Dynamics of Three-Level Laser Pumped by Electron Bombardment

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DECLARATION

I hereby declare that this PhD dissertation is my original work and has not been presented for a degree in any other university, and that all sources of material used for the dissertation have been duly acknowledged.

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Abstract

We have studied the statistical and squeezing properties of the light generated by a three-level laser coupled to a two-mode vacuum reservoir. The three-level atoms available in a closed cavity are pumped from the bottom to the top level by electron bombardment. We have carried out our analysis by taking the noise operators associated with the two-mode vacuum reservoir in arbitrary order. Applying the solutions of the equations of evolution of the cavity mode and atomic operators, we have determined the global mean photon number, the global photon number variance, and the global quadrature squeezing of the one-mode as well as the two-mode cavity light. In addition, we have calculated the local mean photon number, the local photon number variance, and the local quadrature squeezing of the two-mode cavity light.

We have found the maximum global quadrature squeezing of the two-mode cavity light to be 37.5% below the vacuum level. This result happens to be less than that obtained by putting the noise operators in normal order. On the other hand, the global photon number variance calculated by taking the noise operators in arbitrary order turns out to be greater than that obtained by putting the noise operators in normal order. In addition, our analysis shows that a large part of the mean and variance of the photon number are confined in a relatively small frequency interval.

Furthermore, employing the density operator for the superposition of a pair of two-mode cavity light beams, we have calculated the mean photon number, the

photon number variance, and the quadrature squeezing. It so happens that the global (local) mean photon number of the superposed two-mode cavity light beams is the sum of the global (local) mean photon numbers of the component two-mode cavity light beams. On the other hand, the global (local) photon number variance of the superposed two-mode cavity light beams is four times that of the separate two-mode cavity light beams. Moreover, our results show that the global (local) quadrature squeezing of the superposed two-mode cavity light beams is the same as the global (local) quadrature squeezing of the component two-mode cavity light beams.

Contents

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Introduction

Squeezed states of light have received a great deal of interest in recent years [1-4], because in these states the quantum noise in one quadrature is below the vacuum-state level at the expense of enhanced fluctuations in the conjugate quadrature, with the product of the uncertainties in the two quadrature satisfying the uncertainty relation. A light mode with reduced fluctuations in one quadrature has attractive applications in noiseless communications and precision measurements [5-6].

It has been intensively studied by different authors that various quantum optical processes can generate squeezed light [7-33]. Squeezed light can be produced by quantum optical processes such as subharmonic generation [23-33], second harmonic generation [2,4,29], and four wave mixing [2, 29]. It has been also found that squeezed light can be generated by a three-level laser under certain conditions [7-22]. A three-level laser is a quantum optical system in which light is generated by three-level atoms inside a cavity usually coupled to a vacuum reservoir. When a three-level atom makes a transition from the top to bottom level via the intermediate level, two photons are generated. The two photons are highly correlated and this correlation is responsible for the squeezing of the light produced by a three-level laser.

The statistical and squeezing properties of the light generated by a three-level laser have been investigated by several authors. Some authors have studied a three-level laser in which the top and bottom levels of the atoms injected into the cavity are coupled by coherent light [8-14]. In addition, three-level lasers in which three-level atoms initially prepared in a coherent superposition of the top and bottom levels and injected into a cavity have been studied by different authors [14-22]. These studies indicate that the three-level lasers generate under certain conditions squeezed light.

Fesseha Kassahun [7] has studied a three-level laser in which three-level atoms available in a closed cavity are pumped from the bottom to the top level by electron bombardment. He carried out his analysis by putting the vacuum noise operators in normal order which renders the vacuum reservoir to be a noiseless physical entity. He has studied the photon statistics as well as quadrature squeezing of the cavity (output) light. He has found that the light generated by the three-level laser operating under certain conditions is in a squeezed state, with the maximum global quadrature squeezing being 50 % below the vacuum-state level. Moreover, he has shown that the local quadrature squeezing is greater than the global quadrature squeezing. And the quadrature squeezing of the output light is the same as that of the cavity light in any frequency interval. On the basis of this result, he has arrived at the conclusion that the quadrature squeezing of the laser light is an intrinsic property of the individual photons.

Finally, Fesseha Kassahun [4] has considered a three-level laser in which the top and bottom levels of the three-level atoms available in a closed cavity are coupled by coherent light. His study indicates that the three-level laser generates squeezed light under certain conditions.

In this dissertation we seek to study the statistical and squeezing properties of the light generated by three-level atoms available in a closed cavity and pumped from the bottom to the top level by electron bombardment. We carry out our calculation by taking the noise operators associated with the vacuum reservoir in arbitrary order. We first determine the quantum Langevin equations and the equations of evolution of the atomic operators. Applying the solutions of these equations, we calculate the mean photon number, the photon number variances, and the quadrature variances of the single-mode cavity light beams. Moreover, employing the solution of the quantum Langevin equation for the superposition of the two single-mode cavity light beams and the solutions of the equations of evolution of the atomic operators, we determine the global mean photon number and the global photon number variance as well as the global quadrature squeezing of the two-mode light. Using the same solutions, we also calculate the local mean photon number, local photon number variance, and local quadrature squeezing of the two-mode cavity light.

Furthermore, we seek to investigate the squeezing and statistical properties of a pair of superposed two-mode light beams produced by three-level lasers in which the three-level atoms available in closed cavities are pumped from the bottom to the top level by electron bombardment. We thus first determine the Q function for the two-mode light beams. Then using the resulting Q function, we obtain the density operator for a pair of superposed two-mode light beams. Applying this density operator, we calculate the mean photon number, photon number variance, and quadrature squeezing of the superposed two-mode light beams in any frequency interval.

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Operator Dynamics

We consider the case in which N three-level atoms in a cascade configuration are available in a closed cavity and pumped from the bottom to the top level by electron bombardment. We represent the top, intermediate, and bottom levels of a three-level atom by $|a\rangle_k$, $|b\rangle_k$, and $|c\rangle_k$ respectively as shown in Figure 2.1.

We assume the transitions between levels $|a\rangle_k$ and $|b\rangle_k$ and between levels $|b\rangle_k$ and $|c\rangle_k$ to be dipole allowed, with direct transitions between levels $|a\rangle_k$ and $|c\rangle_k$ to be dipole forbidden. We consider the case for which the cavity modes are at resonance with the two transitions $|a\rangle_k \rightarrow |b\rangle_k$ and $|b\rangle_k \rightarrow |c\rangle_k$.

We carry out our analysis by taking the noise operators associated with the vacuum reservoir in an arbitrary order. We first derive the quantum Langevin equations for the cavity mode operators. Moreover, we determine the equations of evolution of the expectation value of the atomic operators in terms of the cavity mode operators. We then obtain approximately valid expressions for the cavity mode operators. Applying this result, we rewrite the equations of evolution of the expectation value of the atomic operators in simplified forms. In addition, we wish to call the light emitted from the top level light mode a_1 and the intermediate level light mode a_2. We finally include the effect of the pumping process on the the dynamics of the atomic operators.

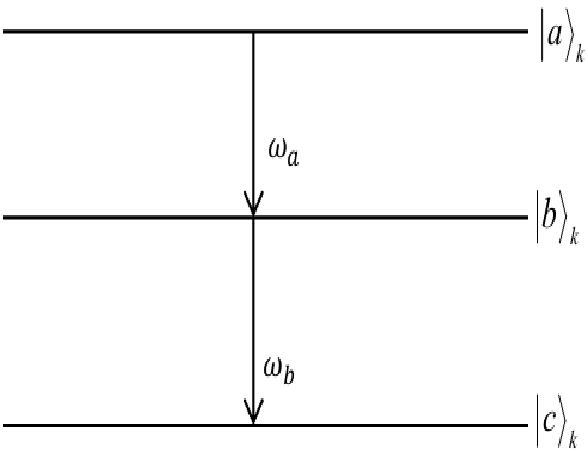

Figure 2.1: Three-level atom in a cascade configuration.

The interaction of the cavity modes with one of the three-level atom can be described at resonance by the Hamiltonian

$$\hat{H} = ig[\hat{\sigma}_a^{\dagger k}\hat{a}_1 - \hat{a}_1^{\dagger}\hat{\sigma_a}^k + \hat{\sigma}_b^{\dagger k}\hat{a}_2 - \hat{a}_2^{\dagger}\hat{\sigma_b}^k], \tag{2.1}$$

where

$$\hat{\sigma_a}^k = |b\rangle_{kk}\langle a| \tag{2.2}$$

and

$$\hat{\sigma_b}^k = |c\rangle_{kk}\langle b| \tag{2.3}$$

are lowering atomic operators, g is the coupling constant betweeen the atom and the cavity modes, and $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operators for the light modes a_1 and a_2. We assume that the cavity modes are coupled to a two-mode vacuum reservoir via a single port mirror. The quantum Langevin equations for the operators $\hat{a}_1$ and $\hat{a}_2$ are given by [34]

$$\frac{d}{dt}\hat{a}_1 = -i[\hat{a}_1, \hat{H}] - \frac{\kappa}{2}\hat{a}_1 + \hat{F}_1(t), \tag{2.4}$$

$$\frac{d}{dt}\hat{a}_2 = -i[\hat{a}_2, \hat{H}] - \frac{\kappa}{2}\hat{a}_2 + \hat{F}_2(t), \tag{2.5}$$

where κ is the cavity damping constant and $\hat{F}_1(t)$ and $\hat{F}_2(t)$ are noise operators associated with the vacuum reservoir and having the following correlation properties [35].

$$\langle \hat{F}_1(t)\rangle = \langle \hat{F}_2(t)\rangle = 0, \tag{2.6}$$

$$\langle \hat{F}_1^\dagger(t)\hat{F}_1(t')\rangle = \langle \hat{F}_2^\dagger(t)\hat{F}_2(t')\rangle = 0, \tag{2.7}$$

$$\langle \hat{F}_1(t)\hat{F}_1^\dagger(t')\rangle = \langle \hat{F}_2(t)\hat{F}_2^\dagger(t')\rangle = \kappa\delta(t-t'), \tag{2.8}$$

$$\langle \hat{F}_1^\dagger(t)\hat{F}_1^\dagger(t')\rangle = \langle \hat{F}_1(t)\hat{F}_1(t')\rangle = \langle \hat{F}_2^\dagger(t)\hat{F}_2^\dagger(t')\rangle = \langle \hat{F}_2(t)\hat{F}_2(t')\rangle = 0. \tag{2.9}$$

Now in view of Eq. (2.1), we can put Eqs. (2.4) and (2.5) in the form

$$\frac{d}{dt}\hat{a}_1(t) = -\frac{\kappa}{2}\hat{a}_1(t) - g\hat{\sigma}_a^k(t) + \hat{F}_1(t), \tag{2.10}$$

$$\frac{d}{dt}\hat{a}_2(t) = -\frac{\kappa}{2}\hat{a}_2(t) - g\hat{\sigma}_b^k(t) + \hat{F}_2(t). \tag{2.11}$$

Adding these two equations, we have

$$\frac{d}{dt}\hat{a}(t) = -\frac{\kappa}{2}\hat{a}(t) - g\hat{\sigma}^k(t) + \hat{F}(t), \tag{2.12}$$

where

$$\hat{a}(t) = \hat{a}_1(t) + \hat{a}_2(t), \tag{2.13}$$

$$\hat{\sigma}^k(t) = \hat{\sigma}_a^k(t) + \hat{\sigma}_b^k(t), \tag{2.14}$$

and

$$\hat{F}(t) = \hat{F}_1(t) + \hat{F}_2(t). \tag{2.15}$$

With $\hat{F}(t)$ satisfying the correlation properties

$$\langle \hat{F}(t)\hat{F}^\dagger(t')\rangle = 2\kappa\delta(t-t'). \tag{2.16}$$

The operator defined by Eq. (2.13) represents the annihilation operator for the superposition of light modes a_1 and a_2. In addition, applying the relation

$$\frac{d}{dt}\langle \hat{A}\rangle = -i\langle[\hat{A},\hat{H}]\rangle \tag{2.17}$$

together with Eq. (2.1), we can readily establish that

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = g\langle\hat{\eta}_b^k\hat{a}_1\rangle - g\langle\hat{\eta}_a^k\hat{a}_1\rangle + g\langle\hat{a}_2^\dagger\hat{\sigma}_c^k\rangle, \tag{2.18}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = g\langle\hat{\eta}_c^k\hat{a}_2\rangle - g\langle\hat{\eta}_b^k\hat{a}_2\rangle - g\langle\hat{a}_1^\dagger\hat{\sigma}_c^k\rangle, \tag{2.19}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle = g\langle\hat{\sigma}_b^k\hat{a}_1\rangle - g\langle\hat{\sigma}_a^k\hat{a}_2\rangle, \tag{2.20}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = g\langle\hat{\sigma}_a^{\dagger k}\hat{a}_1\rangle + g\langle\hat{a}_1^\dagger\hat{\sigma}_a^k\rangle, \tag{2.21}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = g\langle\hat{\sigma}_b^{\dagger k}\hat{a}_2\rangle - g\langle\hat{\sigma}_a^{\dagger k}\hat{a}_1\rangle + g\langle\hat{a}_2^\dagger\hat{\sigma}_b^k\rangle - g\langle\hat{a}_1^\dagger\hat{\sigma}_a^k\rangle, \tag{2.22}$$

$$\frac{d}{dt}\langle\hat{\eta}_c^k\rangle = -g\langle\hat{\sigma}_b^{\dagger k}\hat{a}_2\rangle - g\langle\hat{a}_2^\dagger\hat{\sigma}_b^k\rangle, \tag{2.23}$$

where

$$\hat{\sigma}_c^k = |c\rangle_{kk}\langle a|, \tag{2.24}$$

$$\hat{\eta}_a^k = |a\rangle_{kk}\langle a|, \tag{2.25}$$

$$\hat{\eta}_b^k = |b\rangle_{kk}\langle b|, \tag{2.26}$$

$$\hat{\eta}_c^k = |c\rangle_{kk}\langle c|. \tag{2.27}$$

We notice that Eqs. (2.18), (2.19), (2.20), (2.21), (2.22), and (2.23) are nonlinear and coupled differential equations and hence it is not possible to find exact time-dependent solutions of these equations. We intend to overcome this problem by applying the large-time approximation [36]. Using this approximation scheme, we obtain from Eqs. (2.10) and (2.11) that

$$\hat{a}_1(t) = -\frac{2g}{\kappa}\hat{\sigma}_a^k(t) + \frac{2}{\kappa}\hat{F}_1(t), \tag{2.28}$$

$$\hat{a}_2(t) = -\frac{2g}{\kappa}\hat{\sigma}_b^k(t) + \frac{2}{\kappa}\hat{F}_2(t). \tag{2.29}$$

Therefore on account of Eqs. (2.28) and (2.29), we write Eqs. (2.18)-(2.23) as

$$\frac{d}{dt}\langle\hat{\sigma}_a^k(t)\rangle = -\gamma_c\langle\hat{\sigma}_a^k(t)\rangle + \frac{2g}{\kappa}\bigg(\langle\hat{\eta}_b^k(t)\hat{F}_1(t)\rangle - \langle\hat{\eta}_a^k(t)\hat{F}_1(t)\rangle + \langle\hat{F}_2^\dagger(t)\hat{\sigma}_c^k(t)\rangle\bigg), \tag{2.30}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^k(t)\rangle = -\frac{\gamma_c}{2}\langle\hat{\sigma}_b^k(t)\rangle + \frac{2g}{\kappa}\bigg(\langle\hat{\eta}_c^k(t)\hat{F}_2(t)\rangle - \langle\hat{\eta}_b^k(t)\hat{F}_2(t)\rangle - \langle\hat{F}_1^\dagger(t)\hat{\sigma}_c^k(t)\rangle\bigg), \quad (2.31)$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^k(t)\rangle = -\frac{\gamma_c}{2}\langle\hat{\sigma}_c^k(t)\rangle + \frac{2g}{\kappa}\bigg(\langle\hat{\sigma}_b^k(t)\hat{F}_1(t)\rangle - \langle\hat{\sigma}_a^k(t)\hat{F}_2(t)\rangle\bigg), \quad (2.32)$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k(t)\rangle = -\gamma_c\langle\hat{\eta}_a^k(t)\rangle + \frac{2g}{\kappa}\bigg(\langle\hat{\sigma}_a^{\dagger k}(t)\hat{F}_1(t)\rangle + \langle\hat{F}_1^\dagger(t)\hat{\sigma}_a^k(t)\rangle\bigg), \quad (2.33)$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k(t)\rangle = -\gamma_c\langle\hat{\eta}_b^k(t)\rangle + \gamma_c\langle\hat{\eta}_a^k(t)\rangle + \frac{2g}{\kappa}\bigg(\langle\hat{\sigma}_b^{\dagger k}(t)\hat{F}_2(t)\rangle - \langle\hat{\sigma}_a^{\dagger k}(t)\hat{F}_1(t)\rangle$$

$$+\langle\hat{F}_2^\dagger(t)\hat{\sigma}_b^k(t)\rangle - \langle\hat{F}_1^\dagger(t)\hat{\sigma}_a^k(t)\rangle\bigg), \quad (2.34)$$

$$\frac{d}{dt}\langle\hat{\eta}_c^k(t)\rangle = \gamma_c\langle\hat{\eta}_b^k(t)\rangle - \frac{2g}{\kappa}\bigg(\langle\hat{\sigma}_b^{\dagger k}(t)\hat{F}_2(t)\rangle + \langle\hat{F}_2^\dagger(t)\hat{\sigma}_b^k(t)\rangle\bigg), \quad (2.35)$$

where

$$\gamma_c = \frac{4g^2}{\kappa} \quad (2.36)$$

is the stimulated emission decay constant. We can also write the above equations in the form

$$\frac{d}{dt}\hat{\sigma}_a^k(t) = -\gamma_c\hat{\sigma}_a^k(t) + \frac{2g}{\kappa}\bigg(\hat{\eta}_b^k(t)\hat{F}_1(t) - \hat{\eta}_a^k(t)\hat{F}_1(t) + \hat{F}_2^\dagger(t)\hat{\sigma}_c^k(t)\bigg) + \hat{f}_A(t), \quad (2.37)$$

$$\frac{d}{dt}\hat{\sigma}_b^k(t) = -\frac{\gamma_c}{2}\hat{\sigma}_b^k(t) + \frac{2g}{\kappa}\bigg(\hat{\eta}_c^k(t)\hat{F}_2(t) - \hat{\eta}_b^k(t)\hat{F}_2(t) - \hat{F}_1^\dagger(t)\hat{\sigma}_c^k(t)\bigg) + \hat{f}_B(t), \quad (2.38)$$

$$\frac{d}{dt}\hat{\sigma}_c^k(t) = -\frac{\gamma_c}{2}\hat{\sigma}_c^k(t) + \frac{2g}{\kappa}\bigg(\hat{\sigma}_b^k(t)\hat{F}_1(t) - \hat{\sigma}_a^k(t)\hat{F}_2(t)\bigg) + \hat{f}_C(t), \quad (2.39)$$

$$\frac{d}{dt}\hat{\eta}_a^k(t) = -\gamma_c\hat{\eta}_a^k(t) + \frac{2g}{\kappa}\bigg(\hat{\sigma}_a^{\dagger k}(t)\hat{F}_1(t) + \hat{F}_1^\dagger(t)\hat{\sigma}_a^k(t)\bigg) + \hat{f}_a(t), \quad (2.40)$$

$$\frac{d}{dt}\hat{\eta}_b^k(t) = -\gamma_c\hat{\eta}_b^k(t) + \gamma_c\hat{\eta}_a^k(t) + \frac{2g}{\kappa}\bigg(\hat{\sigma}_b^{\dagger k}(t)\hat{F}_2(t) - \hat{\sigma}_a^{\dagger k}(t)\hat{F}_1(t)$$

$$+\hat{F}_2^\dagger(t)\hat{\sigma}_b^k(t) - \hat{F}_1^\dagger(t)\hat{\sigma}_a^k(t)\bigg) + \hat{f}_b(t), \quad (2.41)$$

$$\frac{d}{dt}\hat{\eta}_c^k(t) = \gamma_c\hat{\eta}_b^k(t) - \frac{2g}{\kappa}\bigg(\hat{\sigma}_b^{\dagger k}(t)\hat{F}_2(t) + \hat{F}_2^\dagger(t)\hat{\sigma}_b^k(t)\bigg) + \hat{f}_c(t), \quad (2.42)$$

where $\hat{f}_A(t)$, $\hat{f}_B(t)$, $\hat{f}_C(t)$, $\hat{f}_a(t)$, $\hat{f}_b(t)$, and $\hat{f}_c(t)$ are atomic noise operators with a vanishing mean.

We next seek to obtain the expectation value of the product of an atomic operator and a noise operator involved in Eqs. (2.30) - (2.35). To this end, the formal solution of Eq. (2.41) can be written as

$$\hat{\eta}_b^k(t) = \hat{\eta}_b^k(0)e^{-\gamma_c t} + \int_0^t e^{-\gamma_c(t-t')}\bigg[\gamma_c\hat{\eta}_a^k(t') + \frac{2g}{\kappa}\bigg(\hat{\sigma}_b^{\dagger k}(t')\hat{F}_2(t')$$

$$-\hat{\sigma}_a^{\dagger k}(t')\hat{F}_1(t') + \hat{F}_2^\dagger(t')\hat{\sigma}_b^k(t') - \hat{F}_1^\dagger(t')\hat{\sigma}_a^k(t')\bigg) + \hat{f}_b(t')\bigg]dt', \tag{2.43}$$

so that multiplying the above equation on the right by $\hat{F}_1(t)$ and taking the expectation value, we find

$$\langle\hat{\eta}_b^k(t)\hat{F}_1(t)\rangle = \langle\hat{\eta}_b^k(0)\hat{F}_1(t)\rangle e^{-\gamma_c t} + \int_0^t e^{-\gamma_c(t-t')}\bigg[\gamma_c\langle\hat{\eta}_a^k(t')\hat{F}_1(t)\rangle$$

$$+\frac{2g}{\kappa}\bigg(\langle\hat{\sigma}_b^{\dagger k}(t')\hat{F}_2(t')\hat{F}_1(t)\rangle - \langle\hat{\sigma}_a^{\dagger k}(t')\hat{F}_1(t')\hat{F}_1(t)\rangle$$

$$+\langle\hat{F}_2^\dagger(t')\hat{\sigma}_b^k(t')\hat{F}_1(t)\rangle - \langle\hat{F}_1^\dagger(t')\hat{\sigma}_a^k(t')\hat{F}_1(t)\rangle\bigg)$$

$$+\langle\hat{f}_b(t')\hat{F}_1(t)\rangle\bigg]dt'. \tag{2.44}$$

It is not possible to evaluate the above integral unless we adopt some approximation scheme. To this end, ignoring the noncommutativity of the atomic and noise operators [37], we see that

$$\langle\hat{F}_2^\dagger(t')\hat{\sigma}_b^k(t')\hat{F}_1(t)\rangle = \langle\hat{\sigma}_b^k(t')\hat{F}_2^\dagger(t')\hat{F}_1(t)\rangle, \tag{2.45}$$

$$\langle\hat{F}_1^\dagger(t')\hat{\sigma}_a^k(t')\hat{F}_1(t)\rangle = \langle\hat{\sigma}_a^k(t')\hat{F}_1^\dagger(t')\hat{F}_1(t)\rangle. \tag{2.46}$$

Then upon neglecting the correlation betweeen the atomic operators and the cavity mode noise operators, assumed to be considerably small [37], we can get the approximately valid expression as

$$\langle\hat{F}_2^\dagger(t')\hat{\sigma}_b^k(t')\hat{F}_1(t)\rangle = \langle\hat{\sigma}_b^k(t')\rangle\langle\hat{F}_2^\dagger(t')\hat{F}_1(t)\rangle, \tag{2.47}$$

$$\langle\hat{F}_1^\dagger(t')\hat{\sigma}_a^k(t')\hat{F}_1(t)\rangle = \langle\hat{\sigma}_a^k(t')\rangle\langle\hat{F}_1^\dagger(t')\hat{F}_1(t)\rangle, \tag{2.48}$$

$$\langle\hat{\sigma}_b^{\dagger k}(t')\hat{F}_2(t')\hat{F}_1(t)\rangle = \langle\hat{\sigma}_b^{\dagger k}(t')\rangle\langle\hat{F}_2(t')\hat{F}_1(t)\rangle, \tag{2.49}$$

$$\langle\hat{\sigma}_a^{\dagger k}(t')\hat{F}_1(t')\hat{F}_1(t)\rangle = \langle\hat{\sigma}_a^{\dagger k}(t')\rangle\langle\hat{F}_1(t')\hat{F}_1(t)\rangle. \tag{2.50}$$

Moreover, upon neglecting the correlation betweeen the two noise operators, we find

$$\langle\hat{F}_2^{\dagger}(t')\hat{F}_1(t)\rangle = \langle\hat{F}_2^{\dagger}(t')\rangle\langle\hat{F}_1(t)\rangle, \tag{2.51}$$

$$\langle\hat{F}_2(t')\hat{F}_1(t)\rangle = \langle\hat{F}_2(t')\rangle\langle\hat{F}_1(t)\rangle, \tag{2.52}$$

so that in view of these results, we write Eqs. (2.47) and (2.49) as

$$\langle\hat{F}_2^{\dagger}(t')\hat{\sigma}_b^{k}(t')\hat{F}_1(t)\rangle = \langle\hat{\sigma}_b^{k}(t')\rangle\langle\hat{F}_2^{\dagger}(t')\rangle\langle\hat{F}_1(t)\rangle, \tag{2.53}$$

$$\langle\hat{\sigma}_b^{\dagger k}(t')\hat{F}_2(t')\hat{F}_1(t)\rangle = \langle\hat{\sigma}_b^{\dagger k}(t')\rangle\langle\hat{F}_2(t')\rangle\langle\hat{F}_1(t)\rangle. \tag{2.54}$$

Now on account of Eqs. (2.6), (2.7), (2.9), Eqs. (2.48), (2.50), (2.53), and (2.54) can be written as

$$\langle\hat{F}_2^{\dagger}(t')\hat{\sigma}_b^{k}(t')\hat{F}_1(t)\rangle = 0, \tag{2.55}$$

$$\langle\hat{F}_1^{\dagger}(t')\hat{\sigma}_a^{k}(t')\hat{F}_1(t)\rangle = 0, \tag{2.56}$$

$$\langle\hat{\sigma}_b^{\dagger k}(t')\hat{F}_2(t')\hat{F}_1(t)\rangle = 0, \tag{2.57}$$

$$\langle\hat{\sigma}_a^{\dagger k}(t')\hat{F}_1(t')\hat{F}_1(t)\rangle = 0. \tag{2.58}$$

