ON DUFFING'S EQUATION WITH DIRICHLET BOUNDARY CONDITION.

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The undersigned hereby certify that they have read and recommend to the department of mathematics for acceptance of this project entitled "ON DUFFING'S EQUATION WITH DIRICHLET BOUNDARY CONDITION" by **Aman Tunshura** in partial fulfillment of the requirements for the degree of Master of Science in mathematics.

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Abstract

We use a direct variational method in order to investigate the dependence on parameter for the solution of the following Duffing type equation with Dirichlet boundary conditions

$$\ddot{x}(t) + r(t)\dot{x}(t) - F_x(t, x(t)) - f(t) = 0$$

$$x(0) = x(1) = 0$$

and we observe that some outline of variational approach, in order to use this method.

Key Words and phrases: Dirichlet boundary problem,Duffing equation and Lower semicontinuity

Notations

a.e	Almost everywhere.
Ω	An open subset of $\mathbb{R}^N$.
$\overline{\Omega}$	The closure of Ω in $\mathbb{R}^N$.
$\partial\Omega$	The boundary of Ω i.e $\partial\Omega = \overline{\Omega} \backslash \Omega$.
$\Omega' \Subset \Omega$	If $\overline{\Omega'} \subset \Omega$ and $\overline{\Omega'}$ is compact.
$\partial_i u$	$u_{x_i} = \dfrac{\partial u}{\partial x_i}$.
$\|\alpha\|$	$\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n$.
D^α	$\dfrac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \dfrac{\partial^{\alpha_n}}{\partial x_1^{\alpha_n}}$ for $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \in \mathrm{N}_0^n$ is multi-index.
D_w^α	The weak derivatives.
$\|\cdot\|$	A seminorm on a Banach space, or an Euclidean norm in $\mathbb{R}^n$.
$\\|\cdot\\|$	A norm on a Banach space
$\\| x \\|_X$	Norm of x in space X
$L^1_{loc}(\Omega)$	Locally integrable function over Ω.
$suppf$	The support of a function $f : \Omega \to \mathbb{R}$ is defined by $\overline{\{x \in \Omega : f(x) \neq 0\}}$.
$C^\infty(\Omega)$	The space of infinitely often continuousely differentiable functions on Ω.
$C_0^\infty(\Omega)$	The space of infinitely often continuousely differentiable functions with compact support on Ω.
$L^p(\Omega)$	$\left\{f : \Omega \to \mathbb{R} \middle\| \int_\Omega \|f(x)\|^p dx < \infty\right\}$
$D(\Omega)$	The set of all test functions.
$W^{k,p}(\Omega)$	The Sobolev space of functions whose distributional derivatives up to k^{th} order belongs to $L^p(\Omega)$.
$W^{1,p}(\Omega)$	First order Sobolev space on Ω.
$W_0^{k,p}(\Omega)$	The closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$.
$H_0^k(\Omega)$	$W_0^{k,2}(\Omega)$.
$H^{-1}(\Omega)$	The dual space of $H_0^1(\Omega)$.
wlsc	weakly lower semicontinuous
lsc	lower semicontinuous

Introduction

The theory of differential equation oftentimes involves the theory of function spaces that are defined in terms of properties of pertinent functions and their derivatives. In this regard, Sobolev space turn out to be one with suitable settings as compared to the classical Banach space of smooth functions $C^n(\Omega)$ and an ordinary differential equation arise in many different context including geometry,mechanics, astronomy, population,etc.In this paper we need to introduce Duffing type equation,which is non linear second order ordinary differential equation used to model certain damped and driven oscillators and we are concerned with the variational formulation for the duffing equation and study dirichlet boundary value problem for duffing equation of the form,

$$\ddot{x}(t) + r(t)\dot{x}(t) + G(t, x(t), u(t))=0$$
$$x(0) = x(1) = 0$$

where $r \in C^1(0,1)$ stands for friction term , $r(\tau) \geq 0$ **for** $\tau \in [0,1]$. Here we do not assume anything about the monotonicity of r, but instead we require that

$$\tfrac{1}{4}r^2 + \tfrac{1}{2}\dot{r}(t) \geq 0 \text{ for all } t \in [0,1]$$

and G is a nonlinear term, satisfying some suitable assumptions, so G can correspond to a restoring force for a string in string-damper system. The Duffings equation was also found applicable for some problems concerning current and flux, thus r and G may as well correspond to its coefficients. In classical variational problem we introduce control function $u \in H_0^1(0,1)$ with only function G dependent on it. Our main result consist of some sufficient conditions for the existence of a solution to the above equation and variational approach was found successful in proving existence of solution to the above problem.In the boundary value problems for differential equations it is also important to know whether the solution, once its existence is proved, depends continuously on a functional parameter. This question has a great impact on future applications of any model since it is desirable to know whether the solution to the small deviation from the model would return, in a continuous way, to the solution of the original model. So using this method we investigate the existence of solution for above eqution and the differential equation with an integral equation,using integration by part to reduce the order of derivatives,the so called **Variational form**. we discuss more in chapter three and four. In the preparation of this paper the following sequence of narratives are adopted. In chapter 1,we will see basic concept. In chapter 2, we will see description of differential operator using multi-index notation followed by the notion of test functions and weak derivatives of the functions. In chapter 3 we see boundary value problem for duffing Equation. Finally, In Chapter 4 we see that weak solution for dirichlet problem,variational formulation for duffing's equation and establish existence of non-trivial weak solution.

Chapter 1

Basic Concepts and Definitions

1.1 Initial Definitions and Basic Theory

Definition 1.1.1. *[16] Let X be a vector space over a field of scalars,then a mapping $\| \cdot \|: X \to [0, \mathbb{R})$ is called **norm** if it has the following properties:*

a) $\| x \| \geq 0$*; the equality holds iff* $x = 0$

b) $\| cx \| = |c| \| x \|$ *for every scalar* c

c) $\| x + y \| \leq \| x \| + \| y \|, \quad \forall x, y \in X$

*A vector space endowed with norm is called **normed space**.That is* $\big(X, \| \cdot \| \big)$ *is normed space*

Definition 1.1.2. *[16] Let X be a normed space and let $\{f_n\}$ be sequence in X.*

a) $\{f_n\}$ ***converges*** *to* $f \in X$ *if,* $lim_{n\to\infty} \| f - f_n \| = 0$ *;that is* $\forall \epsilon > 0$ *,* $\exists N > 0, \forall n \geq N \ \| f - f_n \| < \epsilon$*.In this case,we write* $\lim_{n\to\infty} f_n = f$ *or* $f_n \to f$

b) $\{f_n\}$ *is* ***cauchy*** *if* $\forall \epsilon > 0$ *,*$\exists N > 0$ *such that* $\forall\ m, n \geq N, \| f_m - f_n \| < \epsilon$

Definition 1.1.3. *A normed linear space X which does have that all cauchy sequences are convergent is said to be **complete**. A complete normed space is called a **Banach space**.*

Definition 1.1.4. *[16] Let H be a vector space then H is an inner product space if for every $f, g \in H$ there exists a $< f, g >$ called the **inner product** of f and g ,such that a)* $< f, f >$ *is real and* $< f, f > \ \geq 0$

b) $< f, f > \ = 0 \Leftrightarrow f = 0$

c) $< g, f > \ = \overline{< f, g >}$

d) $< af_1 + bf_2, g > = a < f_1, g > + b < f_2, g >$.

Remark 1.1.1. *Each inner product determines a norm by the formula*

$$\| f \| = < f, f >^{\frac{1}{2}}$$

. *If an inner product space H is complete,then it is called* ***Hilbert space***. *In other words,a Hilbert space is a Banach space whose norm is determined by an inner product.*

Definition 1.1.5. *[16] Let X, Y be normed linear spaces over the same field and suppose that $T : X \to Y$, we write either Tx* ***or*** *$T(x)$ to denote image of an element $x \in X$.*

a) *T is* ***linear*** *if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for every $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$*

b) *T is* ***injective*** *if $T(x) = T(y)$ implies $x = y$*

c) *T is* ***continuous*** *if $f_n \to f \Rightarrow T(f_n) \to T(f)$ that is $\lim_{n\to\infty} \| f - f_n \|_X = 0 \Rightarrow \lim_{n\to\infty} \| T(f) - T(f_n) \|_Y = 0$*

d) *T is* ***bounded*** *if there is a finite real number c so that $\| T(f) \|_Y \leq c \| f \|_X$ for every $f \in X$. The smallest such c is called the* ***operator norm*** *of T and denoted by $\| T \|$. That is $\| T \|$ is the smallest number such that $\forall f \in X$*

Definition 1.1.6. *[5]* ***(Dual Space).*** *If X is a normed linear space then its* ***dual space*** *is the set of all continuous linear functional with domain X. The dual space usually denoted by X' or X^*. That is*
$X' = \{T : X \to \mathbb{C}, T$ is continuous and linear$\}$. A Banach space X is ***reflexive*** *if $(X')' = X$.*

Theorem 1.1.1. *[15]* ***Lebesgue dominated convergence theorem***
Let E be lebesgue measurable set and $\{f_n\}$ are lebesgue measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ exists.Assume there is an integral $g : \mathbb{R} \to [0, \infty]$ with $| f_n | \leq g(x)$ for each $x \in E$. Then f is integrable as f_n for each n, and

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E \lim_{n\to\infty} f_n d\mu = \int_E f d\mu$$

.

proof see [15]

Definition 1.1.7. *[16] Let X be normed space. A sequence $\{x_n\} \subseteq X$ is said to be weakly converges to $x \in X$ written as $x_n \to x$ if $f(x_n) \to f(x), \forall f \in X'$.*

Definition 1.1.8. *[15] Let $\{f_n\}$ be sequence of real valued functions defined on a subset Ω of $\mathbb{R}^n$. Let $x \in \Omega$, the sequence is called* ***Equicontinuous*** *at x ,if $\forall \epsilon > 0 \; \exists \, \delta > 0$, independent of n, such that $||f_n(y) - f_n(x)|| < \epsilon$ for $y \in \Omega$ with $||y - x|| < \delta$.*

Theorem 1.1.2. *[16] ARZELA ASCOLI*
Let $\{f_n\}$ be sequence of real-valued functions defined on a compact subset Ω of $\mathbb{R}^n$. Assume that there is a constant M such that $\| f_n(x) \| \leq M$, $\forall \, n \in \mathbb{R}$ and $\forall \, x \in \Omega$. Moreover, assume that the sequence $\{f_n\}$ is equicontinuous at every point of Ω, then there exists a subsequence which converges uniformly on Ω.

