Duhamel's Principle and the Method of Descent for The Wave Equation

Addis Ababa University

Department of Mathematics

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By Tafesse Hailu

Advisor: Dr. Tadesse Abdi

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Abstract

In this project we present investigation of the linear wave equation with the unknown function u,

$$\boldsymbol{u_{tt} - a\Delta u} = f$$

subject to prescribed initial and/or boundary data, where Δ is n-dimensional Laplacian. In 1d, the solution of IVP is rendered by first reducing it into lower order PDE and then appealing to the method of characteristics, while, for BVP the method of reflection is employed to yield the pertinent solution. In higher dimension, explicit solution of IVP is derived as based on the method of spherical mean and the method of descent. In the sequel, Duhamel's principle is used to get the solution of non-homogeneous wave equation from the associated homogeneous wave equation.

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CHAPTER ONE

Introduction

With the invention of Newtonian mechanics in the late 17th century, it become clear that many laws of physics and engineering are best described in terms of relations involving rates of change. When translated into the language of mechanics, these relations lead to differential equations, since rates of changes are related to derivatives. Consider Newton's second law of motion F = ma. This basic law tells us that if we can describe explicitly the force, then we obtain a differential equation for the position. Here, a stands for acceleration and, as such, it is the second derivative of position.

In a mass-spring system, the mass is subject to a linear restoring force, $F = -kx$ (Hooke's law) in this case, the motion of the mass is determined by the differential equation

$$mx'' + kx = 0.$$

Here, x-represents the position of the mass as a function of the time t. This equation allows us to solve for the specific motion of the mass if we are also supplied with its position and velocity at a given time t_o. This data is called the initial condition and is usually specified at t_o = 0. Thus, ordinary differential equations arise naturally when modeling physical phenomena, such as mechanical, electrical or electromechanical oscillations. If a phenomenon involves functions of more than one variable, then the modeling will typically involve partial derivatives and hence leading to a partial differential equation.

When we integrate an equation with several independent variables, the constant of integration that appears must be allowed to be a function of the remaining variables. Thus, after integrating the equation we get arbitrary function(s) of one variable, in contrast to the case of an ordinary differential equation whereby we get only arbitrary constants.

Now moving to a situation where a partial differential equation is involved, we can cite a real world phenomenon in n-dimensions, to mention but a few, the vibration of string ($n = 1$), membranes ($n = 2$) and elastic solid ($n = 3$). The requisite partial differential equations in these cases are collectively called the wave equation. Broadly speaking, wave equation can be grouped into linear and nonlinear.

1.1. Nonlinear Wave Equation

Wave equations describe time dependent phenomenon and are generally hyperbolic partial differential equations. The study of nonlinear wave phenomena subsumes the understanding of real water waves, optical fiber transmission, traffic flow and the like.

1.1.1 Shallow Water Wave

We recall that, water waves are surface waves that are mixtures of longitudinal and transverse waves. We distinguish between deep and shallow water waves. The distinction between these two waves which has nothing to do with absolute depth of the water is determined by the ratio of the water's depth to the wavelength of the wave. Thus, Shallow water wave arise if the depth of the water is much smaller than the wavelength of the water and is given by the nonlinear system of first order PDEs

$$\begin{cases} h_t + uh_x + hu_x = 0 \\ u_t + gh_x + uu_x = 0 \end{cases}.$$

The above system of nonlinear first order scalar partial differential equations is constructed using principles of conservation laws. In view of this it can be expressed as,

$$V_t + A(V)V_x = 0$$

where, $V = \begin{pmatrix} h \\ u \end{pmatrix}$, $A = \begin{pmatrix} u & h \\ g & u \end{pmatrix}$ u is velocity and h is depth.

1.1.2 Burger's Equation

In the above shallow water wave equation, if the water surface is sufficiently close to the river bed, *i.e.* if $h = 0$ the nonlinear system reduces to the nonlinear partial differential equation,

$$u_t + uu_x = 0$$

often called Inviscid Burger's equation. This has application in traffic flow models, and one can easily come up with closed form of solution as opposed to the nonlinear system, shallow water wave equation. To this end, if we prescribe initial data $u(x,0) = \phi(x)$ then the initial value problem,

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

has a solution of the form $u(x,t) = \phi(x - t\phi(x))$.

1.1.3 Flood Wave Equation

The origin of flood wave equation basically rests in the study of flood waves in rivers within the confines of an approximation theory. In this regard, if we consider flow in a rectangular channel of constant breadth, and the depth/height *h(x,t)* plays the role of density, the conservation equation is then given by the integro-differential equation,

$$\frac{d}{dt}\int_{x_1}^{x_2} h(x,t)dx = q_1 - q_2$$

where, q is the flux. If we assume further that q is C^1-smooth then as $x_2 \to x_1$ we obtain the equation of continuity,

$$h_t + q_x = 0$$

which is a first order partial differential equation. In a flooding river, since the flux q is a function of the depth h, i.e. $q = Q(h)$, we have the *flood wave equation*

$$h_t + c(h)h_x = 0$$

where, $c(h) = \frac{dQ}{dh}$ is the wave speed. It was observed that the speed, v $\alpha \sqrt{h}$. Thus, if the proportionality constant is k then $\mathrm{v} = k\sqrt{h}$ and hence $Q(h) = \mathrm{v}h = kh^{3/2}$. Consequently, we have the relationship between the speed of the flood wave and the speed of the stream,

$$c(h) = \frac{3}{2}k\sqrt{h} = \frac{3}{2}\mathrm{v}$$

This shows that, flood waves move half as fast again as the stream.

1.2. Linear Wave Equation

Linear wave equation is intimately connected to the wave operator,

$$L = \partial_{tt} + \Delta$$

where, $\Delta = \sum_i \partial_{x_i x_i}$ is the n-dimensional Laplacian.

1.2.1 Transverse Wave Propagation

Consider the string stretched along the x-axis between $x = 0$ and $x = L$ and free to vibrate in a fixed plane. The string is taught, so that when in equilibrium the string is a straight flat line. If the string is displaced from the flat line, we will refer to the vertical displacement at a given point $0 \le x \le L$, and at a given time, $t \ge 0$, as $u(x,t)$. Here, the unknown function is a function of two variables in contrast to the simple mass- spring system. Based on this representation velocity of the spring at position x is $\frac{\partial u}{\partial t}$ and is its acceleration there is $\frac{\partial^2 u}{\partial t^2}$. The equation that governs the motion of the stretched string obeys the PDE. This equation is known as the one dimensional wave equation and is given by

$$u_{tt} = c^2 u_{xx}$$

where C is the parameter that depends up on the physical properties of the string, in particular its linear mass density and its tension. The key point to understanding this equation from a physical point of view is to interpret it in light of Newton's second law. The left side represents the acceleration of the small portion of the string centered at a point x , while the right side is telling us that this small portion feels a force whose sign depends on the concavity of the string at that point.

1.2.2 Elastic Membrane (rectangular drum)

In one dimensional wave equation there was only one type of finite domain, an interval. But in two dimensional a finite domain could be any bounded connected region. The solution we are looking for is a time dependent surface $Z = u(x, y, t)$. At any given time, it is just a surface (more specifically a function of x and y). However, it moves with time. At any given point (x_o, y_o), u($x_{0,}$ y_0 , t), is the displacement of the surface from some value defined to be Zero, $u_t(x_o, y_o, t)$ is the velocity in the Z-direction at that point, and $u_{tt}(x_o, y_o, t)$ is the acceleration and so on. If this domain obeys the wave equation on some domain (say a rectangle) then we have

$$u_{tt} - c^2\left(u_{xx} + u_{yy}\right) = 0$$

Once again, we mention that, the **order** of the partial differential equation is the highest order of derivative that appears in the equation and the A PDE is said to be linear if the unknown function and the partial derivatives are of the first degree and at most one of these appears in any given term, otherwise, the equation is called nonlinear.

CHAPTER TWO

Explicit Solution of the Wave Equation

The wave equation is the PDE:

$$\frac{\partial^2}{\partial t^2} \quad u\,(x,t) - \Delta\, u(x,t) = 0\ (homo\ genepous) and$$

$$\frac{\partial^2}{\partial t^2}\, u\,(x,t) - \Delta\, u(x,t) = f(non\ homo\ genepous)$$

Subject to appropriate initial and boundary conditions for t ∈ (O,∞) and $x \in U$ were $U\ C\ R^n$ is open. The unknown is

$$u\colon \tilde{U} \times (o,\infty) \to R, u = u(x,t),$$

we consider t as time and

$$x = (x_1\,, x_2\,....\,x_n)$$

as a special variables.

2.1. Physical interpretation of the wave equation

The wave equation is a simplified model for a vibrating string $(n = 1), membrane\ (n = 2), or\ elasticsolid\ (n = 3)$. In these physical interpretations $u(x,t)$ represents the displacement in some direction of the point x at time $t \geq 0$

Let v represents any smooth sub region of U. the acceleration within v is then

$$\frac{d^2}{dt^2}\int_V u dx = -\int_{\partial V} F.\,v ds$$

Where F denotes the force acting on V through ∂V and the mass density is taken to be Unity. Newton's law asserts the mass times the acceleration equals the net force.

If we differentiate under the integral sign and apply the divergence theorem, we obtain

$$\int_v \frac{\partial^2 u}{\partial t^2} dx = -\int_{\partial v} F.\,v ds = -\int_v divF dx$$

$$\int_V u_{tt} dx = -\int_V divF dx$$

This identity obtains for each sub region v and so

$$U_{tt} = -divF$$

For elastic bodies, F is function of the displacement gradient Du

Hence

$$U_{tt} + divF\ (Du) = 0$$

For small Du, the linearization $F(DU) \simeq$- aDu is often appropriate: and so

$$U_{tt} - a\Delta u = 0$$

This is the wave equation $if\ a = 1$

This physical interpretation suggests it will mathematically appropriate to specify two initial conditions, on the displacement u and the velocity $u_t\ \ at\ time\ t = 0$

2.2. The method of characteristics equation

The general linear second-order partial differential equation in one dependent Variable u may be written as

$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=G$..........................(1)

Where the coefficients A through F are real function of the independent variable x ,y. Define a discriminate Δ(x,y) by $\Delta(x_{o,}y_o)=B^2(x_{o,}y_o)\_4A(x_{o,}y_o)C(x_{o,}y_o)$.We shall assume that the function u and the coefficients are twice continuously differentiable in some domain in R^2. The classification of partial differential equations is suggested by the classification of the quadratic equation of conic sections in analytic geometry.

The equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ represents hyperbola, parabola or Elliptic according as B^2-4AC positive, zero or negative.

The classification of second-order equations is based upon the possibility of reducing equation (1) by coordinate transformation to canonical or standard form at a point.

In the case of two independent variables, a transformation can always be found to reduce the given equation to canonical form in a given domain. However, in the case of several independent variables, it is not in general, possible to find such a transformation.

To transform equation (1) to a canonical form we make a change of independent variables.

Let the new variables be $\varphi=\varphi(x,y)$, $\eta=\eta(x,y)$ ……………………………………(2)

Assuming that φ and η are twice continuously differentiable function of x and y and that the Jacobian

$$J=\begin{vmatrix}\varphi_X & \varphi_Y\\ \eta_X & \eta_Y\end{vmatrix}$$

is nonzero in the region under consideration, then x and y can be determined uniquely from the system (2). Let x and y be twice continuously differentiable functions of φ and η. Then we have

$$u_x=u_\varphi\,\varphi_x+u_\eta\eta_x \qquad u_y=u_\varphi\varphi_y+u_\eta\eta_y$$

$$u_{xx}=u_{\varphi\varphi}{\varphi_x}^2+2u_{\varphi\eta}\varphi_x\eta_x+u_{\eta\eta}{\eta_x}^2+u_\varphi\varphi_{xx}+u_\eta\eta_{xx}$$

$$u_{xy}=u_{\varphi\varphi}\varphi_x\varphi_y+u_{\varphi\eta}(\varphi_x\eta_y+\eta_y\varphi_x)+u_{\eta\eta}\eta_x\eta_y+u_\varphi\varphi_{xy}+u_\eta\eta_{xy}$$

$$u_{yy}=u_{\varphi\varphi}{\varphi_y}^2+2u_{\varphi\eta}\varphi_y\eta_y+u_{\eta\eta}{\eta_y}^2+u_\varphi\varphi_{yy}+u_\eta\eta_{yy}$$

Substituting these values in equation (1) we obtain

$A^*u_{xx}+B^*u_{xy}+C^*u_{yy}+D^*u_x+E^*u_y+F^*u=G^*$…………………………(3)

Where

$$A^*=A{\varphi_x}^2+B\varphi_x\varphi_y+C{\varphi_y}^2$$

$$B^* = 2A\varphi_x\eta_x + B(\varphi_x\eta_y + \varphi_y\eta_x) + 2C\varphi_y\eta_y \quad \dots \quad (4)$$
$$C^* = A\eta_x{}^2 + B\eta_x\eta_y + C\eta_y{}^2$$
$$D^* = A\varphi_{xx} + B\varphi_{xy} + C\varphi_{yy} + D\varphi_x + E\varphi_y$$
$$E^* = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$$
$$F^* = F \qquad G^* = G$$

The resulting equation (3) is in the same form as the original equation (1) under the general transformation (2). The type of the equation (hyperbolic, parabolic or elliptic) will not change under this transformation. The reason for this is that

$$\Delta^* = B^{*2} - 4A^*C^* = J^2(B^2 - 4AC) = J^2\Delta$$

And since J≠0 the sign of $\Delta^*$the same as that of Δ which can be easily verified. It should be noted here that the equation can be of a different type at different points of the domain, but for our purpose we shall assume that the equation under consideration is of the single type in a given domain. The classification of equation (1) depends on the coefficients A(x, y), B (x, y), and C (x, y) at a given point (x, y). We shall, therefore, rewrite equation (1) as

$Au_{xx} + Bu_{xy} + Cu_{yy} = H(x, y, u, u_x, u_y)$……………………………………..(5)

And equation (3) as

$A^*u_{\varphi\varphi} + B^*u_{\varphi\eta} + C^*u_{\eta\eta} = H^*(\varphi, \eta, u, u_\varphi, u_\eta)$……………………………(6)

2.2.1. Canonical Forms

In this section we shall consider the problem of reducing equation (5) to canonical form.
We suppose first that none of A, B, C, is zero. Let φ and η be new variables such that the coefficients A^* and C^*in equation (6) vanish. Thus, from (4), we have

$$A^* = A\varphi_x{}^2 + B\varphi_x\varphi_y + C\varphi_y{}^2$$
$$C^* = A\eta_x{}^2 + B\eta_x\eta_y + C\eta_y{}^2$$

These two equations are of the same type and hence we may write them in the form

$A\zeta_x{}^2 + B\zeta_x\zeta_y + C\zeta_y{}^2 = 0$……………………………………………………(7)

In which ζ stand for either of the functions φ or η. dividing through by $\zeta_y{}^2$ equation (7) becomes

$A\left(\frac{\zeta_x}{\zeta_y}\right)^2 + B\left(\frac{\zeta_x}{\zeta_y}\right) + C = 0$……………………………………………………(8)

Along the curve ζ = constant, we have

$d\zeta = \zeta_x dx + \zeta_y dy = 0$

Thus $\frac{dy}{dx} = -\frac{\zeta_x}{\zeta_y}$

And therefore, equation (8) may be written in the form

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$$

The roots of which are

$\frac{dy}{dx} = B - \frac{\sqrt{B^2-4AC}}{2A}$……………………………………………….(9)

$\frac{dy}{dx} = B + \frac{\sqrt{B^2-4AC}}{2A}$……………………………………………….(10)

These equations, which are known as the characteristic equations, are ordinary differential equations for families of curves in the xy-plane along which φ= constant and η = constant. The integrals of equations (9) and (10) are called the characteristic curves. Since the equations are first-order ordinary differential equations, the solutions may be written as

$\square_1(x,y) = c_1 \quad c_1 = constant$

$\phi_2(x,y) = c_2 \quad c_2$=constant

Hence the transformations $\varphi = \square_1(x,y), \quad \eta = \phi_2(x,y)$ Will transform equation (5) to canonical form.

2.3 D'Alemberts' representation of a solution in 1D

Now consider the one dimensional wave equation

$$u_{tt}\,(x,t)= c^2\, u_{xx}\,(x,t) \quad (x,t) \in R \text{......................}(11)$$

Where C is a positive constant that has the physical interpretation of wave speed. For example if u(x,t) its corresponds to the displacement of an infinite string then $C = \frac{T}{\rho}$ where T is the tension in the string and ρ is the density.

Now consider the linear second order partial differential equation of the form

$$A\,(x,y)\frac{\partial^2 u}{\partial x^2} + \; 2B\,(x,y)\frac{\partial^2 u}{\partial x \partial y} + \; C\,(x,y)\frac{\partial^2 u}{\partial y^2} = f(x,y,u,u_{x,}u_y)$$

Then the slop of the characteristics satisfies

$$\frac{dy}{dx} = B \pm \frac{\sqrt{B^2-4AC}}{2A}$$

Thus the characteristics of one dimensional wave equation $u_{tt} - c^2 u_{xx} = 0$ is determined by

$$\frac{dx}{dt} = \; B \pm \frac{\sqrt{B^2-4AC}}{2A} = 0 \pm \frac{\sqrt{-4(-c^2)}}{2(1)} = \pm \frac{\sqrt{4c^2}}{2} = \pm \frac{2c}{2} = \; \pm c$$

$\frac{dx}{dt} = c \implies dx = cdt$

$\implies x - ct \; = constant$ and $\frac{dx}{dt} = -c$

$$\implies x + ct = constant$$

Now we can use change of variables

$$\eta(x,t) = x + ct$$
$$\psi(x,t) = x - ct$$

Assuming that η and ψ are twice continuously differentiable function of x and t

$$\eta = \eta(x,t)$$
$$\psi = \psi(x,t)$$

$$u(x,t) = V(\eta,\psi) = V(\ \eta(x,t),\psi(x,t))$$

And also the Jacobian J of the transformation defined by

$$J\ = \begin{vmatrix} \eta_x & \eta_t \\ \psi_x & \psi_t \end{vmatrix}\ is\ \ non\ zero$$

Since $\frac{\partial(\eta\ ,\psi)}{\partial(x,t)} = \begin{vmatrix} \eta_x & \eta_t \\ \psi_x & \psi_t \end{vmatrix} = \begin{vmatrix} 1 & c \\ 1 & -c \end{vmatrix} = -c - c = -2c \neq 0$

$$u_x = u_\eta\, \eta_x + u_\psi\, \psi_x =\ u_\eta + u_\psi$$

$$u_{xx} = u_{\eta\eta} + 2u_{\eta\psi} + u_{\psi\psi} \ldots\ldots\ldots\ldots\ldots\ldots..(12)$$

And $$u_t\ = u_\eta \eta_t + u_\psi \psi_t = cu_\eta - cu_\psi$$

$$u_{tt}\ =\ c^2 u_{\eta\eta} - 2c^2 u_{\eta\psi} + c^2 u_{\psi\psi}\ \ \ldots\ldots\ldots..(13)$$

Now substitute equation (12) and (13) in the wave equation (11)

We have, $c^2 u_{\eta\eta} - 2c^2 u_{\eta\psi} + c^2 u_{\psi\psi} - c^2(u_{\eta\eta} + 2u_{\eta\psi} + u_{\psi\psi}\ )$

$$c^2 u_{\eta\eta} - 2c^2 u_{\eta\psi} + c^2 u_{\psi\psi} - c^2 u_{\eta\eta} - 2c^2\ u_{\eta\psi} - c^2 u_{\psi\psi}$$

$$\Rightarrow\ -4c^2\ u_{\eta\psi} = 0$$

$$\Rightarrow u_{\eta\psi} = 0$$

If we integrate this equation with respect to $\psi\ \ keeping\ \ \eta\ \ fixed\ we\ have\ \left(u_\eta\right)_\psi = 0$

$$u_\eta(\eta,\psi)\ = f(\eta)$$

A second integration yields up on keeping ψ fixed

$$u(\eta,\psi) =\ \int f(\eta)d\eta + G(\psi)$$

$$u(\eta,\psi) =\ F(\eta) + G(\psi)$$

$u(x,t) =\ F(x+ct) + G(x-ct)$, Which is called D' Alembert general Solution .

