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Addis Ababa University

School of graduate studies

Department of Mathematics

Graduate seminar report

on

Weak derivatives and Sobolev spaces

For partial fulfillment of Master of Science in Mathematics

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PREFACE

In this seminar report there are three chapters, weak derivatives, Sobolev spaces and Sobolev integral representation in one-dimensional case.

In the first chapter, which is about weak derivatives, it is given three equivalent definitions with proofs of their equivalence, explanation of what weak derivatives under integral sign look like and the commutativity of weak differentiation and mollifiers.

The second chapter, which discusses basic properties of Sobolev spaces. In this chapter it is given the definition of Sobolev spaces, Minkowski's inequality for Sobolev spaces, and continuity with respect to translation of Sobolev spaces.

The third chapter explains about Sobolev integral representation for one-dimensional case.

Lastly I wish to thank my advisor Dr. Tsegaye Gedif who proof read the entire seminar report and provided me valuable comments and reference materials.
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INTRODUCTION

The theory of Sobolev spaces has been originated by the Russian mathematician S.L.Sobolev around 1938.These spaces were not introduced for some theoretical purpose, but for the need of the theory of partial differential equations. They are closely connected with the theory of distributions, since elements of such spaces are special classes of distributions.

Of course, Sobolev spaces being examples of Banach spaces or, sometimes, Hilbert spaces are interesting objects for themselves. But their importance is connected with the fact that the theory of partial differential equations can be, and even most easily, developed just in such a space. The reason is because partial differential operators are very well suited in Sobolev spaces.

Simultaneously, the space of continuous functions is not very suitable for the studies of partial differential equations.

In order to discuss the theory of Sobolev spaces we shall start with some basic notions that are necessary for introducing and discussing these spaces. The first notion to be discussed is weak differentiability

NOTATIONS AND BASIC NOTIONS

$\Re^n$: Euclidean n-space, $n \geq 2$, with points

$$x = (x_1, \ldots, x_n),\ x_i \in \Re$$

$/N$: The set of all natural numbers

$/N_0$: the set of all non-negative integers

$/N_0^n = /N_0 x \ldots x /N_0$: the set of multi-indices (n is the natural number) and

B (x, r): the open ball of radius $r > 0$ centered at the point $x \in \Re$

$$D^\alpha f \equiv \frac{\partial^{\alpha_1 + \ldots + \alpha_n} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$: the ordinary derivative of the function f of order α;

where $\alpha \in /N_0^n$ and $\alpha \neq 0$

$$D_\omega^\alpha f = \left(\frac{\partial^{\alpha_1 + \ldots + \alpha_n} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right)_\omega$$ $\equiv$ The weak derivative f of order α

For an arbitrary non-empty set $\Omega \subset \Re^n$ we shall denote by:

C (Ω): the space of functions continuous on Ω

C^k (Ω):the space of functions having all derivations of order $\leq k$ continuous in

Ω ($k \in /N_0$ or $k = \infty$)

supp (f): the support of f, the closure for the set on which $f \neq 0$

C_0^k (Ω): The set of functions in $C^k(\Omega)$ with compact support in Ω.

$L_p(\Omega)$ ($1 \leq p < \infty$): the Banach space of functions f measurable on Ω such that the norm

$$\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f|^p dx \right)^{1/p} < \infty$$

Ω is a measurable non empty set $\Omega \subset \Re^n$

$L_p^{loc}(\Omega)$ ($1 \le p \le \infty$): the set of functions defined on Ω such that for each compact

$K \subset \Omega$, $f \in L_p(K)$

${}^c\Omega(\Omega \subset \Re^n)$:the complement of Ω in $\Re^n$

$\overline{\Omega}(\Omega \subset \Re^n)$: the closure of Ω

Ω^δ $(\Omega \subset \Re^n)$: the δ-neighborhood of Ω $(\Omega^\delta = \bigcup_{x \in \Omega} B(x,\delta)$

$\partial\Omega$: Boundary of the point set Ω

$\underline{\Omega}$ $(\Omega \subset \Re^n)$: the interior of Ω

$\Omega_\delta = \{x \in \Omega$: dist $(x, \partial\Omega) \ge \delta\}$, $\delta > 0$

$\underline{\Omega_\delta} = \{x \subset \Omega$: dist $(x, \partial\Omega) > \delta\}$

$W_p^\ell(\Omega)$ ($\ell \in /N$, $1 \le p \le \infty$): Sobolev space, which is the Banach space of functions

$f \in L_p(\Omega)$, with the norm

$$\|f\|_{W_p^\ell(\Omega)} = \|f\|_{L_p(\Omega)} + \sum_{|\alpha|=\ell} \|D_\omega^\alpha f\|_{L_p(\Omega)}$$

$w_p^\ell(\Omega)$ ($\ell \in /N$, $1 \le p \le \infty$): the semi-normed Sobolev space, which is the semi-Branch space of functions $f \in L_1^{loc}(\Omega)$ such that $\forall \alpha \in / N_0^n$ where $|\alpha| = \ell$ the weak derivatives $D_\omega^\alpha f$ exist on Ω and $D_\omega^\alpha f \in L_p(\Omega)$, with semi-norm.

$$\|f\|_{W_p^\ell(\Omega)} = \sum_{|\alpha|=\ell} \|D_\omega^\alpha f\|_{L_p(\Omega)}$$

$(W_p^\ell)_0(\Omega)$ ($\ell \in /N$, $1 \le p \le \infty$): the space of function in $W_p^\ell(\Omega)$ compactly supported in Ω.

Let Ω be a measurable set and $1 \le p \le \infty$

Hölder inequality: suppose

$\frac{1}{p}+\frac{1}{p'}=1$, i.e $p'=\frac{p}{p-1}$ for $1<p<\infty$, $p'=\infty$ for $p=1$ and $p'=1$ for $p=\infty$ .If $f\in L_p(\Omega)$ and $g\in L_{p'}(\Omega)$, then $fg\in L_1(\Omega)$ and

$$\|fg\|_{L_1(\Omega)}\le \|f\|_{L_p(\Omega)}\|g\|_{L_{p'}(\Omega)}$$

Minkowski's inequality: If $f,g\in L_p(\Omega)$ then $f+g\in L_p(\Omega)$ and

$$\|f+g\|_{L_p(\Omega)}\le \|f\|_{L_p(\Omega)}+\|g\|_{L_p(\Omega)}$$

Minkowski's inequality for integrals : In addition , let $A\subset\Re^m$ be a measurable set . Suppose that f is measurable on $A\times\Omega$ and $f(.,y)\in L_p(\Omega)$ for almost all $y\in A$ .Then

$$\left\|\int_A f(.,y)dy\right\|_{L_p(\Omega)}\le \int_A \|f(.,y)\|dy$$ if the right hand side is finite.

Mollifiers: <u>Definition</u> : If $\Omega\subset\Re^n$ is measurable set and $\delta>0$, for a function f defined on Ω and such that $f\in L_1(\Omega\cap B)$ for each ball B, the operator, $A_\delta\equiv A_{\delta,\Omega}$ (a mollifier with step (or radius) δ) is defined by the equality: $\forall x\in\Re^n$

$$(A_\delta f)(x)=(\omega_\delta * f_0)(x)=\frac{1}{\delta^n}\int_\Omega \omega(\frac{x-y}{\delta})f(y)dy=\int_{B(0,1)} f_0(x-\delta z)\omega(z)dz$$

where f_0 denotes the extension of f by zero outside Ω: $f_0(x)=0$ for $x\in{}^c\Omega$ moreover for every f under consideration

$A_\delta f\in C^\infty(\Re)$ and for any $\alpha\in /N_0^n$

$$D^\alpha A_\delta f=\delta^{-|\alpha|}(D^\alpha\omega)_\delta * f_0$$

on $\Re^n$ and $\operatorname{supp}(A_\delta f)\subset\overline{(\operatorname{supp}(f)^\delta}$

We note also that on Ω_δ

$$(A_\delta f)(x) = (\omega_\delta * f)(x) = \int_{B(01)} f(x - \delta z)\omega(z)dz$$

and $\forall \alpha \in / N_0^n$

$$D^\alpha A_\delta f = \delta^{-|\alpha|}(D^\alpha \omega)_\delta * f$$

If $\Omega \subset \Re^n$ is an open set and $f \in L_1^{loc}(\Omega)$, then $A_\delta f \in C^\infty(\underline{\Omega_\delta})$ and

$$A_\delta f \to f \text{ a.e on } \Omega$$

as $\delta \to o^+$ (if $f \in C(\Omega)$,then the convergence holds every where on Ω

CHAPTER ONE
WEAK DERIVATIVES

We shall start with the following observation for the one-dimensional case and for an open interval (a, b), $-\infty \leq a < b \leq +\infty$.

The differential operator

$\frac{d}{dx} : C^1(a,b) \subset C(a,b) \to C(a,b)$ Is a closed operator in $C(a,b)$,

i.e. ,if $f_k \in C^1(a,b), k \in /N, f, g \in C(a,b)$ and $f_k \to f$, $\frac{df_k}{dx} \to g$ in $C(a,b)$ as $k \to \infty$,then $f \in C^1(a,b)$ and $\frac{df}{dx} = g$ on (a,b).

