

RAMSEY THEORY



ADDIS ABABA UNIVERSITY
COLLEGE OF NATURAL SCIENCE
DEPARTMENT OF MATHEMATICS

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Abstract

Ramsey Theory is named after Frank Plumpton Ramsey a young man was especially interested in logical and philosophy. Ramsey died at the age of 26 in 1930 the same year that his paper on a problem of formal logic was published. His paper catalyzed the development of the mathematics field now known as Ramsey Theory. Problem in Ramsey Theory typically ask a question of the form: How many people are required at gathering so that there must exist either three mutual acquaintances or three mutual strangers? We can rephrase this question as a problem in Ramsey Theory: How many vertices do you need in edge 2 colored complete graphs for it to be necessary that there be either a red K_3 (people who know each other) or a blue K_3 (people who do not know each other)? While many results in the subjects are published each year, there are many questions whose answer remains elusive. Ramsey Theory has played an important role in a plethora of mathematics development throughout the last century. Ramsey Theory is concerned with the preservation of structure under partition it is the study of unavoidable regularity in large structures. In this project, I explore some of the core ideas underpinning Ramsey Theory and present a variety of problem to which it can provide interesting and elegant solution. Also we have see, the Ramsey number $R(k, l)$ is the smallest natural number n such that in any red and blue coloring of the edges of the complete graph on n vertices, we are guaranteed to find either a red K_k or a blue K_l . Furthermore, we have discussed a multi-color example $R(3, 3, 3) = 17$, generalization of Ramsey Theorem and infinite Ramsey Theory. Some Known Ramsey Theorem for bound on classical Ramsey numbers are included.

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Chapter 1

Preliminary

In this section, we state some key definitions and formalities that are used throughout the remainder of the project.

Definition 1.1. A graph G is an ordered triple $(V(G), E(G), \Psi)$ consisting of a non-empty set $V(G)$ of vertices, a set $E(G)$, disjoint from $V(G)$, of edges, and an incidence function Ψ that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G .

Definition 1.2. The vertex set of a graph is denoted by $V(G)$, and the edge set is denoted by $E(G)$.

Definition 1.3. A simple graph, G , consists of a nonempty set, $V(G)$, called the vertex of G and a collection, $E(G)$, of two-element subsets of $V(G)$. The members of $E(G)$ are called the edge of G .

Definition 1.4. If $\Psi(e) = uu$, then e is known as a loop. Otherwise ($\Psi(e) = uv, u \neq v$), then e is a link.

Definition 1.5. Two edges e_1 and e_2 having the same pair of end vertices ($\Psi(e_1) = \Psi(e_2) = uv$) are known as parallel edges.

Definition 1.6. A graph with no loop or no parallel edges is known as a simple graph.

Definition 1.7. Infinite graph is a graph with infinite number of vertices and finite graph is a graph with finite number of vertices.

Definition 1.8. In a graph $G = (V, E, \Psi)$, if $\Psi(e) = uv$, then we say that e is incident with the vertices u and v and the vertices u and v are said to be adjacent.

Definition 1.9. The *order* of a graph G is the cardinality of its vertex set, and the *size* of a graph is the cardinality of its edge set.

Definition 1.10. The *degree* of a vertex $v \in V(G)$ denoted by $\deg(v)$ is the number of links incident with v plus twice the number of loops at v .

Definition 1.11. A *complete graph* K_n is a simple graph such that $|V| = n$ and $E = \{\{v_1, v_2\} / v_1, v_2 \in V, v_1 \neq v_2\}$. In other words K_n is a graph with n vertices, each of which is adjacent to every other.

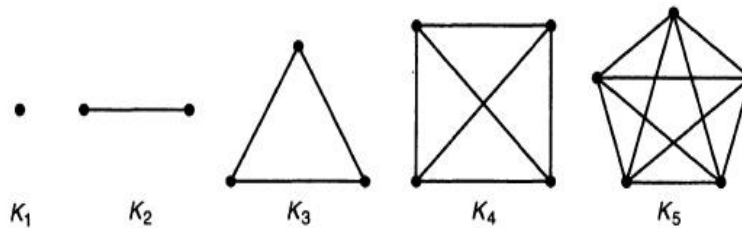


Figure 1.1: Some complete graphs

Definition 1.12. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In this case we write $H \subseteq G$.

Definition 1.13. A graph H is a *spanning subgraph* of G if $H \subseteq G$ and $V(H) = V(G)$.

Definition 1.14. The *complement* \bar{G} of G has $V(\bar{G}) = V(G)$ but $uv \in E(\bar{G})$ if and only if $uv \notin E(G)$.

Definition 1.15. Let $G = (V, E)$ be a graph and $\emptyset \neq V' \subset V$. The subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both end in V' is called the subgraph of G *induced* by V' and is denoted by $G[V']$; we say that $G[V']$ is an induced subgraph of G .

Definition 1.16. A k -vertex (or k -edge) coloring of a loopless graph G is an assignment of k colors, $1, 2, \dots, k$ to the vertices (or edges) of G . The coloring is proper if no two distinct adjacent vertices (or edges) have the same color.

Definition 1.17. G is k -vertex (or k -edge) colorable if G has a proper k -vertex (or k -edge)-coloring.

Definition 1.18. *The chromatic (or edge chromatic) number, $\chi(G)$ or $\chi'(G)$, of a loopless graph G , is the minimum k for which G is k (or k -edge) colorable.*

Definition 1.19. *A 2-colored graph is a graph whose edges have been colored with 2 different colors.*

Definition 1.20. *Monochromatic complete graph is a complete graph all of whose edges are colored the same color.*

Definition 1.21. $[X]^n = \{Y \subseteq X : |Y| = n\}$.

Definition 1.22. *A k -coloring c of $[X]^n$ is a function from $[X]^n$ into a set of size k .*

Definition 1.23. *A clique is a set of pairwise adjacent vertices in the graph and an independent (stable) set is a set of pairwise nonadjacent vertices in the graph.*

Lemma 1.1. *(Handshaking Lemma)*

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Corollary 1.1. *(Corollary of Handshaking Lemma)* In any graph, the number of vertices of odd degree is even.

Definition 1.24. $[n] = \{1, 2, \dots, n\}$ and $\mathbb{Z}_n = \{0, 1, 2, \dots, n - 1\}$.

Chapter 2

FINITE RAMSEY NUMBERS

2.1 Introduction

Ramsey Theory is named after Frank Plumpton Ramsey a young man was especially interested in logical and philosophy. Problem in Ramsey Theory typically ask a question of the form: how many elements of some structure must there be to guarantee that a particular will hold?

A typical result in Ramsey Theory starts with some mathematical structure, which is then cut into pieces. How big must the original structure be, in order to ensure that at least one of the pieces has a given interesting property? In this project, I explore some of the core ideas underpinning Ramsey Theory and present a variety of problem to which it can provide interesting and elegant solution.

We have to see the Ramsey number $R(k, l)$ is the smallest natural number n such that in any red and blue coloring of the edges of the complete graph on n vertices, we are guaranteed to find either a red K_k or a blue K_l . Furthermore, we have to discuss a multi-color example $R(3, 3, 3) = 17$ and generalization of Ramsey Theorem. Some Known Ramsey Theorem for bound on classical Ramsey numbers are included.