Usually interested in solving the wave equation (11) subject to the initial conditions

$$u(x,0) = g(x) \qquad and \qquad u_t\ (x,0) = h(x)$$

2.4. Theorem1: (The solution of the initial value problem)

For the given function g=g(x)∈ $c^2(R)$ and h=h(x)∈ $c^1(R)$

$$u_{tt}(x,t) -\ c^2 u_{xx}(x,t) = 0\ \ \ for\ (x\ t)\epsilon R^2, t \geq 0 \ldots\ldots\ldots\ldots\ldots.(14)$$

With initial conditions

$u(x,0) = g(x)\ \ and\ \ \ u_t\ (x,0) = h(x)$ is given by

$$u(x,t) = \frac{1}{2}\{g(x+ct) + g(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy \ldots\ldots\ldots(15)$$

Proof: from D' Alembert general solution of wave equation we have,

$$u(x,t) = F\ (x+ct) +\ G(x-ct) \ldots\ldots\ldots\ldots(16)$$

Now applying the initial conditions we have,

$$u(x,0) = F\ (x) +\ G(x) =\ g(x) \ldots\ldots\ldots.(17)$$

$$u_t(x,0) =\ CF^{'}(x) -\ CG^{'}(x) = h(x) \ldots\ldots\ldots(18)$$

Now integrating (18) over $[0, x)$ we have

$$\int_0^x \left(CF'(y) - cG'(y)\right) dy = \int_0^x h(y)dy$$

$$\int_0^x F'(y)dy - \int_0^x G'(y)dy = \frac{1}{c}\int_0^x h(y)dy$$

$$(F(y) - G(y)) /_o^x = \frac{1}{C}\int_o^x h(y)dy$$

$$F(x) - G(x) = \frac{1}{C}\int_o^x h(y)dy \dots\dots\dots..(19)$$

From equation (17) and (19) we have

$$\pm \qquad F(x) + G(x) = g(x)$$

$$F(x) - G(x) = \frac{1}{c}\int_0^x h(y)dy$$

Now add the two equations we get

$$F(x) = \frac{1}{2}g(x) + \frac{1}{2c}\int_0^x h(y)dy \dots\dots\dots\dots\dots\dots..(20)$$

And also subtract we get

$$G(x) = \frac{1}{2}g(x) - \frac{1}{2c}\int_0^x h(y)dy \dots\dots\dots\dots\dots\dots\dots.(21)$$

Now substitute equation (20) and (21) in (16)

$$u(x,t) = \frac{1}{2}g(x+ct) + \frac{1}{2}g(x-ct) + \frac{1}{2c}\int_0^{x+ct} h(y)dy - \frac{1}{2c}\int_0^{x-ct} h(y)dy$$

$$u(x,t) = \frac{1}{2}\{g(x+ct) + g(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy.\dots\dots\dots.(22)$$

Which is known as D' Alembert's formula for n=1.

CHAPTER THREE

Method of spherical mean

First we introduce some notations for $X \in \mathrm{I}R$

1) $B(x,r) = ball\ of\ radius\ r\ about\ x$
2) $\partial B(x,r) =\ boundary\ of\ ball\ of\ radius\ r\ about\ x$
3) $\alpha(n) = volume\ of\ unit\ ball\ in\ \mathrm{I}R^{\mathrm{n}}$
4) $n\alpha(n) =\ surface\ area\ of\ unit\ ball\ in\ \mathrm{I}R^{\mathrm{n}}$

With this notations the volume of the ball of radius r about $x \in \mathrm{I}R$,written as $vol\ (B\ (x,r))$ is given by $\alpha(n)r^n$ and the surface area of the ball of radius r about $x \in \mathrm{I}R^n$ written as $S.A\ (B(x,r))$,is given by $n\alpha(n)r^{n-1}$.

For $\boldsymbol{f}: \mathbf{1}\boldsymbol{R}^{\boldsymbol{n}} \to \mathbf{1}\boldsymbol{R}$,

$we\ define\ the\ average\ of\ \boldsymbol{f}\ \boldsymbol{over}\ \boldsymbol{B}(\boldsymbol{x},\boldsymbol{r})$

As ${}_{B(x,r)}f(y)dy \equiv \frac{1}{vol\ (B(x,r))}\int_{B(x,r)}f(y)dy = \frac{1}{\alpha(n)r^n}\int_{B(x,r)}f(y)dy$

Also we define the average of f over $\boldsymbol{\partial}\ \boldsymbol{B}(\boldsymbol{x},\boldsymbol{r})$ **as**

${}_{\partial B(x,r)}f(y)ds(y) \equiv \frac{1}{S.A\ (B(x,r))}\int_{\partial(x,r)}f(y)ds(y) = \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(x,r)}f(y)ds.$

Where ds(y) denotes the surface measure of B(x,r) in R^n.

3.1 Mean Value Property

If $\alpha(n)$ is the n dimensional measure of the unit ball inR^n,then n$\alpha(n)$ is the corresponding n-1 dimensional measure of the unit sphere inR^n.Thus when n=1,2,3 the values of $\alpha(n)$ are $2, \pi, \frac{4}{3}\pi$ while the values of $n\alpha(n) are\ 2{,}2\pi\ {,}4\pi$.

${}_{B(x,r)}f(y)dy = \frac{1}{\alpha(n)r^n}\int_{B(x,r)}f(y)dy = \frac{1}{\alpha(n)}\int_{B(x,1)}f(x+rz)dz\ \ for\ the$ mean value of f over the ball of radius r.

${}_{\partial B(x,r)}f(y)ds(y) = \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(x,r)}f(y)ds(y) = \frac{1}{n\alpha(n)}\int_{\partial B(0,1)}f(x+rz)ds$ for the mean value of f over the sphere of radius r.

3.2. Spherical Average and the Euler-Poisson-Darboux equation.

Now suppose $n \geq 2,\ \ m \geq 2\ and\ u \in c^m\ (R^n \times [0,\infty))$

Solves the initial value problem

$u_{tt} - \Delta u = 0\ in\ R^n \times (0,\infty)\ \ ,\ \ u = g, u_t = h\ on\ R^n \times \{t = 0\}$ ……………………(23i)

The objective is to drive an explicit formula for u in terms of g, h. first study the average of u over certain spheres. These averages, taken as function of the time t and the radius r, turn out to solve the Euler- Poisson-Darboux equation, a PDE which we can for odd n convert in to the ordinary one-dimensional wave equation and then applying d'Alembert's formula it leads to a formula for the solution.

Define:

Let $x \in R^n, t > 0, r > 0$

u(x,t)=u : $R^n \rightarrow R$

$U(x,r,t) = {}_{\partial B(x.r)} u(y,t)ds(y)$…the average of u(.,t) over the sphere

$$\partial B(x,r) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (23)$$

Similarly

$$\begin{cases} (i)\ G(x,r) := {}_{\partial B(x,r)} g(y)ds(y) \\ (ii) H(x,r) := {}_{\partial B(x,r)} h(y)ds(y) \end{cases} \ldots\ldots\ldots\ldots (24)$$

$$u(x,t) = \lim_{r \to 0^+} U(x,r,t)\ , x \epsilon IR$$

Lemma1: (Euler-Poisson- Darboux equation)

Fix a point $x \in R^n$
Let u solves $u_{tt} - \Delta u = 0 \ \ x \in IR^n \ \ \ t \geq 0$
$u = g, u_t \ = h$
Where $\Delta u = \sum_{i=1}^{n} u_{xixi} \ \ , U \in C^m(R_+ \times [0, \infty)) \ \ \ then\ U\ solves$

$$U_{tt} - U_{rr} - \frac{n-1}{r}\ U_r = 0 \qquad 0 < r < \infty, \qquad t \geq 0$$

$$U = G\ , U_t = H, \quad x \epsilon R^n$$

Proof
From the definition we have

$$U(x,r,t) = \ {}_{\partial B(x,r)} u(y,t)ds(y)$$
$$= {}_{\partial B(0,1)} u(\mathrm{x} + \mathrm{rz}, \mathrm{t})\mathrm{ds}(\mathrm{z})$$
$$U_r(x,r,t) = \ {}_{\partial B(0,1)} Du(x + rz, t)zds(z)$$
$$= {}_{\partial B(x,r)} Du(y,t)\frac{y-x}{r} ds(y)$$
$$U_r(x,r,t) = {}_{\partial B(x,r)} \frac{\partial u}{\partial v}(y,t)ds(y)$$
$$= \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial v}\ (y,t)ds(y)$$

$$U_r(x,r,t) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y,t)dy$$

Since u solves the wave equation $u_{tt} - \Delta u = 0$

$$u_{tt} = \Delta u$$

Therefore $U_r(x,r,t) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} u_{tt}(y,t)dy$

$$r^{n-1}U_r = \frac{1}{n\alpha(n)} \int_{B(x,r)} u_{tt}(y,t)dy$$

$$(r^{n-1}U_r)_r = \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt}(y,t)ds(y)$$

$$(r^{n-1}U_r)_r = \frac{r^{n-1}}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} u_{tt}(y,t)ds(y)$$

$$(r^{n-1}U_r)_r = r^{n-1}{}_{\partial B(x,r)} u_{tt}(y,t)ds(y)$$

$$(r^{n-1}U_r)_r = r^{n-1}U_{tt}$$

$$r^{n-1}U_{rr} + (n-1)r^{n-2}U_r = r^{n-1}U_{tt}$$

$$U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0$$

$$And \quad U(x,r,o) = {}_{\partial B(x,r)} u(y,0)ds(y) = {}_{\partial B(x,r)} g(y)ds(y) = G(x,r)$$

$$\Rightarrow U = G \ \ at\ t = 0$$

$$U_t(x,r,o) = {}_{\partial B(x,r)} u_t(y,o)ds(y) = {}_{\partial B(x,r)} h(y)ds(y) = H(x,r)$$

$$\Rightarrow U_t = H, \qquad at\ t = 0 \ \ as\ claimed$$

3.3. Solution for Kirchhoff's formula for n=3

Theorem 2 Let the function g=g(x)∈ $c^3(R^3)$ and h=h(x)∈ $c^2(R^3)$ be given then the initial value problem for the three dimensional wave equation has the unique solution

The overall plan will be to transform the **Euler-Poisson- Darboux** equation in to the usual **one dimensional** wave equation when n=3 and then applying d'Alembert's formula it leads to a formula for the solution.