Suppose now $1 \leq p < \infty$. The following example shows that the differentiation operator

$$\frac{d}{dx} : C^1(a,b) \subset L_p^{loc}(a,b) \to L_p^{loc}(a,b) \text{ is not closed in } L_p^{loc}(a,b).$$

Example 1: Let $(a,b) = (-1,1)$ and $\forall x \in (-1,1)$ set

$f(x) = |x|, f_k(x) = (x^2 + \frac{1}{k})^{1/2}, k \in /N$, $f_k \in C^1(-1,1), f \in L_p^{loc}(a,b)$

Then $f_k \to |x|$, $f_k' \to \operatorname{sgn} x$ even in $L_p(-1,1)$, but $|x| \notin C^1(-1,1)$, since $|x|'$ does not exist on the whole interval (-1,1).

$f_k \to f$ in $C(a,b)$ means $\| f_k - f \|_{C[\alpha,\beta]} \to 0$ as $k \to \infty$ for any closed interval $[\alpha, \beta] \subset (a,b)$

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On the other hand if $f \in C^1(a,b)$ and $\varphi \in C_0^1(a,b)$ then

$$\int_a^b f\varphi' dx = [f\varphi]_a^b - \int_a^b f'\varphi dx$$

Since $\varphi \in C_0^1(a,b)$, $[f\varphi]_a^b = 0$

Thus $\int_a^b f\varphi' dx = -\int_a^b f'\varphi dx$

This equality can also be naturally used to generalize the notion of differentiation, since for some functions (e.g.; $f(x) = |x|$) the ordinary derivative does not exist on (a,b), but a function $g \in L_1^{loc}(a,b)$ exists (in the above example $g(x) = \operatorname{sgn} x$) such that $\forall \varphi \in C_0^1(a,b)$

$$\int_a^b f\varphi' dx = -\int_a^b g\varphi dx$$

We now give the definition for the multidimensional case and for differentiation of arbitrary order.

Definition 1: Let $\Omega \subset \Re^n$ be an open set, $\alpha \in / N_0^n$, $\alpha \neq 0$ and $f, g \in L_1^{loc}(\Omega)$. The function g is a *weak derivative* of the function f of order α on Ω (i.e $g = D_\omega^\alpha f$) if $\forall \varphi \in C_0^\infty(\Omega)$ $\int_\Omega f D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega g\varphi dx$ where $|\alpha| = \alpha_1 + \ldots + \alpha_n$

Lemma 1: Let $\Omega \subset \Re^n$ be an open set, $\alpha \in / N_0^n, \alpha \neq 0$. Moreover let f be a function defined on Ω, which $\forall x \in \Omega$ has (an ordinary) derivative $(D^\alpha f)(x)$ and $D^\alpha f \in C(\Omega)$. Then $D^\alpha f = D_\omega^\alpha f$.

Proof: Integrating by parts α_j, times with respect to the variables $x_j, j = 1, \ldots, n$

$\forall \varphi \in C_0^\infty(\Omega)$

$$\int_\Omega f D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega D^\alpha f \varphi dx$$

That is :

$$\int_{\Omega} fD^{\alpha}\varphi dx = \int_{\Omega} fD^{\alpha_1+...+\alpha_n}\varphi dx$$

$$= (-1)\int_{\Omega} f' D^{(\alpha_1-1)+...+\alpha_n}\varphi dx$$

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$$= (-1)^{\alpha_1}\int_{\Omega} D^{\alpha_1} fD^{\alpha_2+...+\alpha_n}\varphi dx$$

Continuing this way

$$= (-1)^{\alpha_1+...+\alpha_n}\int_{\Omega} D^{\alpha_1+...+\alpha_n} f\varphi dx$$

$$= (-1)^{|\alpha|}\int_{\Omega} D^{\alpha} f\varphi dx$$

Remark 1: The assumption about the continuity of $D^{\alpha}f$ in Lemma 1 is essential. For example, the ordinary derivative of the function $f(x) = x^2 \sin\frac{1}{x^2}$ $(x \neq 0; f(0) = 0)$, which exists everywhere on $\Re$ is not a weak derivative of f on $\Re$ because it is not locally integrable on $\Re$

From the above definition it follows that if $g = D^{\alpha}_{\omega}f$ and the function h is equivalent to g on Ω, then $h = D^{\alpha}_{\omega}f$. Thus the weak derivative is not uniquely defined.

Remark 2: "$g = D_\omega^\alpha f$" means g is a weak derivative of the function f of order α on Ω. We also use $D_\omega^\alpha f$ for any weak derivative of the function f of order α on Ω. Thus, for example, the assertion " the function f has a weak derivative $D_\omega^\alpha f$" means, "the function, denoted by $D_\omega^\alpha f$, is a weak derivative of the function f of order α on Ω". From this point of view the relation

$D_\omega^\alpha f_1 + D_\omega^\alpha f_2 = D_\omega^\alpha (f_1 + f_2)$ Means the following:

If each of $D_\omega^\alpha f_k$, $k = 1,2,$ is a weak derivative (i.e. any of the weak derivatives) of the function f_k, then the function $D_\omega^\alpha f_1 + D_\omega^\alpha f_2$ is a weak derivative of the function $f_1 + f_2$. Finally we assume that $D_\omega^\alpha f = g$ means $g = D_\omega^\alpha f$. This will allow us to write the above relation in the more usual form

$$D_\omega^\alpha (f_1 + f_2) = D_\omega^\alpha f_1 + D_\omega^\alpha f_2$$

Lemma 2: Let $\Omega \subset \Re^n$ be an open set, $\alpha \in / N_0^\alpha, \alpha \neq 0, f, g, h \in L_1^{loc}(\Omega)$ and $g = D_\omega^\alpha f, h = D_\omega^\alpha f$ on Ω. Then $g \sim h$ on Ω.

Proof: $\forall \varphi \in C_0^\infty(\Omega)$

$$\int_\Omega f D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega g \varphi dx$$

and $$\int_\Omega f D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega h \varphi dx$$

$$\Rightarrow (-1)^{|\alpha|} \int_\Omega g\varphi dx = (-1)^{|\alpha|} \int_\Omega h\varphi dx \text{ on } \Omega$$

$$\Rightarrow \int_\Omega (g - h)\varphi = 0 \quad \text{on } \Omega \ \forall \varphi \in C_0^\infty(\Omega)$$

$\Rightarrow g - h = 0$ on Ω, by the main lemma of calculus of variations

$\Rightarrow g \sim h$ on Ω

Remark 3: Note that if a function $f \in L_1^{loc}(\Omega)$ has a weak derivative $D_\omega^\alpha f$ on Ω, then automatically $D_\omega^\alpha f \in L_1^{loc}(\Omega)$

Example 2: (For $n = 1, \Omega = \Re$), $|x|'_\omega = \mathbf{sgn}\, x$

Proof: $\forall \varphi \in C_0^\infty(\Re)$

$$\int_{-\infty}^{\infty} |x|\varphi'(x)dx = -\int_{-\infty}^{0} x\varphi'(x)dx + \int_{0}^{\infty} x\varphi'(x)dx$$

$$= -x\varphi(x)|_{-\infty}^{0} + x\varphi(x)|_0^\infty + \int_{-\infty}^{0}\varphi(x)dx - \int_{0}^{\infty}\varphi(x)dx$$

$$= \int_{-\infty}^{0}\varphi(x)dx - \int_{0}^{\infty}\varphi(x)dx \ldots(1)$$

And $$(-1)\int_{-\infty}^{\infty} \mathrm{sgn}\, x\varphi(x)dx = (-1)\left[-\int_{-\infty}^{0}\varphi(x)dx + \int_{0}^{\infty}\varphi(x)dx\right]$$

$$= \int_{-\infty}^{0}\varphi(x)dx - \int_{0}^{\infty}\varphi(x)dx \ldots(2)$$

Thus from (1) and (2) we can see that $|x|'_\omega = \mathbf{sgn}\, x$

Example 3: $(n = 1, \Omega = \Re)$ The weak derivative $(\mathrm{sgn}\, x)'_x$ does not exist on $\Re$.

Proof: Suppose $g \in L_1^{loc}(\Re)$ is a weak derivative.

Then $\forall \varphi \in C_0^\infty(\Re)$

$$\int_{-\infty}^{\infty} \mathrm{sgn}\, x\varphi'(x)dx = (-1)\int_{-\infty}^{\infty} g\varphi(x)dx$$

$$\Rightarrow -\int_{-\infty}^{0}\varphi'(x)dx + \int_{0}^{\infty}\varphi'(x)dx = -\varphi(0) - \varphi(0) = -2\varphi(0)$$

$$\Rightarrow (-1)\int_{-\infty}^{\infty} g\varphi(x)dx = -2\varphi(0) \ldots\ldots(*)$$

Taking $\varphi(x) = x\psi(x)$ with arbitrary $\psi \in C_0^\infty(\Re)$,

Then from (*)

$$\int gx\varphi(x)dx = 0. \text{ Thus } g \sim 0.$$

Which leads to a contradiction.

