c_1(\xi)\cos\xi ct + c_2(\xi)\sin\xi ct$$

To find $c_1(\xi)$ and $c_2(\xi)$, we note that from the initial condition of equation 3.1 we have
$f_1(x) = u(x,0) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} c_1(\xi) e^{i\xi x} d\xi$

$f_2(x) = \frac{\partial u(x,0)}{\partial t} = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} \xi c c_2(\xi) e^{i\xi x} d\xi$

and hence $F_1(\xi) = c_1(\xi)$ and $F_2(\xi) = \xi c c_2(\xi)$
therefore ,it follows that

$$\hat{u}(\xi,t) = F_1(\xi)\cos\xi ct + \frac{F_2(\xi)}{\xi c}\sin\xi ct$$

From this the fourier representation of the solution is

$$u(x,t) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} [F_1(\xi)\cos\xi ct + \frac{F_2(\xi)}{\xi c}\sin\xi ct] e^{i\xi x} d\xi$$

Now since $\cos\xi = \frac{e^{i\xi}+e^{-i\xi}}{2}$ and $\sin\xi = \frac{e^{i\xi}-e^{-i\xi}}{2i}$ we have

$$\begin{aligned} \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} F_1(\xi)\cos\xi ct e^{i\xi x} dx &= \frac{1}{2}\frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} F_1(\xi)[e^{i\xi ct} + e^{-i\xi ct}] e^{i\xi x} d\xi \\ &= \frac{1}{2}\frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} F_1(\xi)[e^{i\xi(x+ct)} + e^{i\xi(x-ct)}] d\xi \end{aligned}$$

By Fourier transform property 2.4 we have

$$= \frac{1}{2}[f_1(x+ct) + f_2(x-ct)] \tag{3.2}$$

similarly

$$\begin{aligned}
\frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty} F_2(\xi)\frac{\sin \xi ct}{\xi c}e^{i\xi x}dx &= \frac{1}{2}\frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty} F_2(\xi)\frac{e^{i\xi ct} - e^{-i\xi ct}}{i\xi c}e^{i\xi x}d\xi \\
&= \frac{1}{2}\frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty} F_2(\xi)\frac{e^i\xi(x+ct) - e^i\xi(x-ct)}{i\xi c}d\xi \\
&= \frac{1}{2c}\frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty} F_2(\xi)[\int_{x-ct}^{x+ct} e^{i\xi\omega}d\omega]d\xi \\
&= \frac{1}{2c}\int_{x-ct}^{x+ct}[\frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty} e^{i\xi\omega}F_2(\xi)d\xi]d\omega
\end{aligned}$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} f_2(\omega)d\omega \tag{3.3}$$

putting equation 3.2 and 3.3 together yields d'Alembert's formula

$$u(x,t) = \frac{1}{2}[f_1(x-ct) + f_1(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} f_2(\omega)d\omega$$

EXAMPLE 3.1.2. *consider the following problem involving the laplace equation in a half-plane*

$$u_{xx} + u_{yy} = 0, (-\infty < x < \infty), y > 0 \tag{3.4}$$

$u(x,0) = f(x)$,$(-\infty < x < \infty)$
$|u(x,y)| \leq M$,$(-\infty < x < \infty)$,$y > 0$

where the function f is piecewise smooth and absolutely integrable in $(-\infty, \infty)$. If $f(x) \to 0$ as $|x| \to \infty$then we also have the implied boundary conditions $\lim_{|x|\to\infty} u(x,y) = 0$,$\lim_{y\to+\infty} u(x,y) = 0$ for this we use the equation 2.1 and 2.2 we also use the Fourier representation of the solution $U(x,y)$ $u(x,y) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty} \hat{u}(\xi,y)e^{i\xi x}d\xi$ we substitute in to equation 3.4 we get

$$0 = U_{xx} + u_{yy} = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty}[-\xi^2\hat{u}(\xi,y) + \frac{\partial^2 u(\xi,y)}{\partial y^2}]e^{i\xi x}d\xi$$

This $\hat{u}$ must satisfy the ordinary differential equation

$$\frac{\partial^2 \hat{u}}{\partial y^2} = \xi^2 \hat{u}$$

and the inial condition $\hat{u}(\xi, 0) = F(\xi)$ for each ξ .
The general solution of the ordinary differential equation is $c_1 e^{\xi y} + c_2 e^{-\xi y}$.If we impose the initial condition and the bounded ness condition ,the solution becomes

$$\hat{u}(\xi, y) = \begin{cases} F(\omega)e^{-\xi y}, & , \xi \geq 0 \\ F(\omega)e^{\xi y}, & ,\xi < 0 \end{cases} = F(\xi)e^{-|\xi| y}$$

Thus the desired Fourier representation of the solution

$$u(x, y) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} F(\xi) e^{-|\xi| y} e^{i\xi x} d\xi$$

To obtain an explicit representation,we insert the formula for $F(\xi)$ and formally interchange the order of integration,to obtain

$$\begin{aligned} u(x, y) &= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} f(\omega) e^{-i\xi\omega} d\omega] e^{-|\xi| y} e^{i\xi x} dx \\ &= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} e^{i\xi(x-\omega)} e^{-|\xi| y} d\xi] f(\omega) d\omega \end{aligned}$$

