GRADUATE SEMINAR REPORT

ON

THE FINITE ELEMENT METHOD AND ITS APPLICATION IN TWO DIMENSIONAL SPACES

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Preface

When we are interested to have a mathematical modeling for practical problems in science or technology, it is familiar for us to face one or more differential equations. However, the methods that we encounter in pure mathematics can provide us techniques only for problems, which is possible to have their exact solutions. Because of this, emphasis has been placed on obtaining an intuitive understanding of numerical methods for differential equations and thus various methods of solving differential equations has been devised. One of such methods is the finite element method (FEM).

The FEM, historically developed by structural engineers, is recent and important in solving partial differential equations. Moreover its algorithm is most dominantly applicable in many fields of studies of science and technology.

Even though it is popular and has become a study of great importance, it is not by far self-standing,i.e., it depends on various preliminaries and some basic methods that play a great role to master its algorithm. The Rayleigh – Ritz and Galerkin methods are such methods that are studied in broad sense as a variational principle.

This seminar covers most of the standard topics in the FEM, it also explores some of the main underlying theme of the method. Among these are: the approximation of problems by simpler problems, construction of algorithms and providing illustrative examples.

The examples are carefully chosen to develop an understanding of the method, not just to illustrate an algorithm.

In this report three chapters are involved of which the first two are about background material for FEM. Basically the second chapter deals with the Rayleigh – Ritz and Galerkin methods, which are the basis for the FEM. The last chapter is devoted to the main study of the FEM.

I am greatly indebted to my advisor prof. Dr. S.N Murthy without his invaluable advice, guidance, encouragement and unreserved assistance, which started from proposing the title, this seminar report would not have become fruitful and a reality.

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1 Variational Principle

1.1 Introduction

Variational principles occur widely in physical and other problems and approximate methods of solution of such problems are often based on associated, variational principles. Here the quantities of interest in many areas of applied mathematics are often to be found as the solution of certain partial and ordinary differential equations, together with prescribed boundary and/or initial conditions.

The mathematical formulation of a variational principle is that the integral of some typical function has a smaller (larger) value for the actual performance of the system. In this chapter it will be shown that the concept of a functional is required to understand the Rayleigh – Ritz method that will be discussed in chapter two. Thus, one determines the functional so that it takes an extremum value from a set of permissible functions. This is the central idea of a variational principle.

Many problems arising in physics and engineering are modeled by boundary value problems and initial boundary value problems that will be mentioned in section 1.3.Frequently these equations are equivalent to the problem of minimization of a functional which can be interpreted in terms of the total energy of the given system. In any physical situation, therefore, the functional is obtained from a consideration of the total energy explicitly. Mathematically, however, it would be useful to be able to determine the functional from the governing differential equation itself. This is illustrated well in this seminar report.

1.2 The Euler-Lagrange Differential Equation

For our principle we first need to formulate a functional corresponding to the given boundary value problem. To that end we first need an investigation on functional.

Definition: A functional is a mapping that assigns a definite real number to each function belonging to some set. The following is an instance of a functional.

Example 1.1 The length L of a curve y = y(x) between the values $x = x_0$ and $x = x_1$ is given by the integral

$\mathbf{L} = \int (1 + y'^2)^{1/2} dx$; thus the value of L depends on the ''argument''function y(x) ,which may be taken to be an arbitrary continuous function with a piecewise continuous derivative. Therefore the length of the curve is a functional with that real number the value of the integral.

Example1.2 The area of a surface $z = z(x, y)$ lying above the region **G** in the xy −plane, given by the integral

$$\iint (1 + z_x^{\,2} + z_y^{\,2}) dxdy$$

is a functional of the argument $z(x, y)$.

Definitions: i) problems in which it is required to determine the maxima and minima of functionals are called variational problems.

ii) The extremal of a functional is a function at which the functional attains its maximum or minimum.

In order to furnish the means of formulating a variational problem corresponding to a given second order boundary value problem we need to analyse the functional;

$$J[\varphi] = \int_a^b F(x, \varphi, \varphi')dx \text{ for } a < x < b \ldots\ldots \quad \textbf{(1.1a)}$$

$$\varphi(a) = \varphi_a, \varphi(b) = \varphi_b, \quad \textbf{(1.1b)}$$

where $\varphi = \varphi(x)$, $\varphi' = \varphi'(x)$, $\varphi \in c^1[a,b]$, and F is sufficiently smooth with respect to x, φ and φ'. More specifically, we shall investigate the variational problem, which requires the determination of φ, for which

$J[\varphi] \rightarrow$ minimum; when $J[\varphi]$ is as defined in (1.1a).

Definition: The domain of (1.1a) and (1.1b) is called the set of admissible functions.

The following theorem facilitates a necessary condition for a function y(x) to be that function which gives the integral in (1.1a) a value not greater than that given by any other function φ which is in the domain (1.1), i.e., $J[\varphi] \geq J[y]$ for all φ in the domain of (1.1).

Theorem: Given the variational problem

$$\int_a^b F(x, \varphi, \varphi')\, dx \rightarrow \text{minimum}, \quad \textbf{(1.2a)}$$

for $a < x < b$, $\varphi(a) = \varphi_a$, $\varphi(b) = \varphi_b$, $\varphi \in c^1$ [a, b] and F is sufficiently smooth with respect to x, φ and φ'; if y is a solution of (1.2a) then

$$\left.\frac{\partial F}{\partial \varphi}\right|_{\varphi = y} - \frac{d}{dx}\left.\frac{\partial F}{\partial \varphi'}\right|_{\varphi' = y} = 0. \qquad \textbf{(1.2b)}$$

Proof;

Let y be an admissible function that satisfies (1.2a) and (1.2b). Consider the family of function

$$y_\alpha = y + \alpha y,$$

where α is sufficiently small positive number and y is a sufficiently smooth function s atisfying $y_a = y_b = 0$. Then y_α is sufficiently smooth function satisfying

$$y_\alpha(a) = \varphi_a, y_\alpha(b) = \varphi_b.$$

Then the functional $J[y_\alpha] = \int_a^b F(x, y_\alpha, y'_\alpha)dx$ after integration over [a , b] is carried on will be an ordinary function of α.

Thus set

$$\phi(\alpha) = J[y_\alpha]$$

Since by assumption y is a solution of (1.2), we have that

$$\phi(\alpha) = J[y_\alpha] \geq J[y] = \phi(0) \qquad \text{for all } \alpha.$$

Therefore ϕ assumes its minimum value at $\alpha = 0$.

Since $\phi'(o) = 0$, we have

$$[\frac{d}{dx}\int_a^b F(x, y_\alpha, y'_\alpha)dx]_{\alpha=0} = 0,$$

F is sufficiently smooth so we can also write the above as

$$\int_a^b [\frac{d}{d\alpha}F(x, y_\alpha, y'_\alpha)]_{\alpha=0} = 0.$$

From the fact that $\frac{dx}{d\alpha} = 0$, we have

$$\int [\frac{\partial F}{\partial y_\alpha}\frac{\partial y_\alpha}{\partial \alpha} + \frac{\partial F}{\partial y'_\alpha}\frac{\partial y'_\alpha}{\partial \alpha}]_{\alpha=0} dx = 0.$$

This gives

$$\int_a^b \left[\left.\frac{\partial F}{\partial y_\alpha}\right|_{\alpha=0} Y + \left.\frac{\partial F}{\partial y'_\alpha}\right|_{\alpha=0} Y' \right] dx = 0. \qquad (1.3)$$

Applying integration by parts on the second term of the integral we have

$$\int_a^b \left.\frac{\partial F}{\partial y'_\alpha}\right|_{\alpha=0} Y' dx = \left[\left.\frac{\partial F}{\partial y'_\alpha}\right|_{\alpha=0} Y \right]_a^b - \int_a^b \left[\left.\frac{d}{dx}\frac{\partial F}{\partial y'_\alpha} Y\right|_{\alpha=0} \right] dx. \qquad (1.4)$$

Using the fact that $Y_a = Y_b = 0, and$ substituting (1.4) into (1.3) we have

$$\int_a^b \left[\left.\frac{\partial F}{\partial y_\alpha}\right|_{\alpha=0} - \frac{d}{dx}\left.\frac{\partial F}{\partial y_\alpha}\right|_{\alpha=0} \right] Y dx = 0. \qquad (1.5)$$

Equation (1.5) holds for all functions $Y \in C^1[a,b]$, with $y_a = y_b = 0$.

In particular it is true that if we let

$$Y(x) = (x-a)(b-x)\left[\left.\frac{\partial F}{\partial y_\alpha}\right|_{\alpha=0} - \frac{d}{dx}\left.\frac{\partial F}{\partial y'_\alpha}\right|_{\alpha=0} \right]. \qquad (1.6)$$

Using (1.6) in (1.5) yields

$$\int_a^b \left[\left.\frac{\partial F}{\partial y_\alpha}\right|_{\alpha=0} - \frac{d}{dx}\left.\frac{\partial F}{\partial y'_\alpha}\right|_{\alpha=0} \right]^2 (x-a)(b-x) dx = 0.$$

From which we conclude that

$$\left.\frac{\partial F}{\partial y_\alpha}\right|_{\alpha=0} - \frac{d}{dx}\left.\frac{\partial F}{\partial y'_\alpha}\right|_{\alpha=0} = 0.$$

But when $\alpha = 0, y_\alpha = y, and y'_\alpha = y'$.

Hence, we finally obtain

$$\left.\frac{\partial F}{\partial \varphi}\right|_{\varphi=y} - \frac{d}{dx}\left.\frac{\partial F}{\partial \varphi'}\right|_{\varphi'=y'} = 0. \qquad (1.7)$$

Definition: Equation (1.7) is called the Euler-Lagrange differential equation of the variational problem (1.2).

The following is an illustration of the application of the above theorem.

Example 1.3; Let **A** = (a, φ_a) and B = (b, φ_b) be two points on a plane.