Solution for n = 3

$Let\ n = 3$ and suppose that *u* is a solution of

$$u_{tt} - \Delta u = 0\ ,\ t > 0$$

$$u = g, u_t = h \qquad for\ n = 3$$

As before define the function

$$U(x,r,t) = {}_{\partial B(x,r)} u(y,t)ds(y)$$

$$G(x,r) = {}_{\partial B(x,r)} g(y)ds(y)$$

$$H(x,r) = {}_{\partial B(x,r)} h(y)ds(y)$$

set

$$\tilde{U} = rU \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

$$\tilde{G} = rG, \quad \widetilde{H} = rH \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

Then $\tilde{\boldsymbol{U}}$ solves

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 \quad 0 < r < \infty, t \geq 0 \\ \tilde{U} = \tilde{G}, \qquad \tilde{U}_t = \widetilde{H}\ , 0 < r < \infty, t = 0 \\ \tilde{U} = 0\ on\ \{r = 0\ \{\times (0, \infty) \end{cases}$$

Proof: $\tilde{U}_{tt} = rU_{tt}$

$= r[U_{rr} + \frac{2}{r} U_r]$ $\quad since\ n = 3\ , U_{tt} = U_{rr} + \frac{n-1}{r}\ U_r = U_{rr} + \frac{2}{r} U_r$

$$\tilde{U}_{tt} = r(U_{rr+} \frac{2}{r} U_r)$$

$$= rU_{rr} + 2U_r$$

$$= (U + rU_r)_r$$

$$= ((rU)_r)_r$$

$= \tilde{U}_{rr}$There fore $\tilde{U}_{tt} = \tilde{U}_{rr}$

And Ũ = rU

$i.e\ \ \tilde{U}(x, r, o) = rU(x, r, o) = r_{\partial B(x,r)} u(y, o) ds(y)$

$$= r\ _{\partial B(x,r)} g(y) ds(y)$$

$$= rG(x, r) = rG = \tilde{G}$$

$$Therefore \quad \tilde{U} = \tilde{G}\ , at\ t = 0$$

$\tilde{U}_t = rU_t.$

That is $\tilde{U}_t\ (x, r, 0) = rU_t(x, r, o) = r_{\partial B(x,r)} u_t(y, 0) ds(y)$

$$= r_{\partial B(x,r)} h(y) ds(y) = rH(x, r) = rH = \widetilde{H}$$

$\Longrightarrow \tilde{U}_t = \widetilde{H}\ , at\ t = 0$ and $\quad \tilde{U}(x, 0, t) = 0. U\ (x, 0, t) = 0$

There fore $\tilde{U}$ $solves\ the\ one\ dimensional\ wave\ equation$

Since $\tilde{U}$ $solves\ the\ one\ dimensional\ wave\ equation$

For $0 \leq r \leq t, using\ D'Almeberts\ formula\ the\ solution\ of\ \tilde{U}(x, r, t) is\ given\ by$

$$\tilde{U}(x, r, t) = \frac{1}{2}\left[\tilde{G}(r + t) - \tilde{G}(t - r)\right] + \frac{1}{2}\int_{-r+t}^{r+t} \widetilde{H}(y) dy$$

$$Since\ (23) implies \quad u(x, t) = \lim_{r \to 0^+} U\ (x, r, t)$$

From equation (25) we have $\tilde{U} = rU$, U= $\frac{\tilde{U}}{r}$

$$\Rightarrow u(x, t) = \lim_{r \to 0^+} U(x, r, t) = \lim_{r \to 0^+} \tilde{U}\ \frac{(x, r, t)}{r}$$

$$\Rightarrow u(x, t) = \frac{1}{2r} \lim_{r \to 0^+} [(\tilde{G}(r + t) - \tilde{G}(t - r) + \int_{-r+t}^{r+t} \widetilde{H}(y) dy]$$

$$u(x, t) = \lim_{r \to 0^+} [\frac{\tilde{G}(r+t) - \tilde{G}(t-r)}{2r} + \frac{1}{2r} \int_{t-r}^{t+r} \widetilde{H}(y) dy]$$

$$u(x, t) = \tilde{G}'(t) + \widetilde{H}(t)$$

$$But\ \tilde{G} = tG, \quad \widetilde{H} = tH$$

$u\,(x,t) = \frac{d}{dt}(t\ {}_{\partial B(x,t)} g(y)ds(y)) + \mathrm{t}\ {}_{\partial B(x,t)} h(y)ds(y)$………………………..27(i)

$$= \frac{d}{dt} t.\ {}_{\partial B(x,t)} g(y)ds(y) + t.\frac{d}{dt}\ {}_{\partial B(x,t)} g(y)ds(y) + t\ {}_{\partial B(x,t)} h(y)ds(y)$$

$$u(x,t) = {}_{\partial B(x,t)} g(y)ds(y) + t\ \frac{d}{dt}\ {}_{\partial B(x,t)} g(y)ds(y) + t\ {}_{\partial B(x,t)} h(y)ds(y)$$

But ${}_{\partial B(x,t)} g(y)ds(y) = {}_{\partial B(0,1)} g(x+tz)ds(z)$

And so, $\frac{d}{dt}\ {}_{\partial B(x,t)} g(y)ds(y) = \frac{d}{dt}\ {}_{\partial B(0,1)} g(x+tz)ds(z)$

$$= {}_{\partial B(0,1)} Dg(x+tz).\,z\ ds(z)$$

$$\frac{d}{dt}\ {}_{\partial B(x,t)} g(y)ds(y) = {}_{\partial B(x,t)} Dg(y)\frac{y-x}{t}\ ds(y)$$

$$t.\ \frac{d}{dt}\ {}_{\partial B(x,t)} g(y)ds(y) = t_{\partial B(x,t)} Dg(y)\frac{y-x}{t}\ ds(y)$$

$$u(x,t) = {}_{\partial B(x,t)} g(y)\ ds(y) + t_{\partial B(x,t)} Dg(y)\frac{y-x}{t}\ ds(y) + t_{\partial B(x,t)} h(y)\ ds(y)$$

$u(x,t) = {}_{\partial B(x,t)}(g(y) + Dg(y).(y-x) + th(y))ds(y)$……………..27 (ii)

This is Kirchhoff's formula for the solution of the initial value problem

$u_{tt} - \Delta u = 0.$, u=g,$u_{t=}h$ in three dimensions.

We note that inR^3 ${}_{\partial B(x,t)} = \frac{1}{n\alpha(n)t^{n-1}}\int_{\partial B(x,t)} = \frac{1}{4\pi t^2}\int_{\partial B(x,t)}$ we can write (27ii) as

$$\mathrm{u(x,t)} = \frac{1}{4\pi \mathrm{t}^2}\int_{\partial \mathrm{B(x,t)}}(\mathrm{g(y)} + \mathrm{Dg(y).(y-x)} + \mathrm{th(y))ds(y)}$$

Chapter Four

Method of Descent

Theorem 3 Given the function g=g(y)=g($y_{1,}y_2$) ∈ $(c^3(R^2)$ and h=h(y)= h($y_{1,}y_2$) ∈ $c^2(R^2)$ then

The initial value problem for the two dimensional wave equation has the unique

Solution.

No transformation works to convert the Euler-Poisson- Darboux equation in to the one dimensional wave equation when n=2. Instead we will take the initial value problem (23i) of the wave equation for n=2 and simply regard it as a problem for n=3, in which the third spatial variable x_3 does not appear.

In other words we use Kirchhoff's formula for the solution of the wave equation in three dimensional to drive the solution of the wave equation in two dimensional. This technique is known as the method of descent.
Solution for n=2.

$$\begin{cases} u_{tt} - \Delta u = 0 \\ u(x,0) = g(x) \qquad x \in 1R^2, t \geq 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28) \\ u_t(x,0) = h(x) \end{cases}$$