Now the inner integral is

$$\begin{aligned} \int_{-\infty}^{\infty} e^{i\xi(x-\omega)} e^{-|\xi y} d\xi &= 2Re \int_{0}^{\infty} e^{i\xi(x-\omega)} e^{-\xi y} d\xi \\ &= 2Re \int_{0}^{\infty} e^{-\xi[y - i(x-\omega)]} d\xi \\ &= 2Re \frac{1}{y - i(x-\omega)} = \frac{2y}{y^2 + (x - \omega^2)} \end{aligned}$$

Therefore the solution $u(x, y)$ can be explicitly written as

$$u(x, y) = \frac{1}{\Pi} \int_{-\infty}^{\infty} \frac{y}{y^2 + (x - w)^2} f(\omega) d\omega \tag{3.5}$$

This representation 3.5 known as possion's integral.In particular for

$$u(x, 0) = f(x) = \begin{cases} 1, & \text{if } a < x < b \\ 0, & \text{otherwise.} \end{cases}$$

thus equation 3.5 becomes

$$u(x,y) = \frac{1}{\Pi}\int_a^b \frac{y}{y^2+(x-\omega)^2}d\omega = \frac{1}{\Pi}\int_a^b \frac{\frac{d\omega}{y}}{\frac{(x-\omega)^2}{y^2}+1}$$

using substitution $v = \frac{x-\omega}{y}$ we have $d\omega = ydv$ so that

$$\begin{aligned} u(x,y) &= \frac{1}{\Pi}\int_{\frac{(a-x)}{y}}^{\frac{b-x}{y}} \frac{1}{1+v^2}dv \\ &= \frac{1}{\Pi}[\tan^{-1}\frac{b-x}{y} - \tan^{-1}\frac{a-x}{y}] \end{aligned}$$

EXAMPLE 3.1.3. *consider the heat flow problem of an infinitely long thin bar insulated on its lateral surface which is modeled by the following initial value problem.*

$$u_t = c^2u_{xx}, (-\infty,\infty), t>0, c>0 \tag{3.6}$$

u and u_x finite as $|x| \to \infty$,$t > 0$
$u(x,0) = f(x)$ $-\infty < x < \infty$

Where the function f is piecewise smooth and absolutely integrable in $(-\infty,\infty)$ .Let $\hat{u}(\xi,t)$ be the Fourier transform of $u(x,t)$. By 2.1 and 2.2 we have

$$u(x,t) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty} \hat{u}(\xi,t)e^{i\xi x}d\xi$$

$$\hat{u}(\xi,t) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty} u(x,t)e^{-i\xi x}d\xi$$

Let f be continuous and piecewise smooth in$(-\infty,\infty)$ let f approach zero as $|x| \to \infty$ if f f' are absolutely integrable ,by 2.7 and 2.6 we get

$$\frac{\partial u}{\partial t} = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty} \frac{\partial \hat{u}(\xi,t)}{\partial t}e^{-i\xi x}d\xi$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty} \hat{u}(\xi,t)(-i\xi)e^{-i\xi x}d\xi$$

$$\frac{\partial u}{\partial x^2} = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty} \hat{u}(\xi,t)(-i\xi)^2e^{-i\xi x}d\xi$$

In order to satisfy equation 3.6 we have

$$0 = \frac{\partial u}{\partial t} - c^2 \frac{\partial u}{\partial x^2} = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} [\frac{\partial \hat{u}(\xi, t)}{\partial t} + c^2 \xi^2 \hat{u}(\xi, t)] e^{-i\xi x} d\xi$$

Thus,$\hat{u}$ must be the solution of the ordinary differential equation

$$\frac{\partial \hat{u}(\xi, t)}{\partial t} + c^2 \xi^2 \hat{u}(\xi, t) \tag{3.7}$$

From the initial condition equation 3.6 we have

$$\begin{aligned} \hat{u}(\xi, 0) &= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} u(x, 0) e^{-i\xi x} dx \\ &= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx \\ &= F(\xi) \end{aligned}$$

Therefore from equation 3.7 we have

$$\hat{u}(\xi, t) = F(\xi) e^{-\xi^2 c^2 t}$$

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} [F(\xi) e^{-\xi^2 c^2 t}] e^{i\xi x} d\xi \\ &= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} [\frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} f(\omega) e^{-i\omega x} d\omega] e^{-\xi^2 c^2 t} e^{i\xi x} d\xi \\ &= \frac{1}{2\Pi} \int_{-\infty}^{\infty} f(\omega) \int_{-\infty}^{\infty} e^{-\xi^2 c^2 t} e^{-i\xi\omega} e^{i\xi x} d\xi d\omega \\ &= \frac{1}{2\Pi} \int_{-\infty}^{\infty} f(\omega) \int_{-\infty}^{\infty} [e^{-\xi^2 c^2 t} e^{-i\xi(x-\omega)} d\xi] d\omega \end{aligned}$$