The problem is to find the shortest path from point A to point B.

To that end let φ be any smooth curve joining points A and B.

If we let $J[\varphi]$ to be the length of the curve φ joining points A and B, then $J[\varphi] = \int_a^b \sqrt{1+\varphi'^2}\,dx$. To find the shortest path from A to B, we need to find the smooth curve y for which $J[\varphi]$ is minimum.

If y is a solution for the problem

$$J[\varphi] = \int_a^b \sqrt{1+\varphi'^2}\,dx \rightarrow \text{Minimum}.$$

Then by the above theorem,

$$\frac{\partial}{\partial\varphi}\sqrt{1+\varphi'^2}-\frac{d}{dx}\frac{\partial}{\partial\varphi'}\sqrt{1+\varphi'^2}=0.$$

That is, $\frac{d}{dx}\frac{2\varphi}{2\sqrt{1+\varphi'^2}}=0.$ which yields

$$\frac{d}{dx}\frac{\varphi'}{\sqrt{1+\varphi'^2}}=0.$$

From this observe that $\varphi'=$ constant.

Which implies $\varphi(x)=c_1x+c_2,$

where c_1 and c_2 are constants, which can be determined right away from $\varphi(a)=\varphi_a and \varphi(b)=\varphi_b.$

1.3 Boundary Conditions

Consider the second-order partial differential equation

$$L\phi = f, \tag{1.8}$$

where L is the operator defined by

$$L\phi = a\frac{\partial^2\phi}{\partial x^2} + b\frac{\partial^2\phi}{\partial x\partial y} + c\frac{\partial^2\phi}{\partial y^2} + F(x, y,\phi,\phi_x, \phi_y), \tag{1.9}$$

a, b, and c are in general functions of x and y; they may also depend on ϕ itself, in which case the equation is non-linear. Suppose that the function ϕ satisfies (1.8) in a two dimensional region R, i.e.,

$$L\phi = f,$$

where f(x, y) is a given function of position. Suppose also that ϕ satisfies certain given conditions on the boundary C. These conditions are classified as homogeneous and non-homogeneous boundary conditions.

I. Homogeneous Boundary conditions

Usually these boundary conditions are the following types;

Dirichlet boundary condition: $\phi = 0$, **(1.10)**

Neumann boundary condition: $\frac{\partial \phi}{\partial n} = 0$, **(1.11)**

Mixed boundary condition: $\frac{\partial \phi}{\partial n} + \sigma(s)\,\phi = 0$, **(1.12)**

where s is the arc length measured along C, from some fixed point on C and $\frac{\partial}{\partial n}$ represents differentiation along the outward normal to the boundary C.

Π Non-Homogeneous Boundary Conditions

The boundary conditions to be considered are the non-homogeneous counter parts of (1.10), (1.11), and (1.12).

Which are:

Dirichlet boundary conditions

$$\phi = g(s), \qquad \textbf{(1.13)}$$

Neumann boundary condition

$$\frac{\partial \phi}{\partial n} = j(s), \qquad \textbf{(1.14)}$$

Mixed boundary condition

$$\frac{\partial \phi}{\partial n} + \sigma(s)\,\phi = h(s). \qquad \textbf{(1.15)}$$

1.4 Exteremum Formulation of a Boundary value Problem,

We have already stated that the concept of a functional arises in the study of variational problems that occur widely in physical and other problems. Mathematically, it would be useful to be able to determine the functional from the governing differential equations. This is illustrated by considering the boundary value problem both in one-dimensional domain x and in two-dimensional space. We list below two important formulas, which are frequently used in the variational formulation of differential equations.

$$(1): \delta(dx) = d(\delta x)$$

$$(2): \delta \int f(x)dx = \int \delta f(x)dx,$$

where δ is a small variational parameter.

Now consider the integral defined by,

$$I(y) = \int_a^b f(x, y, y')dx. \qquad \textbf{(1.16)}$$

We know that a necessary condition for I(y) to have an extremum is that y(x) must satisfy the Euler-Lagrange differential equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0. \qquad \textbf{(1.17)}$$

This is the consequence of the theorem in page 4.Similarly for functions of the form

$$I(y) = \int_a^b f(x, y, y', y'')dx. \qquad \textbf{(1.18)}$$

Its Euler-Lagrange differential equation becomes

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0. \qquad \textbf{(1.19)}$$

The Euler-Lagrange equation both in equations (1.17) and (1.19) have several solutions and the one which satisfies the boundary condition given in the problem is selected. This is illustrated below by considering the boundary value problem both in one and in two – dimensional problems.

As first illustration we consider the two – point b oundary v alue problem defined by

$$\frac{d^2y}{dx^2} = f(x)\,, \qquad \textbf{(1.20a)}$$

with y(a)=y(b) =0.

We have $\delta \int_a^b f y dx = \int_a^b f \delta y dx$, using (2)

$$= \int_a^b \frac{d^2y}{dx^2}\delta y dx, \text{ on substituting for f from (1.20a).}$$

$$= \left[\frac{dy}{dx}(\delta y)\right]_a^b - \int_a^b \frac{dy}{dx}\frac{d}{dx}(\delta y)dx, \text{ on integration by parts.}$$

$$= -\int_a^b \frac{dy}{dx}\frac{d}{dx}(\delta y)dx, \text{ since } \delta y(a) = \delta y(b) = 0.$$

$$= -\int_a^b \frac{dy}{dx}\delta\left(\frac{dy}{dx}\right)dx, \text{ using (1).}$$

$$= -\left\{\left[y\delta\left(\frac{dy}{dx}\right)\right]_a^b - \int_a^b y\frac{d}{dx}\left[\delta\left(\frac{dy}{dx}\right)\right]dx\right\}$$

$$= -\left[y\delta\left(\frac{dy}{dx}\right)\right]_a^b + \int_a^b y\frac{d}{dx}\left[\delta\left(\frac{dy}{dx}\right)\right]dx$$

$$= \int_a^b y \frac{d}{dx}\left[\delta\left(\frac{dy}{dx}\right)\right]dx \text{ ; since y(a) = y(b) =0}$$

$$= \int_a^b y\delta\left(\frac{d^2y}{dx^2}\right)dx \text{ , again using (1)}$$

$$= \int_a^b \frac{1}{2}\delta\left(y\frac{d^2y}{dx^2}\right)dx .$$

From this it follows

$$\delta\int_a^b fydx = \delta\int_a^b \frac{1}{2}\left(y\frac{d^2y}{dx^2}\right)dx .$$

This implies that

$$\delta\int_a^b (2fy - yy'')dx = 0 . \qquad \textbf{(1.21)}$$

Hence, a unique solution of the problem defined by (1.20) exists at a minimum value of the integral defined by

$$I(y) = \int_a^b y(2f - y'')dx . \qquad \textbf{(1.22)}$$

By definition, therefore, the integral (1.22) represents the required functional of the problem.

In addition to this if we are given a two – point boundary value problem of the type

1.
$$\left.\begin{array}{l} p(x)y''+q(x)y' = f(x) \\ with\ y(a) = y(b) = 0 \end{array}\right\}, \qquad \textbf{(1.23)}$$

the corresponding functional for this problem is given as

$$I(y) = \int_a^b y[2f - (p(x)y')']dx . \qquad \textbf{(1.24)}$$

2.
$$\left.\begin{array}{l} y''+p(x)y+q(x) = 0, a < x < b \\ with\ y(a) = y(b) = 0 \end{array}\right\}, \qquad \textbf{(1.25)}$$

its functional is then

$$I(y) = \int_a^b \left(y'' - py^2 - 2qy\right)dx . \qquad \textbf{(1.26)}$$

As second illustration (in two dimensional space), we consider the Poisson equation;

$$\left.\begin{aligned} -\nabla^2 u &= (x, y) \\ u &= 0 \ on \ the \ boundary \ C \end{aligned}\right\}. \qquad \textbf{(1.27)}$$

And its required functional is given by;

$$I(y) = \iint_S \left[\frac{1}{2}\left(u_x^2 + u_y^2\right) - fu\right]dxdy . \qquad \textbf{(1.28)}$$

2. Methods of Approximation

2.1 Introduction

In this chapter we have two basic approximation methods viz.: Rayleigh – Ritz and Galerkin methods. So far we have dealt with variational principles as well as functionals, which is required here to understand the Rayleigh – Ritz method. Rayleigh developed the method (1842 - 1919) to solve certain vibration problems and Ritz (1875 - 1909) provided a mathematical basis for it and also applied it to more general problems. The second technique is due to Galerkin who proposed it as an error minimization. It belongs to a wider class of methods called Weighted – Residual methods. An advantage of the Galerkin method is that it works with the governing equation of the problem and does not required functional.

Since both methods have a common feature in that they seek an approximate solution in the form of a linear combination of base functions, we consider first a base function:

2.2 Base Functions

Suppose we wish to approximate a real valued function f(x) over a finite interval [a, b]. A usual approach is to divide [a, b] into a number of subintervals $[x_i, x_{i+1}]$, i = 0, 1, 2, …,n, where

$x_0 = a$, $x_n = b$, and to interpolate linearly between the values of f(x) at the end points of each subinterval. In $[x_i, x_{i+1}]$ the linear approximating function is given by

$$l_i(x) = \frac{1}{h_i}[(x_{i+1} - x)\, f_i + x(x - x_i)\, f_{i+1}], \qquad \textbf{(2.1)}$$

Where $h_i = x_{i+1} - x_i$. From this, we construct the piecewise linear interpolating function over $[x_0, x_n]$ by the formula

$$p(x) = \sum_{i=0}^{n} \phi_i(x) f_i, \qquad \textbf{(2.2)}$$

where

$$\left.\begin{aligned}
\phi_0(x) &= \begin{cases} (x_1 - x)/h_0, x_0 \le x \le x_1 \\ 0 \qquad\qquad , x_1 \le x \le x_n \end{cases} \\
\phi_i(x) &= \begin{cases} (x - x_{i-1})/h_{i-1}, x_{i-1} \le x \le x_i \\ (x_{i+1} - x)/h_i \;, x_i \le x \le x_{i+1} \\ 0 \qquad\qquad , x \ge x_{n-1} \end{cases} \\
\phi_n(x) &= \begin{cases} 0 \qquad\qquad , x_0 \le x \le x_{n-1} \\ \phi(x - x_{n-1})/h_{n-1}, x_{n-1} \le x \le x_n \end{cases}
\end{aligned}\right\} \qquad (2.3)$$

The functions ϕ_i (x), i= 1,2, …, n are called base functions or shape functions. It is easily seen that the base functions ϕ_i (x) are identically zero except for the range $[x_{i-1}, x_{i+1}]$ with $\phi_i(x_i) = 1$.