Aim: find a solution in two dimensional causes, by using the solution of the three dimensional.
Let $u(x_1, x_{2,}t)$ be the solution of the two dimensional problem (28)
Define $\bar{u}(x_1, x_{2,}, x_{3,}\, t) := u\,(x_1, x_{2,}t)$ Then (28) implies

$$\bar{u}(x_{1,}x_{2,}x_3, 0) \equiv \mathrm{u}(x_{1,}x_{2,}, 0) = \mathrm{g}(\mathrm{x}_{1,}\mathrm{x}_2)$$

$$\bar{u}_t(x_{1,}x_{2,}x_3, 0) \equiv u_{t(}x_{1,}x_2, 0) = h(x_{1,}x_2)$$

$$and\ \bar{u}_{tt} - \Delta\bar{u} = \bar{u}_{tt} - \bar{u}_{x_1x_1} - \bar{u}_{x_2x_2} - \bar{u}_{x_3x_3}$$

$$= u_{tt} - u_{x_1x_1} - u_{x_2x_2} - 0 \ \ since\ u\ is\ independent\ of\ x_3$$

$$= u_{tt} - \Delta u = 0$$

$\Longrightarrow \bar{u}\,(x_1, x_{2,}, x_{3,}t)$ Solves the three dimensional wave equations with initial data $g(x_1, x_2\,)$ *and* $h(x_1, x_2\,)$. Therefore in $R^3 \times (o, \infty)$ (28) implies;

$$\bar{u}_{tt} - \bar{u}_{x_1x_1} - \bar{u}_{x_2x_2} - \bar{u}_{x_3x_3} = 0$$

$$\bar{\mathrm{U}}(x_{1,}x_{2,}x_3, 0) = \bar{\mathrm{g}}(x_1, x_{2,}x_3) = g(x_1, x_2)\,, \bar{u}_t(x_{1,}x_{2,}x_3, 0) = \bar{h}(x_1, x_{2,}x_3) = h(x_1, x_{2,})$$

$$For\ x = (x_1, x_2) \in R^2\ and\ \bar{x} = (x_1, x_2, 0) \in R^3\ Then$$

Kirchhoff's formula to solve the three dimensional wave equation (27ii) imply,

$$u(x,t) = \bar{u}(\bar{x}, t) = {}_{\partial\bar{B}(\bar{x},t)}[\bar{\mathrm{g}}(\mathrm{y}) + \mathrm{D}\bar{\mathrm{g}}(\mathrm{y}).(\mathrm{y} - \mathrm{x}) + \mathrm{t}\bar{\mathrm{h}}(\mathrm{y})]\mathrm{ds}(\mathrm{y}) \ldots\ldots\ldots\ldots\ldots.. (29)$$

where $\bar{B}(\bar{x}, t)$ *denotes the ball in* R^3 *about the point* $\bar{x} = (x_{1,}x_{2,}0)$, *radius* $t > 0$
From parameterization equation for the surface we have:

$$|\bar{y}-\bar{x}|^2=t^2$$

$$(y_1-x_1)^2+(y_2-x_2)^2+{y_3}^2=t^2$$

$${y_3}^2=t^2-|y-x|^2 \Longrightarrow y=\sqrt{t^2-|y-x|^2}$$

$\gamma(y)=\gamma(y_1,y_2)=\sqrt{t^2-|y-x|^2}$ and $ds=(1+|D\gamma(y)|^2)^{\frac{1}{2}}dy$ for $y\in B(x,t)$.

$$D\gamma(y)=\frac{\partial\gamma}{\partial y_i}=\frac{1}{2}\frac{1}{\sqrt{t^2-|y-x|^2}}(-2(y_i-x_i))=-\frac{(y_i-x_i)}{\sqrt{t^2-|y-x|^2}}=-\frac{(y-x)}{\sqrt{t^2-|y-x|^2}}=D\gamma$$

$$Now\ ds=(1+|D\gamma(y)|^2)^{\frac{1}{2}}=\left(1+\frac{|y-x|^2}{t^2-|y-x|^2}\right)^{\frac{1}{2}}dy=\left(\frac{t^2}{t^2-|y-x|^2}\right)^{\frac{1}{2}}dy=\frac{t}{\sqrt{t^2-|y-x|^2}}dy$$

$$(1+|D\gamma(y)|^2)^{\frac{1}{2}}=\left(\frac{t}{\sqrt{t^2-|y-x|^2}}\right)^{dy}$$

$$since\bar{B}(\bar{x},t) denotes\ the\ ball\ in\ R^3\ about\ the\ point\ \bar{x}=(x_{1,}x_{2,}0), radius\ t>0$$

$$_{\partial\bar{B}(\bar{x},t)}\bar{g}(y)ds(y)=\frac{1}{4\pi t^2}\int_{\partial\bar{B}(\bar{x},t)}\bar{g}(y)ds(y)=\frac{2}{4\pi t^2}\int_{B(x,t)}g(y)(1+|D\gamma(y)|^2)^{\frac{1}{2}}dy$$

$$_{\partial\bar{B}(\bar{x},t)}\bar{g}(y)ds(y)=\frac{1}{2\pi t^2}\int_{B(x,t)}g(y)(1+|D\gamma(y)|^2)^{\frac{1}{2}}dy$$

$Where\ \gamma(y)=(t^2-|y-x|^2)^{\frac{1}{2}}\ for\ y\in\ B(x,t)\ and$

$B(x,t) is\ the\ ball\ in\ R^2\ of\ radius\ t\ about\ the\ point\ x=(x_1,x_2). The factor "2" enters$

$$since\ \partial\bar{B}(\bar{x},t) consists\ of\ two\ hemispheres.$$

$$Thus\ _{\partial\bar{B}(\bar{x},t)}\bar{g}(y)ds(y)=\frac{1}{2\pi t^2}\int_{B(x,t)}\frac{tg(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy$$

Similarly

$$_{\partial\bar{B}(\bar{x},t)}t\bar{h}(y)ds(y)=\frac{1}{2\pi t^2}\int_{B(x,t)}\frac{t^2h(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy$$

$$And_{\partial\bar{B}(\bar{x},t)}D\bar{g}(y)(y-x)ds\,(y)=\frac{1}{2\pi t^2}\int_{B(x,t)}\frac{tDg(y)(y-x)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy$$

$$\boldsymbol{Thus\ u(x,t)=\frac{1}{2\pi t^2}\int_{B(x,t)}\frac{tg(y)+t^2h(y)+tDg(y)(y-x)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy}\ \ldots\ldots\ldots\ldots.\mathbf{(30)}$$

Is the Poisson formula for solution of the initial value problem $u_{tt}-\Delta u=0$

$u=g\ \ u_t\ =h\ \ t>0\ in\ two\ dimensions.$

The trick of solving the problem for n=3 first and then dropping to n=2 is called the method of descent.

4.1. Odd Dimensions

4.1.1. Solution for odd dimensions

Theorem 4 Let the function g=g(x), h=h(x)∈ $c^{\frac{n+3}{2}}(R^n)$ with odd n≥ 3 be given. Then the initial Value problem for the n dimensional wave equation has unique solution.

For the cause of odd dimension we use the method of spherical means as we did n=3

Before solving PDE for odd $n\geq 3$ we first see some useful identities

Lemma 2. (Some useful identities)

Let $\phi: R\to R$ be C^{k+1} then for $k=1,2,\ldots$

i) $\left(\frac{d^2}{dr^2}\right)\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}(r^{2k-1}\phi(r))=\left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}\frac{d\phi}{dr}(r)\right)$…………………(31)

ii) $\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1} (r^{2k-1}\phi(r)) = \sum_{j=0}^{k-1} \beta_j^k \, r^{j+1} \frac{d_j\phi}{dr^j}(r),$

Where the constants β_j^k $(j = 0, \dots, k-1) are\ independent\ of\ \phi$

Further more

iii) $\beta_0^k = 1.3.5 \dots (2k-1)$

Now assume $n \geq 3$ is an odd integer and set $n = 2k+1$ $(k \geq 0)$ and **define** the following **Notation** as:

i) $\tilde{U}(r,t) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1}U(x,r,t)\right)$

ii) $\tilde{G}(r) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1}G(x,r)\right) (r > 0, t \geq 0)$ …(32)

iii) $\tilde{H}(r) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1}H(x,r)\right)$

Then $\tilde{U}(r,0) = \tilde{G}(r), \tilde{U}_t(r,0) = \tilde{H}(r) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. (33)$

Next we combine lemma 1 and the identities provided by lemma 2 to demonstrate that the transformation (32) of U into Ũ converts the Euler- Poisson- Darboux equation into the wave equation.

Lemma 3 (Ũ solves the one dimensional wave equation) of

$$\tilde{U}_{tt} - \tilde{U}_{rr} = 0\ in\ R_+ \times (0,\infty)$$
$$\tilde{U} = \tilde{G}\ , \tilde{U}_t = \tilde{H}\ on\ R_+ \times \{t = 0\} \dots\dots\dots\dots\dots\dots. (34)$$
$$\tilde{U} = 0\ on\ \{r = 0\} \times (0,\infty)$$

Proof if r>0

$$\tilde{U}_{rr} = \left(\frac{\partial^2}{\partial r^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} (r^{2k-1}U)$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k} (r^{2k}U_r) \dots\dots by\ lemma\ 2\ (i)$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \frac{1}{r}(2kr^{2k-1}U_r + r^{2k}U_{rr})$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} (2kr^{2k-2}U_r + r^{2k-1}U_{rr})$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1}[\frac{2k}{r}U_r + U_{rr}]\right)$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1}\left[U_{rr} + \frac{n-1}{r}\ U_r\right]\right), since\ (n = 2k+1)$$

$$\tilde{U}_{rr} = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} (r^{2k-1}U_{tt}) = \tilde{U}_{tt} \dots\dots. by\ Notation \dots .32(i)$$

$$There\ fore\ \tilde{U}_{tt} = \tilde{U}_{rr}$$

Clearly $\tilde{U} = 0\ on\ \{r = 0\}\ and\ \tilde{U} = \tilde{G}, \tilde{U}_t = \widetilde{H}\ at\ t = 0$

This implies $\tilde{U}(r,t)$ is the solution of one dimenational wave equation on the half line with boundary condition implies for $0 \le r \le t$.