Remark 4: For each $f \in L_1^{loc}(\Omega)$ the derivative $D^\alpha f$ exist in the sense of the theory of distributions,

i.e., as a functional in $D'(\Omega)$:

$$\forall \varphi \in C_0^\infty(\Omega) \; (D^\alpha f, \varphi) = (-1)^{|\alpha|}(f, D^\alpha \varphi) = (-1)^{|\alpha|} \int fD^\alpha \alpha\varphi dx$$

Theorem 1: Definition 1 is equivalent to the following once:

i) Let $\Omega \subset \Re^n$ be an open set, $\alpha \in /N_0^n, \alpha \neq 0$ and $f, g \in L_1^{loc}(\Omega)$. The function g is a weak derivative of the function f of order α on Ω (briefly $g = D_\omega^\alpha f$) if there exist $\psi_k \in C^\infty(\Omega), k \in /N$, such that

$$\psi_k \to f, D^\alpha \psi_k \to g \textbf{ In } L_1^{loc}(\Omega) \textbf{ as } k \to \infty$$

ii) Let $\Omega \subset \Re$ be an open set, $\ell \in /N$ and $f, g \in L_1^{loc}(\Omega)$. The function g is a weak derivative of the function f of order ℓ on Ω (briefly $g = D_\omega^\alpha f \cong f_\omega^{(\ell)}$) if there is a function h equivalent to f on Ω, which has a locally absolutely continuous $(\ell - 1)^{th}$ ordinary derivative $h^{(\ell-1)}$ and such that its ordinary derivative $h^{(\ell)}$ is equivalent to g.

Proof: $(1 \Rightarrow (i))$

$$\int_\Omega f D^\alpha \phi dx = \int_\Omega \lim \psi_k D^\alpha \phi dx \quad \text{as } k \to \infty$$

$$= \lim \int_\Omega \psi_k D^\alpha \phi dx = (-1)^{|\alpha|} \lim \int_\Omega \phi D^\alpha \psi_k dx \qquad \text{as } k \to \infty$$

$$= (-1)^{|\alpha|} \int_\Omega \phi g dx$$

$((i) \Rightarrow 1)$:

For $k \in /N$ let χ_k is the characteristic function of set $\{x \in \Omega : |x| < k, \text{dist}(x, \partial\Omega) > 2/k\}$. Functions $\psi_k \in C^\infty(\Omega)$ (and even $\psi_k \in C_0^\infty(\Omega)$) are constructed in the following way: $\psi_k = A_{1/k}(f\chi_k)$ where $A_{1/k}$ is a mollifier.

By property of mollifiers $A_{1/k}(f\chi_k) \to f$ as $k \to \infty$ and $A_{1/k}(f\chi_k) \in C_0^\infty(\Omega)$

To show the other implication it is enough to consider the case in which $\Omega = (a, b)$

$((ii) \Rightarrow 1)$: **Since** $h^{(\ell-1)}$ is locally absolutely continuous on (a, b), it is possible $\forall \varphi \in C_0^\infty(\Omega)$ to integrate by parts ℓ times:

$$\int_a^b f\varphi^{(\ell)} d = \int_a^b h\varphi^{(\ell)} dx = (-1)^\ell \int_a^b h^{(\ell)} \varphi dx = (-1)^\ell \int_a^b g\varphi dx$$

$((i) \Rightarrow (ii))$: Let $\ell = 1$. Since $\psi_k \to f$ in $L_1^{loc}(a,b)$ as $k \to \infty$ there exist a subsequence k_s and a set $G \subset (a, b)$ such that measure$[(a,b) \setminus G] = 0$ and $\psi_{k_s}(x) \to f(x)$ as $s \to \infty$ for each $x \in G$. Choose $z \in G$ and pass to the limit in the equality

$$\psi_{k_s}(x) = \psi_{k_s}(z) + \int_z^x \psi'_{k_s}(y) dy$$

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i.e. $\lim_{k,s\to\infty}\psi_{k_s}(x)=\lim_{k,s\to\infty}[\psi_{k_s}(z)+\int_z^x\psi'_{k_s}(y)dy]$

Then $f(x)=f(z)+\int_z^x g(y)dy\equiv h(x)$ for each $x\in G$.

By the properties of absolutely continuous functions the function h (which is defined on (a,b) and equivalent to f) is locally absolutely continuous on (a,b) and $g\sim h'$.

If $\ell>1$, then apply the averaged Taylor's formula with $a<\alpha<x<\beta<b$ to the function ψ_{k_s}. Write it in the form

$$\psi_{k_s}(x)=\int_\alpha^\beta P(x,y)\psi_{k_s}(y)dy+\frac{1}{(\ell-1)!}\int_\alpha^\beta(x-y)^{\ell-1}(\int_\alpha^\beta\omega(u)du)\psi_{k_s}^{(\ell)}(y)dy$$

$$-\frac{1}{(\ell-1)!}\int_x^\beta(x-y)^{\ell-1}(\int_y^\beta\omega(u)du)\psi_{k_{sa}}(y)dy$$

(Here $P\in C([a,b]\times[a,b]),\forall y\in[a,b]$ $P(.,y)$ is a polynomial of order less than or equal to $\ell-1$ and $\omega\in C_0^\infty(\alpha,\beta)$.)

The notion of a weak derivative, as the notion of an ordinary derivative, is a local notion in the following sense. If the function $g\in L_1^{loc}(\Omega)$ is a weak derivative of the function $f\in L_1^{loc}(\Omega)$ of order $\alpha\in/N_0^n,\alpha\neq0$, on Ω locally, i.e. $\forall x\in\Omega$ there exists a neighborhood u_x of x such that g is a weak derivative of f of order α on u_x, then g is a weak derivative of f of order α on Ω.

Consider for an arbitrary $\varphi\in C_0^\infty(\Omega)$ a finite open covering $\{u_{x_k}\}_{k=1}^s$ of $\operatorname{supp}\varphi$ and a corresponding partition of unity $\{\psi_k\}_{k=1}^s$ i.e., a family of functions $\psi_k\in C_0^\infty(u_{x_k})$ which are such that $\sum_{k=1}^s\psi_k=1$ on $\operatorname{supp}\varphi$. Then $\varphi=\sum_{k=1}^s\varphi\psi_k$ on Ω and

$$\int_\Omega fD^\alpha\varphi dx=\sum_{k=1}^s\int_{u_{x_k}}fD^\alpha(\varphi\psi_k)dx=(-1)^{|\alpha|}\sum_{k=1}^s\int_{u_{x_k}}g\varphi\psi_k dx=(-1)^{|\alpha|}\int_\Omega g\varphi dx$$

For an open set $\Omega \subset \Re^n$ and $\alpha \in / N_0^n\ \alpha \neq 0$, let us denote by $G_\alpha(\Omega)$ the domain of the operator D_ω^α i.e. the subset of $L_1^{loc}(\Omega)$ consisting of all functions $f \in L_1^{loc}(\Omega)$, for which the weak derivative $D_\omega^\alpha f$ exists on Ω

We note that weak differentiation operator

$$D_\omega^\alpha : G_\alpha(\Omega) \to L_1^{loc}(\Omega)$$

is closed, i.e., if the function $f_k \in G_\alpha(\Omega)$ and the function $f, g \in L_1^{loc}(\Omega)$ are such that $f_k \to f$ in $L_1^{loc}(\Omega)$,

$D_\omega^\alpha f_k \to g$ in $L_1^{loc}(\Omega)$,

then $f \in G_\alpha(\Omega)$ and $D_\omega^\alpha f = g$

The operator D_ω^α considered as operator

$D_\omega^\alpha : G_\alpha(\Omega) \cap L_p(\Omega) \to L_p(\Omega)$, Where $1 \leq p \leq \infty$ is also closed.

Lemma 3:(weak differentiation under the integral sign)

Let $\Omega \subset \Re^n$ be an open set, $A \subset \Re^m$ a measurable set and let $\alpha \in / N_0^n, \alpha \neq 0$. Suppose that the function f is defined on $\Omega \times A$, for almost every $y \in A$ $f(.,y) \in L_1^{loc}(\Omega)$ and there exists a weak derivative $D_\omega^\alpha f(.,y)$ on Ω. Moreover, suppose that $f, D_\omega^\alpha f \in L_1(K \times A)$ for each compact $K \subset \Omega$. Then on Ω

$$D_\omega^\alpha \left(\int_A f(x,y)dy \right) = \int_A (D_\omega^\alpha f)(x,y)dy$$

(i.e. the function $\int_A (D_\omega^\alpha f)(x,y)dy$ is a weak derivative of $\int_A f(x,y)dy$ of order α

Proof: For all $\varphi \in C_0^\infty(\Omega)$ the functions $f(x,y)(D^\alpha\varphi)(x)$ and $(D_\omega^\alpha f)(x,y)\varphi(x)$ belongs to $L_1(\Omega \times A)$

Since

$$\int_{\Omega \times A} |f(x,y)(D^\alpha \varphi)(x)| dxdy \leq M \int_{\sup p\varphi \times A} |f|\, dxdy < \infty$$

where $M = \max_{x\in\Omega} |(D^{\alpha}\varphi)(x)|$

Now,

(1)... $\int_{\Omega\times A} (D_{\omega}^{\alpha}f)(x,y)\varphi(x)dxdy = \int_{\Omega} (\int_{A} (D_{\omega}^{\alpha}f)(x,y)dy)\varphi(x)dx$ by

Fubini's theorem

$= \int_{A} (\int_{\Omega} (D_{\omega}^{\alpha}f)(x,y)\varphi(x)dx)dy$ By Fubini's theorem

$= (-1)^{|\alpha|} \int_{A} (\int_{\Omega} f(x,y)(D^{\alpha}\varphi)(x)dx)dy$

(2).... $= (-1)^{|\alpha|} \int_{\Omega}(\int_{A} f(x,y)dy)(D^{\alpha}\varphi)(x)dx$ By Fubini's theorem

From the right side of (1) and (2) we can conclude

$D_{\omega}^{\alpha}(\int_{A} f(x,y)dy) = \int_{A} (D_{\omega}^{\alpha}f)(x,y)dy$

<u>Lemma 4</u>:(commutativity of weak differentiation and the mollifiers).