2.3: The Rayleigh – Ritz Method

The Rayleigh – Ritz method provides an algorithm for minimizing a given functional, and requires the choice of a suitable complete set of linearly independent basis functions φ_i (x), i = 1, 2, …, n. In this method we do not obtain the actual solution but only an approximate solution as nearer the actual solution as the base functions allow. To obtain a good approximation, therefore, the choice of the base functions is important and to improve the approximation, the number of base functions should be increased.

We explain this method by considering second order boundary value problem defined by

$$y'' + p(x)y + q(x) = 0, \qquad \textbf{(2.4a)}$$

$$y(a) = y(b) = 0. \qquad \textbf{(2.4b)}$$

The functional for the above problem (using1.28) is given by

$$I(v) = \int_a^b \left[\left(\frac{dv}{dx} \right)^2 - pv^2 - 2qv \right] dx = 0. \qquad \textbf{(2.5)}$$

From the definition of the functional we know that if y (x), the solution of (2.4a), is substituted in (2.5), then the integral I(v) will be a minimum. i.e. y satisfies the Euler-Lagrange equation. Since we don't know the solution of (2.4a), we try with an approximate solution and determine the parameters of the approximation so that the integral is a minimum. This is the central idea of the Rayleigh-Ritz method.

Now, let

$$v(x) = \sum_{i=1}^{n} \alpha_i \phi_i(x) \qquad \textbf{(2.6)}$$

be an approximate solution where the base functions, $\phi_i(x)$, are linearly independent and satisfy the boundary conditions in (2.4b), i.e. $\phi_i(a) = \phi_i(b) = 0$ and substituting for v in (2.5), we obtain an approximation to the solution y(x) by finding the constants $\alpha_1, \alpha_2, \ldots, \alpha_n$. That is

$$I(\alpha_1,\ldots,\alpha_n) = \int \left[\left\{ \frac{d}{dx} \sum \alpha_i \phi_i(x) \right\}^2 - \{\sum \alpha_i \phi_i(x)\}^2 - 2q \sum \alpha_i \phi_i(x) \right] dx. \qquad \textbf{(2.7)}$$

For a minimum, we have

$$\frac{\partial I}{\partial \alpha_1} \delta\alpha_1 + \frac{\partial I}{\partial \alpha_2} \delta\alpha_2 + \ldots + \frac{\partial I}{\partial \alpha_n} \delta\alpha_n = 0. \qquad \textbf{(2.8)}$$

Since the $\delta\alpha_i^s$ are arbitrary, equation (2.8) gives

$$\frac{\partial I}{\partial \alpha_i} = 0;\ i = 1, 2, \ldots, n. \qquad \textbf{(2.9)}$$

If I is a quadratic function of y and $\frac{dy}{dx}$, then the equation (2.9) will be linear in α_i and can be solved easily.

Next to have an approximation for this method in two-dimensional space; let us consider Poison's equation, $-\nabla^2\phi = f$ with homogeneous Dirchlet or Neumann boundary equations.

The functional is given by equation (1.28) as

$$I(y) = \iint_S \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 - 2\phi f \right] dxdy \qquad \textbf{(2.10)}$$

Using (2.6) and $\phi_i(x) = \phi_i(x, y)$ this functional may be written as

$$I(\alpha_1, \alpha_2, \ldots, \alpha_n) = \iint_S \left\{ \left(\sum_{i=1}^{n} \alpha_i \frac{\partial \phi_i}{\partial x} \right)^2 + \left(\sum_{i=1}^{n} \alpha_i \frac{\partial \phi_i}{\partial y} \right)^2 - 2 \sum_{i=1}^{n} \alpha_i \phi_i f \right\} dxdy,$$

$$= \alpha_i^2 \iint_S \left\{ \left(\frac{\partial \phi_i}{\partial x} \right)^2 + \left(\frac{\partial \phi_i}{\partial y} \right)^2 \right\} dxdy +$$
$$2 \sum_{j \neq i} \alpha_i \alpha_j \iint_S \left(\frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} \right) dxdy$$
$$- 2\alpha_i \iint_S \phi_i f dxdy + terms independent of \alpha_i.$$

Therefore,

$$\frac{\partial I}{\partial \alpha_i} = 2A_{ii} + 2 \sum_{j \neq i} A_{ij} \alpha_j - 2h_i, \qquad \textbf{(2.11)}$$

$$\text{where } A_{ij} = \iint_S \left(\frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} \right) dxdy \qquad \textbf{(2.12)}$$

And $$h_i = \iint_S \phi_i f dxdy. \qquad \textbf{(2.13)}$$

Now the parameters α_j are to be chosen such that $I(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a minimum.

Thus

$$\frac{\partial I}{\partial \alpha_i} = 0, i = 1,2,\ldots n.$$

Equation (2.11) then gives

$$\sum_{i=1}^{n} A_{ij}\alpha_j = h_i, i = 1,\ldots,n$$

Or $$A\alpha = h. \qquad \textbf{(2,14)}$$

Where the elements of the matrices A and h are given by (2.12) and (2.13) and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. The system in (2.14) is a system of linear equation, for the unknown parameters, which has a unique solution provided that A is non – singular.

Example 2.1: Consider the two-point boundary value problem defined by

$$\left.\begin{array}{r} y''+x = 0, \\ \textit{with the boundary conditions} \\ y(0) = y(1) = 0 \end{array}\right\}.$$

Solution: The set of independent basis in (2.6) is frequently taken to be a set of polynomials. Suppose a cubic approximation is taken, then the trial function is of the form

$$v(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3.$$

Two of these coefficients may be evaluated immediately in order that v satisfies the Dirichlet boundary conditions v(0) = v(1) = 0.

Thus $v(x) = ax(1-2x) + bx^2(1-x)$.

Using the one dimensional forms of (2.12) and (2.13) and $-y'' = x$, we have

$$A_{11} = \int_0^1 (1-2x)^2 dx = \frac{1}{3},$$

$$A_{12} = A_{21} = \int_0^1 (1-2x)(2x-3x^2)dx = \frac{1}{6},$$

$$A_{22} = \int_0^1 (2x-3x^2)^2 = \frac{2}{15},$$

$$h_1 = \int_0^1 \left(x-x^2\right)xdx = \frac{1}{12},$$

and $$h_2 = \int \left(x^2-x^3\right)xdx = \frac{1}{20}.$$

Thus the Rayleigh – Ritz equations of (2.14) are

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{15} \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1}{12} \\ \frac{1}{20} \end{pmatrix}$$

This implies

$$4a + 2b = 1$$

$$10a + 8b = 3$$

From this it follows

$$a = b = \frac{1}{2}$$

Thus the Rayleigh – Ritz cubic approximation is given by

v (x) = $\frac{1}{6}x(1-x)^2$. Which is the exact solution of the problem.

Example 2.2: Consider the two-point boundary value problem,

$$-y'' = x^2,\ 0< x < 1$$

$$\phi(0) = \phi(1) = 0$$

Solution: The set of independent basis functions in (2.6) is frequently taken to be a set of polynomials. Suppose $v(x) = \alpha_1 x(1-x) + \alpha_2 x^2(1-x)$ be the trail function, which satisfies the homogeneous Dirichlet boundary conditions $v(0) = v(1) = 0$, with $\phi_1(x) = x(1-x)$ and $\phi_2(x) = x^2(1-x)$.

Using the same procedure as in example (2.1), the value of the unknown parameters is $\alpha_1 = 1/5$ and $\alpha_2 = 1/6$. Hence the Rayleight-ritz cubic approximation is given by

$$v(x) = \frac{x}{30}(1-x)\,(2 + 5x)$$

The exact solution is $v_0 = \frac{x}{12}(1-x^3)$, and a comparison between v and v_0 is shown in the figure 2.1.

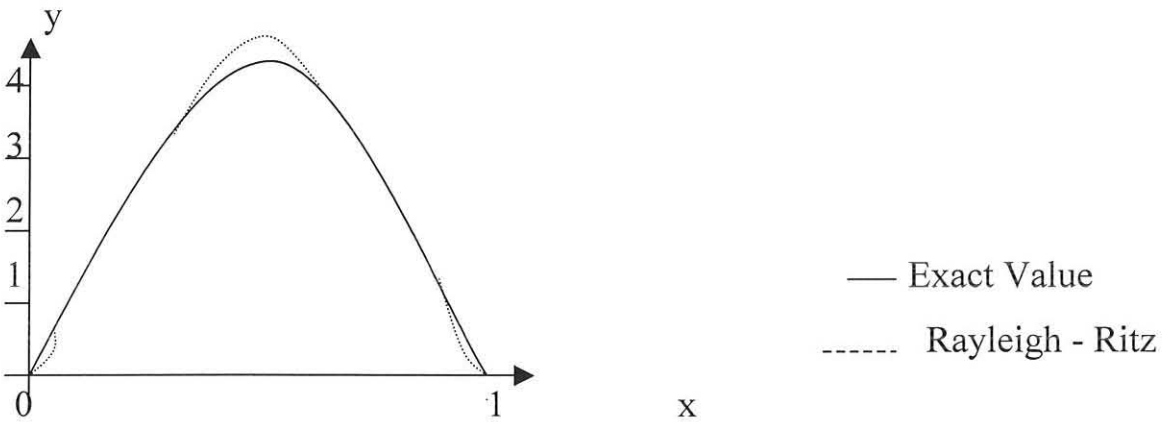

Fig 2.1 Comparison of the Rayleigh-Ritz approximate solution with the exact solution.