Proof: Let $\{X_i\}_{i \in \mathbb{N}}$ be sequence of points that is dense S.
The sequence $f_m(x_1)$ is bounded; hence it has subsequence.i.e,we can choose a sequence m_{1j} such that $f_{m_{1j}}(x_1)$ converges as $j \to \infty$.similarly, we choose subsequence m_{2j} of the sequence m_{1j},such that $f_{m_{2j}}(x_1)$ converges.since m_{2j} subsequence of m_{1j} ,$f_{m_{2j}}(x_1)$ converges as well.
Next, we choose a subsequence m_{3j} of sequence m_{2j} such that $f_{m_{3j}}(x_1)$ converges also at x_3.we proceed in this manner.
Finally,consider the "diagonal" sequence $f_{m_{jj}}(x_1)$. Except for the first $i-1$ terms, m_{jj} is the subsequence of m_{ij} ;hence $f_{m_{jj}}(x_i)$ converges $\forall i \in \mathbb{N}$. To simplfy notation,we shall set $g_i = f_{m_{jj}}$ in the following.
To conclude the proof,we show that the sequence g_m is uniformly cauchy.Let $\epsilon > 0$ be given and g_m,being a subsequence of the f_m, are uniformly equicontinuous on S ;hence there is a $\delta > 0$ such that $| \, g_m(y) - g_m(x) \, | \; < \frac{\epsilon}{3}$ whenever $| \, y - x \, | \; < \delta$. since S is compact, there is a $k \in \mathbb{N}$ such that for every $x \in S$ there exists $| \, x_i - x \, | < \delta$.
Now choose N large enough so that $| \, g_m(x_i) - g_k(x_i) \, | \; < \frac{\epsilon}{3}$ for$m, k > N$ and every $i \in \{1, 2, ..., k\}$ and arbitrary $x \in S$. Now we have
$| \, g_m(x_i) - g_k(x_i) \, | \leq | \, g_m(x) - g_m(x_i) \, | + | \, g_m(x_i) - g_k(x_i) \, | + | \, g_k(x_i) - g_k(x) \, | < \epsilon$
for some $i \in \{1, 2, ..., k\}$. [1]

[1]Let X and Y be sets and $Y \subset X$ we say Y is Dense subset of X, if for every $x \in X$, there is exists a sequence of elements $\{y_n\}_{n \in \mathbb{N}}$ in Y such that $lim_{n\infty} y_n = x$.

1.2 Notions of Convex Function

Definition 1.2.1. *[17] Let X be a banach space.*

a) *A function* $f : X \to \mathbb{R}$ *is said to be convex if*
$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$, $\forall x, y \in X, \forall \lambda \in [0,1]$. *and f strictly convex if*
$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)$

b) *The set defined by* $dom f = \{x \in X : f(x) < \infty\}$ *is called* ***effective domain of*** f.

c) *The* ***epigraph*** *of the function* f *is the set* $epi f = \{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}$

d) *A function* $f : X \to \mathbb{R}$ *is called proper convex if* $f(x) > -\infty; x \in X$ *and* $f(x) \neq +\infty$.

Definition 1.2.2. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set and let* $f : \Omega \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ *then* f *is said to be* ***caratheodory*** *function if*

i) $\xi \to f(x, \xi)$ *is continuous for almost every* $x \in \Omega$

ii) $x \to f(x, \xi)$ *is measurable for every* $\xi \in \mathbb{R}^N$

Theorem 1.2.1. *A function* $f : X \to \mathbb{R}$ *is convex if and only if its epigraph is convex.*

Chapter 2

Overview of Function Spaces

2.1 Lebesgue Spaces

If $\Omega \subset \mathbb{R}^n$ is a domain and $1 \leq p < \infty$, then

$$L^p(\Omega) = \big\{ f : \Omega \to \mathbb{R} : \int_\Omega |f(x)|^p dx < \infty \big\}$$

is space of p-integrable functions on Ω, while for $p = \infty$

$$L^\infty(\Omega) = \big\{ f : \Omega \longrightarrow \mathbb{R} : |f(x)| < M \ a.e \ on \ \Omega, \ M > 0 \big\}$$

is the space of essentially bounded functions on Ω. We recall, for $f \in L^p(\Omega), 1 \leq p \leq \infty$

$$||f||_{L^p(\Omega)} = \begin{cases} \big(\int_\Omega |f(x)|^p dx \big)^{\frac{1}{p}}, \ 1 \leq p < \infty \\ ess \sup_\Omega |f(x)|, \ p = \infty \end{cases}$$

is a normed space which is complete i.e a Banach Space.

Definition 2.1.1. *[5] Locally integrable functions: For $1 \leq p < \infty$, the space of locally integrable functions on Ω, $L^p_{loc} = \{ f : \Omega \to \mathbb{R} s.t \int_K \mid f(x) \mid^p < \infty, \ K \subset\subset \Omega \}$ is the set of functions which are $p-$ integrable on every $K \subset \Omega$ compact.*

Proposition 2.1.1. *For $1 \leq p < \infty$, $L^p(\Omega) \subset L^1_{loc}(\Omega)$*

Proof: Let $f \in L^p(\Omega)$.

$$If \ \amalg_k = \begin{cases} 1, & x \in K \\ 0, & x \notin K \end{cases}$$

is an indicator function of $K \subset\subset \Omega$, then for $1 \leq q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned} \int_K |f(x)|dx &= \int_\Omega \amalg_k |f(x)|dx \\ &\leq \| \amalg_k \|_{L^q} \|f\|_{L^p}, \quad (Hölder's\ inequality) \\ &< \infty \end{aligned}$$

Therefore $f \in L^1_{loc}(\Omega)$

Lemma 2.1.1. *[5]* L^P***-space Inequality***

a) *(Minkowski)*
For $1 \leq p \leq \infty$ *and* $f, g \in L^p(\Omega)$ *then* $f + g \in L^p(\Omega)$
and $\| f + g \|_p = \| f \|_p + \| g \|_p$

b) *(Hölder inequality)*
For,$p, q \in [1, \infty)$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p, g \in L^q$ *implies*
$fg \in L^p$ *and* $\| fg \|\|_1 \leq \| f \|_p \| g \|_q$

Proof: see [5]

Theorem 2.1.1. *[5]* ***(Riesz representation theorem)****: Let* $1 < p < \infty$ *and let* $\varphi \in (L^p)^*$ *Then there exists a unique function* $u \in L^{p'}$ *such that*

$$< \phi, f >= \int uf, \quad \forall f \in L^p.$$

Moreover, $\| u \|_{p'} = \| \phi \|_{(L^p)^*}$.

Remark 2.1.1. *This theorem says that every continuous linear functional on* L^p *with* $1 < p < \infty$ *can be represented concretely as an integral. The mapping* $\phi \to u$*, which is a linear surjective isometry, allows us to identify the abstract space* $(L^p)^*$ *with* $L^{p'}$.

Proof:see [5]

Definition 2.1.2. *The support of the function* f *on* Ω*is*

$$supp\ f = \overline{\{x \in \Omega : f(x) \neq 0\}}$$

Evidently, the space $C_0(\Omega)$ *of functions with compact support on* Ω *is a linear space, that is*

i. $f \in C_0(\Omega) \Longrightarrow \alpha f \in C_0(\Omega), \quad \alpha \in \mathbb{R}$

ii. $f, g \in C_0(\Omega) \Longrightarrow f + g \in C_0(\Omega)$

2.2 Multi-index and Derivatives

The use of multi-index notation for partial differential operators renders an efficient and ultra convenient settings for description of differential equations.

2.2.1 Multi-index

An array $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}_0^n$ of non-negative integers is what we call n-dimensional multi-index. With multi-index α we associate the following scalars,

i. $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$

ii. $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$

The operation of addition on the product set $\mathbb{N}_0^n$ is introduced component wise, that is

$$\alpha + \beta = (\alpha_1 + \beta_1, \cdots, \alpha_n + \beta_n)$$

Proposition 2.2.1. *The relation $\sim$ defined on $\mathbb{N}_0^n$ by*

$$\alpha \sim \beta \;\; iff \;\; \alpha_i \leq \beta_i \; ; \;\; i = 1, 2, \cdots, n$$

is a partial ordering.

Proof: i) Let $\alpha \in \mathbb{N}_0^n$ be arbitrary.
Since, $\alpha_i \leq \alpha_i \;\; for \;\; i = 1, 2, \cdots, n$
we have $\alpha \sim \alpha$
$\Rightarrow$ " $\sim$ " is reflexive.

ii) Let $\alpha, \beta \in \mathbb{N}_0^n$ such that
$\alpha \sim \beta \;\; \wedge \;\; \beta \sim \alpha$
$\Rightarrow \alpha_i \leq \beta_i \;\; \wedge \;\; \beta_i \leq \alpha_i \;\; i = 1, 2, \cdots, n$
$\Rightarrow \alpha_i = \beta_i \; ; \;\; i = 1, 2, \cdots, n$
$\Rightarrow \alpha = \beta$
$\Rightarrow$ " $\sim$ " is antisymmetric.
iii) Let $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ such that
$\alpha \sim \beta \;\; \wedge \;\; \beta \sim \gamma$
$\Rightarrow \alpha_i \leq \beta_i \;\; \wedge \;\; \beta_i \leq \gamma_i \;\; i = 1, 2, \cdots, n$
$\Rightarrow \alpha_i \leq \gamma_i \; ; \;\; i = 1, 2, \cdots, n$
$\Rightarrow \alpha \sim \gamma$
$\Rightarrow$ " $\sim$ " is transitive.

Consequently, " $\sim$ " is partial ordering, and $(\mathbb{N}_0^n, \sim)$ is partial ordered set. One can employ multi-indexes to describe a monomial in $\mathbb{R}^n$ and a polynomial of degree k with n-independent variables, that is if $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$ then a

i) Monomial

$$X^\alpha = \prod_{i=1}^{n} X_i^{\alpha_i}$$

ii) Polynomial of degree k

$$P(x) = \sum_{|\alpha| \leq k} C_\alpha X^\alpha$$

2.2.2 Derivatives and Differential Operators

We write D_k for the usual partial derivative with respect to the k^{th}-independent variable. As a result, the expression

$$D = (D_1, \cdots, D_n)$$

represents gradient.
If $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is n-dimensional multi-index with length $|\alpha| = k$ then the expression

$$D^\alpha u = D_1^{\alpha_1} u \cdots D_n^{\alpha_n} u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

stands for α^{th} derivative of u of order k.

Obviously, $|\alpha| = 0 \quad \Rightarrow \alpha = (0, 0, \cdots, 0)$

$\Rightarrow D^\alpha = D^0 =$ identity operator

$|\alpha| = 1 \quad \Rightarrow \alpha \in \big\{(1, 0, \cdots, 0), (0, 1, \cdots, 0), \cdots, (0, \cdots, 1)\big\}$

$\Rightarrow D^\alpha \in \big\{D_1, D_2, \cdots, D_n\big\}$ that is one of the n first partials.