Let $\Omega \subset \Re^{n}$ be an open set, $\alpha \in / N_{0}^{n}, \alpha \neq 0, f \in L_{1}^{loc}(\Omega)$ and suppose that there exists a weak derivative $D_{\omega}^{\alpha}f$ on Ω.

Then $\forall\delta > 0$

$D^{\alpha}(A_{\delta}f) = A_{\delta}(D_{\omega}^{\alpha}f)$ on $\underline{\Omega_{\delta}}$

<u>Proof</u>: By the property of mollifiers $A_{\delta}(D_{\omega}^{\alpha}f) \in C^{\infty}(\underline{\Omega_{\delta}})$. Moreover,

$\forall x \in \Omega_{\delta}$

$(A_{\delta}f)(x) = \int_{B(0,1)} f(x-\delta z)\omega(z)dz$

Furthermore, $D_{\omega}^{\alpha}(f(.-\delta z)) = (D_{\omega}^{\alpha}f)(.-\delta z)$, on $\underline{\Omega_{\delta}}$, which follows from Definition 1

For $(x,z) \in \underline{\Omega_{\delta}} \times B(0,1)$, let $F(x,z) = f(x-\delta z)\,\omega(z)$

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and $G(x,y)=(D^{\alpha}_{\omega}f)(x-\delta z)\omega(z)$. Then for each compact $K \subset \underline{\Omega_{\delta}}$ the functions F,G belong to $L_1(K\times B(0,1))$, because they are measurable on $\underline{\Omega_{\delta}}\times B(0,1)$ and,

For example

$$\int_K\Big(\int_{B(0,1)}|f(x-\delta z)\omega(z)|\,dz\Big)dx \le M\int_K\Big(\int_{B(0,1)}|f(x-\delta z)dz\Big)dx$$

$$= M\int_K\Big(\int_{B(x,\delta)}|f(y)|\,dy\Big)dx \le M\int_K\Big(\int_{K^{\delta}}|f(y)|\,dy\Big)dx$$

$$= M\,\mathrm{meas}K\int_{K^{\delta}}|f(y)|\,dy < \infty$$

Here $M=\max_{z\in/R^n}|\omega(z)|$ and $\overline{K^{\delta}}\subset\Omega$ (because $K\subset\underline{\Omega_{\delta}}$)

Now the Lemma follows from lemma 1and 3: $\forall x\in\Omega$

$$D^{\alpha}((A_{\delta}f)(x)=D^{\alpha}_{\omega}\Big(\int_{B(0,1)}f(x-\delta z)\omega(z)dz\Big)=\int_{B(0,1)}D^{\alpha}_{\omega}(f(x-\delta z))\omega(z)dz$$

$$=\int_{B(0,1)}(D^{\alpha}_{\omega}f)(x-\delta z)\omega(z)dz=(A_{\delta}(D^{\alpha}_{\omega}f))(x)$$

Corollary 1: For $\Omega=\Re^n$

$$D^{\alpha}_{\omega}A_{\delta}=A_{\delta}D^{\alpha}_{\omega}$$

Corollary 2: For $\gamma\in/N^n_0$ and $\gamma\ge\alpha$, then

$$D^{\gamma}(A_{\delta}f)=\delta^{|\alpha|-|\gamma|}(D^{\gamma-\alpha}\omega)_{\delta}*D^{\alpha}_{\omega}f \quad \text{On } \underline{\Omega_{\delta}}$$

Proof:

$$D^{\gamma}(A_{\delta}f)=D^{\gamma-\alpha}(D^{\alpha}(A_{\delta}f))$$

$$=D^{\gamma-\alpha}(A_{\delta}(D^{\alpha}_{\omega}f)) \quad \text{By the above lemma}$$

$$=\delta^{|\alpha|-|\gamma|}(D^{\gamma-\alpha}\omega)_{\delta}*D^{\alpha}_{\omega}f \quad \text{On } \underline{\Omega_{\delta}}$$

In Definition 1 the weak derivative is defined directly (not by induction as the ordinary derivative). Therefore the question arises as to whether a weak derivative $D^{\beta}_{\omega} f$, where $\beta \leq \alpha, \beta \neq \alpha$, exists when, a weak derivative $D^{\alpha}_{\omega} f$ exists. The answer is negative:

Example 4: Set $\forall (x_1, x_2) \in \Re^2 \ f(x_1, x_2) = \operatorname{sgn} x_1 + \operatorname{sgn} x_2$. The derivatives $(\frac{\partial f}{\partial x_1})_{\omega}$ and $(\frac{\partial f}{\partial x_2})_{\omega}$ do not exist on $\Re^2$.

<u>Lemma 5</u>: Let $\Omega \subset \Re^n$ be an open set, $\ell \in /N, \ell \geq 2, f \in L_1^{loc}(\Omega)$ and suppose that for some $j = \overline{1, n}$ a weak derivative $(\frac{\partial^{\ell} f}{\partial x_j^{\ell}})_{\omega}$ exists on Ω. Then $\forall m \in /N$ satisfying $m < \ell$ a weak derivative $(\frac{\partial^m f}{\partial x_j^m})_{\omega}$ also exists on Ω.

<u>Proof</u>: For sufficiently large $k \in /N$ the functions $f_k = A_{1/k} f \in C^{\infty}(Q)$; Q is any open cube with faces parallel to the coordinate planes, which is such that $\overline{Q} \subset \Omega$.

Since $A_{1/k} \to f$ a.e on Q as $K \to \infty$

i.e. $f_k \to f$ in $L_1(Q)$ and.

$$\frac{\partial^{\ell} f_k}{\partial x_j^{\ell}} = A_{1/k}(\frac{\partial^{\ell} f}{\partial x_j^{\ell}}) \to \frac{\partial^{\ell} f}{\partial x_j^{\ell}} \text{ In } L_1(Q)$$

Moreover, applying the inequality

$$\left[\left\| \frac{\partial^m f}{\partial x_j^m} \right\|_{L_1(Q)} \leq c_1 (\|f\|_{L_1(Q)} + \left\| \frac{\partial^{\ell} f}{\partial x_j^{\ell}} \right\|_{L_1(Q)}) \right], \quad c_1 > 0$$

We have

$$\left\| \frac{\partial^m f_k}{\partial x_j^m} - \frac{\partial^m f_s}{\partial x_j^m} \right\|_{L_1(Q)} \leq c_1 (\|f_k - f_s\|_{L_1(Q)} + \left\| \frac{\partial^{\ell} f_k}{\partial x_j^{\ell}} - \frac{\partial^{\ell} f_s}{\partial x_j^{\ell}} \right\|_{L_1(Q)})$$

Consequently,

$$\lim_{k,s\to\infty}\left\|\frac{\partial^m f_k}{\partial x_j^m}-\frac{\partial^m f_s}{\partial x_j^m}\right\|_{L_1(Q)}=0$$

Because of the completeness of $L_1(Q)$ there exists a function $g_Q \in L_1(Q)$ such that $\frac{\partial^m f_k}{\partial x_j^m} \to g_Q$ in $L_1(Q)$ as $k \to \infty$

Since $f_k \to f$ in $L_1(Q)$ as well, by Definition 2 it follows that g_Q is a weak derivative of order m with respect to x_j on Q.

We note that if Q_1 and Q_2 are any intersecting admissible cubes then $g_{Q_1} = g_{Q_2}$ almost everywhere on $Q_1 \cap Q_2$, since both g_{Q_1} and g_{Q_2} are weak derivatives of f on $Q_1 \cap Q_2$.

Consequently, there exists a function $g \in L_1^{loc}(\Omega)$ such that $g = g_Q$ a.e on each admissible cube Q and g is a weak derivative of f on Q. Hence g is a weak derivative of f of order m with respect to x_j on Ω.

<u>Lemma 6</u>: Let $n \geq 2, \Omega \subset \Re^n$ be an open set, $\ell \in /N, \ell \geq 2$. $f \in L_1^{loc}(\Omega)$ and suppose that $\forall \alpha \in /N_0^n$ satisfying $|\alpha| = \ell$ a weak derivative $D_\omega^\alpha f$ exists on Ω Then $\forall \beta \in /N_0^n$ satisfying $0 < |\beta| < \ell$ a weak derivative $D_\omega^\beta f$ also exists on Ω.