Thus $\tilde{U}(r,t) = \frac{1}{2}\left[\tilde{G}(r+t) - \tilde{G}(t-r)\right] + \frac{1}{2}\int_{t-r}^{t+r} \widetilde{H}(y)dy$. For all $r \in 1R, t \ge 0$, …. (35)

$u(x,t) = \lim_{r \to 0^+} U(x,r,t). Further\ more$ Lemma 2(ii) asserts:

$$\tilde{U}(r,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}U(x,r,t)\right) = \sum_{j=0}^{k-1} \beta_j^k\, r^{j+1}\frac{\partial^j}{\partial r^j}\, U(x,r,t)$$

$$\tilde{U}(r,t) = \beta_0^k\, rU(x,r,t) + \beta_1^k\, r^2\frac{\partial}{\partial r}\, U(x,r,t) + \cdots \ldots + \beta_{k-1}^k\, r^k\frac{\partial^{k-1}}{\partial r^{k-1}}\, U(x,r,t)$$

$$\beta_0^k\, r\, U(x,r,t) = \tilde{U}(r,t) - \beta_1^k\, r^2\frac{\partial}{\partial r}U(x,r,t) - \cdots - \beta_{k-1}^k r^k\frac{\partial^{k-1}}{\partial r^{k-1}}\, U(x,r,t)$$

$$\Rightarrow U(x,r,t) = \frac{\tilde{U}(r,t)}{\beta_0^k r} - \frac{\beta_1^k r^2}{\beta_0^k r}\frac{\partial}{\partial r}\, U(x,r,t) - \cdots - \frac{\beta_{k-1}^k r^k}{\beta_0^k r}\frac{\partial^{k-1}}{\partial r^{k-1}}U(x,r,t)$$

$But\ we\ have\ u(x,t) = \lim_{r \to 0} U(x,r,t) = \lim_{r \to 0}\left[\frac{\tilde{U}(r,t)}{\beta_0^k r} - \frac{\beta_1^k r}{\beta_0^k}\frac{\partial}{\partial r}\, U(x,r,t) - \cdots - \frac{\beta_{k-1}^k r^{k-1}}{\beta_0^k}\frac{\partial^{k-1}}{\partial r^{k-1}}\, U(x,r,t)\right]$

$u(x,t) = \lim_{r \to 0}\frac{\tilde{U}(r,t)}{\beta_0^k r}$.

Thus (35) implies

$u(x,t) = \frac{1}{\beta_0^k}\lim_{r \to 0}\left[\frac{\tilde{G}(t+r) - \tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r}\widetilde{H}(y)dy\right]$

$u(x,t) = \frac{1}{\beta_0^k}\left[\frac{\partial}{\partial r}\,\tilde{G}(t) + \widetilde{H}\,(t)\right]$, $Where\ \beta_0^k = 1.3.5 - - - - (2k-1),$

But we have, $\tilde{G}\,(r) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}(\, r^{2k-1}\, G(x,r)$……………………………………by 32(ii)

$Now\ n = 2k+1, implies\ k = \frac{n-1}{2}\ and\ there\ fore$

$$\tilde{G}(r) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}(\, r^{2k-1}\, G(x,r))$$

$$\tilde{G}(r) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}(\, r^{2k-1}{}_{\partial B(x,t)}\, g(y)ds(y) \quad \ldots\ldots\ldots\ldots .31(i)$$

$$\tilde{G}(t) = \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}(\, t^{n-2}{}_{\partial B(x,t)}\, g(y)ds(y))$$

$$\tilde{G}(t) = \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}(\, t^{n-2}{}_{\partial B(x,t)}\, g(y)ds(y))$$

Similarly

$$\widetilde{H}(r) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}(\, r^{2k-1}H(x,r))$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}(\, r^{2k-1}{}_{\partial B(x,t)}\, h(y)ds(y) \ldots\ldots \text{by } 31(ii)$$

$$\widetilde{H}(t) = \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} (\, t^{n-2}{}_{\partial B(x,t)}\, h(y)ds(y))$$

$We\ have\ u(x,t) = \frac{1}{\beta_0^k}[\frac{\partial}{\partial r}\ \tilde{G}(t) + \tilde{H}\ (t)]$

$$u(x,t) = \frac{1}{\gamma_n}\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} (\, t^{n-2}{}_{\partial B(x,t)}\, g(y)ds(y) +$$

$$\frac{1}{\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} (\, t^{n-2}{}_{\partial B(x,t)}\, h(y)ds(y)) \ldots\ldots\ldots\ldots\ldots(36)$$

$Where\ n\ is\ odd\ and\ \ \gamma_n\ = 1.3.5 \ldots.(n-2)\ for\ x \in R^n, t > 0$

$If\ \gamma_3 = 1\,,(36) agrees\ for\ n = 3\ with\ \ 27(i) and\ thus\ with\ kirchhoff' sformula\ \ 27(ii)$

4.2. Even Dimensions

4.2.1. Solution for Even n

Theorem 5. Let the even positive integer n≥ 2 and the function g=g(x), h=h(x)∈ $c^{\frac{n+4}{2}}(R^n)$ be

given the initial value problem for the n-dimensional wave equation is unique

Suppose $u(x_1, x_2, \ldots.x_{n,} t)$ is a solution of the wave equation (23i) in R^n with initial data

$$u(x_1, x_2, \ldots.x_n, 0) = \ g(x_1, x_2, \ldots.x_n)$$
$$u_t(x_1, x_2, \ldots.x_n, 0) = h(x_1, x_2, \ldots.x_n)$$

Then define

$$\bar{u}(x_1, x_2, \ldots.x_{n+1}, t\,) \equiv \ u(x_1, x_2, \ldots.x_n, t)$$

With initial condition

$$\bar{u} = \bar{g}\ , \bar{u}_t = \bar{h}\ \ on\ \ R^{n+1}\ \times\{t = 0\}$$
$$Where\ \bar{g}\ (x_1, \ldots\ldots\ x_{n+1}) \coloneqq\ g(x_1, x_2, \ldots.x_n)$$
$$\bar{h}\ (x_1, \ldots\ldots\ x_{n+1}) \coloneqq\ h(x_1, x_2, \ldots.x_n)$$

Solves the wave equation in $R^{n+1} \times (0,\infty)\ \ where\ \ n+1$ is odd

Now let us fix $x \in R^n, t > 0, and\ writ\ \bar{x} = (x_1, \ldots\ldots\ x_n, 0) \in R^{n+1}$,. Then (36), with n+1 replacing n, gives

$$\bar{u}(\bar{x},t) = \frac{1}{\gamma_{n+1}}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} (t^{n-1}{}_{\partial\bar{B}(\bar{x},t)}\bar{g}(y)ds(y) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} (t^{n-1}{}_{\partial\bar{B}(\bar{x},t)}\bar{h}ds(y)\right]$$

$Where\ \gamma_{n+1} = 1.3.5 \ldots.(n-1)\ and\ \bar{B}(\bar{x},t)$ is the ball in R^{n+1} of radius t with center $\bar{x} = (x_1, x_2, \ldots.x_n, 0\,)$.

$Now_{\partial\bar{B}(\bar{x},t)}\bar{g}(y)ds(y) = \frac{1}{(n+1)\alpha(n+1)t^n}\ \int_{\partial\bar{B}(\bar{x},t)}\bar{g}(y)ds(y)$ since n=n+1

$But\ \partial\bar{B}(\bar{x},t) \cap \{y_{n+1} \geq 0\} is\ the\ graph\ of\ the\ function\ \gamma(y) \coloneqq (t^2 - |y-x|^2)^{\frac{1}{2}}\ for$
$y \in B(x,t)$ and similarly $\partial\bar{B}(\bar{x},t) \cap \{y_{n+1} \leq 0\}$ is the graph of $-\gamma$ there fore

$\frac{1}{(n+1)\alpha(n+1)t^n} \int_{\partial\bar{B}(\bar{x},t)} \bar{g}(y)ds(y) = \frac{2}{(n+1)\alpha(n+1)t^n} \int_{B(x,t)} g(y)(1+|\nabla\gamma(y)|^2)^{\frac{1}{2}} dy$

The factor "2" interring because $\partial\bar{B}(\bar{x},t)$ comprises two hemispheres

$Now\ (1+|\nabla\gamma(y)|^2)^{\frac{1}{2}} = \frac{t}{(t^2-|y-x|^2)^{\frac{1}{2}}}$

Therefore $_{\partial\bar{B}(\bar{x},t)}\bar{g}(y)ds(y) = \frac{2}{(n+1)\alpha(n+1)t^n} \int_{B(x,t)} \frac{tg(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\, dy$

$$= \frac{2t}{(n+1)\alpha(n+1)t^n} \int_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\, dy$$

$$= \frac{2}{(n+1)\alpha(n+1)t^{n-1}} \int_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\, dy$$

$$= \frac{2t\alpha(n)}{(n+1)\alpha(n+1)t\alpha(n)t^{n-1}} \int_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\, dy$$

$$= \frac{2t\alpha(n)}{(n+1)\alpha(n+1)\alpha(n)t^n} \int_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\, dy$$

$$_{\partial\bar{B}(\bar{x},t)}\bar{g}(y)ds(y) = \frac{2t\alpha(n)}{(n+1)\alpha(n+1)}\ _{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\, dy$$

$$Similarly\ _{\partial\bar{B}(\bar{x},t)}\bar{h}(y)ds(y) = \frac{2t\alpha(n)}{(n+1)\alpha(n+1)}\ _{B(x,t)} \frac{h(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\, dy$$

Therefore our solution formula is give by:

$$u(x,t) = \frac{1}{\gamma_{n+1}}[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} (t^{n-1}\ _{\partial\bar{B}(\bar{x},t)}\bar{g}(y)ds(y) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} (t^{n-1}\ _{\partial\bar{b}(\bar{x},t)}\bar{h}(y)ds(y)]$$

$$u(x,t) = \frac{1}{\gamma_{n+1}} \frac{2\alpha(n)}{(n+1)\alpha(n+1)} [\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} (t^n\ _{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}} +$$

$$\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} (t^n\ _{B(x,t)} \frac{h(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}]\, dy \quad \ldots\ldots(37)$$

$Since\ \gamma_{n+1} = 1.3.5 \ldots.(n-1) and\ \alpha(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$

$We\ can\ compute\ \gamma_n = 2.4 \ldots.(n-2).n\ \ as\ follows$

Now $\Gamma(n)$ is the gamma function defined by

$$\Gamma(n) = \int_0^{\infty} x^{n-1}e^{-x}\, dx \ldots\ldots(38)$$

Having the following properties

1. $\Gamma(n+1) = n\Gamma(n)$
2. $\Gamma(1) = \int_0^{\infty} e^{-x}\, dx = 1$

3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

4. $\Gamma(n+1) = n!$ *when n is a positive integer*

5. $\Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) \dots \dots .\left(\frac{1}{2}\right)\sqrt{\pi}$