For $|\alpha| = 2$, D^α is one of the $\frac{n(n+1)}{2}$ second order partials.

A linear partial differential operator, L of order k can now be expressed as

$$L := \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha, \quad x \in \mathbb{R}^n$$

consequently, a linear PDE of order k in n-independent variables is given by

$$\sum_{|\alpha|\leq k} a_\alpha(x) D^\alpha u = b(x)$$

and the corresponding quasi-linear PDE of order k is

$$\sum_{|\alpha|\leq k} a_\alpha\big(x, (D^\beta u)_{|\beta|\leq k-1}\big) D^\alpha u = b\big(x, (D^\beta u)_{|\beta|\leq k-1}\big)$$

2.3 Test Functions

Given a non-negative integer k, we say that a function

$$f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$$

is k-times continuously differentiable if

i) $D^\alpha f$ exists on Ω

ii) $D^\alpha f$ is continuous

for all multi-indexes $\alpha \in \mathbb{N}_0^n \; with \; |\alpha| \leq k$

Notation:$C^k(\Omega) = \big\{ f : \Omega \to \mathbb{R} : D^\alpha f \in C(\Omega), \;\; |\alpha| \leq k \big\}$
For $\Omega \subseteq \mathbb{R}^n$ a domain, the space of infinitely often continuously differentiable functions on Ω is given by

$$C^\infty(\Omega) = \big\{ f : \Omega \to \mathbb{R} : D^\alpha f \in C(\Omega), \;\; \forall \alpha \in \mathbb{N}_0^n \big\}$$

and the space of infinitely often continuously differentiable functions with compact support on Ω is given by

$$C_0^\infty(\Omega) = \big\{ f : \Omega \to \mathbb{R} : f \in C_0(\Omega) \cap C^\infty(\Omega) \big\}$$

Notation: $D(\Omega) := C_0^\infty(\Omega)$
The set $D(\Omega)$ as above is the set of test functions on Ω
For demonstration, consider the function

$$f(x) = \begin{cases} e^{\frac{-1}{1-\|x\|^2}}, & \| x \| \leq 1 \\ 0 \;, & \| x \| > 1. \end{cases}$$

Claim! $f \in C_0^\infty(\mathbb{R}^n)$
Set $f(x) = g(1- \parallel x \parallel^2)$ where

$$g(t) = \begin{cases} e^{\frac{-1}{t}}, & t > 0 \\ 0\ , & else. \end{cases}$$

For $t > 0, e^{\frac{-1}{t}} \in C^\infty$ and
for $t < 0, g(t) = 0 \in C^\infty$ at $t = 0, g(0) = 0$ and $g(0+h) = e^{\frac{-1}{h}}$

$$\begin{aligned} \Rightarrow \lim_{h\to 0} \frac{g(h) - g(0)}{h} &= \lim_{h\to 0} \frac{e^{\frac{-1}{h}}}{h} \\ &= \lim_{h\to 0} \big(\frac{1}{h}\big)\big(\frac{1}{e^{\frac{1}{h}}}\big) = \infty \cdot 0 \end{aligned}$$

$$\textbf{set} \quad k = \frac{1}{h} \quad \textbf{then} \quad h = \frac{1}{k} \quad and \quad e^{\frac{1}{h}} = e^k$$

$$\begin{aligned} \Rightarrow \lim_{h\to 0} \frac{g(h) - g(0)}{h} &= \lim_{h\to 0} \big(\frac{1}{h}\big)\big(\frac{1}{e^{\frac{1}{h}}}\big) \\ &= \lim_{k\to \infty} \big(k\big)\big(\frac{1}{e^k}\big) \\ &= \lim_{k\to \infty} \frac{k}{e^k} = \big(\frac{\infty}{\infty}\big) \\ &= \lim_{k\to \infty} \frac{1}{e^k} \quad (L^{'}HopitalsRule) \\ &= 0 \end{aligned}$$

$$\Rightarrow g^{'}(0) = 0 \quad \textbf{evidently}, \quad g^{[n]}(0) = 0,\ \forall\ n \in \mathbb{N}$$
$$\Rightarrow \quad g \in C^\infty(\mathbb{R}^n)$$

$$\textbf{If} \quad h(x) = \begin{cases} 1- \parallel x \parallel^2, & \parallel x \parallel < 1 \\ 0\ , & \parallel x \parallel \geq 1. \end{cases} = \begin{cases} 1 - x^2, & \mid x \mid < 1 \\ 0\ , & \mid x \mid \geq 1. \end{cases}$$

$$\textbf{then} \quad h^{'}(x) = \begin{cases} -2x, & 0 < \mid x \mid < 1 \\ 0\ , & \mid x \mid \geq 1. \end{cases} \quad \textbf{at} \quad x = 0 \Rightarrow h(0) = 1 \quad \textbf{and} \quad h(x+k) = h(k) = -k^2$$

$$\begin{aligned} \Rightarrow \lim_{k\to 0} \frac{h(k) - h(0)}{k} &= \lim_{k\to 0} \frac{\big(1 - k^2\big) - 1}{k} \\ &= \lim_{k\to 0} \big(-k\big) = 0 \end{aligned}$$

$$\Rightarrow h^{'}(0)=0 \;\Rightarrow\; h^{'}(x)=\begin{cases}-2x, & 0<\mid x\mid<1\\ 0\;, & x=0\vee\mid x\mid\geq 1.\end{cases} \quad \textbf{and } h^{''}(x)=\begin{cases}-2, & 0<\mid x\mid<1\\ 0\;, & \mid x\mid\geq 1 \textbf{ and at } x=0\end{cases}$$

$$\lim_{k\to 0}\frac{h^{'}(k)-h^{0}}{k}=\lim_{k\to 0}\frac{-2k-0}{k} \;=\; \lim_{k\to 0}(-2)=-2=h^{''}(0)$$

$$\Rightarrow h^{''}(x)\begin{cases}-2, & 0<\mid x\mid<1\\ 0\;, & \mid x\mid\geq 1\end{cases} \quad \Rightarrow h^{'''}(x)=0,\forall\; x \quad \Rightarrow h\in C^{\infty}$$

$$supph=[-1,1]\Rightarrow h\in C_0(\mathbb{R}^n)\Rightarrow supp\; f=supp\; goh=[-1,1]$$

$\therefore f\in C_0^{\infty}(\mathbb{R}^n)=D(\mathbb{R}^n)$

Hence f is a test function.

2.4 Weak Derivatives

In this part, we introduce the notion of weak derivatives, a situation whereby significant weakening of the classical notion of derivatives takes place. The definition of weak derivatives rests on the known concept of integration by parts,

$$\int_{\Omega}u_{x_i}\phi dx=-\int_{\Omega}u\phi_{x_i}dx, \quad \phi\in D(\Omega)$$

Definition 2.4.1. *(Weak Derivatives) Let $f\in L^1_{loc}(\Omega)$ and $\alpha\in\mathbb{N}_0^n$. If there is $g\in L^1_{loc}(\Omega)$ such that*

$$\int_{\Omega}fD^{\alpha}\phi dx=(-1)^{|\alpha|}\int_{\Omega}g\phi dx, \quad \phi\in D(\Omega)$$

then we say that g is the α^{th}-weak derivative of f on Ω.

Notation: $g=D^{\alpha}_w f$

Some of the typical properties of weak derivatives are stated in the lemma that follow.

Lemma 2.4.1. *(Uniqueness)*
Let $f\in L^1_{loc}(\Omega)$ such that $D^{\alpha}_w f$ exists for $\alpha\in\mathbb{N}_0^n$.
If g and h are α^{th}-weak derivative of f on Ω, then $g=h$ a.e on Ω.

Proof. By definition of weak derivatives; we have

$$g=D^{\alpha}_w f \;\;\Rightarrow\int_{\Omega}fD^{\alpha}\phi dx=(-1)^{|\alpha|}\int_{\Omega}g\phi dx,\;\;\forall\phi\in D(\Omega) \tag{2.1}$$

$$h=D^{\alpha}_w f \;\;\Rightarrow\int_{\Omega}fD^{\alpha}\phi dx=(-1)^{|\alpha|}\int_{\Omega}h\phi dx,\;\;\forall\phi\in D(\Omega) \tag{2.2}$$

From (1.1) and (1.2) we have

$$\int_\Omega g(x)\phi(x) - \int_\Omega h(x)\phi(x)dx = 0, \ \ \forall\phi \in D(\Omega)$$

$$\Rightarrow \ \int_\Omega (g(x) - h(x))\phi(x)dx = 0, \ \ \forall\phi \in D(\Omega)$$

$$\Rightarrow \ (g(x) - h(x))\phi = 0 \ a.e \ \ on \ \ \Omega, \ \ \forall\phi \in D(\Omega)$$

$$\Rightarrow \ g(x) - h(x) = 0 \ a.e \ \ on \ \ \Omega$$

$$\Rightarrow \ g(x) = h(x) \ a.e \ \ on \ \ \Omega$$

□

Lemma 2.4.2. *(Linearity)*
Let $f, g \in L^1_{loc}(\Omega)$ and $\alpha \in \mathbb{N}_0^n$. If f and g have α^{th}-weak derivative on Ω, then so does $f + g$, further more,

$$D_w^\alpha(c_1 f + c_2 g) = c_1 D_w^\alpha f + c_2 D_w^\alpha g$$

Proof 2.4.1 (***Proof***)**.** *Suppose $h_1 = D_w^\alpha f$, $h_2 = D_w^\alpha g$, then*

$$\begin{aligned}
\int_\Omega (c_1 f + c_2 g) D^\alpha \phi dx &= \int_\Omega c_1 f D^\alpha \phi dx + \int_\Omega c_2 g D^\alpha \phi dx \ \ \forall\phi \in D(\Omega), \ \forall\phi \in D(\Omega) \\
&= c_1 \int_\Omega f D^\alpha \phi dx + c_2 \int_\Omega g D^\alpha \phi dx, \ \forall\phi \in D(\Omega) \\
&= c_1(-1)^{|\alpha|} \int_\Omega h_1 \phi dx + c_2(-1)^{|\alpha|} \int_\Omega h_2 \phi dx, \ \forall\phi \in D(\Omega) \\
&= c_1 D_w^\alpha f + c_2 D_w^\alpha g
\end{aligned}$$

Lemma 2.4.3. *(Commutativity)*
Let $f \in L^1_{loc}(\Omega)$ such that $D_w^\alpha f$ exists for $|\alpha| = k$ and $\phi \in D(\Omega)$.
If $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha + \beta| = |\alpha| + |\beta| = k$ then

$$D_w^\alpha(D_w^\beta f) = D_w^\beta(D_w^\alpha f)$$

Proof.

$$\begin{aligned}
\int_\Omega D^\alpha(D^\beta f)\phi dx &= (-1)^{|\alpha|}\int_\Omega (D^\beta f)(D^\alpha\phi)dx \\
&= (-1)^k \int_\Omega f D^\beta(D^\alpha\phi)dx \\
&= (-1)^k \int_\Omega f D^{\beta+\alpha}\phi dx \\
&= (-1)^k \int_\Omega f D^\alpha(D^\beta\phi)dx \\
&= (-1)^{k+|\alpha|} \int_\Omega D^\alpha f D^\beta\phi dx \\
&= (-1)^{2k} \int_\Omega D^\beta(D^\alpha f)\phi dx \\
&= \int_\Omega D^\beta(D^\alpha f)\phi dx
\end{aligned}$$

□

Function spaces visited thus far are not adequate for indepth treatment of partial differential equations. Treatment of the theory of partial differential equations, especially, approximation (interior) of solutions by smooth functions requires a space with suitable settings. Hence, the need for Sobolev Space.