Now consider the inner part of above equation

$$\begin{aligned} H(x, t, \xi) &= \int_{-\infty}^{\infty} [e^{-\xi^2 c^2 t} e^{-i\xi(x-\omega)} d\xi \\ &= \int_{-\infty}^{\infty} exp[-t(c\xi - i\frac{(x-\omega)}{2t})^2 - \frac{(x-\omega)^2}{4t}] d\xi \end{aligned}$$

since the exponent satisfies (use completing square method)

$$-\xi^2c^2t+ic\xi(x-\omega)=-t(\xi^2c^2-ic\xi\frac{(x-\omega)}{t})=-t[(c\xi-i\frac{x-\omega}{2t})^2+\frac{(x-\omega)^2}{4t^2}]$$

and set $c\xi - i\frac{x-\omega}{2t}=\frac{s}{\sqrt{t}}$,with $cd\xi = ds$ such that

$$H(x,t,\xi)=\int_{-\infty}^{\infty}[e^{-s^2}exp[-\frac{(x-\omega)}{4t}]\frac{ds}{c\sqrt{t}}=\sqrt{\frac{\Pi}{c^2t}}e^-\frac{(x-\omega)^2}{4t}$$

since by Gaussian function

$$\int_{-\infty}^{\infty}e^{-s^2}ds=\sqrt{\Pi}$$

Therefore

$$\begin{aligned}u(x,t)&=\frac{1}{\sqrt{4c^2\pi t}}\int_{-\infty}^{\infty}f(\omega)exp[-\frac{(x-\omega)^2}{4t}]d\omega\\&=\int_{-\infty}^{\infty}k(x-\omega,t)f(\omega)d\omega\end{aligned}$$

where $k(x,t)=\frac{1}{\sqrt{4c^2\Pi t}}exp[-\frac{x^2}{4t}]$

Chapter 4

THE SPACE $D^{'}$ DISTRIBUTION

4.1 Definitions on Distribution

DEFINITION 4.1.1. *let* $\Omega \subset \Re^n$ *be an open set and* $\varphi : \Omega \to C$ *a function we say that* φ *is a test function if*

1. *support of* φ *is compact*
2. φ *is* c^∞

DEFINITION 4.1.2. *The heaviside function is defined to be equal to zero for every negative values of* x *and unity for every positive values of* x *and is denoted by*

$$H(x) = \begin{cases} 0, & \textit{if } x < 0 \\ 1, & ;x > 0. \end{cases} \tag{4.1}$$

This is also called the unit step function .

DEFINITION 4.1.3. *In physical problem one often encounters idealized concepts such that a force concentrated at a point* ξ *or(an impulsive force that acts instantaneously)These forces are described by the Dirac delta function* $\delta(x-\xi)$ *Which have several significant properties*

$$\delta(x-\xi) = 0, x \neq \xi$$

$$\int_a^b \delta(x-\xi)dx = \begin{cases} 0, & \textit{if } a,b<\xi or \xi<a,b \\ 1, & ;a\le x\le b. \end{cases}$$

And

$$\int_{-\infty}^{\infty} \delta(x-\xi)dx = 1 \tag{4.2}$$

Equation 4.2 is a special case of the general formula

$$\delta * f = \int_{-\infty}^{\infty} \delta(x-\xi)f(x)dx = f(\xi) \tag{4.3}$$

DEFINITION 4.1.4. *A mapping $T : D(\Re^n) \to C$ is called a distribution if*

1. *it is linear in the sense that if $\varphi, \phi \in D(\Re^n)$ and $a, b \in C$ then*

$$T(a\varphi + b\phi) = aT(\varphi) + bT(\phi)$$

2. *it is continuous , in the sense that if $\{\varphi_m\} \to \varphi$ in $D(\Re^n)$ Then $\{T(\varphi_m)\} \to T(\varphi) in\ C$*

OR A continuous linear functional on a space D of test function is called distribution

The space of distribution is denoted by $D^{'}(\Re^n)$ it is also called generalized function. A function on $\Re^n$ is locally integrable if it is integrable on every compact subset of $\Re^n$. Every locally integrable function can be treated as a distribution. Let f be a locally integrable function on $\Re^n$, Then for any $\varphi \in D(\Re^n)$,we can define the integral

$$\langle f, \varphi \rangle = \int_{\Re^n} f(x)\varphi(x)dx$$

TYPES OF DISTRIBUTION

1. Regular distributions: are these distributions which are generated by locally integrable function. In fact let f a locally integrable function on $\Re^n$ Then for any $\varphi \in D(\Re^n)$, we can define the integral

$$\langle f, \varphi \rangle = \int_{\Re^n} f(x)\varphi(x)dx$$

2. singular distribution : are distribution which are not regular.

EXAMPLE 4.1.1. *The heaviside distribution in* $\Re$ *is*

$$\langle H, \phi\rangle = \int_{\Re} \phi(x)dx \tag{4.4}$$

by equation 4.1 equation 4.4 becomes

$$\langle H, \phi\rangle = \int_{0}^{\infty} \phi(x)$$

Since $H(x)$ *is a piecewise continuous function .this is regular distribution*

EXAMPLE 4.1.2. *The Dirac delta distribution in* $\Re^n$ *is*

$$\langle \delta(x-\xi), \phi(x)\rangle = \phi(\xi)$$

For ξ *is fixed point in*$\Re^n$ *the linearity of this functional follows from the relation*

$$\begin{aligned}\langle \delta, c_1\phi_1 + c_2\phi_2\rangle &= c_1\phi_1(\xi) + c_2\phi_2(\xi)\\ &= c_1\langle \delta, \phi_1\rangle + c_2\langle \delta, \phi_2\rangle\end{aligned}$$

where c_1 *and* c_2 *are arbitrary real constants. To prove continuity we observe that*
$\lim_{m\to\infty}\langle(\delta, \phi_m)\rangle = \lim_{m\to\infty}\phi_m(\xi)$ *However if* $\phi_m(x) \to 0$ *then* $\phi_m(\xi)$ *and we have continuity, but it is not locally integrable. The distribution is therefore a singular distribution.*