2.1.1 Pigeonhole Principle

Pigeonhole principle presents the most essential and basic part in the mathematics of counting and sorting. The simplest form of the pigeonhole principle is the following fairly obvious assertion.

Definition 2.1. *Pigeonhole principle*, simple states that if there exists n pigeonholes containing $n+1$ pigeons, then one of the pigeonholes must contain two pigeons.

Generalization of pigeonhole principle

A generalized version of this principle states that, if n discrete objects are to be allocated to m containers, then at least one container must hold no less than $\lceil \frac{n}{m} \rceil$ objects, where $\lceil \frac{n}{m} \rceil$ is the smallest positive integer greater than or equal to $\frac{n}{m}$.

2.2 Classical Ramsey Number

Definition 2.2. Let p and q be positive integers, *The Ramsey number* associated with these integers, denoted by $R(p, q)$, is defined to be the smallest integer n such that every 2-coloring of the edges of K_n either contain a red K_p or a blue K_q as a subgraph.

Example 2.1. We would like to find the value of $R(1, 3)$, According to the definition, $R(1, 3)$ is the least value of n such that 2-coloring of the edges of K_n either contains as a subgraph a K_1 all of whose edges are red, or a K_3 all of whose edges are blue. How many vertices are required before we know that we will have one of these objects in every coloring of a complete graph? If you have just one vertex, then no matter K_1 has no edges and so no edges to color, thus any coloring of K_1 will always contain a red K_1 . Therefore, $R(1, 3) = 1$.

Similar reasoning, $R(k, 1) = R(1, l) = 1$ for every positive integers k and l .

Proposition 2.1. For all $k \in \mathbb{N}$ and $k \geq 2$, $R(k, 2) = k$.

Proof. K_2 is a graph consisting of a single edge. If G is a 2-coloring of K_k in red and blue and G assigns at least one of edges the color blue then we have a monochromatic blue K_2 . But if G assigns no edge the color blue, then all edges are colored red and we have a monochromatic red K_k . So $R(k, 2) \leq k$. On the other hand consider a monochromatic red K_{k-1} . It has neither a red K_k subgraph nor a blue K_2 subgraph. So $R(k, 2) \geq k$. This implies $R(k, 2) = k$. \square

Example 2.2. Six people are waiting in the lobby of a hotel. Prove that there are either three of them who know each other, or three of them who do not know each other.

Proof. Take a K_6 so that each person corresponds to a vertex. Color the edge joining x and y red if x and y know each other and blue if they do not. Do this for all 15 edges of the graph. The claim of example 2.2 will be proved if we can show that there will always be triangle with monochromatic edges in our graph.

Take any vertex v of our bicolored graph. As v has degree five, it must have at least three edges incident to it that have the same color. Assume without loss of generality that this color is red. (By pigeonhole principle v is either incident with at least three red edges or at least three blue edges). Let x , y , and z be the end points of three red edges incident to v . Now if any edge of the triangle xyz (at least one of the edges of triangle xyz) is red, then that edge, and the two edges joining (the end points of) that edge to v are red, so we have a triangle with three red edges. If the triangle xyz does not have a red edge, then it has three blue edges. See fig 2.1 below. Thus, we proved $R(3, 3) \leq 6$. \square

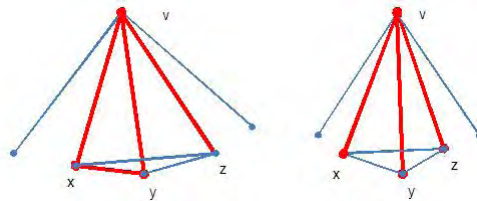


Figure 2.1: The colors of the edge of the triangles xyz are crucial

Indeed, let us see again the party problem asks: What is the smallest number of people attending a party for which we are guaranteed to find 3 people none of whom know the other two or 3 people each of whom know the other two? Assume that knowing is symmetric relationship.

Suppose we have 5 people at a math party: Aman, Aster, Abebe, Kebede, and Ali. If Aman Knows Aster and Ali, Aster knows Aman and Abebe, Abebe knows Aster and Kebede, Kebede knows Abebe and Ali, and Ali knows Kebede and Aman, then no three of the partygoers are mutual non acquaintances or mutual acquaintances(friends). To see this quickly, we encode this information in a graph by taking K_5 and assigning each partygoer a vertex and then coloring the edge between partygoers red if they know each other and blue if they do not know each other. If three partygoers all know each other there will be red K_3 subgraph, and if three partygoers all do not know each other will be a blue K_3 subgraph. Figure 2.2 shows this graph and it has no monochromatic K_3 (This 2-coloring of the edges of K_5 contain neither a red K_3 nor a blue K_3). Thus $R(3, 3) \geq 6$.

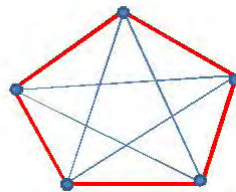


Figure 2.2: 2-Coloring of K_5 .

Therefore $R(3, 3) = 6$.

Proposition 2.2. *The Ramsey number is symmetric, i.e $R(a, b) = R(b, a)$ for $a, b \in \mathbb{N}$.*

Proof. Let $a, b \in \mathbb{N}$. suppose $R(a, b) = n$ then there is a 2-coloring χ of K_{n-1} avoiding red K_a and avoiding blue K_b . Define a coloring δ on G by $\delta(e)$ is red if $\chi(e)$ is blue and $\delta(e)$ is blue if $\chi(e)$ is red. Then δ and χ always swap colors. So δ is a coloring of K_{n-1} that avoids red K_b and blue K_a . Therefore $R(b, a) \geq R(a, b)$. Nothing we did above depends on the order of the inputs. So swapping a and b in the above argument gives $R(a, b) \geq R(b, a)$, hence we must conclude $R(a, b) = R(b, a)$. \square

Theorem 2.1. *(Ramsey Theorem for graphs)*

Let k and l be two positive integers, both of which are at least two, then there exists a minimal(a least) positive integer $R(k, l)$, so that if we color the edges of complete graph with $R(k, l)$ vertices red and blue, then this graph will either have a K_k subgraph with only red edges, or a K_l subgraph with only blue edges.

Proof. We prove the statement by induction on k and l . This induction will run as follow. First, we prove the initial conditions that $R(k, 2)$ and $R(2, l)$ exist for all $k, l \geq 2$. Then we prove the induction step that if $R(k, l - 1)$ exists, and also $R(k - 1, l)$ exists, then $R(k, l)$ also exists. To see that the initial conditions hold, note that $R(k, 2) = k$, and similarly, $R(2, l) = l$, by proposition 1.1.

We prove the induction step by showing that: $R(k, l) \leq R(k, l - 1) + R(k - 1, l)$ for every integer $k, l \geq 3$.