In this particular example, the Rayleigh – Ritz method gives a good approximation to the exact solution with only two parameters α_1 *and* α_2. This is due to the simplicity of the problem and the fact that the exact solution is a quartic polynomial, thus a cubic approximation would be expected to produce reasonable results. In fact the exact solution would be recovered if a quartic trial function was used.

For problems in which the exact solution is a transcendental function, a low order polynomial approximation is not likely to yield a good approximation. In deeded even a trial function which itself is a similar transcendental function will not yield satisfactory results.

2.5 **Galerkin's Method**

The Rayleigh – Ritz method discussed in section 2.3 is a powerful technique for the solution of boundary value problems. It has, however, the disadvantage of requiring the existence of a functional, which is not always possible to obtain. In the case where such a functional does not exist, or cannot be found, techniques of numerical solutions have to be considered. Galerkin's method belongs to a wider class of methods called the Weighted – residual methods.

In this method, an approximating function called the trial function (which satisfies all the boundary conditions) is substituted in the given differential equation and the result is called the residual (the result will not be zero since we have substituted an approximating function). The residual is then weighted and the integral of the product, taken over the domain, is then set

to zero. It can be shown that if the Euler – Lagrange equation corresponds to a functional coincides with the differential equation of the problem, then the Rayleigh – Ritz and Galerkin methods yield the same system of equations.

To explain Galerkin's method, we consider the boundary value problem defined by

$$y'' + p(x)y' + q(x)y = f(x), \quad a < x < b, \tag{2.16a}$$

with the boundary conditions

$$\left.\begin{aligned} p_0 y(a) + q_0 y'(a) &= 1_0 \\ p_1 y(b) + q_1 y'(b) &= 1_1 \end{aligned}\right\} \tag{2.16b}$$

Choose base functions $\phi_i(x)$ as in the Rayleigh-Ritz method to find an approximate solution of the problem in (2.16). Then an approximate solution v (x) is given by

$$v(x) = \sum_{i=1}^{n} \alpha_i \phi_i(x),$$

v (x) will not, in general, satisfy (2.16a) but produces a residual or discrepancy. This is equal to the difference between the left hand and right hand sides of the equation (2.16a) when on the left – side y (x) is replaced by v(x).If R(v) is the residual, then we write

$$R(v) = v'' + p(x)\, v' + q(x)v - f(x) \tag{2.17}$$

Taking the weight function as $\psi_i(x)$, we write

$$\int_a^b \psi_i(x)\, R(v)dx = 0 \tag{2.18}$$

Which yields a system of equations for the unknown parameters α_i and can be solved. In Galerkin method, we usually take $\psi_i = \phi_i$. The method is illustrated with the following example.

Example2.3: We consider a two – point boundary value problem

$$y''+y=-x,0<x<1 \tag{2.3.1}$$

$$y(0)=y(1)=0 \tag{2.3.2}$$

Solution;

As our first approximation we choose,

$$v(x)=\alpha_1\phi_1(x)=\alpha_1 x(1-x) \tag{2.3.3}$$
$$where\ \phi_1(0)=\phi_1(1)=0$$

Substituting for v in (2.3.1), we obtain

$$\mathrm{R(v)}=v''+v+x \tag{2.3.4}$$

Hence using (2.18), we write

$$\int_0^1 (v''+v+x)\phi_1(x)dx=0,\ where\ \psi_1=\phi_1$$

That is

$$\int_0^1 (v''+v+x)x(1-x)dx=0 \tag{2.3.5}$$

$$Now\ \int_0^1 v''x(1-x)dx=[v'x(1-x)]_0^1-\int_0^1 v'(1-2x)dx$$

$$=-\int_0^1 v'(1-2x)dx$$

$$=-v[v(1-2x)]_0^1+\int_0^1-2vdx$$

$$=-2\int_0^1 vdx,\ \sin ce\ v(0)=v(1)=0$$

Hence (2.3.5) becomes

$$\int_0^1 \left[-2v+v(1-2x)+x^2(1-x)\right]dx=0. \tag{2.3.6}$$

From this it follows

$$-2\int_0^1 v\,dx + \int_0^1 vx(1-x)\,dx + \int_0^1 x^2(1-x)\,dx = 0$$

$$-2\alpha_1\int_0^1 x(1-x)\,dx + \alpha_1\int_0^1 x^2(1-x)\,dx + \int_0^1 x^2(1-x)\,dx = 0$$

$$\Rightarrow \alpha_1 = \frac{5}{18}, \Rightarrow v(x) = \frac{5}{18}x(1-x)$$

To obtain a better approximation let us assume

$$v(x) = \alpha_1 x(1-x) + \alpha_2 x^2(1-x) \qquad (2.3.7)$$

so that $v'' = -2\alpha_1 + \alpha_2(2-6x)$

Equation (2.18) *now gives the two equations*

$$\int_0^1 (v''+v+x)x(1-x)\,dx \qquad (2.3.8)$$

and

$$\int_0^1 (v''+v+x)x^2(1-x)\,dx \qquad (2.3.9)$$

Substituting for v in (2.3.8) and (2.3.9) and simplifying, we obtain the two equations

$$2\alpha_1 + \alpha_2 = \frac{5}{9} \qquad (2.4.0)$$

and

$$\frac{3}{20}\alpha_1 + \frac{13}{105}\alpha_2 = \frac{1}{20} \qquad (2.4.1)$$

Solving the above equations, we obtain $\alpha_1 = 0.1924$ *and* $\alpha_2 = 0.1707$

The comparison of two successive approximates of Galerkin and the exact solution is shown in the table 2.1 given below.

X	First app.	Second app.	Exact solution
0	1	0	0
0.1	0.025	0.0188	0.0186
0.2	0.044	0.0362	0.0361
0.3	0.058	0.o512	0.0512
0.4	0.0667	0.0626	0.0628
⋮	⋮	⋮	⋮
1	0	0	0

Table 2.1

The table shows that the second approximation is more appropriate and shows a reasonable result compared to the exact solution.

2.5 Application to two-dimensional Problems

In section (2.3) we have already formulated our Rayleigh –Ritz method both in one-dimensional x and two-dimensional space. Therefore, here we are intended only to develop the procedure for Galerkin's method in two – dimensional space. We use the same procedure as in one-dimensional x.

Consider the boundary-value problem

$$L\phi = f, \text{ in } \Re.... \qquad \textbf{(2.19)}$$

Subject to the boundary conditions $\phi = g(s)$ on some part C_1 of the boundary, and $\frac{\partial \phi}{\partial n} + \sigma(s)\phi = h(s)$ on the remaining boundary C_2. An approximate

solution ϕ will not, in general, satisfy (2.19) exactly, and associated with such an approximation solution is the residual defined by

$$R(\phi) = L\phi - f \tag{2.20}$$

If the exact solution is ϕ_0 then

$$R(\phi_0) = 0 \tag{2.21}$$

As in the Rayleigh – Ritz of section (2.3), a set of basis functions ψ is chosen and an approximation of the following form is made.

$$\phi_n = \sum_{i=1}^{n} c_i \psi_i \tag{2.22}$$

In the weighted residual method the unknown parameters c_i are chosen to minimize the residual in some sense. The integral of the residual, weighted by the basis functions, is set to zero. i.e.,

$$\iint_S R(\phi)\psi_i dxdy = 0, \text{ i= 1, 2, ..., n}$$

This yields the following n equations for the n parameters c_i i.e.

$$\sum_{j=1}^{n} c_j \iint_S \psi_i L \psi_j dxdy = \iint_S f\psi_i dxdy, \text{i} = 1, \ldots, \text{n} \tag{2.23}$$

Now let us define

$$A_{ij} = -\iint_S \psi_i \nabla^2 \psi_j dxdy \tag{2.24}$$

$$h_i = \iint_S \psi_i f dxdy \tag{2.25}$$

In the special case $L = -\nabla^2$, these equations are of the form

$$Ac = h.... \tag{2.26}$$

Where $A = (A_{ij})_{i,j=1,2,...,n}$

$$h = (h_i)_{i=1,..,n},$$

and $$c = (c_j)_{j=1,..,n}$$

Next, we illustrate the application of Rayleigh – Ritz and Galerkin methods with the following.

Example 2.4;Consider Poisson's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = k \quad ,0 \le x, y \le 1,$$

with $\phi = 0$ on the boundary C of the region S.

Solution;

a) **Rayleigh – Ritz method:**

Using the equation (2.12) and (2.13),we obtain

$$A_{ij} = \iint_S \left(\frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} \right) dxdy \text{ and}$$

$$h_i = \iint_S \phi_i f dxdy$$

Let $v(x, y) = \alpha_1 xy(x-1)(y-1)$ be a first approximation to ϕ. Clearly, ϕ' satisfies, the boundary condition.i.e., $\phi' = 0$ on the boundary C.

Here $\phi_1(x, y) = xy(x-1)(y-1)$.