4.2 Some Operations on Distribution

Let $f, g \in D^{'}(\Re^n)$ and $\alpha \in c$ we define

1. The sum of a distributions, $f+g$, $\langle f+g, \varphi\rangle = \langle f, \varphi\rangle + \langle g, \varphi\rangle$, $\forall\varphi \in D$
2. Multiplication by constants by a constant αf as, $\langle \alpha f, \varphi\rangle = \alpha\langle f, \varphi\rangle, \forall\varphi \in D$

3. The shifting of a distribution,$f(t-\tau)$,as

$$\langle f(t-\tau), \varphi\rangle = \langle f, \varphi(t+\tau)\rangle, \forall\varphi \in D$$

That is

$$\begin{aligned}
\langle f(t-\tau), \varphi(t)\rangle &= \int_{\Re^n} f(t-\tau)\varphi(t)dt \\
&= \int_{\Re^n} f(y)\varphi(y+\tau)dy \\
&= \langle f, \varphi(t+\tau)\rangle
\end{aligned}$$

4. The translation of a distribution $f(-t)$ as

$$\langle f(-t), \varphi\rangle = \langle f, \varphi(-t)\rangle, \forall\varphi \in D$$

$$\begin{aligned}
\langle f(-t), \varphi\rangle &= \int_{\Re^n} f(-t)\varphi dt \\
&= \int_{\Re^n} f(y)\varphi(-y) - dy \\
&= \langle f, \varphi(-t)\rangle
\end{aligned}$$

5. The dilation of a distribution $f(at)$,as

$$\begin{aligned}
\langle f(at), \varphi(t)\rangle &= \int_{\Re^n} f(at)(\varphi - t)dt \\
&= \int_{\Re^n} f(y)\varphi(\frac{y}{a})\frac{1}{|a|^n}dy \\
&= \langle f(t), \frac{1}{|a|^n}\varphi(\frac{t}{a})\rangle
\end{aligned}$$

6. The multiplication of a distribution by a smooth function ,gf as

$$\langle gf, \varphi\rangle = \langle f, g\varphi\rangle, \forall\varphi \in D$$

4.3 Differentiation of a Distribution

DEFINITION 4.3.1. *Let f be a distribution in $D^{'}(\Re^n)$ the partial derivatives of f are defined as*

$$\langle \frac{\partial f}{\partial t_i}, \varphi \rangle = \langle f, -\frac{\partial \varphi}{\partial t_i} \rangle, \forall \varphi \in D$$

for every $i \in \{1, ...n\}$

As most of the properties for distribution ,this definition has its orgins in the behaviour of regular distribution.
Let function $f \in D^{'}(\Re)$, for which the function f is smooth then its derivative $f^{'}$ is also smooth ,so it defines a distribution and the integration by parts yields

$$\begin{aligned} \langle f^{'}(t), \varphi \rangle &= \int_{\Re^n} f^{'}(t)\varphi(t)dt \\ &= \int_{\Re^n} (f\varphi)^{'}(t)dt - \int_{\Re^n} f(t)\varphi^{'}(t)dt \end{aligned}$$

But as φ is a test function ,its support is compact ,and thus the fundamental theorem of calculus forces the first integral to be zero and we can write

$$\langle f^{'}(t), \varphi \rangle = -\langle f(t), \varphi^{'} \rangle, \forall \varphi \in D$$

REMARK 4.3.1. *i. Let f be a distribution then the expression for $\frac{\partial f}{\partial t_i}$ in definition defines a distribution*
i.e $\langle \frac{\partial f}{\partial t_i}, \varphi_n \rangle = -\langle f, \frac{\partial \varphi_n}{\partial t_i} \rangle$

ii. Differentiation is a linear operation in the space of distribution ,and it is continuous in the sense that if a sequence of distributions f_n converges to f in $D^{'}$ then $D^k f_n$ converges to $D^k f$ in $D^{'}$
i,e $\langle D^k f_n, \varphi \rangle = (-1)^{|k|}\langle f_n, D^k \varphi \rangle \to (-1)^{|k|}\langle f, D^k \varphi \rangle = \langle f D^k, \varphi \rangle$

For regular distributions which comes from the differentiable function ,the usual derivatives coincides with the distributional derivative ,but we know nothing about the remaining ones,such as the delta function

EXAMPLE 4.3.1.

$$\langle \delta, \varphi \rangle = \varphi(0)$$

so if we write the derivatives with respect to variable t_i $i \in (1, ...n)$ we get

$$\langle \frac{\partial \delta}{\partial t_i}, \varphi \rangle = -\langle \delta, \frac{\partial \varphi}{\partial t_i} \rangle = -\frac{\partial \varphi}{\partial t_i}(0) \tag{4.5}$$

In general,from the above equation 4.5 we immediately get that if $k = (k_1, ..., k_n) \in (\Re^+)^n$,

$$\langle D^k \delta, \varphi \rangle = (-1)^{|k|} D^k \varphi(0)$$

EXAMPLE 4.3.2. *consider a Heaviside function* $H(x) = \begin{cases} 0, & \textit{if } x < 0 \\ 1, & ;x > 0. \end{cases}$ *if we compute its derivatives for* $\varphi \in D(\Re)$,

$$\begin{aligned} \langle H^{'}(t), \varphi \rangle &= -\langle H(t), \varphi^{'} \rangle \\ &= -\int_{\Re} H(t) \varphi^{'}(t) dt \\ &= -\int_0^{\infty} \varphi^{'}(t) dt \end{aligned}$$