There fore $\frac{1}{\gamma_{n+1}} \frac{2\alpha(n)}{(n+1)\alpha(n+1)} = \frac{1}{1.3.5\dots..(n-1)} \frac{\frac{2\pi^{n/2}}{\Gamma(\frac{n+2}{2})}}{\frac{(n+1)\pi^{\frac{(n+1)}{2}}}{\Gamma(\frac{n+3}{2})}}$

$$= \frac{1}{1.3.5\dots..(n-1)(n+1)} \frac{1}{\pi^{1/2}} \frac{\Gamma(\frac{n+3}{2})}{\Gamma(\frac{n+2}{2})}$$

But $\Gamma\left(\frac{n+3}{2}\right) = \Gamma\left(\frac{n}{2}+\frac{3}{2}\right) = \Gamma\left(\frac{n}{2}+\frac{1}{2}+1\right) = \Gamma\left(\frac{n+1}{2}+1\right) = \left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)$

$$\Gamma\left(\frac{n+1}{2}\right) = \Gamma\left(\frac{n}{2}+\frac{1}{2}\right) = \Gamma\left(\frac{n}{2}-\frac{1}{2}+1\right) = \Gamma\left(\frac{n-1}{2}+1\right) = (\frac{n-1}{2})\Gamma\left(\frac{n-1}{2}\right)$$

$$\Gamma\left(\frac{n-1}{2}\right) = \Gamma\left(\frac{n}{2}-\frac{1}{2}\right) = \Gamma\left(\frac{n}{2}-\frac{3}{2}+1\right) = \Gamma\left(\frac{n-3}{2}+1\right) = (\frac{n-3}{2})\Gamma\left(\frac{n-3}{2}\right)$$

$\vdots$

Since n- is even and as in the cause of $n = 2$

$$\Gamma\left(\frac{n+3}{2}\right) = \Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}.\sqrt{\pi}$$

Therefore

$$\Gamma\left(\frac{n+3}{2}\right) = \left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\dots\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

Similarly

$$\Gamma\left(\frac{n+2}{2}\right) = \Gamma\left(\frac{n}{2}+1\right) = \frac{n}{2}\Gamma\left(\frac{n}{2}\right)$$

$$\Gamma\left(\frac{n}{2}\right) = \Gamma\left(\frac{n}{2}-1+1\right) = \Gamma\left(\frac{n-2}{2}+1\right) = \left(\frac{n-2)}{2}\right)\Gamma\left(\frac{n-2}{2}\right)$$

$$\Gamma\left(\frac{n-2}{2}\right) = \Gamma\left(\frac{n}{2}-1\right) = \Gamma\left(\frac{n}{2}-2+1\right) = \Gamma\left(\frac{n-4)}{2}+1\right) = \left(\frac{n-4}{2}\right)\Gamma\left(\frac{n-4}{2}\right)$$

$\vdots$

Since n- is even and As in the cause of n=2

$$\Gamma\left(\frac{n+2}{2}\right) = \Gamma\left(\frac{2+2}{2}\right) = \Gamma\left(\frac{4}{2}\right) = \Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1.1 = 1$$

Therefore

$$\Gamma\left(\frac{n+2}{2}\right)=\left(\frac{n}{2}\right)\left(\frac{n-2}{2}\right)\left(\frac{n-4}{2}\right)\ldots\ldots\ldots\frac{2}{2}$$

$$=\frac{1}{1.3.5\ldots\ldots(n-1)(n+1)}\,\frac{1}{\pi^{1/2}}\quad\frac{\Gamma(\frac{n+3}{2})}{\Gamma\left(\frac{n+2}{2}\right)}$$

$$=\frac{1}{1.5.3\ldots\ldots(n-1)(n+1)}\,\frac{1}{\pi^{1/2}}\quad\frac{\frac{n+1}{2}.\frac{n-1}{2}\ldots.\frac{1}{2}\sqrt{\pi}}{\frac{n}{2}.\frac{n-2}{2}.\frac{n-4}{2}.\frac{n-6}{2}\ldots\ldots\frac{6}{2}\,\frac{4}{2}\,\frac{2}{2}}$$

$$=\frac{1}{1.5.3\ldots\ldots(n-5)(n-3)(n-1)(n-1)}\quad\frac{\frac{(n+1)}{2}\frac{(n-1)}{2}\frac{(n-3)}{2}\frac{(n-5)}{2}\ldots\ldots\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}}{\left(\frac{n}{2}\right)\left(\frac{n-2}{2}\right)\left(\frac{n-4}{2}\right)\ldots\ldots\frac{6}{2}\,\frac{4}{2}\,\frac{2}{2}}$$

$$=\frac{1}{n(n-2)\ldots\ldots6.4.2}$$

$$=\frac{1}{2.4.6\ldots\ldots(n-2)n}$$

And therefore $\frac{1}{\gamma_{n+1}}\,\frac{2\alpha(n)}{(n+1)\alpha(n+1)}=\frac{1}{2.4\ldots..(n-2)n}$

Therefore the solution of the wave equation in even dimensions is givenby

$$u(x,t)=\frac{1}{\gamma_n}\left[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\,\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n\int_{B(x,t)}\frac{g(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy\right)+\left(\frac{1}{t}\,\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n\int_{B(x,t)}\frac{h(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy\right)\right]$$

Where $\gamma_n=2.4\ldots\ldots(n-2)n$

Chapter Five

Solution of the Non-homogeneous wave equation

5.1. Duhamel's principle: It is a way to express the solution of a non homogeneous partial differential equation as an integral of the solution of a homogeneous equation with appropriate initial or boundary conations.

First let us see the following proposition

Proposition (differentiation of an integral).

Let U(x) =$\int_a^x f(x,s)ds$ *where a is a fixed constant .Then*

$$\frac{d}{dx}U(x) = f(x,x) + \int_a^x \frac{\partial}{\partial x} f(x,s)ds$$

Proof *Define a function of two variables*

$$V(x,y) = \int_a^y f(x,s)ds$$

By the fundamental theorem of calculus

$$\frac{\partial}{\partial y}V(x,y) = \frac{\partial}{\partial y}\int_a^y f(x,s)ds = f(x,y)$$

Now also differentiation under the integral sign gives

$\frac{d}{dx}V(x,y) = \frac{\partial}{\partial x}\int_a^y f(x,s)ds = \int_a^y \frac{\partial}{\partial x} f(x,s)ds$

Suppose that y is a function of x and write y=y(x).The chain rule in two dimensions implies that

$\frac{d}{dx}V(x,y(x)) = \frac{\partial}{\partial x}V\big(x,y(x)\big) + \frac{\partial}{\partial y}V(x,y(x))\frac{dy}{dx}$

$= \int_a^{y(x)} \frac{\partial}{\partial x} f(x,s)ds + f(x,y(x))\frac{dy}{dx}$

$= \int_a^x \frac{\partial}{\partial x} f(x,s)ds + f(x,x)\frac{dy}{dx}$

Thus

$$\frac{d}{dx}U(x) = f(x,x) + \int_a^x \frac{\partial}{\partial x} f(x,s)ds$$

Examples: Function defined by an integral

Let U(x)=$\int_0^x \sin(xs)\,ds$. Find $\frac{d}{dx}U(x)$

Solution: a) use preposition 1 with f(x,s)=sin(xs) we obtain

$$\frac{d}{d(x)}U(x) = \sin(x^2) + \int_0^x \frac{d}{dx}(\sin(xs))ds$$

$$= \sin(x^2) + \int_0^x s\cos(xs)ds$$

Integrate by parts, u=s , dv=cos(xs)ds , dv=ds, v=$\frac{1}{x}$sin(xs)

$$= \sin(x^2) + \frac{s}{x}\sin(xs)\,|_0{}^x - \frac{1}{x}\int_0^x \sin(xs)\,ds$$

$$= sinx^2 + sinx^2 + \frac{\cos(x^2)-1}{x^2} = 2\sin(x^2) + \frac{\cos(x^2)-1}{x^2}$$

b) By evaluating the integral

$$U(x) = \int_0^x \sin(xs)\,ds = -\frac{\cos(xs)}{x}|^x{}_0 = \frac{1-\cos(x^2)}{x}$$

Differentiate with respect to x, using the quotient rule

$$\frac{d}{dx} = \left(\frac{1-\cos(x^2)}{x}\right)' = \frac{2x^2\sin(x^2)-(1-\cos(x^2)}{x^2} = 2\sin(x^2) + \frac{\cos(x^2)-1}{x^2}$$

Which matches the answer in (a)

D' Alembert's method tells us that the solution of the wave equation

$$V_{tt} = C^2 V_{xx}\,(-\infty < x < \infty, t > 0)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(39)$$
$$V(x,o) = g(x)$$
$$V_t(x,o) = h(x)$$

is $V(x,t) = ½\{g(x+ct) + g(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy$

Our goal is to use this solution to solve the wave equation in the presence of an additional external force $f(x,t)$ *that depends on* x *and* t

Theorem 6: Duhamel's principle for the wave equation

For a fixed $s > 0$ *suppose V(x, t, s) is a solution of the homogeneous wave equation (39) with the initial conditions V(x, 0, s) =0 and* $V_t(x,0,s) = f(x,s)$ *Then*

$u(x,t) = \int_0^t V(x,t-s,s)ds \quad (-\infty < x < \infty, t > 0)$ *is the solution of the nonhomogeneous wave equation* $u_{tt} = c^2 u_{xx} + f(x,t) \quad (-\infty < x < \infty, t > 0)$ *subject to the initial condtions* $u(x,0) = 0$ *and* $u_t(x,0) = 0$