Thus in view of Eqs. (2.55)-(2.58), we can put Eq. (2.44) in the form

$$\langle\hat{\eta}_b^{k}(t)\hat{F}_1(t)\rangle = \langle\hat{\eta}_b^{k}(0)\hat{F}_1(t)\rangle e^{-\gamma_c t} + \int_0^t e^{-\gamma_c(t-t')}\bigg(\gamma_c\langle\hat{\eta}_a^{k}(t')\hat{F}_1(t)\rangle + \langle\hat{f}_b(t')\hat{F}_1(t)\rangle\bigg)dt'. \tag{2.59}$$

Since the noise operator at a certain time does not affect the atomic operator at an earlier time and the fact that $\hat{f}_b(t)$ and $\hat{F}_1(t)$ are uncorrelated, we have

$$\langle\hat{\eta}_b^{k}(t)\hat{F}_1(t)\rangle = 0. \tag{2.60}$$

We also see that

$$\langle \hat{F}_1^\dagger(t)\hat{\eta}_b^k(t)\rangle = 0. \tag{2.61}$$

Following the same procedure, we readily obtain

$$\langle \hat{\eta}_b^k(t)\hat{F}_2(t)\rangle = \langle \hat{F}_2^\dagger(t)\hat{\eta}_b^k(t)\rangle = 0, \tag{2.62}$$

$$\langle \hat{\eta}_a^k(t)\hat{F}_1(t)\rangle = \langle \hat{F}_1^\dagger(t)\hat{\eta}_a^k(t)\rangle = \langle \hat{\eta}_c^k(t)\hat{F}_2(t)\rangle = \langle \hat{F}_2^\dagger(t)\hat{\eta}_c^k(t)\rangle = 0, \tag{2.63}$$

$$\langle \hat{F}_2^\dagger(t)\hat{\sigma}_c^k(t)\rangle = \langle \hat{\sigma}_c^{\dagger k}(t)\hat{F}_2(t)\rangle = \langle \hat{F}_1^\dagger(t)\hat{\sigma}_c^k(t)\rangle = \langle \hat{\sigma}_c^{\dagger k}(t)\hat{F}_1(t)\rangle = 0, \tag{2.64}$$

$$\langle \hat{\sigma}_a^k(t)\hat{F}_2(t)\rangle = \langle \hat{F}_2^\dagger(t)\hat{\sigma}_a^{\dagger k}(t)\rangle = \langle \hat{\sigma}_b^k(t)\hat{F}_1(t)\rangle = \langle \hat{F}_1^\dagger(t)\hat{\sigma}_b^{\dagger k}(t)\rangle = 0, \tag{2.65}$$

$$\langle \hat{\sigma}_a^{\dagger k}(t)\hat{F}_1(t)\rangle = \langle \hat{F}_1^\dagger(t)\hat{\sigma}_a^k(t)\rangle = \langle \hat{\sigma}_b^{\dagger k}(t)\hat{F}_2(t)\rangle = \langle \hat{F}_2^\dagger(t)\hat{\sigma}_b^k(t)\rangle = 0. \tag{2.66}$$

Finally, with the aid of Eqs. (2.60) - (2.66), Eqs. (2.30)-(2.35) reduce to

$$\frac{d}{dt}\langle \hat{\sigma}_a^k\rangle = -\gamma_c\langle \hat{\sigma}_a^k\rangle, \tag{2.67}$$

$$\frac{d}{dt}\langle \hat{\sigma}_b^k\rangle = -\frac{1}{2}\gamma_c\langle \hat{\sigma}_b^k\rangle, \tag{2.68}$$

$$\frac{d}{dt}\langle \hat{\sigma}_c^k\rangle = -\frac{1}{2}\gamma_c\langle \hat{\sigma}_c^k\rangle, \tag{2.69}$$

$$\frac{d}{dt}\langle \hat{\eta}_a^k\rangle = -\gamma_c\langle \hat{\eta}_a^k\rangle, \tag{2.70}$$

$$\frac{d}{dt}\langle \hat{\eta}_b^k\rangle = -\gamma_c\langle \hat{\eta}_b^k\rangle + \gamma_c\langle \hat{\eta}_a^k\rangle, \tag{2.71}$$

$$\frac{d}{dt}\langle \hat{\eta}_c^k\rangle = \gamma_c\langle \hat{\eta}_b^k\rangle. \tag{2.72}$$

We see that the noise operators associated with the two-mode vacuum reservoir do not affect the dynamics of the atomic operators.

The three-level atoms are pumped from the bottom to the top level by electron bombardment. The pumping process must certainly affect the dynamics of $\langle \hat{\eta}_a^k\rangle$ and $\langle \hat{\eta}_c^k\rangle$. If r_a represents the rate at which a single atom is pumped from the bottom to the top level, then $\langle \hat{\eta}_a^k\rangle$ increases at the rate of $r_a\langle \hat{\eta}_c^k\rangle$ and $\langle \hat{\eta}_c^k\rangle$ decreases at the same rate [4]. Hence taking into account the pumping process, Eqs. (2.70) and (2.72) can be rewritten as

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -\gamma_c\langle\hat{\eta}_a^k\rangle + r_a\langle\hat{\eta}_c^k\rangle \tag{2.73}$$

and

$$\frac{d}{dt}\langle\hat{\eta}_c^k\rangle = \gamma_c\langle\hat{\eta}_b^k\rangle - r_a\langle\hat{\eta}_c^k\rangle. \tag{2.74}$$

We next seek to determine the effect of the pumping process on the dynamics of the atomic operators $\hat{\sigma}_a^k$ and $\hat{\sigma}_b^k$. Thus we rewrite Eqs. (2.67) and (2.68) as

$$\frac{d}{dt}\hat{\sigma}_a^k(t) = -\frac{\mu}{2}\hat{\sigma}_a^k(t) + \hat{f}_1(t), \tag{2.75}$$

$$\frac{d}{dt}\hat{\sigma}_b^k(t) = -\frac{\mu}{2}\hat{\sigma}_b^k(t) + \hat{f}_2(t), \tag{2.76}$$

where $\hat{f}_1(t)$ and $\hat{f}_2(t)$ are noise operators with a vanishing mean and μ is a parameter whose value remains to be determined. Applying the relation

$$\frac{d}{dt}\left\langle\hat{\sigma}_a^{\dagger k}(t)\hat{\sigma}_a^k(t)\right\rangle = \left\langle\frac{d\hat{\sigma}_a^{\dagger k}(t)}{dt}\hat{\sigma}_a^k(t)\right\rangle + \left\langle\hat{\sigma}_a^{\dagger k}(t)\frac{d\hat{\sigma}_a^k(t)}{dt}\right\rangle, \tag{2.77}$$

together with Eq. (2.75), we see that

$$\frac{d}{dt}\left\langle\hat{\sigma}_a^{\dagger k}(t)\hat{\sigma}_a^k(t)\right\rangle = -\mu\langle\hat{\sigma}_a^{\dagger k}(t)\hat{\sigma}_a^k(t)\rangle + \langle\hat{f}_1^\dagger(t)\hat{\sigma}_a^k(t)\rangle + \langle\hat{\sigma}_a^{\dagger k}(t)\hat{f}_1(t)\rangle. \tag{2.78}$$

This can also be put in the form

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -\mu\langle\hat{\eta}_a^k\rangle + \langle\hat{f}_1^\dagger(t)\hat{\sigma}_a^k(t)\rangle + \langle\hat{\sigma}_a^{\dagger k}(t)\hat{f}_1(t)\rangle. \tag{2.79}$$

On the other hand, employing the relation

$$\langle\hat{\eta}_a^k\rangle + \langle\hat{\eta}_b^k\rangle + \langle\hat{\eta}_c^k\rangle = 1, \tag{2.80}$$

we can put Eq. (2.73) in the form

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -\gamma_c\langle\hat{\eta}_a^k\rangle + r_a(1 - \langle\hat{\eta}_a^k\rangle - \langle\hat{\eta}_b^k\rangle). \tag{2.81}$$

Moreover, applying the large-time approximation to Eq. (2.71), we find

$$\langle \hat{\eta}_a^k(t)\rangle = \langle \hat{\eta}_b^k(t)\rangle \tag{2.82}$$

and on substituting this result into Eq. (2.81), we see that

$$\frac{d}{dt}\langle \hat{\eta}_a^k(t)\rangle = -(\gamma_c + 2r_a)\langle \hat{\eta}_a^k(t)\rangle + r_a. \tag{2.83}$$

Thus upon comparing Eqs. (2.79) and (2.83), we have

$$\mu = \gamma_c + 2r_a, \tag{2.84}$$

$$\langle \hat{f}_1^\dagger(t)\hat{\sigma}_a^k(t)\rangle + \langle \hat{\sigma}_a^{\dagger k}(t)\hat{f}_1(t)\rangle = r_a. \tag{2.85}$$

The formal solution of Eq.(2.75)can be written as

$$\hat{\sigma}_a^k(t) = \hat{\sigma}_a^k(0)e^{-\mu t/2} + e^{-\mu t/2}\int_0^t e^{\mu t'/2}\hat{f}_1(t')dt', \tag{2.86}$$

so that multiplying both sides of this equation from the left by $\hat{f}_1^\dagger(t)$ and taking the expectation value, we get

$$\langle \hat{f}_1^\dagger(t)\hat{\sigma}_a^k(t)\rangle = \langle \hat{f}_1^\dagger(t)\hat{\sigma}_a^k(0)\rangle e^{-\mu t/2} + e^{-\mu t/2}\int_0^t e^{\mu t'/2}\langle \hat{f}_1^\dagger(t)\hat{f}_1(t')\rangle dt'. \tag{2.87}$$

In view of the fact that a noise operator at a certain time does not affect the atomic operator at earlier time, we find

$$\langle \hat{f}_1^\dagger(t)\hat{\sigma}_a^k(t)\rangle = e^{-\mu t/2}\int_0^t e^{\mu t'/2}\langle \hat{f}_1^\dagger(t)\hat{f}_1(t')\rangle dt'. \tag{2.88}$$

Following a similar procedure, we easily establish that

$$\langle \hat{\sigma}_a^{\dagger k}(t)\hat{f}_1(t)\rangle = e^{-\mu t/2}\int_0^t e^{\mu t'/2}\langle \hat{f}_1^\dagger(t')\hat{f}_1(t)\rangle dt'. \tag{2.89}$$

Hence substitution of Eqs. (2.88) and (2.89) into Eq. (2.85) leads to

$$e^{-\mu t/2}\int_0^t e^{\mu t'/2}\langle \hat{f}_1^\dagger(t)\hat{f}_1(t')\rangle dt' + e^{-\mu t/2}\int_0^t e^{\mu t'/2}\langle \hat{f}_1^\dagger(t')\hat{f}_1(t)\rangle dt' = r_a. \tag{2.90}$$

Now we assume that

$$\langle \hat{f}_1^\dagger(t)\hat{f}_1(t')\rangle = \langle \hat{f}_1^\dagger(t')\hat{f}_1(t)\rangle. \tag{2.91}$$

Then in view of this, we write Eq. (2.90) as

$$2e^{-\mu t/2}\int_0^t e^{\mu t'/2}\langle \hat{f}_1^\dagger(t)\hat{f}_1(t')\rangle dt' = r_a, \tag{2.92}$$

from which follows

$$\langle \hat{f}_1^\dagger(t)\hat{f}_1(t')\rangle = r_a\delta(t-t'). \tag{2.93}$$

On the other hand, employing the relation

$$\frac{d}{dt}\left\langle \hat{\sigma}_b^{\dagger k}(t)\hat{\sigma}_b^k(t)\right\rangle = \left\langle \frac{d\hat{\sigma}_b^{\dagger k}(t)}{dt}\hat{\sigma}_b^k(t)\right\rangle + \left\langle \hat{\sigma}_b^{\dagger k}(t)\frac{d\hat{\sigma}_b^k(t)}{dt}\right\rangle, \tag{2.94}$$

along with Eq. (2.76), we have

$$\frac{d}{dt}\left\langle \hat{\sigma}_b^{\dagger k}(t)\hat{\sigma}_b^k(t)\right\rangle = -\mu\langle \hat{\sigma}_b^{\dagger k}(t)\hat{\sigma}_b^k(t)\Big\rangle + \langle \hat{f}_2^\dagger(t)\hat{\sigma}_b^k(t)\rangle + \langle \hat{\sigma}_b^{\dagger k}(t)\hat{f}_2(t)\rangle. \tag{2.95}$$

It then follows that

$$\frac{d\langle \hat{\eta}_b^k(t)\rangle}{dt} = -\mu\langle \hat{\eta}_b^k(t)\rangle + \langle \hat{f}_2^\dagger(t)\hat{\sigma}_b^k(t)\rangle + \langle \hat{\sigma}_b^{\dagger k}(t)\hat{f}_2(t)\rangle. \tag{2.96}$$

Now on applying the large-time approximation to Eq. (2.73), we find

$$\langle \hat{\eta}_a^k(t)\rangle = \frac{r_a}{\gamma_c}\langle \hat{\eta}_c^k(t)\rangle. \tag{2.97}$$

Hence on account of this result, we put Eq. (2.71) in the form

$$\frac{d\langle \hat{\eta}_b^k(t)\rangle}{dt} = -\gamma_c\langle \hat{\eta}_b^k\rangle + r_a\langle \hat{\eta}_c^k\rangle \tag{2.98}$$

and in view of Eqs. (2.80) and (2.82), we see that

$$\frac{d\langle \hat{\eta}_b^k(t)\rangle}{dt} = -(\gamma_c + 2r_a)\langle \hat{\eta}_b^k(t)\rangle + r_a. \tag{2.99}$$

Therefore, comparision of Eqs. (2.96) and (2.99) leads to

$$\mu = \gamma_c + 2r_a, \tag{2.100}$$

$$\langle \hat{f}_2^\dagger(t)\hat{\sigma}_b^k(t)\rangle + \langle \hat{\sigma}_b^{\dagger k}(t)\hat{f}_2(t)\rangle = r_a. \tag{2.101}$$

This expression is equivalent to

$$\langle \hat{f}_2^\dagger(t)\hat{f}_2(t')\rangle = r_a\delta(t-t'). \tag{2.102}$$

Finally, upon adding Eqs. (2.75) and (2.76), we have

$$\frac{d}{dt}\hat{\sigma}^k(t) = -\frac{\mu}{2}\hat{\sigma}^k(t) + \hat{f}(t), \tag{2.103}$$

in which

$$\hat{f}(t) = \hat{f}_1(t) + \hat{f}_2(t). \tag{2.104}$$

Upon summing Eqs. (2.67), (2.68), (2.69), (2.71), (2.73), and (2.74) over the N three-level atoms, we get

$$\frac{d}{dt}\langle \hat{m}_a\rangle = -\gamma_c\langle \hat{m}_a\rangle, \tag{2.105}$$

$$\frac{d}{dt}\langle \hat{m}_b\rangle = -\frac{1}{2}\gamma_c\langle \hat{m}_b\rangle, \tag{2.106}$$

$$\frac{d}{dt}\langle \hat{m}_c\rangle = -\frac{1}{2}\gamma_c\langle \hat{m}_c\rangle, \tag{2.107}$$

$$\frac{d}{dt}\langle \hat{N}_a\rangle = -\gamma_c\langle \hat{N}_a\rangle + r_a\langle \hat{N}_c\rangle, \tag{2.108}$$

$$\frac{d}{dt}\langle \hat{N}_b\rangle = -\gamma_c\langle N_b\rangle + \gamma_c\langle \hat{N}_a\rangle, \tag{2.109}$$

$$\frac{d}{dt}\langle \hat{N}_c\rangle = \gamma_c\langle \hat{N}_b\rangle - r_a\langle \hat{N}_c\rangle, \tag{2.110}$$

where

$$\hat{m}_a = \sum_{k=1}^{N}\hat{\sigma}_a^k, \tag{2.111}$$

$$\hat{m}_b = \sum_{k=1}^{N}\hat{\sigma}_b^k, \tag{2.112}$$

$$\hat{m}_c = \sum_{k=1}^{N}\hat{\sigma}_c^k, \tag{2.113}$$

$$\hat{N}_a = \sum_{k=1}^{N}\hat{\eta}_a^k, \tag{2.114}$$

$$\hat{N}_b = \sum_{k=1}^{N} \hat{\eta}_b^k, \tag{2.115}$$

$$\hat{N}_c = \sum_{k=1}^{N} \hat{\eta}_c^k. \tag{2.116}$$

The operators $\hat{N}_a$, $\hat{N}_b$, and $\hat{N}_c$ represent the number of atoms in the top, intermediate, and bottom levels. Moreover, applying the completeness relation described by Eq. (2.80) we readily get

$$\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle = N. \tag{2.117}$$

3

One-Mode Photon Statistics and Quadrature Variance

We now seek to study the photon statistics and the quadrature variances of one-mode light beams produced by a three-level laser coupled to a two-mode vacuum reservoir. The three-level atoms available in a closed cavity are pumped from the bottom to the top level by electron bombardment. Employing the solutions of the equations of evolution of the atomic and the cavity mode operators, we calculate the mean photon number, the variance of the photon number, and the variance of the quadrature operators for light modes a_1 and a_2.

3.1 One-Mode photon statistics

In this section we seek to determine the mean photon number and the variance of the photon number of light modes a_1 and a_2. To this end, making use of Eqs. (2.108) and (2.117), we see that

$$\frac{d}{dt}\langle \hat{N}_a\rangle = -(\gamma_c + r_a)\langle \hat{N}_a\rangle + r_a(N - \langle \hat{N}_b\rangle) \tag{3.1}$$

and applying the large-time approximation to Eqs. (2.108), (2.109), and (2.110), we have

$$\langle \hat{N}_a\rangle = \frac{r_a}{\gamma_c}\langle \hat{N}_c\rangle, \tag{3.2}$$

$$\langle \hat{N}_a\rangle = \langle \hat{N}_b\rangle, \tag{3.3}$$

$$\langle \hat{N}_c \rangle = \frac{\gamma_c}{r_a} \langle \hat{N}_b \rangle. \tag{3.4}$$

Thus taking into account Eq. (3.3), we can rewrite Eq. (3.1) as

$$\frac{d}{dt} \langle \hat{N}_a \rangle = -(\gamma_c + 2r_a) \langle \hat{N}_a \rangle + r_a N. \tag{3.5}$$

The steady-state solution of this equation has the form

$$\langle \hat{N}_a \rangle = \frac{r_a N}{\gamma_c + 2r_a}. \tag{3.6}$$

In addition, applying the relation

$$\frac{d}{dt} \left\langle \hat{a}_1^\dagger(t) \hat{a}_1(t) \right\rangle = \left\langle \frac{d\hat{a}_1^\dagger(t)}{dt} \hat{a}_1(t) \right\rangle + \left\langle \hat{a}_1^\dagger(t) \frac{d\hat{a}_1(t)}{dt} \right\rangle \tag{3.7}$$

along with Eq. (2.10), we have

$$\frac{d}{dt} \left\langle \hat{a}_1^\dagger(t) \hat{a}_1(t) \right\rangle_k = -\kappa \langle \hat{a}_1^\dagger(t) \hat{a}_1(t) \rangle_k - g\Big(\langle \hat{\sigma}_a^{\dagger k}(t) \hat{a}_1(t) \rangle + \langle \hat{a}_1^\dagger(t) \hat{\sigma}_a^k(t) \rangle \Big)$$

$$+ \Big(\langle \hat{F}_1^\dagger(t) \hat{a}_1(t) \rangle + \langle \hat{a}_1^\dagger(t) \hat{F}_1(t) \rangle \Big). \tag{3.8}$$

Next we wish to determine an explicit expression for $\langle \hat{\sigma}_a^{\dagger k}(t) \hat{a}_1(t) \rangle$. Now upon multiplying Eq. (2.28) on the left by $\hat{\sigma}_a^{\dagger k}(t)$ and taking the expectation value of the resulting expression, we see that

$$\langle \hat{\sigma}_a^{\dagger k}(t) \hat{a}_1(t) \rangle = -\frac{2g}{\kappa} \langle \eta_a^k(t) \rangle + \frac{2}{\kappa} \langle \hat{\sigma}_a^{\dagger k}(t) \hat{F}_1(t) \rangle. \tag{3.9}$$

Now on substituting this result and its complex conjugate into Eq. (3.8), we have

$$\frac{d}{dt} \left\langle \hat{a}_1^\dagger(t) \hat{a}_1(t) \right\rangle_k = -\kappa \langle \hat{a}_1^\dagger(t) \hat{a}_1(t) \rangle_k + \gamma_c \langle \hat{\eta}_a^k(t) \rangle - \frac{2g}{\kappa} \Big(\langle \hat{\sigma}_a^{\dagger k}(t) \hat{F}_1(t) \rangle$$

$$+ \langle \hat{F}_1^\dagger(t) \hat{\sigma}_a^k(t) \rangle \Big) + \Big(\langle \hat{F}_1^\dagger(t) \hat{a}_1(t) \rangle + \langle \hat{a}_1^\dagger(t) \hat{F}_1(t) \rangle \Big). \tag{3.10}$$

We next seek to determine the explicit form of $\langle \hat{F}_1^\dagger(t) \hat{a}_1(t) \rangle$. To this end, we note that the solution of Eq. (2.10) can be written as

$$\hat{a}_1(t) = \hat{a}_1(0)e^{-\kappa t/2} + e^{-\kappa t/2}\int_0^t e^{\kappa t'/2}\Big(-g\hat{\sigma}_a^k(t') + \hat{F}_1(t')\Big)dt'. \tag{3.11}$$

Moreover, multiplying this equation on the left by $\hat{F}_1^\dagger(t)$ and taking the expectation value of the resulting expression, we have

$$\langle\hat{F}_1^\dagger(t)\hat{a}_1(t)\rangle = \langle\hat{F}_1^\dagger(t)\hat{a}_1(0)\rangle e^{-\kappa t/2} + e^{-\kappa t/2}\int_0^t e^{\kappa t'/2}\Big(-g\langle\hat{F}_1^\dagger(t)\hat{\sigma}_a^k(t')\rangle$$
$$+\langle\hat{F}_1^\dagger(t)\hat{F}_1(t')\rangle\Big)dt'. \tag{3.12}$$

Since a noise operator at a certain time does not affect a system variable at earlier time, we get

$$\langle\hat{F}_1^\dagger(t)\hat{a}_1(t)\rangle = e^{-\kappa t/2}\int_0^t e^{\kappa t'/2}\langle\hat{F}_1^\dagger(t)\hat{F}_1(t')\rangle dt', \tag{3.13}$$

so that on account of Eq. (2.7), we see that

$$\langle\hat{F}_1^\dagger(t)\hat{a}_1(t)\rangle = 0. \tag{3.14}$$

Therefore, with the help of this result and its complex conjugate, we can rewrite Eq. (3.10) as

$$\frac{d}{dt}\Big\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\Big\rangle_k = -\kappa\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle_k + \gamma_c\langle\hat{\eta}_a^k(t)\rangle - \frac{2g}{\kappa}\Big(\langle\hat{\sigma}_a^{\dagger k}(t)\hat{F}_1(t)\rangle$$
$$+\langle\hat{F}_1^\dagger(t)\hat{\sigma}_a^k(t)\rangle\Big). \tag{3.15}$$

We next proceed to calculate the expectation value of the product of the atomic and noise operators involved in this expression. To this end, multiplying Eq. (2.86) on the left by $\hat{F}_1^\dagger(t)$ and taking the expectation value of the resulting expression, we have

$$\langle\hat{F}_1^\dagger(t)\hat{\sigma}_a^k(t)\rangle = \langle\hat{F}_1^\dagger(t)\hat{\sigma}_a^k(0)\rangle e^{-\mu t/2} + e^{-\mu t/2}\int_0^t e^{\mu t'/2}\langle\hat{F}_1^\dagger(t)\hat{f}_1(t')\rangle dt'. \tag{3.16}$$

Since the two noise operators that appear in Eq. (3.16) are uncorrelated along with the fact that a noise operator at a certain time does not affect system variables at earlier time, we have

$$\langle \hat{F}_1^\dagger(t)\hat{\sigma}_a^k(t)\rangle = 0. \tag{3.17}$$

Now in view of this result and its complex conjugate, we can put Eq. (3.15) in the form

$$\frac{d}{dt}\left\langle \hat{a}_1^\dagger \hat{a}_1 \right\rangle_k = -\kappa\langle \hat{a}_1^\dagger \hat{a}_1\rangle_k + \gamma_c\langle \hat{\eta}_a^k\rangle. \tag{3.18}$$

The the steady-state solution of this equation is

$$\langle \hat{a}_1^\dagger \hat{a}_1\rangle_k = \frac{\gamma_c}{\kappa}\langle \hat{\eta}_a^k\rangle. \tag{3.19}$$

And on summing over all atoms, we have

$$\langle \hat{a}_1^\dagger \hat{a}_1\rangle = \frac{\gamma_c}{\kappa}\langle \hat{N}_a\rangle, \tag{3.20}$$

where

$$\langle \hat{a}_1^\dagger \hat{a}_1\rangle = \sum_{k=1}^{N}\langle \hat{a}_1^\dagger \hat{a}_1\rangle_k. \tag{3.21}$$

Hence making use of Eq. (3.6), the mean photon number of light mode a_1 is expressible as

$$\bar{n}_1 = \frac{\gamma_c}{\kappa}\left(\frac{N}{\eta+2}\right), \tag{3.22}$$

in which

$$\eta = \frac{\gamma_c}{r_a}. \tag{3.23}$$

We define the regime of laser operation with more atoms in the top level than in the bottom level as above threshold, the regime of laser operation with equal number of atoms in the top and bottom levels as threshold, and the regime of laser operation with less atoms in the top level than in the bottom level as below threshold [4]. Thus according to Eq. (3.2) for the laser operating above threshold

$\gamma_c < r_a$, for the laser operating at threshold $\gamma_c = r_a$, and for the laser operating below threshold$\gamma_c > r_a$. We note that for the laser operating well above threshold $(\gamma_c \ll r_a)$, Eq. (3.22) reduces to

$$\bar{n}_1 = \frac{\gamma_c}{2\kappa}N \tag{3.24}$$

and for the laser operating at threshold, we have

$$\bar{n}_1 = \frac{\gamma_c}{3\kappa}N. \tag{3.25}$$

We next proceed to calculate the variance of the photon number of light mode a_1. To this end, the expectation value of Eq. (2.86) is

$$\langle\hat{\sigma}_a^k(t)\rangle = \langle\hat{\sigma}_a^k(0)\rangle e^{-\mu t/2} + e^{-\mu t/2}\int_0^t e^{\mu t'/2}\langle\hat{f}_1(t')\rangle dt', \tag{3.26}$$

with the atoms considered to be initially in the bottom level, this equation reduces to

$$\langle\hat{\sigma}_a^k(t)\rangle = 0. \tag{3.27}$$

In addition, we see that the expectation value of Eq. (3.11) is expressible as

$$\langle\hat{a}_1(t)\rangle = \langle\hat{a}_1(0)\rangle e^{-\kappa t/2} + e^{-\kappa t/2}\int_0^t e^{\kappa t'/2}\Big(-g\langle\hat{\sigma}_a^k(t')\rangle + \langle\hat{F}_1(t')\rangle\Big)dt', \tag{3.28}$$

so that on account of Eqs. (2.6) and (3.27) along with the assumption that the cavity light is initially in a vacuum state, Eq. (3.28) reduces to

$$\langle\hat{a}_1(t)\rangle = 0. \tag{3.29}$$

Hence in view of Eqs. (2.10) and (3.29), we see that $\hat{a}_1(t)$ is a Gaussian variable with zero mean. It can also be established in a similar manner that

$$\langle\hat{a}_2(t)\rangle = 0. \tag{3.30}$$

On account of Eqs. (2.11) and (3.30), we also observe that $\hat{a}_2(t)$ is a Gaussian variable with zero mean.