Now as φ is a compact support ,we can find $k > 0$ such that the support is contained in $[-k, k]$ and

$$-\int_0^{\infty} \varphi^{'}(t) dt = -\int_0^{\infty} \varphi^{'}(t) dt = \int_0^{k} \varphi^{'}(t) dt$$

$$= -\varphi(k) + \varphi(0) = \varphi(0) = \langle \delta, \varphi \rangle$$

Therefore,in terms of distributions,we get

$$H^{'}(t) = \delta(t) \tag{4.6}$$

4.4 The Schwartz Space S

DEFINITION 4.4.1. *Let $\phi : \Re^n \to C$ be a function we say that ϕ is a Schwartz function if*

1. *ϕ is c^{∞}*
2. *for every $m \in z^+$, $k \in z^+$ there exists a constant $c_m, k > 0$ such that*

$$|t|^m |D^k \phi(t)| \leq c_m, k > 0; \forall t \in \Re^n$$

The space of all Schwartz function is called Schwartz space and denoted by $S(\Re^n)$

DEFINITION 4.4.2. *Let* $\{\phi_n\}_{n=1}^{\infty}$ *be a sequence in the Schwartz space* S *we say that the sequence converges to zero in* S *if for every* $m \in z^n$, $k \in (Z^{+)}$ *the sequence* $\{|t|^m D^k \phi_m(t)\}_{n=1}^{\infty}$ *converges to zero uniformly Following this definition ,we say that the sequence* $\{\phi_n\}_{n=1}^{\infty}$ *converges to* ϕ *if the sequence* $\{\phi_n(t) - \phi(t)\}_{n=1}^{\infty}$ *converges to zero.*

4.5 The Space $S^{'}$ of Tempered Distribution

DEFINITION 4.5.1. *A mapping* $T : S(\Re^n) \to C$ *is called a tempered distribution if*

1. *it is linear,in the sense that if* $\phi, \varphi \in S(\Re^n)$ *and* $a, b \in C$ *then*

$$T(a\phi + b\varphi) = aT(\phi) + bT(\varphi)$$

2. *it is continuous ,that is if* $\{\phi_n\} \to \phi$ *in* $S(\Re^n)$ *then* $\{T(\phi_n)\} \to T(\phi)$ *in* C

Properties of Tempered Distribution

1. The sum of a distributions $f + g$, $|\langle f + g, \phi\rangle| \leq |\langle f, \phi\rangle| + |\langle g, \phi\rangle$

2. Multiplication by constants by a constant αf as

$$|\langle \alpha f, \phi\rangle| \leq \|\alpha||\langle f, \phi\rangle|$$

3. The shifting of a tempered distribution,$f(t - \tau)$,as

$$\langle f(t - \tau), \varphi\rangle = \langle f, \phi(t + \tau)\rangle$$

$$|t|^m |D^k \phi_n(t + \tau)| \leq 2^m \{|t + \tau|^m |D^k \phi_n(t + \tau)| + |\tau^m||D^k \phi_n(t + \tau)|\}$$

4. The translation of a distribution $f(-t)$ as in the previous case ,it is enough to see that the sequence $\{\phi_n\}$ converges to zero in S but this is trivial. because $|t| = |-t|$

5. The dilation of a tempered distribution $f(at)$,as

$$|t|^m|D^k\phi_n(\frac{t}{a})| = |a|^m|\frac{t}{a}|^m|D^k\phi_n(\frac{t}{a})|$$

   And right hand side term converges uniformly to zero.

6. The derivative of tempered distribution,$\frac{\partial f}{\partial t_i}$ It is immediate from the definition of convergence in S that the sequence $\{\frac{\partial t}{\partial t_i}\}$ converges to zero in S, so the derivative of a tempered distribution in continuous and thus a tempered distribution.

4.6 Convolution and Direct Product of Generalized Functions

4.6.1 Convolution of Generalized Functions

DEFINITION 4.6.1. *Let f and g be functions locally integrable in $\Re^n$,where the function $h(x)\int|g(x)f(x-y)|dy$ is locally integrable in $\Re^n$ .the function $(f*g)(x) = \int f(y)g(x-y)dy$*

$$= \int g(y)f(x-y)dx = (g*f)(x) \tag{4.7}$$

is known as the convolution $f*g$ of these functions

The function in equation 4.7 is locally integrable in $\Re^n$ and there for defines a regular distribution (generalized function)acting on the test functions $\varphi \in D(\Re^n)$ according to the rule:

$$\begin{aligned}(f*g,\varphi) &= \int(f*g)(\xi)\varphi(\xi)d\xi \\ &= \int[g(y)f(\xi-y)dy]\varphi(\xi)d\xi \\ &= \int g(y)[\int f(\xi-y)\varphi(\xi)d\xi]dy \\ &= \int g(y)[\int f(x)\varphi(x+y)dx]dy\end{aligned}$$

(by virtue of Fubini's theorem) that is

$$(f * g, \varphi) = \int f(x)g(y)\varphi(x+y)dxdy, \varphi \in D(\Re^n)$$

Note: If the convolution $f * g$ exists,then there is also a convolution $g * f$, and they are equal $f * g = g * f$ that is,the convolution is commutative