Then the derivatives are given by

$$\frac{\partial \phi_1}{\partial x} = (y^2 - y)(2x-1), \quad \frac{\partial \phi_1}{\partial y} = (2y-1)(x^2 - x)$$

Thus

$$A_{11} = \iint_S \left\{ \left(\frac{\partial \phi_1}{\partial x} \right)^2 + \left(\frac{\partial \phi_1}{\partial y} \right)^2 \right\} dxdy$$

$$= \int_0^1 \int_0^1 \left\{ \left[(y^2 - y)(2x-1) \right]^2 + \left[(2y-1)(x^2 - x) \right]^2 \right\} dxdy$$

$$= \int_0^1 \left\{ (y^2 - y) \int_0^1 (2x-1)^2 \, dx \right\} dy + \int_0^1 \left\{ (2y-1)^2 \int_0^1 (x^2 - x)^2 \, dx \right\} dy$$

$$= \int_0^1 \frac{(y^2 - y)^2}{3} dy + \int_0^1 \frac{(2y-1)^2}{30} dy$$

$$= \frac{1}{3}\left(\frac{1}{30} \right) + \frac{1}{30}\left(\frac{1}{3} \right)$$

$$= \frac{1}{45}$$

$$h_1 = \int_0^1 \int_0^1 kxy(x-1)(y-1)dxdy = k \int_0^1 \left\{ (y^2 - y) \int_0^1 (x^2 - x)dx \right\} dy = \frac{k}{36}$$

Hence using (2.14), we have

$$\frac{1}{45}\alpha_1 = \frac{k}{36} \Rightarrow \alpha_1 = \frac{5}{4}k .$$

Therefore, the first function ϕ' for ϕ is given by $\phi \approx \phi' = \frac{5}{4} xy(x-1)(y-1)$.

b) **Galerkin method**

Using the same trial function v, we have

$$\phi_1(x, y) = xy(x-1)(y-1) = (x^2 - x)(y^2 - y) .$$

Then the derivatives are given by

$$\frac{\partial \phi_1}{\partial x} = (2x-1)(y^2-y); \frac{\partial \phi_1}{\partial y} = (x^2-x)(2y-1)$$

$$\frac{\partial^2 \phi_1}{\partial x^2} = 2(y^2-y); \quad \frac{\partial^2 \phi_1}{\partial y^2} = 2(x^2-x)$$

and $\nabla^2 \phi_1 = \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2}$. *Using (2.24) and (2.25). we obtain*

$$A_{11} = -\int_0^1 \int_0^1 \phi_1 \nabla^2 \phi_1 dxdy$$

$$= -\int_0^1 \int_0^1 \phi_1 \left[2\left(y^2-1\right)+2\left(x^2-x\right)\right] dxdy; \text{since} \nabla^2 \phi_1 = 2(y^2-y)+2(x^2-x)$$

$$= \frac{1}{45}$$

and

$$h_1 = \int_0^1 \int_0^1 \phi_1 dxdy = \int_0^1 \int_0^1 kxy(x-1)(y-1)dxdy = \frac{k}{36}$$

Hence using (2.26), we obtain $\frac{1}{45}\alpha_1 = \frac{k}{36} \Rightarrow \alpha_1 = \frac{5}{4}k$.

It follows the required approximation is given by

$$\phi \approx \phi' = \frac{5}{4}kxy(x-1)(y-1) \ .$$

This shows that the two methods give the same approximation solution due to the reason that the Euler – Lagrange equation corresponding to a functional coincides with the differential equation of the problem.

3. The Finite Element Method (FEM)

3.1; Introduction

The Rayleigh – Ritz and Galerkin methods discussed in chapter two cannot be applied directly for obtaining the global approximate solutions of engineering problems due to the following two basic reasons:

I. The difficulties associated with the choice of trial functions (satisfying boundary conditions) particularly for complicated boundaries.

II. Even if the approximating functions are available, very high order polynomials have to be used to obtain global solutions with a reasonable accuracy.

In the FEM, the ideas of both Rayleigh – Ritz and Galerkin methods are used in such a way that the above mentioned problems are taken care. In this method the region of interest is subdivided into a finite number of sub regions called ELEMENTS, and over each element the variational formulation of the given differential problem is constructed using simple functions for approximations. The individual elements are then assembled and the equations for the whole problem are formed by a piecewise approximation of the variatinal method. For better accuracy it will not be necessary to increase the order of the functions used, but it would be sufficient to use a finer mesh (sub divisions). In this way, the

difficulties encountered in the direct application of the variational methods are taken care. We shall now present FEM both for one and two-dimensional problems.

3.2: FEM For One Dimensional Problem.

Consider the two-point boundary value problem defined by

$$[p(x)y']' = -f(x), \quad a < x < b \qquad (3.1a)$$

with the boundary conditions

$$y(a) = 0$$

$$[p(x)y']_{x=b} = 0 \qquad (3.1b)$$

We shall now present the basic steps involved in FEM, to understand the boundary value problem defined in (3.1).

Step 1: Discretization of the domain

This is the first step in the FEM. It constitutes the division of the region [a,b] in to a finite number of simpler sub regions, called elements, of unequal length, in general; but more often we use equal length to simplify the problem at hand.

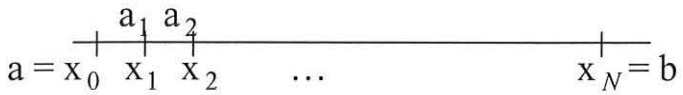

Fig 3.1 Discretization of the region

In the figure, the points $x_0, x_1, \cdots, x_N$, which are the intersection points between the elements and are called nodal points; and the values of y at x_i, for i = 0,1,2, … ,N, denoted by y_i *is called a Nodal value.* In the sequel

we shall use Ω^e to represent the element $[x_{e-1}, x_e]$, and the element Ω^e is assigned the number e. After dividing the domain (region) into such a finite number of elements, we shall find an approximation y_e to the solution of (3.1a) on Ω^e such that

$$y_e(x) = 0, \forall x \notin \Omega^e.$$

The solution y(x) of (3.1a) on [a,b] will then obtained as

$$y(x) = \sum_e y_e(x),$$

where e runs over all the elements.

Step 2: Variatonal formulation on Ω^e

Once the discretization process is completed, an arbitrary element Ω^e will be isolated and a variational approximation of (3.1a) on Ω^e will be formulated over it.

Let v be a function of x that is satisfying boundary conditions given in (3.1b).

From (3.1a), we obtain

$$\int_{x_{e-1}}^{x_e} v[py']' dx = -\int_{x_{e-1}}^{x_e} vfdx,$$

which is written as

$$[vpy']_{x_{e-1}}^{x_e} - \int_{x_{e-1}}^{x_e} (pv'y')dx + \int_{x_{e-1}}^{x_e} vfdx = 0$$

$$\Rightarrow \int_{x_{e-1}}^{x_e} (pv'y')dx = \int_{x_{e-1}}^{x_e} vfdx + v(x_e)D_2^{(e)} + v(x_{e-1})D_1^{(e)} \tag{3.2}$$

$$\left.\begin{aligned} where \quad D_1^{(e)} &= [-py']_{x_{e-1}} \\ D_2^{(e)} &= [py']_{x_e} \end{aligned}\right\} \tag{3.3}$$

Step 3: Rayleigh – Ritz approximation over the element Ω^e

In this step, we use a variational method to approximate equation (3.2). We demonstrate this by using Rayleigh – Ritz method.

Let $y_e(x)$ be an approximation to y(x) over the element Ω^e, so that

$$y_e(x) = \sum_{j=1}^{n} \alpha_j^{(e)} \phi_j(x) \tag{3.4}$$

where the α_j 's are parameters to be determined and $\phi_j(x)$ are approximation functions to be chosen. Substituting (3.4) in (3.2) and letting v = ϕ_i(x), we obtain

$$\sum_{j=1}^{n} \alpha_j^{(e)} \left[\int_{x_{e-1}}^{x_e} p\phi_i' \phi_j' dx \right] = \int_{x_{e-1}}^{x_e} f\phi_i dx + \phi_i(x_e) D_2^{(e)} + \phi_i(x_{e-1}) D_1^{(e)}$$

$$i = 1,2,\cdots,n \tag{3.5}$$

Now defining

$$\left.\begin{aligned} K_{ij}^{(e)} &= \int_{x_{e-1}}^{x_e} p\phi'_i \phi_j' dx \\ F_i^{(e)} &= \int_{x_{e-1}}^{x_e} f\phi_i dx + \phi_i(x_e) D_2^{(e)} + \phi_i(x_{e-1}) D_1^{(e)} \end{aligned}\right\} \tag{3.6}$$

Equation (3.2) can be written as

$$K^{(e)} \alpha^{(e)} = F^{(e)} \tag{3.7}$$

Where

$$\begin{aligned} K^{(e)} &= \left(K_{ij}^{(e)}\right)_{i,j=1,2,\cdots,n} \\ \alpha^{(e)} &= \left(\alpha_j^{(e)}\right)^T_{j=1,2,\cdots,n} \\ F^{(e)} &= \left(F_i^{(e)}\right)^T_{i=1,2,\ldots,n} \end{aligned} \tag{3.8}$$

The matrices $k^{(e)}$ *and* $F^{(e)}$ are called element stiffness matrix and element force vector respectively.