Indeed, take a complete graph with $R(k - 1, l) + R(k - 1, l)$ vertices. Take one of its vertices, and call it v . As v has degree $R(k, l - 1) + R(k - 1, l) - 1$, it has either at least $R(k, l - 1)$ blue edges incident to it, or it has at least $R(k - 1, l)$ red edges incident to it. In the first case, let B denote the $R(k, l - 1)$ element set of the other endpoints of these blue edges. Then by the definition of $R(k, l - 1)$, the set B either contains monochromatic red K_k and we are done, or a monochromatic blue K_{l-1} , which can be completed to a monochromatic blue K_l by adding the vertex v , and we are done again. In the second case, let C denote the $R(k - 1, l)$ -element set of the other endpoints of these red edges. Then again, C either contains a monochromatic blue K_l and we are done or a monochromatic red K_{k-1} , which can be completed to a monochromatic red K_k by adding the vertex v , and we are done again. \square

Example 2.3. Show that $R(4, 3) = 9$

Proof. To prove our claim, we have to show two cases: that all 2-colorings of the edges of K_9 will result in either a red K_4 or a blue K_3 and that the same will not hold for K_8 . We now want to prove that $R(4, 3) \leq 9$, and we will use the facts that $R(3, 3) = 6$ and $R(4, 2) = 4$.

Case 1: Take a K_9 with two colored edges. We claim that there has to be a vertex v so that either:

- i. at least six of the edges incident to v are red or
- ii. at least four of the edges incident to v are blue. If neither statement were true, then all vertices of this K_9 would have five red (or three blue) edges incident to them which is contradiction as the sum of the degree in the subgraph of all red (or all blue) edges must be even. By handshaking Lemma, so it cannot be $9 \times 5 = 45$ or $9 \times 3 = 27$. If there are six red edges incident to v , then denote by A the six-element set of their end points. By example 2.2, $R(3, 3) = 6$, there is either a red triangle, or blue triangle on A . So our K_9 either contain a blue triangle, or together with v , a red K_4 . (see figure 2.3)

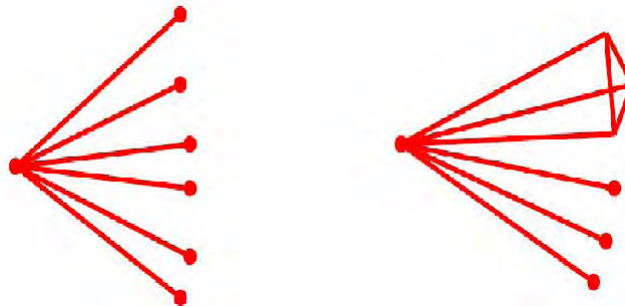


Figure 2.3: Red colored edges.

If, on the other hand, there are four blue edges incident to v , then denote by B the four-element set of their end points. If all edges on B are red, then there is a red K_4 . If at least one of the edges of B is blue, which will form a blue triangle, together with v (see figure 2.4). Thus $R(4, 3) \leq 9$.

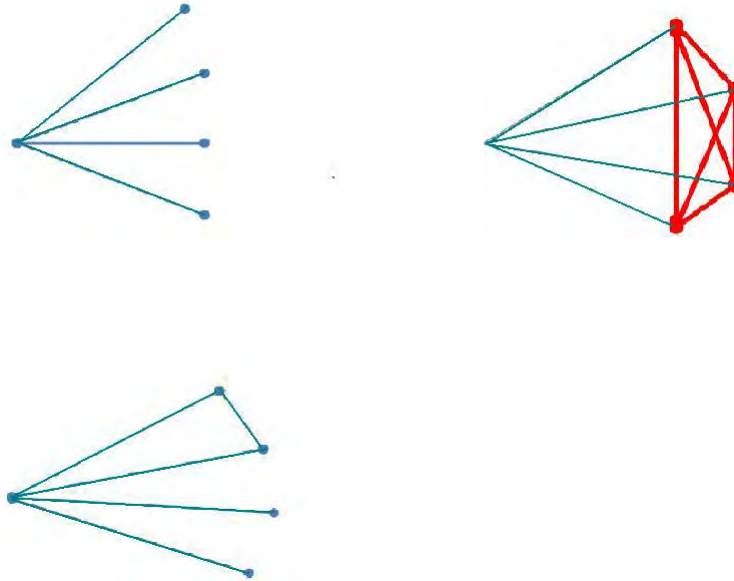


Figure 2.4: Four blue edges incident to v .

Case 2: In order to see that $R(4, 3) \geq 9$. Consider the 2-coloring of the edges of K_8 , and label its vertices by the elements of \mathbb{Z}_8 , in clockwise direction, say let the edge (i, j) with $j > i$, be blue (thin) if $j - i = 1$, or 4 and red (thick) otherwise. (See fig 2.5)

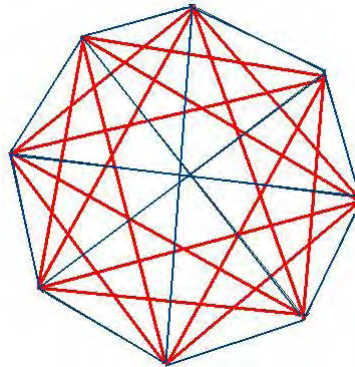


Figure 2.5: 2-Coloring of K_8 .

This graph does not contain red (thick) K_4 and no blue (thin) K_3 . Thus $R(4, 3) \geq 9$. Therefore, $R(4, 3) = 9$. \square

Example 2.4. $R(4, 4) = 18$

Proof. By formula $R(k, l) \leq R(k, l-1) + R(k-1, l)$, it is $R(4, 4) \leq R(4, 3) + R(3, 4) = 9 + 9 = 18$. We want to show $R(4, 4) \geq 18$.

To prove this, we need a red-blue coloring of K_{17} containing no monochromatic K_4 . In general, coloring which establish lower bounds tend to look locally random. Let us assume that the vertices of K_{17} are labeled with \mathbb{Z}_{17} (or modulo 17) and let $G = (\mathbb{Z}_{17}, E)$, where $E = \{\{x, y\} \in \mathbb{Z}_{17}/x - y \in QR\}$, where $QR = \{1, 2, 4, 8, 9, 13, 15, 16\}$. So that the 16 nonzero residues fall into two classes, quadratic residues and quadratic nonresidues.

The set of quadratic residue modulo 17 is:

$QR = \{x^2/x \in \mathbb{Z}_{17}\} = \{1, 2, 4, 8, 9, 13, 15, 16\}$ (Since if we divide the square of an integer by 17, then non-zero remainder will always be one of these eight numbers) and The set of quadratic nonresidues is:

$QN = \{3, 5, 6, 7, 10, 11, 12, 14\}$. Recall that QR is the range of the homomorphism $f : \mathbb{Z}_{17} \rightarrow \mathbb{Z}_{17}, f(x) = x^2$. Both QR and QN are closed under multiplication by -1 (because $-1 = 16 \in QR$), so that $j - i$ has the same quadratic character as $i - j$. Edge ij is colored red if $j - i \in QR$ and blue if $j - i \in QN$. Suppose there is a monochromatic K_4 on vertices a, b, c, d . First we note that the coloring is translation invariant: $(j + k) - (i + k) = j - i$. Therefore we may assume that $a = 0$. Now multiple each vertex by b^{-1} (the multiplicative inverse of b), and note that either no edge changes color (if $b \in QR$) or else every edge changes color (if $b \in QN$). The reason for this is that $b^{-1}j - b^{-1}i = b^{-1}(j - i)$. In either case, we now have a monochromatic K_4 on vertices $0, 1, cb^{-1}, db^{-1}$. Now, because 1 is a quadratic residue, it follow that the differences $cb^{-1}, db^{-1}, cb^{-1} - 1, db^{-1} - 1, db^{-1} - cb^{-1}$ must be quadratic residues. Upon inspection of the elements of QR , we see that this is impossible. Why?