Differentiation of a Convolution

If the convolution $f * g$, $D^\alpha f * g$,$D^\alpha(f * g)$ and $f * D^\alpha g$ exist and more over

$$D^\alpha f * g = D^\alpha(f * g) = f * D^\alpha g \tag{4.8}$$

4.6.2 Direct Product of Generalized Function

DEFINITION 4.6.2. *Let $f \in D^{'}(\Re^n)$ and $g \in D^{'}(\Re^m)$ be any two distribution there direct product is defined as*

$$\langle f(x).g(y), \varphi(x,y)\rangle = \langle f(x), \langle g(y), \varphi(x,y)\rangle\rangle, \forall\varphi \in D^{(}\Re^{n+m})$$

And the direct product of

$$\langle g(y).f(x), \varphi(x,y)\rangle = \langle g(y), \langle f(x), \varphi(x,y)\rangle\rangle, \forall\varphi \in D^{(}\Re^{n+m})$$

it follows that

$$f(x).g(y) = g(y).f(x) \tag{4.9}$$

4.7 The Fourier Transform in S And in $S^{'}$

4.7.1 The Fourier Transform in S

DEFINITION 4.7.1. *The Fourier transform on the Schwartz space $S(\Re^n)$ is an operator $F : S(\Re^n) \to L^\infty$ which assigns to every $\phi \in S(\Re^n)$ the function*

$$F\phi\xi = \hat{\phi}(\xi) = \int_{\Re^n} \phi(x)e^{-i\xi x}dx$$

Some Properties of Fourier Transform in S

proposition 4.7.1 Let $\phi \in S(\Re^n)$ be a Schwartz function ,Then for every $k \in (Z^+)^n$ the following properties holds

1. $D^kF(\phi)(\xi) = (-i)^|k|F(x^k\phi)(\xi)$
2. $\xi^kF(\phi)(\xi) = (-i)^|k|F(D^k\phi)(\xi)$
3. If$\tau \in \Re^n$,then $F(\phi(x-\tau))(\xi) = e^{-i\xi\tau}F\phi(\xi)$
4. If$\lambda \in \Re^n$,then $F(\phi(\lambda x))(\xi) = \frac{1}{|\lambda|^n}(\frac{\xi}{\lambda})$
5. If $\phi, \varphi \in \Re^n$,then $\int_{\Re^n} F\phi(x)\varphi(x)dx = \int_{\Re^n} \phi(x)F\varphi(x)dx$

4.7.2 The Fourier Transform in S'

DEFINITION 4.7.2. *Let $f \in S^{'}(\Re^n)$ be a tempered distribution ,the Fourier transform of f as a functional over $S(\Re^n)$ denoted by Ff or $\hat{f}$ given by*

$$\langle Ff, \varphi\rangle = \langle f, F\phi\rangle, \forall\phi \in S(\Re^n)$$

In fact, if ϕ is a Schwartz function and we treat it as a distribution then for every $\varphi \in S$

$$\langle F\phi, \varphi\rangle = \int_{\Re^n} F(\phi)\varphi(x)dx$$

and by property 5 in the above proposition we can write

$$\int_{\Re^n} F\phi(x)\varphi(x)dx = \int_{\Re^n} \phi(x)F\varphi(x)dx = \langle\phi, F\varphi\rangle$$

DEFINITION 4.7.3. *Let $f \in S^{'}(\Re^n)$ be a tempered distribution .Its inverse Fourier transform denoted by $F^{-1}f$ or f is defined as follows*

$$\langle F^{-1}f, \varphi\rangle = \langle f, F^{-1}\phi\rangle, \forall\phi \in S(\Re^n)$$

REMARK 4.7.1. *Let f be a tempered distribution and denote by ˜ $\tilde{f}$ its transposition. Let also $k \in (Z^+)^n$ be a multi-index.*

i $D^k(Ff) = (-i)^{|k|}F(x^kf)$

ii $x^kFf = (-i)^{|k|}F(D^kf)$

iii $F^2 f = (2\Pi)^n \tilde{f}$

proposition 4.7.3 Let $f \in S^{'}(\Re^n)$ and $g \in S^{'}(\Re^n)$ then

$$\langle (f(x).g(y)(\xi,\eta))\rangle = Ff(\xi).Fg(\eta)$$

PROOF 4.7.1. *consider $\phi \in S(\Re^{n+m})$ by definition ,we have*

$$\langle F(f(x).g(y)(\xi,\eta)),\phi(\xi,\eta)\rangle = \langle (f(x),\langle g(y)F\phi(\xi,\eta))\rangle\rangle$$

In this point ,we need to remark the fact that the Fourier transform can be computed in different variable .This can be easily checked by using definition.This means for example

$$F\phi(x,y) = F_\eta(F_\xi[\phi](x,y)), \tag{4.10}$$

where $F_\xi\phi(x,\eta) = \int \phi(\xi,\eta)e^{-ix\xi}d\xi$
This method will also be used several times this by equation 4.10

$$\begin{aligned}
\langle f(x),\langle g(y)F\phi(x,y))\rangle\rangle &= \langle f(x),\langle g(y)(F_\eta(F_\xi[\phi](x,y))\rangle\rangle \\
&= \langle f(x),\langle (F_g(\eta),F_\phi(x,\xi))\rangle\rangle \\
&= \langle f(x).F_g(\eta),F_\xi(\phi)(x,\eta)\rangle
\end{aligned}$$

we saw in equation 4.9 that the direct product is commutative.Thus

$$\begin{aligned}
\langle f(x).F_g(\eta),F_\xi(\phi)(x,\eta)\rangle &= \langle F_g(\eta).f(x)F_\xi(\phi)(x,\eta)\rangle \\
&= \langle F_g(\eta)\langle F_f,\phi(\xi,\eta)\rangle \\
&= \langle Ff(\xi).Fg(\eta),\phi(\xi,\eta)\rangle
\end{aligned}$$