The variance of the photon number of light mode $\hat{a}_1$ is defined by

$$(\triangle n_1)^2 = \langle \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_1 \rangle - \langle \hat{a}_1^\dagger \hat{a}_1 \rangle^2 \tag{3.31}$$

and on account of the fact that $\hat{a}_1(t)$ is a Gaussian varible with zero mean, we get

$$(\triangle n_1)^2 = \langle \hat{a}_1^\dagger \hat{a}_1 \rangle \langle \hat{a}_1 \hat{a}_1^\dagger \rangle + \langle \hat{a}_1^{\dagger 2} \rangle \langle \hat{a}_1^2 \rangle. \tag{3.32}$$

We next seek to calculate the explicit expressions for $\langle \hat{a}_1 \hat{a}_1^\dagger \rangle$ and $\langle \hat{a}_1^2 \rangle$. To this end, applying the relation

$$\frac{d}{dt}\left\langle \hat{a}_1(t)\hat{a}_1^\dagger(t) \right\rangle = \left\langle \frac{d\hat{a}_1(t)}{dt}\hat{a}_1^\dagger(t) \right\rangle + \left\langle \hat{a}_1(t)\frac{d\hat{a}_1^\dagger(t)}{dt} \right\rangle, \tag{3.33}$$

together with Eq. (2.10), we have

$$\frac{d}{dt}\left\langle \hat{a}_1(t)\hat{a}_1^\dagger(t) \right\rangle_k = -\kappa \langle \hat{a}_1(t)\hat{a}_1^\dagger(t) \rangle_k - g\Big(\langle \hat{\sigma}_a^k(t)\hat{a}_1^\dagger(t) \rangle + \langle \hat{a}_1(t)\hat{\sigma}_a^{\dagger k}(t) \rangle \Big)$$

$$+ (\langle \hat{F}_1(t)\hat{a}_1^\dagger(t) \rangle + \langle \hat{a}_1(t)\hat{F}_1^\dagger(t) \rangle). \tag{3.34}$$

Now on multiplying the adjoint of Eq. (2.28) by $\hat{\sigma}_a^k(t)$ on the left, we see that

$$\langle \hat{\sigma}_a^k(t)\hat{a}_1^\dagger(t) \rangle = -\frac{2g}{\kappa}\langle \hat{\eta}_b^k \rangle + \frac{2}{\kappa}\langle \hat{\sigma}_a^k(t)\hat{F}_1^\dagger(t) \rangle. \tag{3.35}$$

Moreover, multiplying the adjoint of Eq. (3.11) on the left by $\hat{F}_1(t)$, we find

$$\langle \hat{F}_1(t)\hat{a}_1^\dagger(t) \rangle = \langle \hat{F}_1(t)\hat{a}_1^\dagger(0) \rangle e^{-\kappa t/2} + e^{-\kappa t/2}\int_0^t e^{\kappa t'/2}\Big(- g\langle \hat{F}_1(t)\hat{\sigma}_a^{\dagger k}(t') \rangle$$

$$+ \langle \hat{F}_1(t)\hat{F}_1^\dagger(t') \rangle \Big) dt'. \tag{3.36}$$

It then follows that

$$\langle \hat{F}_1(t)\hat{a}_1^\dagger(t) \rangle = e^{-\kappa t/2}\int_0^t e^{\kappa t'/2}\langle \hat{F}_1(t)\hat{F}_1^\dagger(t') \rangle dt'. \tag{3.37}$$

Now on account Eq. (2.8), we write Eq. (3.37) as

$$\langle \hat{F}_1(t)\hat{a}_1^\dagger(t) \rangle = e^{-\kappa t/2}\int_0^t e^{\kappa t'/2}\kappa\delta(t - t')dt' \tag{3.38}$$

and applying the property of the Dirac delta function

$$\int_p^q \delta(x-a)dx = \frac{1}{2} \tag{3.39}$$

for q=a, we find

$$\langle \hat{F}_1(t)\hat{a}_1^\dagger(t)\rangle = \frac{\kappa}{2}. \tag{3.40}$$

Thus substitution of Eq. (3.35) and its complex conjugate as well as Eq. (3.40) and its complex conjugate into Eq. (3.34) results in

$$\frac{d}{dt}\left\langle \hat{a}_1\hat{a}_1^\dagger \right\rangle_k = -\kappa\langle \hat{a}_1\hat{a}_1^\dagger\rangle_k + \gamma_c\langle \hat{\eta}_b^k\rangle - \frac{2g}{\kappa}\left(\langle \hat{\sigma}_a^k(t)\hat{F}_1^\dagger(t)\rangle + \langle \hat{F}_1(t)\hat{\sigma}_a^{\dagger k}(t)\rangle\right) + \kappa. \tag{3.41}$$

Now we seek to obtain the expectation value of the product of atomic and noise operators that appears in this equation. To this end, multiplying Eq. (2.86) on the right by $\hat{F}_1^\dagger(t)$ and taking the expectation value of the resulting expression, we have

$$\langle \hat{\sigma}_a^k(t)\hat{F}_1^\dagger(t)\rangle = \langle \hat{\sigma}_a^k(0)\hat{F}_1^\dagger(t)\rangle e^{-\mu t/2} + e^{-\mu t/2}\int_0^t e^{\mu t'/2}\langle \hat{f}_1(t')\hat{F}_1^\dagger(t)\rangle dt', \tag{3.42}$$

from which follows

$$\langle \hat{\sigma}_a^k(t)\hat{F}_1^\dagger(t)\rangle = 0. \tag{3.43}$$

Thus substitution of this result and its complex conjugate into Eq. (3.41) leads to

$$\frac{d}{dt}\left\langle \hat{a}_1\hat{a}_1^\dagger \right\rangle_k = -\kappa\langle \hat{a}_1\hat{a}_1^\dagger\rangle_k + \gamma_c\langle \hat{\eta}_b^k(t)\rangle + \kappa. \tag{3.44}$$

We see that the steady-state solution is

$$\langle \hat{a}_1\hat{a}_1^\dagger\rangle_k = \frac{\gamma_c}{\kappa}\langle \hat{\eta}_b^k\rangle + 1. \tag{3.45}$$

Now upon summing over all atoms, we have

$$\langle \hat{a}_1\hat{a}_1^\dagger\rangle = \frac{\gamma_c}{\kappa}\langle \hat{N}_b\rangle + N, \tag{3.46}$$

where

$$\langle \hat{a}_1\hat{a}_1^\dagger\rangle = \sum_k \langle \hat{a}_1\hat{a}_1^\dagger\rangle_k. \tag{3.47}$$

Following a similar procedure, we easily obtain

$$\langle \hat{a}_1^2(t)\rangle = \langle \hat{a}_1^{\dagger 2}(t)\rangle = 0. \tag{3.48}$$

Therefore, employing Eqs. (3.20), (3.46), (3.48), and (3.3), the variance of the photon number of light mode a_1 can be put in the form

$$(\triangle n_1)^2 = \bar{n}_1^2 + N\bar{n}_1. \tag{3.49}$$

From this result, we realize that light mode a_1 has super-Poissonian photon statistics.

In a similar manner, we have found the mean photon number and the variance of the photon number of light mode a_2 to be

$$\bar{n}_2 = \frac{\gamma_c}{\kappa}\langle \hat{N}_b\rangle, \tag{3.50}$$

$$(\triangle n_2)^2 = \eta\bar{n}_2^2 + N\bar{n}_2. \tag{3.51}$$

We notice that light mode a_2 just like light mode a_1 exhibits super-Poissonian photon statistics. Moreover, in view of Eq. (3.3), we see that the mean photon number of light mode a_1 and light mode a_2 is the same.

3.2 One-mode quadrature variance

In this section we proceed to calculate the quadrature variances of light modes a_1 and a_2. To this end, applying the definition of the commutator of two operators, we see that

$$[\hat{a}_1, \hat{a}_1^\dagger] = \hat{a}_1\hat{a}_1^\dagger - \hat{a}_1^\dagger\hat{a}_1. \tag{3.52}$$

Upon taking the expectation value of the operators on the right-hand side of this equation, we have

$$[\hat{a}_1, \hat{a}_1^\dagger] = (\langle\hat{a}_1\hat{a}_1^\dagger\rangle - \langle\hat{a}_1^\dagger\hat{a}_1\rangle) \tag{3.53}$$

and in view of Eqs. (3.20), (3.46), and (3.3), we see that

$$[\hat{a}_1, \hat{a}_1^\dagger] = N. \tag{3.54}$$

The squeezing properties of a single-mode light are described by two quadrature operators

$$\hat{a}_{1+} = \hat{a}_1 + \hat{a}_1^\dagger \tag{3.55}$$

and

$$\hat{a}_{1-} = i(\hat{a}_1^\dagger - \hat{a}_1). \tag{3.56}$$

We can write

$$[\hat{a}_{1-}, \hat{a}_{1+}] = [i(\hat{a}_1^\dagger - \hat{a}_1), \hat{a}_1 + \hat{a}_1^\dagger], \tag{3.57}$$

so that employing the properties of commutators, we have

$$[\hat{a}_{1-}, \hat{a}_{1+}] = i\bigg([\hat{a}_1^\dagger, \hat{a}_1] + [\hat{a}_1^\dagger, \hat{a}_1^\dagger] - [\hat{a}_1, \hat{a}_1] - [\hat{a}_1, \hat{a}_1^\dagger]\bigg). \tag{3.58}$$

Since any operator commutes with itself and the fact that

$$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}], \tag{3.59}$$

we find

$$[\hat{a}_{1-}, \hat{a}_{1+}] = -2i[\hat{a}_1, \hat{a}_1^\dagger]. \tag{3.60}$$

Thus substitution of Eq. (3.54) into Eq. (3.60) leads to

$$[\hat{a}_{1-}, \hat{a}_{1+}] = -2iN. \tag{3.61}$$

It then follows that

$$\triangle a_{1-} \triangle a_{1+} \geq N. \tag{3.62}$$

The quadrature variances are defined by

$$(\triangle a_{1\pm})^2 = \langle \hat{a}_{1\pm}^2 \rangle - \langle \hat{a}_{1\pm} \rangle^2. \tag{3.63}$$

This can be put in the form

$$(\triangle a_{1\pm})^2 = \langle \hat{a}_1\hat{a}_1^\dagger\rangle + \langle \hat{a}_1^\dagger\hat{a}_1\rangle \pm \langle \hat{a}_1^2\rangle \pm \langle \hat{a}_1^{\dagger 2}\rangle - \langle \hat{a}_1^\dagger\rangle\langle \hat{a}_1\rangle - \langle \hat{a}_1\rangle\langle \hat{a}_1^\dagger\rangle \mp \langle \hat{a}_1\rangle^2 \mp \langle \hat{a}_1^\dagger\rangle^2 \tag{3.64}$$

and in view of Eqs. (3.29) and (3.48), we have

$$(\triangle a_{1\pm})^2 = \langle \hat{a}_1\hat{a}_1^\dagger\rangle + \langle \hat{a}_1^\dagger\hat{a}_1\rangle. \tag{3.65}$$

Now making use of Eqs. (3.20), (3.46), and (3.3), we find

$$(\triangle a_{1\pm})^2 = 2\frac{\gamma_c}{\kappa}\langle \hat{N}_a\rangle + N. \tag{3.66}$$

This can also be written as

$$(\triangle a_{1\pm})^2 = 2\bar{n}_1 + N. \tag{3.67}$$

This result shows that light mode a_1 is in a chaotic state.

Following a similar procedure, we also find

$$[\hat{a}_2, \hat{a}_2^\dagger] = \frac{\gamma_c}{\kappa}\bigg(\langle \hat{N}_c\rangle - \langle \hat{N}_b\rangle\bigg) + N, \tag{3.68}$$

$$[\hat{a}_{2-}, \hat{a}_{2+}] = 2i\bigg(\frac{\gamma_c}{\kappa}(\langle \hat{N}_b\rangle - \langle \hat{N}_c\rangle) - N\bigg), \tag{3.69}$$

$$\triangle a_{2-}\triangle a_{2+} \geq \bigg|\frac{\gamma_c}{\kappa}(\langle \hat{N}_b\rangle - \langle \hat{N}_c\rangle) - N\bigg|, \tag{3.70}$$

$$(\triangle a_{2\pm})^2 = \frac{\gamma_c}{\kappa}\bigg(\langle \hat{N}_b\rangle + \langle \hat{N}_c\rangle\bigg) + N. \tag{3.71}$$

Now in view of Eqs. (3.3), (3.4), and (3.6), we write Eqs. (3.70) and (3.71) as

$$\triangle a_{2-}\triangle a_{2+} \geq \bigg|\frac{\gamma_c}{\kappa}N\frac{1-\eta}{\eta+2} - N\bigg|, \tag{3.72}$$

$$(\triangle a_{2\pm})^2 = \frac{\gamma_c}{\kappa}N\bigg(\frac{\eta+1}{\eta+2}\bigg) + N. \tag{3.73}$$

Hence for a laser operating well above threshold, we have

$$\triangle a_{2-}\triangle a_{2+} \geq \bigg|\frac{\gamma_c}{2\kappa}N - N\bigg|, \tag{3.74}$$

$$(\triangle a_{2\pm})^2 = \frac{\gamma_c}{2\kappa}N + N. \tag{3.75}$$

On the other hand, Fesseha [4] has found, by putting the noise operators in normal order, the quadrature variances and the product of the uncertainties in the quadrature operators for light mode a_2 for the laser operating well above threshold to be

$$\triangle a_{2-}\triangle a_{2+} \geq \frac{\gamma_c}{2\kappa}N, \tag{3.76}$$

$$(\triangle a_{2\pm})^2 = \frac{\gamma_c}{2\kappa}N. \tag{3.77}$$

We observe that light mode a_2 generated by the three-level laser operating well above threshold is coherent. However, light mode a_2 generated by a three-level laser operating well above threshold is not coherent when we carry out our analysis by taking the noise operators in arbitrary order.

4

Two-Mode Photon Statistics

In this chapter we study the statistical properties of the two-mode cavity light produced by a three-level laser in which the three-level atoms available in a closed cavity are pumped from the bottom to the top level by electron bombardment. We apply the solution of the quantum Langevin equation for the two-mode cavity light and the equations of evolution of the atomic operators to calculate the mean and the variance of the photon number.

4.1 Global mean photon number

Here we seek to calculate the mean photon number for the two-mode cavity light produced by a three-level laser coupled to a two-mode vacuum reservoir. To this end, one can write

$$\frac{d}{dt}\Big\langle \hat{a}^\dagger(t)\hat{a}(t)\Big\rangle_k = \Big\langle \frac{d\hat{a}^\dagger(t)}{dt}\hat{a}(t)\Big\rangle_k + \Big\langle \hat{a}^\dagger(t)\frac{d\hat{a}(t)}{dt}\Big\rangle_k \tag{4.1}$$

and making use of Eq. (2.12), we find

$$\frac{d}{dt}\Big\langle \hat{a}^\dagger(t)\hat{a}(t)\Big\rangle_k = -\kappa\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_k - g\Big(\langle \hat{a}^\dagger(t)\hat{\sigma}^k(t)\rangle + \langle \hat{\sigma}^{\dagger k}(t)\hat{a}(t)\rangle\Big)$$
$$+\Big(\langle \hat{a}^\dagger(t)\hat{F}(t)\rangle + \langle \hat{F}^\dagger(t)\hat{a}(t)\rangle\Big). \tag{4.2}$$

We now proceed to calculate the expectation value of the product of the atomic and cavity mode operators. Application of the large-time approximation to Eq. (2.12) leads to

$$\hat{a}(t) = -\frac{2g}{\kappa}\hat{\sigma}^k(t) + \frac{2}{\kappa}\hat{F}(t), \tag{4.3}$$

so that on multiplying the adjoint of this equation on the right by $\hat{\sigma}^k(t)$, we have

$$\langle \hat{a}^\dagger(t)\hat{\sigma}^k(t)\rangle = -\frac{2g}{\kappa}\Big(\langle \hat{\eta}_a^k(t)\rangle + \langle \hat{\eta}_b^k(t)\rangle\Big) + \frac{2}{\kappa}\langle \hat{F}^\dagger(t)\hat{\sigma}^k(t)\rangle. \tag{4.4}$$

Thus substitution of this equation and its complex conjugates into Eq. (4.2) yields

$$\frac{d}{dt}\Big\langle \hat{a}^\dagger(t)\hat{a}(t)\Big\rangle_k = -\kappa\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_k + \gamma_c\Big(\langle \hat{\eta}_a^k(t)\rangle + \langle \hat{\eta}_b^k(t)\rangle\Big) - \frac{2g}{\kappa}\Big(\langle \hat{F}^\dagger(t)\hat{\sigma}^k(t)\rangle$$

$$+\langle \hat{\sigma}^{\dagger k}(t)\hat{F}(t)\rangle\Big) + \Big(\langle \hat{a}^\dagger(t)\hat{F}(t)\rangle + \langle \hat{F}^\dagger(t)\hat{a}(t)\rangle\Big). \tag{4.5}$$

We next seek to determine the explicit expression for $\langle \hat{a}^\dagger(t)\hat{F}(t)\rangle$. To this end, the solution of Eq. (2.12) can be written as

$$\hat{a}(t) = \hat{a}(0)e^{-\kappa t/2} + e^{-\frac{\kappa t}{2}}\int_0^t e^{\frac{\kappa t'}{2}}\Big(-g\hat{\sigma}^k(t') + \hat{F}(t')\Big)dt'. \tag{4.6}$$

Now multiplying the adjoint of this equation on the right by $\hat{F}(t)$ and taking the expectation value of the resulting expression, we find

$$\langle \hat{a}^\dagger(t)\hat{F}(t)\rangle = \langle \hat{a}^\dagger(0)\hat{F}(t)\rangle e^{-\kappa t/2} + e^{-\kappa t/2}\int_0^t e^{\kappa t'/2}\Big(-g\langle \hat{\sigma}^{\dagger k}(t')\hat{F}(t)\rangle$$

$$+\langle \hat{F}^\dagger(t')\hat{F}(t)\rangle\Big)dt'. \tag{4.7}$$

Because the noise operator at a certain time does not affect a system variables at earlier time along with Eq. (2.7), we see that

$$\langle \hat{a}^\dagger(t)\hat{F}(t)\rangle = 0. \tag{4.8}$$

Hence on substituting this result and its complex conjugate into Eq. (4.5), we have

$$\frac{d}{dt}\left\langle \hat{a}^{\dagger}(t)\hat{a}(t)\right\rangle_k = -\kappa\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle_k + \gamma_c\bigg(\langle\hat{\eta}_a^k(t)\rangle + \langle\hat{\eta}_b^k(t)\rangle\bigg)$$
$$-\frac{2g}{\kappa}\bigg(\langle\hat{F}^{\dagger}(t)\hat{\sigma}^k(t)\rangle + \langle\hat{\sigma}^{\dagger k}(t)\hat{F}(t)\rangle\bigg). \qquad (4.9)$$

We next proceed to determine the expectation value of the product of the atomic and the cavity mode noise operators. To this end, the solution of Eq. (2.103) can be written as

$$\hat{\sigma}^k(t) = \hat{\sigma}^k(0)e^{-\mu t/2} + e^{-\mu t/2}\int_0^t e^{-\mu t'/2}\hat{f}(t')dt' \qquad (4.10)$$

and upon multiplying this expression on the left by $\hat{F}^{\dagger}(t)$, we find

$$\langle\hat{F}^{\dagger}(t)\hat{\sigma}^k(t)\rangle = \langle\hat{F}^{\dagger}(t)\hat{\sigma}^k(0)\rangle e^{-\mu t/2} + e^{-\mu t/2}\int_0^t e^{-\mu t'/2}\langle\hat{F}^{\dagger}(t)\hat{f}(t')\rangle dt'. \qquad (4.11)$$

Since $\hat{F}^{\dagger}(t)$ and $\hat{f}(t')$ are uncorrelated and the fact that a noise operator at a certain time does not affect a system variable at earlier time , we see that

$$\langle\hat{F}^{\dagger}(t)\hat{\sigma}^k(t)\rangle = 0. \qquad (4.12)$$

Therefore, employing this result and its complex conjugate , we can put Eq. (4.9) in the form

$$\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle_k = -\kappa\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle_k + \gamma_c(\langle\hat{\eta}_a^k(t)\rangle + \langle\hat{\eta}_b^k(t)\rangle). \qquad (4.13)$$

The steady-state solution of this equation is

$$\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle_k = \frac{\gamma_c}{\kappa}(\langle\hat{\eta}_a^k\rangle + \langle\hat{\eta}_b^k\rangle) \qquad (4.14)$$

and on summing over all atoms, we find

$$\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle = \frac{\gamma_c}{\kappa}(\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle). \qquad (4.15)$$

On account of Eqs. (3.20) and (3.50), we see that the mean photon number of the two-mode cavity light is the sum of the mean photon numbers of light modes

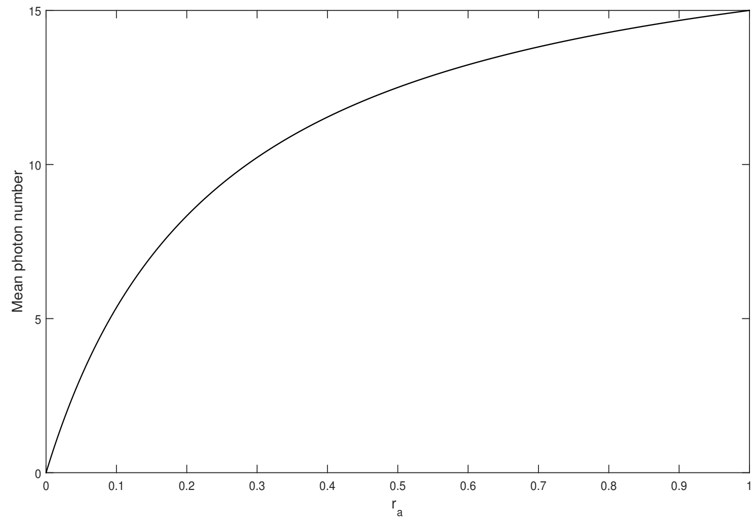

Figure 4.1: Plot of $\bar{n}$ [Eq. (4.16)] versus r_a for $\kappa = 0.8$, N=30, and $\gamma_c = 0.5$.

a_1 and a_2. Moreover, we note that this result is identical to that obtained by Fesseha [7] for the case in which the noise operators associated with the vacuum reservoir are put in normal order. Hence we note that carrying out our analysis by taking the noise operators associated with the vacuum reservoir in arbitrary order do not affect the mean photon number of the two-mode cavity light. Taking into account Eqs. (3.3) and (3.5), the mean photon number of the two-mode cavity light can be put in the form

$$\bar{n} = 2\frac{\gamma_c}{\kappa}\left(\frac{r_a N}{\gamma_c + 2r_a}\right). \tag{4.16}$$

We see from this expression that the mean photon number in the cavity is zero in the absence of the pumping process. This result also shows that the mean photon number in the cavity increases with the number of atoms.