2.5 Sobolev Spaces

If $\Omega \subset \mathbb{R}^n$ is a domain and k is non-negative integer, the space of functions on Ω whose α^{th}-weak derivative exists for various orders (up to k)

$$W^k(\Omega) = \{f : \Omega \to \mathbb{R} \mid \; D_w^\alpha \;\; \textbf{exists} \;\; \forall\alpha \quad \mid \alpha \mid \leq k\}.$$

The space $W^k(\Omega)$ has the following properties,

i) $W^k(\Omega)$ is a linear space.

ii) $C^k(\Omega) \subset W^k(\Omega)$

Property i) is an immediate consequence of the linearity property of weak derivatives while ii) follows from the fact that classical differentiability implies weak differentiability.

For $1 \leq p \leq \infty$ and k a non-negative integer, the space of k-times weakly differentiable functions with p-integrable weak derivatives on Ω,

$$W^{k,p}(\Omega) = \left\{f \in W^k(\Omega) : D_w^\alpha f \in L^p(\Omega), \;\; |\alpha| \leq k\right\}$$

is a linear space.
We define a norm on the space $W^{k,p}(\Omega)$ by

$$||f||_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\le k}\int_\Omega |D_w^\alpha f|^p dx\right)^{\frac{1}{p}}, & 1\le p<\infty \\ \sum_{|\alpha|\le k} ess\sup_\Omega |D_w^\alpha f|, & p=\infty \end{cases}$$

We make the following observation,

i) Since $|D_w^\alpha f| \ge 0, \quad \forall \alpha \in \mathbb{N}_0^n$
we have $||f||_{W^{k,p}(\Omega)} \ge 0$ and $||f||_{W^{k,p}(\Omega)} = 0$ if $f = 0$ a.e on Ω.

ii)By linearity of weak derivative, for any scalar, $D_w^\alpha(\lambda f) = \lambda D_w^\alpha f$.
$\Rightarrow |D_w^\alpha(\lambda f)| = |\lambda||D_w^\alpha f|$
$\Rightarrow ||\lambda f||_{W^{k,p}(\Omega)} = |\lambda|||f||_{W^{k,p}(\Omega)}$

iii) For any $f, g \in W^{k,p}(\Omega)$ and $1 \le p < \infty$,

$$\begin{aligned} ||f+g||_{W^{k,p}(\Omega)} &= \left(\sum_{|\alpha|\le k}\int_\Omega |D_w^\alpha(f+g)|^p dx\right)^{\frac{1}{p}} \\ &= \left(\sum_{|\alpha|\le k}\int_\Omega |D_w^\alpha f + D_w^\alpha g|^p dx\right)^{\frac{1}{p}} \\ &\le \left(\sum_{|\alpha|\le k}\left(||D_w^\alpha f||_{L^p(\Omega)} + ||D_w^\alpha g||_{L^p(\Omega)}\right)^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_{|\alpha|\le k}||D_w^\alpha f||^p_{L^p(\Omega)}\right)^{\frac{1}{p}} + \left(\sum_{|\alpha|\le k}||D_w^\alpha g||^p_{L^p(\Omega)}\right)^{\frac{1}{p}} \\ &= ||f||_{W^{k,p}(\Omega)} + ||g||_{W^{k,p}(\Omega)} \end{aligned}$$

and convinced that $||\cdot||_{W^{k,p}(\Omega)}$ is indeed a norm
As a result, the space

$$\{W^{k,p}(\Omega), ||\cdot||_{W^{k,p}(\Omega)}\}$$

is a normed space. We proceed to show that this space is complete. To this end, let $\{f_n\}_{n=1}^\infty \subseteq W^{k,p}(\Omega)$ be a Cauchy sequence.
$\Rightarrow \{D_w^\alpha(f_n)\}_{n=1}^\infty \subseteq L^p(\Omega)$ is a Cauchy sequence for any α with $|\alpha| \le k$.

$\Rightarrow \lim_{n\to\infty} D_w^\alpha(f_n) = f_\alpha \in L^p(\Omega)$ (Since $L^p(\Omega)$ is complete.)

$\Rightarrow \lim_{n\to\infty} f_n = f \in L^p$; $(\alpha = (0,0,\cdots,0)\ \ D^\alpha = D^0 = identity)$

It remains to show $f \in W^{k,p}(\Omega)$
Now, for $\phi \in D(\Omega)$ fixed,

$$\begin{aligned}\int_\Omega f D^\alpha \phi dx &= \int_\Omega \lim_{n\to\infty} f_n D^\alpha \phi dx \\ &= \lim_{n\to\infty} \int_\Omega f_n D^\alpha \phi dx \\ &= \lim_{n\to\infty} (-1)^{|\alpha|} \int_\Omega D^\alpha f_n \phi dx \\ &= (-1)^{|\alpha|} \int_\Omega \lim_{n\to\infty} D^\alpha f_n \phi dx \\ &= (-1)^{|\alpha|} \int_\Omega f_\alpha \phi dx\end{aligned}$$

$\Rightarrow D_w^\alpha f = f_\alpha \in L^p(\Omega)$

$\Rightarrow f \in W^{k,p}(\Omega)$

Definition 2.5.1. *(Sobolev space) [5] For $1 \le p \le \infty$ and a non-negative integer k, the Banach space of functions on Ω,*

$$\{W^{k,p}(\Omega), ||\cdot||_{W^{k,p}(\Omega)}\}$$

is called a ***Sobolev space****. Sobolev space with $k = 1$ i.e $W^{1,p}(\Omega)$ are often called first order spaces.*

For $1 \le p < n$, the number $p^* = \frac{np}{n-p}$ is the Sobolev conjugate of p. Since $\frac{1}{p^*} = \frac{n-p}{np} = \frac{1}{p} - \frac{1}{n} < \frac{1}{p}$ we have $1 \le p < p^*$

2.5.1 The Sobolev Space $W^{1,p}(I)$ and $W_0^{1,p}(I)$

Let $I = (a, b)$ be an open interval, possibly unbounded, and let $p \in \mathbb{R}$ with $1 \le p \le \infty$

Definition 2.5.2. *[5] The Sobolev space $W^{1,p}(I)$ is defined to be*
$W^{1,p}(I) = \left\{U \in L^p(I); \exists g \in L^p \text{ such that } \int_I U\varphi' = -\int_I g\varphi \quad \forall \varphi \in C_c^1(I)\right\}$

Remark 2.5.1. *For $p = 2$;We set $H^1(I) = W^{1,2}(I)$.*

Definition 2.5.3. *[5] Given $1 \le p < \infty$, denote by $W_0^{1,p}(I)$ the closure of $C^1(I)$ in $W^{1,p}$.*

Remark 2.5.2. *For* $p=2$ *;We set* $H_0^1(I)=W_0^{1,2}(I)$

The space $W_0^{1,p}(I)$ is equipped with the norm of $W^{1,p}(I)$, and the space $H_0^1(I)$ is equipped with the scalar product of $H^1(I)$.
The space $W_0^{1,p}(I)$ is a Banach space. Moreover, it is reflexive for $p>1$. The space $H_0^1(I)$ is a Hilbert space.

Proposition 2.5.1. *[5]* $\big($***Poincaré's Inequality***$\big)$ *Suppose* I *is a bounded interval. Then there exists a constant* C $\big($*depending on* $|I|<\infty\big)$ *such that*

$$\| U \|_{W^{1,p}(I)} \leq C \| U' \|_{L^p(I)} \quad \forall U \in W_0^{1,p}(I)$$

In other words, on $W_0^{1,p}(I)$*, the quantity* $\| U' \|_{L^p(I)}$ *is a norm equivalent to the* $W^{1,p}(I)$ *norm.*

Proof. Let $U \in W_0^{1,p}(I)$ $\big($with $I=(a,b)\big)$. Since $u(a)=0$, we have
$| u(x) | = | u(x)-u(a) | = | \int_a^x U'(t)dt | \leq \| U' \|_{L^1}$.
Thus $\| U \|_{L^\infty(I)} \leq \| U' \|_{L^1(I)}$ and $\| U \|_{W^{1,p}(I)} \leq C \| U' \|_{L^p(I)}$ then follows by Hölders inequality

Remark 2.5.3. *Let* $1 \leq p < q$, $x \in L^q(0,1), f \in L^{\frac{q}{q-p}}(0,1)$ *then*

$$i) \int_0^1 | x |^p | f(t) | \, dt \leq \| x \|^p_{L^q(0,1)} \| f \|_{L^{\frac{q}{q-p}}(0,1)} . \tag{2.3}$$

$$ii) \| x \|_{L^p(0,1)} \leq \| x \|_{L^q(0,1)} \tag{2.4}$$

Lemma 2.5.1. *Let* $I=(0,1)$ $f \in H_0^1(0,1)$

$$\| f \|_{L^2} (0,1) \leq \| \tfrac{df}{dt} \|_{L^2(0,1)}.$$

.In this paper we use this formulation of poincarè inequality.

proof

Since $f \in H_0^1(0,1)$, then $f(0)=0$ then

$$| f(t) | = | f(t)-f(0) | = | \int_0^t \tfrac{df}{dt} dt | \leq \| \tfrac{df}{dt} \|_{L^1(0,1)} \leq \| \tfrac{df}{dt} \|_{L^2(0,1)} \text{ by (2.4)}$$

.