EXAMPLE 4.7.1. *check that for every multi-index $k \in (z^+)^n$ and $\gamma \in (\Re^n$,*

$$F(D^k[\delta(x-\gamma)])(\xi) = (i)^{|}k|\xi^k e^{-i\xi\gamma} \tag{4.11}$$

choosing any Schwartz function ϕ,

$$\begin{aligned}
\langle F(D^k[\delta(x-\gamma)]),\phi\rangle &= (-1)^{|k|}\langle\delta(x-\gamma),D^k\hat{\xi}(x)\rangle \\
&= (-1)^{|k|}(-i)^{|k|}\langle\delta(x-\gamma),F(\xi^k\phi)\rangle \\
&= i^{|k|}F(\xi^k)(\gamma) \\
&= i^{|k|}\int_{\Re^n}\xi^k\phi(\xi)e^{-i(\gamma\xi)}d\xi \\
&= \langle (i)^{|k|}\xi^k e^{-i(\xi\gamma)},\phi\rangle
\end{aligned}$$

In particular ,for $k = 0$,$\gamma = 0$

$$F(\delta) = 1(\xi)$$

EXAMPLE 4.7.2. *The Fourier transform of delta function ,*

$$\begin{aligned}\langle F[\delta(x_1, ..., x_n)], \phi\rangle &= \langle \delta(x), \hat{\phi}\rangle \\ &= \langle \delta(x), \hat{\int}_{-\infty}^{\infty} \phi(y) e^{-ixy} dy\rangle \\ &= \int_{-\infty}^{\infty} \phi(y) dy \\ &= \langle 1, \phi\rangle\end{aligned}$$

Thus

$$\hat{\delta}(x) = 1$$

Chapter 5

Fundamental Solution of Differential operator

5.1 Generalized Solution and Fundamental Solutions

Let the following be a linear differential equation of order m

$$\sum_{|\alpha|=0}^{m} a_\alpha(x) D^\alpha u = f \tag{5.1}$$

in which $f \in D^{'}(\Re^n)$ is distribution and the coefficients a_α and c^∞ functions. To shorten the expressions to be used, we will denote the differential operator

$$L(x, D) = \sum_{|\alpha|=0}^{m} a_\alpha(x) D^\alpha \tag{5.2}$$

thus the equation 5.1 gives,

$$L(x, D)u = f \tag{5.3}$$

In general we have looking for a distribution $u \in D^{'}(\Re^n)$ which satisfies equation 5.3

DEFINITION 5.1.1. *Let $L(x, D)$ be a differential operator , $f \in D^{'}(\Re^n)$ and an open set $A \subseteq \Re^n$ we say that a distribution $u \in D^{'}(\Re^n)$ is a generalized* *solution of the equation* $L(x, D)u = f$ *in the region* A *if*

$$\langle L(x, D)u, \varphi\rangle = \langle f, \varphi\rangle$$

For every testing function $\varphi \in D(\Re^n)$ *whose support is contained in* A

Consider now a differential operator with constant coefficients. If we keep notation as in 5.2 in this case we will write

$$L(x, D) = \sum_{|\alpha|=0}^{m} a_\alpha(x)D^\alpha = \sum_{|\alpha|=0}^{m} a_\alpha D^\alpha = L(D) \tag{5.4}$$

In this situation we will be able to obtain some particular solutions which will be of extreme importance .

DEFINITION 5.1.2. *Let* $L(D)$ *be a differential operator with constant coefficients, we say that a distribution* $E \in D^{'}(\Re^n)$ *is a fundamental solution of the differential operator* $L(D)$ *if* E *satisfies*

$$L(D)E = \delta, in D^{'}(\Re^n)$$

THEOREM 5.1.1. *Let* $L(D)$ *be a differential operator with constant coefficients and* E *a tempered fundamental solution of it . Let also* f *be a Schwartz functions on* $\Re^n$ *then a solution to the equation* $L(D) = f$ *is given by*

$$u = E * f$$

PROOF 5.1.1. *we know that* $L(D)E = \delta$ *as* E *is a fundamental solution of the operator* $L(D)$*. Now,if we consider* $E * f$ *as a candidate solution ,and considering notations in equation 5.4*

$$L(D)(E * f) = \sum_{|\alpha|=0}^{m} a_\alpha D^\alpha(E * f)$$

Recall the rules of differential of the convolution given by 4.8 we can write

$$\sum_{|\alpha|=0}^{m} a_\alpha D^\alpha(E * f) = \sum_{|\alpha|=0}^{m} a_\alpha(D^\alpha E * f)$$

By linearity we get

$$L(D)(E * f) = L(D)E * f = \delta * f = f \tag{5.5}$$

The delta function is the identity element with respect to convolution As the consequence, the distribution $E * f$ *is a solution to equation in the statement*

The reason for bring fundamental solutions in to the limelight is clear now, If we manage to obtain one for an operator,much of the work will be done. In any case,it is important to remark that, in general they are not unique .suppose we have been able to get a solution to homogeneous equation $L(D)u = 0$ If we call it E_0,then

$$L(D)(E + E_0) = L(D)E + L(D)E_0 = \delta + 0 = \delta$$

showing that $E + E_0$ is also a fundamental solution.