<u>Proof</u>: If $Q = (a,b)^n$, then

$$\left\|D^\beta f\right\|_{L_1(Q)} = \left\|\dots\left\|\frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}}\left(\frac{\partial^{\beta_2+\dots+\beta_n} f}{\partial x_2^{\beta_2}\dots\partial x_n^{\beta_n}}\right)\right\|_{L_1(a,b)}\dots\right\|_{L_1(a,b)}$$

$$\leq c_1\left(\left\|\frac{\partial^{\beta_2+\dots+\beta_n} f}{\partial x_2^{\beta_2}\dots\partial x_n^{\beta_n}}\right\|_{L_1(Q)} + \left\|\frac{\partial^\ell f}{\partial x_1^{\ell-\beta_2-\beta_3-\dots-\beta_n}\partial x_2\dots\partial x_n^{\beta_n}}\right\|_{L_1(Q)}\right.$$

$$\leq \dots \leq c_2\left(\|f\|_{L_1(Q)} + \sum_{|\alpha|=1}\left\|D^\alpha f\right\|_{L_1(Q)}\right.$$

By writing f_k for f in this inequality and taking limits we see that it is possible to replace here the ordinary derivatives $D^{\beta}f$, $D^{\alpha}f$ by the weak ones $D^{\beta}_{\omega}f$ and $D^{\alpha}_{\omega}f$ respectively.

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CHAPTER TWO
SOBOLEV SPACES

Definition 1: let $\Omega \subset \Re^n$ be an open set, $\ell \in /N, \ell \le p \le \infty$ .The function f belongs to the Sobolev space $W_p^\ell(\Omega)$ if $f \in L_p(\Omega)$, if it has weak derivative $D_\omega^\alpha f$ on Ω for all $\alpha \in /N_0^n$ satisfying $|\alpha| = \ell$ and

$$\|f\|_{W_p^\ell(\Omega)} = \|f\|_{L_p(\Omega)} + \sum_{|\alpha|=\ell} \|D_\omega^\alpha f\|_{L_p(\Omega)} < \infty$$

Remark 1: In one-dimensional case this definition is by Definition 3 of the previous chapter equivalent to the following. The function f is equivalent to a function h on Ω, for which the (ordinary) derivative $h^{(\ell-1)}$ is locally absolutely continuous on Ω and

$$\|f\|_{W_p^\ell(\Omega)} = \|f\|_{L_p(\Omega)} + \|f_\omega^{(\ell)}\|_{L_p(\Omega)} = \|h\|_{L_p(\Omega)} + \|h^{(\ell)}\|_{L_p(\Omega)} < \infty$$

Moreover, if $\Omega = (a,b)$ is a finite interval, the limits $\lim_{x \to a^+} h(x)$ and $\lim_{x \to b^-} h(x)$ exist and one may define h on $[a,b]$ by setting $h(a)$ and $h(b)$ to be equal to those limits. Then $h^{(s)}, s = 1,\dots,\ell-1$ exists and $h^{(\ell-1)}$ is absolutely continuous on $[a,b]$

This follows from the Taylor expansion

$$h^{(s)}(x) = \sum \frac{h^{(s+k)}(x_0)}{k!}(x - x_0)^k + \frac{1}{(\ell - s - 1)!} \int_{x_0}^{x} (x-u)^{\ell-s-1} h^{(\ell)}(u) du,$$

Where $x, x_0 \in (a,b)$ and $s = 1,\dots,\ell-1$. Since $h^{(\ell)} \in L_p(a,b)$, hence $h^{(\ell)} \in L_1(a,b)$, the limits $\lim_{x \to a^+} h(x)$ and $\lim_{x \to b^-} h(x)$ exist.

Consequently, the right derivatives $h^{(s)}(a)$ and the left derivatives $h^{(s)}(b)$ exist and $h^{(s)}(a) = \lim_{x \to a^+} h(x)$, $\quad h^{(s)}(b) = \lim_{x \to b^-} h(x)$

Finally, since $h^{(\ell-1)}(x) = h^{(\ell-1)}(x_0) + \int_{x_0}^{x} h^{(\ell)}(u)du$ for all $x, x_0 \in [a,b]$ and $h^{(\ell)} \in L_1(a,b)$, it follows that $h^{(\ell-1)}$ is absolutely continuous on $[a,b]$

Remark 2: By lemma 6 of chapter one $D_\omega^\alpha f$ exists also for $|\alpha| < \ell$. Moreover, $D_\omega^\alpha f \in L_p^{loc}(\Omega)$ but in general $D_\omega^\alpha f \notin L_p(\Omega)$

Theorem 1: Let $\Omega \subset \Re^n$ be an open set, $\ell \in /N, 1 \le p \le \infty$. Then $W_p^\ell(\Omega)$ is a Banach space

Proof: Obviously $W_p^\ell(\Omega)$ is a normed space

To prove completeness, let $\{f_k\}_{k \in /N}$ be a Cauchy sequence. Since $L_p(\Omega)$ is complete there

exists $f \in L_p(\Omega)$ and $f_\alpha \in L_p(\Omega)$ where $\alpha \in /N_0^n, |\alpha| = \ell$, such that $f_k \to f$ and $D_\omega^\alpha f_k \to f_\alpha$ in $L_p(\Omega)$

From the closed ness of weak differentiation it follows that $f_\alpha = D_\omega^\alpha f$. Hence $f_k \to f$ in $W_p^\ell(\Omega)$.

Remark 3: The norm

$$\|f\|_{W_p^\ell(\Omega)} = \|f\|_{L_p(\Omega)} + \sum_{|\alpha|=\ell} \|D_\omega^\alpha f\|_{L_p(B)} \text{...........}(*)$$

is equivalent to

$$\|f\|_{W_p^\ell(\Omega)}^{(1)} = \Big(\int_\Omega (|f|^p + \sum_{|\alpha|=\ell} |D_\omega^\alpha f|^p) dx\Big)^{1/p}$$

for $1 \le p < \infty$ and to

$$\|f\|_{W_\infty^\ell(\Omega)}^{(1)} = \max\left\{\|f\|_{L_\infty(\Omega)}, \max_{|\alpha|=\ell} \|D_\omega^\alpha f\|_{L_\infty(\Omega)}\right\}$$

for $p=\infty$,i.e., $\forall f \in W_p^\ell(\Omega)$

$$c_3\|f\|_{W_p^\ell(\Omega)}^{(1)} \le \|f\|_{W_p^\ell(\Omega)} \le c_4\|f\|_{W_p^\ell(\Omega)}^{(1)}$$

Where $c_3, c_4 > 0$ are independent f. This follows, with c_3, c_4 depending only on n, p and ℓ, from Hölder's and Jenssen's inequalities for finite sums.

If $p=2$, then $W_2^\ell(\Omega)$ is a Hilbert space with inner product

$$(f,g)_{W_2^\ell(\Omega)} = \int_\Omega (f\bar{g} + \sum_{|\alpha|=\ell} D_\omega^\alpha f \overline{D_\omega^\alpha g})dx$$

and $\|f\|_{W_2^{\ell 1}(\Omega)}^{(1)}$ is a Hilbert norm, i.e., $\|f\|_{W_2^\ell}^{(1)} = (f,f)_{W_2^\ell(\Omega)}^{\frac{1}{2}}$

Let us consider the weak gradient of order ℓ

$$\nabla_\omega^\ell f = \left(\left(\frac{\partial^\ell f}{\partial x_{i_1} \ldots \partial x_{i_l}}\right)_\omega\right)_{i_1,\ldots,i_l=1}^n$$

Then

$$\left|\nabla_\omega^\ell f\right|^2 = \sum_{i_1,\ldots,i_\ell=1}^n \left|\left(\frac{\partial^\ell f}{\partial x_{i_1} \ldots \partial x_{i_\ell}}\right)_\omega\right|^2 = \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!}\left|D_\omega^\alpha f\right|^2$$

and norm (*) is equivalent to

$$\|f\|_{W_p^\ell(\Omega)}^{(2)} = \left(\int_\Omega (|f|^p + \left|\nabla_\omega^\ell f\right|^p)dx\right)^{1/p}$$

We also note that for even ℓ $\forall f \in C_0^\infty(\Omega)$

$$\int_\Omega \left|\nabla^\ell f\right|^2 dx = \int_\Omega \left|\Delta^{\ell/2} f\right|^2 dx$$

Where Δ is the Laplacian. Hence for such f,

$$\|f\|_{W_2^\ell(\Omega)}^{(2)} = \left(\int_\Omega (|f|^2 + \left|\Delta^{\ell/2} f\right|^2)dx\right)^{1/2}$$

We shall also need the following variant of Sobolev spaces.