Moreover, the plot in Figure 4.1 indicates that the mean photon number of the two-mode cavity light increases with r_a. In Figure 4.2, we plot the mean photon number of the two-mode cavity light versus r_a for different values of γ_c. It is not difficult to see from this figure that the mean photon number of the two-mode light

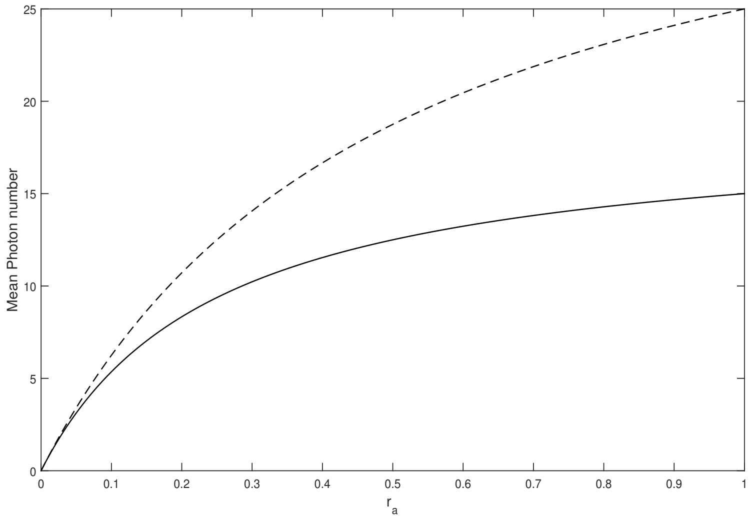

Figure 4.2: Plots of the mean photon number $\bar{n}$ [Eq. (4.16)] versus r_a for $\kappa = 0.8$, N=30, $\gamma_c = 0.5$ (solid curve), and $\gamma_c = 1$ (dashed curve).

increases with γ_c. We see that for the laser operating well above threshold ($\gamma_c \ll r_a$), Eq. (4.16) reduces to

$$\bar{n} = \frac{\gamma_c}{\kappa} N \tag{4.17}$$

and for the laser operating at threshold, we find

$$\bar{n} = \frac{2\gamma_c}{3\kappa} N. \tag{4.18}$$

4.2 Global photon number variance

Next we wish to calculate the variance of the photon number for the two-mode cavity light. To this end, we first obtain the expectation value of the atomic operator $\hat{m}_c$. We assume that the state vector of the three-level atom, put in the form [4]

$$|\psi\rangle_k = C_a|a\rangle_k + C_b|b\rangle_k + C_c|c\rangle_k, \tag{4.19}$$

can be used to evaluate the expectation value of the atomic operator formed by a pair of identical energy levels or by two distinct energy levels between which transition with the emission of a photon is dipole forbidden. We can thus readily establish that

$$\langle\hat{\eta}_a^k\rangle = C_a C_a^*, \tag{4.20}$$

$$\langle\hat{\eta}_c^k\rangle = C_c C_c^*, \tag{4.21}$$

$$\langle\hat{\sigma}_c^k\rangle = C_a C_c^*. \tag{4.22}$$

We then see that

$$|\langle\hat{\sigma}_c^k\rangle|^2 = \langle\hat{\eta}_a^k\rangle\langle\hat{\eta}_c^k\rangle \tag{4.23}$$

and on taking $\langle\hat{\sigma}_c^k\rangle$ to be real, we have

$$\langle\hat{\sigma}_c^k\rangle = \sqrt{\langle\hat{\eta}_a^k\rangle\langle\hat{\eta}_c^k\rangle}. \tag{4.24}$$

Finally on summing over k from 1 upto N, we get [4]

$$\langle\hat{m}_c\rangle = \sqrt{\langle\hat{N}_a\rangle\langle\hat{N}_c\rangle}. \tag{4.25}$$

Moreover, taking into account the expectation value of Eq. (2.13) along with Eqs. (3.29) and (3.30), we see that

$$\langle\hat{a}(t)\rangle = 0. \tag{4.26}$$

Hence in view of Eqs. (2.12) and (4.26), we see that $\hat{a}(t)$ is a Gaussian variable with zero mean. The variance of the photon number for the two-mode cavity light is expressible as

$$(\triangle n)^2 = \langle\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}^\dagger\hat{a}\rangle^2 \tag{4.27}$$

and since $\hat{a}$ is a Gaussian variable with zero mean, the above equation can be put in the form

$$(\triangle n)^2 = \langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^{\dagger 2}\rangle\langle\hat{a}^2\rangle. \tag{4.28}$$

We now proceed to obtain the explicit expressions for $\langle \hat{a}(t)\hat{a}^\dagger(t)\rangle$ and $\langle \hat{a}^2(t)\rangle$. To this end, one can write

$$\frac{d}{dt}\left\langle \hat{a}(t)\hat{a}^\dagger(t)\right\rangle = \left\langle \frac{d\hat{a}(t)}{dt}\hat{a}^\dagger(t)\right\rangle + \left\langle \hat{a}(t)\frac{d\hat{a}^\dagger(t)}{dt}\right\rangle, \tag{4.29}$$

together with Eq. (2.12), we get

$$\frac{d}{dt}\left\langle \hat{a}(t)\hat{a}^\dagger(t)\right\rangle_k = -\kappa\langle \hat{a}(t)\hat{a}^\dagger(t)\rangle_k - g\Big(\langle \hat{\sigma}^k(t)\hat{a}^\dagger(t)\rangle + \langle \hat{a}(t)\hat{\sigma}^{\dagger k}(t)\rangle\Big)$$
$$+\Big(\langle \hat{a}(t)\hat{F}^\dagger(t)\rangle + \langle \hat{F}(t)\hat{a}^\dagger(t)\rangle\Big). \tag{4.30}$$

We next seek to obtain the expectation values of the product of the atomic and the cavity mode operators. Multiplying the adjoint of Eq. (4.3) on the left by $\hat{\sigma}^k(t)$, we see that

$$\langle \hat{\sigma}^k(t)\hat{a}^\dagger(t)\rangle = -\frac{2g}{\kappa}\langle \hat{\sigma}^k(t)\hat{\sigma}^{\dagger k}(t)\rangle + \frac{2}{\kappa}\langle \hat{\sigma}^k(t)\hat{F}^\dagger(t)\rangle, \tag{4.31}$$

from which follows

$$\langle \hat{\sigma}^k(t)\hat{a}^\dagger(t)\rangle = -\frac{2g}{\kappa}\Big(\langle \hat{\eta}_b^k(t)\rangle + \langle \hat{\eta}_c^k(t)\rangle\Big) + \frac{2}{\kappa}\langle \hat{\sigma}^k(t)\hat{F}^\dagger(t)\rangle. \tag{4.32}$$

Thus with the aid of this equation and its complex conjugate, we can rewrite Eq. (4.30) as

$$\frac{d}{dt}\left\langle \hat{a}(t)\hat{a}^\dagger(t)\right\rangle_k = -\kappa\langle \hat{a}(t)\hat{a}^\dagger(t)\rangle_k + \gamma_c\Big(\langle \hat{\eta}_b^k(t)\rangle + \langle \hat{\eta}_c^k(t)\rangle\Big)$$
$$-\frac{2g}{\kappa}\Big(\langle \hat{\sigma}^k(t)\hat{F}^\dagger(t)\rangle + \langle \hat{F}(t)\hat{\sigma}^{\dagger k}(t)\rangle\Big)$$
$$+\Big(\langle \hat{a}(t)\hat{F}^\dagger(t)\rangle + \langle \hat{F}(t)\hat{a}^\dagger(t)\rangle\Big). \tag{4.33}$$

Moreover, multiplying Eq. (4.6) on the right by $\hat{F}^\dagger(t)$ and taking the expectation value of the resulting expression, we find

$$\langle \hat{a}(t)\hat{F}^{\dagger}(t)\rangle = \langle \hat{a}(0)\hat{F}^{\dagger}(t)\rangle e^{-\kappa t/2} + e^{-\kappa t/2}\int_0^t e^{\kappa t'/2}\bigg(- g\langle \hat{\sigma}^k(t')\hat{F}^{\dagger}(t)\rangle$$
$$+\langle \hat{F}(t')\hat{F}^{\dagger}(t)\rangle\bigg)dt', \tag{4.34}$$

from which follows

$$\langle \hat{a}(t)\hat{F}^{\dagger}(t)\rangle = e^{-\kappa t/2}\int_0^t e^{\kappa t'/2}\langle \hat{F}(t')\hat{F}^{\dagger}(t)\rangle dt'. \tag{4.35}$$

Now in view of Eq. (2.16), one can put this equation in the form

$$\langle \hat{a}(t)\hat{F}^{\dagger}(t)\rangle = 2\kappa e^{-\kappa t/2}\int_0^t e^{\kappa t'/2}\delta(t-t')dt' \tag{4.36}$$

and applying the relation described by Eq. (3.39), we get

$$\langle \hat{a}(t)\hat{F}^{\dagger}(t)\rangle = \kappa. \tag{4.37}$$

Therefore, on account this result and its complex conjugate, Eq. (4.33) can be written as

$$\frac{d}{dt}\bigg\langle \hat{a}(t)\hat{a}^{\dagger}(t)\bigg\rangle_k = -\kappa\langle \hat{a}(t)\hat{a}^{\dagger}(t)\rangle_k + \gamma_c\bigg(\langle \hat{\eta}_b^k(t)\rangle + \langle \hat{\eta}_c^k(t)\rangle\bigg) - \frac{2g}{\kappa}\bigg(\langle \hat{\sigma}^k(t)\hat{F}^{\dagger}(t)\rangle$$
$$+\langle \hat{F}(t)\hat{\sigma}^{\dagger k}(t)\rangle\bigg) + 2\kappa. \tag{4.38}$$

We next wish to determine the expectation values of the product of the atomic and the cavity mode noise operators involved in this expression. To this end, on multiplying Eq. (4.10) on the right by $\hat{F}^{\dagger}(t)$ and taking the expectation value of the resulting expression, we arrive at

$$\langle \hat{\sigma}^k(t)\hat{F}^{\dagger}(t)\rangle = 0. \tag{4.39}$$

Finally, making use of Eq. (4.39) and its complex conjugate, we put Eq. (4.38) in the form

$$\frac{d}{dt}\bigg\langle \hat{a}(t)\hat{a}^{\dagger}(t)\bigg\rangle = -\kappa\langle \hat{a}(t)\hat{a}^{\dagger}(t)\rangle_k + \gamma_c(\langle \hat{\eta}_b^k(t)\rangle + \langle \hat{\eta}_c^k(t)\rangle) + 2\kappa. \tag{4.40}$$

The steady-state solution of this equation is

$$\langle \hat{a}(t)\hat{a}^{\dagger}(t)\rangle_k = \frac{\gamma_c}{\kappa}\langle \hat{\eta}_b^k(t) + \hat{\eta}_c^k(t)\rangle + 2, \tag{4.41}$$

and on summing over all atoms, we have

$$\langle \hat{a}(t)\hat{a}^{\dagger}(t)\rangle = \frac{\gamma_c}{\kappa}(\langle \hat{N}_b(t)\rangle + \langle \hat{N}_c(t)\rangle) + 2N. \tag{4.42}$$

Following a similar procedure, we also easily find

$$\langle \hat{a}^2(t)\rangle = \frac{\gamma_c}{\kappa}\langle \hat{m}_c(t)\rangle \tag{4.43}$$

and using Eqs. (4.25) and (3.2), we arrive at

$$\langle \hat{a}^2(t)\rangle = \frac{\gamma_c}{\kappa}\sqrt{\frac{\gamma_c}{r_a}}\langle \hat{N}_a\rangle. \tag{4.44}$$

Hence in view of Eqs. (4.15), (4.42), (4.44), (3.2), and (3.6), the variance of the photon number takes the form

$$(\triangle n)^2 = \left(\frac{\gamma_c N}{\kappa}\right)^2 \frac{(2r_a^2 + 3\gamma_c r_a)}{(\gamma_c + 2r_a)^2} + 4\frac{\gamma_c N^2}{\kappa}\left(\frac{r_a}{\gamma_c + 2r_a}\right). \tag{4.45}$$

We see from this expression that the variance of the photon number of the two-mode cavity light increases with the number of atoms. Moreover, as clearly shown in Figure 4.3 that the variance of the photon number increases with the stimulated emission. The expression in Eq. (4.45) can be rewritten as

$$(\triangle n)^2 = 2N\bar{n} + \frac{\bar{n}^2}{4}\left(2 + 3\eta\right). \tag{4.46}$$

It is not difficult to see from this expression that the two-mode cavity light exhibits super-Poissonian photon statistics. We also note that unlike the mean photon number of the two-mode light, the variance of the photon number of the two-mode cavity light is not the sum of the variances of the photon number of the individual light modes. On the other hand, Fesseha [4] has found the variance of the photon num-

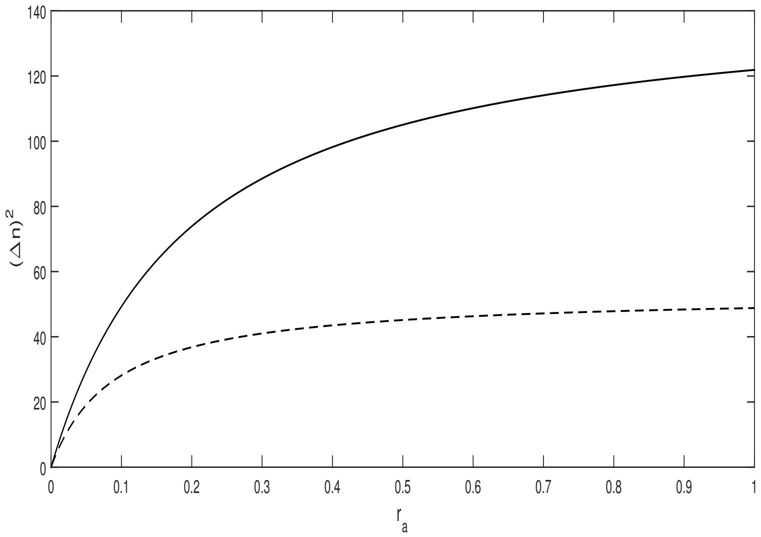

Figure 4.3: Plots of the variance of the phton number [Eq. (4.45)] versus r_a for $N = 10$, $\kappa = 0.8$, $\gamma_c = 0.2$ (dashed curve), and $\gamma_c = 0.5$ (solid curve).

ber of the two-mode cavity light by putting the noise operators associated with the vacuum reservoir in normal order to be

$$(\triangle n)^2 = \frac{\bar{n}^2}{4}(2 + 3\eta). \tag{4.47}$$

The expressions in Eqs. (4.46) and (4.47) clearly indicate that the effect of carrying out our analysis by taking the noise operators associated with the vacuum reservoir in arbitrary order is to increase the variance of the photon number by $2N\bar{n}$.

We note from Eq. (4.46) that the variance of the photon number for the laser operating well above threshold is

$$(\triangle n)^2 = 2N\bar{n} + \frac{\bar{n}^2}{4}, \tag{4.48}$$

in which $\bar{n}$ is given by Eq. (4.17) and for the laser operating at threshold

$$(\triangle n)^2 = 2N\bar{n} + \frac{5}{4}\bar{n}^2, \tag{4.49}$$

where $\bar{n}$ is given by Eq. (4.18).

4.3 Local mean photon number

In this section we seek to obtain the mean photon number of the two-mode cavity light in a given frequency interval. To this end, we first determine the power spectrum of the cavity light. The power spectrum of the superposed light modes a_1 and a_2 having the same central frequency ω_o is expressible as [36]

$$P(\omega) = \frac{1}{\pi} Re \int_0^\infty d\tau \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss} e^{i(\omega-\omega_o)\tau}. \tag{4.50}$$

Now upon integrating both sides of this equation over ω, we have

$$\int_{-\infty}^{\infty} P(\omega)d\omega = \bar{n}, \tag{4.51}$$

where $\bar{n}$ is the steady-state mean photon number of the two-mode cavity light. On the basis of this result, we note that $P(\omega)d\omega$ is the steady-state mean photon number in the interval between ω and $\omega + d\omega$ [4]. Then the mean photon number in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as

$$\bar{n}_{\pm\lambda} = \int_{-\lambda}^{+\lambda} P(\omega')d\omega', \tag{4.52}$$

in which $\omega' = \omega - \omega_o$.

We now proceed to calculate the two-time correlation function that appears in Eq. (4.50). We observe that the solution of Eq. (2.12) can be written as

$$\hat{a}(t+\tau) = \hat{a}(t)e^{-\kappa\tau/2} + e^{-\kappa\tau/2}\int_0^\tau e^{\kappa\tau'/2}\Big[-g\hat{\sigma}^k(t+\tau') + \hat{F}(t+\tau')\Big]d\tau'. \tag{4.53}$$

Now on multiplying Eq. (4.53) on the left by $\hat{a}^\dagger(t)$ and taking the expectation values of the resulting expression, we see that

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_k = \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_k e^{-\kappa\tau/2} + e^{-\kappa\tau/2}\int_0^\tau e^{\kappa\tau'/2}\bigg[-g\langle \hat{a}^\dagger(t)\hat{\sigma}^k(t+\tau')\rangle$$
$$+\langle \hat{a}^\dagger(t)\hat{F}(t+\tau')\rangle\bigg]d\tau' \tag{4.54}$$

and on account of the assertion that a noise operator at a certain time does not affect cavity mode variable at earlier time, we have

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_k = \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_k e^{-\kappa\tau/2} - g e^{-\kappa\tau/2}\int_0^\tau e^{\kappa\tau'/2}\langle \hat{a}^\dagger(t)\hat{\sigma}^k(t+\tau')\rangle d\tau'. \tag{4.55}$$

We next seek to determine the explicit form of $\langle \hat{a}^\dagger(t)\hat{\sigma}^k(t+\tau')\rangle$. The solution of Eq. (2.103) can be put in the form

$$\hat{\sigma}^k(t+\tau') = \hat{\sigma}^k(t)e^{-\mu\tau'/2} + e^{-\mu\tau'/2}\int_0^{\tau'} e^{\mu\tau''/2}\hat{f}(t+\tau'')d\tau''. \tag{4.56}$$

Now on multiplying this equation on both sides from the left by $\hat{a}^\dagger(t)$, we have

$$\langle \hat{a}^\dagger(t)\hat{\sigma}^k(t+\tau')\rangle = \langle \hat{a}^\dagger(t)\hat{\sigma}^k(t)\rangle e^{-\mu\tau'/2} + e^{-\mu\tau'/2}\int_0^{\tau'} e^{\mu\tau''/2}\langle \hat{a}^\dagger(t)\hat{f}(t+\tau'')\rangle d\tau''. \tag{4.57}$$

This equation can also be written as

$$\langle \hat{a}^\dagger(t)\hat{\sigma}^k(t+\tau')\rangle = \langle \hat{a}^\dagger(t)\hat{\sigma}^k(t)\rangle e^{-\mu\tau'/2}. \tag{4.58}$$

Hence substitution of this equation into Eq. (4.55) results in

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_k = \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_k e^{-\kappa\tau/2} - g\langle \hat{a}^\dagger(t)\hat{\sigma}^k(t)\rangle e^{-\kappa\tau/2}\int_0^t e^{(\kappa-\mu)\tau'/2}d\tau' \tag{4.59}$$

and on carrying out the integration, we find

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_k = \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_k e^{-\kappa\tau/2} - \frac{2g}{\kappa-\mu}\langle \hat{a}^\dagger(t)\hat{\sigma}^k(t)\rangle\Big(e^{-\mu\tau/2} - e^{-\kappa\tau/2}\Big). \tag{4.60}$$

Therefore, on account Eq. (4.4), we see that

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t+\tau)\rangle_k = \langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle_k e^{-\kappa\tau/2} + \frac{\gamma_c}{\kappa-\mu}\Big(\langle \hat{\eta}_a^k(t)\rangle + \langle \hat{\eta}_b^k(t)\rangle\Big)$$

$$\times\Big(e^{-\mu\tau/2} - e^{-\kappa\tau/2}\Big) - \frac{4g}{\kappa(\kappa-\mu)}\langle \hat{F}^{\dagger}(t)\hat{\sigma}^k(t)\rangle\Big(e^{-\mu\tau/2} - e^{-\kappa\tau/2}\Big), \quad (4.61)$$

so that making use of Eq. (4.12), we have

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t+\tau)\rangle_k = \langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle_k e^{-\kappa\tau/2} + \frac{\gamma_c}{\kappa-\mu}\Big(\langle \hat{\eta}_a^k(t)\rangle + \langle \hat{\eta}_b^k(t)\rangle\Big)$$

$$\times\Big(e^{-\mu\tau/2} - e^{-\kappa\tau/2}\Big). \quad (4.62)$$

In view of Eq. (4.14), we put this equation in the form

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t+\tau)\rangle_k = \Big(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\Big)\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle_k \quad (4.63)$$

and on summing over all atoms, we obtain

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t+\tau)\rangle = \bar{n}\Big(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\Big), \quad (4.64)$$

in which $\bar{n}$ is given by Eq. (4.16). Thus substitution of this equation into Eq. (4.50) yields

$$P(\omega) = \frac{\bar{n}}{\pi}\Big[\frac{\kappa}{\kappa-\mu}Re\int_0^{\infty} e^{-[\mu/2-i(\omega-\omega_0)]\tau}d\tau - \frac{\mu}{\kappa-\mu}Re\int_0^{\infty} e^{-[\kappa/2-i(\omega-\omega_0)]\tau}d\tau\Big], \quad (4.65)$$

so that on carrying out the integration, the power spectrum of the cavity light turns out to be

$$P(\omega) = \bar{n}\Big[\frac{\kappa}{\kappa-\mu}\frac{\mu/2\pi}{(\mu/2)^2+(\omega-\omega_0)^2} - \frac{\mu}{\kappa-\mu}\frac{\kappa/2\pi}{(\kappa/2)^2+(\omega-\omega_0)^2}\Big]. \quad (4.66)$$

Now on introducing this result into Eq. (4.52), we find

$$\bar{n}_{\pm\lambda} = \bar{n}\Big[\frac{\kappa}{\kappa-\mu}\frac{\mu}{2\pi}\int_{-\lambda}^{+\lambda}\frac{d\omega'}{(\mu/2)^2+\omega'^2} - \frac{\mu}{\kappa-\mu}\frac{\kappa}{2\pi}\int_{-\lambda}^{+\lambda}\frac{d\omega'}{(\kappa/2)^2+\omega'^2}\Big] \quad (4.67)$$

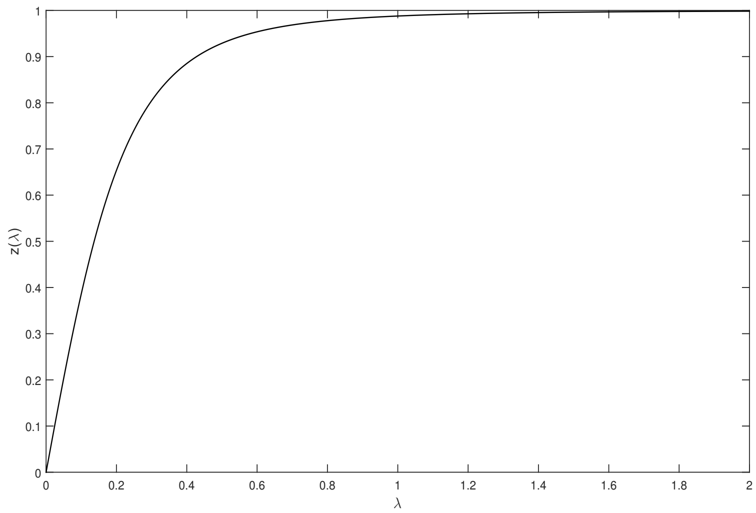

Figure 4.4: Plot of $z(\lambda)$ [Eq. (4.70)] versus λ for $\kappa = 0.8$ and $\mu = 0.5$.

and on carrying out the integration applying the relation

$$\int_{-\lambda}^{+\lambda} \frac{dx}{x^2 + a^2} = \frac{2}{a} tan^{-1}\left(\frac{\lambda}{a}\right), \tag{4.68}$$

we have

$$\bar{n}_{\pm\lambda} = z(\lambda)\bar{n}, \tag{4.69}$$

where $z(\lambda)$ is given by

$$z(\lambda) = \frac{2\kappa/\pi}{\kappa - \mu} tan^{-1}\left(\frac{2\lambda}{\mu}\right) - \frac{2\mu/\pi}{\kappa - \mu} tan^{-1}\left(\frac{2\lambda}{\kappa}\right). \tag{4.70}$$

We note from the plot in Figure (4.4) that $z(0.5) = 0.9288$, $z(1) = 0.9878$, and $z(2) = 0.9983$. Therefore, combination of these results with Eq. (4.69) gives $\bar{n}_{\pm 0.5} = 0.9288\bar{n}$, $\bar{n}_{\pm 1} = 0.9878\bar{n}$, and $\bar{n}_{\pm 2} = 0.9983\bar{n}$. This shows that a large part of the mean photon number is contained in a small frequency interval.

4.4 Local photon number variance

In this section we wish to calculate the photon-number variance of the cavity light in a given frequency interval. To this end, we first obtain the spectrum of the photon number fluctuations of the cavity light. The spectrum of the photon number fluctuations of the two-mode cavity light with central frequency ω_o can be written as

$$J(\omega) = \frac{1}{\pi} Re \int_0^\infty d\tau \langle \hat{n}(t), \hat{n}(t+\tau)\rangle_{ss} e^{i(\omega-\omega_o)\tau}, \tag{4.71}$$

where

$$\hat{n}(t) = \hat{a}^\dagger(t)\hat{a}(t). \tag{4.72}$$

On integrating both sides of Eq. (4.71) over ω, we find

$$\int_{-\infty}^{\infty} J(\omega) d\omega = (\triangle n)^2, \tag{4.73}$$

in which

$$(\triangle n)^2 = \langle \hat{n}(t), \hat{n}(t)\rangle_{ss} \tag{4.74}$$

is the steady-state photon number variance of the two-mode cavity light. On the basis of Eq. (4.73), we note that $J(\omega)d\omega$ is the photon number fluctuations of the cavity light in the interval between ω and $\omega + d\omega$. The photon number fluctuations in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ can then be written as

$$(\triangle n)^2_{\pm\lambda} = \int_{-\lambda}^{\lambda} J(\omega') d\omega'. \tag{4.75}$$

We note that

$$\langle \hat{n}(t), \hat{n}(t+\tau)\rangle = \langle \hat{a}^\dagger(t)\hat{a}(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\rangle - \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle\langle \hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\rangle. \tag{4.76}$$

Because $\hat{a}$ is a Gaussian variable with zero mean, we see that

$$\langle \hat{n}(t), \hat{n}(t+\tau)\rangle = \langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle\langle \hat{a}(t)\hat{a}(t+\tau)\rangle + \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle\langle \hat{a}(t)\hat{a}^\dagger(t+\tau)\rangle. \tag{4.77}$$

Thus on account of Eq. (4.77), we can write Eq. (4.71) as

$$J(\omega) = \frac{1}{\pi} Re \int_0^\infty d\tau \bigg(\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle \langle \hat{a}(t)\hat{a}(t+\tau)\rangle$$

$$+\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle \langle \hat{a}(t)\hat{a}^\dagger(t+\tau)\rangle \bigg) e^{i(\omega-\omega_o)\tau}. \tag{4.78}$$

We proceed to evaluate the two-time correlation function involved in this expression. Now on multiplying the adjoint of Eq. (4.53) on the left by $\hat{a}^\dagger(t)$, we have

$$\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle_k = \langle \hat{a}^\dagger(t)\hat{a}^\dagger(t)\rangle_k e^{-\kappa t/2} + e^{-\kappa\tau/2} \int_0^\tau d\tau' e^{\kappa\tau'/2} \bigg[- g\langle \hat{a}^\dagger(t)\hat{\sigma}^{\dagger k}(t+\tau')\rangle$$

$$+\langle \hat{a}^\dagger(t)\hat{F}^\dagger(t+\tau')\rangle \bigg]. \tag{4.79}$$

Because of the fact that a noise operator at a certain time does not affect cavity mode operator at an earlier time, we get

$$\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle_k = \langle \hat{a}^\dagger(t)\hat{a}^\dagger(t)\rangle_k e^{-\kappa t/2} - g e^{-\kappa\tau/2} \int_0^\tau d\tau' e^{\kappa\tau'/2} \langle \hat{a}^\dagger(t)\hat{\sigma}^{\dagger k}(t+\tau')\rangle. \tag{4.80}$$

We now seek to obtain an explicit expression for $\langle \hat{a}^\dagger(t)\hat{\sigma}^{\dagger k}(t+\tau')\rangle$. Upon multiplying the adjoint of Eq. (4.56) from the left by $\hat{a}^\dagger(t)$, we find

$$\langle \hat{a}^\dagger(t)\hat{\sigma}^{\dagger k}(t+\tau')\rangle = \langle \hat{a}^\dagger(t)\hat{\sigma}^{\dagger k}(t)\rangle e^{-\mu\tau'/2} + e^{-\mu\tau'/2} \int_0^{\tau'} d\tau'' e^{\mu\tau''/2} \langle \hat{a}^\dagger(t)\hat{f}(t+\tau'')\rangle, \tag{4.81}$$

from which follows

$$\langle \hat{a}^\dagger(t)\hat{\sigma}^{\dagger k}(t+\tau')\rangle = \langle \hat{a}^\dagger(t)\hat{\sigma}^{\dagger k}(t)\rangle e^{-\mu\tau'/2}. \tag{4.82}$$

Thus on substituting this result into Eq. (4.80) and carrying out the integration, we get

$$\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle_k = \langle \hat{a}^\dagger(t)\hat{a}^\dagger(t)\rangle_k e^{-\kappa\tau/2} - \frac{2g}{\kappa-\mu} \langle \hat{a}^\dagger(t)\hat{\sigma}^{\dagger k}(t)\rangle \Big(e^{-\mu\tau/2} - e^{-\kappa\tau/2} \Big). \tag{4.83}$$

we wish to obtain the explicit expression for $\langle\hat{a}^\dagger(t)\hat{\sigma}^{\dagger k}(t)\rangle$. To this end, on multiplying the adjoint of Eq. (4.3) on the right by $\hat{\sigma}^{\dagger k}(t)$ and taking the expectation value of the resulting expression, we have

$$\langle\hat{a}^\dagger(t)\hat{\sigma}^{\dagger k}(t)\rangle = -\frac{2g}{\kappa}\langle\hat{\sigma}^{\dagger k}(t)\hat{\sigma}^{\dagger k}(t)\rangle + \frac{2}{\kappa}\langle\hat{F}^\dagger(t)\hat{\sigma}^{\dagger k}(t)\rangle. \tag{4.84}$$