In the Rayleigh – Ritz and Galerkin methods, the system of equation is obtained in terms of the arbitrary parameters α_j, in the FEM, on the other hand, the system of equations is obtained in terms of the nodal values y_i. This is done in the following way. Let

$$y_e(x) = \alpha_1^{(e)} + \alpha_2^{(e)} x \tag{3.9}$$

be an approximation in the element Ω^e. Then we have

$$\left.\begin{aligned} y_e(x_{e-1}) &= \alpha_1^{(e)} + \alpha_2^{(e)} x_{e-1} = y_1^{(e)} \\ y_e(x_e) &= \alpha_1^{(e)} + \alpha_2^{(e)} x_e = y_2^{(e)} \end{aligned}\right\} \tag{3.10}$$

Solving equations in (3.10), we obtain

$$\alpha_1^{(e)} = \frac{y_1^{(e)} x_e - y_2^{(e)} x_{e-1}}{x_e - x_{e-1}} \tag{3.11}$$

and

$$\alpha_2^{(e)} = \frac{y_2^{(e)} - y_1^{(e)}}{x_e - x_{e-1}} \tag{3.12}$$

Equation (3.9) now becomes

$$\begin{aligned} y_e(x) &= \frac{y_1^{(e)} x_e - y_2^{(e)} x_{e-1}}{x_e - x_{e-1}} + \frac{y_2^{(e)} - y_1^{(e)}}{x_e - x_{e-1}} x \\ &= \frac{x_e - x}{x_e - x_{e-1}} y_1^{(e)} + \frac{x - x_{e-1}}{x_e - x_{e-1}} y_2^{(e)} \end{aligned}$$

From this we set

$$y_e(x) = \sum_{i=1}^{2} y_i^{(e)} \phi_i^{(e)} \tag{3.13}$$

Where

$$\left.\begin{aligned} \phi_1^{(e)}(x) &= \frac{x_e - x}{x_e - x_{e-1}} \\ \phi_2^{(e)}(x) &= \frac{x - x_{e-1}}{x_e - x_{e-1}} \end{aligned}\right\} \tag{3.14}$$

The $\phi_i (i = 1,2)$ *in* (3.14) are linearly dependent. Besides, with this choice for ϕ_i, $y_e(x)$ in (3.13) satisfies the essential boundary conditions and

$$y_e(x_{e-1}) = y_1^{(e)}$$
$$y_e(x_e) = y_2^{(e)}$$

And the parameters (Ritz coefficients) are the nodal values $y_i^{(e)}$ *for* $i = 1,2$. This is the type of approximation that we require in FEM.

Instead of equation (3.7) we now have

$$K^{(e)} y_e = F^{(e)} \tag{3.15}$$

Where $K^{(e)}$ *and* $F^{(e)}$ *are given by* (3.8) *and* $y_e = (y_i^{(e)})_{i=1,2,\cdots,n}$

Equation (3.15) gives the system of equations for each element Ω^e. The next step will be assembling the equations to the over all system of equations.

Step 4: Assembly of element equations

For the systematic description of the summation of the solution y(x) of (3.1) from y_e, let us first define

$$y^T := (y_0, y_1, y_2, \cdots, y_N)$$

where y_i, for I = 0,1,2,...,N are the nodal values

Let us set $B_e := (b_{kj}^{(e)})_{k=1,2;\, j=0,1,2,\cdots,N}$ *to be the* $2 \times (N \times 1)$ *matrix defined by*

$$b_{kj}^{(e)} = \begin{cases} 1, \text{if } k = 1 \text{ and } j = e \text{ or } k = 2 \text{ and } j = e+1 \\ 0, \text{otherwise} \end{cases}$$

with this we observe that

$$y_e = B_e y,$$

where y_e *is as in* (3.15) *and* y *is the transpose of* y^T.
The equation (3.15) *may be written as*

$$K^{(e)} B_e y = F^{(e)}.$$

Multiplying both sides from the left by B_e^T, *we have*

$$B_e^T K^{(e)} B_e y = B_e^T F^{(e)}.$$

Adding these over the N elements, we obtain

$$\sum_{e=1}^{N} B_e^T K^{(e)} B_e y = \sum_{e=1}^{N} B_e^T F^{(e)}.$$

Now setting

$$K = \sum_{e=1}^{N} B_e^T K^{(e)} B_e y \text{ and } F = \sum_{e=1}^{N} B_e^T F^{(e)},$$

we obtain

$$Ky = F. \tag{3.16}$$

Which is the system of equations for the global system.

Step 5: Imposition of boundary conditions

The assembly procedure for K and F described in step 4 does not include the enforcement of the essential boundary conditions.

To describe the enforcement of essential boundary conditions, assume as a trial case first that $y_0 = 0$ is given. The overall system has been set up assuming that all the variables $y_i, i = 0,1,2,\cdots,N$, are available for minimization of the functional. But if $y_0 = 0$, it follows that the equation obtained by minimizing with respect to y_0 is identically 0. This means that the first equation Ky = F should be deleted. More over since $y_0 = 0$, all terms in Ky = F that contain y_0 should be zero. Thus the introduction of essential boundary condition $y_0 = 0$ is effected by deleting the first row and the first column in K and the first entry in F. Similarly, if y (a) = $y_0 \neq 0$ the introduction of such a non-homogeneous essential boundary condition is effected by deleting the first equation in (3.16), deleting the first entry in F, and substituting the numerical value for y_0, wherever it appears. A similar treatment can be used if an essential boundary condition y (b) = y_0 is given. After imposing the essential boundary condition, the resulting system of equations will be solved for the remaining unknown nodal values. Once all the nodal values are determined the finite element solution y of (3.1) is finally given by

$$y(x) = \sum_{e=1}^{N} y_e(x)$$

3.3; An illustration of FEM for one dimensional problem

To demonstrate the FEM, we consider the following two point boundary value problem (BVP)

$$-y''(x) = 2 \qquad o < x < 1 \tag{3.3.1}$$
$$y(0) = y'(1) = 0$$

Consider an N element descretization of [0,1] with N+1 nodes as in fig.3.2, where $0 = x_0 < x_1 < \cdots < x_N = 1$

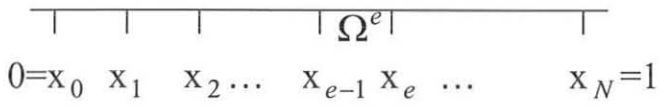

Fig. 3.2 discretization of the domain [0,1]

For an element $\Omega^e = [x_{e-1}, x_e]$, the system of equations as given in (3.15) is

$$K^{(e)} y_e = F^{(e)} \tag{3.3.2}$$

From (3.6) and (3.14) with h = x$_e$-x$e-1$, we have

$$K_{11}^{(e)} = \int_{x_{e-1}}^{x_e} (\phi_1'^{(e)})^2 dx = \frac{1}{h_e}$$

$$K_{12}^{(e)} = K_{21}^{(e)} = \int_{x_{e-1}}^{x_e} \phi_1^{(e)} \phi_2^{(e)} dx = -\frac{1}{h_e}$$

$$K_{22}^{(e)} = \int_{x_{e-1}}^{x_e} (\phi_2'^{(e)})^2 dx = \frac{1}{h_e}$$

$$F_1^{(e)} = \int_{x_{e-1}}^{x_e} 2\phi_1^{(e)} dx + \phi_1^{(e)}(x_{e-1}) D_1^{(e)} + \phi_1^{(e)}(x_e) D_2^{(e)} = h_e + D_1^{(e)}$$

$$F_2^{(e)} = \int_{x_{e-1}}^{x_e} 2\phi_2^{(e)} dx + \phi_2^{(e)}(x_{e-1}) D_1^{(e)} + \phi_2^{(e)}(x_e) D_2^{(e)} = h_e + D_2^{(e)}$$

Thus the matrices $K^{(e)}$ *and* $F^{(e)}$ *are*

$$K^{(e)} = \frac{1}{h_e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \text{ and } F^{(e)} = \frac{1}{h_e}\begin{pmatrix} h_e + D_1^{(e)} \\ h_e + D_2^{(e)} \end{pmatrix}$$

with $y_e = \left(y_1^{(e)}, y_2^{(e)}\right)^T$, *the system* (3.3.2) *will be*

$$\frac{1}{h_e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} y_1^{(e)} \\ y_2^{(e)} \end{pmatrix} = \begin{pmatrix} h_e + D_1^{(e)} \\ h_e + D_2^e \end{pmatrix}$$

If we now consider a special case where the number of elements N = 2, then $y^T = (y_0, y_1, y_2)^T$ (the vector of nodal values). Now the system of equations for the two elements will be as follows

i) for $e=1, x_{e-1}=0, x_e=\frac{1}{2}$ *and* $h_e=\frac{1}{2}$

The system is then

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} y_1^{(1)} \\ y_2^{(1)} \end{pmatrix}=\begin{pmatrix} \frac{1}{2}+D_1^{(1)} \\ \frac{1}{2}+D_2^{(1)} \end{pmatrix}$$

ii)for $e=2, x_{e-1}=\frac{1}{2}, x_e=1$ *and* $h_e=\frac{1}{2}$

The system is then

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} y_1^{(2)} \\ y_2^{(2)} \end{pmatrix}=\begin{pmatrix} \frac{1}{2}+D_1^{(2)} \\ \frac{1}{2}+D_2^{(2)} \end{pmatrix}$$

We can now assemble these equations to the whole system. From the matrix B_e introduced in step 4 (section3.2) we have

$$B_1=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } B_2=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then proceeding as in step 4, equation (3.16) will be of the form Ky = F.

That is:

$$\sum_{e=1}^{2} B_e^T K^{(e)} B_e y=\sum_{e=1}^{2} B_e^T F^{(e)}$$

which is the same as

$$\left(B_1^T K^{(1)} B_1+B_2^T K^{(2)} B_2\right)y=B_1^T F^{(1)}+B_2^T F^{(2)} \qquad (3.3.3)$$

When the actual matrices are substituted in (3.3.3) we obtain

$$\begin{pmatrix} 2 & 2 & 0 \\ -2 & 2+2 & -2 \\ 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + D_1^{(1)} \\ 1 + D_2^{(1)} + D_1^{(2)} \\ \frac{1}{2} + D_2^{(2)} \end{pmatrix} \qquad (3.3.4)$$

By virtue of step 5 of section 3.2, the homogeneous boundary condition $y_0 = y(x_0) = 0$ is effected by deleting the first row and the first column in K and the first entry in f. Moreover, from the boundary condition $y'(1) = 0$, we have $D_2^{(2)} = 0$, and by definition, $D_1^{(2)} + D_2^{(1)} = 0$.

Thus the system (3.3.4) will be

$$4y_1 - 2y_2 = 1$$

$$-2y_1 + 2y_2 = \frac{1}{2}$$

From which we obtain $y_1 = \frac{3}{4}$ *and* $y_2 = 1$.