Definition 2: Let $\Omega \subset \Re^n$ be an open set, $\ell \in /N$, $1 \le p \le \infty$. The function f belongs to the semi-normed Sobolev space $w_p^\ell(\Omega)$ if $f \in L_1^{loc}(\Omega)$, if it has weak derivatives $D_\omega^\alpha f$ on Ω for all $\alpha \in /N_o^n$ satisfying $|\alpha| = \ell$ and

$$\|f\|_{w_p^\ell(\Omega)} = \sum_{|\alpha|=\ell} \|D_\omega^\alpha f\|_{L_p(\Omega)} < \infty$$

The space $w_p^\ell(\Omega)$ is a complete space. We can show the completeness similarly as the $W_p^\ell(\Omega)$ space. Thus $w_p^\ell(\Omega)$ is a Semi-Banach space, because the condition $\|f\|_{w_p^\ell(\Omega)} = 0$ is equivalent to the following one: on each connected component of an open set Ω, f is equivalent to a polynomial of degree less than or equal to $\ell - 1$ (in general different polynomials for different components)

Remark 4: Let $\Omega \subset \Re^n$ be bounded domain and B be a ball such that $B \subset \Omega, \ell \in /N, 1 \le p \le \infty$. We denote by $L_p^\ell(\Omega)$ the Banach space, which is the set $w_p^\ell(\Omega)$, equipped with the norm

$$\|f\|_{L_p^\ell(\Omega)} = \|f\|_{L_1(B)} + \|f\|_{w_p^\ell(\Omega)}$$

(It is a norm, because if $\|f\|_{L_p^\ell(\Omega)} = 0$, then $\|f\|_{w_p^\ell(\Omega)} = 0$ it follows that f is equivalent to a polynomial of degree less than or equal to $\ell - 1$, and from $\|f\|_{L_p(B)} = 0$ it follows that $f \sim 0$ on Ω) .For different balls with closure in Ω these norms are equivalent. One can replace $\|f\|_{L_1(B)}$ by $\|f\|_{L_p(B)}$ and the corresponding norms will again be equivalent. Note that by definition

$$L_p^\ell(\Omega) = w_p^\ell(\Omega)$$

Let F[f] denote the Fourier transform of the function f:

For $f \in L_1(\Re^n)$ and $\forall \xi \in \Re^n$

$$F[f](\xi) = (2\pi)^{-n/2} \int_{\Re^n} e^{-i\pi\xi} f(x)dx$$

For $f \in L_2(\Re^n)$

$$F[f] = \lim_{k \to \infty} F[f\chi_k]$$

Where χ_k is the characteristic function of a ball $B(0,k)$ and the limit is taken in $L_2(\Re^n)$. It exists for each $f \in L_2(\Re^n)$ and

$$\|F[f]\|_{L_2(\Re^n)} = \|f\|_{L_2(\Re^n)} \qquad \text{(Parseval's equality)}$$

<u>Lemma 1</u>: For all $\ell \in /N$ and $f \in W_2^\ell(\Re^n)$

$$\|\nabla_\omega^\ell f\|_{L_2(\Re^n)} = \left\||\xi|^\ell (F[f])(\xi)\right\|_{L_2(\Re^n)} \qquad \dots (1)$$

and $$\|f\|^{(2)}_{w_2^\ell(\Re^n)} = \left\|(1+|\xi|^{2\ell})^{1/2}(Ff)(\xi)\right\|_{L_2(\Re^n)} \qquad \dots (2)$$

<u>Proof</u>: For $f \in L_1(\Re^n) \cap W_2^\ell(\Re^n)$ staring with Definition 3 we can show $F(D_\omega^\alpha f)(\xi) = (i\xi)^\alpha (Ff)(\xi)$ on $\Re^n$.

To obtain (1) and (2) apply (*) and the identity

$$\sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} |\xi^{2\alpha}| = \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha! \cdots \alpha_n!} (\xi_1^2)^{\alpha_1} \dots (\xi_n^2)^{\alpha_n} = |\xi|^{2\ell}.$$

<u>Lemma 2</u>: Let $\Re^n$ be an open set, $M \geq 0$ and suppose that $\forall x, y \in \Omega$

$$|f(x) - f(y)| \leq M|x - y| \qquad \dots (3)$$

Then $f \in w_\infty^1(\Omega)$, the gradient $(\nabla f)(x)$ exists for almost every $x \in \Omega$ and

$$|\nabla f(x)| \leq M \text{ a.e on } \Omega \qquad \dots (4)$$

If, in addition, Ω is a convex set, then the condition (3) is equivalent to if the following:

$f \in C(\Omega) \cap w_\infty^1(\Omega)$ and (4) holds.

Proof: Let $j \in \{1, \ldots, n\}$, $x = (x^{(j)}, x_j)$, $x^{(j)} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$,

$\Omega^{(j)} = \Pr_{x_j=0} \Omega \subset \Re^{n-1}$ and $Dx^{(j)} \in \Omega^{(j)}$ $\Omega_{(j)}(x^{(j)}) = \Pr_{0x_j} \Omega \cap \ell_x^{(j)} \subset \Re$, where $\ell_x^{(j)}$ is a straight line parallel to the axis Ox_j and passing through the point $(x^{(j)}, 0)$.

Deduce from (3) that for almost every $x_j \in \Omega_{(j)}(x^{(j)})$ there exists

$$\frac{\partial f}{\partial x_j}(x) = \frac{\partial f}{\partial xj}(x^{(j)}, x_j) \text{ and } \left|\frac{\partial f}{\partial x_j}(x)\right| \leq M.$$

Integrating by parts

(which is possible because $\forall x^{(j)} \in \Omega^{(j)}$ the function $f(x^{(j)},.)$ is locally absolutely continuous on $\Omega_{(j)}(x^{(j)})$ show that the ordinary derivative $\frac{\partial f}{\partial x_j}$ (existing thus almost everywhere on Ω) is a weak derivative $\left(\frac{\partial f}{\partial x_j}\right)_\omega$ on Ω.

If Ω is convex, then to obtain the converse result use Lemma 4 of chapter one and the inequality:

$$\|A_\delta f\|_{L_p(\Omega)} \leq \|f\|_{L_p(\Omega \cap G^\delta)}. \text{ where } c = \|f\|_{L_p(/R^n)}$$

to prove that $\forall x, y \in \Omega$ and $0 < \delta <$ dist $([x, y], \partial\Omega)$ the following inequalities for the mollifier A_δ with a non-negative kernel are satisfied.

$$|(A_\delta f)(x) - (A_\delta f)(y) \leq \|\nabla A_\delta f\|_{C([x,y])}|x - y|$$
$$= \|A_\delta \nabla_\omega f\|_{C([x,y])} |x - y| \leq \|\nabla_\omega f\|_{L\infty([x,y]^\delta)} |x - y|$$
$$\leq \|\nabla_\omega f\|_{L\infty(\Omega)}|x-y| = \|\nabla f\|_{L\infty(\Omega)} |x - y| \leq M |x - y|$$

(note also that for $f \in C(\Omega) \cap w^1_\infty(\Omega)$ the gradient ∇f exists a.e on Ω and $\nabla f = \nabla_\omega f$ on Ω). Now it is enough to pass, apply the result from the introductory art (i.e for $\Omega \subset \Re^n$, $f \in L_1^{loc}(\Omega)$ then $A_\delta f \in c^\infty(\underline{\Omega}_\delta)$ and $A_\delta f \to f$ a.e. On Ω) to the limit as $\delta \to 0^+$.

Corollary: If $\Omega \subset \Re^n$ is a convex open set, the $g \in w^1_\infty(\Omega)$ if, and only if, it is equivalent to a function f satisfying (3) with some $M \geq 0$. (Given a function g, the function f is defined uniquely.)

Moreover, denote by M^* the minimal possible value of M in (3). Then

$\|\nabla g\|_{L_\infty(\Omega)} = M^*$ and, hence,

$M^* \leq \|g\|_{W^\ell_\infty(\Omega)} \leq nM^*$

Proof: The first statement is just a reformulation of the above lemma for the case of convex open sets. The second one follows from the definition of $\|g\|_{W^\ell_\infty(\Omega)}$ and $\nabla_\omega g$.

Lemma 3: **(Minkowski's inequality for Sobolev spaces):**

Let $\Omega \subset \Re^n$ be an open set and $A \subset \Re^m$ a measurable set, $\ell \in N$, $1 \leq p \leq \infty$. Moreover, suppose that f is a function measurable on $\Omega x A$ and that $f(.,y) \in W^\ell_p(\Omega)$ for almost every $y \in A$. Then

$$\left\| \int_A f(x,y)dy \right\|_{W^\ell_p(\Omega)} \leq \int_A \|f(x,y)\|_{W^\ell_p(\Omega)} dy \qquad ....(1)$$

(the norm $\|f(x,y)\|_{W^\ell_p(\Omega)}$ is calculated with respect to x).

Proof: Let the right-hand side of (1) be finite, then by Holder's inequality for each compact $K \subset \Omega$

$$\int_A \left(\int_K |f(x,y)| dx \right) dy < \infty \text{ and } \int_A \left(\int_K |D^\alpha_\omega f(x,y)| dx \right) dy < \infty$$

$\forall \alpha \in / N^n_0$ Where $|\alpha| = \ell$.