This equation can be rewritten as

$$\langle\hat{a}^\dagger(t)\hat{\sigma}^{\dagger k}(t)\rangle = -\frac{2g}{\kappa}\langle\hat{\sigma}_c^{\dagger k}(t)\rangle + \frac{2}{\kappa}\langle\hat{F}^\dagger(t)\hat{\sigma}^{\dagger k}(t)\rangle. \tag{4.85}$$

In addition, upon multiplying the adjoint of Eq. (4.10) on the left by $\hat{F}^\dagger(t)$, we see that

$$\langle\hat{F}^\dagger(t)\hat{\sigma}^{\dagger k}(t)\rangle = \langle\hat{F}^\dagger(t)\hat{\sigma}^{\dagger k}(0)\rangle e^{-\mu t/2} + e^{-\mu t/2}\int_0^t e^{\mu t'/2}\langle\hat{F}^\dagger(t)\hat{f}^\dagger(t')\rangle dt', \tag{4.86}$$

from which follows

$$\langle\hat{F}^\dagger(t)\hat{\sigma}^{\dagger k}(t)\rangle = 0. \tag{4.87}$$

Hence in view of this result, we can put Eq. (4.85) in the form

$$\langle\hat{a}^\dagger(t)\hat{\sigma}^{\dagger k}(t)\rangle = -\frac{2g}{\kappa}\langle\hat{\sigma}_c^{\dagger k}(t)\rangle. \tag{4.88}$$

Now on substituting this result into Eq. (4.83), we find

$$\langle\hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle_k = \langle\hat{a}^{\dagger 2}(t)\rangle_k e^{-\kappa\tau/2} + \frac{\gamma_c}{\kappa-\mu}\langle\hat{\sigma}_c^{\dagger k}\rangle\left(e^{-\mu\tau/2} - e^{-\kappa\tau/2}\right) \tag{4.89}$$

and on summing over all atoms, we get

$$\langle\hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle = \langle\hat{a}^{\dagger 2}(t)\rangle e^{-\kappa\tau/2} + \frac{\gamma_c}{\kappa-\mu}\langle\hat{m}_c^\dagger(t)\rangle\left(e^{-\mu\tau/2} - e^{-\kappa\tau/2}\right). \tag{4.90}$$

Taking into account Eq. (4.43), this expression can be put in the form

$$\langle\hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle = \langle\hat{a}^{\dagger 2}(t)\rangle\left(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right). \tag{4.91}$$

Following the same procedure, we can readily established that

$$\langle \hat{a}(t)\hat{a}(t+\tau)\rangle = \langle \hat{a}^2(t)\rangle\bigg(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\bigg), \tag{4.92}$$

$$\langle \hat{a}(t)\hat{a}^\dagger(t+\tau)\rangle = \langle \hat{a}(t)\hat{a}^\dagger(t)\rangle\bigg(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\bigg)$$

$$-\frac{2\kappa}{\kappa-\mu}\bigg(e^{-\mu\tau/2} - e^{-\kappa\tau/2}\bigg). \tag{4.93}$$

On account of Eqs. (4.64), (4.91), (4.92), and (4.93), we can put Eq. (4.78) in the form

$$J(\omega) = \frac{(\triangle n)^2}{\pi}Re\int_0^\infty d\tau\bigg(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\bigg)^2 e^{i(\omega-\omega_o)\tau}$$

$$+\frac{\bar{n}}{\pi}Re\int_0^\infty d\tau\bigg[-2\bigg(\frac{\kappa}{\kappa-\mu}\bigg)^2 e^{-\mu\tau} + \frac{2\kappa\mu}{(\kappa-\mu)^2}e^{-(\kappa+\mu)\tau/2}$$

$$+2\bigg(\frac{\kappa}{\kappa-\mu}\bigg)^2 e^{-(\kappa+\mu)\tau/2} - \frac{2\kappa\mu}{(\kappa-\mu)^2}e^{-\kappa\tau}\bigg]e^{i(\omega-\omega_0)\tau} \tag{4.94}$$

and on carrying out the integration, we arrive at

$$J(\omega) = \frac{(\triangle n)^2}{\pi}\bigg[\bigg(\frac{\kappa}{\kappa-\mu}\bigg)^2\frac{\mu}{\mu^2+(\omega-\omega_o)^2} - 2\bigg(\frac{\kappa}{\kappa-\mu}\bigg)\bigg(\frac{\mu}{\kappa-\mu}\bigg)$$

$$\times\frac{(\mu+\kappa)/2}{(\mu+\kappa)^2/4+(\omega-\omega_o)^2} + \bigg(\frac{\mu}{\kappa-\mu}\bigg)^2\frac{\kappa}{\kappa^2+(\omega-\omega_o)^2}\bigg]$$

$$+\frac{\bar{n}}{\pi}\bigg[-2\bigg(\frac{\kappa}{\kappa-\mu}\bigg)^2\frac{\mu}{\mu^2+(\omega-\omega_0)^2} + \frac{2\kappa\mu}{(\kappa-\mu)^2}\frac{(\kappa+\mu)/2}{(\kappa+\mu)^2/4+(\omega-\omega_0)^2}$$

$$+2\bigg(\frac{\kappa}{\kappa-\mu}\bigg)^2\frac{(\kappa+\mu)/2}{(\kappa+\mu)^2/4+(\omega-\omega_0)^2} - \frac{2\kappa\mu}{(\kappa-\mu)^2}\frac{\kappa}{\kappa^2+(\omega-\omega_0)^2}\bigg]. \tag{4.95}$$

Now on introducing this equation into Eq. (4.75), we find

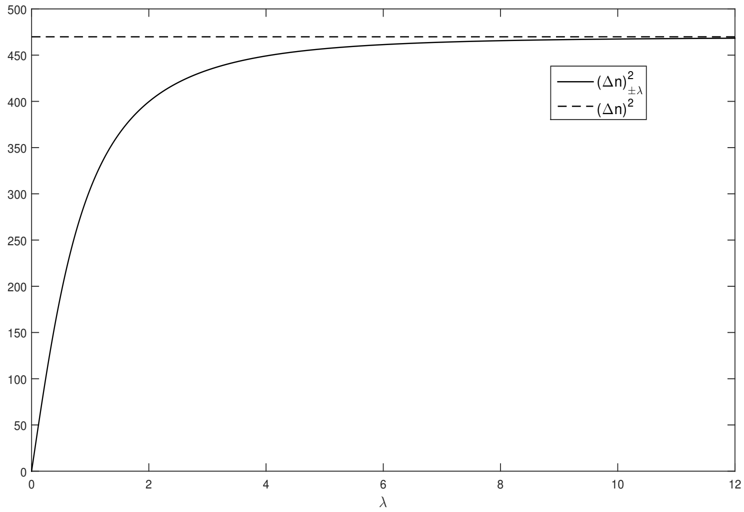

Figure 4.5: Plots of local photon number variance [Eq. (4.98), solid curve] and global photon number variance [Eq. (4.46), dashed curve] versus λ for $\kappa = 0.8$, N=30, $r_a = 5$, and $\gamma_c = 0.2$.

$$
\begin{aligned}
(\triangle n)^2_{\pm\lambda} = \frac{(\triangle n)^2}{\pi}\bigg[&\left(\frac{\kappa}{\kappa-\mu}\right)^2 \mu \int_{-\lambda}^{+\lambda} \frac{d\omega'}{\mu^2+\omega'^2} \\
&-\left(\frac{\kappa}{\kappa-\mu}\right)\left(\frac{\mu}{\kappa-\mu}\right)(\mu+\kappa)\int_{-\lambda}^{+\lambda}\frac{d\omega'}{(\mu+\kappa)^2/4+\omega'^2} \\
&+\left(\frac{\mu}{\kappa-\mu}\right)^2 \kappa \int_{-\lambda}^{+\lambda}\frac{d\omega'}{\kappa^2+\omega'^2}\bigg] \\
&+\frac{\bar{n}}{\pi}\bigg[-2\left(\frac{\kappa}{\kappa-\mu}\right)^2 \mu \int_{-\lambda}^{\lambda}\frac{d\omega'}{\mu^2+\omega'^2} \\
&+\frac{2\kappa\mu}{(\kappa-\mu)^2}(\kappa+\mu)/2\int_{-\lambda}^{+\lambda}\frac{d\omega'}{(\kappa+\mu)^2/4+\omega'^2} \\
&+\left(\frac{\kappa}{\kappa-\mu}\right)^2(\kappa+\mu)\int_{-\lambda}^{+\lambda}\frac{d\omega'}{(\kappa+\mu)^2/4+\omega'^2} \\
&-\frac{2\kappa\mu}{(\kappa-\mu)^2}\kappa\int_{-\lambda}^{+\lambda}\frac{d\omega'}{\kappa^2+\omega'^2}\bigg]. \qquad (4.96)
\end{aligned}
$$

And on performing the integration, we have

$$(\triangle n)^2_{\pm\lambda} = (\triangle n)^2\bigg[\frac{2}{\pi}\bigg(\frac{\kappa}{\kappa-\mu}\bigg)^2 tan^{-1}\bigg(\frac{\lambda}{\mu}\bigg) - \frac{4}{\pi}\bigg(\frac{\kappa}{\kappa-\mu}\bigg)\bigg(\frac{\mu}{\kappa-\mu}\bigg)tan^{-1}\bigg(\frac{2\lambda}{\mu+\kappa}\bigg)$$

$$+\frac{2}{\pi}\bigg(\frac{\mu}{\kappa-\mu}\bigg)^2 tan^{-1}\bigg(\frac{\lambda}{\kappa}\bigg)\bigg] + \bar{n}\bigg[-\frac{4}{\pi}\bigg(\frac{\kappa}{\kappa-\mu}\bigg)^2 tan^{-1}\bigg(\frac{\lambda}{\mu}\bigg)$$

$$+\frac{4\kappa\mu/\pi}{(\kappa-\mu)^2}tan^{-1}\bigg(\frac{2\lambda}{\kappa+\mu}\bigg) + \frac{4}{\pi}\bigg(\frac{\kappa}{\kappa-\mu}\bigg)^2 tan^{-1}\bigg(\frac{2\lambda}{\kappa+\mu}\bigg)$$

$$-\frac{4}{\pi}\frac{\kappa\mu}{(\kappa-\mu)^2}tan^{-1}\bigg(\frac{\lambda}{\kappa}\bigg)\bigg]. \tag{4.97}$$

Applying Eq. (2.100), one can get

$$(\triangle n)^2_{\pm\lambda} = (\triangle n)^2\bigg[\frac{2}{\pi}\bigg(\frac{\kappa}{\kappa-(\gamma_c+2r_a)}\bigg)^2 tan^{-1}\bigg(\frac{\lambda}{\gamma_c+2r_a}\bigg)$$

$$-\frac{4}{\pi}\bigg(\frac{\kappa}{\kappa-(\gamma_c+2r_a)}\bigg)\bigg(\frac{\gamma_c+2r_a}{\kappa-(\gamma_c+2r_a)}\bigg)tan^{-1}\bigg(\frac{2\lambda}{(\gamma_c+2r_a)+\kappa}\bigg)$$

$$+\frac{2}{\pi}\bigg(\frac{\gamma_c+2r_a}{\kappa-(\gamma_c+2r_a)}\bigg)^2 tan^{-1}\bigg(\frac{\lambda}{\kappa}\bigg)\bigg] + \bar{n}\bigg[-\frac{4}{\pi}\bigg(\frac{\kappa}{\kappa-(\gamma_c+2r_a)}\bigg)^2$$

$$\times tan^{-1}\bigg(\frac{\lambda}{\gamma_c+2r_a}\bigg) + \frac{4\kappa(\gamma_c+2r_a)/\pi}{[\kappa-(\gamma_c+2r_a)]^2}tan^{-1}\bigg(\frac{2\lambda}{\kappa+\gamma_c+2r_a}\bigg)$$

$$+\frac{4}{\pi}\bigg(\frac{\kappa}{\kappa-(\gamma_c+2r_a)}\bigg)^2 tan^{-1}\bigg(\frac{2\lambda}{\kappa+(\gamma_c+2r_a)}\bigg) - \frac{4}{\pi}\frac{\kappa(\gamma_c+2r_a)}{[\kappa-(\gamma_c+2r_a)]^2}$$

$$\times tan^{-1}\bigg(\frac{\lambda}{\kappa}\bigg)\bigg]. \tag{4.98}$$

We observe from the plot in Figure 4.5 that as λ increases, the local photon number variance approaches to the global photon-number variance. We also note that a large part of the photon number variance is confined in a relatively small frequency interval.

5

Quadrature Squeezing

We seek here to study the quadrature squeezing of the two-mode cavity light produced by a three-level laser coupled to a two-mode vacuum reservoir. The three-level atoms available in a closed cavity are pumped from the bottom to the top level by electron bombardment. Using the solutions of the quantum Langevin equation for the two-mode cavity light and the equations of evolution of the atomic operators, we calculate the global as well as the local quadrature squeezing.

5.1 Quadrature variance

Here we wish to calculate the quadrature variances of the two-mode cavity light. To this end, employing the definition of the commutator of two operators, we have

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} \tag{5.1}$$

and on taking the expectation value of the operators on the right hand side of this equation, we see that

$$[\hat{a}, \hat{a}^\dagger] = \langle\hat{a}\hat{a}^\dagger\rangle - \langle\hat{a}^\dagger\hat{a}\rangle. \tag{5.2}$$

Thus in view of Eqs. (4.15) and (4.42), we find

$$[\hat{a}, \hat{a}^\dagger] = 2N + \frac{\gamma_c}{\kappa}(\langle\hat{N}_c\rangle - \langle\hat{N}_a\rangle). \tag{5.3}$$

The squeezing properties of the two-mode cavity light are described by two quadrature operators

$$\hat{a}_+ = \hat{a} + \hat{a}^\dagger \tag{5.4}$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \tag{5.5}$$

The quadrature variances are defined by

$$(\triangle a_\pm)^2 = \langle \hat{a}_\pm^2 \rangle - \langle \hat{a}_\pm \rangle^2. \tag{5.6}$$

This can also be put in the form

$$(\triangle a_\pm)^2 = \pm \langle [\hat{a}^\dagger \pm \hat{a}]^2 \rangle \mp [\langle \hat{a}^\dagger \rangle \pm \langle \hat{a} \rangle]^2 \tag{5.7}$$

and in view of Eq. (4.26), there follows

$$(\triangle a_\pm)^2 = \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}\hat{a}^\dagger \rangle \pm \langle \hat{a}^{\dagger 2} \rangle \pm \langle \hat{a}^2 \rangle. \tag{5.8}$$

Now substitution of Eqs. (4.15), (4.42), and (4.44) into Eq. (5.8) leads to

$$(\triangle a_\pm)^2 = \frac{\gamma_c}{\kappa}\bigg(\langle \hat{N}_a \rangle + 2\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle\bigg) + 2N \pm 2\frac{\gamma_c}{\kappa}\sqrt{\frac{\gamma_c}{r_a}}\langle \hat{N}_a \rangle, \tag{5.9}$$

so that on making use of Eq. (2.117), we get

$$(\triangle a_\pm)^2 = 2N + \frac{\gamma_c}{\kappa}N + \frac{\gamma_c}{\kappa}\langle \hat{N}_b \rangle \pm 2\frac{\gamma_c}{\kappa}\sqrt{\frac{\gamma_c}{r_a}}\langle \hat{N}_a \rangle. \tag{5.10}$$

Now on account of Eqs. (3.3) and (3.6), we see that

$$(\triangle a_\pm)^2 = 2N + \frac{\gamma_c}{\kappa}N + \frac{\gamma_c}{\kappa}N\bigg(\frac{1 \pm 2\sqrt{\eta}}{\eta + 2}\bigg). \tag{5.11}$$

This can also be written as

$$(\triangle a_+)^2 = 2N + \frac{\gamma_c}{\kappa}N + \frac{\gamma_c}{\kappa}N\bigg(\frac{1 + 2\sqrt{\eta}}{\eta + 2}\bigg), \tag{5.12}$$

$$(\triangle a_-)^2 = 2N + \frac{\gamma_c}{\kappa}N + \frac{\gamma_c}{\kappa}N\bigg(\frac{1 - 2\sqrt{\eta}}{\eta + 2}\bigg). \tag{5.13}$$

Expressions (5.12) and (5.13) represent the variances of the quadrature operators $\hat{a}_+$ and $\hat{a}_-$ for the two-mode cavity light. We note that the squeezing occurs in the minus quadrature.

Furthermore, employing Eqs. (5.4) and (5.5), we see that

$$[\hat{a}_-, \hat{a}_+] = i\bigg([\hat{a}^\dagger, \hat{a}] - [\hat{a}, \hat{a}^\dagger]\bigg). \tag{5.14}$$

Thus employing Eq. (5.3) together with the relation described by Eq. (3.59), we have

$$[\hat{a}_+, \hat{a}_-] = 2i\bigg(\frac{\gamma_c}{\kappa}\langle\hat{N}_a\rangle - \frac{\gamma_c}{\kappa}\langle\hat{N}_c\rangle - 2N\bigg), \tag{5.15}$$

from which follows

$$\triangle a_+ \triangle a_- \geq \bigg|\frac{\gamma_c}{\kappa}\langle\hat{N}_a\rangle - \frac{\gamma_c}{\kappa}\langle\hat{N}_c\rangle - 2N\bigg|. \tag{5.16}$$

5.2 Global quadrature squeezing

We next seek to calculate the quadrature squeezing of the two-mode cavity light in the entire frequency interval. On account of Eq. (4.16), the two-mode cavity light is in a vacuum state for $r_a = 0$ and the variances of the quadrature operators for this case are

$$(\triangle a_-)^2_{vac} = (\triangle a_+)^2_{vac} = 2N + \frac{\gamma_c}{\kappa}N. \tag{5.17}$$

We define the quadrature squeezing of the cavity light relative to the quadrature variance of the vacuum state by [4]

$$S = \frac{(\triangle a_-)^2_{vac} - (\triangle a_-)^2}{(\triangle a_-)^2_{vac}}. \tag{5.18}$$

Thus with the aid of Eqs. (5.13) and (5.17), one can put Eq. (5.18) in the form

$$S = \frac{\gamma_c}{\gamma_c + 2\kappa}\left(\frac{2\sqrt{\frac{\gamma_c}{r_a}} - 1}{\frac{\gamma_c}{r_a} + 2}\right). \tag{5.19}$$

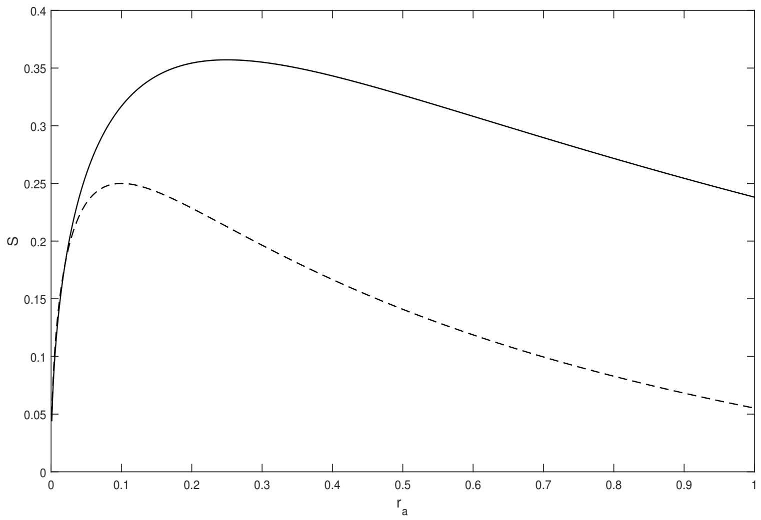

Figure 5.1: Plots of quadrature squeezing S [Eq. (5.19)] versus r_a for $\kappa = 0.2$, $\gamma_c = 0.4$ [dashed curve], and $\gamma_c = 1$ [solid curve].

We observe from this result that the amount of squeezing depends on the cavity damping constant (κ), the stimulated emission decay constant(γ_c), and the rate at which an atom is pumped to the top level (r_a). However, the quadrature squeezing does not depend on the number of atoms in the cavity. Figure 5.1 shows the global quadrature squeezing versus r_a for different values of γ_c. For fixed κ, the amount of squeezing increases with the stimulated decay constant. In addition, the amount of squeezing increases from zero to its maximum value and then starts to decrease as r_a increases. The maximum squeezing is 37.5 $\%$ below the vacuum level for $\kappa = 0.2$ and $\gamma_c = 1.2$. We can also put Eq. (5.19) in the form

$$S = \frac{\gamma_c}{\gamma_c + 2\kappa}\left(\frac{2\sqrt{\eta} - 1}{\eta + 2}\right). \tag{5.20}$$

On the other hand, Fesseha[4] has found the quadrature squeezing of the cavity light by putting the noise operator associated with the vacuum reservoir in normal

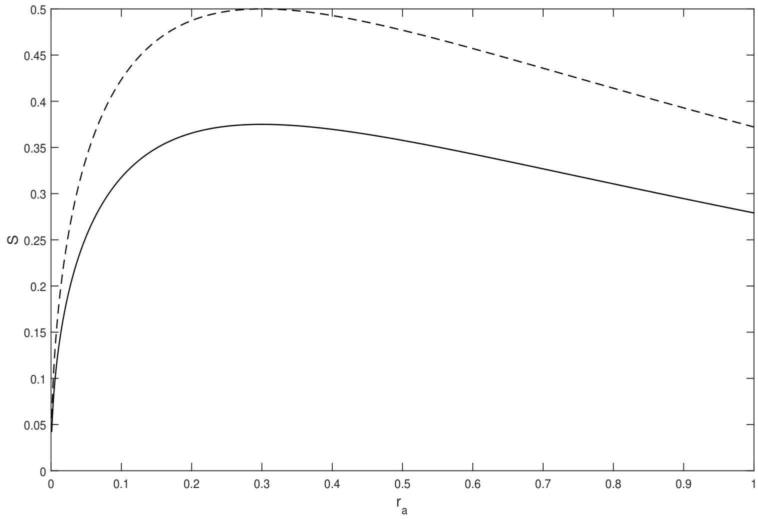

Figure 5.2: Plots of quadrature squeezing S [Eq. (5.21), dashed curve] and [Eq. (5.20), solid curve] versus r_a for $\gamma_c = 1.2$ and $\kappa = 0.2$.

order to be

$$S = \frac{2\sqrt{\eta} - 1}{\eta + 2}. \tag{5.21}$$

We note from Figure 5.2 that the two-mode cavity light is in a squeezed state for $r_a > 0$ and the maximum quadrature squeezing occurs when the three-level laser is operating below threshold at $r_a = 0.3$. Moreover, the plots in Figure 5.2 indicate that the quadrature squeezing is less when we put the noise operators associated with the vacuum reservoir in an arbitrary order than in normal order.