Because suppose that $K = \{0, b, c, d\}$; by definition of G , it holds that $R = \{b, c, d, b - c, b - d, c - d\} \subseteq QR$. Since \mathbb{Z}_{17} is field, b^{-1} exists. We define $x = cb^{-1}$ and $y = db^{-1}$; these are distinct numbers and difference from 1, since b, c , and d are distinct. Because $b^{-1} = (n^2)^{-1}$ for some $n \in \mathbb{Z}_{17}$, by multiplying all elements R by b^{-1} , we get that $\{1, x, y, x - 1, y - 1, x - y\} \subseteq QR$. on the other hand, suppose that $I \subseteq \mathbb{Z}_{17}$ is an independent set in G (or QN) with 4 elements. Again, we may assume that $I = \{0, b, c, d\}$. We have now that $J = \{b, c, d, b - c, b - d, c - d\} \subseteq QN$. It can be easily verified by testing all possible cases that if $u, v \in QN$, then $uv \in QR$. Thus multiplying all the element of J by b^{-1} we see that $\{1, x, y, x - 1, y - 1, x - y\} \subseteq QR$ where $x = cb^{-1}$ and $y = db^{-1}$.

We have seen that if there is a red or a blue monochromatic K_4 with 4 vertices in G , then there are distinct number $x, y \in QR - \{1\}$ such that $\{1, x, y, x - 1, y - 1, x - y\} \subseteq QR$

We have that

$2 - 1 = 1 \in QR, 4 - 1 = 3 \notin QR, 8 - 1 = 7 \notin QR, 9 - 1 = 8 \in QR, 13 - 1 = 12 \notin QR, 15 - 1 = 14 \notin QR$ and $16 - 1 = 15 \in QR$ and

thus $x, y \in \{2, 9, 16\}$. However, $x = 2, y = 9 \Rightarrow x - y = 10 \notin QR$ and $x = 9, y = 2 \Rightarrow x - y = 7 \notin QR$ and $x = 2, y = 16 \Rightarrow x - y = 3 \notin QR$ and $x = 16, y = 2 \Rightarrow x - y = 14 \notin QR$ and $x = 9, y = 16 \Rightarrow x - y = 10 \notin QR$ and $x = 16, y = 9 \Rightarrow x - y = 7 \notin QR$.

It follows that set $\{1, x, y, x - 1, y - 1, x - y\}$ cannot be a subset of QR . Thus, existence of a red or a blue monochromatic K_4 with 4 vertices would lead to contradiction and is therefore impossible. Since G is a graph with no red or blue monochromatic K_4 with 4 vertices (see figure 2.6). Thus, $R(4, 4) \geq 18$. Therefore, $R(4, 4) = 18$.

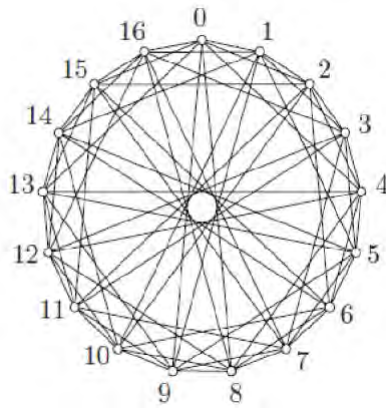


Figure 2.6: Red-Coloring of K_{17} .

□

Proposition 2.3. *The Ramsey numbers are monotone.*

Proof. Let $u_1 > u_2$ and $v_1 > v_2$ then if n is large enough to guarantee the existence of either a red K_{u_1} or a blue K_{v_1} then n also guarantees the existence of either a red K_{u_2} or a blue K_{v_2} , as K_{u_2} is a subgraph of K_{u_1} and K_{v_2} is a subgraph of K_{v_1} . Hence $R(u_1, v_1) \geq R(u_2, v_2)$. □

Some known Ramsey numbers are listed below.

$R(1, k) = R(k, 1) = 1$ for every positive integer k .

$R(2, k) = R(k, 2) = k$ for any integer $k \geq 2$.

$R(3, 3) = 6, R(3, 4) = 9, R(3, 5) = 14, R(3, 6) = 18, R(3, 7) = 23,$

$R(3, 8) = 28, R(3, 9) = 36, R(4, 4) = 18, R(4, 5) = 25.$

2.3 Bound on Ramsey Numbers

Determining exact values of Ramsey numbers is extremely difficult in general. In fact, the list given above is not only a list of some known Ramsey numbers.

It is a list of all known Ramsey numbers. However, there has been progress in finding bounds, and we state some important ones:

2.3.1 Upper Bound for Ramsey Number $R(k, l)$

The following Theorems yield different ways of determining upper bound of Ramsey numbers. Erdős and Szekeres are given credit for proving Theorem 2.2.

Theorem 2.2. *Let k and l be positive integers. Then $R(k, l) \leq \binom{k+l-2}{k-1}$.*

Proof. By induction on $k + l$. Using $R(1, l) = R(k, 1) = 1$ for every positive integer k and l . $R(k, 2) = k$ and $R(2, l) = l$ for every integers $k, l \geq 2$. We see that the theorem holds when $k + l \leq 5$. Let m and n be positive integers and assume that the theorem is valid for all positive integers k and l such that $5 \leq k + l < m + n$. Then Theorem 2.1, for any two integers $m \geq 2$ and $n \geq 2$, $R(m, n) \leq R(m, n-1) + R(m-1, n)$ and the induction hypothesis, we want show that: $R(m, n-1) + R(m-1, n) \leq \binom{m+n-2}{m-1}$ Hence, $R(m, n) \leq R(m, n-1) + R(m-1, n) \leq \binom{m+n-3}{m-1} + \binom{m+n-3}{m-2}$

$$\begin{aligned}
&= \frac{(m+n-3)!}{(m-1)!(m+n-3-(m-1))!} + \frac{(m+n-3)!}{(m-2)!(m+n-3-(m-2))!} \\
&= \frac{(m+n-3)!}{(m-1)!(n-2)!} + \frac{(m+n-3)!}{(m-2)!(n-1)!} \\
&= \frac{(n-1)(m+n-3)! + (m-1)(m+n-3)!}{(m-1)!(n-1)!} \\
&= \frac{(m+n-3)!(n-1+m-1)}{(m-1)!(n-1)!} \\
&= \frac{(m+n-3)!(n+m-2)}{(m-1)!(n-1)!} \\
&= \frac{(m+n-2)!}{(m-1)!(n-1)!} \\
&= \binom{m+n-2}{m-1}
\end{aligned}$$

Thus, $R(m, n) \leq \binom{m+n-2}{m-1} = \binom{m+n-2}{n-1}$.