Hence by Fubini's theorem the function f, being measurable on K x A, belongs to $L_1(K x A)$. Now the inequality (1) follows from our previous Lemma 3 of Chapter one and Minkowski's inequality for $L_p(\Omega)$:

$$\left\| \int_A f(x,y)dy \right\|_{W_p^\ell(\Omega)} = \left\| \int_A f(x,y)dy \right\|_{L_p(\Omega)} + \sum_{|\alpha|=\ell} \left\| D_\omega^\alpha \int_A f(x,y)dy \right\|_{L_p(\Omega)}$$

$$\leq \int_A \|f(x,y)\|_{L_p(\Omega)} dy + \sum_{|\alpha|=\ell} \int_A \|D_\omega^\alpha f(x,y)\|_{L_p(\Omega)} dy = \int_A \|f(x,y)\|_{W_p^\ell(\Omega)} dy$$

Lemma 4: **(Multiplication by C_0^∞-functions):** Let $\Omega \subset \Re^n$ be an open set, $\ell \in /N$, $1 \leq p \leq \infty$. Then $\forall \varphi \in C_0^\infty(\Omega)$ there exists $c_\varphi > 0$ such that $\forall f \in W_p^\ell \Omega)$

$$\|\varphi f\|_{W_p^\ell(\Omega)} \leq c_\varphi \|f\|_{W_p^\ell(\Omega)} \quad \cdots \quad (1)$$

Proof: Let $\alpha \in / N_0^n$ satisfy $|\alpha| = \ell$. By Lemma 6 $\forall \beta \in / N_0^n$ where $|\beta| \leq \ell$ there exist $D_\omega^\beta f$, therefore on Ω Leibnitz' formula holds:

$$D_\omega^\alpha(\varphi f) = \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} D^{\alpha-\beta}\varphi \, D_\omega^\beta f \qquad \ldots(2)$$

Let $Q_j \subset \Omega$, $j = 1, \ldots, s$, be open cubes with faces parallel to the coordinate planes such that $\operatorname{supp}\varphi \subset \bigcup_{j=1}^{s} Q_j$. Then applying the inequality

$$\left(\|D_\omega^\beta f\|_{L_p(Q)} \leq M \left(\|f\|_{L_p(Q)} + \sum_{|\alpha|=\ell} \|D_\omega^\alpha f\|_{L_p(Q)} \right) \right) \text{twice}$$

we get

$$\|D_\omega^\alpha(\varphi f)\|_{L_p(Q)} \leq 2^\ell \max \|D^r \varphi\|_{C(\operatorname{supp} p\varphi)} \sum_{|\beta| \leq \ell} \|D_\omega^\beta f\|_{L_p(\operatorname{sup} p\varphi)}$$

$$\leq 2^\ell \left(\sum_{|r| \leq \ell} \sum_{j=1}^{s} \|D^r \varphi\|_{C(Q_j)} \right) \left(\sum_{|\beta| \leq \ell} \sum_{j=1}^{s} \|D_\omega^\beta \varphi\|_{L_p(Q_j)} \right)$$

$\leq M \|\varphi\|_{C^\ell(\Omega)} \|f\|_{W_p^\ell(\Omega)}$ where M depends only on ℓ, Ω and supp φ.

Lemma 5: Let $\ell \in /N$, $1 \leq p < \infty$, $\eta \in C_0^\infty(\Re^n)$ be a function of "cap shaped" type such that

$\eta = 1$ on $B(0, 1)$ and $\forall s \in /N$,

$$\forall x \in /R^n \quad \eta_s(x) = \eta\left(\frac{x}{s}\right). \text{ Then } \forall f \in W_p^\ell(\Re^n)$$

$$\eta_s f \to f \text{ in } W_p^\ell(\Re^n) \text{ as } s \to \infty$$

Proof: First of all $\forall g \in L_p(\Re^n)$ where $1 \leq p < \infty$

$$\|(\eta_s - 1)\, g\|_{L_p(\Re^n)} \leq \|g\|_{L_p({}^c B(0,s))} \to 0$$

as $s \to \infty$.

From (2) it follows that $\forall\, \alpha \in / N_0^n$ where $|\alpha| = \ell$

$$\|D_\omega^\alpha(\eta_s f - f)\|_{L_p(\Re^n)}$$

$$\leq \|(\eta_s - 1) D_\omega^\alpha f\|_{L_p(\Re^n)} + \sum_{0 \leq \beta \leq \alpha, \beta \neq 0} \binom{\alpha}{\beta} \|D^{\alpha-\beta}\eta_s D_\omega^\beta f\|_{L_p(\Re^n)}$$

$$\leq \|\eta_s D_\omega^\alpha f - D_\omega^\alpha f\|_{L_p(\Re^n)} + \frac{M}{s} \sum_{0 \leq \beta \leq \alpha, \beta \neq 0} \|D_\omega^\beta f\|_{L_p(\Re^n)}$$

Remark 5: For $p = \infty$ the above lemma does not hold because, for instance, for $f = 1$ on $/R^n$ $\forall s \in / N$ $\|\eta_s f - f\|_{L_\infty(\Re^n)} = 1$. However, $\eta_s f \to f$ a.e, in $\Re^n$ and $\|\eta_s f\|_{W_\infty^\ell(\Re^n)} \to \|f\|_{W_\infty^\ell(\Re^n)}$ as $s \to \infty$

If $\Omega \subset \Re^n$ is a measurable set and $1 \leq p < \infty$, then each function $f \in L_p(\Omega)$ is continuous with respect to translation i.e.,

$$\lim_{h \to 0} \|f_0(x+h) - f(x)\|_{L_p(\Omega)} = 0 \qquad \ldots(\alpha)$$

The analogous result is valid for Sobolev spaces.

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<u>Lemma 6</u>: (continuity with respect to translation for Sobolev spaces)

Let $\Omega \subset \Re^n$ be an open set, $\ell \in N$, $1 \le p < \infty$. Then $\forall f \in W_p^\ell(\Omega)$

$$\lim_{h\to 0} \|f_0(x+h) - f(x)\|_{W_p^\ell(\Omega_{\{h\}})} = 0 \qquad \ldots \qquad (\beta)$$

where $h \in \Re^n$, $\Omega_{\{h\}} = \{x \in \Omega: x + h \in \Omega\}$, and $\forall f \in \left(W_p^\ell\right)_o(\Omega)$

$$\lim_{h\to 0} \|f_0(x+h) - f(x)\|_{W_p^\ell(\Omega)} = 0$$

<u>Proof</u>: equation (β) follows from (α) because

$$\|f(x+h) - f(x)\|_{w_p^\ell(\Omega\{h\})}$$

$$= \|f(x+h) - f(x)\|_{L_p(\Omega\{h\})} + \sum_{|\alpha|=\ell} \left\|\left(D_\omega^\alpha f\right)(x+h) - \left(D_\omega^\alpha f\right)(x)\right\|_{L_p(\Omega_{\{h\}})}$$

$$\le \|f_o(x+h) - f(x)\|_{L_p(\Omega)} + \sum_{|\alpha|=\ell} \left\|\left(D_\omega^\alpha f\right)_0(x+h) - \left(D_\omega^\alpha f\right)(x)\right\|_{L_p(\Omega)}.$$

If $f \in \left(W_p^\ell\right)_o(\Omega)$, then $\forall \alpha \in / N_0^n$ satisfying $|\alpha| = \ell$ we have $\left(D_\omega^\alpha f\right)_o = D_\omega^\alpha(f_o)$ on $\Re^n$, which easily follows from Definition 1, and thus $f_o \in W_p^\ell(\Re^n)$. Therefore

$$\|f_0(x+h) - f(x)\|_{w_p^\ell(\Omega)} \le \|f_o(x+h) - f(x)\|_{w_p^\ell(\Re^n)}$$

then the theorem follows.

CHAPER THREE

SOBOLEV'S INTEGRAL REPRESENTATION (one dimensional case)

Let $-\infty < a < b < \infty$,

$$\omega \in L_1(a, b)\ ,\ \int_a^b \omega dx = 1$$

and suppose that the function f is absolutely continuous on [a, b]. Then the derivative f' exists almost everywhere on [a, b], $f' \in L_1(a, b)$ and $\forall x, y \in [a, b]$ we have $f(x) = f(y) + \int_y^x f'(u)du$. Multiplying this equality by $\omega(y)$ and integrating with respect to y from a to b we get

$$f(x) = \int_a^b f(y)\omega(y)dy + \int_a^b \left(\int_y^x f'(u)du \right) \omega(y)dy.$$

Interchanging the order of integration we obtain

$$\int_a^b \left(\int_y^x f'(u)du \right) \omega(y)dy = \int_a^x \left(\int_y^x f'(u)du \right) \omega(y)dy - \int_x^b \left(\int_x^y f'(u)du \right) \omega(y)dy$$

$$= \int_a^x \left(\int_a^u \omega(y)dy \right) f'(u)du - \int_x^b \left(\int_u^b \omega(y)dy \right) f'(u)du = \int_a^b \Lambda(x, y)f'(y)dy$$

$$\text{where } \Lambda(x, y) = \begin{cases} \int_a^y \omega(u)dy, & a \le y \le x \le b \\ -\int_y^b \omega(u)du, & a \le x < y \le b \end{cases} \qquad \ldots\ \ldots.(1)$$

Hence $\forall x \in (a, b)$

$$f(x) = \int_a^b f(y)\omega(y)dy + \int_a^b \Lambda(x, y)f'(y)dy \quad \ldots.\ \ldots(2)$$

Note that Λ is bounded:

$$\forall_{x,y} \in [a, b],\ |\Lambda(x, y)| \le \|\omega\|_{L_1[a,b]}$$

Let us consider two limiting cases of (2). The first one corresponds to ω = const, hence, $\forall x \in (a, b)$ we have $\omega(x) = (b-a)^{-1}$.