5.3 Local quadrature squeezing

In this section we wish to calculate the quadrature squeezing of the two-mode cavity light in a given frequency interval. We define the spectrum of quadrature fluctuations for the two-mode light with central frequency ω_o by [4]

$$S_{\pm}(\omega) = \frac{1}{\pi} Re \int_0^{\infty} \langle \hat{a}_{\pm}(t), \hat{a}_{\pm}(t+\tau) \rangle_{ss} e^{i(\omega - \omega_o)\tau} d\tau. \tag{5.22}$$

Upon integrating both sides of Eq. (5.22) over ω, we get

$$\int_{-\infty}^{\infty} S_{\pm}(\omega)d\omega = (\triangle a_{\pm})^2, \tag{5.23}$$

in which

$$(\triangle a_{\pm})^2 = \langle \hat{a}_{\pm}(t), \hat{a}_{\pm}(t)\rangle_{ss} \tag{5.24}$$

is the global quadrature variance of the two-mode cavity light at steady state. On the basis of Eq. (5.23), we observe that $S_{\pm}(\omega)d\omega$ is the quadrature variance of the light mode in the interval between ω and $\omega + d\omega$ [4]. The variance of the minus quadrature in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ can then be expressed as

$$(\triangle a_{-})^2_{\pm\lambda} = \int_{-\lambda}^{\lambda} S_{-}(\omega')d\omega'. \tag{5.25}$$

On account of Eq. (4.26), one can write

$$\langle \hat{a}_{\pm}(t), \hat{a}_{\pm}(t+\tau)\rangle_{ss} = \langle \hat{a}_{\pm}(t)\hat{a}_{\pm}(t+\tau)\rangle. \tag{5.26}$$

We now proceed to determine the two-time correlation function that appears in Eq. (5.26). To this end, making use of Eqs. (4.64), (4.91), (4.92), and (4.93), we get

$$\langle \hat{a}_{\pm}(t), \hat{a}_{\pm}(t+\tau)\rangle_{ss} = (\Delta a_{\pm})^2\bigg(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\bigg)$$

$$-\frac{2\kappa}{\kappa-\mu}(e^{-\mu\tau/2} - e^{-\kappa\tau/2}). \tag{5.27}$$

Thus in view of this result, we can put Eq. (5.22) in the form

$$S_{-}(\omega) = \frac{1}{\pi}\bigg[\bigg(\frac{\kappa}{\kappa-\mu}(\Delta a_{-})^2 - \frac{2\kappa}{\kappa-\mu}\bigg)Re\int_0^{\infty} d\tau e^{-[\mu/2-i(\omega-\omega_o)]\tau}$$

$$+\bigg(\frac{2\kappa}{\kappa-\mu} - \frac{\mu}{\kappa-\mu}(\Delta a_{-})^2\bigg)Re\int_0^{\infty} d\tau e^{-[\kappa/2-i(\omega-\omega_o)]\tau}\bigg] \tag{5.28}$$

and on performing the integration, we find the spectrum of the minus quadrature fluctuations to be

$$S_{-}(\omega) = \frac{1}{\pi}\bigg[\bigg(\frac{\kappa}{\kappa-\mu}(\Delta a_{-})^{2} - \frac{2\kappa}{\kappa-\mu}\bigg)\frac{\mu/2}{(\mu/2)^{2}+(\omega-\omega_{o})^{2}}$$

$$+\bigg(\frac{2\kappa}{\kappa-\mu} - \frac{\mu}{\kappa-\mu}(\Delta a_{-})^{2}\bigg)\frac{\kappa/2}{(\kappa/2)^{2}+(\omega-\omega_{o})^{2}}\bigg]. \tag{5.29}$$

Now on introducing this result into Eq. (5.25), we have

$$S_{-}(\omega) = \frac{1}{\pi}\bigg[\bigg(\frac{\kappa}{\kappa-\mu}(\Delta a_{-})^{2} - \frac{2\kappa}{\kappa-\mu}\bigg)\mu/2\int_{-\lambda}^{+\lambda}\frac{d\omega'}{(\mu/2)^{2}+(\omega')^{2}}$$

$$+\bigg(\frac{2\kappa}{\kappa-\mu} - \frac{\mu}{\kappa-\mu}(\Delta a_{-})^{2}\bigg)\kappa/2\int_{-\lambda}^{+\lambda}\frac{d\omega'}{(\kappa/2)^{2}+(\omega')^{2}}\bigg]. \tag{5.30}$$

And on carrying out the integration, applying the relation described by Eq. (4.68), we get

$$(\triangle a_{-})^{2}_{\pm\lambda} = \frac{\bigg(2\kappa(\Delta a_{-})^{2} - 4\kappa\bigg)/\pi}{\kappa-\mu}tan^{-1}\bigg(\frac{2\lambda}{\mu}\bigg)$$

$$+\frac{\bigg(4\kappa - 2\mu(\Delta a_{-})^{2}\bigg)/\pi}{\kappa-\mu}tan^{-1}\bigg(\frac{2\lambda}{\kappa}\bigg), \tag{5.31}$$

where $(\triangle a_{-})^{2}$ is given by Eq. (5.13). Using Eq. (2.84), we can put Eq. (5.31) in the form

$$(\triangle a_{-})^{2}_{\pm\lambda} = \frac{\bigg(2\kappa(\Delta a_{-})^{2} - 4\kappa\bigg)/\pi}{\kappa-(\gamma_{c}+2r_{a})}tan^{-1}\bigg(\frac{2\lambda}{\gamma_{c}+2r_{a}}\bigg)$$

$$+\frac{\bigg(4\kappa - 2(\gamma_{c}+2r_{a})(\Delta a_{-})^{2}\bigg)/\pi}{\kappa-(\gamma_{c}+2r_{a})}tan^{-1}\bigg(\frac{2\lambda}{\kappa}\bigg). \tag{5.32}$$

Moreover, the local quadrature variance of the vacuum state can be obtained by setting $r_{a}=0$ in Eq. (5.32) and is given by

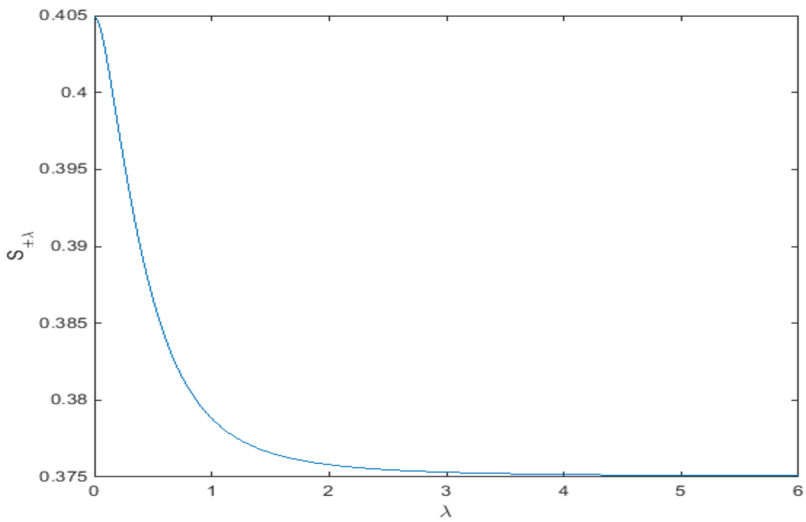

Figure 5.3: Plots of the local quadrature squeezing S [Eq. (5.35)] versus λ for $\gamma_c = 1.2$, $\kappa = 0.2$, $r_a = 0.3$, and $N = 20$.

$$(\triangle a_-)^2_{vac\pm\lambda} = \frac{\left(2\kappa(\Delta a_-)^2_{vac} - 4\kappa\right)/\pi}{\kappa - \gamma_c} tan^{-1}\left(\frac{2\lambda}{\gamma_c}\right)$$

$$+\frac{\left(4\kappa - 2\gamma_c(\Delta a_-)^2_{vac}\right)/\pi}{\kappa - \gamma_c} tan^{-1}\left(\frac{2\lambda}{\kappa}\right). \quad (5.33)$$

We define the quadrature squeezing of the two-mode cavity light in the $\lambda_\pm$ frequency interval by

$$S_{\pm\lambda} = \frac{(\triangle a_-)^2_{vac\pm\lambda} - (\triangle a_-)^2_{\pm\lambda}}{(\triangle a_-)^2_{vac\pm\lambda}}. \quad (5.34)$$

On account of Eqs. (5.32) and (5.33), we then see that

$$S_{\pm\lambda} = 1 - \frac{\frac{\left(2\kappa(\Delta a_-)^2 - 4\kappa\right)/\pi}{\kappa-(\gamma_c+2r_a)} tan^{-1}\left(\frac{2\lambda}{\gamma_c+2r_a}\right) + \frac{\left(4\kappa-2(\gamma_c+2r_a)(\Delta a_-)^2\right)/\pi}{\kappa-(\gamma_c+2r_a)} tan^{-1}\left(\frac{2\lambda}{\kappa}\right)}{\frac{\left(2\kappa(\Delta a_-)^2_{vac} - 4\kappa\right)/\pi}{\kappa-\gamma_c} tan^{-1}\left(\frac{2\lambda}{\gamma_c}\right) + \frac{\left(4\kappa-2\gamma_c(\Delta a_-)^2_{vac}\right)/\pi}{\kappa-\gamma_c} tan^{-1}\left(\frac{2\lambda}{\kappa}\right)}. \quad (5.35)$$

This represents the local quadrature squeezing of the two-mode cavity light. We note from Figure 5.3 that the local quadrature squeezing is greater than the global quadrature squeezing. We also see from the same plot in Figure 5.3 that the maximum local quadrature squeezing is 40.25 $\%$ below the vacuum-state level and occurs in the ± 0.01 frequency interval. Moreover, the local quadrature squeezing approaches the global quadrature squeezing as λ increases.

6

Superposed Two-Mode Light Beams

In this chapter we investigate the squeezing and statistical properties of a pair of superposed two-mode light beams produced by three-level lasers coupled to a two-mode vacuum reservoir. With the aid of the antinormally ordered characterstic function defined in the Heisenberg picture, we first obtain the Q function of a two-mode light. We then determine the density operator for a pair of superposed two-mode light beams in terms of the Q functions. Applying the resulting density operator, we wish to study the squeezing and statistical properties of a pair of superposed two-mode light beams.

6.1 The Q function

Here we seek to obtain, applying the antinormally ordered characterstic function defined in the Heisenberg picture, the Q function of the two-mode light produced by a three-level laser in which the three-level atoms available in a closed cavity are pumped from the bottom to the top level by electron bombardment. To this end, the antinormally ordered characterstic function is defined by

$$\phi_a(z) = Tr(\hat{\rho} e^{-z^*\hat{a}} e^{z\hat{a}^\dagger}). \tag{6.1}$$

On the other hand, for operators subject to the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = \lambda, \tag{6.2}$$

in which

$$\lambda = \frac{\gamma_c}{\kappa}(\hat{N}_c - \hat{N}_a) + 2N, \tag{6.3}$$

the completeness relation and the action of $\hat{a}$ on the coherent states are respectively given by [4]

$$\hat{I} = \frac{\lambda}{\pi}\int d^2\beta|\beta\rangle\langle\beta| \tag{6.4}$$

and

$$\hat{a}|\beta\rangle = \lambda\beta|\beta\rangle. \tag{6.5}$$

In view of Eqs. (6.4) and (6.5), we write Eq. (6.1) as

$$\phi_a(z) = \int d^2\beta\lambda Q(\lambda\beta)exp(\lambda z\beta^* - \lambda z^*\beta), \tag{6.6}$$

where

$$Q(\lambda\beta) = \frac{1}{\pi}\langle\beta|\hat{\rho}|\beta\rangle \tag{6.7}$$

is the Q function. Introducing a new variable $\alpha = \lambda\beta$, the antinormally ordered characterstic function can also be written as

$$\phi_a(z) = \int d^2\alpha\frac{Q(\alpha)}{\lambda}exp(z\alpha^* - z^*\alpha). \tag{6.8}$$

Since $\frac{Q(\alpha)}{\lambda}$ is the inverse Fourier transform of the antinormally ordered characterstic function, we have

$$Q(\alpha) = \frac{\lambda}{\pi^2}\int d^2z\phi_a(z)exp(z^*\alpha - z\alpha^*). \tag{6.9}$$

We next seek to obtain the explicit form of the antinormally ordered characterstic function for the system under considerations. Upon replacing the atomic operators by their expectation values in Eq. (6.3), we have

$$\lambda = \frac{\gamma_c}{\kappa}(\langle\hat{N}_c\rangle - \langle\hat{N}_a\rangle) + 2N \tag{6.10}$$

and applying the Baker-Hausdorff identity to Eq. (6.1), we write the antinormally ordered characterstic function as

$$\phi_a(z,t) = e^{-(\lambda z^* z)/2}\langle e^{z\hat{a}^\dagger(t) - z^*\hat{a}(t)}\rangle. \tag{6.11}$$

Since $\hat{a}$ is a Guassian variable with zero mean, this equation can also be put in the form

$$\phi_a(z,t) = e^{-(\lambda z^* z)/2} exp\bigg[\frac{1}{2}\langle\bigg(z\hat{a}^\dagger(t) - z^*\hat{a}(t)\bigg)^2\rangle\bigg], \tag{6.12}$$

from which follows

$$\phi_a(z,t) = e^{-(\lambda z^* z)/2} exp\bigg[\frac{1}{2}\langle\bigg(z^2\hat{a}^{\dagger 2}(t) - zz^*\hat{a}^\dagger(t)\hat{a}(t) - z^* z\hat{a}(t)\hat{a}^\dagger(t) + z^{*2}\hat{a}^2(t)\bigg)\rangle\bigg]. \tag{6.13}$$

In view of Eqs. (4.15), (4.42), and (4.44) along with Eq. (6.10), we see that

$$\phi_a(z,t) = exp[-az^* z + b(z^2 + z^{*2})/2], \tag{6.14}$$

where

$$a = \frac{\gamma_c}{\kappa}(\langle\hat{N}_b\rangle + \langle\hat{N}_c\rangle) + 2N \tag{6.15}$$

and

$$b = \frac{\gamma_c}{\kappa}\sqrt{\frac{\gamma_c}{r_a}}\langle\hat{N}_a\rangle. \tag{6.16}$$

On introducing Eq. (6.14) into Eq. (6.9), we have

$$Q(\alpha) = \frac{\lambda}{\pi}\int\frac{d^2 z}{\pi} exp[-az^* z + b(z^2 + z^{*2})/2 + z^*\alpha - z\alpha^*]. \tag{6.17}$$

Now employing the relation

$$\int\frac{d^2 z}{\pi} exp(-az^* z + bz + cz^* + Az^2 + Bz^{*2})$$

$$= \bigg[\frac{1}{a^2 - 4AB}\bigg]^{\frac{1}{2}} exp\bigg(\frac{abc + Ac^2 + Bb^2}{a^2 - 4AB}\bigg), \tag{6.18}$$

we find

$$Q(\alpha) = \frac{\lambda}{\pi}\frac{1}{(a^2 - b^2)^{\frac{1}{2}}} exp\bigg[\frac{-a\alpha^*\alpha + \frac{b}{2}(\alpha^2 + \alpha^{*2})}{a^2 - b^2}\bigg]. \tag{6.19}$$

This represents the Q function of a two-mode cavity light.

6.2 The density operator

We next proceed to determine the density operator for a pair of superposed two-mode light beams produced by three-level lasers in wich the three-level atoms available in a closed cavity are pumped from the bottom to the top level by electron bombardment. Let $\hat{\rho}'(\hat{a}^\dagger, \hat{a})$ be the density operator for one of the two-mode light beams. Then upon expanding the density operator in the normal order and employing the completeness relation for coherent states, we have

$$\hat{\rho}' = \frac{\lambda}{\pi}\int d^2\gamma \sum_{kl} c_{kl}(\lambda\gamma^*)^k|\gamma\rangle\langle\gamma|\hat{a}^l. \tag{6.20}$$

Employing the relation [4]

$$|\gamma\rangle\langle\gamma|\hat{a}^l = (\lambda\gamma + \frac{\partial}{\partial\gamma^*})^l|\gamma\rangle\langle\gamma|, \tag{6.21}$$

we put Eq. (6.20) in the form

$$\hat{\rho}' = \lambda\int d^2\gamma Q(\lambda\gamma^*, \lambda\gamma + \frac{\partial}{\partial\gamma^*})|\gamma\rangle\langle\gamma|. \tag{6.22}$$

We can also write Eq. (6.22) as

$$\hat{\rho}' = \lambda\int d^2\gamma Q(\lambda\gamma^*, \lambda\gamma + \frac{\partial}{\partial\gamma^*})D(\gamma)|0\rangle\langle 0|D(-\gamma). \tag{6.23}$$

We then realize that the density operator for the superposition of the first light beam and another one is expressible as

$$\hat{\rho} = \lambda\int d^2\eta Q(\lambda\eta^*, \lambda\eta + \frac{\partial}{\partial\eta^*})\hat{D}(\eta)\hat{\rho}'\hat{D}(-\eta) \tag{6.24}$$

and in view of Eq. (6.22), we find

$$\hat{\rho} = \lambda^2\int d^2\gamma d^2\eta Q(\lambda\gamma^*, \lambda\gamma + \frac{\partial}{\partial\gamma^*})Q(\lambda\eta^*, \lambda\eta + \frac{\partial}{\partial\eta^*})|\gamma + \eta\rangle\langle\eta + \gamma|. \tag{6.25}$$

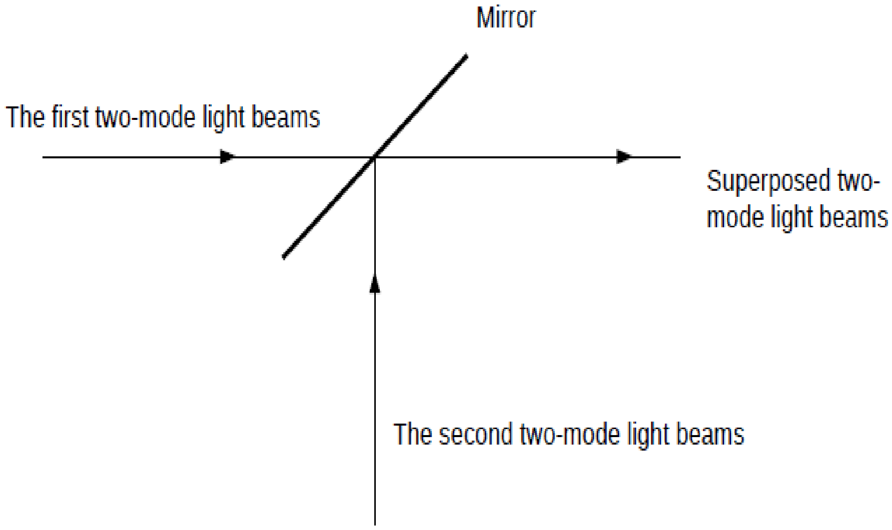

Figure 6.1: The superposed two-mode light beams, with $\kappa = 1$ and $\kappa = 0$ for the upper and lower surfaces of the mirror respectively.

6.3 Global photon statistics

In this section we wish to obtain the global mean and variance of the photon number applying the density operator for the pair of superposed two-mode light beams.

6.3.1 Global mean photon number

We next seek to obtain the mean photon number for the superposed two-mode light beams. To this end, applying Eq. (6.22), the expectation value of a given operator function $\hat{A}(\hat{a}^\dagger, \hat{a})$ can be written as

$$\langle \hat{A} \rangle = \lambda \int d^2\gamma Q(\lambda\gamma^*, \lambda\gamma + \frac{\partial}{\partial\gamma^*})A_n(\lambda\gamma^*, \lambda\gamma), \tag{6.26}$$

where

$$A_n(\lambda\gamma^*, \lambda\gamma) = \sum_{lm} C_{lm}(\lambda\gamma^*)^l(\lambda\gamma)^m \tag{6.27}$$

is the c-number function corresponding to $\hat{A}$ in the normal order. Introducing a variable $\alpha = \lambda\gamma$, we can put Eq. (6.26) in the form

$$\langle\hat{A}\rangle = \frac{1}{\lambda}\int d^2\alpha Q(\alpha^*, \alpha + \lambda\frac{\partial}{\partial\alpha^*})A_n(\alpha^*, \alpha). \tag{6.28}$$

Now we write the mean photon number of the superposed two-mode light beams employing Eq. (6.25) as

$$\bar{n}_s = \lambda^2 \int d^2\gamma d^2\eta Q(\lambda\gamma^*, \lambda\gamma + \frac{\partial}{\partial\gamma^*})Q(\lambda\eta^*, \lambda\eta + \frac{\partial}{\partial\eta^*})Tr\Big[|\gamma+\eta\rangle\langle\gamma+\eta|\hat{c}^\dagger\hat{c}\Big], \tag{6.29}$$

where

$$\hat{c} = \hat{a} + \hat{b}. \tag{6.30}$$

It then follows that

$$\bar{n}_s = \lambda^4 \int d^2\gamma d^2\eta Q(\lambda\gamma^*, \lambda\gamma + \frac{\partial}{\partial\gamma^*})Q(\lambda\eta^*, \lambda\eta + \frac{\partial}{\partial\eta^*})(\gamma^*\gamma + \eta^*\eta + \gamma^*\eta + \eta^*\gamma) \tag{6.31}$$

and using the new variables $\alpha = \lambda\gamma$ and $\beta = \lambda\eta$, we get

$$\bar{n}_s = \frac{1}{\lambda^2}\int d^2\alpha d^2\beta Q(\alpha^*, \alpha + \lambda\frac{\partial}{\partial\alpha^*})Q(\beta^*, \beta + \lambda\frac{\partial}{\partial\beta^*})(\alpha^*\alpha + \beta^*\beta + \alpha^*\beta + \beta^*\alpha). \tag{6.32}$$

The above equation can also be put in the form

$$\bar{n}_s = \frac{1}{\lambda}\int d^2\alpha Q(\alpha^*, \alpha + \lambda\frac{\partial}{\partial\alpha^*})\alpha^*\alpha$$
$$+\frac{1}{\lambda}\int d^2\beta Q(\beta^*, \beta + \lambda\frac{\partial}{\partial\beta^*})\beta^*\beta$$
$$+\frac{1}{\lambda}\int d^2\alpha Q(\alpha^*, \alpha + \lambda\frac{\partial}{\partial\alpha^*})\alpha^*$$
$$\times\frac{1}{\lambda}\int d^2\beta Q(\beta^*, \beta + \lambda\frac{\partial}{\partial\beta^*})\beta$$
$$+\frac{1}{\lambda}\int d^2\alpha Q(\alpha^*, \alpha + \lambda\frac{\partial}{\partial\alpha^*})\alpha$$
$$\times\frac{1}{\lambda}\int d^2\beta Q(\beta^*, \beta + \lambda\frac{\partial}{\partial\beta^*})\beta^*. \tag{6.33}$$

Now in view of Eq. (6.28), we find

$$\bar{n}_s = \langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{b}\rangle + \langle\hat{a}^\dagger\rangle\langle\hat{b}\rangle + \langle\hat{a}\rangle\langle\hat{b}^\dagger\rangle. \tag{6.34}$$

Since the Q function of the two identical two-mode light beams have the same form, we have

$$\langle\hat{a}^\dagger\hat{a}\rangle = \langle\hat{b}^\dagger\hat{b}\rangle \tag{6.35}$$

and

$$\langle\hat{a}\rangle = \langle\hat{b}\rangle. \tag{6.36}$$

Thus the mean photon number of the superposed two-mode light beams can be written as

$$\bar{n}_s = 2\langle\hat{a}^\dagger\hat{a}\rangle + 2\langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle. \tag{6.37}$$

Hence in view of Eqs. (4.15) and (4.26), the mean photon number of the superposed two-mode light beams take the form

$$\bar{n}_s = 2\frac{\gamma_c}{\kappa}\bigg(\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle\bigg). \tag{6.38}$$

We see that the mean photon number of the superposed two-mode light beams is the sum of the mean photon number of the component two-mode light beams.

6.3.2 Global photon number variance

Next we wish to calculate the variance of the photon number for the pair of superposed two-mode light beams. To this end, the variance of the photon number is expressible as

$$(\triangle n)_s^2 = \langle \hat{c}^\dagger \hat{c} \hat{c}^\dagger \hat{c} \rangle - \bar{n}_s^2, \tag{6.39}$$

where $\hat{c}$ denotes the annihilation operator for the superposition of light modes a and b. Applying the relation

$$[\hat{c}, \hat{c}^\dagger] = 2\lambda, \tag{6.40}$$

the photon number variance can also be written as

$$(\triangle n)_s^2 = \langle \hat{c}^{\dagger 2} \hat{c}^2 \rangle + 2\lambda \bar{n}_s - \bar{n}_s^2. \tag{6.41}$$

Thus applying the density operator given by Eq. (6.25), we have

$$\langle \hat{c}^{\dagger 2} \hat{c}^2 \rangle = \lambda^6 \int d^2\gamma d^2\eta Q(\lambda\gamma^*, \lambda\gamma + \frac{\partial}{\partial \gamma^*}) Q(\lambda\eta^*, \lambda\eta + \frac{\partial}{\partial \eta^*})$$

$$\times(\gamma^2\gamma^{*2} + \gamma^2\eta^{*2} + 2\gamma^2\gamma^*\eta^* + \gamma^{*2}\eta^2 + 2\gamma^{*2}\gamma\eta$$

$$+2\gamma^*\eta^*\eta^2 + 2\gamma\eta^{*2}\eta + 4\gamma^*\gamma\eta^*\eta + \eta^{*2}\eta^2). \tag{6.42}$$

Then we can put this equation in terms of the variables $\alpha = \lambda\gamma$ and $\beta = \lambda\eta$ as

$$\langle \hat{c}^{\dagger 2} \hat{c}^2 \rangle = \frac{1}{\lambda^2} \int d^2\alpha d^2\beta Q(\alpha^*, \alpha + \lambda\frac{\partial}{\partial \alpha^*}) Q(\beta^*, \beta + \lambda\frac{\partial}{\partial \beta^*})$$

$$(\alpha^2\alpha^{*2} + \alpha^2\beta^{*2} + 2\alpha^2\alpha^*\beta^* + \alpha^{*2}\beta^2 + 2\alpha\alpha^{*2}\beta$$

$$+2\alpha^*\beta^*\beta^2 + 2\alpha\beta\beta^{*2} + 4\alpha^*\alpha\beta^*\beta + \beta^{*2}\beta^2). \tag{6.43}$$

Now on account of Eq. (6.28), we have

$$\langle \hat{c}^{\dagger 2}\hat{c}^2\rangle = \langle \hat{a}^{\dagger 2}\hat{a}^2\rangle + \langle \hat{a}^2\rangle\langle \hat{b}^{\dagger 2}\rangle + 2\langle \hat{a}^{\dagger}\hat{a}^2\rangle\langle \hat{b}^{\dagger}\rangle + \langle \hat{a}^{\dagger 2}\rangle\langle \hat{b}^2\rangle + 2\langle \hat{a}^{\dagger 2}\hat{a}\rangle\langle \hat{b}\rangle$$

$$+2\langle \hat{a}^{\dagger}\rangle\langle \hat{b}^{\dagger}\hat{b}^2\rangle + 2\langle \hat{a}\rangle\langle \hat{b}^{\dagger 2}\hat{b}\rangle + 4\langle \hat{a}^{\dagger}\hat{a}\rangle.\langle \hat{b}^{\dagger}\hat{b}\rangle + \langle \hat{b}^{\dagger 2}\hat{b}^2\rangle. \tag{6.44}$$

We note that

$$\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle = \langle \hat{b}^{\dagger 2}\hat{b}^2\rangle \tag{6.45}$$

and

$$\langle \hat{a}^2\rangle = \langle \hat{b}^2\rangle. \tag{6.46}$$

Thus employing Eqs. (4.26), (6.45), and (6.46), we can put Eq. (6.44) in the form

$$\langle \hat{c}^{\dagger 2}\hat{c}^2\rangle = 2\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle + 2\langle \hat{a}^{\dagger 2}\rangle\langle \hat{a}^2\rangle + 4\langle \hat{a}^{\dagger}\hat{a}\rangle^2. \tag{6.47}$$

Hence in view of this equation, the variance of the photon number can be written as

$$(\triangle n)_s^2 = 2\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle + 2\langle \hat{a}^{\dagger 2}\rangle\langle \hat{a}^2\rangle + 2\lambda\bar{n}_s. \tag{6.48}$$

Since $\hat{a}$ is a Guassian variable with zero mean, one can write

$$(\triangle n)_s^2 = 4\langle \hat{a}^{\dagger 2}\rangle\langle \hat{a}^2\rangle + \bar{n}_s^2 + 2\lambda\bar{n}_s. \tag{6.49}$$

Now in view of Eq. (4.44), we get

$$(\triangle n)_s^2 = \bar{n}_s(\bar{n}_s + \frac{\eta}{4}\bar{n}_s + 2\lambda) \tag{6.50}$$

and making use of Eq. (6.10), we have

$$(\triangle n)_s^2 = 4N\bar{n}_s + \frac{\bar{n}_s^2}{4}(2 + 3\eta). \tag{6.51}$$

We notice that unlike the mean photon number, the variance of the photon number of a pair of superposed two-mode light beams is not equal to the sum of the photon number variance of the individual two-mode light beams. We easily observe that the global variance of the the photon number of the superposed two-mode light beams is four times that of the component two-mode light beams.

6.4 Local photon statistics

We now proceed to obtain the local mean and variance of the photon number of the superposed two-mode light beams.