5.1.1 Fundamental Solution of Heat operator

The very well known heat equation models the evolution of the temperature in a certain space, which could be a stick or a plane,and even objects in spaces of greater dimension, Consider variable $x \in \Re^n$s to be representative of space and variable $t \in \Re$ of time. Then,the heat equation is given as

$$\frac{\partial u}{\partial t}(x,t) = c^2 \triangle_x u(x,t) \tag{5.6}$$

where $\triangle_x$ is the laplace operator concerning variable x and c is a positive constant.Basing on the heat equation 5.6 the heat operator $L_H(D)$ is

$$L_H(D) = D_t - C^2 \sum_{i=0}^{n} D_{x_i}^2 \tag{5.7}$$

where D_{x_i} represents the partial derivatives with respect to the variable x_i. We know that if we want to obtain a fundamental solution of $L_H(D)$ we need to solve the equation

$$L_H(D)E = D_t E(x,t) - C^2 \triangle_x E(x,t) = \delta(x,t)$$

Instead of applying the general Fourier transform as we did in section (5.1) we will only consider the Fourier transform on the variable x, For for that,

$$F_x(\frac{\partial E}{\partial t}) - C^2 F_x(\triangle_x E) = F_x(\delta) \tag{5.8}$$

By properties of derivatives Fourier transform

$$F_x(\frac{\partial E}{\partial t}) = \frac{\partial}{\partial t} F_x(E)(\xi, t) \tag{5.9}$$

Also observe that

$$F_x(\triangle_x) = \sum_{i=1}^{n} F_x(D_{x_i}^2 E)$$

And by properties in remark (4.7.1)

$$F_x(D_{x_i}^2 E) = i^2 \xi_i^2 F_x(E) = -\xi_i^2 F_x(E)$$

This means that

$$F_x(\triangle_x) = -\sum_{i=1}^{n} \xi_i^2 F_x(E) = |\xi|^2 F_x(E) \tag{5.10}$$

It remains to handle the right hand side of equation 5.8

$$F_x(\delta(x,t)) = F_x(\delta(x).\delta(t)) = F_x(\delta(x)).\delta(t)$$

we also know that

$$F_x(\delta(x)) = 1(\xi), so F_x(\delta(x,t)) = 1(\xi).\delta(t) \tag{5.11}$$

combining 5.8, 5.9, 5.10 and 5.11 we obtain

$$\frac{\partial}{\partial t} F_x(E)(\xi,t) + c^2 |\xi|^2 F_x(E)(\xi,t) = 1(\xi).\delta(t) \tag{5.12}$$

observe that we have a differential equation with respect to variable t,

$$\frac{\partial}{\partial t} F(\xi,t) + k(\xi) F(\xi,t) = 1(\xi).\delta(t) \tag{5.13}$$

which in turn if we fix the value of ξ is similar to

$$\frac{d}{dt} G(t) + kG(t) = \delta(t)$$

If G where a function,it could be expressed in terms of an exponential,Nevertheless this is not a big set back as we know that the delta function is the derivatives of the Heaviside function,H so we will be able to give a solution interms of distributions as

$$G(t) = H(t)e^{-kt}$$

Indeed$G^{'}(t) = \delta(t)e^{-kt} - kG(t)$ and observe that $\delta(t)e^{-kt} = \delta(t)$ following this idea, an identical calculation shows that a solution to equation 5.13 is given by

$$F(\xi, t) = (1(\xi).H(t))e^{-c^2|\xi|^2t}$$

Where the only variation is that $D_t(1(\xi).H(t)) = 1(\xi).D_tH(t)$ we can simplify that expression. In fact

$$\langle 1(\xi).H(t))e^{-c^2|\xi|^2t}, \varphi(\xi, t)\rangle = \langle H(t), \langle 1(\xi), e^{-c^2|\xi|^2t}\varphi(\xi, t)\rangle\rangle$$

$$= \int_{\Re}\int_{\Re^n} H(t), e^{-c^2|\xi|^2t}\varphi(\xi, t)d\xi dt$$

$$= \langle H(t)e^{-c^2|\xi|^2t}, \varphi(\xi, t)\rangle$$

from this

$$F(\xi, x) = H(t)e^{-c^2|\xi|^2t}$$

In short if the fundamental solution we are looking for is E then by inverse Fourier transform ,

$$E = F_{\xi}^{-1}(H(t)e^{-c^2|\xi|^2t})$$

the transformation no effect on H this can be generalized to distributions

$$E = \frac{H(t)}{(2\Pi)^n}F_{\xi}(e^{-c^2|\xi|^2t})$$

By Fourier transform of Gaussian

$$E(x, t) = \frac{H(t)}{(2\Pi)^n}(\sqrt{\frac{\Pi}{c^2t}})^n e^{\frac{-|x|^2}{4c^2t}}$$

$$= \frac{H(t)}{(2c\sqrt{\Pi t})^n} e^{\frac{-|x|^2}{4c^2t}}$$

Bibliography

[1] V.S.Vladimirov.Equation Of mathematical Physics.MARCEL DEKKER,INC,New York,1971 pp. 63-193.

[2] Ram P.Kanwal.Genralized Functions,Theory and technique,volume 171,Acadamic press,New York,$1983, 75-194$

[3] T. Muthukumar, Partial Differential Equation.tmk@iitk.ac.in $April 18, 2014,$