Therefore, $R(k, l) \leq \binom{k+l-2}{k-1} = \binom{k+l-2}{l-1}$. □

Corollary 2.1. *For all positive integer k , $R(k, k) \leq 4^{k-1}$.*

Proof. For diagonal Ramsey numbers, the inequality of theorem 2.2, we obtain $R(k, k) \leq \binom{2k-2}{k-1}$ and we can determine a nice asymptotic estimate. one of the great open problems of Ramsey theory is to calculate $\lim_{k \rightarrow \infty} R(k, k)^{\frac{1}{k}}$ (if it exists). Using binomial expansion (Theorem), $(x+y)^{2n} = \sum_{m=0}^{2n} \binom{2n}{m} x^m y^{2n-m}$. Let $k-1 = n$ and $x = y = 1$, then $(1+1)^{2n} = \binom{2n}{0} 1^0 1^{2n} + \binom{2n}{1} 1^1 1^{2n-1} + \dots + \binom{2n}{n} 1^n 1^n + \dots + \binom{2n}{2n} 1^{2n} 1^0 = \binom{2n}{0} + \binom{2n}{1} + \dots + \binom{2n}{n} + \dots + \binom{2n}{2n} = \sum_{m=0}^{2n} \binom{2n}{m}$. Thus $\binom{2n}{n} < 2^{2n}$. From Theorem 2.2 and Binomial expansion, it follows that $R(k, k) \leq \binom{2k-2}{k-1} < 2^{2k-2} = 4^{k-1}$. \square

2.3.2 A lower Bound for Ramsey Number $R(k, k)$

A lower bound for Ramsey number $R(k, k)$ obtained by means of a powerful technique known as the probabilistic method. The probabilistic method is essentially a crude counting argument. It can be applied to assert the existence of a graph with certain specified properties.

Proposition 2.4. *If G is simple graph of order n and size m , then it has 2^m spanning sub graphs (simple graphs with vertex set $\{v_1, v_2, \dots, v_n\}$).*

Proof. Let $|G| = n$ and $||G|| = m$. G has spanning sub graph of size 0 is $\binom{m}{0}$, size 1 is $\binom{m}{1}$, size 2 is $\binom{m}{2}$, size 3 is $\binom{m}{3}$..., size m is $\binom{m}{m}$. Then G has $\binom{m}{0} + \binom{m}{1} + \dots + \binom{m}{m} = \sum_{k=0}^m \binom{m}{k} = (1+1)^m = 2^m$ spanning subgraphs (by Binomial Theorem). \square

Theorem 2.3. (Erdős, 1947): $R(k, k) \geq 2^{\frac{k}{2}}$, for positive integer $k \geq 2$.

Proof. Since $R(1, 1) = 1$, we may assume that $k \geq 2$. Denote by Y_n the set of simple graphs with vertex set $\{v_1, v_2, \dots, v_n\}$, and by Y_n^k the set of those graph in Y_n that have a clique of k vertices and let G be complete graph, then $m = \binom{n}{2}$ and G has $2^{\binom{n}{2}}$ spanning subgraphs, or $|Y_n| = 2^{\binom{n}{2}}$.

Since each subset of the $\binom{n}{2}$ possible edge $v_i v_j$ determines a graph in Y_n . Similarly, the number of graph in Y_n having a particular set of k vertices as a clique is $2^{\binom{n}{2} - \binom{k}{2}}$. Since there are $\binom{n}{k}$ distinct k -element subset of $\{v_1, v_2, \dots, v_n\}$ we have $|Y_n^k| \leq \binom{n}{k} 2^{\binom{n}{2} - \binom{k}{2}}$. \square

Example 2.5. In case K_3 (see fig 2.8), let $n = 3$, $k = 2$, $V = \{v_1, v_2, v_3\}$ and $E = \{e_1, e_2, e_3\}$.

$|y_n| = 2^{\binom{3}{2}}$. Thus G has 8 spanning subgraphs (see fig 2.7)

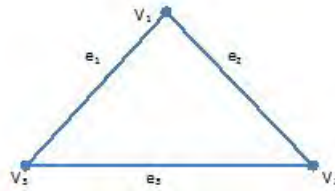


Figure 2.7: The vertices and edges of K_3 .

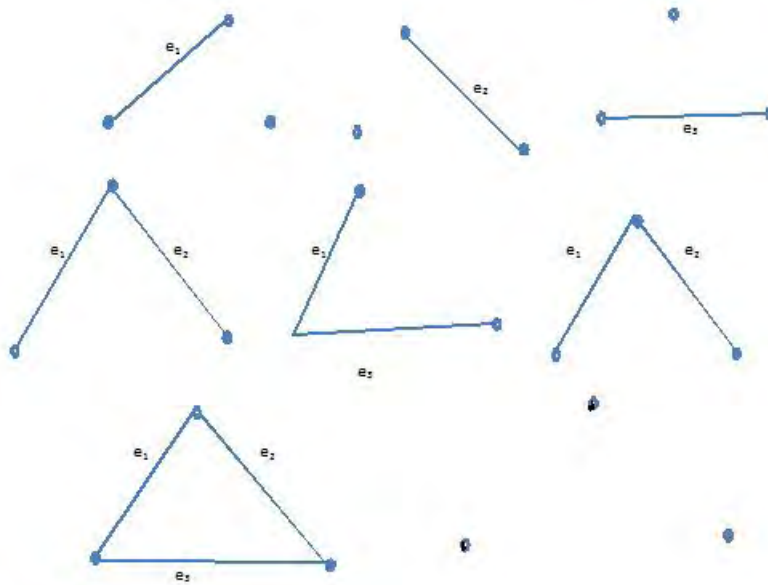


Figure 2.8: The spanning subgraphs of K_3 .

From figure 2.8 spanning subgraph of $G = K_3$ each e_1 , e_2 , and e_3 has 4 times.
 $y_3^2 \leq \binom{3}{2} 2^{\binom{3}{2} - \binom{2}{2}} = 3 \times 4 = 12$.

Example 2.6. In case K_4 , let $n = 4$, $k = 3$, $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and from K_4 the set of K_3 is $\{e_1e_2e_5, e_1e_4e_6, e_2e_3e_6, e_3e_4e_5\}$.

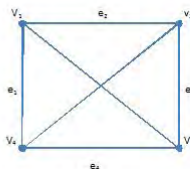


Figure 2.9: Vertices and edges of K_4

Suppose Y_4 is the set of spanning subgraph of K_4 and Y_4^3 is the set of spanning subgraph of K_4 that have a clique of 3 vertices. Thus G has $|y_4| = 2^{\binom{4}{2}} = 2^6 = 64$ is the number of the set of spanning subgraphs of K_4 . From this spanning sub graphs size 3 is $\binom{6}{3} = 20$ spanning sub graphs, Size 4 is $\binom{6}{4} = 15$ spanning subgraphs, size 5 is $\binom{6}{5} = 6$ spanning subgraphs, and size 6 is $\binom{6}{6} = 1$ spanning subgraph.

Thus, from K_4 the set spanning subgraph of size 3 is:

$\{e_1e_2e_3, e_1e_2e_4, e_1e_2e_5, e_1e_2e_6, e_1e_3e_4, e_1e_3e_5, e_1e_3e_6, e_1e_4e_5, e_1e_4e_6, e_1e_5e_6, e_2e_3e_4, e_2e_3e_5, e_2e_3e_6, e_2e_4e_5, e_2e_4e_6, e_2e_5e_6, e_3e_4e_5, e_3e_4e_6, e_3e_5e_6, e_4e_5e_6\}$. From this set of spanning sub graphs, we get each triangles has one time. That is $\{e_1e_2e_5, e_1e_4e_6, e_2e_3e_6, e_3e_4e_5\}$.