Thus the finite element solution of (3.3.1) with two elements is

$$y(x) = \begin{cases} y_0\phi_1^{(1)}(x) + y_1\phi_2^{(1)}(x) & for\ o \le x \le \frac{1}{2} \\ y_1\phi_1^{(2)}(x) + y_2\phi_2^{(2)}(x) & for\ \frac{1}{2} \le x \le 1 \end{cases}$$

$$= \begin{cases} \frac{3}{2}x & for\ 0 \le x \le \frac{1}{2} \\ \frac{x+1}{2} & for\ \frac{1}{2} \le x \le 1. \end{cases}$$

If we increase the number of elements to four, a similar procedure gives the finite element solution as

$$y(x) = \begin{cases} \frac{7x}{4} & for\ 0 \le x \le \frac{1}{4} \\ \frac{10x+1}{8} & for\ \frac{1}{4} \le x \le \frac{1}{2} \\ \frac{6x+3}{8} & for\ \frac{1}{2} \le x \le \frac{3}{4} \\ \frac{x+3}{4} & for\ \frac{3}{4} \le x \le 1 \end{cases}$$

The following figure gives the comparison of the two and four element solution of (3.3.1) with the exact solution y (x) = 2x - x^2. We observe that the refined mesh with four elements gives a better approximation than the two-element mesh. Thus the more the elements we use the better the approximation will be.

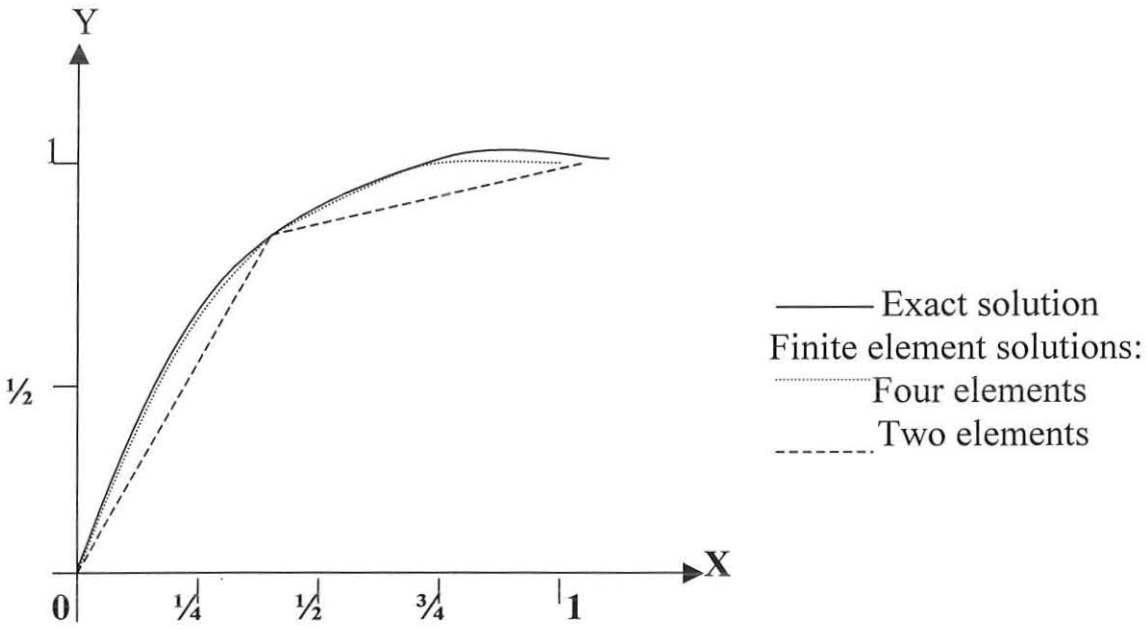

Fig.3.3 Comparison of the finite element solutions with the exact solution.

3.4: Application in two – dimensional problems

The finite element method described in the previous section can be extended to two – dimensional problems. Since the two – dimensional problems are modeled by partial differential equations, the analysis will be complicated and we demonstrate this application by considering the Poisson equation. Moreover we consider triangular elements only, although the geometric shapes chosen could be rectangles or quadrilaterals.

Consider the Poisson equation

$$-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f(x,y) \qquad (3.17a)$$

$$with\ the\ conditon\ u=0$$

on the boundary of the square $a\le x\le b, a\le y\le b$.

Step 1: Variational formulation of the problem

The variational form of (3.17a) using (1.28) is given by

$$\iint_{R_e}\left[\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)-vf\right]dxdy-\int_{C_e} vq_n ds=0 \qquad (3.18)$$

Where $q_n=\eta_x\frac{\partial u}{\partial x}+\eta_y\frac{\partial u}{\partial y}$, η_x *and* η_y being the direction cosines of a unit normal $\hat{n}$ on the boundary C and ds is an arc length of an infinitesimal element along the boundary, and v = v(x,y) is a test function.

Step 2: Setting up a finite element model for the given equation

To do this, we approximate u by the expression

$$u = \sum_{j=1}^{n} u_j \varphi_j \tag{3.19}$$

where $u_j = (x_j, y_j)$ *and* ϕ_j *have the property*

$$\phi_i(x_j, y_j) = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \tag{3.20}$$

Substituting (3.19) in (3.18) and putting v=ϕ_i, we obtain

$$0 = \sum_{j=1}^{n} \iint_{R_e} \left(\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y} \right) u_j \, dxdy - \iint_{R_e} f\phi_i dxdy - \int_{C_e} \phi_i q_n ds$$

$$i = 1,2,\cdots,n \tag{3.21}$$

Equation (3.21) can be written in the form

$$\sum_{j=1}^{n} K_{ij}^{(e)} u_j^{(e)} = F_i^{(e)}, \; i = 1,2,\cdots,n \tag{3.22}$$

where

$$K_{ij}^{(e)} = \iint_{R_e} \left[\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial y} + \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial y} \right] dxdy \tag{3.23}$$

and

$$F_i^{(e)} = \iint_{R_e} f\phi_i dxdy + \int_{C_e} q_n \phi_i ds \tag{3.24}$$

Equation (3.22) represents the finite element model of the Poisson equation.

Step 3: Interpolating linearly functions for the triangular elements

Consider a triangular element (see fig. 3.4) in which the nodes are numbered in the counter clockwise direction and derive the interpolating functions for it. We assume the interpolating polynomial in such a way that the number of

terms in it equals the number of nodes in the triangular element.

Accordingly, we assume

$$u(x, y) = a_1 + a_2 x + a_3 y \tag{3.25}$$

as the required approximation.

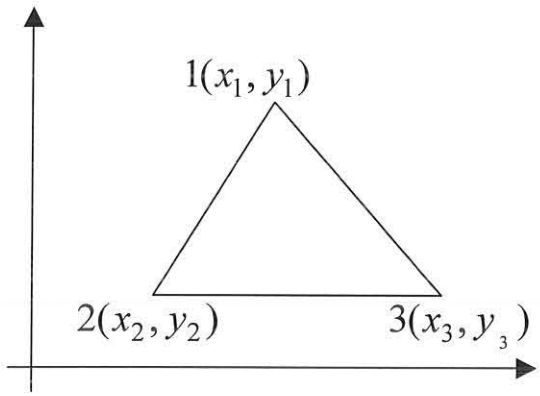

Fig.3.4 representation of triangular nodes.

We also set

$$u(x_i, y_i) = u_i, \quad i = 1,2,3 \tag{3.26}$$

Where $(x_i, y_i), i = 1,2,3$ denote the three vertices of the triangle.

Substituting (3.26) in (3.25) we obtain

$$\left.\begin{aligned} u_1 &= a_1 + a_2 x_1 + a_3 y_1 \\ u_2 &= a_1 + a_2 x_2 + a_3 y_2 \\ u_3 &= a_1 + a_2 x_3 + a_3 y_3 \end{aligned}\right\} \tag{3.27}$$

Solving equations (3.27) we obtain

$$\left.\begin{aligned} a_1 &= \frac{1}{2\Delta_e}\begin{vmatrix} u_1 & u_2 & u_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \\ a_2 &= \frac{1}{2\Delta_e}\begin{vmatrix} u_1 & u_2 & u_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} \\ a_3 &= \frac{1}{2\Delta_e}\begin{vmatrix} u_1 & u_2 & u_3 \\ 1 & 1 & 1 \\ x_1 & x_2 & x_3 \end{vmatrix} \end{aligned}\right\} \tag{3.28}$$

Where Δ_e = area of the triangle = $\frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$ (3.29)

Substituting for a_1, a_2, a_3 in (3.25) and simplifying, we obtain

$$\begin{aligned} u(x,y) = {} & \frac{1}{2\Delta_e}\left[u_1(x_2y_3 - x_3y_2) + u_2(x_3y_1 - x_1y_3) + u_3(x_1y_2 - x_2y_1)\right] \\ & + \frac{1}{2\Delta_e}\left[u_1(y_2 - y_1) + u_2(y_3 - y_1) + u_3(y_1 - y_2)\right]x \\ & + \frac{1}{2\Delta_e}\left[u_1(x_3 - x_2) + u_2(x_1 - x_3) + u_3(x_2 - x_1)\right] \end{aligned} \tag{3.30}$$

Collecting the coefficient of u_1, u_2, and u_3 in the above, equation (3.30) can be written in the form

$$u(x,y) = \sum_{i=1}^{3} u_i \phi_i^{(e)}(x,y) \tag{3.31}$$

Where the $\phi_i^{(e)}$ are linearly interpolating functions for the triangular elements under consideration and are given by

$$\phi_1^{(e)} = \frac{1}{2\Delta_e}\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$\phi_2^{(e)} = \frac{1}{2\mathrm{X}_e}\begin{vmatrix} 1 & x & y \\ 1 & x_3 & y_3 \\ 1 & x_1 & y_1 \end{vmatrix} \qquad (3.32)$$

$$\phi_3^{(e)} = \frac{1}{2\mathrm{X}_e}\begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix}$$

From formula (3.32), it is easily verified that

$$\left.\begin{aligned} \phi_i^{(e)}(x_j, y_j) &= \begin{cases} 1, \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \\ \text{and} \quad \sum_{i=1}^{3} \phi_i^{(e)}(x, y) &= 1 \end{aligned}\right\} \qquad (3.33)$$

We also have

$$\left.\begin{aligned} \frac{\partial \phi_1^{(e)}}{\partial x} &= \frac{y_2 - y_3}{2\Delta_e}, \frac{\partial \phi_1^{(e)}}{\partial y} = \frac{x_3 - x_2}{2\Delta_e} \\ \frac{\partial \phi_2^{(e)}}{\partial x} &= \frac{y_3 - y_1}{2\Delta_e}, \frac{\partial \phi_2^{(e)}}{\partial y} = \frac{x_1 - x_3}{2\Delta_e} \\ \frac{\partial \phi_3^{(e)}}{\partial x} &= \frac{y_1 - y_2}{2\Delta_e}, \frac{\partial \phi_3^{(e)}}{\partial y} = \frac{x_2 - x_1}{2\Delta_e} \end{aligned}\right\} \qquad (3.34)$$

Using equations (3.34), the element matrices $K_{ij}^{(e)}$ and $F_i^{(e)}$ in (3.22) can then be easily computed. These computations will be demonstrated through a simple example.