This leads us to very important estimation, that we will use a lot in this paper:

$$\| f \|_{L^\infty(0,1)} \leq \| \frac{df}{dt} \|_{L^2(0,1)} \tag{2.5}$$

From 2.4 we have that for 2 and $n > 2$ the following holds

$$\| f \|_{L^2(0,1)} \leq \| f \|_{L^n(0,1)}$$

By taking $n \to \infty$ and using known property that $\| x \|_{L^\infty(0,1)} = \lim_{p\to\infty} \| f \|_{L^p(0,1)}$; we obtain

$$\| f \|_{L^2(0,1)} \leq \| f \|_{L^\infty(0,1)} \leq \| \frac{dx}{dt} \|_{L^2(0,1)} \text{ by (2.5)}$$

Since Poincarè inequality holds we shall use the following norm in $H_0^1(0,1)$ space

$$\| f \|^2_{H_0^1(0,1)} := \int_0^1 \left(\frac{df}{dt}(t)\right)^2 dt$$

.

2.5.2 Embeddings of Sobolev Spaces

The importance of Sobolev spaces lies in their connections with the spaces of continuous and uniformly continuous functions.

Definition 2.5.4. *Let V and W be two Banach spaces with $V \subset W$. We say the space V is continuously embedded in W and write $V \hookrightarrow W$, if*

$$\| v \|_W \leq c \| v \|_V$$

Definition 2.5.5. *Let X and Y be banach spaces, $X \subset Y$, we say that X is compactly embedded in Y written as $X \subset\subset Y$ if*

i) *X is embedded in Y, and*

ii) *each bounded sequence in X is compact in Y*

Theorem 2.5.1. *[5] **(Gagliardo-Nirenberg-Sobolev inequality)***
Assume $1 \leq p < n$. There exists a constant C depending only on p and n such that

$$\| u \|_{L^{\frac{np}{n-p}}(\Omega)} \leq C \| Du \|_{L^p(\Omega)} \quad , \forall\ u \in C_0^1(\Omega). \text{ for all } u \in C_0^1(\Omega).$$

Proof: see [5]

Chapter 3

Boundary Value Problem For Duffing's Equation

3.1 Motivation

Consider the problem

$$\begin{cases} -\ddot{x}(t) + b(t)\dot{x} + c(t)x(t) = f(t) \;\; in \; \Omega = (a, b) \\ x(a) = x(b) = 0 \end{cases} \tag{3.1}$$

where b , c , f are continuous function. The boundary condition $x(a) = x(b) = 0$ is called **Dirichlet Boundary Condition.**

Definition 3.1.1. *[5] A classical solution of (3.1) is a function $x \in C^2(\Omega)$ satisfying (3.1) in the usual sense.*

Assume that a classical solution exists, i.e, a twice continuously differentiable function x that satisfies(3.1). Then multiplying (3.1) by arbitrary continuous function y and integrating, we have

$$\int_I \big(\ddot{x} + b\dot{x} + cx\big) y dt = \int_I fy; \quad \forall y \in C(\tilde{\Omega}). \tag{3.2}$$

If a function x satisfies equation (3.2) for all $y \in D(\Omega)$ then x is called **weak solution** of the differential equation (3.1). Then integrating (3.2)using integration by parts we obtain

$$-\int_I \big[\dot{x}\dot{y} + (b\dot{x} + cx - f)y\big] dt = 0 \tag{3.3}$$

which is called **variational problem (equation)** for differential equation (3.1).

3.2 Existence Results For Variational Equation

In this section ,we are concerned with fundamental existence for variation equation in Hilbert space. We assume that H is a Hilbert space with inner product $(\cdot,\cdot)_H$ and associated norm $\|\cdot\|_H$ and we refer to $H^\star$ as its dual. Considr a bilinear form $B(\cdot,\cdot): H\times H \to \mathbb{R}$ and an element $f \in H^\star$ i.e a bounded linear functional on H we consider the varational equation:
find $u \in H$ such that $B(u,v) = (f,v)$, $\forall v \in H$.

Definition 3.2.1. *[17] Let H be real Hilbert space.*

i. $B : H\times H \to \mathbb{R}$ *is called bilinear form if*

$$a.\ B(\alpha u + \beta v, w) = \alpha B(u,w) + \beta B(v,w)$$

$$b.\ B(w, \alpha u + \beta v) = \alpha B(w,u) + \beta B(w,v)$$

for all $u, v, w \in H$ *and* $\alpha, \beta \in \mathbb{R}$

Definition 3.2.2. *[17] A bilinear form* $B(\cdot,\cdot): H\times H \to \mathbb{R}$ *is said to be* ***Bounded****, if there exists a constant* $c \geq 0$ *such that*

$$| B(u,v) | \leq c \| u \|_H \| v \|_H, \quad u, v \in H.$$

Theorem 3.2.1. *[1](Lax-Milgram)*
Let $B : H\times H \to \mathbb{R}$ *be bilinear form. Suppose there exist positive constants* α *and* β *such that*

i. $|B(u,v)| \leq \alpha ||u|| ||v||$, *and*

ii. $B(u,u) \geq \beta ||u||^2$

Then for every $f \in H^*$ *there exists a unique* $u \in H$ *such that*

$$B(u,v) = (f,v), \ \forall v \in H$$

See the proof of this theorem on [1, 2].

3.3 Critical Point,Palais-smale(PS) Conditions

Hereafter H denotes a real Hilbert space, with norm $||\cdot||$ and inner product $(\cdot,\cdot)$. Let $J : H \to \mathbb{R}$ be non-linear functional on H.

Definition 3.3.1. *[17] We say J is differentiable at $u \in H$ if there exists $v \in H$ such that*

$$J(w) = J(u) + (v, w - u) + o(||w - u||), \quad w \in H$$

the element v, if it exists, is unique. We then write

$$J'(u) = v$$

Definition 3.3.2. *[17] Let $U \subset \mathbb{R}^n$ be open. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be Lipschitz continuous on U if and only if there exists $L \in \mathbb{R}$ such that*

$$||f(x) - f(y)|| \leq L||x - y||, \quad \forall x, y \in U$$

Definition 3.3.3. *[17] We say J belongs to $C^1(H, \mathbb{R})$ if $J'(u)$ exists for each $u \in H$, and the mapping $J' : H \to H$ is continuous.*

Notation: i. We denotes by $\mathcal{C}$ the collection of functions $J \in C^1(H, \mathbb{R})$ and satisfy $J' : H \to H$ is Lipschits continuous on bounded subsets of H.
ii. If $c \in \mathbb{R}$, we write

$$\begin{aligned} k_c &= \{u \in H : J(u) = c, J'(u) = 0\} \\ A_c &= \{u \in H : J(u) \leq c\} \end{aligned}$$

Definition 3.3.4. *i. We say $u \in H$ is critical point if $J'(u) = 0$*
ii. The real number c is critical value if $k_c \neq \emptyset$

Definition 3.3.5. *[1] A functional $J \in C^1(H, \mathbb{R})$ satisfies the palais-smale compactness condition if each sequence $\{u_k\}_{k=1}^{\infty} \subset H$ such that*
i. $\{J(u_k)\}_{k=1}^{\infty}$ is bounded
ii.$J'(u_k) \to 0$ in H

Lemma 3.3.1. *(Deformation).[1] Let $J : H \longrightarrow \mathbb{R}$ be a C^1 functional satisfying palais-smale(PS) conditions. Suppose also $k_c = \emptyset$.*
Then there exist an $\epsilon > 0$, there exists a constant $0 < \delta < \epsilon$ and a continuous function $\eta : [0, 1] \times H \longrightarrow H$ such that the mapping

$$\eta_t(u) = \eta(t, u), \quad \forall t \in [0, 1], \quad u \in H$$

satisfying the following conditions
i. $\eta_0(u) = u, \quad \forall\, u \in H$

ii. $\eta_t(u) = u, \quad \forall\, t \in [0,1]\ , u \notin I^{-1}\big([c-\epsilon, c+\epsilon]\big)$

iii. $J(\eta_t(u)) \leq J(u), \quad \forall\, t \in [0,1], u \in H$

iv. $\eta_1(A_{c+\epsilon}) \subset A_{c-\epsilon}$

See the proof of this lemma on [1].

Remark: *The above lemma shows that if* c *is not critical value, then the set* $A_{c+\epsilon}$ *is deformed in to* $A_{c-\epsilon}$ *for some* $\epsilon > 0$

Theorem 3.3.1. *[17] (The Mountain pass theorem) Assume* $J \in \mathcal{C}$ *satisfies Palais-Smale condition. Suppose that*

i.$J(0) = 0$

ii.There exist $r, \rho > 0$ *such that* $J(u) \geq \rho$ *for all* $\|u\| = r$

iii.There exist $v \in H$ *such that* $\|v\| > r$

Set $K = \{p : [0,1] \longrightarrow H :\ p(0) = 0, p(1) = v\}$ *and let*

$$c = \inf_{p \in k} \sup_{t \in [0,1]} J(p(t)),$$

then c *is the critical value of* J.

Proof. If, by contradiction, there is no critical point at level c, then $k_c = \emptyset$ if we choose ϵ small enough so that $0 < \delta < \epsilon$. Let $p \in k$ and define the path

$\beta : [0,1] \to H$ by $\beta(t) = \eta_1(p(t))$.

Since $p(0) = 0$ *and* $p(1) = u$, it follows by choice of ϵ, that $\beta(0) = \eta_1(0) = 0,\ \beta(1) = \eta_1(u) = u$ using condition (ii) of lemma.

Now we can choose $p \in k$ such that

$$\max_{t \in [0,1]} I(p(t)) < c + \epsilon,\ p(t) \in A_{c+\epsilon}$$

by condition (iv) of lemma, $p(t) \in A_{c-\epsilon}$.

Thus

$$\max_{t \in [0,1]} J(p(t)) \leq c - \epsilon.$$

Which is contradiction to the definition of c , Hence $k_c \neq \emptyset$. Therefore c is the critical value of J.

$\square$

Definition 3.3.6. *We denote by* $H^{-1}(\Omega)$ *the dual space to* $H_0^1(\Omega)$. *In other words* f *belongs to* $H^{-1}(\Omega)$ *provided* f *is bounded linear functional on* $H_0^1(\Omega)$.

Notation: *We will write* $\langle\cdot,\cdot\rangle$ *to denote the pairing between* $H^{-1}(\Omega)$ *and* $H_0^1(\Omega)$.