6.4.1 Local mean photon number

In this section we wish to calculate the mean photon number of the pair of superposed two-mode light beams in a given frequency interval. The power spectrum of the pair of superposed two-mode light beams with central frequency ω_0 is defined by

$$P_s(\omega) = \frac{1}{\pi} Re \int_0^\infty d\tau \langle \hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle_{ss} e^{i(\omega-\omega_0)\tau}. \tag{6.52}$$

Upon integrating both sides of Eq. (6.52) over ω, we find

$$\int_{-\infty}^{\infty} P_s(\omega) d\omega = \bar{n}_s, \tag{6.53}$$

where $\bar{n}_s$ is the steady-state mean photon number of the superposed two-mode light beams. On the basis of Eq. (6.53), we assert that $P_s(\omega)d\omega$ is the steady-state mean photon number in the interval between ω and $\omega + d\omega$. We can then write the mean photon number in the interval between $\omega' = -\lambda$ and $\omega' = +\lambda$ as

$$\bar{n}_{s\pm\lambda} = \int_{-\lambda}^{+\lambda} P_s(\omega') d\omega'. \tag{6.54}$$

We next seek to obtain the two time correlation function associated with Eq. (6.52) for the pair of superposed two-mode light beams. To this end, in view of Eq. (2.12), the quantum Langevin equations for light modes a and b can be written as

$$\frac{d}{dt}\hat{a}(t) = -\frac{\kappa}{2}\hat{a}(t) - g\hat{\sigma}_A^k(t) + \hat{F}_A(t), \tag{6.55}$$

$$\frac{d}{dt}\hat{b}(t) = -\frac{\kappa}{2}\hat{b}(t) - g\hat{\sigma}_B^k(t) + \hat{F}_B(t). \tag{6.56}$$

Here we replace $\hat{\sigma}^k(t)$ by $\hat{\sigma}_A^k(t)$ and $\hat{\sigma}_B^k(t)$ for light modes a and b respectively. Now upon adding Eqs. (6.55) and (6.56), the equation of evolution of the cavity mode for the pair of superposed two-mode light beams is

$$\frac{d}{dt}\hat{c}(t) = -\frac{\kappa}{2}\hat{c}(t) - g\Big(\hat{\sigma}_A^k(t) + \hat{\sigma}_B^k(t)\Big) + \Big(\hat{F}_A(t) + \hat{F}_B(t)\Big). \tag{6.57}$$

The solution of this equation can be written as

$$\hat{c}(t+\tau) = \hat{c}(t)e^{-\kappa\tau/2} + e^{-\kappa\tau/2}\int_0^\tau d\tau' e^{\kappa\tau'/2}\Big[-g\Big(\hat{\sigma}_A^k(t+\tau') + \hat{\sigma}_B^k(t+\tau')\Big) + \Big(\hat{F}_A(t+\tau') + \hat{F}_B(t+\tau')\Big)\Big]. \tag{6.58}$$

Now upon multiplying this equation from the left by $\hat{c}^\dagger(t)$ and taking the expectation value of the resulting expression, we have

$$\langle\hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle_k = \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle_k e^{-\kappa\tau/2} + e^{-\kappa\tau/2}\int_0^\tau d\tau' e^{\kappa\tau'/2} \times\Big[-g\Big(\langle\hat{c}^\dagger(t)\hat{\sigma}_A^k(t+\tau')\rangle + \langle\hat{c}^\dagger(t)\hat{\sigma}_B^k(t+\tau')\rangle\Big) + \Big(\langle\hat{c}^\dagger(t)\hat{F}_A(t+\tau')\rangle + \langle\hat{c}^\dagger(t)\hat{F}_B(t+\tau')\rangle\Big)\Big]. \tag{6.59}$$

Because of the fact that a noise operator at a certain time does not affect system variables at earlier time, we find

$$\langle\hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle_k = \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle_k e^{-\kappa\tau/2} - ge^{-\kappa\tau/2}\int_0^\tau d\tau' e^{\kappa\tau'/2} \times\Big(\langle\hat{c}^\dagger(t)\hat{\sigma}_A^k(t+\tau')\rangle + \langle\hat{c}^\dagger(t)\hat{\sigma}_B^k(t+\tau')\rangle\Big). \tag{6.60}$$

Next we wish to determine the explicit form of $\langle\hat{c}^\dagger(t)\hat{\sigma}_A^k(t+\tau')\rangle$ and $\langle\hat{c}^\dagger(t)\hat{\sigma}_B^k(t+\tau')\rangle$. Now on the basis of Eq. (4.56), we can write

$$\langle\hat{c}^\dagger(t)\hat{\sigma}_A^k(t+\tau')\rangle = \langle\hat{c}^\dagger(t)\hat{\sigma}_A^k(t)\rangle e^{-\mu\tau'/2} + e^{-\mu\tau'/2}\int_0^{\tau'} d\tau'' e^{\mu\tau''/2}\langle\hat{c}^\dagger(t)\hat{f}(t+\tau'')\rangle, \tag{6.61}$$

from which follows

$$\langle \hat{c}^\dagger(t)\hat{\sigma}_A^k(t+\tau')\rangle = \langle \hat{c}^\dagger(t)\hat{\sigma}_A^k(t)\rangle e^{-\mu\tau'/2}. \tag{6.62}$$

It can also be verified in a similar manner that

$$\langle \hat{c}^\dagger(t)\hat{\sigma}_B^k(t+\tau')\rangle = \langle \hat{c}^\dagger(t)\hat{\sigma}_B^k(t)\rangle e^{-\mu\tau'/2}. \tag{6.63}$$

Thus substitution of Eqs. (6.62) and (6.63) into Eq. (6.60) yields

$$\langle \hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle_k = \langle \hat{c}^\dagger(t)\hat{c}(t)\rangle_k e^{-\kappa\tau/2} - g\bigg(\langle \hat{c}^\dagger(t)\hat{\sigma}_A^k(t)\rangle + \langle \hat{c}^\dagger(t)\hat{\sigma}_B^k(t)\rangle\bigg)$$

$$\times e^{-\kappa\tau/2}\int_0^\tau d\tau' e^{(\kappa-\mu)\tau'/2} \tag{6.64}$$

and on carrying out the integration, we get

$$\langle \hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle_k = \langle \hat{c}^\dagger(t)\hat{c}(t)\rangle_k e^{-\kappa\tau/2} - \frac{2g}{\kappa-\mu}\bigg(\langle \hat{c}^\dagger(t)\hat{\sigma}_A^k(t)\rangle + \langle \hat{c}^\dagger(t)\hat{\sigma}_B^k(t)\rangle\bigg)$$

$$\times\bigg(e^{-\mu\tau/2} - e^{-\kappa\tau/2}\bigg). \tag{6.65}$$

On the other hand, application of the large-time approximation scheme to Eq. (6.57) yields

$$\hat{c}(t) = -\frac{2g}{\kappa}\bigg(\hat{\sigma}_A^k(t) + \hat{\sigma}_B^k(t)\bigg) + \frac{2}{\kappa}\bigg(\hat{F}_A(t) + \hat{F}_B(t)\bigg), \tag{6.66}$$

so that multiplying the adjoint of this result from the right by $\hat{\sigma}_A^k(t)$, we find

$$\langle \hat{c}^\dagger(t)\hat{\sigma}_A^k(t)\rangle = -\frac{2g}{\kappa}\bigg(\langle \hat{\sigma}_A^{\dagger k}(t)\hat{\sigma}_A^k(t)\rangle + \langle \hat{\sigma}_B^{\dagger k}(t)\hat{\sigma}_A^k(t)\rangle\bigg)$$

$$+\frac{2}{\kappa}\bigg(\langle \hat{F}_A^\dagger(t)\hat{\sigma}_A^k(t)\rangle + \langle \hat{F}_B^\dagger(t)\hat{\sigma}_A^k(t)\rangle\bigg). \tag{6.67}$$

It then follows that

$$\langle \hat{c}^\dagger(t)\hat{\sigma}_A^k(t)\rangle = -\frac{2g}{\kappa}(\langle \hat{\eta}_a^k(t)\rangle + \langle \hat{\eta}_b^k(t)\rangle) + \frac{2}{\kappa}\bigg(\langle \hat{F}_A^\dagger(t)\hat{\sigma}_A^k(t)\rangle + \langle \hat{F}_B^\dagger(t)\hat{\sigma}_A^k(t)\rangle\bigg). \tag{6.68}$$

It can also be established in a similar manner that

$$\langle\hat{c}^{\dagger}(t)\hat{\sigma}_{B}^{k}(t)\rangle = -\frac{2g}{\kappa}(\langle\hat{\eta}_{a}^{k}(t)\rangle + \langle\hat{\eta}_{b}^{k}(t)\rangle) + \frac{2}{\kappa}\bigg(\langle\hat{F}_{A}^{\dagger}(t)\hat{\sigma}_{B}^{k}(t)\rangle + \langle\hat{F}_{B}^{\dagger}(t)\hat{\sigma}_{B}^{k}(t)\rangle\bigg). \tag{6.69}$$

Therefore, substitution of Eqs. (6.68) and (6.69) into Eq. (6.65) leads to

$$\langle\hat{c}^{\dagger}(t)\hat{c}(t+\tau)\rangle_{k} = \langle\hat{c}^{\dagger}(t)\hat{c}(t)\rangle_{k}e^{-\kappa\tau/2} + \frac{2\gamma_{c}}{\kappa-\mu}\bigg(\langle\hat{\eta}_{a}^{k}(t)\rangle + \langle\hat{\eta}_{b}^{k}(t)\rangle\bigg)$$

$$\times\bigg(e^{-\mu\tau/2} - e^{-\kappa\tau/2}\bigg) - \frac{4g}{\kappa(\kappa-\mu)}\bigg(\langle\hat{F}_{A}^{\dagger}(t)\hat{\sigma}_{A}^{k}(t)\rangle + \langle\hat{F}_{B}^{\dagger}(t)\hat{\sigma}_{A}^{k}(t)\rangle$$

$$+\langle\hat{F}_{A}^{\dagger}(t)\hat{\sigma}_{B}^{k}(t)\rangle + \langle\hat{F}_{B}^{\dagger}(t)\hat{\sigma}_{B}^{k}(t)\rangle\bigg)\bigg(e^{-\mu\tau/2} - e^{-\kappa\tau/2}\bigg). \tag{6.70}$$

We next seek to determine the expectation value of the product of the atomic and the noise operators involved in this expression. To this end, taking into account Eq. (4.10), one can write

$$\langle\hat{F}_{A}^{\dagger}(t)\hat{\sigma}_{A}^{k}(t)\rangle = \langle\hat{F}_{A}^{\dagger}(t)\hat{\sigma}_{A}^{k}(0)\rangle e^{-\mu t/2} + e^{-\mu t/2}\int_{0}^{t} e^{\mu t'/2}\langle\hat{F}_{A}^{\dagger}(t)\hat{f}(t')\rangle dt'. \tag{6.71}$$

Snice the cavity and the atomic noise operators are uncorrelated and the fact that the noise operator at a certain time does not affect the system varibles at the earlier time, there follows

$$\langle\hat{F}_{A}^{\dagger}(t)\hat{\sigma}_{A}^{k}(t)\rangle = 0. \tag{6.72}$$

Following a similar procedure, one can also readily establish that

$$\langle\hat{F}_{B}^{\dagger}(t)\hat{\sigma}_{A}^{k}(t)\rangle = 0, \tag{6.73}$$

$$\langle\hat{F}_{A}^{\dagger}(t)\hat{\sigma}_{B}^{k}(t)\rangle = 0, \tag{6.74}$$

$$\langle\hat{F}_{B}^{\dagger}(t)\hat{\sigma}_{B}^{k}(t)\rangle = 0. \tag{6.75}$$

Thus making use of Eqs. (6.72), (6.73), (6.74), and (6.75), we write Eq. (6.70) as

$$\langle\hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle_k = \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle_k e^{-\kappa\tau/2} + \frac{2\gamma_c}{\kappa-\mu}\bigg(\langle\hat{\eta}_a^k(t)\rangle + \langle\hat{\eta}_b^k(t)\rangle\bigg)$$
$$\times\bigg(e^{-\mu\tau/2} - e^{-\kappa\tau/2}\bigg) \tag{6.76}$$

and on summing over all atoms, we find

$$\langle\hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle = \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle e^{-\kappa\tau/2} + \frac{2\gamma_c}{\kappa-\mu}\bigg(\langle\hat{N}_a(t)\rangle + \langle\hat{N}_b(t)\rangle\bigg)$$
$$\times\bigg(e^{-\mu\tau/2} - e^{-\kappa\tau/2}\bigg). \tag{6.77}$$

This equation can also be put in the form

$$\langle\hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle = \bar{n}_s\bigg(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\bigg), \tag{6.78}$$

so that substitution of this equation into Eq. (6.52) gives

$$P_s(\omega) = \frac{\bar{n}_s}{\pi}\bigg[\frac{\kappa}{\kappa-\mu}Re\int_0^\infty d\tau e^{-[\mu/2-i(\omega-\omega_0)]\tau} - \frac{\mu}{\kappa-\mu}Re\int_0^\infty d\tau e^{-[\kappa/2-i(\omega-\omega_0)]\tau}\bigg]. \tag{6.79}$$

Now on performing the integration, we arrive at

$$P_s(\omega) = \frac{\kappa\bar{n}_s}{\kappa-\mu}\bigg(\frac{\mu/2\pi}{(\mu/2)^2+(\omega-\omega_0)^2}\bigg) - \frac{\mu\bar{n}_s}{\kappa-\mu}\bigg(\frac{\kappa/2\pi}{(\kappa/2)^2+(\omega-\omega_0)^2}\bigg). \tag{6.80}$$

Therefore, upon substituting Eq. (6.80) into Eq. (6.54), we have

$$\bar{n}_{s\pm\lambda} = \frac{\kappa\bar{n}_s}{\kappa-\mu}\frac{\mu}{2\pi}\int_{-\lambda}^{+\lambda}\frac{d\omega'}{(\mu/2)^2+\omega'^2} - \frac{\mu\bar{n}_s}{\kappa-\mu}\frac{\kappa}{2\pi}\int_{-\lambda}^{+\lambda}\frac{d\omega'}{(\kappa/2)^2+\omega'^2}. \tag{6.81}$$

And on carrying out the integration, we find

$$\bar{n}_{s\pm\lambda} = \bar{n}_s z(\lambda), \tag{6.82}$$

where $z(\lambda)$ is given by Eq. (4.70). This equation can also be written as

$$\bar{n}_{s\pm\lambda} = 2\bar{n}_{\pm\lambda}. \tag{6.83}$$

We thus see that the local mean photon number of the pair of superposed two-mode light beams is the sum of the local mean photon number of the individual two-mode light beams.

6.4.2 Local photon number variance

Next we wish to calculate the photon number variance of the pair of superposed two-mode light beams in a given frequency interval. The photon number variance of the pair of superposed two-mode light beams with central frequency ω_0 is given by

$$J_s(\omega) = \frac{1}{\pi} Re \int_0^{\infty} d\tau \langle \hat{n}_s(t), \hat{n}_s(t+\tau)\rangle_{ss} e^{i(\omega-\omega_0)\tau}. \tag{6.84}$$

On integrating both sides of Eq. (6.84) over ω, we get

$$\int_{-\infty}^{+\infty} J_s(\omega) d\omega = \langle \hat{n}_s(t), \hat{n}_s(t)\rangle_{ss}. \tag{6.85}$$

We note that $\langle \hat{n}_s(t), \hat{n}_s(t)\rangle_{ss}$ is the steady-state photon number variance of the pair of superposed two-mode light beams. On the basis of Eq. (6.85), we realize that the photon number variance of the pair of superposed two-mode light beams in the interval between ω and $\omega + d\omega$ is $J_s(\omega)d\omega$. Then the photon number variance in the interval between $-\lambda$ and $+\lambda$ is expressed as

$$(\triangle n)^2_{s\pm\lambda} = \int_{-\lambda}^{+\lambda} J_s(\omega) d\omega. \tag{6.86}$$

We note that

$$\langle \hat{n}_s(t), \hat{n}_s(t+\tau)\rangle = \langle \hat{c}^\dagger(t)\hat{c}(t)\hat{c}^\dagger(t+\tau)\hat{c}(t+\tau)\rangle - \langle \hat{c}^\dagger(t)\hat{c}(t)\rangle\langle \hat{c}^\dagger(t+\tau)\hat{c}(t+\tau)\rangle. \tag{6.87}$$

Now on taking the expectation value of Eq. (6.30), we have

$$\langle \hat{c}(t)\rangle = \langle \hat{a}(t)\rangle + \langle \hat{b}(t)\rangle \tag{6.88}$$

and in view of Eqs. (4.26) and (6.36), we see that

$$\langle \hat{c}(t)\rangle = 0. \tag{6.89}$$

Thus on account of Eqs. (6.57) and (6.89), we see that $\hat{c}(t)$ is a Gaussian variable with zero mean. Therefore, we can put Eq. (6.87) in the form

$$\langle \hat{n}_s(t), \hat{n}_s(t+\tau)\rangle = \langle \hat{c}^\dagger(t)\hat{c}^\dagger(t+\tau)\rangle\langle \hat{c}(t)\hat{c}(t+\tau)\rangle + \langle \hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle\langle \hat{c}(t)\hat{c}^\dagger(t+\tau)\rangle. \quad (6.90)$$

We next seek to evaluate the two time correlation function involved in this expression. Now on account of Eq. (6.58), one can write

$$\langle \hat{c}^\dagger(t)\hat{c}^\dagger(t+\tau)\rangle_k = \langle \hat{c}^{\dagger 2}(t)\rangle_k e^{-\kappa\tau/2} + e^{-\kappa\tau/2}\int_0^\tau d\tau' e^{\kappa\tau'/2}$$
$$\times\Bigg[-g\bigg(\langle \hat{c}^\dagger(t)\hat{\sigma}_A^{\dagger k}(t+\tau')\rangle + \langle \hat{c}^\dagger(t)\hat{\sigma}_B^{\dagger k}(t+\tau')\rangle\bigg)$$
$$+\bigg(\langle \hat{c}^\dagger(t)\hat{F}_A^\dagger(t+\tau')\rangle + \langle \hat{c}^\dagger(t)\hat{F}_B^\dagger(t+\tau')\rangle\bigg)\Bigg]. \quad (6.91)$$

On account of the fact that the noise operator at a certain time does not affect the system variables at an earlier time, we have

$$\langle \hat{c}^\dagger(t)\hat{c}^\dagger(t+\tau)\rangle_k = \langle \hat{c}^{\dagger 2}(t)\rangle_k e^{-\kappa\tau/2} - g e^{-\kappa\tau/2}\int_0^\tau d\tau' e^{\kappa\tau'/2}\bigg(\langle \hat{c}^\dagger(t)\hat{\sigma}_A^{\dagger k}(t+\tau')\rangle$$
$$+\langle \hat{c}^\dagger(t)\hat{\sigma}_B^{\dagger k}(t+\tau')\rangle\bigg). \quad (6.92)$$

We now proceed to evaluate an explicit expression for $\langle \hat{c}^\dagger(t)\hat{\sigma}_A^{\dagger k}(t+\tau')\rangle$ and $\langle \hat{c}^\dagger(t)\hat{\sigma}_B^{\dagger k}(t+\tau')\rangle$. To this end, taking into account Eq. (4.56), we can write

$$\langle \hat{c}^\dagger(t)\hat{\sigma}_A^{\dagger k}(t+\tau')\rangle = \langle \hat{c}^\dagger(t)\hat{\sigma}_A^{\dagger k}(t)\rangle e^{-\mu\tau'/2} + e^{-\mu\tau'/2}\int_0^{\tau'} e^{\mu\tau''/2}\langle \hat{c}^\dagger(t)\hat{f}(t+\tau'')\rangle d\tau'', \quad (6.93)$$

from which follows

$$\langle \hat{c}^\dagger(t)\hat{\sigma}_A^{\dagger k}(t+\tau')\rangle = \langle \hat{c}^\dagger(t)\hat{\sigma}_A^{\dagger k}(t)\rangle e^{-\mu\tau'/2}. \quad (6.94)$$

One can also easily establish in a similar manner that

$$\langle \hat{c}^\dagger(t)\hat{\sigma}_B^{\dagger k}(t+\tau')\rangle = \langle \hat{c}^\dagger(t)\hat{\sigma}_B^{\dagger k}(t)\rangle e^{-\mu\tau'/2}. \quad (6.95)$$

Now on substituting Eqs. (6.94) and (6.95) into Eq. (6.92), we find

$$\langle \hat{c}^{\dagger}(t)\hat{c}^{\dagger}(t+\tau)\rangle_k = \langle \hat{c}^{\dagger 2}(t)\rangle_k e^{-\kappa\tau/2} - g(\langle \hat{c}^{\dagger}(t)\hat{\sigma}_A^{\dagger k}(t)\rangle + \langle \hat{c}^{\dagger}(t)\hat{\sigma}_B^{\dagger k}(t)\rangle)$$

$$\times e^{-\kappa\tau/2}\int_0^{\tau} d\tau' e^{(\kappa-\mu)\tau'/2} \tag{6.96}$$

and carrying out the integration, we get

$$\langle \hat{c}^{\dagger}(t)\hat{c}^{\dagger}(t+\tau)\rangle_k = \langle \hat{c}^{\dagger 2}(t)\rangle_k e^{-\kappa\tau/2} - \frac{2g}{\kappa-\mu}(\langle \hat{c}^{\dagger}(t)\hat{\sigma}_A^{\dagger k}(t)\rangle + \langle \hat{c}^{\dagger}(t)\hat{\sigma}_B^{\dagger k}(t)\rangle)$$

$$\times (e^{-\mu\tau/2} - e^{-\kappa\tau/2}). \tag{6.97}$$

Next we seek to evaluate the explicit form of $\langle \hat{c}^{\dagger}(t)\hat{\sigma}_A^{\dagger k}(t)\rangle$ and $\langle \hat{c}^{\dagger}(t)\hat{\sigma}_B^{\dagger k}(t)\rangle$. Now making use of Eq. (6.66), we have

$$\langle \hat{c}^{\dagger}(t)\hat{\sigma}_A^{\dagger k}(t)\rangle = -\frac{2g}{\kappa}\bigg(\langle \hat{\sigma}_A^{\dagger k}(t)\hat{\sigma}_A^{\dagger k}(t)\rangle + \langle \hat{\sigma}_B^{\dagger k}(t)\hat{\sigma}_A^{\dagger k}(t)\rangle\bigg)$$

$$+\frac{2}{\kappa}\bigg(\langle \hat{F}_A^{\dagger}(t)\hat{\sigma}_A^{\dagger k}(t)\rangle + \langle \hat{F}_B^{\dagger}(t)\hat{\sigma}_A^{\dagger k}(t)\rangle\bigg). \tag{6.98}$$

On the other hand, multiplying the adjoint of Eq. (4.10) from the left by $\hat{F}_A^{\dagger}(t)$, we see that

$$\langle \hat{F}_A^{\dagger}(t)\hat{\sigma}_A^{\dagger k}(t)\rangle = \langle \hat{F}_A^{\dagger}(t)\hat{\sigma}_A^{\dagger k}(0)\rangle e^{-\mu t/2} + e^{-\mu t/2}\int_0^t e^{\mu t'/2}\langle \hat{F}_A^{\dagger}(t)\hat{f}^{\dagger}(t')\rangle dt', \tag{6.99}$$

from which follows

$$\langle \hat{F}_A^{\dagger}(t)\hat{\sigma}_A^{\dagger k}(t)\rangle = 0. \tag{6.100}$$

Following the same procedure, one can easily find that

$$\langle \hat{F}_B^{\dagger}(t)\hat{\sigma}_A^{\dagger k}(t)\rangle = 0. \tag{6.101}$$

Thus on introducing Eqs. (6.100) and (6.101) into Eq. (6.98), we see that

$$\langle \hat{c}^{\dagger}(t)\hat{\sigma}_A^{\dagger k}(t)\rangle = -\frac{2g}{\kappa}\langle \hat{\sigma}_c^{\dagger k}(t)\rangle. \tag{6.102}$$

One can also easily verify in a similar manner that

$$\langle \hat{c}^\dagger(t)\hat{\sigma}_B^{\dagger k}(t)\rangle = -\frac{2g}{\kappa}\langle \hat{\sigma}_c^{\dagger k}(t)\rangle. \tag{6.103}$$

Now on account of Eqs. (6.102) and (6.103), we put Eq. (6.97) in the form

$$\langle \hat{c}^\dagger(t)\hat{c}^\dagger(t+\tau)\rangle_k = \langle \hat{c}^{\dagger 2}(t)\rangle_k e^{-\kappa\tau/2} + 2\frac{\gamma_c}{\kappa-\mu}\langle \hat{\sigma}_c^{\dagger k}(t)\rangle(e^{-\mu\tau/2} - e^{-\kappa\tau/2}) \tag{6.104}$$

and on summing over all atoms, we find

$$\langle \hat{c}^\dagger(t)\hat{c}^\dagger(t+\tau)\rangle = \langle \hat{c}^{\dagger 2}(t)\rangle e^{-\kappa\tau/2} + 2\frac{\gamma_c}{\kappa-\mu}\langle \hat{m}_c^\dagger(t)\rangle(e^{-\mu\tau/2} - e^{-\kappa\tau/2}). \tag{6.105}$$

This equation can also be written as

$$\langle \hat{c}^\dagger(t)\hat{c}^\dagger(t+\tau)\rangle = \langle \hat{c}^{\dagger 2}(t)\rangle\bigg(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\bigg). \tag{6.106}$$

Following a similar procedure, we readily establish that

$$\langle \hat{c}(t)\hat{c}(t+\tau)\rangle = \langle \hat{c}^2(t)\rangle\bigg(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\bigg), \tag{6.107}$$

$$\langle \hat{c}(t)\hat{c}^\dagger(t+\tau)\rangle = \langle \hat{c}(t)\hat{c}^\dagger(t)\rangle\bigg(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\bigg)$$

$$-4\frac{\kappa}{\kappa-\mu}\bigg(e^{-\mu\tau/2} - e^{-\kappa\tau/2}\bigg). \tag{6.108}$$

Hence substituting Eqs. (6.78), (6.106), (6.107), and (6.108) into Eq.(6.90), we find

$$\langle \hat{n}_s(t), \hat{n}_s(t+\tau)\rangle = \langle \hat{c}^{\dagger 2}(t)\rangle\langle \hat{c}^2(t)\rangle\bigg(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\bigg)^2$$

$$+\langle \hat{c}^\dagger(t)\hat{c}(t)\rangle\langle \hat{c}(t)\hat{c}^\dagger(t)\rangle\bigg(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\bigg)^2 - 4\langle \hat{c}^\dagger(t)\hat{c}(t)\rangle$$

$$\times\bigg(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\bigg)\bigg(\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\kappa}{\kappa-\mu}e^{-\kappa\tau/2}\bigg) \tag{6.109}$$

or

$$\langle \hat{n}_s(t), \hat{n}_s(t+\tau)\rangle = (\Delta n)_s^2 \left(\frac{\kappa}{\kappa-\mu} e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu} e^{-\kappa\tau/2}\right)^2 - 4\bar{n}$$

$$\times \left(\frac{\kappa}{\kappa-\mu} e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu} e^{-\kappa\tau/2}\right)\left(\frac{\kappa}{\kappa-\mu} e^{-\mu\tau/2} - \frac{\kappa}{\kappa-\mu} e^{-\kappa\tau/2}\right). \quad (6.110)$$