From K_4 the set of spanning subgraphs of size 4 is:

$\{e_1e_2e_3e_4, e_1e_2e_3e_5, e_1e_2e_3e_6, e_1e_2e_4e_5, e_1e_2e_4e_6, e_1e_2e_5e_6, e_1e_3e_4e_5, e_1e_3e_4e_6, e_1e_3e_5e_6, e_1e_4e_5e_6, e_2e_3e_4e_5, e_2e_3e_4e_6, e_2e_3e_5e_6, e_2e_4e_5e_6, e_3e_4e_5e_6\}$. From this set of spanning subgraphs of size 4, we get each K_3 has three times.

From K_4 the set of spanning subgraphs of size 5 is:

$\{e_1e_2e_3e_4e_5, e_1e_2e_3e_4e_6, e_1e_2e_3e_5e_6, e_1e_2e_4e_5e_6, e_1e_3e_4e_5e_6, e_2e_3e_4e_5e_6\}$. From the set of spanning subgraphs of size 5: from $e_1e_2e_3e_4e_5$ we get triangles $e_1e_2e_5$ and $e_3e_4e_5$, from $e_1e_2e_3e_4e_6$ we get triangles $e_1e_4e_6$ and $e_2e_3e_6$, from $e_1e_2e_3e_5e_6$ we get triangles $e_1e_2e_5$ and $e_2e_3e_6$, from $e_1e_2e_4e_5e_6$ we get $e_1e_2e_5$ and $e_1e_4e_6$, from $e_1e_3e_4e_5e_6$ we get $e_1e_4e_6$ and $e_3e_4e_5$, from $e_2e_3e_4e_5e_6$ we get $e_2e_3e_6$ and $e_3e_4e_5$. Thus from spanning subgraphs of size 5 we get each triangle has three times, and from K_4 the set of spanning subgraph of size 6 is $\{K_4\}$. From K_4 , we get each triangles has one times. Therefore, we get each triangles(or K_3) has eight times.

Therefore, $|Y_4^3| \leq \binom{4}{3} 2^{\binom{4}{2} - \binom{3}{2}} = 4 \times 8 = 32$ is the number of the set of spanning subgraphs of K_4 that have a clique of 3 vertices.

Hence, $|\frac{Y_n^k}{Y_n}| \leq \frac{\binom{n}{k} 2^{\binom{n}{2} - \binom{k}{2}}}{2^{\binom{n}{2}}} = \binom{n}{k} 2^{-\binom{k}{2}} < \frac{n^k 2^{-\binom{k}{2}}}{k!}$.

Suppose that $n < 2^{\frac{k}{2}}$ and Stirling approximation, valid for large values of k

. It is: $k! \approx \left(\frac{k}{e}\right)^k \sqrt{2\pi k}$. It follows that $|\frac{Y_n^k}{Y_n}| < \frac{2^{\frac{k^2}{2}} 2^{-\binom{k}{2}}}{k!} = \frac{2^{\frac{k}{2}}}{k!} < \frac{1}{2}$.

Therefore, fewer than half of the graph in Y_n contain a clique of k vertices. Also, because $Y_n = \{G/G^c \in Y_n\}$ fewer than half of the graphs in Y_n contain an independent set of k vertices. Hence some graph in Y_n contain neither a clique of k vertices nor an independent set of k vertices. Because this holds for any $n < 2^{\frac{k}{2}}$, we have $R(k, k) \geq 2^{\frac{k}{2}}$.

The known values and bounds for Ramsey number $R(k, l)$, $k \leq 10$, $l \leq 15$ are list in the table below:

L K	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6	9	14	18	23	28	36	40 43	46 51	52 59	59 69	66 78	73 88
4		18	25 41	35 49	49 61	56 84	73 115	92 149	98 191	128 238	133 291	141 349	153 417
5			43 49	58 87	80 143	101 216	126 316	144 442	171 633	191 848	213 1139	239 1461	265 1878
6				102 165	113 298	132 495	169 780	179 1171	253 1804	263 2566	317 3705		401 6911
7					205 540	217 1031	241 1713	289 2826	405 4553	417 6954	511 10581		22116
8						282 1870	317 3583	6090	10630	16944	817 27490	41525	861 63625
9							565 6588	581 12677	22325	39025	64871	89203	
10								798 23556		81200			1265

2.3.3 Generalizations of Ramsey's Theorem

What we can do with two colors, we can do with an arbitrary number as the following generalization of Ramsey's theorem shows.

Definition 2.3. Let $p, q, r \in \mathbb{N}$ and $p, q, r \geq 2$, then there exists the least positive integer n such that $K_n \rightarrow K_p, K_q, K_r$ means $R(p, q, r) = n$. This means, if each of the edges of K_n is colored red, yellow, or green, then either there is a red K_p , or a yellow K_q , or a green K_r . The smallest integer n for which this assertion holds is the Ramsey number $R(p, q, r)$.

Example 2.7. Any two of seventeen students are corresponding with each other on one of three subjects. Prove that there are three among them so that any two of the three of them correspond with each other on the same subject.

This example generalize example 2.2, in the hotel lobby in a major aspect. Now each of 17 students talked with every other student. They all talked about three different subjects. Each pair of students talked about one subject. We will go to show that there are three students that talked about the same subject among themselves. So if we represent our students by a K_{17} ,

then we have to color the edges of this K_{17} by three colors.

Example 2.11 simple says, show that Ramsey number $R(3, 3, 3) = 17$.

Proof. we have to show that, if we color each of the edges of a K_{17} either red, or yellow, or green, there will always be a triangle with monochromatic edges. Choose any vertex v of our K_{17} . As v has degree 16, it follows by pigeonhole principle that there is a color so that at least six of the edges incident to v have the same color, say green. Let V_g be the set of the other endpoints of these green edges. If there is any green edge between two vertices of V_g , then we are done as those two vertices of V_g and v span a green triangle. If not, then all the edges among the vertices of V_g are red, or yellow. However, V_g has at least six elements, so it follows, example 2.2, $R(3, 3) = 6$, then the vertices of V_g span either a red triangle, or a yellow triangle. Thus $R(3, 3, 3) \leq 17$. To see that $R(3, 3, 3) \geq 17$, it suffices to draw an edge coloring on the complete graph on 16 vertices with 3 colors, which avoid monochromatic triangles. Consider the following fig.2.10, 3-coloring of the edges of K_{16} .

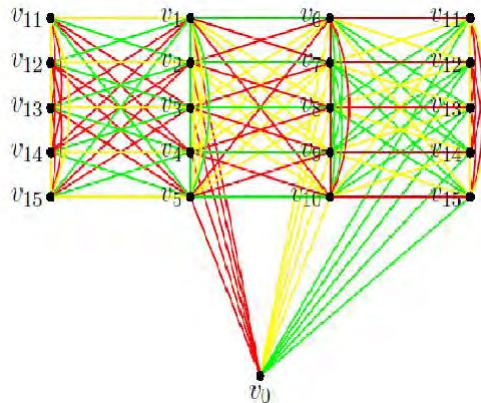


Figure 2.10: The 3-colorings of K_{16} with no monochromatic K_3 .

From the above fig 2.10, let us see each three colored edges graph has no monochromatic triangle.