Definition 3.3.7. *If* $f \in H^{-1}(\Omega)$, *we define the norm*

$$||f||_{H^{-1}(\Omega)} = \sup\left\{\langle f, u\rangle : u \in H_0^1(\Omega), ||u||_{H_0^1(\Omega)} \leq 1\right\}$$

3.4 Dirichlet Problem for a Forced Duffing equation

Consider the following Dirichlet problem for a forced Duffing type equation with a functional parameter u.

$$\begin{cases} \ddot{x}(t) + r(t)\dot{x}(t) + G(t, x(t), u(t)) = 0 \\ x(0) = x(1) = 0 \end{cases} \tag{3.4}$$

where $r \in C^1(0,1)$ stands for a friction term ;$r(\tau) \geq 0$ for $\tau \in [0,1]$.
Here we do not assume anything about the monotonicity of r, but instead we require that

$$\tfrac{1}{4}r^2(t) + \tfrac{1}{2}\dot{r}(t) > 0 \qquad \forall t \in [0,1]$$

Let us denote $\omega(t) = \frac{1}{4}r^2(t) + \frac{1}{2}\dot{r}(t)$.
Of course, when r is nondecreasing we obviously have $\frac{1}{4}r^2(t) + \frac{1}{2}\dot{r}(t) > 0$ and G is a nonlinear term,satisfying some suitable assumptions, so G can correspond to a restoring force for a string in string-damper system. The Duffings equation was also found applicable for some problems concerning current and flux, thus r and G may as well corresponds to its coefficients.By a solution to (3.4) we mean a function $x \in H_0^1(0,1)$ such that x satisfies the boundary conditions and the equation (3.4) is satisfied a.e. in (0,1) and $u \in H_0^1(0,1)$ is control function with only the function G depend on it.
For simplicitys sake we denote $R(t) = e^{\int_0^t \frac{1}{2}r(\tau)d\tau}$. Since $r(\tau) \geq 0$ on [0,1] we set that

$$R_{\max} = e^{\max r(\tau)} \geq R(t) \geq R(0) = 1 \text{ for } \tau \in [0,1].$$

and multiply (3.3) by the function $R(t)$.we put $y = R(t)x(t)$ and obtain for y an equivalent dirichlet problem;

$$\begin{cases} -\ddot{y}(t) + \omega(t)y(t) = R(t)G(t, \frac{y(t)}{R(t)}, u(t)) \\ y(0) = y(1) = 0 \qquad\qquad \text{where } \omega(t) = \frac{1}{4}r^2(t) + \frac{1}{2}\dot{r}(t) \end{cases} \tag{3.5}$$

Chapter 4

Weak solution of Dirichlet problem

4.1 Variational Framework

In this section we need to investigate the classical variational problem for duffing type equation and we are concerned with the variational formulation for the Duffings equation, but we apply the different schema. Namely we consider inclusion instead of equality in (3.4)

$$\begin{cases} \ddot{x}(t) + r(t)\dot{x}(t) - F_x(t, x(t)) - f(t) = 0 \\ x\,(0) = x\,(1) = 0 \end{cases} \tag{4.1}$$

under the assumptions that $r \in L^{\infty}(0,1)$ and $f \in L^1(0,1)$. Solutions to above are investigated in $H_0^1(0,1)$ and these are the weak solutions. We shall show that by the Fundamental Lemma of the Calculus of Variations,any weak solutions to (4.1) is classical one, i.e.

$$x \in H_0^1(0,1) \cap\ W^{1,2}(0,1):$$

[1] The equation (4.1) is not in a variational form i.e. there is no suitable functional J for which (4.1) corresponds to its critical points. Then by putting $h = \frac{dx}{dt} = \dot{x}(t)$, we may consider the following auxiliary problem

$$\ddot{x}(t) + r(t)h(t) - F_x(t, x(t)) - f(t) = 0 \tag{4.2}$$

Therefore instead of 4.1 in this paper we will investigate 4.2.
For the study of weak solution of this DE multiply both sides of this equation by
$v \in H_0^1(0,1)$.

$$\ddot{x}(t)v(t) + r(t)h(t)v(t) - F_x(t, x(t))v(t) - f(t)v(t) = 0$$

[1] Fundamental Lemma Of the Calculus Varaition: Let $f \in L^p(\Omega)$such that $\int f\varphi^{'} = 0, \forall\varphi \in D(\Omega)$ then $f = 0$ a.e Ω.

taking $v = -x$, we have

$$-\ddot{x}(t)x(t) - r(t)h(t)x(t) + F_x(t, x(t))x(t) + f(t)x(t) = 0$$

Then integrating both sides we have;

$$\int_0^1 \big[- \ddot{x}(t)x(t) - r(t)h(t)x(t) + F_x(t, x(t))x(t) + f(t)x(t)\big] dt = 0$$

By integrating parts we obtain:

$$\int_0^1 \big[\big(\tfrac{dx}{dt}\big)^2 + [f(t) - r(t)h(t)]x(t) + F(t, x(t))\big] dt = 0$$

which is **variational problem** of equation 4.2.
Let us define a functional $J : H_0^1(0, 1) \to \mathbb{R}$ given by the following integral

$$J(x) = \int_0^1 \big[\frac{1}{2}\big(\frac{dx}{dt}\big)^2 + \big(f(t) - r(t)h(t)\big)x(t) + F(t, x)\big] dt \tag{4.3}$$

We will see that the weak solutions of 4.2 are critical points of J.

Definition 4.1.1. *[5] Replacing the differential equation with an integral equation,using integration by parts to reduce the order of the derivatives is called* ***variational(weak) form*** *of the equation*

Definition 4.1.2. *The weak (variational) formulation of (4.1) is: Find* $x \in H_0^1(0, 1)$ *such that*

$$B(x, v) = l(v) \quad \forall\ v \in H_0^1(0, 1) \tag{4.4}$$

where $B(\cdot, \cdot)$ *is the bilinear functional* $B(x, v) := \frac{1}{2} < x', v' > + < rx, v >$ *and* $l(v) = (f, v)$.

Definition of Weak solution is closely connected with variational formulation of dirichlet problem for Duffing's equation. To explain this connection we frist summarize some definition of the differentiability of a functional,weakly lower semicontinuty of functionals and coercitivity of functionals acting on a Banach space.

Definition 4.1.3. *[5] A functional* $J : X \to \mathbb{R}$ *on a Banach space* X *is differentiable at* $x \in X$ *, if there is a bounded linear functional* $T : X \to \mathbb{R}$ *such that*

$$\lim_{\|h\|_X \to 0} \frac{|\ J(x + h) - J(x) - hT\ |}{\|\ h\ \|_X} = 0$$

If T *exists,then it is unique and it is called the* ***derivative*** *of* J *at* x *and denoted by* $DJ(x$.

This definition expresses the basic idea of a differentiable function as one which can be approximated locally by a linear map. If J is differentiable at every point of X , then $DJ : X \to X^{\star}$ maps $x \in X$ to the linear functional $DJ(x) \in X^{\star}$ that approximates J near x . A weaker notion of differentiability is the existence of directional derivatives:

$$\delta J(x;h) = \lim_{\epsilon\to 0}\Big[\frac{J(x+\epsilon h)-J(x)}{\epsilon}\Big] = \frac{d}{d\epsilon}J(x+\epsilon h)\Big|_{\epsilon=0}$$

If the directional derivative at x exists for every $h \in X$ and is a bounded linear functional on h, then $\delta J(x;h) = \delta J(x)h$ where $\delta J(x) \in X$. We call $\delta J(x)$ the **Gâteaux derivative** (GD) of J at x .

Definition 4.1.4. *[17]* ***Lower Semicontinuous Functional***
Let X be a Banach space. A functional $J : X \to \mathbb{R}$ is called ***Lower semicontinuous*** *[**weakly lower semi-continuous**] on X , if for every $u \in X$ and every sequence $\{u_n\}$ is strong convergent [weakly convergent] respectively to u in X such that*

$$J(u) \leq \lim_{n\to\infty}\inf J(u_n)$$

.

Definition 4.1.5. *[17] A functional $J : X \to \mathbb{R}$ is called* ***Coercive*** *if*

$$\lim_{\|X\|\to\infty} J(x) = +\infty$$

Under the assumptions that $h \in L^{\infty}(0,1), r \in L^{\infty}(0,1)$, and some growth requirements on F we can prove that problem (4.2) has at least one solution.
To prove this, its sufficient to show that:

1. functional J is differentiable in sense of Gâteaux
2. functional J is coercive
3. functional J is weakly lower semi continuous.

Notation: The weak lower semi-continuity and coercivity of the functional J imply the existence of a critical point of the functional J. When solutions to (4.2) are obtained for any $h \in L^{\infty}(0,1)$, we will apply the iterative procedure assuming that

$$\int_0^1 |F_x(t,x(t)) - F_x(t,y(t))|dt \leq L\|x-y\|_{H_0^1(\Omega)}$$

For any, $y \in H_0^1(\Omega)$, $L < 1$ independent of x, y and $\frac{\|r\|}{1-L} < 1$. This will provide solutions to 4.1 Moreover function F , F_x : $[0,\ 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ will be a Caratheodory

function, satisfying the conditions below with a given $p \in (1,\ 2)$:
(H_1). $F(.,0) \in L^1(0\ ,1)$ $\forall_{d>0}\forall_x \in [-\mathrm{d},\ \mathrm{d}]$ $\exists\, f_{\mathrm{d}} \in L^1(0\ ,1)$ Such that

$$|F_x(t, x(t))| \leq f_{\mathrm{d}}(\mathrm{t}\ )$$

And we shall consider two versions of assumptions that will provide different results.
(H_2). Convex version: $x \to F(t,\ x)$ $for\ a.e.\ t \in [0,\ 1]$ is convex in x .
(H_3). Bounded version. There exists constants $A \in \mathbb{R}\backslash\{0\}, B, C \in \mathbb{R}$ such that $F(t,\ x) \geq A|x|^2 + B|x| + C$ for all $x \in \mathbb{R}$, almost everywhere $t \in [0,\ 1]$.
We shall prove that this functional is well defined, is Gâteaux differentiable and it's critical points are the weak solutions to (4.2). We will also prove that the regularity class of this solution is higher than $H_0^1(0,1)$ we would like to compute Gâteaux derivatives, but first we have to ensure that we can differentiate under integration sign. We see the following properties:

Lemma 4.1.1. *under assumption* (H_1) *the following equality holds for any* $x \in H_0^1(0,1)$ *and* $g \in H_0^1(0,1)$ *then* $\lim_{\lambda\to 0}\int_0^1 \frac{F(t,x+\lambda g)-F(t,x)}{\lambda}dt = \int_0^1 \lim_{\lambda\to 0} \frac{F(t,x+\lambda g)-F(t,x)}{\lambda}\, dt$