Then $\forall x \in [a, b]$

$$f(x) = \frac{1}{b-a}\int_a^b f(y)dy + \int_a^x \frac{y-a}{b-a} f'(y)dy - \int_x^b \frac{b-y}{b-a} f'(y)dy \quad \ldots(3)$$

To obtain another limiting case we take

$$\omega = \frac{1}{2m}\left(\chi_{(a,a+\frac{1}{m})} + \chi_{\left(b-\frac{1}{m},b\right)}\right),$$ where $\chi_{(\alpha,\beta)}$ denotes the characteristic function of an interval (α, β), $m \in /N$ and $m \geq 2(b-a)^{-1}$. Letting $m \to \infty$ we find: $\forall x \in [a, b]$

$$f(x) = \frac{f(a)+f(b)}{2} + \frac{1}{2}\int_a^b \mathbf{sgn}(x-y)f'(y)dy \quad \ldots\ldots\ldots\ldots (4)$$

from (3) , it follows that

$$|f(x)| \leq \frac{1}{b-a}\int_a^b |f'|dy + \int_a^b |f'|dy \text{ for all } x \in [a, b] \ldots\ldots(5)$$

If $f \in (W_1^1)^{loc}(a, b)$, then f is equivalent to a function, which is locally absolutely continuous on (a, b) (its ordinary derivative, which exists almost every where on (a,b), is a weak derivative f'_ω of f). Consequently, (2), (3) and (5) hold for almost every $x \in (a, b)$ if f' to replaced by f'_ω.

Let now $-\infty \leq a < b \leq \infty$, $x_0 \in (a, b)$, $\ell \in /N$ and suppose that the derivative $f^{(\ell-1)}$ exists and is locally absolutely continuous on (a, b). Then the derivative $f^{(\ell)}$ exists almost everywhere on (a, b), $f^{(\ell)} \in L_1^{loc}$ (a, b) and by Taylor's formula with the remainder written in an integral form $\forall x, x_0 \in (a, b)$.

$$f(x) = \sum_{k=0}^{\ell-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{(\ell-1)!}\int_{x_0}^{x}(x-u)^{\ell-1}f^{(\ell)}(u)du.$$

$$= \sum_{k=0}^{\ell-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{(x-x_0)^\ell}{(\ell-1)!}\int_0^1 (1-t)^{\ell-1} f^{(\ell)}(x_0+t(x-x_0))dt \ldots..(6)$$

Theorems 1: Let $\ell \in /N_0$, $-\infty \le a < \alpha < \beta \le \infty$ and

$$\omega \in L_1(\Re),\ \operatorname{supp}\omega \subset [\alpha, \beta],\ \int_{\Re} \omega dx = 1$$

Moreover, suppose that the derivative $f^{(\ell-1)}$ exists and is locally absolutely continuous on (a, b).

Then $\forall x \in (a, b)$

$$f(x) = \sum_{k=0}^{\ell-1} \frac{1}{k!}\int_a^b f^{(k)}(y)(x-y)^k \omega(y)dy + \frac{1}{(\ell-1)!}\int_a^b (x-y)^{\ell-1}\Lambda(x,y) f^{(\ell)}(y)dy \quad \ldots (7)$$

$$= \sum_{k=0}^{\ell-1} \frac{1}{k!}\int_\alpha^\beta f^{(k)}(y)(x-y)^k \omega(y)dy + \frac{1}{(\ell-1)!}\int_{a_x}^{b_x} (x-y)^{\ell-1}\Lambda(x,y) f^{(\ell)}(y)dy;$$

where $a_x = x$, $b_x = \beta$ for $x \in (a, \alpha]$; $a_x = \alpha$, $b_x = \beta$ for $x \in (\alpha, \beta)$;...........(8)

$a_x = \alpha$, $b_x = x$ for $x \in [\beta, b)$.

Proof: The integrated remainder in (6) takes the form in (7) after interchanging the order of integration:

$$\int_a^b \left(\int_y^x (x-u)^{\ell-1} f^{(\ell)}(u)du \right) \omega(y)dy = \int_a^x \omega(y) \left(\int_y^x (x-u)^{\ell-1} f^{(\ell)}(u)du \right) dy$$

$$- \int_x^b \omega(y) \left(\int_y^x (x-u)^{\ell-1} f^{(\ell)}(u)du \right) dy = \int_a^x (x-u)^{\ell-1} \left(\int_a^u \omega(y)dy) f^{(\ell)}(u)du \right)$$

$$- \int_x^b (x-u)^{\ell-1} \left(\int_u^b \omega(y)dy) f^{(\ell)}(u)du \right) = \int_a^b (x-y)^{\ell-1}\Lambda(x,y) f^{(\ell)}(u)dy$$

Finally, since $\operatorname{supp}\omega \subset [\alpha, \beta]$, it follows that $\Lambda(x,y) = 0$ if $y \in (a, a_x) \cup (bx, b)$ and, hence (8) holds.

Theorem 2: Let $\ell \in N$, $-\infty \le a < \alpha < \beta \le b \le \infty$.

$$\omega \in L_1(\Re),\ \operatorname{supp}\omega \subset [x, \beta],\ \int_{/R} \omega dx = 1$$

and $f \in \left(W_1^{\ell}\right)^{loc}(a, b)$. Then for almost every $x \in (a, b)$

$$f(x) = \sum_{k=0}^{\ell-1} \frac{1}{k!} \int_{\alpha}^{\beta} f_{\omega}^{(k)}(y)\,(x-y)^k\,\omega(y)\,dy + \frac{1}{(\ell-1)!} \int_{a_x}^{b_x} (x-y)^{\ell-1} \Lambda(x,y) f^{(\ell)} \omega(y) dy \ldots (9)$$

where a_x and b_x are as defined in Theorem 1.

Proof: set $a(\delta) = \max\{a + \delta, -\frac{1}{\delta}\}$, $b(\delta) = \min\{b - \delta, \frac{1}{\delta}\}$ for sufficiently small $\delta > 0$,

then $[\alpha, \beta] \subset (a(\delta), b(\delta))$ and $(a(\delta))_x = a_x$, $(b(\delta))_x = b_x$

for each $x \in (a(\delta), b(\delta))$, we have $\forall x \in (a(\delta), b(\delta))$

$$(A_\gamma f)(x) = \sum_{k=0}^{\ell-1} \frac{1}{k!} \int_{\alpha}^{\beta} \left(A_\gamma f)^{k}\right)(y)\,(x-y)^k\,\omega(y)dy, \text{ where } 0 < \gamma < \delta.$$

$$= \frac{1}{(\ell-1)!} \int_{a_x}^{b_x} (x-y)^{\ell-1} \Lambda(x,y) (A_\gamma f)^{(\ell)}(y) dy$$

But $f_{\omega}^{(k)}$ exists on (a, b) by Lemma 5 of chapter one , where $k = 1, \ldots, \ell - 1$

and by Lemma 4 $(A_\gamma f)^{(k)} = A_\gamma\left(f_{\omega}^{(k)}\right)$ on $(a(\delta), b(\delta))$ where $k = 1, \ldots, \ell$.

consequently, $\forall x \in (a(\delta), b(\delta))$

$$\left| \int_{\alpha}^{\beta} (A_\gamma f)^{(k)}(y)(x-y)^k \omega(y) dy - \int_{\alpha}^{\beta} f_{\omega}^{(k)}(y)(x-y)^k \omega(y) dy \right|$$

$$\le \int_{\alpha}^{\beta} \left| A_\gamma(f_{\omega}^{(k)})(y) - f_{\omega}^{(k)}(y) \right| \left| (x-y)^k \omega(y) \right| dy$$

$$\le M_1 \int_{\alpha}^{\beta} \left| A_\gamma(f_{\omega}^{(k)}) - f_{\omega}^{(k)} \right| dy \to 0$$

as $\gamma \to 0^+$, where $k = 1, \ldots, \ell - 1$ and M_1 is independent of γ and x_0.

Analogously $\forall x \in (a(\delta), b(\delta))$

$$\left|\int_{a_x}^{b_x}(x-y)^{\ell-1}\Lambda(x,y)(A_\gamma f)^{(1)}(y)dy - \int_{a_x}^{b_x}(x-y)^{\ell-1}\Lambda(x,y)f_\omega^{(\ell)}(y)dy\right|$$

$$\leq M_2 \int_{a(\delta)}^{b(\delta)} \left|A_\gamma(f_\omega^{(\ell)}) - f_\omega^{(\ell)}\right| dy \to 0$$

as $\gamma \to 0^+$, where M_2 is in depend of γ and x

Finally $A_\gamma f \to f$ almost everywhere on $(a(\delta), b(\delta))$. Thus (9) is valid almost everywhere on $(a(\delta), b(\delta))$ and, hence, on (a, b) since $\bigcup_{\delta>0}(a(\delta), b(\delta)) = (a, b)$

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