Thus on introducing this equation into Eq. (6.84), we have

$$\begin{aligned} J_s(\omega) = \quad & \frac{(\triangle n)_s^2}{\pi}\left[\left(\frac{\kappa}{\kappa-\mu}\right)^2 Re\int_0^\infty d\tau e^{-[\mu - i(\omega-\omega_0)]\tau}\right. \\ & -2\left(\frac{\kappa}{\kappa-\mu}\right)\left(\frac{\mu}{\kappa-\mu}\right) Re\int_0^\infty d\tau e^{-[\frac{\mu+\kappa}{2} - i(\omega-\omega_0)]\tau} \\ & \left. +\left(\frac{\mu}{\kappa-\mu}\right)^2 Re\int_0^\infty d\tau e^{-[\kappa - i(\omega-\omega_0)]\tau}\right] \\ & -4\frac{\bar{n}_s}{\pi}\left[(\frac{\kappa}{\kappa-\mu})^2 Re\int_0^\infty e^{-[\mu-i(\omega-\omega_0)]\tau} d\tau\right. \\ & -(\frac{\kappa}{\kappa-\mu})(\frac{\mu}{\kappa-\mu}) Re\int_0^\infty e^{-[(\mu+\kappa)/2-i(\omega-\omega_0)]\tau} d\tau \\ & -(\frac{\kappa}{\kappa-\mu})^2 Re\int_0^\infty e^{-[(\mu+\kappa)/2-i(\omega-\omega_0)]\tau} d\tau \\ & \left. +(\frac{\kappa}{\kappa-\mu})(\frac{\mu}{\kappa-\mu}) Re\int_0^\infty e^{-[\kappa-i(\omega-\omega_0)]\tau} d\tau\right], \end{aligned} \quad (6.111)$$

so that on carrying out the integration, we get

$$\begin{aligned} J_s(\omega) = \frac{(\Delta n)_s^2}{\pi}\Bigg[& (\frac{\kappa}{\kappa-\mu})^2 \frac{\mu}{\mu^2+(\omega-\omega_0)^2} - 2(\frac{\kappa}{\kappa-\mu})(\frac{\mu}{\kappa-\mu}) \frac{(\mu+\kappa)/2}{[(\mu+\kappa)/2]^2+(\omega-\omega_0)^2} \\ & +(\frac{\mu}{\kappa-\mu})^2 \frac{\kappa}{\kappa^2+(\omega-\omega_0)^2}\Bigg] - 4\frac{\bar{n}_s}{\pi}\Bigg[(\frac{\kappa}{\kappa-\mu})^2 \frac{\mu}{\mu^2+(\omega-\omega_0)^2} \\ & -(\frac{\kappa}{\kappa-\mu})(\frac{\mu}{\kappa-\mu}) \frac{(\mu+\kappa)/2}{[(\mu+\kappa)/2]^2+(\omega-\omega_0)^2} \\ & -(\frac{\kappa}{\kappa-\mu})^2 \frac{(\kappa+\mu)/2}{[(\kappa+\mu)/2]^2+(\omega-\omega_0)^2} + (\frac{\mu}{\kappa-\mu})(\frac{\kappa}{\kappa-\mu}) \frac{\kappa}{\kappa^2+(\omega-\omega_0)^2}\Bigg]. \end{aligned} \quad (6.112)$$

Therefore, on introducing this equation into Eq. (6.86), we find

$$(\Delta n)_{s\pm\lambda} = \frac{(\Delta n)_s^2}{\pi}\Bigg[(\frac{\kappa}{\kappa-\mu})^2\mu\int_{-\lambda}^{+\lambda}\frac{d\omega'}{\mu^2+\omega'^2}$$
$$-2(\frac{\kappa}{\kappa-\mu})(\frac{\mu}{\kappa-\mu})(\mu+\kappa)/2\int_{-\lambda}^{+\lambda}\frac{d\omega'}{[(\mu+\kappa)/2]^2+\omega'^2}$$
$$+(\frac{\mu}{\kappa-\mu})^2\kappa\int_{-\lambda}^{+\lambda}\frac{d\omega'}{\kappa^2+\omega'^2}\Bigg]$$
$$-4\frac{\bar{n}_s}{\pi}\Bigg[(\frac{\kappa}{\kappa-\mu})^2\mu\int_{-\lambda}^{+\lambda}\frac{d\omega'}{\mu^2+\omega'^2}$$
$$-(\frac{\kappa}{\kappa-\mu})(\frac{\mu}{\kappa-\mu})(\mu+\kappa)/2\int_{-\lambda}^{+\lambda}\frac{d\omega'}{[(\mu+\kappa)/2]^2+\omega'^2}$$
$$-(\frac{\kappa}{\kappa-\mu})^2(\kappa+\mu)/2\int_{-\lambda}^{+\lambda}\frac{d\omega'}{[(\kappa+\mu)/2]^2+\omega'^2}$$
$$+(\frac{\mu}{\kappa-\mu})(\frac{\kappa}{\kappa-\mu})\kappa\int_{-\lambda}^{+\lambda}\frac{d\omega'}{\kappa^2+\omega'^2}\Bigg]. \tag{6.113}$$

And on performing the integration, we have

$$(\triangle n)_{s\pm\lambda} = (\triangle n)_s^2\Bigg[\frac{2}{\pi}\left(\frac{\kappa}{\kappa-\mu}\right)^2 tan^{-1}(\frac{\lambda}{\mu}) - \frac{4}{\pi}\left(\frac{\kappa}{\kappa-\mu}\right)\left(\frac{\mu}{\kappa-\mu}\right)tan^{-1}\left(\frac{2\lambda}{\kappa+\mu}\right)$$
$$+\frac{2}{\pi}\left(\frac{\mu}{\kappa-\mu}\right)^2 tan^{-1}\left(\frac{\lambda}{\kappa}\right)\Bigg] + \bar{n}_s\Bigg[-\frac{8}{\pi}\left(\frac{\kappa}{\kappa-\mu}\right)^2 tan^{-1}(\frac{\lambda}{\mu})$$
$$+\frac{8}{\pi}\left(\frac{\kappa}{\kappa-\mu}\right)\left(\frac{\mu}{\kappa-\mu}\right)tan^{-1}\left(\frac{2\lambda}{\kappa+\mu}\right) + \frac{8}{\pi}\left(\frac{\kappa}{\kappa-\mu}\right)^2 tan^{-1}\left(\frac{2\lambda}{\kappa+\mu}\right)$$
$$-\frac{8}{\pi}\left(\frac{\kappa}{\kappa-\mu}\right)(\frac{\mu}{\kappa-\mu}\Bigg)tan^{-1}\left(\frac{\lambda}{\kappa}\right)\Bigg]. \tag{6.114}$$

This can also be put in the form

$$(\Delta n)^2_{s\pm\lambda} = 4(\Delta n)^2_{\pm\lambda}. \tag{6.115}$$

This indicates that unlike the local mean photon number, the local photon number variance of the pair of superposed two-mode light beams is not equal to the sum of the local photon number variance of the individual two-mode light beams. We also notice that the local photon number variance of the pair of superposed two-mode light beam is four times that of the separate two-mode light beam.

6.5 Quadrature Squeezing

Applying the density operator for the pair of superposed two-mode light beams and using the relation described by Eq. (6.28), we calculate the global and local quadrature squeezing.

6.5.1 Global quadrature squeezing

Here we calculate the global quadrature squeezing of the pair superposed two-mode light beams. To this end, the quadrature operators for the pair of superposed two-mode light beams are defined by

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c} \tag{6.116}$$

and

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}). \tag{6.117}$$

Now the commutation relation of these two quadrature operators is

$$[\hat{c}_+, c_-] = 2i[\hat{c}, \hat{c}^\dagger] \tag{6.118}$$

and on account of Eq. (6.40), we see that

$$[\hat{c}_+, c_-] = 4\lambda i. \tag{6.119}$$

It then follows that

$$\Delta c_+ \Delta c_- \geq 2\lambda. \tag{6.120}$$

The quadrature variances are defined by

$$(\triangle c_\pm)^2 = \langle \hat{c}_\pm^2 \rangle - \langle \hat{c}_\pm \rangle^2. \tag{6.121}$$

This can also be written in the form

$$(\triangle c_\pm)^2 = \langle \hat{c}^\dagger \hat{c} \rangle + \langle \hat{c}\hat{c}^\dagger \rangle \pm \langle \hat{c}^{\dagger 2} \rangle \pm \langle \hat{c}^2 \rangle - 2\langle \hat{c}^\dagger \rangle \langle \hat{c} \rangle \mp \langle \hat{c}^\dagger \rangle^2 \mp \langle \hat{c} \rangle^2. \tag{6.122}$$

We next seek to calculate an explicit expressions for $\langle\hat{c}\hat{c}^\dagger\rangle$, $\langle\hat{c}^2\rangle$, and $\langle\hat{c}\rangle$. Now applying Eq. (6.25), we have

$$\langle\hat{c}\hat{c}^\dagger\rangle = \lambda^2 \int d^2\gamma d^2\eta Q(\lambda\gamma^*, \lambda\gamma + \frac{\partial}{\partial\gamma^*})Q(\lambda\eta^*, \lambda\eta + \frac{\partial}{\partial\eta^*})\langle\eta+\gamma|\hat{c}\hat{c}^\dagger|\gamma+\eta\rangle \tag{6.123}$$

and using the relation described by Eq. (6.40), we get

$$\langle\hat{c}\hat{c}^\dagger\rangle = \lambda^2 \int d^2\gamma d^2\eta Q(\lambda\gamma^*, \lambda\gamma + \frac{\partial}{\partial\gamma^*})Q(\lambda\eta^*, \lambda\eta + \frac{\partial}{\partial\eta^*})\langle\eta+\gamma|(\hat{c}^\dagger\hat{c}+2\lambda)|\gamma+\eta\rangle. \tag{6.124}$$

It then follows that

$$\langle\hat{c}\hat{c}^\dagger\rangle = \lambda^4 \int d^2\gamma d^2\eta Q(\lambda\gamma^*, \lambda\gamma + \frac{\partial}{\partial\gamma^*})Q(\lambda\eta^*, \lambda\eta + \frac{\partial}{\partial\eta^*})(\gamma^*\gamma + \gamma^*\eta + \eta^*\gamma + \eta^*\eta)$$

$$+2\lambda^3 \int d^2\gamma d^2\eta Q(\lambda\gamma^*, \lambda\gamma + \frac{\partial}{\partial\gamma^*})Q(\lambda\eta^*, \lambda\eta + \frac{\partial}{\partial\eta^*}). \tag{6.125}$$

On account of the variables $\alpha = \lambda\gamma$ and $\beta = \lambda\eta$, we have

$$\langle\hat{c}\hat{c}^\dagger\rangle = \frac{1}{\lambda^2} \int d^2\alpha d^2\beta Q(\alpha^*, \alpha + \lambda\frac{\partial}{\partial\alpha^*})Q(\beta^*, \beta + \lambda\frac{\partial}{\partial\beta^*})(\alpha^*\alpha + \alpha^*\beta + \beta^*\alpha + \beta^*\beta)$$

$$+\frac{2}{\lambda} \int d^2\alpha d^2\beta Q(\alpha^*, \alpha + \lambda\frac{\partial}{\partial\alpha^*})Q(\beta^*, \beta + \lambda\frac{\partial}{\partial\beta^*}). \tag{6.126}$$

Now in view of Eq. (6.28), one can write

$$\langle\hat{c}\hat{c}^\dagger\rangle = \langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\rangle\langle\hat{b}\rangle + \langle\hat{b}^\dagger\rangle\langle\hat{a}\rangle + \langle\hat{b}^\dagger\hat{b}\rangle + 2\lambda. \tag{6.127}$$

Thus employing Eqs. (4.26), (6.35), and (6.36), we have

$$\langle\hat{c}\hat{c}^\dagger\rangle = 2\langle\hat{a}^\dagger\hat{a}\rangle + 2\lambda. \tag{6.128}$$

Following a similar procedure, we can establish that

$$\langle\hat{c}\rangle = \langle\hat{a}\rangle + \langle\hat{b}\rangle = 0, \tag{6.129}$$

$$\langle\hat{c}^2\rangle = \langle\hat{a}^2\rangle + 2\langle\hat{a}\rangle\langle\hat{b}\rangle + \langle\hat{b}^2\rangle = 2\langle\hat{a}^2\rangle. \tag{6.130}$$

Now on account of Eqs. (6.37), (6.122), (6.128), (6.129), and (6.130), the quadrature variances can be written as

$$(\triangle c_{\pm})^2 = 4\langle \hat{a}^\dagger \hat{a}\rangle + 2\lambda \pm 2\langle \hat{a}^2\rangle \pm 2\langle \hat{a}^{\dagger 2}\rangle. \tag{6.131}$$

Finally, using Eqs. (4.15), (4.44), and (6.10), the quadrature variances of the pair of superposed two-mode light beams can be put in the form

$$(\triangle c_{\pm})^2 = 2\left[\frac{\gamma_c}{\kappa}(N + \langle \hat{N}b\rangle) + 2N \pm 2\frac{\gamma_c}{\kappa}\sqrt{\frac{\gamma_c}{r_a}}\langle \hat{N}_a\rangle\right]. \tag{6.132}$$

This shows that the quadrature variances of the pair of superposed two-mode light beams are equal to the sum of the quadrature variance of the separate two-mode light beams.

We seek here to obtain the quadrature squeezing of the pair of superposed two-mode light beams. To this end, we see that for $r_a = 0$, Eq. (6.132) reduces to

$$(\triangle c_{\pm})^2_{vac} = 2\frac{\gamma_c}{\kappa}N + 4N. \tag{6.133}$$

We observe that this is the quadrature variance of the vacuum state. We define the quadrature squeezing of the pair of superposed two-mode light beams relative to the vacuum state by

$$S_s = \frac{(\triangle c_-)^2_{vac} - (\triangle c_-)^2}{(\triangle c_-)^2_{vac}}. \tag{6.134}$$

Now on account of Eqs. (6.132)and (6.133), the quadrature squeezing can be written as

$$S_s = \frac{\gamma_c}{\gamma_c + 2\kappa}\left(\frac{2\sqrt{\eta} - 1}{\eta + 2}\right). \tag{6.135}$$

The expressions in Eqs. (5.20) and (6.135) indicate that the quadrature squeezing of the pair of superposed two-mode light beams is the same as the quadrature squeezing of the individual two-mode light beams.

6.5.2 Local quadrature squeezing

In this section we seek to obtain the quadrature squeezing of the pair of superposed two-mode light beams in a given frequency interval. To this end, the spectrum of quadrature fluctuations for the pair of superposed two-mode light beams are defined by

$$S_{s\pm}(\omega) = \frac{1}{\pi} Re \int_0^\infty d\tau \langle \hat{c}_\pm(t), \hat{c}_\pm(t+\tau)\rangle_{ss} e^{i(\omega-\omega_0)\tau}, \tag{6.136}$$

where ω_0 is the central frequency of the superposed two-mode light beams. Upon integrating both sides of Eq. (6.136) over ω, we get

$$\int_{-\infty}^{+\infty} S_{s\pm}(\omega) d\omega = (\Delta c_\pm)^2, \tag{6.137}$$

in which

$$(\Delta c_\pm)^2 = \langle \hat{c}_\pm(t), \hat{c}_\pm(t)\rangle_{ss} \tag{6.138}$$

is the global quadrature variance of the superposed light modes at steady state. From this result, we observe that $S_{s\pm}(\omega)d\omega$ is the steady-state quadrature variance of the superposed light modes in the interval between ω and $\omega + d\omega$. The variance of the minus quadrature in the interval between $\omega' = -\lambda$ to $\omega' = +\lambda$ is then given by

$$(\triangle c_-)^2_{\pm\lambda} = \int_{-\lambda}^{+\lambda} S_{s-}(\omega') d\omega'. \tag{6.139}$$

We now proceed to evaluate the two-time correlation function that appears in Eq. (6.136). Thus in view of Eq. (6.129), we see that

$$\langle \hat{c}_\pm(t), \hat{c}_\pm(t+\tau)\rangle_{ss} = \langle \hat{c}_\pm(t)\hat{c}_\pm(t+\tau)\rangle_{ss}. \tag{6.140}$$

This can also be written as

$$\langle \hat{c}_{\pm}(t), \hat{c}_{\pm}(t+\tau)\rangle_{ss} = \langle \hat{c}^{\dagger}(t)\hat{c}(t+\tau)\rangle + \langle \hat{c}(t)\hat{c}^{\dagger}(t+\tau)\rangle$$

$$\pm\langle \hat{c}(t)\hat{c}(t+\tau)\rangle \pm \langle \hat{c}^{\dagger}(t)\hat{c}^{\dagger}(t+\tau)\rangle. \tag{6.141}$$

Hence upon substituting Eqs. (6.78), (6.106), (6.107), and (6.108) into Eq. (6.141), we find

$$\langle \hat{c}_{\pm}(t), \hat{c}_{\pm}(t+\tau)\rangle_{ss} = (\triangle c_{\pm})^2 \left(\frac{\kappa}{\kappa-\mu} e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu} e^{-\kappa\tau/2}\right)$$

$$-4\frac{\kappa}{\kappa-\mu}(e^{-\mu\tau/2} - e^{-\kappa\tau/2}). \tag{6.142}$$

Now on account of this equation, we put Eq. (6.136) in the form

$$S_{s\pm}(\omega) = \frac{1}{\pi}\left(\frac{\kappa}{\kappa-\mu}(\Delta c_{\pm})^2 - \frac{4\kappa}{\kappa-\mu}\right) Re \int_0^{\infty} d\tau e^{-[\mu/2 - i(\omega-\omega_o)]\tau}$$

$$+\frac{1}{\pi}\left(\frac{4\kappa}{\kappa-\mu} - \frac{\mu}{\kappa-\mu}(\Delta c_{\pm})^2\right) Re \int_0^{\infty} d\tau e^{-[\frac{\kappa}{2} - i(\omega-\omega_o)]\tau} \tag{6.143}$$

and on carrying out the integration, we have

$$S_{\pm}(\omega) = \left(\frac{\kappa}{\kappa-\mu}(\Delta c_{\pm})^2 - \frac{4\kappa}{\kappa-\mu}\right)\frac{\mu/2\pi}{(\mu/2)^2 + (\omega-\omega_o)^2}$$

$$+\left(\frac{4\kappa}{\kappa-\mu} - \frac{\mu}{\kappa-\mu}(\Delta c_{\pm})^2\right)\frac{\kappa/2\pi}{(\kappa/2)^2 + (\omega-\omega_o)^2}. \tag{6.144}$$

Now on substituting this equation into Eq. (6.139), we have

$$(\triangle c_-)^2_{\pm\lambda} = \left(\frac{\kappa}{\kappa-\mu}(\Delta c_-)^2 - \frac{4\kappa}{\kappa-\mu}\right)\frac{\mu}{2\pi}\int_{-\lambda}^{+\lambda}\frac{d\omega'}{(\mu/2)^2+\omega'^2}$$

$$+\left(\frac{4\kappa}{\kappa-\mu} - \frac{\mu}{\kappa-\mu}(\Delta c_-)^2\right)\frac{\kappa}{2\pi}\int_{-\lambda}^{+\lambda}\frac{d\omega'}{(\kappa/2)^2+(\omega-\omega_o)^2}. \tag{6.145}$$

so that on performing the integration using the relation described by Eq. (4.68), there follows

$$(\triangle c_-)^2_{\pm\lambda} = \frac{2}{\pi}\left(\frac{\kappa}{\kappa-\mu}(\Delta c_-)^2 - \frac{4\kappa}{\kappa-\mu}\right)tan^{-1}\left(\frac{2\lambda}{\mu}\right)$$

$$+\frac{2}{\pi}\left(\frac{4\kappa}{\kappa-\mu} - \frac{\mu}{\kappa-\mu}(\Delta c_-)^2\right)tan^{-1}\left(\frac{2\lambda}{\kappa}\right). \tag{6.146}$$

Now in view of Eq. (2.84), we write Eq. (6.146) as

$$(\triangle c_-)^2_{\pm\lambda} = \frac{2}{\pi}\left(\frac{\kappa}{\kappa-(\gamma_c+2r_a)}(\Delta c_-)^2 - \frac{4\kappa}{\kappa-(\gamma_c+2r_a)}\right)tan^{-1}\left(\frac{2\lambda}{\gamma_c+2r_a}\right)$$

$$+\frac{2}{\pi}\left(\frac{4\kappa}{\kappa-(\gamma_c+2r_a)} - \frac{\gamma_c+2r_a}{\kappa-(\gamma_c+2r_a)}(\Delta c_-)^2\right)tan^{-1}\left(\frac{2\lambda}{\kappa}\right). \tag{6.147}$$

We then see that the local quadrature variance of the two-mode vacuum state is

$$(\triangle c_-)^2_{vac\pm\lambda} = \frac{2}{\pi}\left(\frac{\kappa}{\kappa-\gamma_c}(\Delta c_-)^2_{vac} - \frac{4\kappa}{\kappa-\gamma_c}\right)tan^{-1}\left(\frac{2\lambda}{\gamma_c}\right)$$

$$+\frac{2}{\pi}\left(\frac{4\kappa}{\kappa-\gamma_c} - \frac{\gamma_c}{\kappa-\gamma_c}(\Delta c_-)^2_{vac}\right)tan^{-1}\left(\frac{2\lambda}{\kappa}\right). \tag{6.148}$$

We define the quadrature squeezing of the pair of superposed two-mode light beams in the $\lambda_\pm$ frequency interval

$$S_{s\pm\lambda} = \frac{(\triangle c_-)^2_{vac\pm\lambda} - (\triangle c_-)^2_{\pm\lambda}}{(\triangle c_-)^2_{vac\pm\lambda}}. \tag{6.149}$$

On substituting Eqs. (6.147) and (6.148) into Eq. (6.149), we find

$$S_{s\pm\lambda} = 1 - \frac{\frac{2}{\pi}\left(\frac{\kappa(\Delta c_-)^2-4\kappa}{\kappa-(\gamma_c+2r_a)}\right)tan^{-1}\left(\frac{2\lambda}{\gamma_c+2r_a}\right)}{\frac{2}{\pi}\left(\frac{\kappa(\Delta c_-)^2_{vac}-4\kappa}{\kappa-\gamma_c}\right)tan^{-1}\left(\frac{2\lambda}{\gamma_c}\right)+\frac{2}{\pi}\left(\frac{4\kappa-\gamma_c(\Delta c_-)^2_{vac}}{\kappa-\gamma_c}\right)tan^{-1}\left(\frac{2\lambda}{\kappa}\right)}$$

$$-\frac{\frac{2}{\pi}\left(\frac{4\kappa-(\gamma_c+2r_a)(\Delta c_-)^2}{\kappa-(\gamma_c+2r_a)}\right)tan^{-1}\left(\frac{2\lambda}{\kappa}\right)}{\frac{2}{\pi}\left(\frac{\kappa(\Delta c_-)^2_{vac}-4\kappa}{\kappa-\gamma_c}\right)tan^{-1}\left(\frac{2\lambda}{\gamma_c}\right)+\frac{2}{\pi}\left(\frac{4\kappa-\gamma_c(\Delta c_-)^2_{vac}}{\kappa-\gamma_c}\right)tan^{-1}\left(\frac{2\lambda}{\kappa}\right)}. \quad (6.150)$$

We recall that

$$(\triangle c_-)^2 = 2(\triangle a_-)^2, \quad (6.151)$$

$$(\triangle c_-)^2_{vac} = 2(\triangle a_-)^2_{vac}. \quad (6.152)$$

Finally, in view of Eqs. (6.151) and (6.152), we can write Eq. (6.150) as

$$S_{s\pm\lambda} = 1 - \frac{\frac{2}{\pi}\left(\frac{\kappa(\Delta a_-)^2-2\kappa}{\kappa-(\gamma_c+2r_a)}\right)tan^{-1}(\frac{2\lambda}{\gamma_c+2r_a})}{\frac{2}{\pi}\left(\frac{\kappa(\Delta a_-)^2_{vac}-2\kappa}{\kappa-\gamma_c}\right)tan^{-1}(\frac{2\lambda}{\gamma_c})+\frac{2}{\pi}\left(\frac{2\kappa-\gamma_c(\Delta a_-)^2_{vac}}{\kappa-\gamma_c}\right)tan^{-1}(\frac{2\lambda}{\kappa})}$$

$$-\frac{\frac{2}{\pi}\left(\frac{2\kappa-(\gamma_c+2r_a)(\Delta a_-)^2}{\kappa-(\gamma_c+2r_a)}\right)tan^{-1}(\frac{2\lambda}{\kappa})}{\frac{2}{\pi}\left(\frac{\kappa(\Delta a_-)^2_{vac}-2\kappa}{\kappa-\gamma_c}\right)tan^{-1}(\frac{2\lambda}{\gamma_c})+\frac{2}{\pi}\left(\frac{2\kappa-\gamma_c(\Delta a_-)^2_{vac}}{\kappa-\gamma_c}\right)tan^{-1}(\frac{2\lambda}{\kappa})}. \quad (6.153)$$

This expression is identical to the expression in Eq. (5.35). This indicates that the local quadrature squeezing of the pair of superposed two-mode light beams is equal to that of the local quadrature squeezing of the separate two-mode light beams.

7

Conclusion

In this dissertation we have studied the squeezing and statistical properties of the light produced by a three-level laser coupled to a two-mode vacuum reservior. The three-level atoms available in a closed cavity are pumped from the bottom to the top level by electron bombardment. We have carried out our calculation by taking the noise operators associated with the vacuum reservior in arbitrary order.

Applying the solutions of the equations of evolution of the atomic and cavity mode operators, we have calculated the mean photon number, the variance of the photon number, and the quadrature squeezing of the one-mode and two-mode cavity light. We have seen that both light mode a_1 and light mode a_2 exhibit super-Poissonian photon statistics. We have also found that the global mean photon number of the two-mode cavity light is the sum of the global mean photon numbers of light modes a_1 and a_2. Moreover, the global photon number variance, calculated by taking the noise operators in arbitrary order, turns out to be greater than that obtained by putting the noise operators in normal order. It has been also found that as the frequency interval increases the local mean and local variance of the photon number approach to the global mean and global variance of the photon number. In addition, our analysis shows that a large part of the mean and variance of the photon number is confined in a relatively small frequency interval. We have also seen that both the mean and variance of the photon number depend on the total number of atoms.

The two-mode cavity light is in a squeezed state for $r_a > 0$ and the maximum quadrature squeezing occured at $r_a = 0.3$. Moreover, we have shown the maximum global quadrature squeezing to be 37.5 % below the vacuum level for $\kappa = 0.2$ and $\gamma_c = 1.2$. This result happens to be less than that obtained by putting the vacuum noise operators in normal order. It has been also found that unlike the mean and the variance of the photon number, the quadrature squeezing does not depend on the number of atoms. In addition, we have seen that the local quadrature squeezing, in general, is greater than the global quadrature squeezing. We have also observed that the maximum local quadrature squeezing is 40.25 % below the vacuum state level and occurs in the $\pm$ 0.01 frequency interval.

Furthermore, employing the density operator for the superposition of a pair of two-mode cavity light beams, we have calculated the mean and the variance of the photon number as well as the quadrature squeezing. It so happens that the global (local) mean photon number of the superposed two-mode cavity light beams is the sum of the global (local) mean photon number of the component two-mode cavity light beams. On the other hand, the global (local) photon number variance of the superposed two-mode cavity light beams is four times that of the separate two-mode cavity light beams. Furthermore, our results show that the global (local) quadrature squeezing of the superposed two-mode cavity light beams is the same as the global (local) quadrature squeezing of the separate two-mode cavity light beams. It is also found that superposing squeezed two-mode cavity light beams leads to a more bright and squeezed light.

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