Proof. To show this we have to show that

$$\lim_{\lambda\to 0}\int_0^1 \frac{F(t, x+\lambda g) - F(t,x)}{\lambda}dt \leq \int_0^1 \lim_{\lambda\to 0} \frac{F(t, x+\lambda g) - F(t,x)}{\lambda}\, dt$$

and

$$\int_0^1 \lim_{\lambda\to 0} \frac{F(t, x+\lambda g) - F(t,x)}{\lambda}\, dt \leq \lim_{\lambda\to 0}\int_0^1 \frac{F(t, x+\lambda g) - F(t,x)}{\lambda}dt$$

To show the first let (x_n, λ_n) be a sequence such that:

$$\lim_{\lambda\to 0}\int_0^1 \frac{F(t, x+\lambda g) - F(t,x)}{\lambda}dt = \lim_{n\longrightarrow\infty}\int_0^1 \frac{F(t, x_n+\lambda_n g) - F(t,x_n)}{\lambda_n}\, dt \tag{1}$$

Since (H_1) and Lebesgue's Dominated Convergence Theorem RHS of (1).thus if we replace (x_n, λ_n) with *limits*, we get the following inequality

$$\lim_{\lambda\to 0}\int_0^1 \frac{F(t, x+\lambda g) - F(t,x)}{\lambda}dt \leq \int_0^1 \lim_{\lambda\to 0} \frac{F(t, x+\lambda g) - F(t,x)}{\lambda}\, dt \tag{2}$$

Similarly, again (x_n, λ_n) be a sequence such that:

$$\lim_{\lambda\to 0} \frac{F(t, x+\lambda g) - F(t,x)}{\lambda} = \lim_{n\longrightarrow\infty} \frac{F(t, x_n+\lambda_n g) - F(t,x_n)}{\lambda_n} \quad (3$$

Since $(H1)$ holds, once again Lebesgue's Dominated Convergence Theorem can be applied to RHS of (2). Thus if we replace sequence (x_n, λ_n) with limit the integral will be greater or equal

$$\int_0^1 \lim_{\lambda \to 0} \frac{F(t, x + \lambda g) - F(t, x)}{\lambda}\, dt \leq \lim_{\lambda \to 0} \int_0^1 \frac{F(t, x + \lambda g) - F(t, x)}{\lambda} dt \tag{4}$$

Thus from (2) and (4) implies

$$\lim_{\lambda \to 0} \int_0^1 \frac{F(t, x + \lambda g) - F(t, x)}{\lambda} dt = \int_0^1 \lim_{\lambda \to 0} \frac{F(t, x + \lambda g) - F(t, x)}{\lambda}\, dt$$

Remark. To simplify in notation we introduce $\overline{F}: \ H_0^1(\Omega) \longrightarrow \mathbb{R}$ given by

$$\overline{F}: \ H_0^1(\Omega) \ni x \to \int_0^1 F(t, x(t))dt$$

Theorem 4.1.1. *[18] Functional* (4.3), $J\!: \ H_0^1(0,1) \longrightarrow \mathbb{R}$ *is well defined under assumptions* (H_1) *.Also the functional* (4.3) *is differentiable in sense of Gâteaux and its derivative is equal to* $\delta J(x, g) = \int_0^1 \frac{dx}{dt}\frac{dg}{dt} + [f(t) - r(t)h(t) + F_x(t,\ x)]\, g(t)\, dt$ *For each* $g \in H_0^1(0,1)$

Proof 4.1.1. *we see that*

$$\int_0^1 \tfrac{1}{2}\dot{x}^2 + (f(t) - r(t)h(t))x(t)dt$$

is well-defined and by H_1 *for any* x *we see*

$$\begin{aligned}\int_0^1 | F(t,x) |\, dt &\leq \int_0^1 | F(t,0) |\, dt + \int_0^1 f_d(t) \mid X(t) \mid dt \\ &\leq \| F(t,0) \|_{L^1(0,1)} + \| f_d(t) \|_{L^1(0,1)} \| x \|_{L^1(0,1)}\ .\end{aligned}$$

Thus J *is well defined.*

From the consequence of lemma (4.1.1)we have

$$\begin{aligned}
\delta J(x)g &= \lim_{\alpha\to 0}\frac{J(x+\alpha g)-J(x)}{\alpha}, (by\ \ definition\ of\ GD)\\
&= \frac{d}{d\alpha}J(x+\alpha g)\big|_{\alpha=0}\\
&= \frac{d}{d\alpha}\int_0^1\big[\frac{1}{2}(\frac{d(x+\alpha g)}{dt})^2+(f(t)-r(t)h(t))(x+\alpha g)+F(t,x+\alpha g)\big]dt\big|_{\alpha=0}\\
&= \int_0^1\frac{d}{d\alpha}\big[\frac{1}{2}(\frac{d(x+\alpha g)}{dt})^2+(f(t)-r(t)h(t))(x+\alpha g)+F(t,x+\alpha g)\big]\big|_{\alpha=0}dt\\
&= \int_0^1\big[\frac{dx}{dt}\frac{dg}{dt}+[(f(t)-r(t)h(t)+F_x(t,x)]g(t)\big]dt
\end{aligned}$$

Remark 4.1.1. *Every $x\in H_0^1(0,1)$ for which that satisfies the following equality $\forall g\in\ H_0^1(0,1);\ \ \delta J(x,g)=0$ a is* ***weak solution****(Ws).*

We shall now prove that weak solution (WS) for functional (4.3) is a classical solution. Then we shall see that functional critical points to J are the weak solutions to (4.2).

Lemma 4.1.2. *[17]* ***du Bois-Raymond Lemma*** *[17,pp 6]*
Let $v\in L^2(0,1)$, $\omega\in L^1(0,1)$ be such functions that $\int_0^1 v(x)h'(x)dx=-\int_0^1\omega(x)h(x)dx$, $\forall h\in H_0^1(0,1)$. Then there exists constant $c\in\mathbb{R}$, such that $v(x)=\int_0^1\omega(s)ds+c$ for almost every $t\in[0,1]$.

Lemma 4.1.3. *Let x be a weak solution (WS).If $(H_1)$is satisfied, then this solution is classical solution to* 4.2

Proof. Let x be a solution to (WS), i.e. $\langle\xi,g\rangle=0$ for all $\xi\in\partial J(x)$ and $g\in H_0^1(0,1)$. By theorem (4.1.1) we know that there exists at least single $\psi\in\partial\overline{F}(x)$ such that

$$0=\int_0^1\frac{dx}{dt}\frac{dg}{dt}+[f(t)-r(t)h(t)]g(t)dt+\ \langle\psi,g\rangle.$$

By Riesz representation theorem there exists a function $\overline{\psi}\in L^2(0,1)$ that $\langle\psi,g\rangle=\int_0^1\overline{\psi}(t)g(t)dt$ We know that $f-rh+\overline{\psi}$ is integrable by lemma 4.1.1. Applying du Bois-Raymond Lemma for $g=\frac{dx}{dt}$ and $\omega=f-rh+\overline{\psi}$ then the solution of the problem $\langle\xi,g\rangle=0$, $g\in\ H_0^1(0,1)$ is of a class $H^2(0,1)$ and thus is a classical one.

4.2 The Existence of a Solution

In this section we prove the existence of solution to (4.2)

Lemma 4.2.1. *[18] The functional* J *given by formula* (4.3) *is weakly lower semi-continuous under* (H_1).

Proof. By norm continuity, the first part of (4.3) is weakly lower semi-continuous or to claim that weakly lower semicontinuity of norm,we need a conclusion of the famous Hahn-Banach Theorem.

Theorem 4.2.1. *[17] (**Hahn-Banach-Continuation for normed spaces**): Let* $(X, \|\cdot\|)$ *be a normed space and* f *a linear continuous functional on a linear subspace* $L \subset X$. *Then there exists a linear continuous functional* $\bar{f}$ *on* X *(i.e.* $f \in X^\star$*) which is a continuation of* f *from* L *to* X *keeping the norm* $\bar{f}(x) = f(x), \forall x \in L$ *and* $\| \bar{f} \|_{X^\star} = \| f \|_L$*:*

Conclusion: Let $(X, \|\cdot\|)$ be a linear normed space, $0 \neq x_0 \in X$ any element. Then there exists a linear continuous functional $\bar{f} = x^\star \in X$ on X such that $\| x^\star \|_{X^\star} = 1$ and $< X^\star, x_0 > = \| x_0 \|$.
Using this conclussion we can show that norm is weakly lower semicontinuity as follows: Considering a sequence$x_n \rightharpoonup x_0$, we show that $\| x_0 \| \leq \liminf_{n\to\infty} \| x_n \|$.
Let us suppose the contrary, i.e.$\| x_0 \| > \liminf_{n\to\infty} \| x_n \|$.
Consequently there exists $c \in \mathbb{R}$ such that

$$\| x_0 \| > c > \liminf_{n\to\infty} \| x_n \| \ .$$

By definition of the limit there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\| x_0 \| > c > \| x_{n_k} \| \ .$$

From the above Conclusion follows the existence of a functional $x^\star \in X^\star$ such that

$$\| x^\star \| = 1 \ \ \textbf{and} \ \ (x^\star, x_0) = \| x_0 \| > c$$

.On the other hand, $\| x^\star x_{n_k} \| \ x^\star \|_{X^\star} \| x_{n_k} \| = \| x_{n_k} \| < c$.so

$$(X^\star, x_0) = lim_{n_k\to\infty}(x^\star, x_{n_k}) > c$$

(weak convergence): This contradicts the result obtained above, so the norm is weak-lower semicontinuous indeed. And the second part of J is also $w.l.s.c$ since

$$\int_0^1 \left(f\left(t\right) - r\left(t\right) h\left(t\right)\right)(\cdot)(t)dt$$

is linear and continuous.
For F function we need to apply some additional theory in order to prove its w.l.s.c.
Let's consider weakly converged sequence in $H_0^1(0,1)$, $x_n \rightharpoonup x$.
By Arzela-Ascoli Theorem there exists such subsequence that converge uniformly in $C(0,1)$. Then for sufficiently large d, the below condition holds:

$$max\,|x_n(t)| \leq d$$

for sufficiently large n. By Lebesgue's dominated convergence Theorem we obtain:

$$\int_0^1 F(t, x_n)\,dt \to \int_0^1 F(t,x)\,dt\ , n \to \infty$$

Then it is proved that functional (4.3) is w.l.s.c.