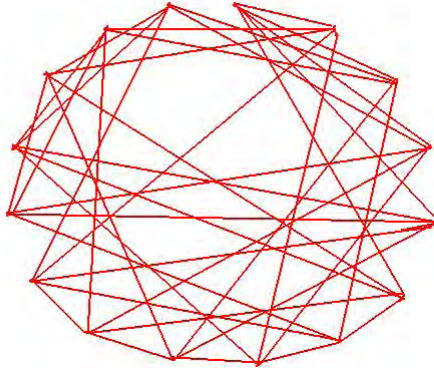


Figure 2.11: No monochromatic triangle red colored edged graph of three colors of graph K_{16} .

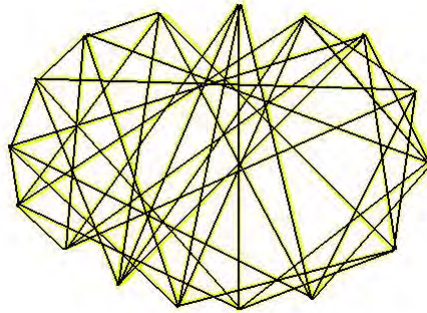


Figure 2.12: No monochromatic triangle yellow colored edged graph of three colors of graph K_{16} .

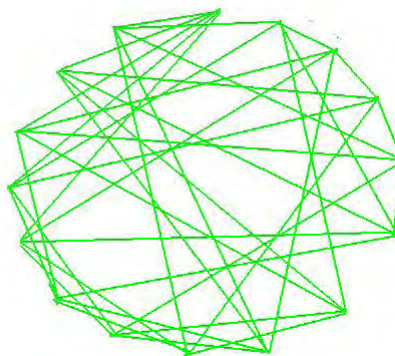


Figure 2.12: No monochromatic triangle green colored edged graph of three colors of graph K_{16} .

The above 3- coloring graph avoid monochromatic triangles, so that $R(3, 3, 3) \geq 17$. Therefore, $R(3, 3, 3) = 17$. □

Definition 2.4. A graph is r -colored if we color each edge of the graph with one of r colors.

Definition 2.5. An r -coloring χ of G is (G_1, G_2, \dots, G_r) -good provided for each i in $[r]$ χ colors no subgraph isomorphic to G_i monochromatically in color i .

Definition 2.6. Vertex v 's deleted neighborhood $n'(v) = \{u \in V : uv \in E\}$ is the set of vertices adjacent to v , and v 's neighborhood $n(v) = n'(v) \cup \{v\}$.

Theorem 2.4. The r -color Ramsey numbers satisfy $R(I_1, I_2, \dots, I_r) \leq 2 + (\sum_{i=1}^r R(I_1, I_2, \dots, I_i - 1, \dots, I_r) - 1)$.

Proof. As in Theorem 2.1, if $n \geq 2 + (\sum_{i=1}^r R(I_1, I_2, \dots, I_i - 1, \dots, I_r) - 1)$, than any $v \in K_n$ has degree $\geq 1 + (\sum_{i=1}^r R(I_1, I_2, \dots, I_i - 1, \dots, I_r) - 1)$. So that in some color j , v has j -degree $\geq R(I_1, I_2, \dots, I_j - 1, \dots, I_r)$ when either for $i \neq j$, $n'(v)$ contains a monochromatic K_{I_i} in color i or $n'(v)$ contains a monochromatic j -colored K_{I_j-1} which when v is added becomes a monochromatic j -colored K_{I_j} . \square

Proposition 2.5. For all $I_1, I_2, \dots, I_r \in \mathbb{N}$, $R(I_1, I_2, \dots, I_r) = R(I_1, I_2, \dots, I_r, 2)$.

Proof. As proposition 1.1, any $(I_1, I_2, \dots, I_r, 2)$ -good $r + 1$ coloring does not use color $r + 1$, so this coloring is a (I_1, I_2, \dots, I_r) -good r -coloring. Also (I_1, I_2, \dots, I_r) -good r -colorings are $(I_1, I_2, \dots, I_r, 2)$ -good $r + 1$ colorings. \square

Theorem 2.5. (Ramsey theorem for r colors)

For any $I_1, I_2, \dots, I_r \geq 2 \in \mathbb{N}$, $R(I_1, I_2, \dots, I_r)$ exists.

Proof. Let $r, m \in \mathbb{N}$. As in Theorem 2.1 induct on $m = \sum_{i=1}^r I_i$. We know $R(2, 2) = 2$. Repeated application of Proposition 2.5, gives $R(r; 2) = 2$, setting our base case of $m = 2r$. Assume $m \geq 2r$ that whenever $I_1, I_2, \dots, I_r \geq 2 \in \mathbb{N}$ and $\sum_{i=1}^r I_i = m$ that $R(I_1, I_2, \dots, I_r)$ exists. Consider $I_1, I_2, \dots, I_r \geq 2 \in \mathbb{N}$ and $\sum_{i=1}^r I_i = m + 1$, Theorem 2.4 gives $R(I_1, I_2, \dots, I_r) \leq 2 + (\sum_{i=1}^r R(I_1, I_2, \dots, I_i - 1, \dots, I_r) - 1)$ where the inductive hypothesis gives that $R(I_1, I_2, \dots, I_i - 1, \dots, I_r)$ exists for each i in the sum, as the inputs to $m + 1 - 1 = m$. So $R(I_1, I_2, \dots, I_r)$ is bounded above and finite. \square

Chapter 3

INFINITE RAMSEY THEORY

3.1 Introduction

In combinatorics, Ramsey Theory considers partitions of some mathematics objects and asks the following question: how large must the original object be in order to guarantee that at least one of the part in the partition exhibits some property? Perhaps the most familiar case is the well-known pigeonhole principle: suppose that n infinite pigeons have been housed in m finite pigeonholes. How big must n be before we can be sure that at least one pigeonhole houses at least one infinite pigeons? The answer is the following infinite pigeonhole principle.

3.2 Infinite Pigeonhole Principle

The pigeonhole principle, simple states that if there exists n pigeonhole containing $n + 1$ pigeons, one of the pigeonholes must contain two pigeons. This can be generalized to say that if there are an infinite number of pigeons then at least one of the pigeonholes must contain an infinite number of pigeons. Suppose X is the set of pigeons, Y is the set of pigeonholes, and f is the function we obtain the following theorem.

Theorem 3.1. *suppose X is infinite; Y is finite, and $f : X \rightarrow Y$ is a function. Then there is an element $y \in Y$ such that the set $\{x \in X / f(x) = y\}$ is infinite.*

Proof. Let X , Y , and f be as in the hypothesis of the theorem, In particular, Let $Y = \{y_0, y_1, y_2, \dots, y_n\}$. Suppose, by way of contradiction, that for each

$y_i \in Y$, the set $X_i = \{x \in X / f(x) = y_i\}$ has z_i elements. Because X can be written as $X = X_0 \cup X_1 \cup X_2 \cup \dots \cup X_n$. We see that $\sum_{i=0}^n z_i$ is the size of X . Thus X is finite which is a contradiction. \square

3.3 The infinite case of Ramsey's Theorem for two colors

Definition 3.1. $K_{\mathbb{N}}$ is the complete graph whose vertex set is countably infinite.