3.5: Illustration of FEM for two-dimensional problems

We consider a particular case of the problem defined by equations (3.17a) and (3.17b), viz., the Poisson equation

$$-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=2,0\le x\le 1,\ 0\le y\le 1 \qquad (3.5.1)$$

$$\textit{with the condition } u=0 \qquad (3.5.2)$$

on the boundary of the square $0\le x\le 1,\ 0\le y\le 1$

We divide the square region along the line of symmetry x = y and then consider only the lower triangular part. We again subdivide the lower triangular part into four triangular elements as shown in fig.3.5. Let the elements be numbered as shown in the figure and it is seen that elements a,b, and d are symmetrical. Hence the element matrices for these elements will all be of the same type.

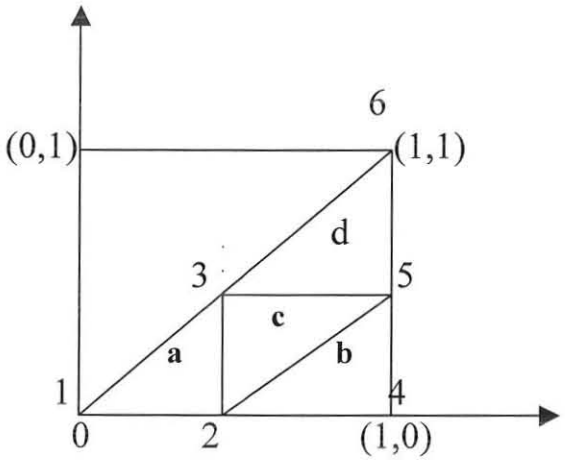

Fig.3.5 division of the region in to triangular elements.

Now the vertices 1, 2 and 3 of the element a are given by (0,0), (1/2,0), (1/2,1/2) respectively. For this element, we obtain $\Delta_e = 1/8$ and equations (3.32) give

$$\left.\begin{aligned} \phi_1^{(a)} &= 1-2x \\ \phi_2^{(a)} &= 2(x-y) \\ and\ \phi_3^{(a)} &= 2y \end{aligned}\right\} \qquad (3.5.3)$$

It is easy to see that $\phi_1^{(a)} + \phi_2^{(a)} + \phi_3^{(a)} = 1$, thus verifying (3.33). The element matrices $K^{(e)}$ *and* $F^{(e)}$ can now be computed easily using (3.23) and (3.24).

We first obtain the derivatives

$$\left.\begin{aligned} \frac{\partial \phi_1^{(a)}}{\partial x} &= 2, \frac{\partial \phi_1^{(a)}}{\partial y} = 0 \\ \frac{\partial \phi_2^{(a)}}{\partial x} &= 2, \frac{\partial \phi_2^{(a)}}{\partial y} = -2 \\ \frac{\partial \phi_3^{(a)}}{\partial x} &= 0, \frac{\partial \phi_3^{(a)}}{\partial y} = 2 \end{aligned}\right\} \qquad (3.5.4)$$

Equation (3.23) now gives

$$\left.\begin{aligned} K_{11}^{(a)} &= \iint_{\Delta_{123}} 4dxdy = \frac{1}{2}, K_{12}^{(a)} = \iint_{\Delta_{123}} -4dxdy = -1/2 = K_{21}^{(a)} \\ K_{13}^{(a)} &= 0 = K_{31}^{(a)}, K_{22}^{(a)} = 1, K_{23}^{(a)} = -1/2 = K_{32}^{(a)}, K_{33}^{(a)} = 1/2 \end{aligned}\right\} \qquad (3.5.5)$$

Similarly equation (3.24) yields

$$\left.\begin{aligned} F_1^{(a)} &= \frac{1}{12} + l_1^{(a)}, where\ l_1^{(a)} = \int_{C_{123}} q_n(1-2x)ds \\ F_2^{(a)} &= \frac{1}{12} + l_2^{(1)}, where\ l_2^{(a)} = \int_{C_{123}} q_n(2x-2y)ds \\ F_3^{(a)} &= \frac{1}{12} + l_3^{(a)}, where\ l_3^{(a)} = \int_{C_{123}} q_n 2y ds \end{aligned}\right\} \qquad (3.5.6)$$

Let the global nodes be $U_1, U_2, U_3, U_4, U_5, and\ U_6$ corresponding to the nodes $u_1, u_2, u_3, u_4, u_5\ and\ u_6$ at the respective vertices. As there are six nodes, the corresponding matrices will be of order 6. Hence, we obtain for element a:

$$K^{(a)} = \begin{bmatrix} 1/2 & -1/2 & 0 & 0 & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} and\ F^{(a)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} l_1^{(1)} \\ l_2^{(1)} \\ l_3^{(1)} \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (3.5.7)$$

Since the elements b and d are similar to a, their element matrices will be of the same type as those of a given in (3.5.7). Thus, for element b,

$$K^{(b)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} and F^{(b)} = \frac{1}{12}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ l_1^{(2)} \\ 0 \\ l_2^{(2)} \\ l_3^{(2)} \\ 0 \end{bmatrix} \qquad (3.5.8)$$

Similarly for element d,

$$K^{(d)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 & -1/2 & 1/2 \end{bmatrix} and F^{(d)} = \frac{1}{12}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ l_1^{(4)} \\ 0 \\ l_2^{(4)} \\ l_3^{(4)} \end{bmatrix} \quad (3.5.9)$$

Similarly, for element c, we note that the correspondence between its vertices and those of a is given by 5 - > 1, 3->2 and 2->3. Hence, we have

$$K^{(c)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 1 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} and\, F^{(c)} = \frac{1}{12}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ l_1^{(3)} \\ l_1^{(3)} \\ 0 \\ l_1^{(3)} \\ 0 \end{bmatrix} \quad (3.6.0)$$

Assembling the element matrices in (3.5.7), (3.5.8), (3.5.9) and (3.6.0) and simplifying, we obtain the matrix equation

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -2 & -1 & 0 & 0 \\ 0 & -2 & 4 & 0 & -2 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -2 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} l_1^{(a)} \\ l_2^{(a)} + l_1^{(b)} + l_3^{(c)} \\ l_3^{(a)} + l_2^{(c)} + l_1^{(d)} \\ l_{2'}^{(2)} \\ l_3^{(b)} + l_1^{(c)} + l_2^{(d)} \\ l_3^{(d)} \end{bmatrix} \quad (3.6.1)$$

From the boundary conditions, we have (see fig.3.5)

$$u_1 = u_2 = u_4 = u_5 = u_6 = 0 \tag{3.6.2}$$

Hence, equation (3.4.1) gives

$$-u_3 = 1/4 + l_1^{(b)} + l_2^{(a)} + l_3^{(a)} \tag{3.6.3}$$

$$2u_3 = 1/4 + l_3^{(a)} + l_2^{(c)} + l_1^{(d)} \tag{3.6.4}$$

$$-u = 1/4 + l_3^{(b)} + l_1^{(c)} + l_2^{(d)} \tag{3.6.5}$$

From (3.4.4) we obtain

$$u_3 = 1/8 + 1/2(l_3^{(a)} + l_1^{(d)} + l_2^{(c)}) \tag{3.6.6}$$

But

$$\begin{aligned} l_3^{(a)} &= \int_{C_{123}} q_n^{(a)} .2y ds \\ &= \int_0^{0.5} \left[q_n^{(a)} .2y\right]_{y=0} dx + \int_0^{0.5} \left[q_n^{(1)} .2y\right]_{x=0} dy + \int_{0.5}^{0} \left[q_n^{(a)} .2y\right]_{y=x} \\ &= 0, \end{aligned}$$

This is true since the first integral vanishes and the remaining two integrals cancel each other.

In a similar manner it can be shown that $l_2^{(3)} = l_1^{(4)} = 0$. Hence, it follows that u=1/8=0.125.From the analytical solution, the exact value is 0.14734 and hence the finite element solution obtained above has an error of 15%. This accuracy is obtained for coarse mesh used above; but if a finer mesh is used, i.e., if the number of elements is increased, then the accuracy of the finite element solution can be improved.

3.6; Concluding remarks

We have attempted, in this report, to discuss some basic numerical methods, which are indispensable for the current scientific research. Many methods have been excluded since they are broad and can be studied and produced as a solid seminar report by themselves. There is indeed much more to include. Unfortunately, the limitations of the space and the credit given for it have forced us to exclude many important topics such as eigen value problems in differential equations, linear and non – linear programming, convergence and stability criteria for partial differential equations and numerical solution of singular integral equations. Our motivation throughout has been to present the FEM in a very simple way so as to enable the reader to understand and apply them to solve the specific problems arising in his/her work.

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