Lemma 4.2.2. *[18] Functional J is coercive if (H_1) and F satisfies (H_3) with A satisfying: $|A| < \frac{1}{2}$ holds.*

Proof. First we observe that

$$\int_0^1 \frac{1}{2}\left(\frac{dx}{dt}\right)^2 dt = \frac{1}{2}\|x\|^2_{H_0^1\ (0,1)}.$$

We see $\int_0^1 (f(t) - r(t)\,h(t))\,x(t)\,dt \geq -\|f - r\cdot h\|_{L^1\ (0,1)}\|x\|_{H_0^1\ (0,1)}$.
If the condition(H_3) holds so there exists such $A \in \mathbb{R} \setminus \{0\}\,, B, C \in \mathbb{R}$ for which the following holds.

$$F(t,x) \geq A|x|^2 + B\,|x| + C \geq A|x|^2 - B\,|x| - C$$

implies $F(t,x) \geq A|x|^2 - B\,|x| - C$. Then Integrating the both sides we get
$\int_0^1 F(t,x(t))dt \geq \int_0^1 (A|x|^2 - B\,|x| - C)dt \geq A\|x\|^2_{L^2(0,1)} - |B|\|x\|_{L^2(0,1)} - |C|$
We should consider two cases:
1. If sequence of norms x_n diverges in $H_0^1\ (0,1)$ it may still converge in $L^2(0,1)$.
In such case $\|x_n\|_{L^2(0,1)} \leq \|x_n\|_{H_0^1\ (0,1)}$
2. In opposite case, the same inequality holds since Poincare inequality is applicable.
Thus $\int_0^1 F(t,x(t))dt \geq -A\|x\|^2_{H_0^1\ (0,1))} - (|B| + |C|)\|x\|_{H_0^1\ (0,1)}$
Then functional (4.3) is coercive since $|A| < \frac{1}{2}$ for unbounded case. Together with bonded case this proves lemma.
The theorem below proves the existence of solution.

Theorem 4.2.2. *[18] Let E be reflexive Banach space and let the functional $f : E \longrightarrow \mathbb{R}$ be w.l.s.c. and coercive.Then there exist a function that minimizes f on E .*

[2] **Proof:** First we show f is bounded from below. Suppose to the contrary that f is not bounded from below.Then there exist a sequence $\{x_n\} \in E$ such that

$$f(x_n) < -n, \quad \forall\, n.$$

Now since E is bounded $\{x_n\}$ has a weakly convergent subsequence x_{n_k}, such that $x_{n_k} \rightharpoonup x^\star$.Moreover, E is weakly closed and hence $x^\star \in E$.
Then, since f is Weakly lsc , we have $f(x^\star) \leq lim\ inf\ f(x_{n_k}) = -\infty$ which is a contradiction.
Hence, f is bounded from below.
[3] Next, we show the existence of a minimizer.Let $\{x_n\} \in E$ be a minimizing Sequence for f; that is $f(x_n) \to inf\ f(x)$. Let $\alpha = inf\ f(x)$. Since E is bound and weakly closed it follows that $\{x_n\}$ has a weakly convergent subsequence $x_{n_k} \rightharpoonup x^\star$. Next, since f is weakly lsc we have

$$\alpha \leq f(x^\star) \leq\ lim\ inf\ f(x_{n_k}) = lim\ f(x_{n_k}) = \alpha$$

Hence, $f(x^\star) = \alpha$ and the theorem is proved.
Then we have the following

Theorem 4.2.3. *[14] There exists at least one solution to* (4.2) *if* (H_1) *is satisfied and one of the following holds:*
1. F Satisfies (H_2)
2. F satisfies (H_3) with A satisfying: $|A| < \frac{1}{2}$

Proof. By Lemmas 4.2.1 and 4.2.2, and reflexiveness of $H_0^1(0,1)$ we see that assumptions of Theorem 4.2.2 are satisfied. Then there exists solutions in functional critical Points problem. By Lemma 4.1.3 this solution is a classical solution to 4.2.
Notation we denote by $H^{-1}(\Omega)$ the dual space to $H_0^1(\Omega)$.In other words f belongs to $H^{-1}(\Omega)$ provided f is bounded linear functional on $H_0^1(\Omega)$.

[2]Function $u_0 \in X$ is a minimize (or global minimize) of the functional $f(u)$ on a set X if $f(u_0) \leq f(u)$ for all $u \in X$:
[3]A nonempty subset D of X is called weakly closed , if for every weakly convergent sequence $x_n \rightharpoonup x, x_n \in D$ follows $x \in D$

4.3 Iterative Scheme Framework

In this section we shall prove that using equation (4.2) we may provide the solution of (4.1).

Theorem 4.3.1. *[18] If H_1 is satisfied and if one of below conditions holds*

1. *F is convex H_2*
2. *F is bounded H_3 and $\mid A \mid < \frac{1}{2}$*

and moreover

$$\int_0^1 \mid F_x(t,x) - F_x(t,y) \mid dt \leq L \parallel x - y \parallel_{H_0^1(0,1)} \tag{4.5}$$

For any $x, y \in H_0^1(0,1), L < 1$ independent of x, y and $\frac{\|r\|_{L^\infty(0,1)}}{1-L} < 1$ then the problem (4.1) has at least one solution.

Proof:
Let h be an arbitrary taken function $h \in H_0^1(0,1)$ and
Let us define a sequence $x_n \in H_0^1(0,1) \cap W^{2,1}(0,1), n \in \mathbb{N}$.
We consider the following formula

$$\begin{cases} \ddot{x}_n(t) + r(t)\dot{x}_{n-1}(t) - F_x - f = 0. \\ x_n := h \in H_0^1(0,1) \end{cases} \tag{4.6}$$

We shall prove that $\{x_n\}$ is Cauchy sequence in $H_0^1(0,1)$ with respect to norm. Since the solution in understood in weak sense,we do the following.
Let $n, m \in \mathbb{N}$, then equation (4.6) for n and m is multiplied by $(x_n - x_m)$ and then integrated with respect to $t \in [0,1]$.

$$-\int_0^1 \ddot{x}_n(t)(x_n - x_m)dt = \int_0^1 \big(r\dot{x}_{n-1}(t) - F_x(t,x_n) - f\big)(x_n - x_m)dt$$

$$-\int_0^1 \ddot{x}_m(t)(x_n - x_m)dt = \int_0^1 \big(r\dot{x}_{m-1}(t) - F_x(t,x_m) - f\big)(x_n - x_m)dt$$

After subtracting the sides and integrating by parts

$$\|x_n - x_m\|^2_{H_0^1(0,1)} = \int_0^1 \big(r\dot{x}_{n-1}(t) - F_x(t,x_n) - f\big)(x_n - x_m)dt$$

$$-\int_0^1 \big(r\dot{x}_{m-1}(t) - F_x(t,x_m) - f\big)(x_n - x_m)dt$$

Thus by equation (2.5) for $0 \neq x_n - x_m \in H_0^1(0,1)$ we have that

$$\|x_n - x_m\|_{H_0^1(0,1)} \leq \int_0^1 \big|(r\dot{x}_{n-1}(t) - F_x(t,x_n) - f\big)(x_n - x_m) - \big(r\dot{x}_{m-1}(t) + F_x(t,x_m) + f\big)(x_n - x_m)\big| dt$$

and by (4.5) we have that

$$\|x_n - x_m\|_{H_0^1(0,1)} \leq \|r\|_{L^\infty(0,1)} \| x_{n-1} - x_{m-1}\|_{H_0^1(0,1)} + L\| x_{n-1} - x_{m-1}\|_{H_0^1(0,1)}.$$

Thus we have that:

$$\|x_n - x_m\|_{H_0^1(0,1)} \leq \frac{\|r\|_{L^\infty(0,1)}}{1-L} \| x_{n-1} - x_{m-1}\|_{H_0^1(0,1)}.$$

Since $\frac{\|r\|_{L^\infty(0,1)}}{1-L} < 1$ we have that (x_n) is Cauchy sequence with respect to $H_0^1(0,1)$ norm.
Thus limit function solves problem (4.2).

4.4 Illustration

We conclude this section with examples of nonlinearities satisfying our assumptions. Let see some example for the clear of this paper

Example 1. *[18] The above scheme can be applied for the following equation*

a) $\ddot{x}(t) + 0.25e^{\frac{-t^2}{2}}\dot{x}(t) - \frac{1}{2}e^{-t}x(t) = t + 1$

b) $\ddot{x}(t) + 0.25e^{\frac{-t^2}{2}}\dot{x}(t) + \frac{1}{4}\frac{x(t)}{1+x^2(t)}$ $\arcsin(t)\; u_n(t) - \frac{1}{2}e^{-t}x(t) = t + 1$

$$\textit{where} \quad u_n^{'}(t) = \begin{cases} 1, & t \in [0, \frac{1}{n}] \\ 0, & t \in (\frac{1}{n}, 1] \quad \textit{is} \quad \textit{control function.} \end{cases}$$

To show that the above equation has at least one solution let us check that three assumption are satisfied
Indeed, H_1 is confirmed since

$$F(\cdot, x) := \frac{1}{2}e^{\frac{-\cdot}{2}}x^2 \in L^1(0,1)$$

and for any $d > 0$ and $x \in [-d, d]$ we have that

$$F_x(t, x) = e^{-t}x \leq e^{-t}d \in L^1(0, 1)$$

H_2 is also satisfied, that is $F(\cdot, x) := \frac{1}{2}e^{\frac{-\cdot}{2}}x^2(t)$ is convex with respect to its second variable.
we can observe for F_x that

$$\mid F_x(t, x) - F_x(t, y) \mid = \mid e^{-t}(x - y) \mid$$

After integrating sides with respect to $t \in [0, 1]$, and knowing that

$$(x - y) \leq \| x - y \|_{L^\infty(0,1)} \leq \| x - y \|_{H_0^1(0,1)}$$

we obtain:

$$\int_0^1 \mid F_x(t, x) - F_x(t, y) \mid dt \leq \| x - y \|_{H_0^1(0,1)} \int_0^1 e^{-t}dt = \frac{e-1}{e} \| x - y \|_{H_0^1(0,1)}$$

which jointly implies that:

$$\int_0^1 \mid F_x(t, x) - F_x(t, y) \mid dt \leq \frac{e-1}{e} \| x - y \|_{H_0^1(0,1)} \leq 0.32 \| x - y \|_{H_0^1(0,1)} \text{ with } L = 0 \cdot 63 < 1.$$

since $\| r \|_{L^\infty(0,1)} = \| 0.25e^{\frac{-t^2}{2}} \|_{L^\infty(0,1)} = 0 \cdot 25$ and $\frac{\|r\|_{L^\infty(0,1)}}{1-L} = \frac{0 \cdot 25}{1-0 \cdot 63} = 0 \cdot 397 < 1$
then by theorem 4.3.1 we conclude that problem given on the given equation has at least one solution.Similarly we can show that the problem given on (b) has at least on solution.

Conclusion

The Duffing equations are processes more complicated non-linearity, for example, it is non-homogeneous, so in the discussions some special techniques will be needed. and using variable exponent theory of Lebesgue and Sobolev spaces combined with variational method , we show the existence of non trivial weak solution of problem 4.1.

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