Proof: Apply the above proposition

$$U(x,t) = \int_0^t V(x,t-s,s)ds$$
$$u_t(x,t) = V(x,t-s,s)|_{s=t} + \int_0^t V_t(x,t-s,s)ds$$
$$= V(x,0,s) + \int_0^t V_t(x,t-s,s)ds$$
$$u_t(x.t) = \int_0^t V_t(x,t-s,s)ds$$

$$u_{tt}(x,t) = V_t(x,t-s,s)|_{s=t} + \int_0^t V_{tt}(x,t-s,s)ds$$

$$= V_t(x,0,s) + \int_0^t V_{tt}(x,t-s,s)ds$$

$$u_{tt}(x,t) = f(x,t) + \int_0^t V_{tt}(x,t-s,s)ds$$

And also differentiate under the integral sine twice

$$u_x(x,t) = V(x,t-s,s)|_{s=t} + \int_0^t V_x(x,t-s,s)ds$$

$$u_x(x,t) = V(x,0,s) + \int_0^t V_x(x,t-s,s)ds$$

$$u_x(x,t) = \int_0^t V_x(x,t-s,s)ds$$

and $u_{xx}(x,t) = V_x(x,t-s,s)|_{s=t} + \int_0^t V_{xx}(x,t-s,s)ds$

$$u_{xx}(x,t) = V_x(x,0,s) + \int_0^t V_{xx}(x,t-s,s)ds$$

$$u_{xx}(x,t) = \int_0^t V_{xx}(x,t-s,s)ds$$

thus $u_{tt} - c^2u_{xx} = f(x,t) + \int_0^t V_{tt}((x,t-s,s) - c^2V_{xx}(x,t-s,s))ds$

But $\int_0^t V_{tt}((x,t-s,s) - c^2V_{xx}(x,t-s,s))ds = 0$

Since V is a solution of (39)

There fore $u_{tt} - c^2u_{xx} = f(x,t)$

Examples: wave equation and Duhamel's principle

Solve $u_{tt} = u_{xx} + e^{-t}\cos(x)$ $(t > 0, -\infty < x < \infty)$ subject to u(x,0)=0 and $u_t(x,0) = 0$

Solution; According to the theorem we consider the wave equation on the real line $V_{tt} = V_{xx}$ subject to V(x,0,s)=0 and $V_t(x,0,s) = e^{-s}cosx$ where x is fixed $t > 0, and -\infty < x < \infty$. Applying d'Alembert's solution (39) we find

$$V(x,t,s) = \frac{e^{-s}}{2}\int_{x-t}^{x+t} coszdz = \frac{e^{-s}}{2}[\sin(x+t) - \sin(x-t)]$$

$$= e^{-s}sintcosx$$

Then the solution is

$$U(x,t)=\int_0^t e^{-s}\sin(t-s)\,cosx\,ds$$

$$=\cos x\int_0^t e^{-s}\sin(t-s)ds$$

$$=\frac{1}{2}cosx[e^{-t}-cost+sint]$$

Evaluated the last integral using the formula

$$\int e^{-s}\sin(t-s)\,ds=\frac{e^{-s}}{2}[\cos(t-s)-\sin(t-s)]$$

5.2. Energy method

Theorem 7. (Uniqueness theorem)

There exists at most one solution of the wave equation

$$u_{tt}=C^2u_{xx}\,,\qquad 0<x<l,t>0$$

Satisfying the initial conditions

$$u(x,o)\;=\;f(x)$$

$$u_t(x,0)=g(x),0\leq x\leq L$$

And the boundary conditions

$$u(0,t)\;=\;0$$

$$u(L,t)=0\,,t\geq 0$$

Where u(x,t) is a twice continuously differentiable function with respect to both $x\ and\ t$

Proof: suppose that there are two solutions $u_1\ \ and\ u_2\ \ and\ let\ \ V=u_1-u_2$. it can readily see that $V(x,t)$ is the solution of the problem.

$$V_{tt}=C^2V_{xx}\ \ ,0<x<L,t>0$$

$$V(0,t)=0\ \ ,t\geq 0$$

$$V(L,t)=0\ \ ,t\geq 0$$

$$V(x,0)=0\ \ ,0\leq x\leq L$$

$$V_t(x,0)=0\ \ ,0\leq x\leq L$$

We shall prove that the function $V(x,t)$ is identically zero. To do so, consider the energy integral $E(t) = \frac{1}{2}\int_0^L (C^2 V_x^2 + V_t^2)dx$

The first term represents the total potential energy of the string, while the second term represents the total kinetic energy, which physically represents the total energy of the vibrating string at time t

Since the function $V(x,t)$ is twice continuously differentiable, we differentiate $E(t)$ with respect to t, thus

$$\frac{dE}{dt} = \int_0^L (C^2 V_x V_{xt} + V_t V_{tt})dx = \int_0^L C^2 V_x V_{xt} dx + \int_0^L (V_t V_{tt})dx = \dots\dots (47)$$

Integrating the first integral in (45) by parts we have

$$\int_0^L C^2 V_x V_{xt} dx = [C^2 V_x V_t]_0^L - \int_0^L C^2 V_t V_{xx} dx$$

But from the condition $v(0,t) = 0$, we have $V_t(0,t) = 0, and\ similarly,$

$$V_t(L,t) = 0\ for\ x = L$$

Hence, the expression in the square brackets vanishes, and equation (47) becomes

$$\frac{dE}{dt} = \int_o^L V_t V_{tt} dx + \int_0^L C^2 V_{xt} dx$$

$$= \int_o^L (V_t V_{tt} - C^2 V_t V_{xx})dx$$

$$\frac{dE}{dt} = \int_o^L V_t (V_{tt} - C^2 V_{xx})\, dx \dots\dots (48)$$

Since $V_{tt} - C^2 V_{xx} = 0, equation (46) reduces\ to$

$$\frac{dE}{dt} = 0$$

$$Which\ means\ E(t) = constant = C$$

Since $V(x,0) = 0$ we have $V_x(x,0) = 0. Taking\ in\ to$ account the condition $V_t(x,0) = 0, we\ evaluate\ C$ to obtain $E(0) = C = \frac{1}{2}\int_0^L [C^2 V_x^2 + V_t^2]_{t=0} dx = 0$

This implies that $E(t) = 0$ which can happen only when

$$V_x = 0 \ and \ V_t = 0 \quad for \ t > 0$$

To satisfy both of this condition's, we must have $V(x,t) =$ constant employing the condition $V(x,0) = 0$ we then find $V(x,t) = 0$

Therefore, $u_1(x,t) = u_2(x,t)$ and solution $u(x,t)$ is unique.

Summary

The physical interpretation of the wave equation strongly suggests that the solution of the initial value problem $u_{tt} - \Delta u = 0$ for x$\in R^n, t \geq 0$, is mathematically appropriate to specify two initial conditions, on the displacement u and the velocity u_t at time t=0.

By solving the Euler- Poisson-Darboux equation, a PDE which we can for odd n convert in to the ordinary one-dimensional wave equation and then applying D'Alembert's formula it leads to a formula for the solution. But no transformation works to convert the Euler-Poisson- Darboux equation in to the one dimensional wave equation when $n = 2$. Instead we will take the initial value problem of wave equation for $n = 2$ and use aformula $for\ n = 3$,in which the third special variables x_3 does not appear.

That means to compute the solution u(x,t) for odd n, we need only have information on g , h and their derivatives on the sphere $\partial B(x, t)$, and not on the entire ball B (x,t).In contrast to this to compute u(x,t) for even n we need information on u=g , $u_t = h$ on all of B(x,t), and not just on $\partial B(x, t)$.

Appendix

It is convenient to calculate integrals in polar coordinates. Thus

$$\int_{R^n} f(x)dx = \int_0^{\infty}\int_{\partial B(x_{0,}r)} f(x)dsdr$$

Hear ds represents surface measure on the n-1 dimensional sphere $\partial B(x_{0,}r)$of radius r centered at x_0. The total surface measure of the sphere is proportional to r^{n-1} and the proportionality constant will be taken so that it is by definition $n\alpha(n)r^{n-1}$.

For example $n\alpha(n)$ in dimensions n=1,2,3 has the values $2, 2\pi, 4\pi$. In dimensions $n = 1, 2, 3$ These numbers represents the count of two points, the length of the unit circle and the area of the unit sphere.

$$\textit{As an example take } f(x) = e^{-x^2} \textit{ and } x_0 = 0$$

Then $\int e^{-x^2}dx = \int_0^{\infty}\int_{\partial B(0,r)} e^{-x^2}dsdr$

$$\int e^{-x^2}dx = n\alpha(n)\int_0^{\infty} e^{-r^2}r^{n-1}dr$$

$$\textit{Here the tota surface mearsure of the ball is defind to be } n\alpha(n)r^{n-1}.$$

$$n\alpha(n)\int_0^{\infty} e^{-r^2}r^{n-1}dr$$

$$\textit{Let } u = r^2 \textit{ then } \frac{du}{dr} = 2r, \frac{du}{2r} = dr \textit{ then we get:}$$

$$= n\alpha(n)\frac{1}{2}\int_0^{\infty} e^{-u}u^{\left(\frac{n}{2}-1\right)} du$$

$$= n\alpha(n)\frac{1}{2}\int_0^{\infty} u^{\left(\frac{n}{2}-1\right)}e^{-u} du$$

$$= n\alpha(n)\frac{1}{2}\Gamma(\frac{n}{2})$$

When n=2 this says that the value of the integral is π.When n=3 the value of the integral is $\pi^{\frac{3}{2}}$. It follows by factoring the exponential that for arbitrary dimensions the value of the integral is $\pi^{\frac{n}{2}}$.

This implies $$\pi^{\frac{n}{2}} = n\alpha(n)\frac{1}{2}\Gamma(\frac{n}{2})$$

$$n\alpha(n) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

$$\textit{This proves is the basic fact that the area of the unit } n-1 \textit{ sphere is}$$

$$n\alpha(n) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

$$\textit{The function } \Gamma(z) \textit{ mentiond is the usual Gamma function}$$

$$\Gamma(z) = \int_0^{\infty} u^{z-1}e^{-u} du$$

$$Its\ main\ properties\ are$$
$$\Gamma(z+1) = z\Gamma(z)$$
$$\Gamma(1) = 1$$
$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$
$$We\ can\ also\ compute\ the\ volume\ of\ the\ unit\ ball\ \alpha(n)\ \ as$$
$$\alpha(n) = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})} = \frac{\pi^{\frac{n}{2}}}{\frac{n}{2}\Gamma(\frac{n}{2})} = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$$

Therefore

$$\alpha(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$$

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