Theorem 3.2. *Every 2-colored $K_{\mathbb{N}}$ must contain a countably infinite monochromatic complete graph.*

Proof. Fix a 2-coloring of the edge of the complete graph, $K_{\mathbb{N}}$. We label each vertex with an element from \mathbb{N} and take the vertex, x , which we have labeled 1, we now consider all the edges incident with x . Since the graph is infinite, using the pigeonhole principle, there must be an infinite set of red (or blue) edges incident with x . Define X to be the infinite set vertices connected to x by a red (or blue) edge. Now consider a vertex within X , say $y > 1$. Again because the set X is infinite there must be an infinite set of blue (or red) edges incident with y and some vertex in X . Define $Y \subset X$ to be the infinite set of vertices which are connected to y by a blue (or red) edge. Now consider a vertex within Y , say z , where $z > y$. Again because the set Y is infinite there must be infinite number of red (or blue) edges connecting z to vertices in Y . Define $Z \subset Y$ to be the infinite set of vertices which are connected to z by a red (or blue) edge. We can continue picking successive vertices indefinitely since our graph is infinite, this will result in a set of vertices $V = \{x, y, z, \dots\} \subseteq K_{\mathbb{N}}$. We define E to be the set of edges connecting the vertices in V so E is $\{xy, xz, yz, \dots\}$. From this definition of the set E it is clear that the color of any edge in E is determined by the smaller of its end vertices. That is, if we assume that the color of each edge in the set of edges $\{x\bar{x} : \bar{x} \in X\}$, is red, each edge in the set of edges $\{y\bar{y} : \bar{y} \in Y\}$, is blue and each edge in the set of edges $\{z\bar{z} : \bar{z} \in Z\}$, is red. Then any edge xv for $v \in V$ must be red, any edge yv for $v \in V - \{x\}$ must be blue and any edge zv for $v \in V - \{x, y\}$ must be red. We can now produce a 2-coloring of V . We color any vertex in V , say p , red if any edge in $\{p\bar{p} : \bar{p} \in P\}$, is colored red, where P is defined in the same way that X, Y, Z were. Similarly, we color any vertex in V , say q , blue if every edge in $\{q\bar{q} : \bar{q} \in Q\}$, is color blue, where Q is defined in the same way that X, Y , and Z were. In our case the coloring of V is $\{x, y, z, \dots\}$. Since V consists of infinitely many vertices, colored with

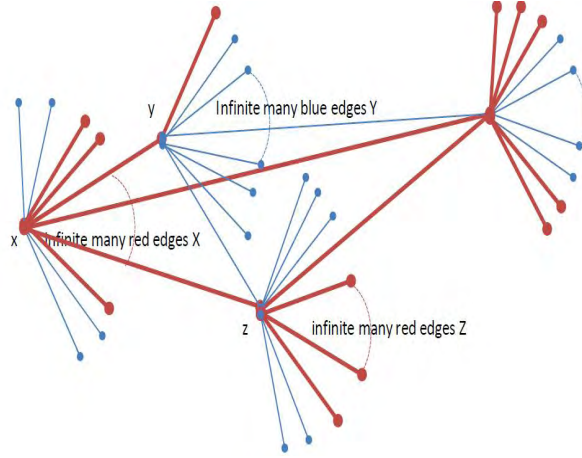


Figure 3.1: A 2-coloring of a countably infinite monochromatic complete graph

only two colors, the pigeonhole principle allows us to conclude there must be an infinite monochromatic set within V , we call this set M . This infinite set of monochromatic vertices induces an infinite monochromatic complete subgraph of $K_{\mathbb{N}}$. Each vertex in M is adjacent to every other vertex in M . Every vertex in M is the same color, so every edge in the graph induced by M must have the same color. Thus the graph induced by the vertex set M is a countably infinite monochromatic complete graph. \square

Theorem 3.3. *Every r -colored $K_{\mathbb{N}}$ must contain a countably infinite monochromatic complete graph, where $1 \leq r < \infty$.*

Proof. Suppose that $K_{\mathbb{N}}$ is colored with r colors, say $k_1, k_2, k_3, \dots, k_r$. We may produce a graph $K_{\mathbb{N}}^1$ by changing the coloring of $K_{\mathbb{N}}$. Each edge of $K_{\mathbb{N}}^1$ that was colored with k_1 in $K_{\mathbb{N}}$ is colored with c_1 in $K_{\mathbb{N}}^1$. Each edge that was colored with one of $k_1, k_2, k_3, \dots, k_{r-1}$ or k_r in $K_{\mathbb{N}}$ is colored with c_2 in $K_{\mathbb{N}}^1$. From Theorem 3.2, since $K_{\mathbb{N}}^1$ is a complete countably infinite graph colored using only two colors, c_1 and c_2 , $K_{\mathbb{N}}^1$ must contain a countably infinite monochromatic complete graph colored with c_1 or c_2 . If $K_{\mathbb{N}}^1$ contains a countably infinite monochromatic complete graph colored with c_1 we are done. Since $K_{\mathbb{N}}$ must then also contain this countably infinite monochromatic complete graph. If $K_{\mathbb{N}}^1$ contains a countably infinite monochromatic complete graph colored with c_2 , then we produce the graph $K_{\mathbb{N}}^2$. We define $K_{\mathbb{N}}^2$ to be the countably infinite monochromatic complete graph colored with $c_2 \in K_{\mathbb{N}}^1$. Each edge of $K_{\mathbb{N}}^2$ that was colored with $k_2 \in K_{\mathbb{N}}$ is colored with $m_1 \in K_{\mathbb{N}}^2$. Each edge of $K_{\mathbb{N}}^2$ that was colored with any of k_3, \dots, k_{r-1} or k_r in $K_{\mathbb{N}}$ is

colored with $m_2 \in K_{\mathbb{N}}^2$. From Theorem 3.2 since $K_{\mathbb{N}}^2$ is a countably infinite complete graph colored using only two colors, m_1 and m_2 , $K_{\mathbb{N}}^2$ must contain a countably infinite monochromatic complete graph colored with either m_1 or m_2 . If $K_{\mathbb{N}}^2$ contains a countably infinite monochromatic complete graph colored with m_1 we are done. Since $K_{\mathbb{N}}$ must then also contain this countably infinite monochromatic complete graph colored with m_2 , then we produce the graph $K_{\mathbb{N}}^3$. We may define $K_{\mathbb{N}}^3$ in exactly the same way we define $K_{\mathbb{N}}^2$ using $K_{\mathbb{N}}$. We may continue in this way, however, since $K_{\mathbb{N}}$ is only colored with a finite number of colors at some step we must find either a countable infinite monochromatic complete graph colored with one of $k_1, k_2, k_3, \dots, k_{r-2}$, or we shall define $K_{\mathbb{N}}^{r-1}$ must be a complete infinite graph, colored using only K_{r-1} and K_r . Again from Theorem 3.2 $K_{\mathbb{N}}^{r-1}$ contains accountably infinite monochromatic complete graph colored with either K_{r-1} or K_r . Since both the sets of edges colored with K_{r-1} and K_r in $K_{\mathbb{N}}^{r-1}$ are also in $K_{\mathbb{N}}$ we must have that $K_{\mathbb{N}}$ contains some countably infinite monochromatic complete graph. \square

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